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# ON THE KERNEL CURVES ASSOCIATED WITH WALKS IN THE QUARTER PLANE

THOMAS DREYFUS, CHARLOTTE HARDOUIN, JULIEN ROQUES, AND MICHAEL F. SINGER

ABSTRACT. The *kernel method* is an essential tool for the study of generating series of *walks* in the quarter plane. This method involves equating to zero a certain polynomial - the *kernel polynomial* - and using properties of the curve - the *kernel curve* - this defines. In the present paper, we investigate the basic properties of the *kernel curve* (irreducibility, singularities, genus, uniformization, etc).

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## INTRODUCTION

Consider a *walk* with small steps in the positive quadrant  $\mathbb{Z}_{\geq 0}^2 = \{0, 1, 2, \dots\}^2$  starting from  $P_0 := (0, 0)$ , that is a succession of points

$$P_0, P_1, \dots, P_k,$$

where each  $P_n$  lies in the quarter plane, where the moves (or steps)  $P_{n+1} - P_n$  belong to  $\{0, \pm 1\}^2$ , and the probability to move in the direction  $P_{n+1} - P_n = (i, j)$  may be interpreted as some *weight-parameter*  $d_{i,j} \in [0, 1]$ , with  $\sum_{(i,j) \in \{0, \pm 1\}^2} d_{i,j} = 1$ . The step set or the *model* of the *walk* is the set of directions with nonzero *weights*, that is

$$\mathcal{S} = \{(i, j) \in \{0, \pm 1\}^2 \mid d_{i,j} \neq 0\}.$$

The following picture is an example of such path:

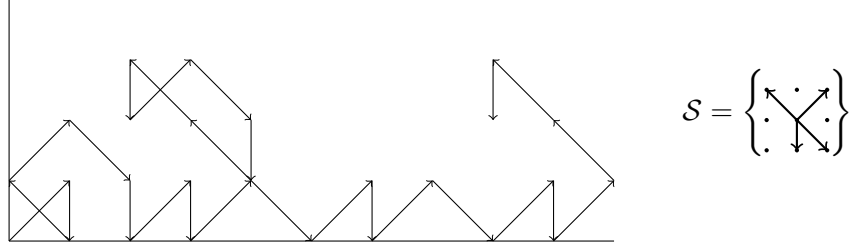
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Such objects are very natural both in combinatorics and probability theory: they are interesting for themselves and also because they are strongly related to other discrete structures, see [BMM10, DW15] and references therein.

If  $d_{0,0} = 0$  and if the nonzero  $d_{i,j}$  all have the same value, we say that the model is *unweighted*.

The *weight of a given walk* is defined to be the product of the *weights* of its component steps. For any  $(i, j) \in \mathbb{Z}_{\geq 0}^2$  and any  $k \in \mathbb{Z}_{\geq 0}$ , we let  $q_{i,j,k}$  be the sum of the *weights* of all *walks* reaching the position  $(i, j)$  from the initial position  $(0, 0)$  after  $k$  steps. We introduce the corresponding trivariate generating series\*

$$Q(x, y, t) := \sum_{i,j,k \geq 0} q_{i,j,k} x^i y^j t^k.$$

The study of the nature of this generating series has attracted the attention of many authors, see for instance [BvHK10, BRS14, BBMR15, BBMR17, BMM10, DHRS18, DHRS20, DR19, DH19, KR12, Mis09, MR09, MM14, Ras12]. The typical questions are: is  $Q(x, y, t)$  rational, algebraic, holonomic, etc? The starting point of most of these works is the following functional equation, see for instance [DHRS20, Lemma 1.1], and [BMM10] for the *unweighted* case

$$K(x, y, t)Q(x, y, t) = xy + K(x, 0, t)Q(x, 0, t) + K(0, y, t)Q(0, y, t) + td_{-1,-1}Q(0, 0, t)$$

where

$$K(x, y, t) = xy(1 - tS(x, y))$$

with

$$S(x, y) = \sum_{(i,j) \in \{0,\pm 1\}^2} d_{i,j} x^i y^j.$$

The polynomial  $K(x, y, t)$  is called the *kernel polynomial* and is the main character of the *kernel method*.

Roughly speaking, the first step of the *kernel method* consists in “eliminating” the left hand side of the above functional equation by restricting our attention to the  $(x, y)$  such that  $K(x, y, t) = 0$ . The set  $E_t$  made of the  $(x, y)$  such that  $K(x, y, t) = 0$  is called the *kernel curve*:

$$E_t = \{(x, y) \in \mathbb{C} \times \mathbb{C} \mid K(x, y, t) = 0\}.$$

Thus, for  $(x, y) \in E_t$ , one has

$$(1) \quad 0 = xy + K(x, 0, t)Q(x, 0, t) + K(0, y, t)Q(0, y, t) + td_{-1,-1}Q(0, 0, t),$$

provided that the various series can be evaluated at the given points.

\*In several papers it is not assumed that  $\sum_{i,j} d_{i,j} = 1$ . But after a rescaling of the  $t$  variable, we may always reduce to the case  $\sum_{i,j} d_{i,j} = 1$ .

The second step of the *kernel method* is to exploit certain involutive birational transformations  $\iota_1, \iota_2$  (they are called  $\zeta, \eta$  in [FIM17]) of the *kernel curve*  $E_t$  of the form

$$\iota_1(x, y) = (x, y') \text{ and } \iota_2(x, y) = (x', y)$$

in order to deduce from (1) some functional equations for  $Q(x, 0, t)$  and  $Q(0, y, t)$ . Hence  $\iota_1$  and  $\iota_2$  switch the roots of the degree two polynomials  $y \mapsto K(x, y, t)$  and  $x \mapsto K(x, y, t)$  respectively. Concretely, the birational transformations  $\iota_1, \iota_2$  are induced by restriction to the curve of the involutive birational transformations  $i_1, i_2$  of  $\mathbb{C}^2$  given by

$$i_1(x, y) = \left( x, \frac{A_{-1}(x)}{A_1(x)y} \right) \text{ and } i_2(x, y) = \left( \frac{B_{-1}(y)}{B_1(y)x}, y \right)$$

where the  $A_i(x) \in x^{-1}\mathbb{Q}[x]$  and the  $B_i(y) \in y^{-1}\mathbb{Q}[y]$  are defined by

$$S(x, y) = A_{-1}(x)\frac{1}{y} + A_0(x) + A_1(x)y = B_{-1}(y)\frac{1}{x} + B_0(y) + B_1(y)x,$$

see [BMM10, Section 3], [KY15, Section 3] or [FIM17]. These  $i_1$  and  $i_2$  are the generators of the group of the *walk*; see [BMM10] for details. Note that although  $i_1$  and  $i_2$  do not depend on  $t$ , the group generated by the induced involutions  $\iota_1$  and  $\iota_2$  may depend on  $t$ , since the order of  $\iota_2 \circ \iota_1$  may depend upon  $t$ , see Remark 4.13.

The third step of the *kernel method* is to use the above mentioned functional equations of  $Q(x, 0, t)$  and  $Q(0, y, t)$  to continue these series as multivalued meromorphic functions. To perform this step, we need an explicit uniformization of the *kernel curve*.

The aim of the present paper is to study the *kernel curve*  $E_t$  and the birational transformations  $\iota_1, \iota_2$ . Note that a similar study has been done in the case  $t = 1$  in [FIM17] and in the *unweighted* case in [KR12]. The goal of the present paper is to extend these works to the *weighted* case when  $t \in ]0, 1[$  is transcendental over  $\mathbb{Q}(d_{i,j})$ . Although many results are similar to [FIM17], the proofs are different. The assumptions we make on  $t$  are crucial in many parts of the proof and it is not clear how the proofs of [FIM17] exactly pass to this context.

We could expect to have classification of the geometric properties of  $\overline{E_t}$  involving configurations of *weights* independent of  $t$ . Hopefully, this paper has been followed by [DR19] where the case for general  $t \in ]0, 1[$  has been considered. The proofs of the latter paper use continuity arguments with respect the parameter  $t$  that permit to deduce many results for algebraic values of  $t$ . Such reasoning needs to be very cautious, and it is not trivial to deduce the results for general  $t \in ]0, 1[$  from the  $t = 1$  case. We will mention explicitly every time if the results are correct for arbitrary values of  $t \in ]0, 1[$ .

The paper is organized as follows. In Section 1, we describe the *nondegenerate models of walks*. In Section 2, we determine the singularities and the genus of the *kernel curve*. In Section 3, we establish the basic properties of  $\iota_1$  and  $\iota_2$ . Finally, in Section 4, we give an explicit uniformization of the *kernel curve*.

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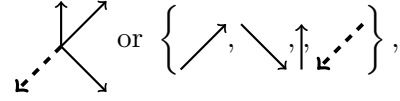
## 1. NONDEGENERATE WALKS

From now on, we fix  $t \in ]0, 1[$ , that is transcendental over the field  $\mathbb{Q}(d_{i,j})$ . We start by recalling the notion of *degenerate walks introduced in [FIM17]*.

**Definition 1.1.** A *model of walk* is called *degenerate* if one of the following holds:

- $K(x, y, t)$  is reducible as an element of the polynomial ring  $\mathbb{C}[x, y]$ ,
- $K(x, y, t)$  has  $x$ -degree less than or equal to 1,
- $K(x, y, t)$  has  $y$ -degree less than or equal to 1.

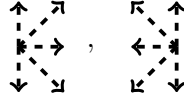
In what follows we will sometimes represent a *model of walks* with arrows. We will also use dashed arrows for a family of models. For instance, the family of models represented by



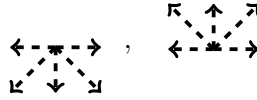
correspond to models with  $d_{1,1}, d_{1,-1}, d_{0,1} \neq 0$ ,  $d_{1,0} = d_{0,-1} = d_{-1,1} = d_{-1,0} = 0$ , and where nothing is assumed on  $d_{-1,-1}$  and  $d_{0,0}$ . In the following results, the behavior of the *kernel curve* never depends on  $d_{0,0}$ . This is the reason why, to reduce the amount of notations, we have decided to not associate an arrow to  $d_{0,0}$ . The following result is the analog of [FIM17, Lemma 2.3.2], that focuses on the case  $t = 1$ . Our proof differs from the proof of [FIM17, Lemma 2.3.2], which only considered factorization over  $\mathbb{R}[x, y]$ , while in this paper, we need to prove the absence of factorization over  $\mathbb{C}[x, y]$ .

**Proposition 1.2.** A *model of walk* is *degenerate* if and only if at least one of the following holds:

- (1) There exists  $i \in \{-1, 1\}$  such that  $d_{i,-1} = d_{i,0} = d_{i,1} = 0$ . This corresponds to the following families of models of walks



- (2) There exists  $j \in \{-1, 1\}$  such that  $d_{-1,j} = d_{0,j} = d_{1,j} = 0$ . This corresponds to the following families of models of walks



- (3) All the weights are 0 except maybe  $\{d_{1,1}, d_{0,0}, d_{-1,-1}\}$  or  $\{d_{-1,1}, d_{0,0}, d_{1,-1}\}$ . This corresponds to the following families of models of walks



*Proof.* This proof is organized as follows. We begin by showing that (1) (resp. (2)) corresponds to  $K(x, y, t)$  having  $x$ -degree  $\leq 1$  or  $x$ -valuation  $\geq 1$  (resp.  $y$ -degree  $\leq 1$  or  $y$ -valuation  $\geq 1$ ). In these cases, the *model of the walk* is clearly *degenerate*. Assuming (1) and (2) do not hold, we then show that (3) holds if and only if  $K(x, y, t)$  is reducible.

Cases (1) and (2). It is clear that  $K(x, y, t)$  has  $x$ -degree  $\leq 1$  if and only if  $d_{1,-1} = d_{1,0} = d_{1,1} = 0$ . Similarly,  $K(x, y, t)$  has  $y$ -degree  $\leq 1$  if and only if we have  $d_{-1,1} = d_{0,1} = d_{1,1} = 0$ . Furthermore,  $d_{-1,-1} = d_{-1,0} = d_{-1,1} = 0$  if and only if  $K(x, y, t)$  has  $x$ -valuation  $\geq 1$ . Similarly,  $d_{-1,-1} = d_{0,-1} = d_{1,-1} = 0$  if and only if  $K(x, y, t)$  has  $y$ -valuation  $\geq 1$ . In these cases, *the model of the walk* is clearly *degenerate*.

Case (3). We now assume that cases (1) and (2) do not hold. This implies that the model belongs to the family of models  $\left\{ \begin{array}{c} \nearrow \\ \nwarrow \end{array} \right\}$  if and only if it belongs to the family of models  $\left\{ \begin{array}{c} \nearrow \\ \swarrow \end{array} \right\}$ . The same holds for the anti-diagonal configuration. If the model belongs to the family of models  $\left\{ \begin{array}{c} \nearrow \\ \swarrow \end{array} \right\}$ , then the *kernel*

$$K(x, y, t) = -d_{-1,-1}t + xy - d_{0,0}txy - d_{1,1}tx^2y^2 \in \mathbb{C}[xy]$$

is a degree two polynomial in  $xy$ . Thus it may be factorized in the following form  $K(x, y, t) = -d_{1,1}t(xy - \alpha)(xy - \beta)$  for some  $\alpha, \beta \in \mathbb{C}$ . If the model belongs to the family of models  $\left\{ \begin{array}{c} \nwarrow \\ \searrow \end{array} \right\}$ , then

$$K(x, y, t) = -d_{-1,1}ty^2 + xy - d_{0,0}txy - d_{1,-1}tx^2.$$

In this situation,  $K(x, y, t)y^{-2} \in \mathbb{C}[x/y]$  may be factorized in the ring  $\mathbb{C}[x/y]$ , proving that  $K(x, y, t)$  may be factorized in  $\mathbb{C}[x, y]$  as well.

Conversely, let us assume that *the model of the walk* is *degenerate*. Recall that we have assumed that cases (1) and (2) do not hold, so  $K(x, y, t)$  has  $x$ - and  $y$ -degree two,  $x$ - and  $y$ -valuation 0, and is reducible. We have to prove that the model belongs to one of the family of models  $\left\{ \begin{array}{c} \nearrow \\ \swarrow \end{array} \right\}$  or  $\left\{ \begin{array}{c} \nwarrow \\ \searrow \end{array} \right\}$ . Let us write a factorization

$$K(x, y, t) = -f_1(x, y)f_2(x, y),$$

with  $f_1(x, y), f_2(x, y) \in \mathbb{C}[x, y]$  not constant. Let us now prove several lemmas on the the polynomials  $f_1(x, y), f_2(x, y) \in \mathbb{C}[x, y]$ .

**Lemma 1.3.** *Both  $f_1(x, y)$  and  $f_2(x, y)$  have bidegree (1, 1).*

*Proof of Lemma 1.3.* Suppose to the contrary that  $f_1(x, y)$  or  $f_2(x, y)$  does not have bidegree (1, 1). Since  $K$  is of bidegree at most (2, 2) then at least one of the  $f_i$ 's has degree 0 in  $x$  or  $y$ . Up to interchange of  $x$  and  $y$  and  $f_1$  and  $f_2$ , we may assume that  $f_1(x, y)$  has  $y$ -degree 0 and we denote it by  $f_1(x)$ . Since we are not in Cases (1) and (2) of Proposition 1.2, the polynomials  $d_{-1,-1}t + d_{0,-1}tx + d_{1,-1}tx^2$  and  $d_{-1,0}t + (d_{0,0}t - 1)x + d_{1,0}tx^2$  are nonzero. By  $K(x, y, t) = -f_1(x)f_2(x, y)$ , we find in particular that  $f_1(x)$  is a common factor of the nonzero polynomials  $d_{-1,-1}t + d_{0,-1}tx + d_{1,-1}tx^2$  and  $d_{-1,0}t + (d_{0,0}t - 1)x + d_{1,0}tx^2$ . Since  $t$  is nonzero, we find that the roots of  $d_{-1,-1}t + d_{0,-1}tx + d_{1,-1}tx^2 = 0$  are algebraic over  $\mathbb{Q}(d_{i,j})$ . On the other hand, since  $t$  is transcendental over  $\mathbb{Q}(d_{i,j})$ , if  $x$  is a root of  $d_{-1,0}t + (d_{0,0}t - 1)x + d_{1,0}tx^2 = 0$  that is algebraic over  $\mathbb{Q}(d_{i,j})$ , then the constant term in  $t$  has to be zero, proving that  $x = 0$ . Therefore, they are polynomials with only zero

as a potential common roots. So the only potential root of  $f_1(x)$  is zero. This means that either  $f_1(x)$  has degree 0, i.e.  $f_1(x) \in \mathbb{C}$ , or  $x$  divides  $f_1(x)$ . In the latter case,  $x$  divides  $K(x, y, t)$ , and we are in Case 1. In both cases, this is a contradiction and proves the lemma.  $\square$

**Lemma 1.4.** *The polynomials  $f_1(x, y)$  and  $f_2(x, y)$  are irreducible in the ring  $\mathbb{C}[x, y]$ .*

*Proof of Lemma 1.4.* To the contrary, suppose that we can find a factorization  $f_1(x, y) = (ax - b)(cy - d)$  for some  $a, b, c, d \in \mathbb{C}$ . Since  $f_1(x, y)$  has bidegree  $(1, 1)$ , we have  $ac \neq 0$ . We then have that

$$0 = K(b/a, y, t) = \frac{b}{a}y - t(\tilde{A}_{-1}(\frac{b}{a}) + \tilde{A}_0(\frac{b}{a})y + \tilde{A}_1(\frac{b}{a})y^2)$$

where  $\tilde{A}_i = xA_i \in \mathbb{Q}[x]$ . Note that  $\tilde{A}_1(x)$  is nonzero because  $K(x, y, t)$  has bidegree  $(2, 2)$ . Equating the  $y^2$ -terms we find that  $\tilde{A}_1(\frac{b}{a}) = 0$  so  $\frac{b}{a}$  is algebraic over  $\mathbb{Q}(d_{i,j})$ . Equating the  $y$ -terms, we obtain that  $\frac{b}{a} - t\tilde{A}_0(\frac{b}{a}) = 0$ . Using the fact that  $t$  is transcendental over  $\mathbb{Q}(d_{i,j})$  and  $\frac{b}{a}$  is algebraic over  $\mathbb{Q}(d_{i,j})$ , we deduce  $\frac{b}{a} = 0$ . Therefore  $b = 0$ . This contradicts the fact that  $K$  has  $x$ -valuation 0. A similar argument shows that  $f_2(x, y)$  is irreducible.  $\square$

**Lemma 1.5.** *Let  $\bar{f}_i(x, y)$  denote the polynomial whose coefficients are the complex conjugates of those of  $f_i(x, y)$ . We may reduce to the case where one of the following cases hold:*

- there exists  $\epsilon \in \{\pm 1\}$  such that  $\bar{f}_1(x, y) = \epsilon f_2(x, y)$ ,
- $\bar{f}_1(x, y) = f_1(x, y) \in \mathbb{R}[x, y]$  and  $\bar{f}_2(x, y) = f_2(x, y) \in \mathbb{R}[x, y]$ .

*Proof of Lemma 1.5.* Unique factorization of polynomials implies that since  $-K(x, y, t) = f_1(x, y)f_2(x, y) = \bar{f}_1(x, y)\bar{f}_2(x, y)$ , there exists  $\lambda \in \mathbb{C}^*$  such that

- either  $\bar{f}_1(x, y) = \lambda f_2(x, y)$  and  $\bar{f}_2(x, y) = \lambda^{-1} f_1(x, y)$ ;
- or  $\bar{f}_1(x, y) = \lambda f_1(x, y)$  and  $\bar{f}_2(x, y) = \lambda^{-1} f_2(x, y)$ .

In the former case, we have  $f_1(x, y) = \bar{\lambda} \bar{f}_2(x, y) = \bar{\lambda} \lambda^{-1} f_1(x, y)$  and so  $\bar{\lambda} \lambda^{-1} = 1$ . This implies that  $\lambda$  is real and replacing  $\bar{f}_1(x, y)$  by  $|\lambda|^{-1/2} f_1(x, y)$  and  $f_2(x, y)$  by  $|\lambda|^{1/2} f_2(x, y)$ , we can assume that either  $\bar{f}_1(x, y) = f_2(x, y)$  and  $\bar{f}_2(x, y) = f_1(x, y)$  or  $\bar{f}_1(x, y) = -f_2(x, y)$  and  $\bar{f}_2(x, y) = -f_1(x, y)$ .

A similar computation in the latter case shows that  $|\lambda| = 1$ . Letting  $\mu$  be a square root of  $\lambda$  we have  $\mu^{-1} = \bar{\mu}$  so  $\lambda = \mu/\bar{\mu}$ . Replacing  $\bar{f}_1(x, y)$  by  $\mu f_1(x, y)$  and  $f_2(x, y)$  by  $\bar{\mu} f_2(x, y)$ , we can assume that  $\bar{f}_1(x, y) = f_1(x, y)$  and  $\bar{f}_2(x, y) = f_2(x, y)$ .  $\square$

Let us continue the proof of Proposition 1.2. For  $i = 1, 2$ , let us write

$$f_i(x, y) = (\alpha_{i,4}x + \alpha_{i,3})y + (\alpha_{i,2}x + \alpha_{i,1}),$$

with  $\alpha_{i,j} \in \mathbb{C}$ . Equating the terms in  $x^i y^j$  with  $-1 \leq i, j \leq 1$ , in  $f_1(x, y)f_2(x, y) = -K(x, y, t)$ , we find (recall that  $d_{i,j} \in [0, 1]$ ,  $t \in ]0, 1[$ )

term	coefficient in $f_1(x, y)f_2(x, y)$	coefficient in $-K(x, y, t)$
1	$\alpha_{1,1}\alpha_{2,1}$	$d_{-1,-1}t \geq 0$
$x$	$\alpha_{1,2}\alpha_{2,1} + \alpha_{1,1}\alpha_{2,2}$	$d_{0,-1}t \geq 0$
$x^2$	$\alpha_{1,2}\alpha_{2,2}$	$d_{1,-1}t \geq 0$
$y$	$\alpha_{1,3}\alpha_{2,1} + \alpha_{1,1}\alpha_{2,3}$	$d_{-1,0}t \geq 0$
$xy$	$\alpha_{1,4}\alpha_{2,1} + \alpha_{1,3}\alpha_{2,2} + \alpha_{1,2}\alpha_{2,3} + \alpha_{1,1}\alpha_{2,4}$	$d_{0,0}t - 1 < 0$
$x^2y$	$\alpha_{1,4}\alpha_{2,2} + \alpha_{1,2}\alpha_{2,4}$	$d_{1,0}t \geq 0$
$y^2$	$\alpha_{1,3}\alpha_{2,3}$	$d_{-1,1}t \geq 0$
$xy^2$	$\alpha_{1,4}\alpha_{2,3} + \alpha_{1,3}\alpha_{2,4}$	$d_{0,1}t \geq 0$
$x^2y^2$	$\alpha_{1,4}\alpha_{2,4}$	$d_{1,1}t \geq 0$

Let us treat separately two cases.

**Case 1:**  $f_1(x, y), f_2(x, y) \notin \mathbb{R}[x, y]$ . So, in this case we have either  $\overline{f_1}(x, y) = f_2(x, y)$  or  $\overline{f_1}(x, y) = -f_2(x, y)$ .

Let us first assume that  $\overline{f_1}(x, y) = f_2(x, y)$ . Then, evaluating the equality  $K(x, y, t) = -f_1(x, y)f_2(x, y)$  at  $x = y = 1$ , we get the following equality  $K(1, 1, t) = -f_1(1, 1)f_2(1, 1) = -|f_1(1, 1)|^2$ . But this is impossible because the left-hand term  $K(1, 1, t) = 1 - t \sum_{i,j \in \{-1, 0, 1\}^2} d_{i,j} = 1 - t$  is  $> 0$  whereas the right-hand term  $-|f_1(1, 1)|^2$  is  $\leq 0$ .

Let us now assume that  $\overline{f_1}(x, y) = -f_2(x, y)$ . Equating the constant terms in the equality  $f_1(x, y)f_2(x, y) = -K(x, y, t)$ , we get  $-|\alpha_{1,1}|^2 = d_{-1,-1}t$ , so  $\alpha_{1,1} = \alpha_{2,1} = d_{-1,-1} = 0$ . Equating the coefficients of  $x^2$  in the equality  $f_1(x, y)f_2(x, y) = -K(x, y, t)$ , we get  $-|\alpha_{1,2}|^2 = d_{1,-1}t$ , so  $\alpha_{1,2} = \alpha_{2,2} = d_{1,-1} = 0$ . It follows that the  $y$ -valuation of  $f_1(x, y)f_2(x, y) = -K(x, y, t)$  is  $\geq 2$ , whence a contradiction.

**Case 2:**  $f_1(x, y), f_2(x, y) \in \mathbb{R}[x, y]$ . We first claim that, after possibly replacing  $f_1(x, y)$  by  $-f_1(x, y)$  and  $f_2(x, y)$  by  $-f_2(x, y)$ , we may assume that  $\alpha_{1,4}, \alpha_{2,4}, \alpha_{1,3}, \alpha_{2,3} \geq 0$ .

Let us first assume that  $\alpha_{1,4}\alpha_{2,4} \neq 0$ . Since  $\alpha_{1,4}\alpha_{2,4} = d_{1,1}t \geq 0$ , we find that  $\alpha_{1,4}, \alpha_{2,4}$  belong simultaneously to  $\mathbb{R}_{>0}$  or  $\mathbb{R}_{<0}$ . After possibly replacing  $f_1(x, y)$  by  $-f_1(x, y)$  and  $f_2(x, y)$  by  $-f_2(x, y)$ , we may assume that  $\alpha_{1,4}, \alpha_{2,4} > 0$ . Since  $\alpha_{1,3}\alpha_{2,3} = d_{-1,1}t \geq 0$ , we have that  $\alpha_{1,3}, \alpha_{2,3}$  belong simultaneously to  $\mathbb{R}_{\geq 0}$  or  $\mathbb{R}_{\leq 0}$ . Then, the equality  $\alpha_{1,4}\alpha_{2,3} + \alpha_{1,3}\alpha_{2,4} = d_{0,1}t \geq 0$  implies that  $\alpha_{1,3}, \alpha_{2,3} \geq 0$ .

We can argue similarly in the case  $\alpha_{1,3}\alpha_{2,3} \neq 0$ .

It remains to consider the case  $\alpha_{1,4}\alpha_{2,4} = \alpha_{1,3}\alpha_{2,3} = 0$ . After possibly replacing  $f_1(x, y)$  by  $-f_1(x, y)$  and  $f_2(x, y)$  by  $-f_2(x, y)$ , we may assume that  $\alpha_{1,4}, \alpha_{2,4} \geq 0$ . The case  $\alpha_{1,4} = \alpha_{1,3} = 0$  is impossible because, otherwise, we would have  $d_{1,1} = d_{-1,1} = d_{0,1} = 0$ , which is excluded. Similarly, the case  $\alpha_{2,4} = \alpha_{2,3} = 0$  is impossible. So, we are left with the cases  $\alpha_{1,4} = \alpha_{2,3} = 0$  or  $\alpha_{2,4} = \alpha_{1,3} = 0$ . In both cases, the equality  $\alpha_{1,4}\alpha_{2,3} + \alpha_{1,3}\alpha_{2,4} = d_{0,1}t \geq 0$  with  $\alpha_{1,4}, \alpha_{2,4} \geq 0$ , implies that  $\alpha_{1,4}, \alpha_{2,4}, \alpha_{1,3}, \alpha_{2,3} \geq 0$ .

Arguing as above, we see that  $\alpha_{1,2}, \alpha_{2,2}, \alpha_{1,1}, \alpha_{2,1}$  all belong to  $\mathbb{R}_{\geq 0}$  or all belong to  $\mathbb{R}_{\leq 0}$ . Using the equation of the  $xy$ -coefficients, we find that  $\alpha_{1,2}, \alpha_{2,2}, \alpha_{1,1}, \alpha_{2,1}$  are all in  $\mathbb{R}_{\leq 0}$ .



Now, equating the coefficients of  $x^2y$  in the equality  $f_1(x, y)f_2(x, y) = -K(x, y, t)$  we get  $\alpha_{1,4}\alpha_{2,2} + \alpha_{1,2}\alpha_{2,4} = d_{1,0}t$ . Using the fact that  $\alpha_{1,4}\alpha_{2,2}, \alpha_{1,2}\alpha_{2,4} \leq 0$  and that  $d_{1,0}t \geq 0$ , we get  $\alpha_{1,4}\alpha_{2,2} = \alpha_{1,2}\alpha_{2,4} = d_{1,0}t = 0$ . Similarly, using the coefficients of  $y$ , we get  $\alpha_{1,3}\alpha_{2,1} = \alpha_{1,1}\alpha_{2,3} = d_{-1,0} = 0$ .

So, we have

$$\alpha_{1,4}\alpha_{2,2} = \alpha_{1,2}\alpha_{2,4} = \alpha_{1,3}\alpha_{2,1} = \alpha_{1,1}\alpha_{2,3} = 0.$$

The fact that  $K(x, y, t)$  has  $x$ - and  $y$ -degree 2 and  $x$ - and  $y$ -valuation 0 implies that, for any  $i \in \{1, 2\}$ , none of the vectors  $(\alpha_{i,4}, \alpha_{i,3}), (\alpha_{i,2}, \alpha_{i,1}), (\alpha_{i,4}, \alpha_{i,2})$  and  $(\alpha_{i,3}, \alpha_{i,1})$  is  $(0, 0)$ . Since  $\alpha_{1,4}\alpha_{2,2} = 0$ , we have  $\alpha_{1,4} = 0$  or  $\alpha_{2,2} = 0$ . If  $\alpha_{1,4} = 0$ , from what precedes, we find

$$\alpha_{1,4} = \alpha_{2,4} = \alpha_{2,1} = \alpha_{1,1} = 0.$$

If  $\alpha_{2,2} = 0$  we obtain

$$\alpha_{2,2} = \alpha_{1,2} = \alpha_{1,3} = \alpha_{2,3} = 0.$$

In the first case, the model belongs to the family of models  $\left\{ \begin{array}{l} \swarrow, \searrow \\ \nearrow, \nwarrow \end{array} \right\}$ . In the second case, we find that the model belongs to the family of models  $\left\{ \begin{array}{l} \swarrow, \nwarrow \\ \nearrow, \searrow \end{array} \right\}$ . This completes the proof.  $\square$

*Remark 1.6.* The fact  $d_{i,j} \in [0, 1]$  are probabilities is crucial in the proof of Proposition 1.2. We do not expect this result to be correct for general  $d_{i,j} \in \mathbb{C}$ .

*Remark 1.7.* The “degenerate models of walks” are called “singular” by certain authors, e.g., in [FIM99, FIM17]. Note also that, in [KR12], “singular walks” has a different meaning and refers to *models of walks* such that the associated *kernel* defines a genus zero curve.

*Remark 1.8.* In [DR19, Proposition 3], the authors show that Proposition 1.2 extends mutatis mutandis to the case when  $t \in ]0, 1[$  is algebraic over  $\mathbb{Q}(d_{i,j})$ . Their proof relies on Proposition 1.2 and uses a continuity argument of the parameter  $t$  to deduce that Proposition 1.2 stays correct for general values of  $t \in ]0, 1[$ .

From now on, we will only consider *nondegenerate models of walks*. In terms of *models of walks*, this only discards one dimensional problems and *models of walks* in the half-plane restricted to the quarter plane that are easier to study, as explained in [BMM10, Section 2.1].

## 2. SINGULARITIES AND GENUS OF THE KERNEL CURVE

The *kernel curve*  $E_t$  is the complex affine algebraic curve defined by

$$E_t = \{(x, y) \in \mathbb{C} \times \mathbb{C} \mid K(x, y, t) = 0\}.$$

We shall now consider a compactification of this curve. We let  $\mathbb{P}^1(\mathbb{C})$  be the complex projective line, which is the quotient of  $(\mathbb{C} \times \mathbb{C}) \setminus \{(0, 0)\}$  by the equivalence relation  $\sim$  defined by

$$(x_0, x_1) \sim (x'_0, x'_1) \Leftrightarrow \exists \lambda \in \mathbb{C}^*, (x'_0, x'_1) = \lambda(x_0, x_1).$$

The equivalence class of  $(x_0, x_1) \in (\mathbb{C} \times \mathbb{C}) \setminus \{(0, 0)\}$  is denoted by  $[x_0 : x_1] \in \mathbb{P}^1(\mathbb{C})$ . The map  $x \mapsto [x : 1]$  embeds  $\mathbb{C}$  inside  $\mathbb{P}^1(\mathbb{C})$ . The latter map is not surjective: its image

is  $\mathbb{P}^1(\mathbb{C}) \setminus \{[1 : 0]\}$ ; the missing point  $[1 : 0]$  is usually denoted by  $\infty$ . Now, we embed  $E_t$  inside  $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$  via  $(x, y) \mapsto ([x : 1], [y : 1])$ . The *kernel curve*  $\overline{E}_t$  is the closure of this embedding of  $E_t$ . In other words, the *kernel curve*  $\overline{E}_t$  is the algebraic curve defined by

$$\overline{E}_t = \{([x_0 : x_1], [y_0 : y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \mid \overline{K}(x_0, x_1, y_0, y_1, t) = 0\}$$

where  $\overline{K}(x_0, x_1, y_0, y_1, t)$  is the following bihomogeneous polynomial

$$(2.1) \quad \overline{K}(x_0, x_1, y_0, y_1, t) = x_1^2 y_1^2 K\left(\frac{x_0}{x_1}, \frac{y_0}{y_1}, t\right) = x_0 x_1 y_0 y_1 - t \sum_{i,j=0}^2 d_{i-1,j-1} x_0^i x_1^{2-i} y_0^j y_1^{2-j}.$$

Although it may seem more natural to take the closure of  $\overline{E}_t$  in  $\mathbb{P}^2(\mathbb{C})$ , the above definition allows us to extend the involutions of  $E_t$  of Section 3 in a natural way as well as allowing us to avoid unnecessary singularities.

We shall now study the singularities and compute the genus of  $\overline{E}_t$ . Recall that since the *model of walk* under consideration is *nondegenerate*, the polynomial  $K(x, y, t)$  is irreducible. We recall that by definition  $P = ([a : b], [c : d]) \in \overline{E}_t$  is called a singularity of the irreducible kernel  $\overline{E}_t$  if

$$\frac{\partial \overline{K}(a, b, c, d, t)}{\partial x_0} = \frac{\partial \overline{K}(a, b, c, d, t)}{\partial x_1} = \frac{\partial \overline{K}(a, b, c, d, t)}{\partial y_0} = \frac{\partial \overline{K}(a, b, c, d, t)}{\partial y_1} = 0.$$

Note that one can check this condition in any affine neighborhood of a point. For example, if  $b, d \neq 0$ , the bihomogeneity of  $\overline{K}$  implies

$$\begin{aligned} 0 = 2\overline{K}(a/b, 1, c/d, 1, t) &= \frac{a}{b} \frac{\partial \overline{K}(a/b, 1, c, /d, 1, t)}{\partial x_0} + \frac{\partial \overline{K}(a/b, 1, c, /d, 1, t)}{\partial x_1} \\ &= \frac{c}{d} \frac{\partial \overline{K}(a/b, 1, c, /d, 1, t)}{\partial y_0} + \frac{\partial \overline{K}(a/b, 1, c, /d, 1, t)}{\partial y_1}. \end{aligned}$$

Therefore the point  $P = ([a : b], [c : d]) \in \overline{E}_t$  is a singular point if and only if

$$\frac{\partial \overline{K}(a/b, 1, c/d, 1, t)}{\partial x_0} = \frac{\partial \overline{K}(a/b, 1, c/d, 1, t)}{\partial y_0} = 0.$$

If  $P = ([a : b], [c : d]) \in \overline{E}_t$  is not a singularity of  $\overline{E}_t$ , then it is called a smooth point of  $\overline{E}_t$ .

We also recall that  $\overline{E}_t$  is called singular if it has at least one singular point. Otherwise, we say that  $\overline{E}_t$  is nonsingular or smooth.

The genus of an algebraic curve is a classical notion in algebraic geometry. It is a nonnegative integer that we may attach to a curve, see [Ful84, Section 8.3] for a definition. The study of the genus of  $\overline{E}_t$  has been considered in [FIM17]. Proposition 2.1 below shows that the smoothness of  $\overline{E}_t$  is intimately related to the value of the genus of  $\overline{E}_t$ . We define the genus of the *weighted model of walk*, as the genus of its *kernel curve*  $\overline{E}_t$ .

Remind the following notations from the introduction :

$$\begin{aligned} K(x, y, t) &= xy - txA_{-1}(x) - txA_0(x)y - txA_1(x)y^2, \\ &= xy - tyB_{-1}(y) - tyB_0(y)x - tyB_1(y)x^2, \end{aligned}$$

where  $x A_i(x) \in \mathbb{Q}[x]$  and  $y B_i(y) \in \mathbb{Q}[y]$ . Then we may write

$$\begin{aligned} \overline{K}(x_0, x_1, y_0, y_1, t) &= \overline{C}_1(x_0, x_1, t) y_1^2 + \overline{B}_1(x_0, x_1, t) y_0 y_1 + \overline{A}_1(x_0, x_1, t) y_0^2 \\ &= \overline{C}_2(y_0, y_1, t) x_1^2 + \overline{B}_2(y_0, y_1, t) x_0 x_1 + \overline{A}_2(y_0, y_1, t) x_0^2. \end{aligned}$$

For any  $[x_0 : x_1]$  and  $[y_0 : y_1]$  in  $\mathbb{P}^1(\mathbb{C})$ , we denote by  $\Delta_1([x_0 : x_1])$  and  $\Delta_2([y_0 : y_1])$  the discriminants of the degree two homogeneous polynomials given by  $y \mapsto \overline{K}(x_0, x_1, y, t)$  and  $x \mapsto \overline{K}(x, y_0, y_1, t)$  respectively, i.e.

$$\begin{aligned} \Delta_1([x_0 : x_1]) &= \overline{B}_1(x_0, x_1, t)^2 - 4\overline{A}_1(x_0, x_1, t)\overline{C}_1(x_0, x_1, t) \\ &= (x_0 x_1 - t^2) \left( (d_{-1,0} x_1^2 - \frac{1}{t} x_0 x_1 + d_{0,0} x_0 x_1 + d_{1,0} x_0^2)^2 \right. \\ &\quad \left. - 4(d_{-1,1} x_1^2 + d_{0,1} x_0 x_1 + d_{1,1} x_0^2)(d_{-1,-1} x_1^2 + d_{0,-1} x_0 x_1 + d_{1,-1} x_0^2) \right) \end{aligned}$$

and

$$\begin{aligned} \Delta_2([y_0 : y_1]) &= \overline{B}_2(y_0, y_1, t)^2 - 4\overline{A}_2(y_0, y_1, t)\overline{C}_2(y_0, y_1, t) \\ &= t^2 \left( (d_{0,-1} y_1^2 - \frac{1}{t} y_0 y_1 + d_{0,0} y_0 y_1 + d_{0,1} y_0^2)^2 \right. \\ &\quad \left. - 4(d_{1,-1} y_1^2 + d_{1,0} y_0 y_1 + d_{1,1} y_0^2)(d_{-1,-1} y_1^2 + d_{-1,0} y_0 y_1 + d_{-1,1} y_0^2) \right). \end{aligned}$$

As we will see in the sequel, see Remark 2.3,  $\Delta_1([x_0 : x_1])$  has a double root if and only if  $\Delta_2([y_0 : y_1])$  has a double root.

**Proposition 2.1.** *For nondegenerate models, the following facts are equivalent:*

- (1) *the curve  $\overline{E}_t$  is a genus zero curve;*
- (2) *the curve  $\overline{E}_t$  is singular;*
- (3) *the curve  $\overline{E}_t$  has exactly one singularity  $\Omega \in \overline{E}_t$ ;*
- (4) *there exists  $([a : b], [c : d]) \in \overline{E}_t$  such that the discriminants  $\Delta_1([x_0 : x_1])$  and  $\Delta_2([y_0 : y_1])$  have a root  $[a : b] \in \mathbb{P}^1(\mathbb{C})$  and  $[c : d] \in \mathbb{P}^1(\mathbb{C})$  respectively;*
- (5) *there exists  $([a : b], [c : d]) \in \overline{E}_t$  such that the discriminants  $\Delta_1([x_0 : x_1])$  and  $\Delta_2([y_0 : y_1])$  have a double root  $[a : b] \in \mathbb{P}^1(\mathbb{C})$  and  $[c : d] \in \mathbb{P}^1(\mathbb{C})$  respectively.*

*If these properties are satisfied, then the singular point is  $\Omega = ([a : b], [c : d])$  where  $[a : b] \in \mathbb{P}^1(\mathbb{C})$  is a double root of  $\Delta_1([x_0 : x_1])$  and  $[c : d] \in \mathbb{P}^1(\mathbb{C})$  is a double root of  $\Delta_2([y_0 : y_1])$ . If the previous properties are not satisfied, then  $\overline{E}_t$  is a smooth curve of genus one.*

*Proof.* By [Dui10, Section 3.3.1], the following formula gives the genus of  $\overline{E}_t$ ,

$$(2.2) \quad g(\overline{E}_t) = 1 - \sum_{P \in \text{Sing}} \frac{m(P)(m(P) - 1)}{2},$$

where  $m(P)$  is a positive integer standing for the multiplicity of a point  $P$ , that is, some partial derivative of order  $m(P)$  does not vanish while for every  $\ell < m(P)$ , the partial derivatives of order  $\ell$  vanish at  $P$ . Since the genus is a nonnegative integer, the above formula shows that  $g(\overline{E}_t)$  is equal to 0 or 1. This formula shows more precisely that  $\overline{E}_t$  is smooth if and only if  $g(\overline{E}_t) = 1$ . Moreover (2.2) shows that if  $\overline{E}_t$  is singular, then there is exactly one singular point that is a double point, and the curve has genus

zero. This proves the equivalence between (1), (2) and (3), and the last statement of the proposition.

Let us prove (4)  $\Rightarrow$  (3). Assume that the discriminant  $\Delta_1([x_0 : x_1])$  (resp.  $\Delta_2([y_0 : y_1])$ ) has a root in  $[a : b] \in \mathbb{P}^1(\mathbb{C})$  (resp.  $[c : d] \in \mathbb{P}^1(\mathbb{C})$ ). Let us write

$$\begin{aligned} & \overline{K}(x_0, x_1, y_0, y_1, t) \\ = & e_{-1,1}(dy_0 - cy_1)^2 + e_{0,1}(bx_0 - ax_1)(dy_0 - cy_1)^2 + e_{1,1}(bx_0 - ax_1)^2(dy_0 - cy_1)^2 \\ + & e_{-1,0}(dy_0 - cy_1) + e_{0,0}(bx_0 - ax_1)(dy_0 - cy_1) + e_{1,0}(bx_0 - ax_1)^2(dy_0 - cy_1) \\ + & e_{-1,-1} + e_{0,-1}(bx_0 - ax_1) + e_{1,-1}(bx_0 - ax_1)^2. \end{aligned}$$

Since  $([a : b], [c : d]) \in \overline{E}_t$ , we have by definition that  $\overline{K}(a, b, c, d, t) = 0$ , i.e.  $e_{-1,-1} = 0$ . Since  $\Delta_1([x_0 : x_1])$  has a root in  $[a : b] \in \mathbb{P}^1(\mathbb{C})$ ,  $K(a, b, y_0, y_1)$  has a double root at  $[c, d]$  and so  $e_{-1,0} = 0$ . Similarly, the fact that  $\Delta_2([y_0 : y_1])$  has a root in  $[c : d] \in \mathbb{P}^1(\mathbb{C})$  implies  $e_{0,-1} = 0$ . This shows that

$$\frac{\partial \overline{K}(a, b, c, d, t)}{\partial x_0} = \frac{\partial \overline{K}(a, b, c, d, t)}{\partial x_1} = \frac{\partial \overline{K}(a, b, c, d, t)}{\partial y_0} = \frac{\partial \overline{K}(a, b, c, d, t)}{\partial y_1} = 0,$$

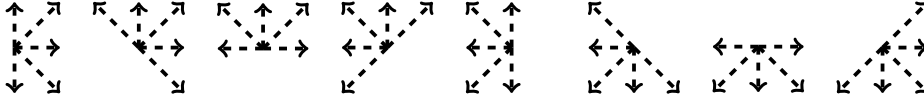
and, hence,  $([a : b], [c : d])$  is the singular point of  $\overline{E}_t$ .

Let us prove (3)  $\Rightarrow$  (5). If  $\Omega = ([a : b], [c : d])$  is the singular point of  $\overline{E}_t$ , then  $e_{-1,-1} = e_{-1,0} = e_{0,-1} = 0$ , and the discriminants  $\Delta_1([x_0 : x_1])$  and  $\Delta_2([y_0 : y_1])$  have a double root in  $[a : b] \in \mathbb{P}^1(\mathbb{C})$  and  $[c : d] \in \mathbb{P}^1(\mathbb{C})$  respectively.

The implication (5)  $\Rightarrow$  (4) is obvious. □

Our next aim is to describe the genus zero *models of walks*.

**Lemma 2.2.** *The discriminant  $\Delta_2([y_0 : y_1])$  has a double zero if and only if the model of the walk is included in a closed half plane whose boundary passes through  $(0, 0)$ . In other word, this correspond to models of the walks that belong to one of the following eight families*



*Remark 2.3.* As the statement of Lemma 2.2 is symmetric with respect to  $x$  and  $y$  we deduce that the same holds for  $\Delta_1([x_0 : x_1])$ . We then deduce that  $\Delta_1([x_0 : x_1])$  has a double root if and only if  $\Delta_2([y_0 : y_1])$  has a double root.

*Remark 2.4.* In the case  $t = 1$ , it is proved in [FIM17, Lemma 2.3.10] that, besides the models listed in Lemma 2.2, any *nondegenerate* model such that the drift is zero, i.e.

$$(\sum_i id_{i,j}, \sum_j jd_{i,j}) = (0, 0),$$

has a curve  $\overline{E}_t$  of genus 0.

*Proof.* The computations seem to be too complicated to be performed by hand, so we used MAPLE<sup>†</sup>. We are going to prove the result with two strategies. This first one is to write the discriminant of the discriminant  $\Delta_2([y_0 : y_1])$  and study when the latter is 0. The second strategy consists in decomposing the radical of an ideal into its prime

<sup>†</sup>The maple worksheet is available at [https://singer.math.ncsu.edu/ms\\_papers.html](https://singer.math.ncsu.edu/ms_papers.html).

components.

Let us first consider the situation where the double root is at  $(a, b)$  where  $b$  is not zero. Let us set  $y_1 = 1$  and  $y_0 = y$  to obtain the specialization  $\Delta_2([y : 1])$  of  $\Delta_2([y_0 : y_1])$ .

The following MAPLE code calculates the discriminant of the discriminant, its degree and order of vanishing in  $t$ , and then sets the coefficients of powers of  $t$  equal to zero. Solving these equations yields the 8 solutions  $\mathbf{S}[i]$ ,  $i = 1, \dots, 8$  corresponding to the 8 stepsets listed in Lemma 2.2.

```
> K := expand(x*y*(1-t*(add(add(d[i, j]*x^i*y^j, i = -1 .. 1), j = -1 .. 1))));
> DX := discrim(K, x);
> DD := discrim(discrim(K,x),y);
> ldegree(DD,t); degree(DD,t);
```

4  
12

```
> S := solve({seq(coeff(DD,t,i), i=4..12)}, [seq(seq(d[i, j], i=-1..1), j=-1..1)]);
> nops(S);
```

8

```
> S[1];S[2];S[3];S[4];S[5];S[6];S[7];S[8];
```

An alternate approach is to use the *PolynomialIdeals* package

```
> with(PolynomialIdeals):
```

and consider the prime decomposition of the radical of the ideal

```
> J := <seq(coeff(DD,t,i), i=4..12)>;
> PrimeDecomposition(J);
```

$$\begin{array}{lll} \langle d_{-1,-1}, d_{-1,0}, d_{-1,1} \rangle, & \langle d_{-1,-1}, d_{-1,0}, d_{0,-1} \rangle, & \langle d_{-1,-1}, d_{0,-1}, d_{1,-1} \rangle, \\ \langle d_{-1,0}, d_{-1,1}, d_{0,1} \rangle, & \langle d_{-1,1}, d_{0,1}, d_{1,1} \rangle, & \langle d_{0,-1}, d_{1,-1}, d_{1,0} \rangle, \\ \langle d_{0,1}, d_{1,0}, d_{1,1} \rangle, & \langle d_{1,-1}, d_{1,0}, d_{1,1} \rangle. & \end{array}$$

The *PrimeDecomposition* command lists a set of prime ideals whose intersection is the radical of the original ideal. In particular, these ideals have the property that any zero of the original ideal is a zero of one of the listed ideals and vice versa, see [CLO97, Chapter 4, Section 6]. These again correspond to the eight step sets listed in Lemma 2.2.

We now consider the case where the double root is at  $(a, b)$  where  $b = 0$ , that is, at  $(1, 0)$ . We will show that this case leads to *models of walks* already mentioned above.

```
> DDX := expand(z^4*subs(y = 1/z, DX));
```

If  $z = 0$  is a double root then the coefficient of 1 and  $z$  must be zero

```
> coeff(DDX, z, 0); coeff(DDX, z, 1);
```

$$-4t^2d_{-1,1}d_{1,1} + t^2d_{0,1}^2$$

$$-4t^2d_{-1,0}d_{1,1} - 4t^2d_{-1,1}d_{1,0} + 2t^2d_{0,0}d_{0,1} - 2td_{0,1}$$

Taking into account that  $t$  is transcendental over  $\mathbb{Q}(d_{i,j})$ , we are led to three cases, corresponding to three of the step sets in Lemma 2.2.

$$[d_{0,1} = 0, d_{-1,1} = 0, d_{-1,0} = 0]$$

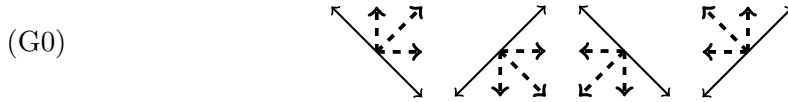
$$[d_{0,1} = 0, d_{-1,1} = 0, d_{1,1} = 0]$$

$$[d_{0,1} = 0, d_{1,1} = 0, d_{1,0} = 0] \quad \square$$

*Remark 2.5.* The proof of Proposition 1.2 proceeds by a direct “hand calculation” while the proof of Lemma 2.2 follows from a simple MAPLE calculation. It would be interesting to have a simple MAPLE based proof of Proposition 1.2 and a hand calculation proof of Lemma 2.2.

**Corollary 2.6.** *The following holds:*

- (1) *The nondegenerate genus zero models of walks are the nondegenerate models whose step set is included in an half space whose boundary passes through  $(0, 0)$ . More precisely, they are nondegenerate models belonging to one of the following families*



- (2) *The nondegenerate genus one models of walks are the models whose step set is not included in any half space whose boundary passes through  $(0, 0)$ .*

*Remark 2.7.* See also [DR19, Proposition 9] for an extension of Corollary 2.6 to the case when  $t \in ]0, 1[$  is algebraic over  $\mathbb{Q}(d_{i,j})$ . Their proof relies on the results of the present section and uses a continuity argument with respect to the parameter  $t$  to deduce that Corollary 2.6 stays correct for general values of  $t \in ]0, 1[$ .

*Proof.* We use Proposition 2.1. We have to determine when there exists  $([a : b], [c : d]) \in \overline{E}_t$  such that the discriminants  $\Delta_1([x_0 : x_1])$  and  $\Delta_2([y_0 : y_1])$  have a double root  $[a : b] \in \mathbb{P}^1(\mathbb{C})$  and  $[c : d] \in \mathbb{P}^1(\mathbb{C})$ . Lemma 2.2 provides such configurations. By Proposition 1.2, the configurations number 1, 3, 5, 7 are dismissed since they led to *singular walks*. Then, if we are considering *nondegenerate genus zero models of walks*, we are in the four families of models considered in (G0). Furthermore, if the step set is not included in any half space whose boundary passes through  $(0, 0)$ , the configurations of Proposition 1.2 and Lemma 2.2 are excluded and by Proposition 2.1, we are in the genus 1 situation.

Conversely, it remains to prove that if the *models of walks* are in the four families of *models* considered in (G0), the kernel has genus 0. Thanks to Proposition 2.1 it suffices to prove that the discriminants have a common zero in that case. This is the goal of the following Lemma 2.8 and Remark 2.9.  $\square$

Let us write

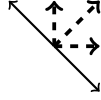
$$\Delta_1([x_0 : x_1]) = \sum_{i=0}^4 \alpha_i(t)x_0^i x_1^{4-i}, \quad \text{and} \quad \Delta_2([y_0 : y_1]) = \sum_{i=0}^4 \beta_i(t)y_0^i y_1^{4-i}.$$

where

$$\begin{aligned}
\alpha_0(t) &= t^2 d_{-1,0}^2 - 4t^2 d_{-1,1} d_{-1,-1} \\
\alpha_1(t) &= 2t^2 d_{-1,0} d_{0,0} - 2t d_{-1,0} - 4t^2 d_{-1,1} d_{0,-1} - 4t^2 d_{0,1} d_{-1,-1} \\
\alpha_2(t) &= t^2 d_{0,0}^2 - 2t d_{0,0} + 1 + 2t^2 d_{-1,0} d_{1,0} - 4t^2 d_{-1,1} d_{1,-1} - 4t^2 d_{0,1} d_{0,-1} - 4t^2 d_{1,1} d_{-1,-1} \\
\alpha_3(t) &= -2t d_{1,0} + 2t^2 d_{0,0} d_{1,0} - 4t^2 d_{1,1} d_{0,-1} - 4t^2 d_{0,1} d_{1,-1} \\
\alpha_4(t) &= t^2 d_{1,0}^2 - 4t^2 d_{1,1} d_{1,-1} \\
\beta_0(t) &= t^2 d_{0,-1}^2 - 4t^2 d_{1,-1} d_{-1,-1} \\
\beta_1(t) &= 2t^2 d_{0,-1} d_{0,0} - 2t d_{0,-1} - 4t^2 d_{1,-1} d_{-1,0} - 4t^2 d_{1,0} d_{-1,-1} \\
\beta_2(t) &= t^2 d_{0,0}^2 - 2t d_{0,0} + 1 + 2t^2 d_{0,-1} d_{0,1} - 4t^2 d_{1,-1} d_{-1,1} - 4t^2 d_{1,0} d_{-1,0} - 4t^2 d_{1,1} d_{-1,-1} \\
\beta_3(t) &= -2t d_{0,1} + 2t^2 d_{0,0} d_{0,1} - 4t^2 d_{1,1} d_{-1,0} - 4t^2 d_{1,0} d_{-1,1} \\
\beta_4(t) &= t^2 d_{0,1}^2 - 4t^2 d_{1,1} d_{-1,1}.
\end{aligned}$$

Note that  $\Delta_1([x_0 : x_1])$  (resp.  $\Delta_2([y_0 : y_1])$ ) is of degree 4 and so has four roots counted with multiplicities  $a_1, a_2, a_3, a_4$  (resp.  $b_1, b_2, b_3, b_4$ ) in  $\mathbb{P}^1(\mathbb{C})$ . If the discriminant  $\Delta_1([x_0 : x_1])$  (resp.  $\Delta_2([y_0 : y_1])$ ) has a double root; up to renumbering, we can assume that  $a_1 = a_2$  (resp.  $b_1 = b_2$ ).

**Lemma 2.8.** *Assume that the model of the walk is nondegenerate and belongs to the first family of (G0)*



Then, the walk has genus zero and the singular point of  $\overline{E}_t$  is  $\Omega = ([0 : 1], [0 : 1])$ , that is,  $a_1 = a_2 = [0 : 1]$  (resp.  $b_1 = b_2 = [0 : 1]$ ) is a double root of  $\Delta_1([x_0 : x_1])$  (resp.  $\Delta_2([y_0 : y_1])$ ). The other roots are distinct from one another and from the double root and are given by

	$a_1 = a_2$	$b_1 = b_2$
	$[0 : 1]$	$[0 : 1]$
	$a_3$	$a_4$
$\alpha_4(t) \neq 0$	$[-\alpha_3(t) - \sqrt{\alpha_3(t)^2 - 4\alpha_2(t)\alpha_4(t)} : 2\alpha_4(t)]$	$[-\alpha_3(t) + \sqrt{\alpha_3(t)^2 - 4\alpha_2(t)\alpha_4(t)} : 2\alpha_4(t)]$
$\alpha_4(t) = 0$	$[1 : 0]$	$[-\alpha_2(t) : \alpha_3(t)]$
	$b_3$	$b_4$
$\beta_4(t) \neq 0$	$[-\beta_3(t) - \sqrt{\beta_3(t)^2 - 4\beta_2(t)\beta_4(t)} : 2\beta_4(t)]$	$[-\beta_3(t) + \sqrt{\beta_3(t)^2 - 4\beta_2(t)\beta_4(t)} : 2\beta_4(t)]$
$\beta_4(t) = 0$	$[1 : 0]$	$[-\beta_2(t) : \beta_3(t)]$

*Remark 2.9.* We can extend Lemma 2.8 to the other configurations in (G0) by using the following remarks:

- (1) Replacing  $([x_0 : x_1], [y_0 : y_1])$  by  $([x_0 : x_1], [y_1 : y_0])$ , which corresponds to the variable change  $(x, y) \mapsto (x, y^{-1})$ , amounts to consider a *weighted model of walk* with *weights*  $d'_{i,j} := d_{i,-j}$ . This can be used to extend Lemma 2.8 to the second configuration of (G0); for instance, the singular point of  $\overline{E}_t$  is  $\Omega = ([0 : 1], [1 : 0])$  in that case.
- (2) Replacing  $([x_0 : x_1], [y_0 : y_1])$  by  $([x_1 : x_0], [y_1 : y_0])$  amounts to consider a *weighted model of walk* with *weights*  $d'_{i,j} := d_{-i,-j}$ . This can be used to extend Lemma 2.8 to the third configuration of (G0); for instance, the singular point of  $\overline{E}_t$  is  $\Omega = ([1 : 0], [1 : 0])$  in that case.
- (3) Replacing  $([x_0 : x_1], [y_0 : y_1])$  by  $([x_1 : x_0], [y_0 : y_1])$  amounts to consider a *weighted model of walk* with *weights*  $d'_{i,j} := d_{-i,j}$ . This can be used to extend Lemma 2.8 to the fourth configuration of (G0); for instance, the singular point of  $\overline{E}_t$  is  $\Omega = ([1 : 0], [0 : 1])$  in that case.

*Remark 2.10.* Note that if we consider the  $x_3, x_4$  (resp.  $y_3, y_4$ ) defined in [FIM17, Chapter 6], we have the equality of sets  $\{a_3, a_4\} = \{x_3, x_4\}$  and  $\{b_3, b_4\} = \{y_3, y_4\}$ , but do not have necessarily  $a_i = x_i, b_j = y_j$ , with  $3 \leq i, j \leq 4$ .

*Proof of Lemma 2.8.* We shall prove the lemma for  $\Delta_2([y_0 : y_1])$ , the proof for  $\Delta_1([x_0 : x_1])$  being similar. By assumption,  $d_{-1,-1} = d_{-1,0} = d_{0,-1} = 0$ . Then,  $\alpha_0(t) = \alpha_1(t) = 0$ . Therefore, the discriminant  $\Delta_2([y_0 : y_1])$  has a double root at  $[0 : 1]$  and we can write

$$\Delta_2([y : 1]) = \beta_2(t)y^2 + \beta_3(t)y^3 + \beta_4(t)y^4.$$

Since  $t$  is transcendental over  $\mathbb{Q}(d_{i,j})$ , we see that the coefficient of  $y^2$  is nonzero. Therefore  $[0 : 1]$  is precisely a double root of  $\Delta_2([y_0 : y_1])$ . To see that  $b_3$  and  $b_4$  are distinct, we calculate the discriminant of  $\Delta_2([y : 1])/y^2$ , which is almost the same as the one we considered in the proof of Lemma 2.2. This is a polynomial of degree four in  $t$  with the following coefficients:

term	coefficient
$t^4$	$-16(4d_{-1,1}d_{1,-1}d_{1,1} - d_{1,-1}d_{1,0}^2 - d_{0,0}^2d_{1,1} + d_{0,0}d_{0,1}d_{1,0} - d_{0,1}^2d_{1,-1})d_{-1,1}$
$t^3$	$-16(2d_{0,0}d_{1,1} - d_{0,1}d_{1,0})d_{-1,1}$
$t^2$	$16d_{-1,1}d_{1,1}$
$t$	$0$
$1$	$0$

Let us prove that if  $\Delta_2([y_0 : y_1])$  has a double root different from  $[0 : 1]$ , all the above coefficients must be zero. Recalling that  $d_{-1,1}d_{1,-1} \neq 0$ , from the coefficient of  $t^2$ , we must have  $d_{1,1} = 0$ . From the coefficient of  $t^3$ , we have that  $d_{0,1} = 0$  or  $d_{1,0} = 0$ . From the coefficient of  $t^4$ , we get in both cases  $d_{0,1} = d_{1,0} = 0$ . This implies that the model of the *walk* would be *degenerate*, a contradiction. The formulas for  $b_3$  and  $b_4$  follow from the quadratic formula.  $\square$

### 3. INVOLUTIVE AUTOMORPHISMS OF THE KERNEL CURVE

Following [BMM10, Section 3], [KY15, Section 3] or [FIM17], we consider the involutive rational functions

$$i_1, i_2 : \mathbb{C}^2 \dashrightarrow \mathbb{C}^2$$



given by

$$i_1(x, y) = \left( x, \frac{A_{-1}(x)}{A_1(x)y} \right) \text{ and } i_2(x, y) = \left( \frac{B_{-1}(y)}{B_1(y)x}, y \right).$$

Note that  $i_1, i_2$  are “only” rational functions in the sense that they are a priori not defined when the denominators vanish. The dashed arrow notation used above and in the rest of the paper is a classical notation for rational functions.

The rational functions  $i_1, i_2$  induce involutive rational functions

$$\iota_1, \iota_2 : \overline{E}_t \dashrightarrow \overline{E}_t$$

given by

$$\begin{aligned} \iota_1([x_0 : x_1], [y_0 : y_1]) &= \left( [x_0 : x_1], \left[ \frac{A_{-1}\left(\frac{x_0}{x_1}\right)}{A_1\left(\frac{x_0}{x_1}\right)\frac{y_0}{y_1}} : 1 \right] \right), \\ \text{and } \iota_2([x_0 : x_1], [y_0 : y_1]) &= \left( \left[ \frac{B_{-1}\left(\frac{y_0}{y_1}\right)}{B_1\left(\frac{y_0}{y_1}\right)\frac{x_0}{x_1}} : 1 \right], [y_0 : y_1] \right). \end{aligned}$$

Again, these functions are a priori not defined where the denominators vanish. However, the following result shows that, actually, this is only an “apparent problem”:  $\iota_1$  and  $\iota_2$  can be extended into endomorphisms of  $\overline{E}_t$ . We recall that a rational map  $f : \overline{E}_t \dashrightarrow \overline{E}_t$  is an endomorphism if it is regular at any  $P \in \overline{E}_t$ , i.e. if  $f$  can be represented in suitable affine charts containing  $P$  and  $f(P)$  by a rational function with nonvanishing denominator at  $P$ . More generally, given  $X$  and  $Y$  algebraic varieties, we say that  $f : X \dashrightarrow Y$  is a morphism if  $f$  can be represented by two suitable affine charts containing  $P$  and  $f(P)$  respectively, by a rational function with nonvanishing denominator at  $P$ .

**Proposition 3.1.** *The rational maps  $\iota_1, \iota_2 : \overline{E}_t \dashrightarrow \overline{E}_t$  can be extended into involutive automorphisms of  $\overline{E}_t$ .*

*Proof.* Note that  $\iota_1(x, y)$  is well-defined if the  $x_i$  and the  $y_j$  are nonzero and if  $A_1\left(\frac{x_0}{x_1}\right)\frac{y_0}{y_1}$  is nonzero. This excludes at most finitely many  $(x, y) \in \overline{E}_t$  and, hence, there exists a finite set  $\mathcal{S}_0 \subset \overline{E}_t$  such that  $\iota_1$  is well defined on  $\overline{E}_t \setminus \mathcal{S}_0$ . The map  $\iota_1$  induces a bijection from  $\overline{E}_t \setminus \mathcal{S}_0$  to  $\overline{E}_t \setminus \mathcal{S}_1$ , where  $\mathcal{S}_1$  is a finite set. The same holds for  $\iota_2$ . We have to prove that  $\iota_1, \iota_2 : \overline{E}_t \dashrightarrow \overline{E}_t$  can be extended into endomorphisms of  $\overline{E}_t$ . According to Proposition 2.1, if the curve  $\overline{E}_t$  has genus one, then it is smooth and the result follows from [Har77, Proposition 6.8, p. 43].

It remains to study the case when  $\overline{E}_t$  has genus zero. In that case, Proposition 2.1 ensures that  $\overline{E}_t$  has a unique singularity  $\Omega$ . It follows from [Har77, Proposition 6.8, p. 43] that  $\iota_1$  and  $\iota_2$  can be uniquely extended into morphisms  $\overline{E}_t \setminus \{\Omega\} \rightarrow \overline{E}_t$  still denoted by  $\iota_1$  and  $\iota_2$ . It remains to study  $\iota_1$  and  $\iota_2$  at  $\Omega$ . Let us first assume that the walk under consideration belongs to the family of the first configuration of (G0). Lemma 2.8 ensures that  $\Omega = ([0 : 1], [0 : 1])$ . For  $([x : 1], [y : 1]) \in \overline{E}_t$ , the equation  $K(x, y, t) = 0$  implies that

$$(3.1) \quad \frac{A_{-1}(x)}{A_1(x)y} = \frac{1}{tA_1(x)} - \frac{A_0(x)}{A_1(x)} - y = \frac{x}{t\tilde{A}_1(x)} - \frac{\tilde{A}_0(x)}{\tilde{A}_1(x)} - y$$

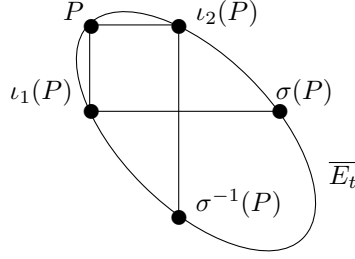


FIGURE 1. The maps  $\iota_1, \iota_2$  restricted to the kernel curve  $\overline{E}_t$

where  $\tilde{A}_0(x) = xA_0(x) = d_{-1,0} + d_{0,0}x + d_{1,0}x^2$  and  $\tilde{A}_1(x) = xA_1(x) = d_{-1,1} + d_{0,1}x + d_{1,1}x^2$ . Since  $d_{-1,1} \neq 0$ ,  $\tilde{A}_1(x)$  does not vanish at  $x = 0$ . Since  $d_{-1,0} = 0$ ,  $\tilde{A}_0(x)$  vanishes at  $x = 0$ . So, (3.1) shows that  $\iota_1$  is regular at  $\Omega$  and that  $\iota_1(\Omega) = \Omega$ . The argument for  $\iota_2$  is similar.

The other cases listed in (G0) can be treated similarly using a reduction argument as in Remark 2.9.  $\square$

We also consider the automorphism of  $\overline{E}_t$  defined by

$$\sigma = \iota_2 \circ \iota_1.$$

It is easily seen that  $\iota_1$  and  $\iota_2$  are the vertical and horizontal switches of  $\overline{E}_t$  (see Figure 1), i.e. for any  $P = (x, y) \in \overline{E}_t$ , we have

$$\{P, \iota_1(P)\} = \overline{E}_t \cap (\{x\} \times \mathbb{P}^1(\mathbb{C})) \text{ and } \{P, \iota_2(P)\} = \overline{E}_t \cap (\mathbb{P}^1(\mathbb{C}) \times \{y\}).$$

We now give a couple of lemmas for later use.

**Lemma 3.2.** *A point  $P = ([x_0 : x_1], [y_0 : y_1]) \in \overline{E}_t$  is fixed by  $\iota_1$  if and only if  $\Delta_1([x_0 : x_1]) = 0$ . A point  $P = ([x_0 : x_1], [y_0 : y_1]) \in \overline{E}_t$  is fixed by  $\iota_2$  if and only if  $\Delta_2([x_0 : x_1]) = 0$ .*

*Proof.* Assume that  $P$  is fixed by  $\iota_1$ . Then, the polynomial  $[y_0 : y_1] \mapsto \overline{K}(x_0, x_1, y_0, y_1, t)$  has a double root, meaning that the discriminant is zero. This is exactly  $\Delta_1([x_0 : x_1]) = 0$ . Conversely,  $\Delta_1([x_0 : x_1]) = 0$  implies that  $[y_0 : y_1] \mapsto \overline{K}(x_0, x_1, y_0, y_1, t)$  has a double root and therefore  $P$  is fixed by  $\iota_1$ . The proof for  $\iota_2$  is similar.  $\square$

The fixed points of  $\iota_1$  have  $y$ -coordinates that are the double roots of  $y \mapsto \overline{K}(x_0, x_1, y, t)$ , i.e. they are the roots of the discriminant. By Lemma 2.8 and Remark 2.9, there are 3 points of  $\overline{E}_t$  that are fixed by  $\iota_1$ . A similar statement holds for  $\iota_2$ . As is shown in the following lemma, one of them plays a particular role.

**Lemma 3.3.** *Let  $P \in \overline{E}_t$ . The following statements are equivalent:*

- (1)  $P$  is fixed by  $\iota_1$  and  $\iota_2$ ;
- (2)  $P$  is a singular point of  $\overline{E}_t$ ;
- (3)  $P$  is fixed by  $\sigma = \iota_2 \circ \iota_1$ .

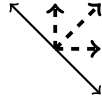
*Proof.* Let  $P = ([a : b], [c : d]) \in \overline{E}_t$ . From Proposition 2.1, we have that  $P$  is a singular point if and only if  $\Delta_1([x_0 : x_1])$  and  $\Delta_2([y_0 : y_1])$  vanish at  $[a : b]$  and  $[c : d]$  respectively. We conclude with Lemma 3.2, that (1) is equivalent to (2).

Clearly, (1) implies (3). It remains to prove that (3) implies (1). Assume that  $P = (a_1, b_1)$  is fixed by  $\sigma$ . After writing  $\iota_1(P) = (a_1, b'_1)$  and  $\iota_2(\iota_1(P)) = (a'_1, b'_1)$ , it is clear that  $\sigma(P) = P$  implies successively  $\iota_1(P) = P$  and  $\iota_2(P) = P$ .  $\square$

#### 4. UNIFORMIZATION OF THE KERNEL CURVE

We still consider a *weighted model of nondegenerate walk*. The aim of this section is to give an explicit uniformization of  $\overline{E}_t$ . Thanks to Proposition 2.1, the latter may have genus zero or one. Although there are algorithms to compute such uniformizations, see for instance [vH97, SWPD08], our presentation of explicit uniformizations allows us to understand in detail the pull-backs of  $\sigma$ ,  $\iota_1$  and  $\iota_2$  and therefore their effect on the generating series of the models of walks. Let us start with the genus zero case.

**4.1. Genus zero case.** Let us consider a *nondegenerate weighted model of walks* of genus zero. Thank to Corollary 2.6 combined with Remark 2.9, it suffices to consider the situation where the *nondegenerate model of walk* arises from the following family



Genus zero curves may be parametrized with maps  $\phi : \mathbb{P}^1(\mathbb{C}) \rightarrow \overline{E}_t$  which are bijective outside a finite set. The aim of this subsection, achieved with Proposition 4.7, is to find such a parametrization explicitly. Although we could have just written down the formula for this parametrization and verified its properties, we have preferred to explain how the formula arises. This requires a preliminary study of the automorphisms of  $\mathbb{P}^1(\mathbb{C})$  obtained by pulling back the maps  $\sigma$ ,  $\iota_1$  and  $\iota_2$  by  $\phi$ , which is done with a series of lemmas preceding Proposition 4.7.

According to Lemma 2.8,  $\overline{E}_t$  has a unique singular point  $\Omega = (a_1, b_1) = ([0 : 1], [0 : 1])$ . Moreover  $\Delta_1([x_0 : x_1])$  has degree four with a double root at  $a_1 = [0 : 1]$  and the remaining two roots  $a_3, a_4$  are distinct. We let  $S_3$  and  $S_4$  be the points of  $\overline{E}_t$  with first coordinate  $a_3$  and  $a_4$  respectively. Similarly,  $\Delta_2([y_0 : y_1])$  has degree four with a double root at  $b_1 = [0 : 1]$  and the remaining two roots  $b_3, b_4$  are distinct. We let  $S'_3$  and  $S'_4$  be the points of  $\overline{E}_t$  with second coordinates  $b_3$  and  $b_4$  respectively.

Since  $\overline{E}_t$  has genus zero, there is a rational parametrization of  $\overline{E}_t$  [Ful89, Page 198, Ex.1], i.e. there exists a birational map

$$\begin{aligned} \phi = (\check{x}, \check{y}) : \mathbb{P}^1(\mathbb{C}) &\dashrightarrow \overline{E}_t \\ s &\mapsto (\check{x}(s), \check{y}(s)). \end{aligned}$$

To simplify the notation, we will abusively denote  $(\check{x}, \check{y})$  by  $(x, y)$ . We will now follow the ideas contained in [FIM17] to produce an explicit uniformization of  $\overline{E}_t$  in Proposition 4.7. If we set  $t = 1$ , we recover the uniformization of [FIM17, Section 6.4.3]. However, it is not clear if their proof can be simply modified to hold in our context, so we preferred to give proofs here with a slightly different strategy.

**Lemma 4.1.** *The map  $\phi$  is surjective and induces a bijection between  $\mathbb{P}^1(\mathbb{C}) \setminus \phi^{-1}(\Omega)$  and  $\overline{E}_t \setminus \{\Omega\}$ .*

*Proof.* As any nonconstant rational map from  $\mathbb{P}^1(\mathbb{C})$  to a projective curve,  $\phi$  is actually a surjective morphism of curves, see [Ful89, Corollary 1, Page 160]. Since  $\Omega$  is the unique singular point of  $\overline{E}_t$ , the result follows.  $\square$

The maps  $x, y : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$  are surjective morphisms of curves as well.

We let  $s_3, s_4 \in \mathbb{P}^1(\mathbb{C})$  (resp.  $s'_3, s'_4 \in \mathbb{P}^1(\mathbb{C})$ ) be such that  $S_3 = \phi(s_3)$  and  $S_4 = \phi(s_4)$  (resp.  $S'_3 = \phi(s'_3)$  and  $S'_4 = \phi(s'_4)$ ).

We will need to know the cardinality of  $x^{-1}(P)$  (resp.  $y^{-1}(P)$ ) for  $P \in \mathbb{P}^1(\mathbb{C})$ . This quantity might depend on  $P$  but it is a general fact about morphisms of curves that the cardinality of  $x^{-1}(P)$  (resp.  $y^{-1}(P)$ ) is constant for  $P$  outside a finite subset of  $\mathbb{P}^1(\mathbb{C})$ . This common value is called the degree of  $x$  (resp.  $y$ ). Inside this finite set, the cardinality can only fall, so is less than the degree.

**Lemma 4.2.** *The morphisms  $x, y : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$  have degree two.*

*Proof.* We will see that this is a consequence of the fact that  $\overline{E}_t$  is a biquadratic curve. Observe that by Lemma 4.1,  $\phi$  induces a bijection between  $\mathbb{P}^1(\mathbb{C}) \setminus \phi^{-1}(\Omega)$  and  $\overline{E}_t \setminus \{\Omega\}$ . Any  $(a, b) \in \overline{E}_t$  with  $a \neq a_1$  cannot be  $\Omega$  and therefore has a unique preimage by  $\phi$ . Additionally, let  $Z$  be the finite set of zeros of the discriminant  $\Delta_1$ . Then, for any  $a \notin Z$ , the cardinality  $(\{a\} \times \mathbb{P}^1(\mathbb{C})) \cap \overline{E}_t$  is two. Since  $x^{-1}(a) = \phi^{-1}((\{a\} \times \mathbb{P}^1(\mathbb{C})) \cap \overline{E}_t)$  and  $a_1 \in Z$ , it follows that if  $a \notin Z$ , the cardinality of  $x^{-1}(a)$  is two. So,  $x$  has degree two. The argument for  $y$  is similar.  $\square$

Since  $\phi$  is a birational map, the involutive automorphisms  $\iota_1, \iota_2$  of  $\overline{E}_t$  induce involutive automorphisms  $\tilde{\iota}_1, \tilde{\iota}_2$  of  $\mathbb{P}^1(\mathbb{C})$  via  $\phi$ . Similarly,  $\sigma$  induces an automorphism  $\tilde{\sigma}$  of  $\mathbb{P}^1(\mathbb{C})$ . In other words, we have the commutative diagrams

$$\begin{array}{ccc} \overline{E}_t & \xrightarrow{\iota_k} & \overline{E}_t & \text{and} & \overline{E}_t & \xrightarrow{\sigma} & \overline{E}_t \\ \phi \uparrow & & \uparrow \phi & & \uparrow \phi & & \uparrow \phi \\ \mathbb{P}^1(\mathbb{C}) & \xrightarrow{\tilde{\iota}_k} & \mathbb{P}^1(\mathbb{C}) & & \mathbb{P}^1(\mathbb{C}) & \xrightarrow{\tilde{\sigma}} & \mathbb{P}^1(\mathbb{C}) \end{array}$$

Note that since by Lemma 4.1  $\phi$  induces a bijection between  $\mathbb{P}^1(\mathbb{C}) \setminus \phi^{-1}(\Omega)$  and  $\overline{E}_t \setminus \{\Omega\}$  and  $\Omega$  is fixed by  $\iota_1, \iota_2$ , see Lemma 3.3, the group generated by  $\iota_1$  and  $\iota_2$  is isomorphic to the group generated by  $\tilde{\iota}_1$  and  $\tilde{\iota}_2$ . Thus we recover the same group as in [BMM10] for instance. Note that although the cardinal of the group may depends upon  $t$ , see Remark 4.13, since the maps  $\iota_1, \iota_2$  are defined on  $\mathbb{Q}(d_{i,j})(t)$ , two distinct values of  $t$  transcendental over  $\mathbb{Q}(d_{i,j})$  lead to isomorphic groups. We summarize some remarks in the following lemmas.

**Lemma 4.3.** *We have  $x = x \circ \tilde{\iota}_1$  and  $y = y \circ \tilde{\iota}_2$ .*

*Proof.* We obtain  $x = x \circ \tilde{\iota}_1$  by equating the first coordinates in the equality  $\phi \circ \tilde{\iota}_1 = \iota_1 \circ \phi$  and we obtain  $y = y \circ \tilde{\iota}_2$  by equating the second coordinates in the equality  $\phi \circ \tilde{\iota}_2 = \iota_2 \circ \phi$ .  $\square$

**Lemma 4.4.** *Let  $P = \phi(s) \in \overline{E}_t$  and let  $k \in \{1, 2\}$ . We have:*

- if  $\tilde{\iota}_k(s) = s$ , then  $\iota_k(P) = P$ ;
- if  $P \neq \Omega$  and  $\iota_k(P) = P$ , then  $\tilde{\iota}_k(s) = s$ .

Furthermore the map  $\tilde{\iota}_1$  (resp.  $\tilde{\iota}_2$ ) has exactly two fixed points, namely  $s_3$  and  $s_4$  (resp.  $s'_3$  and  $s'_4$ ).

*Proof.* We have  $\iota_k(P) = \iota_k(\phi(s)) = \phi(\tilde{\iota}_k(s))$ . The first assertion is now clear, and the second one follows from the fact that  $\phi$  is injective on  $\overline{E_t} \setminus \phi^{-1}(\Omega)$ . Since  $S_3, S_4 \neq \Omega$  are fixed by  $\iota_1$ , this shows that  $s_3$  and  $s_4$  are fixed by  $\tilde{\iota}_1$ . Similar proof holds for  $\tilde{\iota}_2$ .

It remains to prove that there are exactly two points fixed by  $\tilde{\iota}_k$ . To the contrary, assume that there is a third point fixed by  $\tilde{\iota}_k$ . Since  $\tilde{\iota}_k$  is an automorphism of  $\mathbb{P}^1(\mathbb{C})$ , i.e. an homography, with three fixed points, it is the identity. This is a contradiction and concludes the proof of the lemma.  $\square$

**Lemma 4.5.** *The preimage of  $\Omega$  by  $\phi$  has two elements.*

*Proof.* We know that  $x, y : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$  have degree two, so  $\phi^{-1}(\Omega)$  has one or two elements. Suppose that  $\phi^{-1}(\Omega)$  has exactly one element, say  $s_1$ . In virtue of  $\phi(s_3) = S_3$  and  $\phi(s_4) = S_4$ ,  $s_1$  is different to  $s_3, s_4$ . Since  $\phi(\tilde{\iota}_1(s_1)) = \iota_1(\phi(s_1)) = \iota_1(\Omega) = \Omega$ , we have  $\tilde{\iota}_1(s_1) = s_1$ . This contradicts Lemma 4.4. Hence,  $\phi^{-1}(\Omega)$  has two elements.  $\square$

From now on, we define  $s_1 \neq s_2$  the two preimages of  $\Omega$  by  $\phi$ , that is

$$\{s_1, s_2\} := \phi^{-1}(\Omega).$$

**Lemma 4.6.** *The map  $\tilde{\iota}_1$  (resp.  $\tilde{\iota}_2$ ) interchanges  $s_1$  and  $s_2$ . The map  $\tilde{\sigma}$  has exactly two distinct fixed points:  $s_1$  and  $s_2$ .*

*Proof.* We have  $\phi(s) = \Omega$  if and only if  $s = s_1$  or  $s_2$  and the equality  $\iota_1(\phi(s)) = \phi(\tilde{\iota}_1(s))$  shows that  $\tilde{\iota}_1$  induces a permutation of  $\phi^{-1}(\Omega) = \{s_1, s_2\}$ . By Lemma 4.4,  $s_1$  is not fixed by  $\tilde{\iota}_1$ , showing that the permutation is not the identity, i.e.  $\tilde{\iota}_1$  interchanges  $s_1$  and  $s_2$ .

The proof for  $\tilde{\iota}_2$  is similar.

As any homography which is not the identity,  $\tilde{\sigma}$  has at most two fixed points in  $\mathbb{P}^1(\mathbb{C})$ . It only remains to prove that  $s_1$  and  $s_2$  are fixed by  $\tilde{\sigma}$ , and this is indeed the case because  $\tilde{\sigma} = \tilde{\iota}_2 \circ \tilde{\iota}_1$  and  $\tilde{\iota}_1, \tilde{\iota}_2$  interchange  $s_1$  and  $s_2$ .  $\square$

We are now ready to give an explicit expression of  $\phi$ . The coefficients  $\alpha_i, \beta_i$  of the discriminants in this situation are given by the formulas

$$\begin{aligned} \alpha_0(t) &= \alpha_1(t) = 0 \\ \beta_0(t) &= \beta_1(t) = 0 \\ \alpha_2(t) &= \beta_2(t) = 1 - 2td_{0,0} + t^2d_{0,0}^2 - 4t^2d_{-1,1}d_{1,-1} \\ \alpha_3(t) &= 2t^2d_{1,0}d_{0,0} - 2td_{1,0} - 4t^2d_{0,1}d_{1,-1} \\ \beta_3(t) &= 2t^2d_{0,1}d_{0,0} - 2td_{0,1} - 4t^2d_{1,0}d_{-1,1} \\ \alpha_4(t) &= t^2(d_{1,0}^2 - 4d_{1,1}d_{1,-1}) \\ \beta_4(t) &= t^2(d_{0,1}^2 - 4d_{1,1}d_{-1,1}). \end{aligned}$$

Note that for  $k = 3, 4$ ,  $\beta_k(t)$ , may be obtained from  $\alpha_k(t)$  by interchanging  $d_{1,0}$  with  $d_{0,1}$  and  $d_{1,-1}$  with  $d_{-1,1}$ .

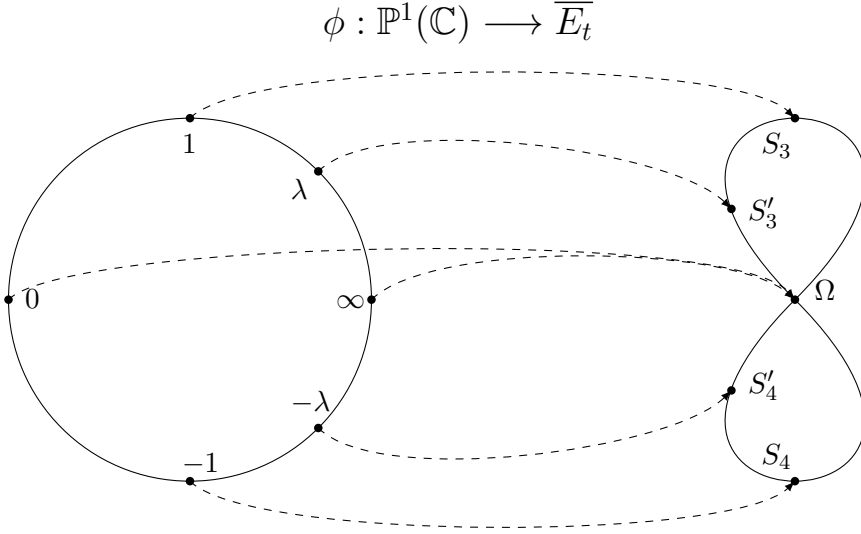


FIGURE 2. An idealized representation of the uniformization map used in Proposition 4.7. The left hand side represents the complex Riemann sphere and the right hand side the curve  $\overline{E}_t$ , seen as an abstract complex algebraic curve.

**Proposition 4.7.** *An explicit parametrization  $\phi : \mathbb{P}^1(\mathbb{C}) \rightarrow \overline{E}_t$  such that*

$$\tilde{t}_1(s) = \frac{1}{s}, \quad \tilde{t}_2(s) = \frac{\lambda^2}{s} = \frac{q}{s} \quad \text{and} \quad \tilde{\sigma}(s) = qs$$

for a certain  $\lambda \in \mathbb{C}^*$  is given by

$$\phi(s) = \left( \frac{4\alpha_2(t)}{\sqrt{\alpha_3(t)^2 - 4\alpha_2(t)\alpha_4(t)}(s + \frac{1}{s}) - 2\alpha_3(t)}, \frac{4\beta_2(t)}{\sqrt{\beta_3(t)^2 - 4\beta_2(t)\beta_4(t)}(\frac{s}{\lambda} + \frac{\lambda}{s}) - 2\beta_3(t)} \right).$$

Moreover, we have, see Figure 2

$$\begin{aligned} x(0) = x(\infty) = a_1, \quad x(1) = a_3, \quad x(-1) = a_4, \\ y(0) = y(\infty) = b_1, \quad y(\lambda) = b_3, \quad y(-\lambda) = b_4. \end{aligned}$$

*Proof.* According to Lemma 4.6,  $\tilde{t}_1$  is an involutive homography with fixed points  $s_3$  and  $s_4$ , so there exists an homography  $h$  such that  $h(s_3) = 1$ ,  $h(s_4) = -1$  and  $h \circ \tilde{t}_1 \circ h^{-1}(s) = 1/s$ . Up to replacing  $\phi$  by  $\phi \circ h$ , we can assume that  $s_3 = 1$ ,  $s_4 = -1$  and  $\tilde{t}_1(s) = \frac{1}{s}$ . Since  $s_1 \neq s_2$ , we can assume up to renumbering that  $s_1 \neq \infty$ . Let us consider the homography  $k(s) = \frac{s-s_1}{-s_1s+1}$ . Note that  $k$  commutes with  $s \mapsto 1/s$ , so changing  $\phi$  by  $\phi \circ k$  does not affect  $\tilde{t}_1$ , and we can also assume that  $s_1 = [0 : 1]$  and  $s_2 = [1 : 0]$ . Lemma 4.2 and Lemma 4.3 ensure that the morphism  $x : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$  has degree two and satisfies  $x(s) = x(1/s)$  for all  $s \in \mathbb{P}^1(\mathbb{C})$ . Since the morphism  $x : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$  has degree two, see Lemma 4.2, it follows that

$$x(s) = \frac{a(s + 1/s) + b}{c(s + 1/s) + d}$$

for some  $a, b, c, d \in \mathbb{C}$ . We have  $x(s_1) = x([0 : 1]) = a_1 = 0$ ,  $x(s_2) = x([1 : 0]) = a_1 = 0$ ,  $x(s_3) = x([1 : 1]) = a_3$  and  $x(s_4) = x([-1 : 1]) = a_4$ . The equality  $x([1 : 0]) = 0$  implies  $a = 0$ . The equalities  $x([1 : 1]) = a_3$  and  $x([-1 : 1]) = a_4$  imply

$$x(s) = \frac{4a_3a_4}{(a_4 - a_3)(s + \frac{1}{s}) + 2(a_3 + a_4)}.$$

The known expressions for  $a_3$  and  $a_4$  given in Lemma 2.8 lead to the expected expression for  $x(s)$ .

According to Lemma 4.6,  $\tilde{t}_2$  is an homography interchanging  $[0 : 1]$  and  $[1 : 0]$ , so  $\tilde{t}_2(s) = \frac{\lambda^2}{s}$  for some  $\lambda \in \mathbb{C}^*$ . Up to renumbering, we have  $s'_3 = \lambda$  and  $s'_4 = -\lambda$ . Using the fact that the morphism  $y : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$  has degree two and is invariant by  $\tilde{t}_2$ , and arguing as we did above for  $x$ , we see that there exist  $\alpha, \beta, \gamma, \eta \in \mathbb{C}$  such that

$$y(s) = \frac{\alpha(\frac{s}{\lambda} + \frac{\lambda}{s}) + \beta}{\gamma(\frac{s}{\lambda} + \frac{\lambda}{s}) + \eta}.$$

The equality  $y([1 : 0]) = 0$  implies  $\alpha = 0$ . Using the equalities  $y(s'_3) = y(\lambda) = b_3$  and  $y(s'_4) = y(-\lambda) = b_4$ , and arguing as we did above for  $x$ , we obtain the expected expression for  $y(s)$ .  $\square$

*Remark 4.8.* (1) The uniformization is not unique. More precisely, the possible uniformizations are of the form  $\phi \circ h$ , where  $h$  is an homography. However, if one requires that  $h$  fixes setwise  $\{[0 : 1], [1 : 0]\}$  then  $q$  is uniquely defined up to inversion.  
(2) The real  $q$  or  $q^{-1}$  specializes for  $t = 1$  to the real  $\rho^2$  in [FIM17, Page 178].

The following proposition determines  $q$  up to its inverse. We include this for completeness.

**Proposition 4.9** ([DHRS20], Proposition 1.7, Corollary 1.10). *One of the two complex numbers  $q$  or  $q^{-1}$  is equal to*

$$\frac{-1 + d_{0,0}t - \sqrt{(1 - d_{0,0}t)^2 - 4d_{1,-1}d_{-1,1}t^2}}{-1 + d_{0,0}t + \sqrt{(1 - d_{0,0}t)^2 - 4d_{1,-1}d_{-1,1}t^2}}.$$

Furthermore,  $q \in \mathbb{R} \setminus \{\pm 1\}$ .

*Remark 4.10.* This implies that  $\sigma$  and  $\tilde{\sigma}$  have infinite order (see also [BMM10, FR11]). Because  $\sigma$  is induced on  $\overline{E_t}$  by  $i_1 \circ i_2$ , we find that  $i_1 \circ i_2$  has infinite order as well. It follows that the group of the walk introduced in [BMM10], which is by definition the group generated by  $i_1$  and  $i_2$ , has infinite order. Note that this was proved in [BMM10] using a valuation argument.

*Remark 4.11.* We stress the fact that since  $\phi(s)$ ,  $q$  and  $\overline{E_t}$  depend continuously on  $t$  and the set of transcendental number over  $\mathbb{Q}(d_{i,j})$  in  $]0, 1[$  is dense in  $]0, 1[$ , we deduce that Proposition 4.7 and Proposition 4.9 stay valid for every  $t \in ]0, 1[$ .

4.2. **Genus one case.** In this section, we consider the uniformization problem in the genus one context. This problem has been solved in [DR19]. We recall below the main result of [DR19], for the sake of completeness. Let us consider a *nondegenerate model of walk* of genus one. By Proposition 2.1,  $\overline{E}_t$  is a smooth curve of genus one and, by Corollary 2.6, this corresponds to the situation where the step set is not included in any half plane whose boundary passes through  $(0,0)$ . By [WW96, Chapter XX], it is biholomorphic to  $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$  for some lattice  $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$  of  $\mathbb{C}$  via some  $(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ -periodic holomorphic map

$$(4.1) \quad \begin{aligned} \Lambda : \mathbb{C} &\rightarrow \overline{E}_t \\ \omega &\mapsto (\mathfrak{q}_1(\omega), \mathfrak{q}_2(\omega)), \end{aligned}$$

where  $\mathfrak{q}_1, \mathfrak{q}_2$  are rational functions of  $\wp$  and its derivative  $d\wp/d\omega$ , and  $\wp$  is the Weierstrass function associated with the lattice  $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ :

$$(4.2) \quad \wp(\omega) = \wp(\omega; \omega_1, \omega_2) := \frac{1}{\omega^2} + \sum_{(\ell_1, \ell_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left( \frac{1}{(\omega + \ell_1\omega_1 + \ell_2\omega_2)^2} - \frac{1}{(\ell_1\omega_1 + \ell_2\omega_2)^2} \right).$$

Then, the field of meromorphic functions on  $\overline{E}_t$  is isomorphic to the field of meromorphic functions on  $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ , which is itself isomorphic to the field of meromorphic functions on  $\mathbb{C}$  that are  $(\omega_1, \omega_2)$ -periodic. The latter field is equal to  $\mathbb{C}(\wp, \wp')$ , see [WW96].

The maps  $\iota_1, \iota_2$  and  $\sigma$  may be lifted to the  $\omega$ -plane  $\mathbb{C}$ . We denote these lifts by  $\tilde{\iota}_1, \tilde{\iota}_2$  and  $\tilde{\sigma}$  respectively. So we have the commutative diagrams

$$\begin{array}{ccc} \overline{E}_t & \xrightarrow{\iota_k} & \overline{E}_t \\ \uparrow \Lambda & & \uparrow \Lambda \\ \mathbb{C} & \xrightarrow{\tilde{\iota}_k} & \mathbb{C} \end{array} \quad \begin{array}{ccc} \overline{E}_t & \xrightarrow{\sigma} & \overline{E}_t \\ \uparrow \Lambda & & \uparrow \Lambda \\ \mathbb{C} & \xrightarrow{\tilde{\sigma}} & \mathbb{C} \end{array}$$

The following result has been proved

- in [FIM17, Section 3.3] when  $t = 1$ ,
- in [Ras12] in the *unweighted* case for general  $0 < t < 1$ , not necessarily transcendental over  $\mathbb{Q}(d_{i,j})$ ,
- in [DR19, Proposition 18] in the *weighted* case, with general  $0 < t < 1$ , not necessarily transcendental over  $\mathbb{Q}(d_{i,j})$ .

In what follows, we set  $D(\omega) = \Delta_1([\omega : 1])$ . Let us introduce  $z = 2A(x)y + B(x)$ , where  $A(x) = t(d_{-1,1} + d_{0,1}x + d_{1,1}x^2)$ , and  $B(x) = t(d_{-1,0} - \frac{1}{t}x + d_{0,0}x + d_{1,0}x^2)$ .

**Proposition 4.12.** *An explicit uniformization  $\Lambda : \mathbb{C} \rightarrow \overline{E}_t$  such that*

$$\tilde{\iota}_1(\omega) = -\omega, \quad \tilde{\iota}_2(\omega) = -\omega + \omega_3 \quad \text{and} \quad \tilde{\sigma}(\omega) = \omega + \omega_3,$$

for a certain  $\omega_3 \in \mathbb{C}^*$  is given by

$$\Lambda(\omega) = (x(\omega), y(\omega))$$



where  $x(\omega)$  and  $y(\omega)$  are given by

	$x(\omega)$	$z(\omega)$
$a_4 \neq [1:0]$	$\left[ a_4 + \frac{D'(a_4)}{\wp(\omega) - \frac{1}{6}D''(a_4)} : 1 \right]$	$\left[ \frac{D'(a_4)\wp'(\omega)}{2(\wp(\omega) - \frac{1}{6}D''(a_4))^2} : 1 \right]$
$a_4 = [1:0]$	$[\wp(\omega) - \alpha_2/3 : \alpha_3]$	$[-\wp'(\omega) : 2\alpha_3]$

A suitable choice for the periods  $(\omega_1, \omega_2)$  is given by the elliptic integrals

$$(4.3) \quad \omega_1 = \mathbf{i} \int_{a_3}^{a_4} \frac{dx}{\sqrt{|D(x)|}} \in \mathbf{i}\mathbb{R}_{>0} \quad \text{and} \quad \omega_2 = \int_{a_4}^{a_1} \frac{dx}{\sqrt{D(x)}} \in \mathbb{R}_{>0}.$$

Note that, according to [DR19, Section 2],

$$\omega_3 = \int_{a_4}^{X_{\pm}(b_4)} \frac{dx}{\sqrt{D(x)}} \in ]0, \omega_2[.$$

*Remark 4.13.* Contrary to the genus zero situation, the map  $\sigma$  may have finite order.

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INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE, U.M.R. 7501 UNIVERSITÉ DE STRASBOURG  
ET C.N.R.S. 7, RUE RENÉ DESCARTES 67084 STRASBOURG, FRANCE

*Email address:* `dreyfus@math.unistra.fr`

UNIVERSITÉ PAUL SABATIER - INSTITUT DE MATHÉMATIQUES DE TOULOUSE, 118 ROUTE DE NARBONNE, 31062 TOULOUSE.

*Email address:* `hardouin@math.univ-toulouse.fr`

UNIV LYON, UNIVERSITÉ CLAUDE BERNARD LYON 1, CNRS UMR 5208, INSTITUT CAMILLE JORDAN, 43 BLVD. DU 11 NOVEMBRE 1918, F-69622 VILLEURBANNE CEDEX, FRANCE

*Email address:* `roques@math.univ-lyon1.fr`

DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, BOX 8205, RALEIGH, NC 27695-8205, USA

*Email address:* `singer@ncsu.edu`