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# Convergence of a discretization of the Maxwell-Klein-Gordon equation based on Finite Element Methods and Lattice Gauge Theory

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## Abstract

In this article we study a semi-discrete numerical scheme for the Maxwell-Klein-Gordon equation in two spatial dimensions, based on Finite Elements and Lattice Gauge Theory (LGT). The discretization procedure from LGT, together with a numerical quadrature, ensures gauge invariance of the scheme. The gauge invariance implies that the scheme is constraint preserving. We combine this with energy conservation to prove convergence of the scheme towards a weak solution, for initial data of finite energy. Finally, we propose and implement a numerical scheme to assess our results.

## 1 Introduction

The Standard Model of physics and Einstein’s theory of Gravitation, which describe the fundamental forces and particles in nature, involve nonlinear partial differential equations. In the Standard Model the particles are either fermions or bosons. The fermions are described by spinors, and the forces between them by connections on certain vector bundles. The connections model bosons. In Quantum Electrodynamics (QED) these are photons. In particular, in scalar QED, one models spin-0 bosons that interacts with photons.

The equations are in both cases derived through a variational principle from a Lagrangian function. The form of the Lagrangian is determined by the symmetry group (gauge symmetry) which the theory is invariant under. If, in addition, one demands the theory to be renormalizable, the maximal order of derivatives in the Lagrangian is reduced to one, see Ref. [22].

Euler-Lagrange (EL) equations stemming from Lagrangians constructed from a local gauge group always contain certain constraint equations. However, through Noether’s theorem (see Ref. [21]) one can show that these constraints are preserved on the solutions of the evolution EL-equations, making the theory consistent.

Since the gauge symmetry makes the theory consistent, one should strive to preserve this symmetry when discretizing the model. This is the spirit of the field of structure preserving discretization in related areas of numerical analysis (see *e.g.* [3, 4, 12, 13, 16, 17] and references therein). This is also of interest in applied physics too (see *e.g.* [26] and references therein). The idea is that the preservation of the underlying geometric structure will reveal itself through both stability and good *qualitative* properties of the solutions.

The coupling between the gauge connections and scalar fields/spinors arises through covariant derivatives. The covariant derivative applied to a scalar/spinor field is given as a linear combination

of the gradient and the gauge field acting on the scalar/spinor field. Conventional methods, such as standard Finite Difference Methods (FDM) or Finite Element Methods (FEM), approximate the gradient part and the part containing the gauge field separately. By doing this, the resulting approximation of the covariant derivative has no local transformation law, and the gauge symmetry is broken at the discrete level.

The key to preserve the gauge symmetry is to approximate the covariant derivative directly, and not as the sum of the gradient and the product of the gauge field with the scalar field/spinor. This was achieved by Kenneth Wilson [29] in 1974 in the field of high energy physics. He was doing calculations on quarks on a lattice, and the theory he developed is now known as Lattice Gauge Theory (LGT). LGT approximates the covariant derivative in a consistent way and at the same time preserves the local gauge symmetry. The essence in the procedure is to localize the nonlocal terms arising in finite difference methods : the nonlocal terms which are to be compared, are parallel transported to a common reference point with the gauge potential. By doing this, the discrete theory becomes gauge invariant.

In two other articles Ref. [5, 6], two of the authors analyzed LGT discretization of Maxwell-type equations.

In Ref. [5] a LGT scheme and a standard finite difference discretization for the Maxwell-Klein-Gordon (MKG) equations have been compared. Concerning the LGT-scheme, the constraint, which is a statement about charge conservation, is preserved on the solution of the evolution equations, as a consequence of a discrete version of Noether's theorem. This makes, as indicated earlier, the equations to be solved compatible. The standard finite difference scheme does not preserve the constraint. However no proof of convergence has been provided in that paper.

In Ref. [6], the convergence of LGT scheme applied to pure Maxwell theory has been studied. This was done by comparing the LGT-scheme with the converging Yee-scheme [30].

With a more applicative perspective, in [26], the same kind of lattice schemes is used (with a Yee-type scheme for the electromagnetic part) to investigate some numerical simulations for the massive Maxwell-Klein-Gordon equation in the context of QED.

Apart from LGT type discretizations, and regarding numerical analysis works in this context, in [8], two of the authors proposed a Finite-Element semi-discrete energy preserving scheme in the temporal gauge where gauge symmetry is lost but the constraint preservation is recovered using a Lagrange multiplier. The complete semi-discrete numerical analysis is provided. Let us also mention the recent study [14] considering Maxwell-Klein-Gordon equations in the Coulomb Gauge. There, a discretization framework is proposed, based on Finite Elements in space and a modified Crank-Nicolson scheme in time that is energy preserving but not constraint preserving. A complete convergence study and some academical test cases are provided. To the best knowledge of the authors, these are the only studies where a complete numerical analysis work is tackled in the precise context of Maxwell Klein Gordon equations.

In this article we propose a semi-discrete numerical scheme for the massive and renormalizable MKG equations in the temporal Gauge, based on conforming Finite Elements (FE) and LGT. The Klein-Gordon (KG) part is discretized as in Ref. [5] with LGT techniques, while the Maxwell part is discretized using conforming Nédélec Finite Elements.

In this work, we prove the convergence of the semi-discrete scheme in two dimensions. This is done with some inspiration from the methodology used in [8]. We furthermore propose a fully discrete scheme based on a leap-frog time integration scheme and give some academical numerical tests cases for validation.

The paper is organized as follows: In §2, the continuous model is introduced from a variational point of view. In §3, we set the discretization in space using the lowest order Nédélec elements [20] on rectangles. The discrete gauge invariant Lagrangian is developed through both LGT and FE. Constraint and energy conservation are shown leading to the proof of the convergence of the scheme in §4. Finally, section §5 numerically assess the fully discrete scheme.

## 2 The Maxwell-Klein-Gordon equation

We first set the equations in a quite general setting of differential forms. Let  $M$  be a compact Riemannian manifold without boundary. The space of real-valued  $k$ -forms will be denoted  $\Omega^k(M)$ . We will often identify one-forms and vector-fields. The real valued  $L^2$ -product on differential forms on  $M$  is denoted  $\langle \cdot, \cdot \rangle$ , and the associated  $L^2$ -norm  $\| \cdot \|$ . Similar notations will be used for complex valued forms. All adjoints, denoted  $(\cdot)^*$ , will be taken with respect to these  $L^2$  products.

### 2.1 Formulation

#### 2.1.1 The Klein Gordon action

The unknown of the Klein-Gordon theory is a complex scalar field,  $t \mapsto \phi(t)$  (an element in  $\Omega^0(M) \otimes \mathbb{C}$ ), and the corresponding action functional is given by

$$S[\phi] = \frac{1}{2} \int_0^T \left( \|\dot{\phi}\|^2 - \|d\phi\|^2 - \mathfrak{m}^2 \|\phi\|^2 - \frac{\gamma}{2} \|\phi\|^4 \right) dt, \quad (2.1)$$

where the dot represents time derivative and  $d : \Omega^k(M) \otimes \mathbb{C} \rightarrow \Omega^{k+1}(M) \otimes \mathbb{C}$  is the exterior derivative acting on complex valued forms. The third term is the mass term ( $\mathfrak{m} \geq 0$  is the mass) and the fourth term is the self coupling term with  $\gamma \geq 0$ .

**Remark 2.1.** *The case  $\gamma < 0$  will not be envisaged in this work and should deserve a specific study, since it leads to non-positive energy.*

#### 2.1.2 The Maxwell Klein Gordon action

In this section, we formally explain the classical physical steps to achieve the gauge invariant MKG action. The MKG-equation is obtained by imposing a local  $U(1)$ -symmetry, i.e. by demanding the action (or more precisely the Lagrangian) to be invariant under the transformation

$$\phi \rightsquigarrow e^{i\beta} \phi, \quad (2.2)$$

where  $\beta$  is a real valued function on  $M$ ,  $\beta \in \Omega^0(M)$ . The Lagrangian given in equation (2.1) is clearly not invariant under this transformation.

This is resolved by replacing the usual derivatives with covariant derivatives, i.e.  $\partial_t \rightsquigarrow D_\alpha := \partial_t + iq\alpha$  and  $d \rightsquigarrow D_A := d + iqA$ , with  $q$  the coupling constant. Here  $t \mapsto \alpha(t)$  is a real valued function on  $M$ , i.e.  $\alpha \in \Omega^0(M)$ , and  $t \mapsto A(t)$  is a real valued one-form on  $M$ , i.e.  $A \in \Omega^1(M)$ .

The function  $\alpha$  is usually called the *electric potential* while  $A$  is called the *magnetic potential*. They are related to the electric and magnetic fields by the following equations

$$\mathbf{E} = -\dot{A} + d\alpha, \quad \mathbf{B} = dA,$$

and they transform as

$$\begin{aligned} \alpha &\rightsquigarrow \alpha - \dot{\beta}, \\ A &\rightsquigarrow A - d\beta, \end{aligned}$$

simultaneously with (2.2). This constitutes the gauge transformation of the field  $(\phi, \alpha, A)$  given as  $(\phi, \alpha, A) \rightsquigarrow (e^{i\beta} \phi, \alpha - \dot{\beta}, A - d\beta)$ .

It is then easy to check that the following action

$$S_{KG}[\phi, \alpha, A] = \frac{1}{2} \int_0^T \left( \|D_\alpha \phi\|^2 - \|D_A \phi\|^2 - \mathfrak{m}^2 \|\phi\|^2 - \frac{\gamma}{2} \|\phi\|^4 \right) dt,$$

is locally  $U(1)$ -invariant.

To complete the action we add the  $U(1)$ -invariant Maxwell action given by

$$S_M[\alpha, A] = \frac{1}{2} \int_0^T (\|\dot{A} - d\alpha\|^2 - c^2 \|dA\|^2) dt,$$

with  $c$  the speed of propagation. The full MKG-action is then given by (cf. Ref. [22, 25, 15, 23])

$$S_{MKG}[\phi, \alpha, A] = S_{KG}[\phi, \alpha, A] + S_M[\alpha, A].$$

One can finally check that this action is invariant under the gauge transformation.

### 2.1.3 Euler-Lagrange equations

The stationary points of this action with respect to the different fields are given by the following Euler-Lagrange equations

$$\begin{aligned} \langle D_\alpha D_\alpha \phi, \phi' \rangle + \langle D_A \phi, D_A \phi' \rangle + \mathbf{m}^2 \langle \phi, \phi' \rangle + \gamma \langle |\phi|^2 \phi, \phi' \rangle &= 0, & \forall \phi' \in \Omega^0(M) \otimes \mathbb{C}, \\ \langle \dot{\mathbf{E}}, A' \rangle - c^2 \langle dA, dA' \rangle + \langle i\phi^* D_A \phi, A' \rangle &= 0, & \forall A' \in \Omega^1(M), \\ \langle \mathbf{E}, d\alpha' \rangle - \langle i\phi^* D_\alpha \phi, \alpha' \rangle &= 0, & \forall \alpha' \in \Omega^0(M). \end{aligned} \quad (2.3)$$

In a strong formulation, the Euler-Lagrange equations are also given by

$$D_\alpha D_\alpha \phi + D_A^* D_A \phi + \mathbf{m}^2 \phi + \gamma |\phi|^2 \phi = 0, \quad (2.4)$$

$$\dot{\mathbf{E}} - c^2 d^* dA + i \frac{1}{2} (\phi^* D_A \phi - \phi (D_A \phi)^*) = 0, \quad (2.5)$$

$$d^* \mathbf{E} - i \frac{1}{2} (\phi^* D_\alpha \phi - \phi (D_\alpha \phi)^*) = 0, \quad (2.6)$$

where  $d^*$  is the adjoint to  $d$  and  $D_A^* = d^* - iqA$ .

We see that the Euler-Lagrange-equations consist of two evolution equations given by (2.4)-(2.5) and a constraint equation given by equation (2.6). Due to the local gauge invariance, Noether's second theorem can be applied to conclude that the constraint is preserved on the solution of the evolution equations (see [5]), which makes the equations consistent. See Ref. [21] for the continuous version(s) of Noether's theorem(s).

These equations together with the differential Bianchi identity (cf. Ref. [28])

$$d\mathbf{B} = 0, \quad \dot{\mathbf{B}} = -d\mathbf{E},$$

which is satisfied by construction of the electromagnetic field from a gauge potential, constitute the complete set of the MKG-equations.

In the rest of the paper, we consider unitary constants  $q = 1$ ,  $c = 1$ . For the mass  $\mathbf{m}$  and self-coupling constant  $\gamma$ , we will consider that they are either 1 or 0 in the mathematical proofs to allow for variations on the system of equations considered. All the proofs are of course valid for non unitary cases.

## 2.2 Definition of weak solution

We now focus on domains in  $\mathbb{R}^2$ . From this section, we also leave the differential forms notation behind (namely  $d$ ) and use rather the usual notation  $\text{grad} = \nabla$ ,  $\text{curl} = \nabla \times$ ,  $\text{div} = \nabla \cdot$ .

### 2.2.1 Notations

We let  $S$  be a bounded contractible domain in  $\mathbb{R}^2$  with  $\mathcal{C}^1$  boundary.

We use the classical notations for  $L^p(S)$  spaces and Sobolev spaces  $W^{1,s}(S)$ ,  $H^1(S)$ ,  $H_0^1(S)$  (with semi-norm  $|\cdot|_{H^1(S)}$ ), and  $H(\text{curl}, S)$  is the space of vector potentials in  $\mathbb{R}^2$  considered as vector fields or one forms, with square integrable curl; the analogue space for the divergence will be denoted  $H(\text{div}, S)$ . We also denote

$$H_0(\text{curl}, S) := \{\mathbf{A} \in H(\text{curl}, S) | \gamma_\tau \mathbf{A} = 0 \text{ on } \partial S\},$$

where  $\gamma_\tau \mathbf{A}$  is the tangential component of  $\mathbf{A}$  on  $\partial S$ , and

$$V := \{v \in H_0(\text{curl}, S) | \text{div } v = 0 \text{ in } \Omega\},$$

Time dependent spaces are defined as follows.

For closed intervals  $I \subseteq [0, T]$ ,  $\mathcal{C}(I; X)$  is the space of continuous functions from  $I$  to  $X$ , and  $\mathcal{C}(0, T; X)$  will denote  $\mathcal{C}([0, T]; X)$ .

We also define for  $1 \leq p \leq +\infty$ , the Bochner spaces  $L^p(0, T; X)$  for  $X$  a Banach space as in [27].

### 2.2.2 Definition of weak solutions in the temporal gauge

We use a similar notion as in [8], and we restrict the equations to the temporal gauge,  $\alpha = 0$ , allowed by the gauge symmetry.

**Definition 2.2.**  $(E, A, \psi, \phi)$  is said to be a **weak solution** of (5.11) in the temporal gauge, if

- There exists  $q < 2$ , such that

- $E \in L^\infty(0, T; L^2(S))$ ,
- $A \in \mathcal{C}(0, T; L^2(S)) \cap L^\infty(0, T; H_0(\text{curl}, S) \cap W^{1,q}(S)^2)$ ,
- $\psi \in L^\infty(0, T; L^2(S))$ ,
- $\phi \in \mathcal{C}(0, T; L^2(S)) \cap L^\infty(0, T; H_0^1(S))$ .

- $\begin{cases} \dot{A} &= -E, \\ \dot{\phi} &= -\psi. \end{cases}$

- For every  $(E', \psi') \in \mathcal{C}_c^\infty([0, T] \times S)^2 \times \mathcal{C}_c^\infty([0, T] \times S)$ , there holds

$$\begin{aligned} - \int_0^T \langle E, \dot{E}' \rangle dt - \int_0^T \langle \psi, \dot{\psi}' \rangle dt &= \int_0^T \langle \nabla \times A, \nabla \times E' \rangle dt + \int_0^T \langle D_A \phi, i\phi E' \rangle dt + \int_0^T \langle D_A \phi, D_A \psi' \rangle dt \\ &\quad + \mathfrak{m}^2 \int_0^T \langle \phi, \psi' \rangle dt + \gamma \int_0^T \langle |\phi|^2 \phi, \psi' \rangle dt. \end{aligned} \tag{2.7}$$

We now turn to the discretization of this equation.

## 3 Semi-discrete setting

### 3.1 Finite Element discretization and gauge invariance

#### 3.1.1 Finite element discretization

We discretize the spatial part of the continuous action. Let  $h > 0$ . We assume  $S$  to be a rectangular domain with a cartesian mesh  $\mathcal{T}_h$ , and we will assume homogeneous boundary conditions. Furthermore, for  $(k, l) \in \mathbb{N} \times \mathbb{N}$ ,  $\mathcal{Q}_{k,l}(\mathbb{C})$  is defined as the space of polynomials with complex

coefficients in two variables  $(x_1, x_2) \in \mathbb{R}^2$  with maximum degree  $k$  with respect to  $x_1$  and  $l$  with respect to  $x_2$ .

The discretization is based on three finite dimensional spaces  $Z_h^0$ ,  $Z_h^1$ , and  $Z_h^2$  defined as (see [20, 19] for properties of these spaces)

$$\begin{cases} Z_h^0 = \{u_h \in H_0^1(S) \mid \forall K \in \mathcal{T}_h, u_h|_K \in \mathcal{Q}_{1,1}(\mathbb{C})\} \text{ the space of piecewise } \mathcal{Q}_{1,1}(\mathbb{C}) \text{ continuous scalar} \\ \text{functions on } S. \\ Z_h^1 = \{v_h \in H_0(\text{curl}, S) \mid \forall K \in \mathcal{T}_h, v_h|_K \in \mathcal{Q}_{0,1}(\mathbb{C}) \times \mathcal{Q}_{1,0}(\mathbb{C})\}, \\ Z_h^2 = \{v_h \in L^2(S) \mid \forall K \in \mathcal{T}_h, v_h|_K \in \mathcal{Q}_{0,0}(\mathbb{C})\}. \end{cases} \quad (3.1)$$

We denote by  $Y_h^0$ ,  $Y_h^1$ ,  $Y_h^2$  (respectively) the analogues of  $Z_h^0$ ,  $Z_h^1$ ,  $Z_h^2$  constructed with real valued functions. These spaces are equipped with real basis functions  $(w_n^h)$ ,  $(w_e^h)$  and  $(w_f^h)$  respectively, which we choose as the tensor product of the one dimensional Whitney forms [20, 19]. With these choices of basis functions, scalar fields  $\phi_h$  have degrees of freedom at the nodes of the mesh,  $\phi_n^h$ , edge-vector-fields/one-forms  $A_h$  have degrees of freedom at the edges of the mesh,  $\mathbf{A}_e^h$ , while face-vector-fields/two-forms  $B_h$  have degrees of freedom at the faces of the mesh.

Moreover, the *grad* and *curl* operators in 2D relate the finite dimensional spaces  $Z_h^0$ ,  $Z_h^1$ , and  $Z_h^2$ , so that we have a complex :

$$Z_h^0 \xrightarrow{\nabla} Z_h^1 \xrightarrow{\nabla \times} Z_h^2.$$

They also induce matrices  $G = (G_{en})$  and  $R = (R_{fe})$  in the chosen bases, such that

$$\mathbb{R}^N \xrightarrow{G} \mathbb{R}^E \xrightarrow{R} \mathbb{R}^F,$$

where  $N, E$  and  $F$  are the sets of vertices, edges and faces respectively.

**Remark 3.1.** Since  $\text{curl} \circ \text{grad} = 0$ , we analogously have  $RG = 0$ .

Thus, for a node element function  $\phi_h$  and an edge element function  $A_h$  we can write

$$\begin{aligned} \phi_h &= \sum_n \phi_n^h w_n^h, & \nabla \phi_h &= \sum_e (G\phi^h)_e w_e^h, & \phi_n^h, (G\phi^h)_e &\in \mathbb{C}, \\ A_h &= \sum_e \mathbf{A}_e^h w_e^h, & \nabla \times A_h &= \sum_f (R\mathbf{A}^h)_f w_f^h, & \mathbf{A}_e^h, (R\mathbf{A}^h)_f &\in \mathbb{R}. \end{aligned}$$

Here  $\phi_n^h$  and  $(G\phi^h)_e$  are vertex and edge degrees of freedom, while  $\mathbf{A}_e^h$  and  $(R\mathbf{A}^h)_f$  are edge and face degrees of freedom. The quantity  $(G\phi^h)_e$  represents the differential of the scalar field along the edge  $e$ , and has the form  $(\phi_m^h - \phi_n^h)$ , where the edge  $e$  goes from node  $m$  to node  $n$ . Since we are considering rectangles, a natural orthogonal coordinate system can be associated to the mesh, and in such a coordinate system the expression  $(\phi_m^h - \phi_n^h)$  represents the differential in one of the two directions (see [5] for a more explicit formulation).

The notation  $e = \{m, n\}$  will denote the edge  $e$  which goes from node  $m$  to node  $n$ .

### 3.1.2 Scalar products and norms

The information about the shape and size of the rectangles is encoded in the *mass matrices*. For two edges  $e$  and  $e'$ , we define

$$(M_1^h)_{ee'} = \int_S w_e^h \cdot w_{e'}^h,$$

where  $\cdot$  denotes the scalar product in  $\mathbb{R}^2$ . We also define

$$(M_k^h)_{ss'} = \int_S w_s^h w_{s'}^h,$$

in the cases  $k = 0$  and  $k = 2$ , where  $s$  and  $s'$  are respectively two nodes or two faces of the mesh.

We note that the matrices  $M_k^h$  are square, symmetric and positive definite. We also note that the non-zero entries in  $M_0^h, M_1^h, M_2^h$  are of the order  $h^2, h^0, h^{-2}$ , where  $h$  denotes the maximum diameter of the elements in the mesh.

These matrices  $M_k^h$  are the representative matrices of the  $L^2$ -products in the corresponding bases  $w^h$ . Thus, for example, for two edge element fields  $u_h$  and  $v_h$  written as

$$u_h = \sum_e \mathbf{u}_e^h w_e^h, \quad v_h = \sum_e \mathbf{v}_e^h w_e^h,$$

where  $\mathbf{u}_e^h$  and  $\mathbf{v}_e^h$  are the edge degrees of freedom (DoF), we have

$$\begin{aligned} \langle u_h, v_h \rangle &= \frac{1}{2} \sum_{e, e' \in E} (\overline{\mathbf{u}_e^h} \mathbf{v}_{e'}^h + \mathbf{u}_e^h \overline{\mathbf{v}_{e'}^h}) \int_S w_e^h \cdot w_{e'}^h \\ &= \Re(\overline{\mathbf{u}^h}^T M_1^h \mathbf{v}^h) = (M_1^h \mathbf{u}^h, \mathbf{v}^h). \end{aligned}$$

We use  $(\cdot, \cdot)$  to denote the real valued scalar product of vectors.

### 3.1.3 Norm estimates

One can establish estimates between norms and degrees of freedom as in the following Lemmas

**Lemma 3.2.** *If  $u_h \in Y_h^1$ , then*

$$|\mathbf{u}_e^h| \leq C \|u_h\|_{L^2(S)}.$$

Furthermore, with  $p > 2$ ,  $q > 2$  and  $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ , we have :

$$|\mathbf{u}_e^h| \leq C h^{\frac{2}{q}} \|u_h\|_{L^p(S)}.$$

*Proof.* Let  $u_h \in Y_h^1$ . There exists a constant  $C$ , independent of  $u$  and  $h$  such that

$$\frac{1}{C} (\mathbf{u}^h)^T \mathbf{u}^h \leq \|u_h\|_{L^2}^2 = (M_1^h \mathbf{u}^h, \mathbf{u}^h) \leq C (\mathbf{u}^h)^T \mathbf{u}^h, \quad (\mathbf{u}^h)^T \mathbf{u}^h = \sum_e (\mathbf{u}_e^h)^2.$$

Thus

$$|\mathbf{u}_e^h| \leq \sqrt{\sum_{e' \in K} (\mathbf{u}_{e'}^h)^2} \leq C \|u_h\|_{L^2(K)} \leq C \|u_h\|_{L^2(S)}.$$

This estimate together with Hölder's inequality gives

$$|\mathbf{u}_e^h| \leq C h^{\frac{2}{q}} \|u_h\|_{L^p(S)}.$$

□

### 3.1.4 Gauge symmetry

We can redo the formal derivation of the gauge invariant action at the discrete level. This amounts to understand how the natural analogue of the action at the discrete level would transform under the gauge and correct the terms to obtain a gauge invariant action.

With the notation developed in the previous sections, we can express the discrete version of the Lagrangian from (2.1) as

$$\langle \dot{\phi}_h, \dot{\phi}_h \rangle - \langle \nabla \phi_h, \nabla \phi_h \rangle = (M_0^h \dot{\phi}^h, \dot{\phi}^h) - (M_1^h G \phi^h, G \phi^h).$$



where  $\dot{\phi}^h$  and  $\phi^h$  are degrees-of-freedom vectors.

As in the continuous model we want to impose a local  $U(1)$  gauge symmetry. We would like the theory to be invariant under the set of transformations

$$\phi_h \rightsquigarrow e^{i\beta_h} \phi_h, \quad (3.2)$$

where  $t \mapsto \beta_h(t) \in Y_h^0$  is a real valued function, but with  $\phi_h \in Z_h^0$ ,  $e^{i\beta_h} \phi_h \notin Z_h^0$ .

We thus modify  $\langle \dot{\phi}_h, \dot{\phi}_h \rangle = (M_0^h \dot{\phi}^h, \dot{\phi}^h)$  in two steps.

- (a) First, as in the continuous case, we replace the ordinary time derivative by the covariant time derivative, i.e.  $\partial_0 \rightsquigarrow D_{\alpha_h} = \partial_0 + i\alpha_h$ , where  $\alpha_h$  is a real valued function,  $\alpha_h \in Y_h^0$ . Furthermore,  $\alpha_h \rightsquigarrow \alpha_h - \dot{\beta}_h$  under a gauge transformation, implying that  $D_{\alpha_h} \phi_h$  transforms as

$$D_{\alpha_h} \phi_h \rightsquigarrow e^{i\beta_h} D_{\alpha_h} \phi_h.$$

- (b) Next, we replace the mass matrix  $M_0^h$  with a mass lumped version  $H_0^h$ , which is both diagonal and positive definite. Its entries are given by

$$(H_0^h)_{nm} = \begin{cases} \sum_k (M_0^h)_{nk}, & n = m, \\ 0, & n \neq m. \end{cases}$$

Let  $u, v$  be continuous scalar functions, and  $\mathbf{u}, \mathbf{v}$  their vectors of nodal degrees-of-freedom. We define the associated bilinear form by  $\langle \cdot, \cdot \rangle_{0,h}$  (which gives the scalar product associated to the matrix  $H_0^h$  on  $Z_h^0$ ) by

$$\begin{aligned} \langle u, v \rangle_{0,h} &= \sum_{n \in N} \Re(u_n (H_0^h)_{nn} \bar{v}_n) = (H_0^h \mathbf{u}, \mathbf{v}) \\ &= \Re \left( \int_S \Pi_{0,h}(u \bar{v}) dS \right) = \sum_{K \in \mathcal{T}_h} \frac{|K|}{4} \sum_{x \in K} \Re(u(x) \bar{v}(x)), \end{aligned} \quad (3.3)$$

where the points  $x$  appearing in the sum are the vertices of  $K$  and  $(\cdot, \cdot)$  denotes the real valued scalar product for nodal DoF vectors.

*Consistency.*  $H_0^h$  is consistent with  $M_0^h$ . More precisely, we have the following error estimate (see [10, 9]).

For real  $l > 1$ , and all  $u \in H^l(S)$  and  $v_h \in Z_h^0$ , there exists a constant  $C$  depending on  $l$  such that

$$|\langle u, v_h \rangle_{0,h} - \langle u, v_h \rangle| \leq Ch \|u\|_{H^l(S)} \|v_h\|_{L^2(S)}. \quad (3.4)$$

As a conclusion, we have modified the term  $\langle \dot{\phi}_h, \dot{\phi}_h \rangle$  as  $\langle D_{\alpha_h} \phi_h, D_{\alpha_h} \phi_h \rangle_{0,h}$ .

As for the term  $\langle \dot{\phi}_h, \dot{\phi}_h \rangle = (M_0^h \dot{\phi}^h, \dot{\phi}^h)$ , we get a gauge invariant expression approximating  $\langle \nabla \phi_h, \nabla \phi_h \rangle = (M_1^h G \phi^h, G \phi^h)$  in two steps.

- (a) The mass matrix  $M_1^h$  defines the  $L^2$ -product for fields with edge degrees-of-freedom. We lump this matrix with the same numerical quadrature as we used for  $M_0^h$  in equation (3.3), as follows.

Define the scalar product  $\langle \cdot, \cdot \rangle_{1,h}$  on  $Z_h^1 \times Z_h^1$  by a diagonal matrix  $H_1^h$  in the basis  $w_e^h$  indexed over the edges in the mesh. Let  $u, v$  be continuous vector fields/one-forms, and  $\mathbf{u}, \mathbf{v}$  their edge degrees-of-freedom. Then

$$\begin{aligned} \langle u, v \rangle_{1,h} &= \Re \left( \int_S \Pi_{0,h}(u \cdot \bar{v}) dS \right) = \sum_{K \in \mathcal{T}_h} \frac{|K|}{4} \sum_{x \in K} \Re(u(x) \bar{v}(x)) \\ &= \sum_{e \in E} \Re(u_e H_{1,ee}^h \bar{v}_e) = (H_1^h \mathbf{u}, \mathbf{v}). \end{aligned}$$

Here  $(\cdot, \cdot)$  denotes the real valued scalar product for edge DoF vectors,  $(\mathbf{u}, \mathbf{v}) = \sum_{e \in E} \Re(\mathbf{u}_e \bar{\mathbf{v}}_e)$ . We observe that  $H_1^h$  is both symmetric and positive-definite.

*Consistency.* We have the following error estimate (see [10, 9]). For real  $l > 1$ , and all  $u \in H^l(S)$  and  $v_h \in Z_h^1$ ,

$$|\langle u, v_h \rangle - \langle u, v_h \rangle_{1,h}| \leq C(l)h \|u\|_{H^l(S)} \|v_h\|_{L^2(S)}. \quad (3.5)$$

*Hölder type inequality.* Furthermore, for  $p$  and  $p'$  such that  $\frac{1}{p} + \frac{1}{p'} = 1$ ,

$$|\langle u, v \rangle_{1,h}| \leq \|u\|_{h,p} \|v_h\|_{h,p'}, \quad (3.6)$$

where for  $p > 1$ , one defines

$$\|u\|_{h,p} = \left( \int_S \Pi_{0,h}(|u|^p) dS \right)^{\frac{1}{p}},$$

which is uniformly equivalent to the true  $L^p$  norm [2], i.e. there exists a constant  $C$  independent of  $u$  such that

$$\frac{1}{C} \|u\|_{L^p(S)} \leq \|u\|_{h,p} \leq C \|u\|_{L^p(S)}.$$

**Remark 3.3.** We use the same notation,  $\|u\|_{h,p}$ , for vectorial and scalar  $u$ .

Henceforth, the mass matrix  $M_1^h$  is replaced with the above described mass lumped version  $H_1^h$  in the inner products.

However, we observe that

$$(G\phi^h)_{e=\{m,n\}} = \phi_n^h - \phi_m^h \rightsquigarrow e^{i\beta_n^h} \phi_n^h - e^{i\beta_m^h} \phi_m^h$$

under a gauge transformation, so the expression  $(H_1^h G\phi^h, G\phi^h)$  is not gauge invariant.

- (b) This can be resolved by inspiration from Lattice Gauge Theory (LGT) [29, 24, 11], i.e. we make the replacement

$$(G\phi^h)_{e=\{m,n\}} \longrightarrow (G_{\mathbf{A}^h} \phi^h)_{e=\{m,n\}} = \phi_n^h - U^h(m,n) \phi_m^h, \quad U^h(m,n) = e^{-i \int_m^n A_h},$$

where  $U^h(m,n)$  is called a link variable.

The vector-field  $A_h \in Y_h^1$  transforms as

$$A_h \rightsquigarrow A_h - \nabla \beta_h,$$

under a gauge transformation, implying that  $(G_{\mathbf{A}^h} \phi^h)_{e=\{m,n\}} \rightsquigarrow e^{i\beta_n^h} (G_{\mathbf{A}^h} \phi^h)_{e=\{m,n\}}$ .

We will denote  $G_{\mathbf{A}^h} \phi^h$  the vector of degrees of freedom  $(\phi_n^h - U^h(m,n) \phi_m^h)_{e=\{m,n\}}$  and  $G_{A_h} \phi_h$  the corresponding element of  $Z_h^1$  (i.e. that has the vector  $G_{\mathbf{A}^h} \phi^h$  of degrees of freedom). Furthermore, we define  $U_h \in Z_h^1$  that has vector of degrees of freedom  $(e^{-iA_e})_{e=\{m,n\}}$ .

In conclusion, we have thus replaced  $\langle \nabla \phi_h, \nabla \phi_h \rangle$  with the expression  $\langle G_{A_h} \phi_h, G_{A_h} \phi_h \rangle_{1,h}$ .

## 3.2 Discrete formulation of the Maxwell-Klein-Gordon equation

### 3.2.1 Gauge invariant discrete Maxwell-Klein-Gordon action

The LGT inspired discretely gauge invariant Klein-Gordon action is therefore given by

$$S_h^{KG}[\phi_h, \alpha_h, A_h] = \frac{1}{2} \int_0^T \left( \|D_{\alpha_h} \phi_h\|_{h,2}^2 - \|G_{A_h} \phi_h\|_{h,2}^2 - \mathfrak{m}^2 \|\phi_h\|_{h,2}^2 - \frac{\gamma}{2} \|\phi_h\|_{h,2}^2 \right) dt.$$

To complete the construction of the Maxwell-Klein-Gordon action, we add the Maxwell action

$$S^M[\alpha_h, A_h] = \frac{1}{2} \int_0^T \left( \|\dot{A}_h - \nabla \alpha_h\|_{L^2(S)}^2 - \|\nabla \times A_h\|_{L^2(S)}^2 \right) dt. \quad (3.7)$$

The discretization of the Maxwell part of the action is well understood (see e.g. Ref. [1, 18, 19, 20]), so the gauge invariant action we are going to use is

$$S_h^{MKG}[\phi_h, \alpha_h, A_h] = S_h^M[\alpha_h, A_h] + S_h^{KG}[\phi_h, \alpha_h, A_h]. \quad (3.8)$$

With the above considerations,  $S_h^{MKG}$  is invariant under the discrete *gauge transformation*  $\mathcal{G}_{\beta_h} : (\phi_h, \alpha_h, A_h) \mapsto (\Pi_{0,h}(e^{i\beta_h}\phi_h), \alpha_h - \beta_h, A_h - \nabla\beta_h)$ , with  $t \mapsto \beta_h(t) \in Y_h^0$  and  $\Pi_{0,h}$  the nodal interpolant onto  $Z_h^0$ .

### 3.2.2 Weak formulation of the discretized equations

The variation of  $S_h^{MKG}$  at  $(\phi_h, \alpha_h, A_h)$  in the direction  $(\phi'_h, \alpha'_h, A'_h)$  is given by

$$\begin{aligned} DS_h^{MKG}[\phi_h, \alpha_h, A_h](\phi'_h, 0, 0) &= \int_0^T \left( \langle D_\alpha \phi_h, D_\alpha \phi'_h \rangle_{0,h} - \langle G_{A_h} \phi_h, G_{A_h} \phi'_h \rangle_{1,h} \right) dt \\ &\quad + \mathbf{m}^2 \int_0^T \left( \langle \phi_h, \phi'_h \rangle_{0,h} + \gamma \langle |\phi_h|^2 \phi_h, \phi'_h \rangle_{0,h} \right) dt \end{aligned} \quad (3.9)$$

$$DS_h^{MKG}[\phi_h, \alpha_h, A_h](0, \alpha'_h, 0) = \int_0^T \left( -\langle \dot{A}_h - \nabla \alpha_h, \nabla \alpha'_h \rangle + \langle D_\alpha \phi_h, i\alpha'_h \phi_h \rangle_{0,h} \right) dt, \quad (3.10)$$

$$\begin{aligned} DS_h^{MKG}[\phi_h, \alpha_h, A_h](0, 0, A'_h) &= \\ \int_0^T \left( \langle \dot{A}_h - \nabla \alpha_h, \dot{A}'_h \rangle - \langle \nabla \times A_h, \nabla \times A'_h \rangle - \langle G_{A_h} \phi_h, i\mathcal{R}_h(U_h, \phi_h, A'_h) \rangle_{1,h} \right) dt, \end{aligned} \quad (3.11)$$

where  $\mathcal{R}_h : Z_h^1 \times Z_h^0 \times Y_h^1 \rightarrow Z_h^1$ , and for all  $(\tilde{U}, \tilde{\phi}, \tilde{A})$  in  $Z_h^1 \times Z_h^0$ ,  $\mathcal{R}_h(\tilde{U}, \tilde{\phi}, \tilde{A})$  is the edge element uniquely defined by its edge degrees of freedom

$$(\mathcal{R}_h(\tilde{U}, \tilde{\phi}, \tilde{A}))_e := \tilde{U}_e \tilde{\phi}_m \tilde{A}_e,$$

for  $e = \{m, n\}$ .

The Euler-Lagrange equations are given by the stationarity of the action, i.e.

$$DS_h^{MKG}[\phi_h, \alpha_h, A](\phi'_h, \alpha'_h, A'_h) = 0.$$

By defining the electric field  $E_h$  as

$$E_h = \nabla \alpha_h - \dot{A}_h, \quad (3.12)$$

and by a partial integration in time, the Euler-Lagrange equations read

$$\begin{aligned} \langle D_\alpha^2 \phi_h, \phi'_h \rangle_{0,h} + \langle G_{A_h} \phi_h, G_{A_h} \phi'_h \rangle_{1,h} + \mathbf{m}^2 \langle \phi_h, \phi'_h \rangle_{0,h} + \gamma \langle |\phi_h|^2 \phi_h, \phi'_h \rangle_{0,h} &= 0, & \forall \phi'_h \in Z_h^0, \\ \langle \dot{E}_h, A'_h \rangle - \langle \nabla \times A_h, \nabla \times A'_h \rangle - \langle G_{A_h} \phi_h, i\mathcal{R}_h(U_h, \phi_h, A'_h) \rangle_{1,h} &= 0, & \forall A'_h \in Y_h^1, \end{aligned} \quad (3.13)$$

and

$$\langle E_h, \nabla \alpha'_h \rangle + \langle D_\alpha \phi_h, i\alpha'_h \phi_h \rangle_{0,h} = 0, \quad \forall \alpha'_h \in Y_h^1. \quad (3.14)$$

We see that these equations consist of two evolution equations, Eq. (3.13), and one constraint equation, Eq. (3.14) (corresponding to equation (5.11)).

We suppose that the following initial conditions,  $A^0 \in H^1(S)$ ,  $E^0 \in L^2(S)$ ,  $\phi^0 \in H_0^1(S)$ ,  $\psi^0 \in L^2(S)$  are given.

Then we consider the following discrete initial conditions

$$\begin{aligned} A_h(0, \cdot) &= A_h^0 \in Y_h^1, \\ E_h(0, \cdot) &= E_h^0 \in Y_h^1, \\ \phi_h(0, \cdot) &= \phi_h^0 \in Z_h^0, \\ \psi_h(0, \cdot) &= \psi_h^0 \in Z_h^0. \end{aligned} \tag{3.15}$$

Furthermore, we suppose that they are chosen such that

$$A_h^0 \xrightarrow{h \rightarrow 0} A^0 \text{ in } H_0(\text{curl}, S) \cap L^q, \quad E_h^0 \xrightarrow{h \rightarrow 0} E^0 \text{ in } L^2(S), \tag{3.16}$$

$$\phi_h^0 \xrightarrow{h \rightarrow 0} \phi^0 \text{ in } H_0^1(S), \quad \psi_h^0 \xrightarrow{h \rightarrow 0} \psi^0 \text{ in } L^2(S). \tag{3.17}$$

In the rest of section 3.2, for simplicity of notations we drop the indices  $h$ , and consider the situation where  $h$  is fixed.

### 3.2.3 Constraint preservation

One important feature of this scheme concerns the constraint equation (3.14). The discrete MKG action (3.8) is gauge invariant, since both terms are. One can therefore use a discrete Noether's theorem to prove constraint preservation, in a similar manner as in [5].

We can also show this by a direct calculation.

**Theorem 1.** *Suppose  $(\mathbf{E}, \mathbf{A}, \alpha, \phi)$  solves equation (3.13) on a time interval  $[0, T]$ . Suppose furthermore that the constraint (3.14) is satisfied at  $t = 0$ . Then the constraint (3.14) is satisfied for all  $t \in [0, T]$ .*

*Proof.* We start out by a differentiation in time of the left hand side of equation (3.14), denoted  $\kappa$ . This gives

$$\dot{\kappa} = \langle \dot{E}, \nabla \alpha' \rangle + \langle D_\alpha \dot{\phi}, i\alpha' \phi \rangle_{0,h} + \langle D_\alpha \phi, i\alpha' \dot{\phi} \rangle_{0,h} + \langle i\dot{\alpha} \phi, i\alpha' \phi \rangle_{0,h}. \tag{3.18}$$

By the evolution equation for the electric field, equation (3.13), we have

$$\langle \dot{E}, \nabla \alpha' \rangle = \langle G_A \phi, i\mathcal{R}(U, \phi, \nabla \alpha') \rangle_{1,h}. \tag{3.19}$$

Furthermore,

$$D_\alpha^2 \phi = D_\alpha \dot{\phi} + i\alpha D_\alpha \phi + i\dot{\alpha} \phi, \tag{3.20}$$

and we can apply the nodal interpolator to both sides of this equality. Then (3.18) can be rewritten

$$\dot{\kappa} = \langle G_A \phi, i\mathcal{R}(U, \phi, \nabla \alpha') \rangle_{1,h} + \langle D_\alpha^2 \phi, i\alpha' \phi \rangle_{0,h} + \langle D_\alpha \phi, i\alpha' \dot{\phi} \rangle_{0,h} - \langle i\alpha D_\alpha \phi, i\alpha' \phi \rangle_{0,h}. \tag{3.21}$$

The evolution equation for the Klein-Gordon scalar field gives

$$\begin{aligned} \dot{\kappa} &= \langle G_A \phi, i\mathcal{R}(U, \phi, \nabla \alpha') \rangle_{1,h} - \langle G_A \phi, G_A(i\Pi_{0,h}(\alpha' \phi)) \rangle_{1,h} + \mathbf{m}^2 \langle \phi, i\Pi_{0,h}(\alpha' \phi) \rangle_{0,h} + \gamma \langle |\phi|^2 \phi, i\Pi_{0,h}(\alpha' \phi) \rangle_{0,h} \\ &\quad + \langle D_\alpha \phi, i\alpha' \dot{\phi} \rangle_{0,h} - \langle i\alpha D_\alpha \phi, i\alpha' \phi \rangle_{0,h}. \end{aligned} \tag{3.22}$$

Since our scalar product is real valued, we obtain

$$\langle D_\alpha \phi, i\alpha' \dot{\phi} \rangle_{0,h} - \langle i\alpha D_\alpha \phi, i\alpha' \phi \rangle_{0,h} = \langle D_\alpha \phi, i\alpha' D_\alpha \phi \rangle_{0,h} = 0 \tag{3.23}$$

In a same manner, since

$$\langle |\phi|^2 \phi, i\Pi_{0,h}(\alpha' \phi) \rangle_{0,h} = \langle |\phi|^2 \phi, i\alpha' \phi \rangle_{0,h}, \tag{3.24}$$

one has

$$\langle |\phi|^2 \phi, i\Pi_{0,h}(\alpha' \phi) \rangle_{0,h} = 0, \tag{3.25}$$

and

$$\langle \phi, i\Pi_{0,h}(\alpha' \phi) \rangle_{0,h} = \langle \phi, i\alpha' \phi \rangle_{0,h} = 0 \quad (3.26)$$

This gives

$$\dot{\kappa} = \langle G_A \phi, i\mathcal{R}(U, \phi, \nabla \alpha') \rangle_{1,h} - \langle G_A \phi, G_A(i\Pi_{0,h}(\alpha' \phi)) \rangle_{1,h}. \quad (3.27)$$

where,

$$(G_A(i\Pi_{0,h}(\alpha' \phi)))_e = i\alpha'_n \phi_n - iU_e \alpha'_m \phi_m \quad (3.28)$$

$$= i\alpha'_n (\phi_n - U_e \phi_m) - iU_e (G\alpha')_e \phi_m, \quad (3.29)$$

$$= i\alpha'_n (G_A \phi)_e - iU_e (G\alpha')_e \phi_m \quad (3.30)$$

This gives that

$$\langle G_A \phi, G_A(i\Pi_{0,h}(\alpha' \phi)) \rangle_{1,h} = \langle G_A \phi, i\mathcal{R}(U, \nabla \alpha', \phi) \rangle. \quad (3.31)$$

Thus  $\dot{\kappa} = 0$ .

This concludes the proof.  $\square$

### 3.2.4 Energy conservation

We define the energy of the system at any time with

$$\mathcal{H}(t) = \frac{1}{2} \left( \|D_\alpha \phi\|_{0,h}^2 + \|G_A \phi\|_{1,h}^2 + \|E\|^2 + \|\nabla \times A\|^2 + \mathbf{m}^2 \|\phi\|_{0,h}^2 + \frac{\gamma}{2} \|\phi\|^2_{0,h} \right).$$

We will show through a direct formal calculation that this energy is preserved by the flow.

**Proposition 1.** *Suppose  $(E, A, \alpha, \phi)$  solves the evolution equations (3.12, 3.13). Then the energy is conserved.*

*Proof.* The proof is a mere calculation. We have

$$\begin{aligned} \dot{\mathcal{H}}(t) &= \langle D_\alpha \dot{\phi}, D_\alpha \phi \rangle_{0,h} + \langle i\dot{\alpha} \phi, D_\alpha \phi \rangle_{0,h} + \langle G_A \phi, G_A \dot{\phi} \rangle_{1,h} + \langle G_A \phi, \mathcal{R}(iU, \phi, \dot{A}) \rangle_{1,h} \\ &\quad + \langle \dot{E}, E \rangle + \langle \nabla \times A, \nabla \times \dot{A} \rangle + \mathbf{m}^2 \langle \phi, \dot{\phi} \rangle_{0,h} + \gamma \langle |\phi|^2 \phi, \dot{\phi} \rangle_{0,h}. \end{aligned} \quad (3.32)$$

Using (3.20), we deduce that

$$\langle D_\alpha^2 \phi, D_\alpha \phi \rangle_{0,h} = \langle D_\alpha \dot{\phi}, D_\alpha \phi \rangle_{0,h} + \langle i\dot{\alpha} \phi, D_\alpha \phi \rangle_{0,h}. \quad (3.33)$$

So that

$$\begin{aligned} \dot{\mathcal{H}}(t) &= \langle D_\alpha^2 \phi, D_\alpha \phi \rangle_{0,h} + \langle G_A \phi, G_A \dot{\phi} \rangle_{1,h} + \langle G_A \phi, \mathcal{R}(iU, \phi, \dot{A}) \rangle_{1,h} + \langle \dot{E}, E \rangle + \langle \nabla \times A, \nabla \times \dot{A} \rangle \\ &\quad + \mathbf{m}^2 \langle \phi, \dot{\phi} \rangle_{0,h} + \gamma \langle |\phi|^2 \phi, \dot{\phi} \rangle_{0,h}. \end{aligned} \quad (3.34)$$

Using (3.13), we find

$$\begin{aligned} \dot{\mathcal{H}}(t) &= -\langle G_A \phi, G_A \Pi_{0,h}(D_\alpha \phi) \rangle_{0,h} - \mathbf{m}^2 \langle \phi, D_\alpha \phi \rangle_{0,h} - \gamma \langle |\phi|^2 \phi, D_\alpha \phi \rangle_{0,h} \\ &\quad + \langle G_A \phi, G_A \dot{\phi} \rangle_{1,h} + \langle G_A \phi, \mathcal{R}(iU, \phi, \dot{A}) \rangle_{1,h} \\ &\quad - \langle G_A \phi, i\mathcal{R}(U, \phi, \dot{A}) \rangle_{1,h} + \langle G_A \phi, i\mathcal{R}(U, \phi, \nabla \alpha) \rangle_{1,h} \\ &\quad + \mathbf{m}^2 \langle \phi, \dot{\phi} \rangle_{0,h} + \gamma \langle |\phi|^2 \phi, \dot{\phi} \rangle_{0,h}. \end{aligned} \quad (3.35)$$

Since the scalar product is real, one has that  $\langle \phi, i\alpha \phi \rangle_{0,h}$  and  $\langle |\phi|^2 \phi, i\alpha \phi \rangle_{0,h}$  vanish. This gives

$$\dot{\mathcal{H}}(t) = -\langle G_A \phi, G_A(i\alpha \phi) \rangle_{0,h} + \langle G_A \phi, i\mathcal{R}(U, \phi, \nabla \alpha) \rangle_{1,h} \quad (3.36)$$

Using the computation done in the proof of theorem 1, we conclude that  $\dot{\mathcal{H}} \equiv 0$ , so that the energy is preserved in time.  $\square$

### 3.2.5 Choice of gauge and existence

We choose to work in the temporal gauge, i.e.  $\alpha \equiv 0$  (as in definition 2.2) and, for the discretization,  $\alpha_h \equiv 0$ . This implies

$$\begin{aligned}\dot{A}_h &= -E_h, \\ D_{\alpha_h} \phi_h &= \dot{\phi}_h.\end{aligned}$$

Let  $T > 0$ . Since we are working on a finite dimensional space, we have local existence of solutions of (3.13), (3.14) and (3.15). Conservation of energy assures that the local solution is a global one: the discrete solutions are defined on the whole interval  $[0, T]$ .

## 4 Convergence of the semi-discrete scheme

In the rest of the paper,  $C$  will denote a generic constant (independent of  $t$  and  $h$ ). In the proof of convergence, we will need some results concerning the convergence of approximations. We state them here and postpone their proofs to the appendix A.1.

### 4.1 Preliminary results

**Lemma 4.1.** *Let  $\Pi_{1,h}$  be the edge interpolant. For  $p > 2$ , the following inequalities hold. There exists  $C > 0$  such that for all  $(F_h, \zeta_h) \in Z_h^1 \times Z_h^0$ ,*

$$(i) \quad \|F_h \zeta_h - \Pi_{1,h}(F_h \zeta_h)\|_{L^2(S)} \leq Ch^{1-\frac{2}{p}} \|F_h\|_{L^p(S)} |\zeta_h|_{H^1(S)}. \quad (4.1)$$

Furthermore there exists  $C > 0$  such that for all  $t \in [0, T]$ ,

$$(ii) \quad \sqrt{\sum_{K \in \mathcal{T}_h} |A_h|_{H^1(K)}^2} \leq C \left( \|\nabla \times A_h\|_{L^2(S)} + \sup_{r_h \in Y_h^0} \frac{|\langle A_h, \nabla r_h \rangle|}{\|r_h\|_{L^2(S)}} \right), \quad (4.2)$$

$$(iii) \quad \sqrt{\sum_{K \in \mathcal{T}_h} |\Pi_{1,h}(A_h \phi_h)|_{H^1(K)}^2} \leq Ch^{-\frac{2}{p}} \left[ \left( \|\nabla \times A_h\|_{L^2(S)} + \sup_{r_h \in Y_h^0} \frac{|\langle A_h, \nabla r_h \rangle|}{\|r_h\|_{L^2(S)}} \right) \|\phi_h\|_{L^p(S)} + \right. \\ \left. + \|A_h\|_{L^p(S)} |\phi_h|_{H^1(S)} \right]. \quad (4.3)$$

The following result allows, by the constraint (3.14), to control the weak divergence of  $A_h$  appearing in (4.2) and (4.3).

**Lemma 4.2.** *Let  $p > 2$ . There exists  $C > 0$  such that for all  $t \in [0, T]$ :*

$$\sup_{r_h \in Y_h^0} \frac{|\langle A_h, \nabla r_h \rangle|}{\|r_h\|_{L^2(S)}} \leq Ch^{-\frac{2}{p}} \|\phi_h\|_{L^\infty(0,T,L^p(S))} \|\dot{\phi}_h\|_{L^\infty(0,T,L^2(S))}.$$

With these results at hand, we are ready to prove the convergence of the weak solution of (3.13).

## 4.2 Study of convergence

### 4.2.1 Boundedness in the energy norm

The initial energy is bounded uniformly in  $h$ , as can be seen from (3.16), (3.17), and since it is conserved in time, we can immediately conclude that  $E_h$  and  $\nabla \times A_h$  are bounded in  $L^\infty(0, T; L^2(S))$ , i.e.

$$\begin{aligned} \|E_h\|_{L^\infty(0, T, L^2(S))} &\leq C, \\ \|\nabla \times A_h\|_{L^\infty(0, T, L^2(S))} &\leq C. \end{aligned} \quad (4.4)$$

We can also conclude that  $\phi_h$ ,  $\dot{\phi}_h$  and  $G_{A_h}\phi_h$  are bounded in time in the following sense.

$$\begin{aligned} \sup_{[0, T]} \|\dot{\phi}_h\|_{h, 2} &\leq C, \\ \sup_{[0, T]} \|G_{A_h}\phi_h\|_{h, 2} &\leq C. \end{aligned} \quad (4.5)$$

Furthermore if  $\mathbf{m} \neq 0$ ,

$$\sup_{[0, T]} \|\phi_h\|_{h, 2} \leq C, \quad (4.6)$$

and if  $\gamma > 0$ ,

$$\sup_{[0, T]} \|\phi_h\|_{h, 4} \leq C, \quad (4.7)$$

### 4.2.2 Convergence of $\phi_h$

This is obtained in three steps. First, one bounds the  $H^1(S)$ -norm of  $|\phi_h|$ , then one obtains a bound on the  $L^p(S)$  ( $p > 2$ ) norm of the gauge potential  $A_h$ . In a third step, this is used to conclude that  $\phi_h$  is bounded in  $H^1$ .

*Boundedness of the  $H^1$ -norm of  $|\phi_h|$ .* We would like to deduce that

$$\|G_{A_h}\phi_h\|_{h, 2}^2 \leq C \text{ implies } \|\nabla|\phi_h|\|_{L^2}^2 \leq C.$$

One has

$$(G_{\mathbf{A}^h}\phi^h)_{e=\{m, n\}} = \phi_m^h - U^h(m, n)\phi_n^h,$$

with

$$U^h(m, n) = \exp(-i \int_m^n A_h) = \exp(-i \mathbf{A}_e^h),$$

where  $\mathbf{A}_e^h$  is the degree of freedom relative to the edge  $e$ .

Since  $i\mathbf{A}_e^h$  is purely imaginary, we have the following estimate

$$|\phi_m| - |\phi_n| \leq |(G_{\mathbf{A}^h}\phi^h)_{e=\{m, n\}}|,$$

so that by positivity of the diagonal matrix  $H_1^h$ , one can conclude that

$$\|\nabla \Pi_{0, h}|\phi_h|\|_{h, 2}^2 \leq \|G_{A_h}\phi_h\|_{h, 2}^2.$$

We recall that  $\nabla \Pi_{0, h}|\phi_h|$  is the edge element vector field whose degrees of freedom are given by the vector  $G|\phi^h|$ . By previous estimates,

$$\|\nabla \Pi_{0, h}|\phi_h|\|_{L^2(S)}^2 \leq C \|\nabla \Pi_{0, h}|\phi_h|\|_{h, 2}^2.$$

This implies

$$\|\nabla \Pi_{0,h}|\phi_h|\|_{L^\infty(0,T,L^2(S))}^2 \leq C,$$

which gives that  $\Pi_{0,h}|\phi_h|$  is bounded in  $L^\infty(0,T,H_0^1(S))$ .

In order to extract estimates on the  $H^1$ -norm of  $\phi_h$  rather than on the  $H^1$ -norm of its modulus  $|\phi_h|$ , one needs a control of the  $L^p$ -norm of  $A_h$ .

*Boundedness of the gauge potential.* Along the same lines as in [8], one can obtain a bound on  $A_h$  in the  $L^p$  norm. To this aim, we consider the discrete Helmholtz decomposition of  $A_h$

$$A_h = \mathring{A}_h + \nabla p_h. \quad (4.8)$$

We bound each part in  $L^p$ .

*Bound on the discrete divergence-free part.* The discrete divergence free part  $\mathring{A}_h$  is bounded in  $L^p$  by the  $L^2$ -norm of the curl of the gauge potential,

$$\|\mathring{A}_h\|_{L^\infty(0,T,L^p(S))} \leq C \|\nabla \times A_h\|_{L^\infty(0,T,L^2(S))}. \quad (4.9)$$

We won't give any details on this estimate as it can be extracted exactly as in Proposition 2.5. of [8].

*Bound on the gradient part.* One has from the constraint equation (3.14)

$$\langle \nabla p_h, \nabla v_h \rangle = \langle u_h, v_h \rangle_{0,h}, \quad (4.10)$$

with

$$u_h(t) = \int_0^t \dot{\phi}_h \bar{\phi}_h dt.$$

We would like to bound  $p_h$  in  $L^\infty(0,T;W^{1,p}(S))$  for  $p > 2$ . By classical estimates that can be found in [2], we have

$$\|p_h\|_{W^{1,p}(S)} \leq C \sup_{v_h \in Y_h^0} \frac{|\langle u_h, v_h \rangle_{0,h}|}{\|v_h\|_{W^{1,p'}(S)}} \text{ at any } t \in [0,T],$$

where  $\frac{1}{p} + \frac{1}{p'} = 1$ , with  $p > 2$  and  $p' < 2$ .

Let us choose  $q'$  such that  $\frac{1}{q'} = \frac{1}{p'} - \frac{1}{2}$ . For any  $q$  such that  $\frac{1}{q} + \frac{1}{q'} = 1$ , we have

$$|\langle u_h, v_h \rangle_{0,h}| \leq \|u_h\|_{L^q(S)} \|v_h\|_{L^{q'}(S)}.$$

Furthermore,

$$\|u_h\|_{L^q(S)} \leq C \int_0^t \|\dot{\phi}_h \bar{\phi}_h\|_{L^q(S)} \leq C \int_0^t \|\dot{\phi}_h\|_{L^2(S)} \|\phi_h\|_{L^p(S)}.$$

Since  $\frac{1}{2} + \frac{1}{p} = \frac{1}{q}$ , from the energy bound (4.7), we deduce that  $\|\dot{\phi}_h\|_{L^2(S)} \leq C$  and  $\|\phi_h\|_{L^p(S)} \leq C$ . This implies that

$$|\langle u_h, v_h \rangle_{0,h}| \leq C \|v_h\|_{L^{q'}(S)},$$

independently of  $t \in [0,T]$ .

By the Sobolev embeddings  $W^{1,p'}(S) \hookrightarrow L^{q'}(S)$ , we deduce

$$|\langle u_h, v_h \rangle_{0,h}| \leq C \|v_h\|_{W^{1,p'}(S)},$$

which implies that  $p_h$  is bounded in  $W^{1,p}(S)$ , and

$$\|A_h\|_{L^\infty(0,T,L^p(S))} \leq C.$$



**Remark 4.3.** If  $\gamma > 0$ , we can also directly prove (without having to use the boundedness in the  $H^1$  norm of  $|\phi_h|$ ) that  $A_h$  is bounded in  $L^\infty(0, T, L^4(S))$ , using that in this case  $\phi_h$  is bounded in  $L^\infty(0, T, L^4(S))$ .

*Convergence of  $\phi_h$ .* From this, we deduce that  $\phi_h$  is bounded in  $H^1(S)$ . Indeed,

$$(G_{\mathbf{A}^h} \phi^h)_e = \frac{1 + \exp(-i\mathbf{A}_e^h)}{2} (G\phi^h)_e + \frac{1 - \exp(-i\mathbf{A}_e^h)}{2} (\phi_m^h + \phi_n^h),$$

so that

$$(G\phi^h)_e = \frac{2}{1 + \exp(-i\mathbf{A}_e^h)} (G_{\mathbf{A}^h} \phi^h)_e - \frac{1 - \exp(-i\mathbf{A}_e^h)}{1 + \exp(-i\mathbf{A}_e^h)} (\phi_m^h + \phi_n^h),$$

which implies

$$|(G\phi^h)_e| \leq \frac{1}{\left| \cos\left(\frac{\mathbf{A}_e^h}{2}\right) \right|} \left| (G_{\mathbf{A}^h} \phi^h)_e \right| + \frac{\left| \sin\left(\frac{\mathbf{A}_e^h}{2}\right) \right|}{\left| \cos\left(\frac{\mathbf{A}_e^h}{2}\right) \right|} \left( |\phi_m^h| + |\phi_n^h| \right).$$

In the last section we obtained

$$\|A_h\|_{L^\infty(0, T, L^p(S))} \leq C,$$

which yields

$$|\mathbf{A}_e^h| \leq Ch^{\frac{2}{q}}, \quad (4.11)$$

where  $\frac{1}{q} + \frac{1}{p} = \frac{1}{2}$ . This means that for  $h$  sufficiently small,  $h \leq \varepsilon_0$  with  $\varepsilon_0 > 0$  given,

$$|\cos(\frac{|\mathbf{A}_e^h|}{2})| \geq C > 0.$$

We also make use of the following inequality

$$\sin(\frac{|\mathbf{A}_e^h|}{2}) \leq \frac{|\mathbf{A}_e^h|}{2}.$$

As a consequence

$$|(G\phi^h)_e| \leq C|(G_{\mathbf{A}^h} \phi^h)_e| + C|\mathbf{A}_e^h|(|\phi_m^h| + |\phi_n^h|).$$

Furthermore,  $\mathbf{A}_e^h(\frac{|\phi_m^h| + |\phi_n^h|}{2})$  are the degrees of freedom of the product  $A_h \Pi_{0,h}(|\phi_h|)$ , and one can then conclude that

$$\|\nabla \phi_h\|_{h,2}^2 \leq C\|G_{\mathbf{A}^h} \phi_h\|_{h,2}^2 + C\|\Pi_{1,h}(A_h \Pi_{0,h}(|\phi_h|))\|_{h,2}^2.$$

Using Lemma 4.1 to  $A_h \Pi_{0,h}(|\phi_h|)$  and bounds obtained on both  $A_h$  and  $|\phi_h|$ , we can conclude that

$$\|\nabla \phi_h\|_{h,2}^2 \leq C.$$

Finally since  $\|\nabla \phi_h\|_{L^2(S)} \leq C\|\nabla \phi_h\|_{h,2}$ , it follows that

$$\|\phi_h\|_{L^\infty(0, T, H_0^1(S))} \leq C.$$

**Remark 4.4.** Following remark 4.3, if  $\gamma > 0$ , we can also directly obtain this result (without having to use the boundedness in the  $H^1$  norm of  $|\phi_h|$ ) using the boundedness of  $A_h$  in  $L^\infty(0, T, L^4(S))$ .

By following [8] and using the bound on the energy, we arrive the convergence of  $\phi_h$  in  $L^\infty(0, T; L^p(S))$  (up to a subsequence) using the compactness result from [27] and interpolation estimates on  $L^p(S)$  spaces.

#### 4.2.3 Convergence of the gauge potential $A_h$

The convergence of the gauge potential is obtained by considering the discrete divergence free part and the gradient part of the discrete Helmholtz decomposition (4.8) separately.

*Discrete divergence free part.* We follow again [8]. By a Kikuchi type result, one obtains the convergence of  $\hat{A}_h$  in  $L^\infty(0, T, L^2(S))$ . Interpolation estimates then give the convergence in  $L^\infty(0, T, L^p(S))$ , since one has (4.9).

*Gradient part.* The equation we are considering is (equation(4.10))

$$\langle \nabla p_h, \nabla v_h \rangle = \langle u_h, v_h \rangle_{0,h} =: l_h(v_h), \quad \forall v_h \in Y_h^0.$$

We have  $l_h \in (Y_h^0)^*$ , and one can then find, by the Riesz representation theorem,  $f_h \in Y_h^0$  such that  $\forall v_h \in Y_h^0$ ,

$$\langle f_h, v_h \rangle = l_h(v_h).$$

Let  $q > 2$  be given. We choose  $r > 0$  such that  $\frac{1}{2} < \frac{1}{r} < \frac{1}{2} + \frac{1}{q}$ . One has

$$\|f_h\|_{L^r(S)} = \sup_{v \in L^{r'}(S)} \frac{|\langle f_h, v \rangle|}{\|v\|_{L^{r'}(S)}} = \sup_{v \in L^{r'}(S)} \frac{|\langle f_h, v_h \rangle|}{\|v\|_{L^{r'}(S)}} = \sup_{v \in L^{r'}(S)} \frac{|l_h(v_h)|}{\|v\|_{L^{r'}(S)}} \leq C \|u_h\|_{h,r},$$

where  $\frac{1}{r} + \frac{1}{r'} = 1$ , and  $v_h = P_h(v)$  is the  $L^2$  orthogonal projection on  $Y_h^0$  (which is stable in  $L^{r'}(S)$ ). As already shown in a previous section, this implies

$$\|f_h\|_{L^\infty(0,T;L^r(S))} \leq C \|\dot{\phi}_h\|_{L^\infty(0,T;L^2(S))} \|\phi_h\|_{L^\infty(0,T;L^q(S))}.$$

Furthermore,  $\forall t \in [0, T]$ ,

$$\langle \dot{f}_h, v_h \rangle = \langle \dot{u}_h, v_h \rangle_{0,h},$$

which means that

$$\|\dot{f}_h\|_{L^\infty(0,T;L^r(S))} \leq C \|\dot{\phi}_h\|_{L^\infty(0,T;L^2(S))} \|\phi_h\|_{L^\infty(0,T;L^q(S))}.$$

From this one concludes that there exists  $w \in L^\infty(0, T; W^{-1,q}(S))$  such that

$$f_h \xrightarrow[h \rightarrow 0]{} w \text{ in } L^\infty(0, T; W^{-1,q}(S)),$$

where we used the compact embedding from  $L^r(S)$  into  $W^{-1,q}(S)$ , and the compactness result of [27].

Define  $p \in W^{1,q}(S)$  as the unique solution of

$$\langle \nabla p, \nabla v \rangle = \langle w, v \rangle =: l(v), \quad \forall v \in W^{1,q'}(S), \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

We would like to prove that

$$\|p_h - p\|_{L^\infty(0,T;W^{1,q}(S))} \xrightarrow[h \rightarrow 0]{} 0.$$

In order to prove this, we use a version of the Strang lemma, i.e.

$$\sup_{v_h \in Z_h^0} \frac{|l_h(v_h) - l(v_h)|}{\|v_h\|_{W^{1,q'}(S)}} \xrightarrow[h \rightarrow 0]{} 0,$$

which is verified by construction of  $l$ , and we can conclude that  $p_h \xrightarrow[h \rightarrow 0]{} p$  in  $L^\infty(0, T; W^{1,q}(S))$  for all  $q > 2$ .

### 4.3 The limit equation

#### 4.3.1 Summary of convergences obtained

(a) Convergence obtained for  $A_h$ . We have that

$$A_h \xrightarrow{h \rightarrow 0} A \text{ in } L^\infty(0, T; L^p(S)), \forall 2 < p,$$

and from energy bound we directly have

$$\nabla \times A_h \rightharpoonup \nabla \times A \text{ in } L^\infty(0, T; L^2(S)) \text{ weak-}^*,$$

and

$$\dot{A}_h \rightharpoonup \dot{A} \text{ in } L^\infty(0, T; L^2(S)) \text{ weak-}^*.$$

(b) Convergence of  $\phi_h$ . We have that

$$\phi_h \xrightarrow{h \rightarrow 0} \phi \text{ in } L^\infty(0, T; L^p(S)), \forall p < +\infty,$$

and from energy bound, we directly have

$$\phi_h \rightharpoonup \phi \text{ in } L^\infty(0, T; H_0^1(S)) \text{ weak-}^*,$$

and

$$\dot{\phi}_h \rightharpoonup \dot{\phi} \text{ in } L^\infty(0, T; L^2(S)) \text{ weak-}^*.$$

#### 4.3.2 Limit equations

Let  $\phi' \in \mathcal{C}_c^\infty([0, T[ \times S)$  and  $A' \in (\mathcal{C}_c^\infty([0, T[ \times S))^2$ . Define  $\phi'_h = \Pi_{0,h}(\phi') \in \mathcal{C}_c^\infty(0, T; Z_h^0)$  and  $A'_h = \Pi_{1,h}(A') \in \mathcal{C}_c^\infty(0, T; Y_h^1)$ . We denote by  $\phi'^h$  and  $\mathbf{A}'^h$  the vectors of the degrees of freedom associated to  $\phi'_h$  and  $A'_h$  respectively.

In the temporal gauge, the semi-discrete equations (3.13) read

$$\int_0^T \langle \ddot{\phi}_h, \phi'_h \rangle_{0,h} + \int_0^T \langle G_{A_h} \phi_h, G_{A_h} \phi'_h \rangle_{1,h} + \mathbf{m}^2 \int_0^T \langle \phi_h, \phi'_h \rangle_{0,h} + \gamma \int_0^T \langle |\phi_h|^2 \phi_h, \phi'_h \rangle_{0,h} = 0, \quad (4.12)$$

$$\int_0^T \langle \dot{E}_h, A'_h \rangle - \int_0^T \langle \nabla \times A_h, \nabla \times A'_h \rangle - \int_0^T \langle G_{A_h} \phi_h, i\mathcal{R}_h(U_h, \phi_h, A'_h) \rangle_{1,h} = 0. \quad (4.13)$$

*Study of equation (4.12).* We have  $\int_0^T \langle \ddot{\phi}_h, \phi'_h \rangle_{0,h} = - \int_0^T \langle \dot{\phi}_h, \dot{\phi}'_h \rangle_{0,h} = - \int_0^T \langle \dot{\phi}_h, \dot{\phi}' \rangle_{0,h}$ , and by weak-\* convergence of  $\dot{\phi}_h$ ,

$$\int_0^T \langle \dot{\phi}_h, \dot{\phi}' \rangle \xrightarrow{h \rightarrow 0} \int_0^T \langle \dot{\phi}, \dot{\phi}' \rangle.$$

Furthermore from (3.4) and uniform  $L^2$  bound on  $\dot{\phi}_h$

$$\left| \int_0^T \langle \dot{\phi}_h, \dot{\phi}' \rangle_{0,h} - \int_0^T \langle \dot{\phi}_h, \dot{\phi}' \rangle \right| \leq \int_0^T Ch \|\dot{\phi}_h\|_{L^2(S)} \|\dot{\phi}'\|_{H^1(S)} \leq Ch.$$

The convergence of the terms  $\int_0^T \langle \phi_h, \phi' \rangle_{0,h}$  and  $\int_0^T \langle |\phi_h|^2 \phi_h, \phi' \rangle_{0,h}$  directly follows from the convergence of  $\phi_h$  in  $L^\infty(0, T; L^p(S))$  and the convergence for the test functions. We now study

the convergence of the term  $\int_0^T \langle G_{A_h} \phi_h, G_{A_h} \phi'_h \rangle_{1,h}$  to  $\int_0^T \langle D_A \phi, D_A \phi' \rangle$ . It will be obtained in several steps.

First, we decompose the quantity of interest  $\langle G_{A_h} \phi_h, G_{A_h} \phi'_h \rangle_{1,h} - \langle D_A \phi, D_A \phi' \rangle$  into three terms as

$$\begin{aligned} \langle G_{A_h} \phi_h, G_{A_h} \phi'_h \rangle_{1,h} - \langle D_A \phi, D_A \phi' \rangle &= \underbrace{\langle G_{A_h} \phi_h, G_{A_h} \phi'_h \rangle_{1,h} - \langle \Pi_{1,h}(D_{A_h} \phi_h), G_{A_h} \phi'_h \rangle_{1,h}}_{I_1} \\ &\quad + \underbrace{\langle \Pi_{1,h}(D_{A_h} \phi_h), G_{A_h} \phi'_h \rangle_{1,h} - \langle \Pi_{1,h}(D_{A_h} \phi_h), \Pi_{1,h}(D_{A_h} \phi'_h) \rangle_{1,h}}_{I_2} \\ &\quad + \underbrace{\langle \Pi_{1,h}(D_{A_h} \phi_h), \Pi_{1,h}(D_{A_h} \phi'_h) \rangle_{1,h} - \langle D_A \phi, D_A \phi' \rangle}_J \end{aligned} \quad (4.14)$$

We first concentrate on  $I_1$  and  $I_2$ . We need the two following Lemma. Their proofs are postponed to appendix B.

**Lemma 4.5.** *There exist  $\eta > 0$  and  $C > 0$  such that if  $h < \eta$ ,*

$$\sup_{[0,T]} \|G_{A_h} \phi_h - \Pi_{1,h} D_{A_h} \phi_h\|_{h,2}^2 \leq C h^{2-4/p},$$

so that

$$\sup_{[0,T]} \|G_{A_h} \phi_h - \Pi_{1,h} D_{A_h} \phi_h\|_{h,2} \xrightarrow{h \rightarrow 0} 0.$$

Furthermore, from the bound on the energy we directly have the following.

**Lemma 4.6.** *There exists  $C > 0$  such that*

$$\sup_{[0,T]} \|G_{A_h} \phi_h\|_{h,2} \leq C. \quad (4.15)$$

We can in a same manner obtain the analogous Lemma with  $\phi_h$  replaced by  $\phi'_h$ . Using Lemma 4.5, 4.6 and 4.1 (and their analogous counter part for  $\phi'_h$ ), one proves that  $I_1$  and  $I_2$  converges uniformly in time to 0 as  $h \rightarrow 0$ .

The estimation of  $J$  rely on the following decomposition

$$J = \underbrace{\langle \Pi_{1,h}(D_{A_h} \phi_h), \nabla \phi'_h \rangle_{1,h} - \langle \Pi_{1,h}(D_{A_h} \phi_h), \nabla \phi' \rangle}_{J_1} \quad (4.16)$$

$$+ \underbrace{\langle \nabla \phi_h, i A_h \phi'_h \rangle_{1,h} - \langle \nabla \phi_h, i A_h \phi'_h \rangle}_{J_2} \quad (4.17)$$

$$+ \underbrace{\langle \nabla \phi_h, \Pi_{1,h}(i A_h \phi'_h) - i A_h \phi'_h \rangle_{1,h}}_{J_3} \quad (4.18)$$

$$+ \underbrace{\langle \Pi_{1,h}(D_{A_h} \phi_h), i A_h \phi'_h \rangle - \langle \Pi_{1,h}(D_{A_h} \phi_h), i A \phi' \rangle}_{J_4} \quad (4.19)$$

$$+ \underbrace{\langle \Pi_{1,h}(i A_h \phi_h), \Pi_{1,h}(i A_h \phi'_h) \rangle_{1,h} - \langle \Pi_{1,h}(i A_h \phi_h), \Pi_{1,h}(i A_h \phi'_h) \rangle}_{J_5} \quad (4.20)$$

$$+ \underbrace{\langle \Pi_{1,h}(i A_h \phi_h), \Pi_{1,h}(i A_h \phi'_h) \rangle - \langle \Pi_{1,h}(i A_h \phi_h), i A_h \phi'_h \rangle}_{J_6} \quad (4.21)$$

$$+ \underbrace{\langle \Pi_{1,h}(D_{A_h} \phi_h), \nabla \phi' \rangle - \langle D_A \phi, D_A \phi' \rangle + \langle \Pi_{1,h}(D_{A_h} \phi_h), i A \phi' \rangle}_{J_7}. \quad (4.22)$$

To estimate these terms, one needs the four following Lemma. To ease the reading, we postponed their proofs to the Appendix B.

**Lemma 4.7.**

$$\|\langle \Pi_{1,h}(D_{A_h}\phi_h), \nabla\phi'_h \rangle_{1,h} - \langle \Pi_{1,h}(D_{A_h}\phi_h), \nabla\phi' \rangle\|_{L^\infty(0,T)} \xrightarrow{h \rightarrow 0} 0.$$

**Lemma 4.8.**

$$\|\langle \nabla\phi_h, A_h\phi'_h \rangle_{1,h} - \langle \nabla\phi_h, A_h\phi'_h \rangle\|_{L^\infty(0,T)} \xrightarrow{h \rightarrow 0} 0.$$

**Lemma 4.9.**

$$\|\langle \Pi_{1,h}(A_h\phi_h), \Pi_{1,h}(A_h\phi'_h) \rangle_{1,h} - \langle \Pi_{1,h}(A_h\phi_h), \Pi_{1,h}(A_h\phi'_h) \rangle\|_{L^\infty(0,T)} \xrightarrow{h \rightarrow 0} 0.$$

**Lemma 4.10.**

$$\int_0^T \langle \Pi_{1,h}D_{A_h}\phi_h, \Pi_{1,h}D_{A_h}\phi'_h \rangle dt \xrightarrow{h \rightarrow 0} \int_0^T \langle D_A\phi, D_A\phi' \rangle dt.$$

Let us briefly describe how the seven terms of the decomposition of  $J$  are treated. The first term  $J_1$  is estimated using Lemma 4.7. Lemma 4.8 gives an estimation of  $J_2$ . The terms  $J_3$ ,  $J_4$  and  $J_6$  are estimated using 4.1 and bounds on the discrete solution. Lemma 4.9 gives an estimation of  $J_5$ . Finally, since  $J_7 = \langle \Pi_{1,h}(D_{A_h}\phi_h), D_A\phi' \rangle - \langle D_A\phi, D_A\phi' \rangle$ , we deduce that this terms converges to 0 as  $h \rightarrow 0$  with the help of Lemma 4.1 and the convergences obtained and summarized at the beginning of section 4.3.

*Study of equation (4.13).* The two first terms in equation (4.13) are classical and are treated in a same manner as for equation (4.12). Thus, the only remaining term to estimate is the non linear term of equation (4.13). We use the following Lemma

**Lemma 4.11.**

$$\langle \mathcal{R}_h(U_h, \phi_h, A'_h) - \Pi_{1,h}(A'\phi), \mathcal{R}_h(U_h, \phi_h, A'_h) - \Pi_{1,h}(A'\phi) \rangle_{1,h} \xrightarrow{h \rightarrow 0} 0.$$

The proof is postponed to the appendix B.

We write

$$\begin{aligned} \langle G_{A_h}\phi_h, i\mathcal{R}_h(U_h, \phi_h, A'_h) \rangle_{1,h} - \langle D_A\phi, i\phi A' \rangle &= \underbrace{\langle G_{A_h}\phi_h, i\mathcal{R}_h(U_h, \phi_h, A_h) \rangle_{1,h} - \langle D_{A_h}\phi_h, i\Pi_{1,h}(\phi_h A'_h) \rangle}_I \\ &\quad + \langle D_{A_h}\phi_h, i\Pi_{1,h}(\phi_h A'_h) \rangle - \langle D_A\phi, i\phi A' \rangle \end{aligned} \quad (4.23)$$

Using convergences summarized at the beginning of Section 4.3 and Lemma 4.1, we can prove that

$$\int_0^T \langle D_{A_h}\phi_h, i\Pi_{1,h}\phi_h A'_h \rangle dt \xrightarrow{h \rightarrow 0} \int_0^T \langle D_A\phi, i\phi A' \rangle dt,$$

Furthermore,

$$\begin{aligned}
I &= \underbrace{\langle G_{A_h} \phi_h, i\mathcal{R}_h(U_h, \phi_h, A_h) - i\Pi_{1,h}(\phi_h A'_h) \rangle_{1,h}}_{I_1} \\
&+ \underbrace{\langle G_{A_h} \phi_h - \Pi_{1,h}(D_{A_h} \phi_h), i\Pi_{1,h}(\phi_h A'_h) \rangle_{1,h}}_{I_2} \\
&+ \underbrace{\langle \Pi_{1,h}(D_{A_h} \phi_h), i\Pi_{1,h}(\phi_h A'_h) - i\phi_h A'_h \rangle_{1,h}}_{I_3} \\
&+ \underbrace{\langle \Pi_{1,h}(D_{A_h} \phi_h), i\phi_h(A'_h - A') \rangle_{1,h}}_{I_4} \\
&+ \underbrace{\langle \Pi_{1,h}(D_{A_h} \phi_h), i(\phi_h - \phi)A' \rangle_{1,h}}_{I_5} \\
&+ \underbrace{\langle \Pi_{1,h}(D_{A_h} \phi_h), i\phi A' \rangle_{1,h} - \langle \Pi_{1,h}(D_{A_h} \phi_h), i\phi A' \rangle}_{I_6} \\
&+ \underbrace{\langle (\Pi_{1,h}(D_{A_h} \phi_h) - D_{A_h} \phi_h), i\phi A' \rangle}_{I_7} \\
&+ \underbrace{\langle (D_{A_h} \phi_h - D_A \phi), i\phi A' \rangle}_{I_8}. \tag{4.24}
\end{aligned}$$

Each of the terms converges to 0 as  $h \rightarrow 0$ . Indeed, for

- $I_1$ , we use the energy norm estimate and Lemma 4.11,
- $I_2$ , we use Lemma 4.7, Lemma 4.1 (i) and estimates on  $A'_h$  and  $\phi_h$ ,
- $I_3$ , we use estimates on  $A_h$ ,  $\phi_h$ ,  $A'_h$ , Lemma 4.1 and the fact that  $\|\Pi_{1,h}(D_{A_h} \phi_h)\|_{1,h}$  is bounded (combination of Lemma 4.1 and bounds on the fields),
- $I_4$ , we use the estimates on  $A_h$ ,  $\phi_h$ , the convergence properties of  $A'_h$  and bound on  $\|\Pi_{1,h}(D_{A_h} \phi_h)\|_{1,h}$ ,
- $I_5$ , we use the estimates on  $A_h$ ,  $\phi_h$ , the convergence properties of  $\phi'_h$  and bound on  $\|\Pi_{1,h}(D_{A_h} \phi_h)\|_{1,h}$ ,
- $I_6$ , we use the consistency estimate (3.5),
- $I_7$ , we use Lemma (4.1) and estimates on  $A_h$ ,  $\phi_h$ ,
- $I_8$ , we use the convergence properties of  $A_h$  and  $\phi_h$ .

Thus we can state that

$$\int_0^T \langle G_{A_h} \phi_h, i\mathcal{R}_h(U_h, \phi_h, A'_h) \rangle_{1,h} \xrightarrow{h \rightarrow 0} \int_0^T \langle D_A \phi, i\phi A' \rangle.$$

*The limit equation.* To conclude,

$$\begin{aligned}
& - \int_0^T \langle \dot{\phi}, \dot{\phi}' \rangle + \int_0^T \langle D_A \phi, D_A \phi' \rangle = 0, \\
& - \int_0^T \langle E, \dot{A}' \rangle - \int_0^T \langle dA, dA' \rangle - \int_0^T \langle D_A \phi, i\phi A' \rangle = 0,
\end{aligned} \tag{4.25}$$

which means that  $(A, \phi)$  is a weak solution of the Maxwell-Klein-Gordon equation in the sense of Definition 2.2.

## 5 Numerical results

In this section, we provide some numerical results to assess the theory. To this end, we first propose a fully discrete scheme and then study two types of test cases in two dimensions.

### 5.1 Fully discrete setting

We consider a time-discretization that consists of a uniform subdivision of  $N_T + 1$  ( $N_T \in \mathbb{N}^*$ ) points of the interval  $[0, T]$ . The time step will be denoted  $\Delta t := \frac{T}{N_T}$ . We propose the following simple time discretization of leap-frog type.

For all  $k \in \{0, \dots, N_T\}$ , find  $(\psi_h^k, \phi_h^k, E_h^k, A_h^k) \in Z_h^0 \times Z_h^0 \times Y_h^1 \times Y_h^1$  such that for all  $(\phi_h', A_h') \in Z_h^0 \times Y_h^0$ ,

$$\begin{cases} \langle d_t \psi_{0,h}^k, \psi_h' \rangle_{0,h} - \langle G_{A_h^k} \phi_h^k, G_{A_h^k} \phi_h' \rangle_{1,h} - \mathbf{m}^2 \langle \phi_h^k, \phi_h' \rangle_{0,h} - \gamma \langle |\phi_h^k|^2 \phi_h^k, \phi_h' \rangle_{0,h} = 0. \\ \langle d_t E_h^k, A_h' \rangle - \langle \text{curl } A_h^k, \text{curl } A_h' \rangle - \langle G_{A_h^k} \phi_h^k, U_h^k \phi_h^k A_h' \rangle_{1,h} = 0 \end{cases}$$

with

$$\begin{aligned} \bullet \quad E_h^{k+\frac{1}{2}} &:= -\frac{A_h^{k+1} - A_h^k}{\Delta t} \quad \text{and} \quad \psi_h^{k+\frac{1}{2}} := -\frac{\phi_h^{k+1} - \phi_h^k}{\Delta t} \\ \bullet \quad d_t E_h^k &:= \frac{E_h^{k+\frac{1}{2}} - E_h^{k-\frac{1}{2}}}{\Delta t} \quad \text{and} \quad d_t \psi_h^k := \frac{\psi_h^{k+\frac{1}{2}} - \psi_h^{k-\frac{1}{2}}}{\Delta t} \\ \bullet \quad E_h^k &:= \frac{E_h^{k+\frac{1}{2}} + E_h^{k-\frac{1}{2}}}{2} \quad \text{and} \quad \psi_h^k := \frac{\psi_h^{k+\frac{1}{2}} + \psi_h^{k-\frac{1}{2}}}{2} \end{aligned}$$

For a given sequence  $B^k$ , we will also use the following notations  $d_t B^k = \frac{B^{k+1} - B^k}{\Delta t}$ .

We initialize the algorithm with values  $A_h^0, A_h^1, \psi_h^0, \psi_h^1$ .

In this work, we focus on first numerical results and postpone a more thorough fully-discrete numerical analysis for a future work.

### 5.2 Constraint preservation

One can straightforwardly check that the constraint is verified.

**Proposition 5.1.** *If  $\langle E_h^{\frac{1}{2}}, \nabla \beta' \rangle = \langle \psi_h^{\frac{1}{2}}, \phi_h^1 \beta' \rangle_{0,h}$ , the constraint is verified i.e.*

$$\langle E_h^{k-\frac{1}{2}}, \nabla \beta_h' \rangle = \langle \psi_h^{k-\frac{1}{2}}, \phi_h^k \beta_h' \rangle_{0,h}, \quad \forall k \in \{0, \dots, N_T\} \quad (5.1)$$

*Proof.* If one expresses the discrete differential, one obtains

$$d_t \left( \langle E_h^{k-\frac{1}{2}}, \nabla \beta_h' \rangle - \langle \psi_h^{k-\frac{1}{2}}, \phi_h^k \beta_h' \rangle_{0,h} \right) = \frac{1}{\Delta t} \left[ \langle E_h^{k+\frac{1}{2}}, \nabla \beta_h' \rangle - \langle E_h^{k-\frac{1}{2}}, \nabla \beta_h' \rangle - \langle \psi_h^{k+\frac{1}{2}}, \phi_h^{k+1} \beta_h' \rangle_{0,h} + \langle \psi_h^{k-\frac{1}{2}}, \phi_h^k \beta_h' \rangle_{0,h} \right] \quad (5.2)$$

But from the definition  $\psi_h^{k+\frac{1}{2}}$  and the scalar product,

$$\begin{aligned} \frac{1}{\Delta t} \left[ \langle \psi_h^{k+\frac{1}{2}}, \phi_h^{k+1} \beta_h' \rangle_{0,h} - \langle \psi_h^{k-\frac{1}{2}}, \phi_h^k \beta_h' \rangle_{0,h} \right] &= \left\langle \frac{\psi_h^{k+\frac{1}{2}} - \psi_h^{k-\frac{1}{2}}}{\Delta t}, \phi_h^k \beta_h' \right\rangle_{0,h} + \langle \psi_h^{k+\frac{1}{2}}, \frac{(\phi_h^{k+1} - \phi_h^k)}{\Delta t} \beta_h' \rangle_{0,h}, \\ &= \langle d_t \psi_h^k, \phi_h^k \beta_h' \rangle_{0,h} \end{aligned} \quad (5.3)$$

So that

$$d_t (\langle E_h^{k-\frac{1}{2}}, \nabla \beta' \rangle - \langle \psi_h^{k-\frac{1}{2}}, \phi_h^k \beta' \rangle_{0,h}) = \langle d_t E^k, \nabla \beta' \rangle - \langle d_t \psi^k, \phi_h^k \beta' \rangle_{0,h} \quad (5.4)$$

Similar arguments as in Section 3.2.3 apply to prove that the constraint is preserved even in discrete time. This implies that

$$\langle E^{k-\frac{1}{2}}, \nabla \beta' \rangle - \langle \psi^{k-\frac{1}{2}}, \phi^k \beta' \rangle_{0,h} = 0 \quad (5.5)$$

if the constraint is verified at initial time, i.e. for  $k = 1$ .  $\square$   $\square$

### 5.3 Numerical tests

We have implemented the proposed fully-discrete scheme using a dedicated module that we developed using the Finite Element library Firedrake<sup>1</sup>. Let us point out that these results are preliminary and used as first assessments of the method. We consider a two dimensional domain given by a square  $[0, 1] \times [0, 1]$ . The scheme defined in the previous section is explicit. Thus, as expected in each of our test cases, we observe that the stability of the scheme is guaranteed by a CFL type condition  $\Delta t \leq \text{CFL } h$ . The CFL constant has been empirically fixed according to our tests to 0.25.

#### 5.3.1 Artificial test case and convergence order

One does not have access to an exact solution as such. In order to assess our implementation, we propose to design an artificial test case as explained in the following. We consider for all  $(t, x, y) \in [0, T] \times [0, 1] \times [0, 1]$ ,

$$\phi_{art}(t, x, y) = \sin(\omega t) \sin(\pi x) \sin(\pi y), \quad (5.6)$$

$$A_{art}(t, x, y) = (\cos(\sqrt{2}\pi t) \cos(\pi x) \sin(\pi y), -\cos(\sqrt{2}\pi t) \sin(\pi x) \cos(\pi y)), \quad (5.7)$$

with  $\omega = \sqrt{2\pi^2 + \mathbf{m}^2}$ . Then we define

$$J_{\phi_{art}} = \partial_{tt} \phi_{art} + (D_{A_{art}})^* D_{A_{art}} \phi_{art} + \mathbf{m}^2 \phi_{art} + \gamma |\phi_{art}|^2 \phi_{art} \quad (5.8)$$

$$J_{A_{art}} = \partial_{tt} A_{art} + d^* d A_{art} - i \phi_{art}^* D_{A_{art}} \phi_{art} \quad (5.9)$$

$$(5.10)$$

In this way,  $(A_{art}, \phi_{art})$  is a solution of the following set of equations:

$$\begin{aligned} \partial_{tt} \phi + (D_A)^* D_A \phi + \mathbf{m}^2 \phi + \gamma |\phi|^2 \phi &= J_{\phi_{art}}, \\ \partial_{tt} A + d^* d A - i \phi^* D_A \phi &= J_{A_{art}}. \end{aligned} \quad (5.11)$$

with initial conditions

$$\phi(0, \cdot) = \phi_{art}(0, \cdot), \quad (5.12)$$

$$A(0, \cdot) = A_{art}(0, \cdot). \quad (5.13)$$

We then use the discretization proposed above, add the external currents  $(J_{A_{art}}, J_{\phi_{art}})$  on the right hand side, and compute the corresponding solution  $(A_h^n, \phi_h^n)_{n \in \{0, \dots, N_T\}}$ . Doing so, we are able to compute the  $L^\infty L^2$  error as

$$\text{err}_h^{L^2} = \max_{n \in \{0, \dots, N_T\}} \left( \|\phi_h^n - \Pi_{0,h}(\phi_{art}(t_n, \cdot))\|_{L^2(S)}^2 + \|A_h^n - \Pi_{1,h}(A_{art}(t_n, \cdot))\|_{L^2(S)}^2 \right)^{\frac{1}{2}} \quad (5.14)$$

We computed the solution for 4 types of  $(m, \gamma)$  couples:  $(0, 0)$ ,  $(1, 0)$ ,  $(0, 1)$  and  $(1, 1)$ . In Figure 1, we plot for several values of the number of points, the values of the respective error defined in (5.14). The error analysis is performed using mesh parameters that fulfill  $\Delta t = \text{CFL } h$ . Doing so, we compute the minimum order in time and space of the scheme.

In Figure 2, we plot the results in logarithm scales. We observe an order of convergence of 1.

<sup>1</sup><https://www.firedrakeproject.org/>



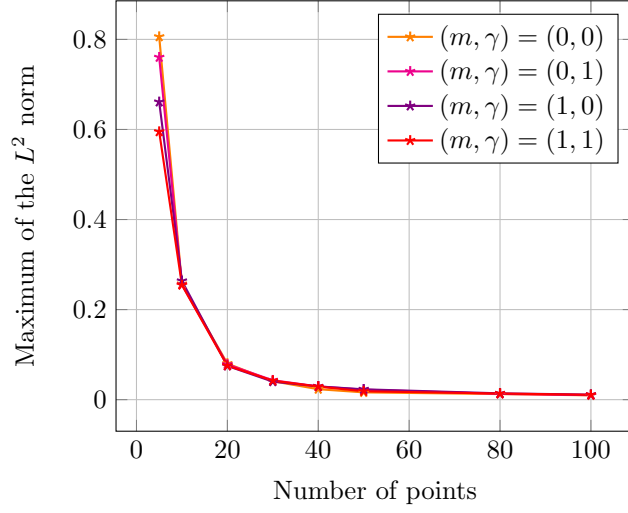


Figure 1: Maximum in time of  $L^2$  errors *v.s.* number of points

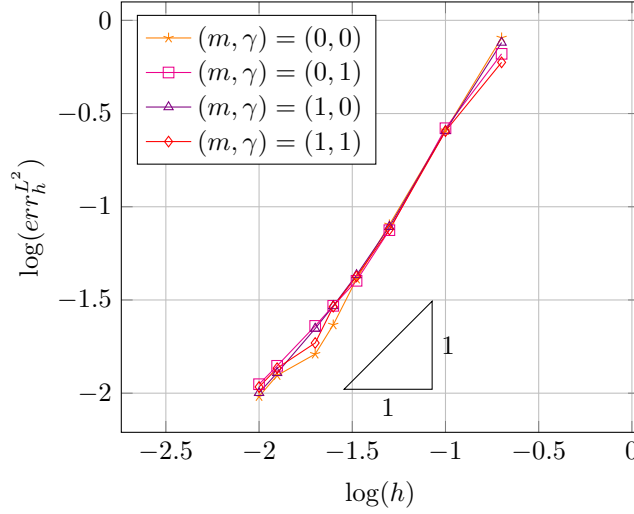


Figure 2: Convergence order

Although this is out of the scope of this work to obtain a theoretical proof of the convergence orders, one can however try to find, through our analysis, some indication of first order convergence for this test case using regular fields. Indeed the approximation of the discrete covariant derivative  $D_{A_h} \phi_h$  by the LGT gauge compatible discretization come at the price of the estimate such as (4.5) where we do not hope to obtain more than first order of approximation of the covariant derivative.

Regarding the constraint  $\kappa$ , in this case, one has instead of (3.14), the following equation.

$$\langle \kappa(t), \alpha' \rangle = \langle \kappa(0), \alpha' \rangle + \int_0^t (\langle J_{\phi_{art}}, \Pi_{0,h}(-i\alpha' \phi) \rangle + \langle J_{A_{art}}, \nabla \alpha' \rangle) ds. \quad (5.15)$$

Hence, except if currents are chosen such that the right hand side vanishes, we do not expect preservation of the constraint in this artificial case. We will not compute the constraint for this precise test case and postpone the tests to the next section.

### 5.3.2 Second test case and preservation of the constraint

For this second test case, we propose to initialize the system with an electromagnetic plane wave for  $A$  and a 2D-Gaussian initial profile for  $\phi$  centered at the center of the domain. In other words, we consider a focalized density profile for the modulus of the complex Klein-Gordon field. In figure 3, we observe that the constraint is preserved up to machine precision. This is in accordance with the theory developed in previous sections. The energy is not preserved by the time-integration

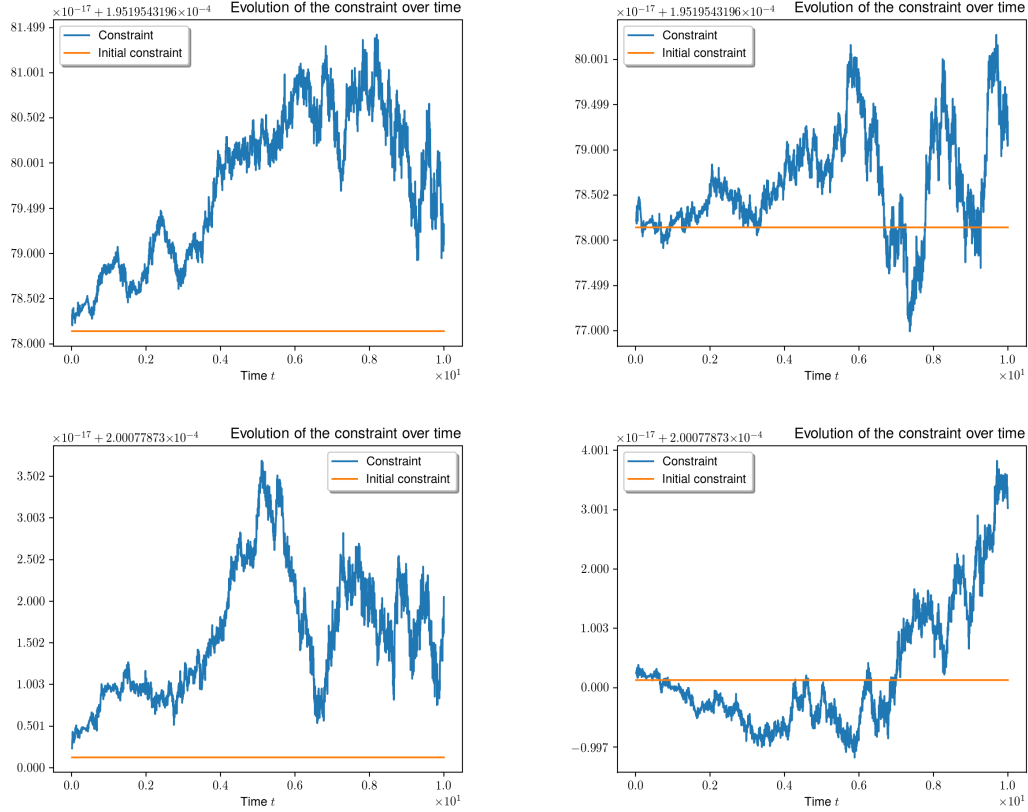


Figure 3: Evolution of the constraint in time for several couples  $(m, \gamma)$ :  $(0, 0)$  upper left,  $(0, 1)$  upper right,  $(1, 0)$  bottom left,  $(1, 1)$  bottom right. The initial constraint is non zero and is represented by the orange horizontal line.

scheme, however relative variations of the energy from its initial value are relatively small (see Figure 4) and this variation gets consistently smaller as the discretization parameters decrease (see Figure 5).

To conclude, we represent in Figure 7 some  $\phi$  modulus profiles and their evolution with time (the initial profile is represented in Figure 6). At this stage of our investigations, we cannot really interpret these profiles since we work on a toy academic model. This will be part of future works to improve the physical relevance of the setting considered here and the corresponding test cases. The way we use these profiles here is to show the effect of the introduction of mass and self-coupling term. Even if, again, the physical relevance of the values chosen are not discussed here, we propose to test the use of non unitary values of the mass and self-coupling term.

What we observe is that the effect of mass and/or self-coupling occurs after some elapsed time and that the effect of self-coupling seems to be weak. We therefore also tested a situation where self-coupling is ten times bigger than mass. In this case, we significantly see self-coupling effects over mass on the profiles patterns.

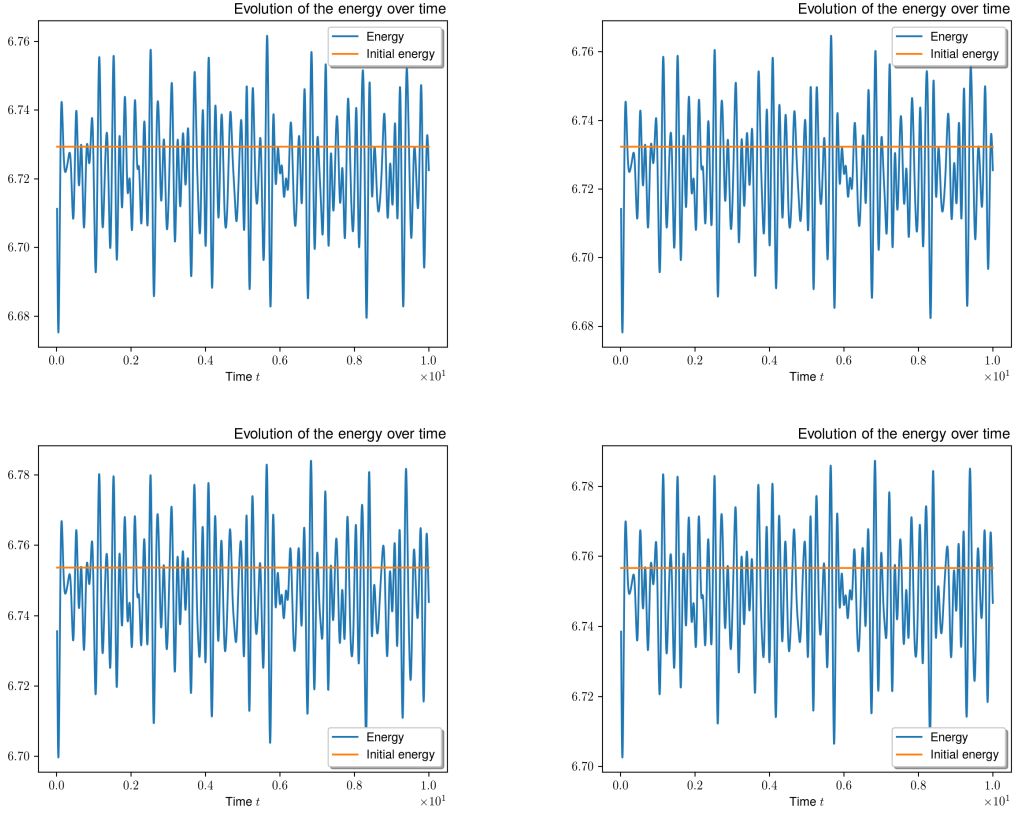


Figure 4: Evolution of the energy over time for several couples  $(\mathfrak{m}, \gamma)$ :  $(0,0)$  upper left,  $(0,1)$  upper right,  $(1,0)$  bottom left,  $(1,1)$  bottom right. The initial energy is represented by the orange horizontal line.

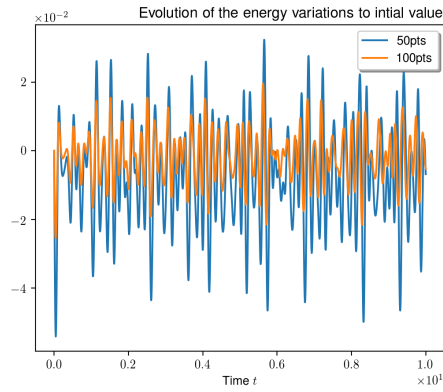


Figure 5: Absolute variations of the energy with respect to its initial value over time for a space discretization made of 50pts and 100pts.

## 6 Conclusion

In this work, we studied the Maxwell-Klein-Gordon equation in dimension two and focused on the

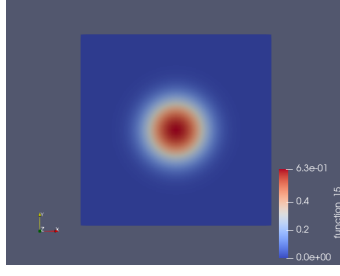


Figure 6: Initial profile of the modulus of the Klein Gordon complex field.

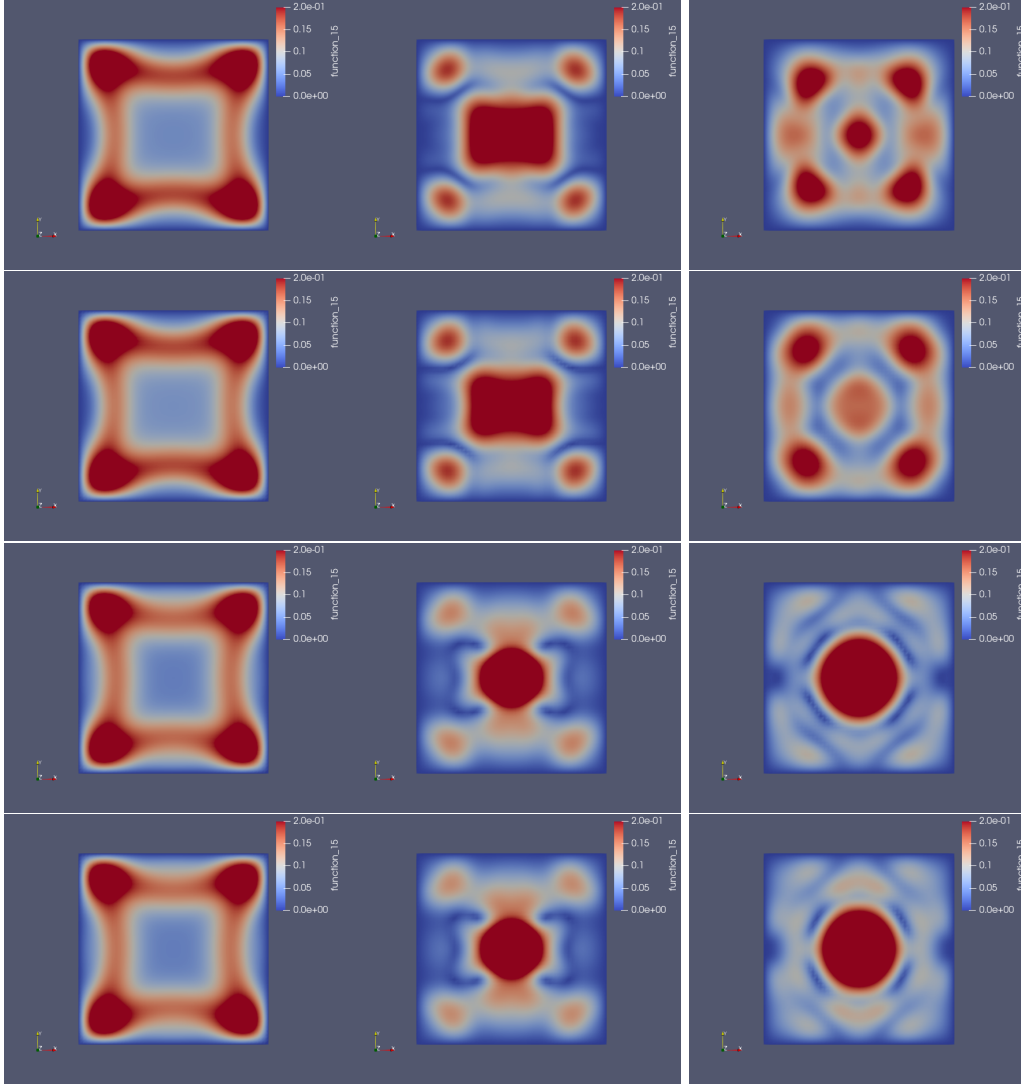


Figure 7: Evolution of the profile of the modulus of the Klein Gordon complex field with  $(\mathbf{m}, \gamma) = (0, 0)$  (first row),  $(\mathbf{m}, \gamma) = (0, 10)$  (second row),  $(\mathbf{m}, \gamma) = (10, 0)$  (third row) and  $(\mathbf{m}, \gamma) = (10, 10)$  (fourth row):  $t = 0.125$  left,  $t = 0.225$  middle,  $t = 0.985$  right, with  $t = 1$  being the end of simulation time.

semi-discrete analysis of a discretization scheme based on Lattice Gauge Theory (LGT) combined with Nédélec finite elements. The special feature of this scheme is that it ensures gauge invariance

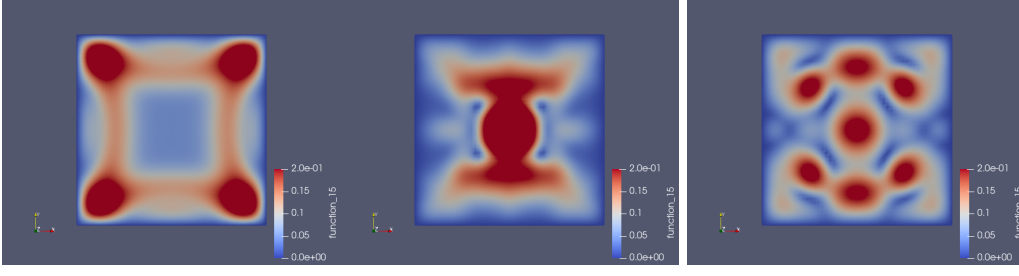


Figure 8: Evolution of the profile of the modulus of the Klein Gordon complex field with  $(\mathbf{m}, \gamma) = (10, 100)$ :  $t = 0.125$  left,  $t = 0.225$  center,  $t = 0.985$  right, with  $t = 1$  being the end of simulation time.

at the discrete level therefore respecting the geometrical structure of the original equation through the preservation of the constraint. Our analysis is not an a priori analysis, but concentrates on sequential compactness arguments without the a priori knowledge of a solution of the continuous equations. We use the discrete energy principle combined to constraint preservation to extract bounds in the appropriate spaces and extract convergent subsequences. At last, we implement a fully discrete numerical scheme based on a leap frog type time integration. We provide first academical test cases to extract convergence orders and assess the method. Future works include more relevant physical test cases and the fully discrete analysis of the proposed scheme.

## Acknowledgment

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## A Proofs of the preliminary Lemma 4.1 and A.2.

**Lemma A.1.** *Let  $\Pi_{1,h}$  be the edge interpolant. For  $p > 2$ , the following inequalities hold. There exists  $C > 0$  such that for all  $(F_h, \zeta_h) \in Y_h^1 \times Z_h^0$*

$$(i) \quad \|F_h \zeta_h - \Pi_{1,h}(F_h \zeta_h)\|_{L^2(S)} \leq Ch^{1-\frac{2}{p}} \|F_h\|_{L^p(S)} |\zeta_h|_{H^1(S)}, \quad (A.1)$$

Furthermore there exists  $C > 0$  such that

$$(ii) \quad \sqrt{\sum_{K \in \mathcal{T}_h} |A_h|_{H^1(K)}^2} \leq C \left( \|\nabla \times A_h\|_{L^2(S)} + \sup_{r_h \in Y_h^0} \frac{|\langle A_h, \nabla r_h \rangle|}{\|r_h\|_{L^2(S)}} \right), \quad (A.2)$$

$$(iii) \quad \sqrt{\sum_{K \in \mathcal{T}_h} |\Pi_{1,h}(A_h \phi_h)|_{H^1(K)}^2} \leq Ch^{-\frac{2}{p}} \left[ \left( \|\nabla \times A_h\|_{L^2(S)} + \sup_{r_h \in Y_h^0} \frac{|\langle A_h, \nabla r_h \rangle|}{\|r_h\|_{L^2(S)}} \right) \|\phi_h\|_{L^p(S)} + \|A_h\|_{L^p(S)} |\phi|_{H^1(S)} \right]. \quad (A.3)$$

*Proof. Proof of (i).* Let  $\hat{\Pi}$  be the projector on the reference domain  $\hat{K}$  corresponding to  $\Pi_{1,h}$  on  $Z_h^1$ , and  $\hat{F}_h, \hat{\zeta}_h$  the corresponding functions classically defined on the reference element  $\hat{K}$ . One has

$$\|\hat{F}_h \hat{\zeta}_h - \hat{\Pi} \hat{F}_h \hat{\zeta}_h\|_{L^2(\hat{K})} \leq C \inf_{p \in Q} \|\hat{F}_h \hat{\zeta}_h - p\|_{H^1(\hat{K})},$$

where  $Q = \mathcal{Q}_{0,1}(\mathbb{C}) \times \mathcal{Q}_{1,0}(\mathbb{C})$  is the space of polynomials we will consider on  $\hat{K}$ . Furthermore

$$\inf_{p \in Q} \|\hat{F}_h \hat{\zeta}_h - p\|_{H^1(\hat{K})} \leq \inf_{p \in \mathbb{P}_0} \|\hat{F}_h(\hat{\zeta}_h - p)\|_{H^1(\hat{K})},$$

where  $\mathbb{P}_0$  are complex constant functions on  $\hat{K}$ . In addition,

$$\begin{aligned} & \inf_{p \in \mathbb{P}_0} \|\hat{F}_h(\hat{\zeta}_h - p)\|_{H^1(\hat{K})} \\ & \leq C \left[ \|\hat{F}_h\|_{L^p(\hat{K})} \inf_{p \in \mathbb{P}_0} \|\hat{\zeta}_h - p\|_{H^1(\hat{K})} + |\hat{F}_h|_{H^1(\hat{K})} \inf_{p \in \mathbb{P}_0} \|\hat{\zeta}_h - p\|_{H^1(\hat{K})} + |\hat{\zeta}_h|_{H^1(\hat{K})} \|\hat{F}_h\|_{L^p(\hat{K})} \right], \end{aligned}$$

which gives

$$\begin{aligned} \inf_{p \in \mathbb{P}_0} \|\hat{F}_h(\hat{\zeta}_h - p)\|_{H^1(\hat{K})} & \leq C \|\hat{F}_h\|_{L^p(\hat{K})} |\hat{\zeta}_h|_{H^1(\hat{K})} + |\hat{F}_h|_{H^1(\hat{K})} |\hat{\zeta}_h|_{H^1(\hat{K})} + |\hat{\zeta}_h|_{H^1(\hat{K})} \|\hat{F}_h\|_{L^p(\hat{K})} \\ & \leq C \|\hat{F}_h\|_{L^p(\hat{K})} |\hat{\zeta}_h|_{H^1(\hat{K})}. \end{aligned}$$

So on any rectangle  $K$ , one deduces by scaling

$$\|F_h \zeta_h - \Pi_{1,h}(F_h \zeta_h)\|_{L^2(K)} \leq Ch^{1-\frac{2}{p}} \|F_h\|_{L^p(K)} |\zeta_h|_{H^1(K)} \leq Ch^{1-\frac{2}{p}} \|F_h\|_{L^p(S)} |\zeta_h|_{H^1(K)},$$

and by summing up the squares of each part of this inequality (i) is proved.

By inverse inequality one also has

$$\|F_h \zeta_h - \Pi_{1,h}(F_h \zeta_h)\|_{H^1(K)} \leq Ch^{-\frac{2}{p}} \|F_h\|_{L^p(K)} |\zeta_h|_{H^1(K)}.$$

Furthermore, on the reference square  $\hat{K}$ ,

$$|\hat{F}_h \hat{\zeta}_h|_{H^1(\hat{K})} \leq C \left[ |\hat{F}_h|_{H^1(\hat{K})} \|\hat{\zeta}_h\|_{L^p(\hat{K})} + |\hat{\zeta}_h|_{H^1(\hat{K})} \|\hat{F}_h\|_{L^p(\hat{K})} \right],$$

which gives on any rectangle  $K$ ,

$$|F_h \zeta_h|_{H^1(K)} \leq Ch^{-\frac{2}{p}} \left[ \|F_h\|_{H^1(K)} \|\zeta_h\|_{L^p(K)} + |\zeta_h|_{H^1(K)} \|F_h\|_{L^p(K)} \right].$$

This implies

$$\sqrt{\sum_{K \in \mathcal{T}_h} |\Pi_{1,h}(F_h \zeta_h)|_{H^1(K)}^2} \leq Ch^{-\frac{2}{p}} \left[ \left( \sqrt{\sum_{K \in \mathcal{T}_h} |F_h|_{H^1(K)}^2} \right) \|\zeta_h\|_{L^p(S)} + \|F_h\|_{L^p(S)} |\phi|_{H^1(S)} \right]. \quad (\text{A.4})$$

Turning to the proof of (ii). The discrete Helmholtz decomposition of  $A_h$  is written (see [19])

$$A_h = \hat{A}_h + \nabla p_h, \quad (\text{A.5})$$

where

- $\hat{A}_h$  is discrete divergence free, i.e.  $\hat{A}_h \in V_h := \{u_h \in Y_h^1 | \langle u_h, \nabla \beta_h \rangle = 0, \forall \beta_h \in Y_h^0\}$ ,
- $p_h \in Y_h^0$ .

First, we consider the term  $\mathring{A}_h$ . We would like to prove that

$$\sqrt{\sum_{K \in \mathcal{T}_h} |\mathring{A}_h|_{H^1(K)}^2} \leq C \|\nabla \times \mathring{A}_h\|_{L^2(S)}. \quad (\text{A.6})$$

Let us define  $P_V$  the  $L^2$  projection on the space of divergence free vectors fields. By inverse inequality

$$\sqrt{\sum_{K \in \mathcal{T}_h} |\mathring{A}_h - P_V \mathring{A}_h|_{H^1(K)}^2} \leq Ch^{-1} |\mathring{A}_h - P_V \mathring{A}_h|_{L^2(S)},$$

and following [8] (the proof of proposition 2.5.), one has

$$|\mathring{A}_h - P_V \mathring{A}_h|_{L^2(S)} \leq Ch \|\nabla \times \mathring{A}_h\|_{L^2(S)}.$$

This yields

$$\sqrt{\sum_{K \in \mathcal{T}_h} |\mathring{A}_h - P_V \mathring{A}_h|_{H^1(K)}^2} \leq C \|\nabla \times \mathring{A}_h\|_{L^2(S)}.$$

Furthermore  $P_V(\mathring{A}_h) \in H^1(S)$  and one has

$$|P_V \mathring{A}_h|_{H^1(S)} \leq C \|\nabla \times P_V \mathring{A}_h\|_{L^2(S)} = C \|\nabla \times \mathring{A}_h\|_{L^2(S)},$$

and (A.6) is proved. Let us study the gradient part. From the expression of the constraint (3.14), there exists, from Riesz representation theorem,  $u_h \in Y_h^0$  such that

$$\langle \nabla p_h, \nabla r_h \rangle = \langle u_h, r_h \rangle, \forall r_h \in Y_h^0,$$

and

$$\|u_h\|_{L^2(S)} = \sup_{r_h \in Y_h^0} \frac{|\langle \nabla p_h, \nabla r_h \rangle|}{\|r_h\|_{L^2(S)}}.$$

Define  $p \in H_0^1(S)$  as the unique solution of the following equation

$$\langle \nabla p, \nabla r \rangle = \langle u_h, r \rangle, \forall r \in H_0^1(S).$$

Since  $u_h \in L^2(S)$ , then  $p \in H^2(S) \cap H_0^1(S)$  and the following inequality holds

$$|p|_{H^2(S)} \leq C \|u_h\|_{L^2(S)}.$$

Furthermore, by standard estimates

$$|p_h - \Pi_{0,h} p|_{H^1(S)} \leq Ch |p|_{H^2(S)}.$$

We would like to prove that

$$\sqrt{\sum_{K \in \mathcal{T}_h} |\nabla p_h|_{H^1(K)}^2} \leq C \|u_h\|_{L^2(S)}.$$

One has

$$\sqrt{\sum_{K \in \mathcal{T}_h} |\nabla p_h|_{H^1(K)}^2} \leq \sqrt{\sum_{K \in \mathcal{T}_h} |\nabla p_h - \nabla \Pi_{0,h} p|_{H^1(K)}^2} + \sqrt{\sum_{K \in \mathcal{T}_h} |\nabla \Pi_{0,h} p|_{H^1(K)}^2} \quad (\text{A.7})$$

$$\leq Ch^{-1} \sqrt{\sum_{K \in \mathcal{T}_h} |\nabla p_h - \nabla \Pi_{0,h} p|_{L^2(K)}^2} + \sqrt{\sum_{K \in \mathcal{T}_h} |\nabla \Pi_{0,h} p|_{H^1(K)}^2} \quad (\text{A.8})$$

$$\leq C |p|_{H^2(S)} + C |\nabla p|_{H^1(S)} \quad (\text{A.9})$$

$$\leq C \|u_h\|_{L^2(S)}, \quad (\text{A.10})$$

where we used inverse inequalities and the continuity arguments.

Using the estimates established, one concludes that

$$\sqrt{\sum_{K \in \mathcal{T}_h} |A_h|_{H^1(K)}^2} \leq C \left( \|\nabla \times A_h\|_{L^2(S)} + \sup_{r_h \in Y_h^0} \frac{|\langle A_h, \nabla r_h \rangle|}{\|r_h\|_{L^2(S)}} \right),$$

and (ii) is proved.

Inequality (iii) follows directly from (ii) and (A.4).  $\square$

The following result allows by the constraint (3.14) to control the weak divergence of  $A_h$  appearing in (4.2) and (4.3).

**Lemma A.2.** *For any  $p > 2$ ,*

$$\sup_{r_h \in Y_h^0} \frac{|\langle A_h, \nabla r_h \rangle|}{\|r_h\|_{L^2(S)}} \leq Ch^{-\frac{2}{p}} \|\phi_h\|_{L^\infty(0,T,L^p(S))} \|\dot{\phi}_h\|_{L^\infty(0,T,L^2(S))}.$$

*Proof.* One has from the constraint (3.14)

$$|\langle A_h, \nabla r_h \rangle| = \left| \left\langle \int_0^t \dot{\phi}_h \bar{\phi}_h, r_h \right\rangle_{0,h} \right|, \quad \forall r_h \in Y_h^0,$$

implying that

$$\begin{aligned} \sup_{r_h \in Y_h^0} \frac{|\langle A_h, \nabla r_h \rangle|}{\|r_h\|_{L^2(S)}} &= \sup_{r_h \in Y_h^0} \frac{|\langle \int_0^t \dot{\phi}_h \bar{\phi}_h, r_h \rangle_{0,h}|}{\|r_h\|_{L^2(S)}} \leq \sup_{r_h \in Y_h^0} \frac{\|\int_0^t \dot{\phi}_h \bar{\phi}_h\|_{0,h} \|r_h\|_{0,h}}{\|r_h\|_{L^2(S)}} \\ &\leq C \left\| \int_0^t \dot{\phi}_h \bar{\phi}_h \right\|_{0,h} \leq C \int_0^t \|\dot{\phi}_h\|_{L^p(S)} \|\phi_h\|_{L^q(S)} \\ &\leq Ch^{-\frac{2}{p}} \|\phi_h\|_{L^\infty(0,T,L^p(S))} \|\dot{\phi}_h\|_{L^\infty(0,T,L^2(S))}, \end{aligned}$$

with  $1/p + 1/q = 1/2$ .  $\square$

## B Proof of Lemma 4.5 to Lemma 4.11

We will use the following decomposition of  $G_{A_h} \phi_h$  in terms of its gradient part and its non linear part, that upon requiring  $h$  to be sufficiently small, is close to  $iA_h \phi_h$ .

Let  $e = \{m, n\}$  denote an oriented edge of the mesh. The following decomposition holds

$$(G_{A_h} \phi^h)_e = \frac{1 + \exp(-i\mathbf{A}_e^h)}{2} (\phi_n^h - \phi_m^h) + \frac{1 - \exp(-i\mathbf{A}_e^h)}{2} (\phi_m^h + \phi_n^h) \quad (\text{B.1})$$

Let us denote

- $N_h \in Z_h^1$  the edge element such that its edge degrees of freedom are  $\mathbf{N}_e^h = 1 - \exp(-i\mathbf{A}_e^h)$ .
- $\mathbf{N}\phi^h$  the vector of the degrees of freedom of the vector  $N_h \phi_h$ , i.e.  $(\mathbf{N}\phi^h)_e = \mathbf{N}_e^h (\frac{\phi_m^h + \phi_n^h}{2})$ .
- $\mathbf{P}^h$  the vector such that  $\mathbf{P}_e^h := -\frac{1}{2} \mathbf{N}_e^h (G\phi^h)_e$ , and  $P_h \in Z_h^1$  its associated edge element vector field.



This gives

$$(G_{\mathbf{A}^h} \phi^h)_e = (G\phi^h)_e + \mathbf{P}_e^h + (\mathbf{N}\phi^h)_e.$$

We first prove the following

**Lemma B.1.** *There exist  $\eta > 0$  and  $C > 0$  such that if  $h < \eta$ ,*

$$\sup_{[0,T]} \|G_{A_h} \phi_h - \Pi_{1,h}(D_{A_h} \phi_h)\|_{h,2}^2 \leq Ch^{2-4/p},$$

so that

$$\sup_{[0,T]} \|G_{A_h} \phi_h - \Pi_{1,h}(D_{A_h} \phi_h)\|_{h,2} \xrightarrow{h \rightarrow 0} 0.$$

*Proof.* Let  $e = \{m, n\}$  be an oriented edge of the mesh, one has,

$$(G_{\mathbf{A}^h} \phi^h - \Pi_{1,h}(D_{\mathbf{A}^h} \phi^h))_e = (\mathbf{P}^h)_e + (\mathbf{N}_e^h - i\mathbf{A}_e^h) \frac{\phi_m^h + \phi_n^h}{2},$$

and

$$\|P_h + (N_h - iA_h)\phi_h\|_{1,h} \leq \|P_h\|_{1,h} + \|(N_h - iA_h)\phi_h\|_{1,h}.$$

Furthermore,

$$\begin{aligned} \|P_h\|_{h,2}^2 &= \sum_e (H_1^h)_{ee} \left| \frac{\exp(-i\mathbf{A}_e^h) - 1}{2} (G\phi^h)_e \right|^2 \\ &\leq \sum_e (H_1^h)_{ee} \frac{|\mathbf{A}_e^h|^2}{4} \left| (G\phi^h)_e \right|^2 \\ &\leq Ch^{2-\frac{4}{p}} \|A_h\|_{L^\infty(0,T,L^p(S))}^2 \sum_e (H_1^h)_{ee} \left| (G\phi^h)_e \right|^2 \\ &\leq Ch^{2-\frac{4}{p}} \|\nabla \phi_h\|_{h,2}^2 \leq Ch^{2-\frac{4}{p}} \|\nabla \phi_h\|_{L^\infty(0,T;L^2(S))}^2 \leq Ch^{2-\frac{4}{p}}, \end{aligned}$$

since  $\nabla \phi_h$  is uniformly bounded in  $L^\infty(0,T,L^2(S))$ . In addition,

$$\|(N_h - iA_h)\phi_h\|_{h,2}^2 = \sum_e (H_1^h)_{ee} \left| \frac{1 - \exp(-i\mathbf{A}_e^h) - i\mathbf{A}_e^h}{2} (\phi_m^h + \phi_n^h) \right|^2$$

There exists  $\eta > 0$ , such that  $|\mathbf{A}_e^h|$  is sufficiently small (from (4.11)) so that the following inequalities hold for  $h < \eta$

$$\begin{aligned} \|(N_h - iA_h)\phi_h\|_{h,2}^2 &\leq C \sum_e (H_1^h)_{ee} |\mathbf{A}_e^h|^2 \left| \frac{\mathbf{A}_e^h}{2} (\phi_m^h + \phi_n^h) \right|^2 \\ &\leq Ch^{2-\frac{4}{p}} \|A_h\|_{L^\infty(0,T,L^p(S))}^2 \sum_e (H_1^h)_{ee} \left| \frac{\mathbf{A}_e^h}{2} (\phi_m^h + \phi_n^h) \right|^2 \\ &\leq Ch^{2-\frac{4}{p}} \|A_h\|_{L^\infty(0,T,L^p(S))}^2 \|\Pi_{1,h}(A_h \phi_h)\|_{h,2}^2 \\ &\leq Ch^{2-\frac{4}{p}} \|A_h\|_{L^\infty(0,T,L^p(S))}^2 \|\Pi_{1,h}(A_h \phi_h)\|_{L^\infty(0,T,L^2(S))}^2 \\ &\leq Ch^{2-\frac{4}{p}}, \end{aligned}$$

where we used Lemma A.1, and uniform bounds on the  $L^p$ -norm of  $A_h$  and the  $H^1$ -norm on  $\phi_h$ .  $\square$

The following Lemma is a direct consequence of the estimate on the energy.

**Lemma B.2.** *There exists  $C > 0$  such that*

$$\sup_{[0,T]} \|G_{A_h} \phi_h\|_{h,2} \leq C. \quad (\text{B.2})$$

**Lemma B.3.**

$$\|\langle \Pi_{1,h}(D_{A_h} \phi_h), \nabla \phi'_h \rangle_{1,h} - \langle \Pi_{1,h}(D_{A_h} \phi_h), \nabla \phi' \rangle\|_{L^\infty(0,T)} \xrightarrow{h \rightarrow 0} 0.$$

*Proof.* From the consistency estimate, one has

$$|\langle \Pi_{1,h}(D_{A_h} \phi_h), \nabla \phi'_h \rangle_{1,h} - \langle \Pi_{1,h}(D_{A_h} \phi_h), \nabla \phi' \rangle| \leq Ch \|\Pi_{1,h}(D_{A_h} \phi_h)\|_{L^2(S)} \|\nabla \phi'\|_{H^1(S)},$$

for  $l > 1$ , and from bounds established in Lemma A.1, one deduces for  $p > 2$  a bound on  $\|\Pi_{1,h}(D_{A_h} \phi_h)\|_{L^2(S)}$  in terms of  $\|A_h\|_{L^\infty(0,T;L^p(S))}$  and  $\|\phi_h\|_{L^\infty(0,T;H^1(S))}$  which concludes the proof using the obtained bounds and convergence of discrete test functions.  $\square$

**Lemma B.4.**

$$\|\langle \nabla \phi_h, A_h \phi'_h \rangle_{1,h} - \langle \nabla \phi_h, A_h \phi'_h \rangle\|_{L^\infty(0,T)} \xrightarrow{h \rightarrow 0} 0.$$

*Proof.* Mapping consistency estimates from the reference square to any rectangle  $K$  of the mesh (as is done in [9]) gives

$$\begin{aligned} & |\langle \nabla \phi_h, A_h \phi'_h \rangle_{1,h} - \langle \nabla \phi_h, A_h \phi'_h \rangle| \\ & \leq C \left( h \sqrt{\sum_{K \in \mathcal{T}_h} |A_h \phi'_h|_{H^1(K)}^2} + h^2 \sqrt{\sum_{K \in \mathcal{T}_h} |A_h \phi'_h|_{H^2(K)}^2} \right) \|\nabla \phi_h\|_{L^2(S)}. \end{aligned}$$

The two terms  $\sqrt{\sum_{K \in \mathcal{T}_h} |A_h \phi'_h|_{H^1(K)}^2}$  and  $\sqrt{\sum_{K \in \mathcal{T}_h} |A_h \phi'_h|_{H^2(K)}^2}$  can be estimated by standard techniques on the reference simplex as in Lemma A.1. It will not be detailed here since arguments are similar to those already used in previous proofs.

One obtains

$$h \sqrt{\sum_{K \in \mathcal{T}_h} |A_h \phi'_h|_{H^1(K)}^2} + h^2 \sqrt{\sum_{K \in \mathcal{T}_h} |A_h \phi'_h|_{H^2(K)}^2} \leq Ch^{1-\frac{2}{p}} \sqrt{\sum_{K \in \mathcal{T}_h} |A_h|_{H^1(K)}^2} \|\phi'_h\|_{H^1(S)}.$$

Using Lemma A.1 and A.2 and previously obtained bounds, we obtain

$$\|\langle \nabla \phi_h, A_h \phi'_h \rangle_{1,h} - \langle \nabla \phi_h, A_h \phi'_h \rangle\|_{L^\infty(0,T)} \leq Ch^{1-\frac{4}{p}}.$$

Choosing  $p > 4$  gives the result.  $\square$

**Lemma B.5.**

$$\|\langle \Pi_{1,h}(A_h \phi_h), \Pi_{1,h}(A_h \phi'_h) \rangle_{1,h} - \langle \Pi_{1,h}(A_h \phi_h), \Pi_{1,h}(A_h \phi'_h) \rangle\|_{L^\infty(0,T)} \xrightarrow{h \rightarrow 0} 0.$$

*Proof.* From error estimates in [10], one deduces that

$$\begin{aligned} & |\langle \Pi_{1,h}(A_h \phi_h), \Pi_{1,h}(A_h \phi'_h) \rangle_{1,h} - \langle \Pi_{1,h}(A_h \phi_h), \Pi_{1,h}(A_h \phi'_h) \rangle| \\ & \leq Ch \sum_{K \in \mathcal{T}_h} (\|\Pi_{1,h}(A_h \phi_h)\|_{L^2(K)} |\Pi_{1,h}(A_h \phi'_h)|_{H^1(K)} \\ & \quad + \|\Pi_{1,h}(A_h \phi'_h)\|_{L^2(K)} |\Pi_{1,h}(A_h \phi_h)|_{H^1(K)}). \end{aligned}$$

So by Lemma A.1, Lemma A.2 and the bounds on  $A_h$ ,  $\phi_h$  and  $\phi'_h$  the result follows.  $\square$

**Lemma B.6.**

$$\int_0^T \langle \Pi_{1,h}(D_{A_h}\phi_h), \Pi_{1,h}(D_{A_h}\phi'_h) \rangle dt \xrightarrow{h \rightarrow 0} \int_0^T \langle D_A\phi, D_A\phi' \rangle dt.$$

*Proof.* This follows from Lemma A.1, weak-\* convergence of  $\nabla\phi_h$  to  $\nabla\phi$  in  $L^\infty(0, T; L^2(S))$  and strong convergence of  $A_h$  and  $\phi_h$  in  $L^\infty(0, T; L^p(S))$ .  $\square$

Last

**Lemma B.7.**

$$\|\mathcal{R}_h(U_h, \phi_h, A'_h) - \Pi_{1,h}(A'_h\phi_h)\|_{1,h} \xrightarrow{h \rightarrow 0} 0.$$

*Proof.* For  $h$  sufficiently small

$$\begin{aligned} & \|\mathcal{R}_h(U_h, \phi_h, A'_h) - \Pi_{1,h}(A'_h\phi_h)\|_{1,h}^2 \\ &= \sum_e (H_1^h)_{ee} \left| \mathbf{A}_e'^h \exp(-i\mathbf{A}_e^h) \phi_m^h - \mathbf{A}_e'^h \frac{\phi_m^h + \phi_n^h}{2} \right|^2 \\ &= \sum_e (H_1^h)_{ee} \left| \mathbf{A}_e'^h \exp(-i\mathbf{A}_e^h) \frac{\phi_n^h + \phi_m^h}{2} - \mathbf{A}_e'^h \exp(-i\mathbf{A}_e^h) \frac{\phi_n^h - \phi_m^h}{2} - \mathbf{A}_e'^h \frac{\phi_m^h + \phi_n^h}{2} \right|^2 \\ &= \sum_e (H_1^h)_{ee} \left| \mathbf{A}_e'^h (1 - \exp(-i\mathbf{A}_e^h)) \frac{\phi_n^h + \phi_m^h}{2} - \mathbf{A}_e'^h \exp(-i\mathbf{A}_e^h) \frac{\phi_n^h - \phi_m^h}{2} \right|^2 \\ &\leq C \sum_e (H_1^h)_{ee} \left( |\mathbf{A}_e'^h|^2 |\mathbf{A}_e^h|^2 \left| \frac{\phi_n^h + \phi_m^h}{2} \right|^2 + |\mathbf{A}_e'^h|^2 \left| \frac{\phi_n^h - \phi_m^h}{2} \right|^2 \right) \\ &\leq Ch^{2-\frac{4}{p}} \sum_e (H_1^h)_{ee} \left( |\mathbf{A}_e^h|^2 \left| \frac{\phi_n^h + \phi_m^h}{2} \right|^2 + |\phi_n^h - \phi_m^h|^2 \right) \\ &\leq Ch^{2-\frac{4}{p}} \left( \|\Pi_{1,h}(A_h\phi_h)\|_{L^\infty(0,T;L^2(S))}^2 + \|\nabla\phi_h\|_{L^\infty(0,T;L^2(S))}^2 \right). \end{aligned}$$

The same arguments as in Lemma 4.9 gives the result.  $\square$

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