Coupling of External Electric Circuits with Computational Domains

Riccardo Scorretti

Article

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Riccardo Scorretti[1]

[1] Univ Lyon, Université Claude Bernard Lyon 1, Ecole Centrale de Lyon, INSA Lyon, CNRS, Ampère, Villeurbanne, France; riccardo.scorretti@univ-lyon1.fr

Abstract: Coupling of electrical circuits with 2D and 3D computational domains is very important for practical applications. To this aim, the notions of "electrical current" and "voltage" must be defined, and linked with local quantities (i.e. fields and potentials) in the computational domain. The definition of voltage is more complex than it may appear at a first glance, and usually tainted by unspoken hypothesis. The purpose of this work twofold: on one hand, to shed light on the definition and on the physical meaning of voltage in the case of time varying quasi-static fields, and on the other hand to show that existing coupling formulas can be rewritten within a more general framework, basing upon the power balance.

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1. Introduction

Coupling of electrical circuits with Finite Element (FE) computational domains is very important for practical applications, and it is investigated since a long time [1–15]. Maxwell's equations are solved in the computational domain by using various formulations, the unknown of which are local quantities like potentials, or directly the electric or magnetic fields. Currents and voltages are global quantities, which are applied to the computational domain through "ports", that is interfaces between internal regions, or surfaces on the external boundary of the domain.

To this aim, the notions of "current" and "voltage" must be precisely defined, and linked with the electric field $\mathbf{e}$ and current density $\mathbf{j}$ in the computational domain. The definition of the electrical current I which flows across a given surface Σ is easily expressed, and it depends uniquely on the current density:

$$I = -\iint_{\Sigma} \mathbf{n} \cdot \mathbf{j} \tag{1}$$

where $\mathbf{n}$ is the unit normal vector, with outward orientation with respect of the computational domain. Conversely, the understanding of the physical significance of voltage is not as trivial as it may appears [16], and it is worth to be clarified. In the case of static fields (electrostatics, continuous currents) the electric field is conservative, and the voltage U between two equipotential surfaces Σ_a and Σ_b can be uniquely defined as:

$$U = \int_{\gamma} \mathbf{e} \cdot \mathbf{dl} = v_a - v_b \tag{2}$$

where γ is any path which goes from Σ_a to Σ_b, and v is an electric scalar potential such that $\mathbf{e} = -\operatorname{grad} v$. However, this definition of voltage cannot be truly satisfying in that it doesn't apply, and cannot be easily extended to the case of time varying fields. In fact in this case the electric field is no more conservative and thus the choice of the path γ matters.

In the case of time varying fields one may define the voltage as the circulation of the "electrostatic component" ($-\operatorname{grad} v$) of the electric field:

$$\mathbf{e} = -\partial_t \mathbf{a} - \operatorname{grad} v \tag{3}$$

However, in the case of bounded domains there is no such a thing like *the* electrostatic component: at most, one may speak of *an* electrostatic component. In fact Helmholtz's theorem [17] ensures that such a decomposition (3) exists, but in the case of bounded domains it is not unique [18] unless appropriate additional boundary conditions are imposed. Even assuming that such a definition is well posed from the mathematical point of view – that is, if one assumes that the value of the voltage hereby defined is independent on the gauge of $(\mathbf{a}, v)$, which is indeed the case – the question of the physical significance arises.

One observes that the definition (1) is not tainted by any of these issues: the current I is defined basing on the current density only, and the definition is unambiguous in the case of time varying fields, and whatever the surface Σ.

The question of the well-posedness of the definition of voltage in the case of time varying fields is seldom addressed in undergraduate courses in physics and electrical engineering (because in practice "things work") but also in most of the scientific literature. In the author's opinion, any effort to shed light on this topic is worth to be explored.

The case of stranded coils [1,2,19**?**] is considerably simpler. In fact, isopotential surfaces Σ_a and Σ_b reduce to points, so that in practice the integration path γ is unique, and it is imposed by the geometry of the electrical wires. In this case the induced electromotive force can be expressed through the time derivative of the magnetic flux.

The case of solid conductors is more complex. In some cases [6] the voltage between two "isopotential" surfaces is mathematically defined as the difference of an electric scalar potential v, but the question of its physical significance is eluded. As already observed, this scalar potential is not unique and it is not clear at all why the voltage has the same value whatever the gauge of $(\mathbf{a}, v)$. The very same notion of "isopotential" surfaces is disturbing, because in the case of bounded domains it depends on a particular choice of the gauge, whereas the voltage ought to depend uniquely on the electric field.

The same considerations apply to [5,12,15] where a 2D modelling is taken on. In this case Coulomb's gauge $\operatorname{div} \mathbf{a} = 0$ is arbitrarily fixed by imposing that the vector potential $\mathbf{a}$ is perpendicular to the xy plane, thus a voltage between the front and the rear part of the domain can be uniquely defined as:

$$U = \ell(-\mathbf{n} \cdot \operatorname{grad} v) \tag{4}$$

where ℓ is the depth of the domain, and $\mathbf{n}$ is the unit vector perpendicular to the plane.

In [7,20] the following relationship is derived for a two-ports conductor in the electrokinetic case (aka continuous current):

$$U = -\int_{\Omega} \mathbf{j_0} \cdot \operatorname{grad} v \tag{5}$$

where $\mathbf{j_0}$ is any current density corresponding to a net current of 1 A. This equation is then generalized in various "flavours" [7,14,20] to the case of time varying fields by rewriting $\operatorname{grad} v = -\partial_t \mathbf{a} - \mathbf{e}$:

$$\begin{aligned} U &= \int_{\Omega_c} \mathbf{j_0} \cdot \mathbf{e} + \int_{\Omega_c} \mathbf{j_0} \cdot \partial_t \mathbf{a} && (6) \\ &= \int_{\Omega_c} \mathbf{j_0} \cdot \mathbf{e} + \int_{\Omega_c} \mathbf{t_0} \cdot \partial_t \mathbf{b} && (7) \\ &= RI + \partial_t \int_{\Omega_c} \mathbf{t_0} \cdot \mathbf{b} && (8) \end{aligned}$$

where $\mathbf{j_0} = \operatorname{curl} \mathbf{t_0}$ (see the aforementioned works for details). However, by doing so one forgets that the starting equation (5) holds under the hypothesis that fields don't vary with time. In spite of the fact that the derivation of (6)–(8) is somehow simplistic, all these equations are found to be correct.

A completely different approach is to postulate that the current flows out of special thin "generator regions" $\Omega_{emf,i}$, where a source of electromotive force exists [8–11]. Inside the generator regions the electric field is conservative and hence (2) is well posed, hence the voltage between the electrodes of a generator writes:

$$U = \int_{\gamma_i} \mathbf{e_i} \cdot \mathbf{dl} \tag{9}$$

where γ_i is a path which joins the electrodes, and $\mathbf{e_i}$ is the electric field inside the generator. In practice the integral in (9) is never computed explicitly. Generators are removed from the computational domain, so that their surfaces become new boundaries of the domain, over which appropriate boundary conditions must be imposed. Depending on the formulation, the voltage can be imposed either strongly (**a** formulation) or weakly (**h** formulation) – see details in [8,11].

This approach removes the practical difficulty of numerically defining voltages and has a correct power balance (all the electrical power which exits from generators enters in the domain), but it is not completely satisfying from the point of view of physics significance: voltage ought to be defined independently on how it is applied – that is, basing exclusively on the electric field **e** *inside* the computational domain.

Finally, one observes that numerical modelling of the coupling of electrical circuits with computational domains is addressed in specific ways for each formulation. This gives rise to a large number of different formulas which express the link between global (currents and voltages) and local quantities (fields and potentials). This is a source of complication – including from the pedagogical point of view.

The purpose of this work is twofold. First, we aim at clarifying the notion of "voltage" in the case of time varying fields, and to expressing it by using exclusively the electric field inside the domain, without resorting to any potential. An operator $\mathcal{U}_{ij}[\mathbf{e}]$ which express the voltage between two ports i, j for a given electric field **e** will be defined (section 2). As side product, a similar operator $\mathcal{I}_n[\mathbf{h}]$ for the current is obtained (section 3). On the other hand, to show that existing formulas to express currents and voltages can be derived as particular case of a unique general formula (section 4). Some pedagogical considerations conclude the article (section 5).

2. Definition of voltage

In [21] Hiptmair and Sterz pointed out the difficulties to define rigorously what voltage is, and suggested it could be defined through the notion of electric power. Basing on this idea, it will be shown that it is possible to give a precise meaning to the notion of "voltage" in the case of time varying fields, and for an arbitrary number of ports. The proposed definition of voltage respects the power balance, and it is coherent with all previous works. Moreover, the proposed definition of voltage relies exclusively on the electric field, and it is therefore independent of any formulation or numerical method. To go further, we need to demonstrate the following theorem:

Theorem 1 (fake power)**.** *Let* $\mathbf{e}_1$ *and* $\mathbf{h}_2$ *be an electric and a magnetic field, defined in a domain* Ω_c *and* Ω*, where* $\Omega_c \subset \Omega$ *is the conductive part of* Ω*. Let* $\mathbf{b}_1$ *and* $\mathbf{j}_2$ *the magnetic flux density and he current density associated respectively with* $\mathbf{e}_1$ *and* $\mathbf{h}_2$*, and assume that displacement currents are neglected:*

$$\operatorname{curl} \mathbf{e}_1 = -\partial_t \mathbf{b}_1 \tag{10}$$

$$\operatorname{curl} \mathbf{h}_2 = \mathbf{j}_2 \tag{11}$$

Assume that the boundary $\partial\Omega = \Gamma_h \cup \Gamma_e$ *can be parted in two surfaces* Γ_e *and* Γ_h *where the following boundary conditions hold:*

$$\text{on } \Gamma_e: \quad \mathbf{n} \times \mathbf{e}_1 = 0 \quad (\Rightarrow \ \mathbf{n} \cdot \partial_t \mathbf{b}_1 = 0) \tag{12}$$

$$\text{on } \Gamma_h: \quad \mathbf{n} \times \mathbf{h}_2 = 0 \quad (\Rightarrow \ \mathbf{n} \cdot \mathbf{j}_2 = 0) \tag{13}$$

and that it exists a couple of potentials $(\mathbf{a}_1, v_1)$ *such that:*

$$\mathbf{e}_1 = -\partial_t \mathbf{a}_1 - \operatorname{grad} v_1 \tag{14}$$

$$\mathbf{n} \times \partial_t \mathbf{a}_1|_{\Gamma_e} = 0 \tag{15}$$

Under these hypothesis we have:

1. *the scalar potential* v_1 *is equipotential on each of the N connected components* $\Gamma_{e,n}$ *of* $\Gamma_e \cap \partial\Omega_c$*:*

$$v_1|_{\Gamma_{e,n}} = V_{1,n} \tag{16}$$

   Moreover, for a given electric field $\mathbf{e}_1$ *the set of potentials* $\{V_{1,n}\}$ *is unique up to an arbitrary constant value.*

2. *The following equality holds:*

$$\mathcal{P}[\mathbf{e}_1, \mathbf{h}_2] = \int_{\Omega_c} \mathbf{j}_2 \cdot \mathbf{e}_1 + \int_{\Omega} \mathbf{h}_2 \cdot \partial_t \mathbf{b}_1 = \int_{\Omega_c} \operatorname{curl} \mathbf{h}_2 \cdot \mathbf{e}_1 - \int_{\Omega} \mathbf{h}_2 \cdot \operatorname{curl} \mathbf{e}_1 \tag{17}$$

$$= -\int_{\Omega_c} \mathbf{j}_2 \cdot \operatorname{grad} v_1 \tag{18}$$

3. *Let* $I_{2,n}$ *be the total current which enters in* Ω_c *through* $\Gamma_{e,n}$*:*

$$I_{2,n} = -\int_{\Gamma_{e,n}} \mathbf{n} \cdot \mathbf{j}_2 \tag{19}$$

   where $\mathbf{n}$ *is the unitary vector normal to* $\Gamma_{e,n}$ *oriented outwardly. Then, the following equality holds:*

$$\mathcal{P}[\mathbf{e}_1, \mathbf{h}_2] = \sum_n V_{1,n} I_{2,n} \tag{20}$$

Before giving the demonstration, some remarks are mandatory:

- We stress that the hypothesis of the theorem don't require that constitutive laws are enforced (they could eventually be, but this is not necessary):

$$\mathbf{j}_2 \neq \sigma \mathbf{e}_1 \quad ; \quad \mathbf{b}_1 \neq \mu \mathbf{h}_2 \tag{21}$$

  In this case, the E-side and the H-side of Tonti's diagram are unlinked, thus $\mathbf{e}_1$ and $\mathbf{h}_2$ cannot be both the true electric field $\mathbf{e}$ and magnetic fields $\mathbf{h}$:

$$\begin{array}{lccccccc} \text{E-side:} & v_1 & \xrightarrow{\text{grad}} & \mathbf{e}_1, \mathbf{a}_1 & \xrightarrow{\text{curl}} & \mathbf{b}_1 & \xrightarrow{\text{div}} & 0 \\ & & & \vdots & & \vdots & & \\ \text{H-side:} & 0 & \xleftarrow{\text{div}} & \mathbf{j}_2 & \xleftarrow{\text{curl}} & \mathbf{h}_2 & & \end{array} \tag{22}$$

  Hence the quantity $\mathcal{P}[\mathbf{e}_1, \mathbf{h}_2]$ defined by (17), which is homogeneous to a power, has no physical significance. For this reason in the general case $\mathcal{P}[\mathbf{e}_1, \mathbf{h}_2]$ will be hereafter called "fake" power. Conversely, when constitutive laws are enforced, then $\mathbf{e}_1 = \mathbf{e}$ and $\mathbf{h}_2 = \mathbf{h}$ are the true electric and magnetic fields, and $\mathcal{P}[\mathbf{e}, \mathbf{h}]$ is the true, physically significant, instantaneous power which is provided to the domain by the generators.

- The boundary Γ_e writes:

$$\Gamma_e = \underbrace{\left(\bigcup_n \Gamma_{e,n}\right)}_{\subseteq \partial\Omega_c} \cup \Gamma_b \tag{23}$$

The surfaces $\Gamma_{e,n}$ are the ports which connect the computational domain with the external electric circuit. As such, they are also part of the boundary of $\partial\Omega_c$, where the scalar potential v_1 is defined, hence. The surface Γ_b is the remaining part of Γ_e, where the scalar potential are not defined (Figure 1). The boundary of Ω_c writes:

$$\partial\Omega_c = \left(\bigcup_n \Gamma_{e,n}\right) \cup \Gamma_j \tag{24}$$

Γ_j is the interface between Ω_c and the remaining (insulator) part of Ω, where $\mathbf{n} \cdot \mathbf{j} = 0$.

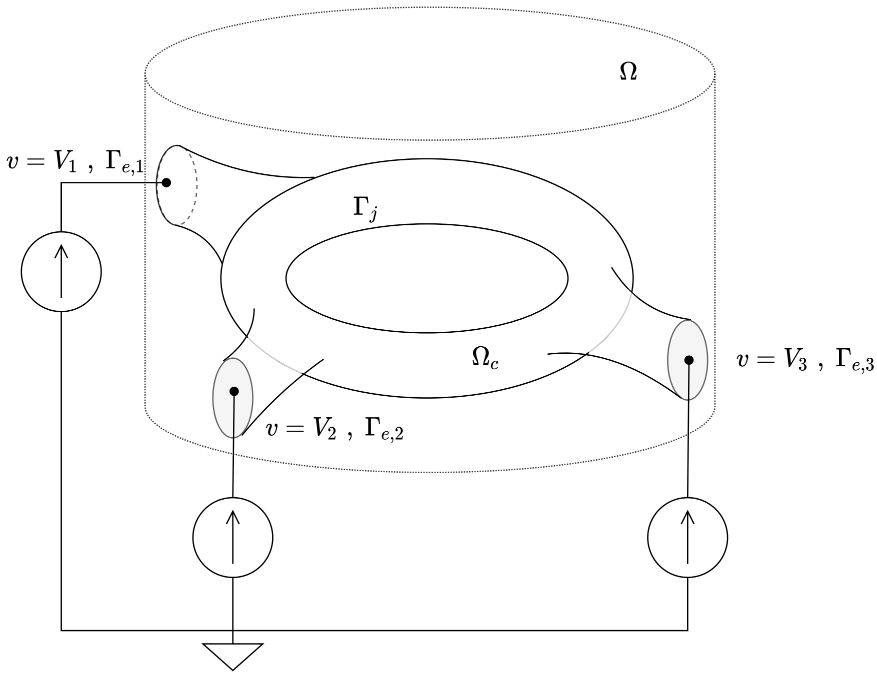

Figure 1. Computational domain Ω composed of a conductive part Ω_c. The generators feed the conductive part through the three ports $\Gamma_{e,1\ldots3}$. The boundary of Ω_c is composed of the ports $\Gamma_{e,1\ldots3}$ and of Γ_j.

Proof of Theorem 1. By writing $\operatorname{grad} v_1 = -\mathbf{e}_1 - \mathbf{a}_1$ and by using (15) on each port $\Gamma_{e,n}$ we have:

$$\mathbf{n} \times \operatorname{grad} v_1|_{\Gamma_{e,n}} = -\mathbf{n} \times \partial_t \mathbf{a}_1|_{\Gamma_{e,n}} - \mathbf{n} \times \mathbf{e}_1|_{\Gamma_{e,n}} = 0 \tag{25}$$

Hence v_1 is constant on each port $\Gamma_{e,n}$. By using (14) the first integral of (17) writes:

$$\int_{\Omega_c} \mathbf{j}_2 \cdot \mathbf{e}_1 = -\int_{\Omega_c} \mathbf{j}_2 \cdot \partial_t \mathbf{a}_1 - \int_{\Omega_c} \mathbf{j}_2 \cdot \operatorname{grad} v_1 \tag{26}$$

By integrating by parts, the second integral of (17) writes:

$$\int_\Omega \mathbf{h}_2 \cdot \partial_t \mathbf{b}_1 = \int_\Omega \mathbf{h}_2 \cdot \partial_t \operatorname{curl} \mathbf{a}_1 \tag{27}$$

$$= \int_\Omega \mathbf{j}_2 \cdot \partial_t \mathbf{a}_1 + \int_{\partial\Omega} \mathbf{h}_2 \cdot \mathbf{n} \times \partial_t \mathbf{a}_1 \tag{28}$$

$$= \int_\Omega \mathbf{j}_2 \cdot \partial_t \mathbf{a}_1 \tag{29}$$

where the boundary integral vanishes due to the boundary conditions (12) and (13). By using together (26) and (29) one obtains (18). Finally, by using integration by parts, (18) writes:

$$-\int_{\Omega_c} \mathbf{j}_2 \cdot \operatorname{grad} v_1 = \int_{\Omega_c} v_1 \operatorname{div} \mathbf{j}_2 - \int_{\partial\Omega_c} v_1 \mathbf{n} \cdot \mathbf{j}_2 \tag{30}$$

The first integral on the right hand side vanishes because $\operatorname{div} \mathbf{j}_2 = \operatorname{div} \operatorname{curl} \mathbf{h}_2 = 0$. The boundary of Ω_c can be parted as:

$$\partial\Omega_c = \left(\bigcup_n \Gamma_{e,n} \right) \cup \Gamma_j \tag{31}$$

where Γ_j is the interface between the conductive and insulator part of the domain $\Omega_i = \Omega \backslash \Omega_c$. The current cannot flow in or out from Γ_j, that is:

$$\mathbf{n} \cdot \mathbf{j}_2|_{\Gamma_j} = 0 \tag{32}$$

hence the boundary integral in (30) writes:

$$\mathcal{P}[\mathbf{e}_1, \mathbf{h}_2] = \sum_n -\int_{\Gamma_{e,n}} v_1 \mathbf{n} \cdot \mathbf{j}_2 = \sum_n V_n \left(-\int_{\Gamma_{e,n}} \mathbf{n} \cdot \mathbf{j}_2 \right) = \sum_n V_n I_n \tag{33}$$

Finally, the uniqueness of the set $\{V_{1,n}\}$ can be demonstrated by observing that these values depend uniquely on the electric field $\mathbf{e}_1$. Hence, a set of $N-1$ independent equations can be obtained from (33) by selecting an arbitrary port $\Gamma_{e,ref}$ (namely $\Gamma_{e,ref} = \Gamma_{e,N}$) and by considering $N-1$ magnetic fields $\mathbf{h}_{2,n}$ such that $I_{2,ref} = -I_{2,n} = 1$ A.

$$\begin{cases} V_{1,1} - V_{1,ref} & = \mathcal{P}[\mathbf{e}_1, \mathbf{h}_{2,1}] \\ V_{1,2} - V_{1,ref} & = \mathcal{P}[\mathbf{e}_1, \mathbf{h}_{2,2}] \\ \vdots \\ V_{1,N-1} - V_{1,ref} & = \mathcal{P}[\mathbf{e}_1, \mathbf{h}_{2,N-1}] \end{cases} \tag{34}$$

□

In order to define the voltage U_{ij} between two ports $\Gamma_{e,i}$ and $\Gamma_{e,j}$, we define the set $\mathcal{H}_{ij}$ of "test" magnetic field:

$$\mathcal{H}_{ij} = \{\mathbf{h}' \in \mathbf{H}(\operatorname{curl}, \Omega) : \mathbf{n} \times \mathbf{h}' = 0 \text{ on } \Gamma_h\, , I'_i = -I'_j = 1 \text{ A}\, , I'_{n \neq i,j} = 0\} \tag{35}$$

where I'_n is the net current due to $\mathbf{j}' = \operatorname{curl} \mathbf{h}'$ which enters in Ω through the n^{th} port $\Gamma_{e,n}$. The following theorem provides a well posed definition of the voltage U_{ij}:

Theorem 2 (definition of voltage)**.** *Let* $\mathbf{e}$ *be the electric field in* Ω_c*, and* $\Gamma_{e,i}$ *and* $\Gamma_{e,j}$ *two ports. There is one and only one value* $U_{ij}[\mathbf{e}]$ *such that:*

$$U_{ij}[\mathbf{e}] = \frac{\mathcal{P}[\mathbf{e}, \mathbf{h}']}{1\,A} \tag{36}$$

for any $\mathbf{h}' \in \mathcal{H}_{ij}$.

Proof of Theorem 2. The theorem 1 holds with $\mathbf{e}_1 = \mathbf{e}$ and $\mathbf{h}_2 = \mathbf{h}'$, $\forall \mathbf{h}' \in \mathcal{H}_{ij}$, thus the fake power writes:

$$\mathcal{P}[\mathbf{e}, \mathbf{h}'] = V_i I_i' + V_j I_j' = (V_i - V_j) \times 1\ \mathrm{A} \tag{37}$$

In order to demonstrate that this value is unique we must show that it does not depend on $\mathbf{h}'$, neither on the gauge of $(\mathbf{a}, v)$. The left side hand term of (37) does not depend on the gauge because of (17), and the right side hand of (37) does not depend on $\mathbf{h}'$. Hence $\mathcal{P}[\mathbf{e}, \mathbf{h}']$ is independent of both $\mathbf{h}'$ and the gauge of $(\mathbf{a}, v)$. Therefore, $\mathcal{P}[\mathbf{e}, \mathbf{h}']$ depends exclusively on the electric field $\mathbf{e}$, which proves its uniqueness. Hence $U_{ij}[\mathbf{e}] = V_i - V_j$ is the only value which satisfies (36) whatever $\mathbf{h}' \in \mathcal{H}_{ij}$. □

This definition of voltage is well posed because U_{ij} depends exclusively on the electric field $\mathbf{e}$, and it holds as well for static and time varying fields. One observes that when $V_n = v|_{\Gamma_{e,n}}$ are the voltages of generators which feed the device (Figure 1), and $\mathbf{e}_1 = \mathbf{e}$ and $\mathbf{h}_2 = \mathbf{h}$ are the true electric and magnetic fields, the power balance is verified in that (20) writes:

$$\mathcal{P}[\mathbf{e}, \mathbf{h}] = \int_{\Omega_c} \mathbf{j} \cdot \mathbf{e} + \int_{\Omega} \mathbf{h} \cdot \partial_t \mathbf{b} = \sum_n V_n I_n \tag{38}$$

This result is important because it provides a precise physical significance to "voltages", that is the unique set of values $U_{ij} = V_i - V_j$ such that the power balance (38) is respected. In particular, in the case of a simple two-ports domain the voltage can be defined like the ratio between the instantaneous power injected into the domain and the instantaneous current. One observes that the electric vector potential $\mathbf{t_0}$ defined beforehand belongs to $\mathcal{H}_{ij}$: this shows that indeed (5) and (7) are particular cases of (36).

3. Expression of the current

Now that we have given a precise definition of voltage, one observes that the roles of $\mathbf{e}_1$ and $\mathbf{h}_2$ can be exchanged, so as to provide a useful expression of the net current which enters in the domain through electrodes. To this aim, let's define the set $\mathcal{E}_n$ of "test" electric fields:

$$\mathcal{E}_n = \{\mathbf{e}' \in \mathbf{H}(\mathrm{curl}, \Omega_c) : \mathbf{n} \times \mathbf{e}' = 0 \text{ on } \Gamma_e\,,\ \mathbf{e}' = -\partial_t \mathbf{a}' - \mathrm{grad}\, v' \\ \text{with } v' \equiv 1\ \mathrm{V} \text{ on } \Gamma_{e,n}\,,\ v' \equiv 0 \text{ on } \Gamma_{e,\neq n}\} \tag{39}$$

The following theorem provides a useful formula to compute the electric current which enters in the domain through a port:

Theorem 3 (computation of current)**.** *Let* $\mathbf{h}$ *be the magnetic field in* Ω*, and* $\Gamma_{e,n}$ *a port. The electric current* I_n *which enters in the domain through* $\Gamma_{e,n}$ *is equal to:*

$$I_n = \frac{\mathcal{P}[\mathbf{e}', \mathbf{h}]}{1\ V} \tag{40}$$

for any $\mathbf{e}' \in \mathcal{E}_n$.

Proof of Theorem 3. The theorem 1 holds with $\mathbf{e}_1 = \mathbf{e}'$ and $\mathbf{h}_2 = \mathbf{h}$, $\forall \mathbf{e}' \in \mathcal{E}_n$, thus the fake power writes:

$$\mathcal{P}[\mathbf{e}', \mathbf{h}] = v'|_{\Gamma_{e,n}} \times I_n = 1\ \mathrm{V} \times I_n \tag{41}$$

□

In this case there is no need to prove the uniqueness of I_n because currents are correctly defined. Nevertheless an argument similar to the one which we used to prove the Theorem 2 applies as well (that is: $\mathcal{P}[\mathbf{e}', \mathbf{h}]$ doesn't depend on the gauge, I_n doesn't depend on $\mathbf{e}'$).

One observes that even if (40) express the current basing on $\mathbf{h}$, only its curl matters:

$$I_n = -\int_{\Gamma_{e,n}} \mathbf{n} \cdot \mathbf{j} = -\int_{\Gamma_{e,n}} \mathbf{n} \cdot \operatorname{curl} \mathbf{h} = \frac{\mathcal{P}[\mathbf{e}', \mathbf{h}]}{1\ \mathrm{V}} \tag{42}$$

4. Formulations

Hereafter several Finite Element formulations are reviewed, with the purpose to show that for all of them the coupling equations with electrical circuits can be easily derived by using (36) and (40). In order to somehow simplify the notation hereafter the divisions by 1 A or 1 V will be dropped, and the expressions of the voltage and current will write simply:

$$\mathcal{U}_{ij}[\mathbf{e}] : \mathbf{e} \mapsto \mathcal{P}[\mathbf{e}, \mathbf{h}'] \quad \forall \mathbf{h}' \in \mathcal{H}_{ij} \tag{43}$$

$$\mathcal{I}_n[\mathbf{h}] : \mathbf{h} \mapsto \mathcal{P}[\mathbf{e}', \mathbf{h}] \quad \forall \mathbf{e}' \in \mathcal{E}_n \tag{44}$$

For each formulation, particular test fields $\mathbf{h}' \in \mathcal{H}_{ij}$ or $\mathbf{e}' \in \mathcal{E}_n$ will be chosen in order to simplify the writing of formulations at the discrete level. The purpose of this section is to show that known expressions to compute voltages and currents can be view as particular case of (36) and (40) respectively.

4.1. Electrokinetics

The scalar potential electrokinetic formulation writes:

$$\operatorname{div}(\sigma \operatorname{grad} v) = 0 \tag{45}$$

that is $\operatorname{div} \mathbf{j} = 0$, with $\mathbf{e} = -\operatorname{grad} v$, with σ = electrical conductivity. The weak form writes:

$$\int_{\Omega_c} \sigma \operatorname{grad} v \cdot \operatorname{grad} v' + \int_{\partial\Omega_c} (\mathbf{n} \cdot \mathbf{j}) \cdot v' = 0 \tag{46}$$

where v and v' belong to appropriate function spaces.

At the discrete level the domain $\Omega = \Omega_c$ is meshed, and the set of nodes $\mathbb{N}$ is parted as:

$$\mathbb{N} = \left[\bigcup_n \mathbb{N}_{i,n} \right] \cup \mathbb{N}_v \tag{47}$$

where $\mathbb{N}_{i,n}$ is the set of nodes on the n^{th} port $\Gamma_{e,n}$, and $\mathbb{N}_v$ is the set of the remaining nodes. Classical nodal shape functions $s_n(\mathbf{x})$ are associated with each nodes of $\mathbb{N}_v$, whereas special isopotential shape functions $f_n(\mathbf{x})$ are associated with each port $\Gamma_{e,n}$ [8,9,11]:

$$f_n(\mathbf{x}) = \sum_{k \in \mathbb{N}_{i,n}} v_k(\mathbf{x}) \tag{48}$$

The support of f_n is bound to the layer of elements which touch $\Gamma_{e,n}$. One observes that $f_n \equiv 1$ on $\Gamma_{e,n}$ and $f_n \equiv 0$ on any other port, thus $\mathbf{e}'_n = -\operatorname{grad} f_n \in \mathcal{E}_n$. By using (40) the net current I_n which enters in Ω_c through the n^{th} writes:

$$I_n = \int_{\Omega_c} \mathbf{j} \cdot (-\operatorname{grad} f_n) = \int_{\Omega_c} \sigma \operatorname{grad} v \cdot \operatorname{grad} f_n = -\int_{\partial\Omega_c} (\mathbf{n} \cdot \mathbf{j}) \cdot f_n \tag{49}$$

where the last equality comes from (46). One observes that the latter equation writes

$$I_n = \mathcal{I}_n[\mathbf{h}] = \mathcal{P}[-\operatorname{grad} f_n, \mathbf{h}] \tag{50}$$

where $\mathbf{h}$ is the magnetic field (which is not computed by using this formulation) and $\mathbf{j} = \operatorname{curl} \mathbf{h}$.

The discrete approximation of v writes:

$$v(\mathbf{x}) = \sum_{n\in\mathbb{N}_v} v_n s_n(\mathbf{x}) + \sum_k V_k f_k(\mathbf{x}) \tag{51}$$

With this formulation, the potential V_n on the port $\Gamma_{e,n}$ can be imposed strongly, as it belongs directly to the set of unknowns. When Galerkin's method is used the same shape functions are used both to expand the unknowns and as test functions, hence the coupling equation which allows to impose the current I_n appears naturally for $v' = f_n$:

$$\int_{\Omega_c} \sigma \operatorname{grad} v \cdot \operatorname{grad} f_n - I_n = 0 \tag{52}$$

4.2. Eddy current formulations

A very large number of formulations have been developed to model eddy currents. In general, either the current or the voltage can be imposed strongly, whereas the other quantity has to be imposed weakly. Hereafter some of the most used formulations are reviewed, but similar considerations can be taken on for any other formulation.

4.2.1. $\mathbf{t} - \mathbf{t}_0 - \phi$ formulation

With the $\mathbf{t} - \mathbf{t}_0 - \phi$ formulation the equations on the E side of Tonti's diagram ($\operatorname{curl}\mathbf{e} = -\partial_t \mathbf{b}$, $\operatorname{div}\mathbf{b} = 0$) are imposed weakly:

$$\operatorname{curl}(\rho \mathbf{j}) = -\partial_t(\nu \mathbf{h}) \tag{53}$$

with $\rho = 1/\sigma$ = electrical resistivity and $\nu = 1/\mu$ = magnetic reluctivity. The magnetic field $\mathbf{h}$ and the current density $\mathbf{j}$ write respectively [14,20]:

$$\mathbf{h} = \begin{cases} \mathbf{t} + \sum_n I_n \mathbf{t}_{0,n} - \operatorname{grad}\phi & \text{in } \Omega_c \\ \sum_n I_n \mathbf{t}_{0,n} - \operatorname{grad}\phi & \text{in } \Omega\backslash\Omega_c \end{cases} \tag{54}$$

$$\mathbf{j} = \operatorname{curl}\mathbf{t} + \sum_n I_n \mathbf{t}_{0,n} \text{in } \Omega_c \tag{55}$$

where ϕ is a magnetic scalar potential and $\mathbf{t}_{0,n}$ are precomputed electric vector potentials defined on $\Omega_0 \supset \Omega_c$. For each two-ports solid conductor, a potential $\mathbf{t}_{0,n}$ is precomputed in such a way that the net current associated with $\mathbf{j}_{0,n} = \operatorname{curl}\mathbf{t}_{0,n} = 1$ A, thus $\mathbf{t}_{0,n} \in \mathcal{H}_n$ (in order to simplify the notation, where two-ports cases are concerned we write n instead of i, j).

Currents in coils I_n belong to the set of unknowns, and are therefore imposed strongly. By taking $\mathbf{h}' = \mathbf{t}_{0,n}$ one finds with (36) the expression of the voltage U_n associated with each coil:

$$U_n = \mathcal{P}[\mathbf{e}, \mathbf{t}_{0,n}] = \int_{\Omega_c} \mathbf{j}_{0,n} \cdot \mathbf{e} + \int_{\Omega_0} \mathbf{t}_{0,n} \cdot \partial_t \mathbf{b} \tag{56}$$

In the case of non simply connected conductors, additional potentials $\mathbf{t}_{\mathbf{0},k}$ and associated unknowns I_k must be added to cope with the so-called connexity problem (that is, to be able to impose Ampère's theorem). The corresponding voltages U_k are set to 0 by using (56):

$$U_k = \int_{\Omega_c} \mathbf{j}_{0,k} \cdot \mathbf{e} + \int_{\Omega_0} \mathbf{t}_{0,k} \cdot \partial_t \mathbf{b} = 0 \tag{57}$$

By the way, one observes that in the case of static fields ($\partial_t \equiv 0$) one obtains seamlessly the dual electrokinetic formulation [22] with respect of the aforementioned v formulation by imposing $\operatorname{curl} \mathbf{e} = 0$:

$$\operatorname{curl}[\rho \operatorname{curl}(\mathbf{t} + \sum_n \mathbf{t}_{0,n} I_n)] = 0 \tag{58}$$

Voltages can be expressed by (36) as:

$$U_n = \mathcal{P}[\mathbf{e}, \mathbf{t}_{0,n}] = \int_{\Omega_c} \mathbf{j}_{0,n} \cdot \mathbf{e} \tag{59}$$

145 4.2.2. $\mathbf{a} - v$ formulation

With the $\mathbf{a} - v$ formulation the equations on the H side of Tonti's diagram ($\operatorname{curl} \mathbf{h} = \mathbf{j}$, $\operatorname{div} \mathbf{j} = 0$) are imposed weakly:

$$\operatorname{curl}(\nu \mathbf{b}) = \sigma \mathbf{e} \tag{60}$$

The electric field $\mathbf{e}$ and the flux density $\mathbf{b}$ write respectively

$$\mathbf{e} = -\partial_t \mathbf{a} - \operatorname{grad} v \quad \text{in } \Omega_c \tag{61}$$

$$\mathbf{b} = \operatorname{curl} \mathbf{a} \tag{62}$$

where $\mathbf{a}$ and v are respectively a magnetic vector potential and an electric scalar potential. By using (36) together with (20) one observes that voltages is determined by imposing the scalar potential v only. In fact for any $\mathbf{h}' \in \mathcal{H}_{ij}$ we have:

$$U_{ij} = \mathcal{P}[\mathbf{e}, \mathbf{h}'] = V_i - V_j \tag{63}$$

For instance in [9] this is accomplished by discretizing the scalar potential v as:

$$v = \sum_n V_i v_{0,n}(\mathbf{x}) \tag{64}$$

where the function $v_{0,n}$ is any potential which corresponds to a voltage of 1 V for the n^{th} solid conductor, and hence $-\operatorname{grad} v_{0,n} \in \mathcal{E}_n$ (in [9] only two-ports conductors are considered). It is found that the current I_n which flows through the n^{th} conductor writes:

$$I_n = \int_{\Omega_c} -\sigma(\partial_t \mathbf{a} + \operatorname{grad} v) \cdot (-\operatorname{grad} v_{0,n}) = \int_{\Omega_c} \mathbf{j} \cdot \mathbf{e}' \tag{65}$$

where $\mathbf{e}' = -\operatorname{grad} v_{o,n}$. It seems that in order to retrieve (40) the term $\int_\Omega \mathbf{h} \cdot \partial_t \mathbf{b}$ is missing. But in fact, notice that $-\operatorname{grad} v_{o,n}$ is defined only on Ω_c and it is curl-free. It is possible to find a curl-free extension of $-\operatorname{grad} v_{o,n}$ over all the domain:

$$\mathbf{e}' = \begin{cases} -\operatorname{grad} v_{o,n} & \text{in } \Omega_c \\ -\operatorname{grad} \tilde{v} & \text{in } \Omega \backslash \Omega_c \end{cases} \tag{66}$$

where $\tilde{v}$ is any scalar potential which enforce the continuity with v on $\partial\Omega_c$. The thereby defined test electric field $\mathbf{e}'$ is curl-free and it is defined over all the domain, thus:

$$\partial_t \mathbf{b}' = -\operatorname{curl} \mathbf{e}' \equiv 0 \tag{67}$$

Hence the integral $\int_\Omega \mathbf{h} \cdot \partial_t \mathbf{b}'$ vanishes, and the expression (40) is found again. Notice that the real electric field $\mathbf{e}$ is not curl-free, but test electric fields $\mathbf{e}'$ are allowed to.

4.2.3. $\mathbf{h} - \phi$ formulation

With the $\mathbf{h} - \phi$ formulation the equations on the E side of Tonti's diagram ($\operatorname{curl} \mathbf{e} = -\partial_t \mathbf{b}$, $\operatorname{div} \mathbf{b} = 0$) are imposed weakly. The computational domain is parted as $\Omega =$

$\Omega_c \cup \Omega_c^C$, and a number of "cuts" have to be done to make Ω_c^C (insulators) simply connected. Assume that only two-ports solid conductors are present, some of which are coils. The field is impressed by generators through thin sections in the coils. The weak form of the formulation writes [10]:

$$\int_\Omega \partial_t(\mu \mathbf{h}) \cdot \mathbf{h}' + \int_{\Omega_c} \rho \,\mathrm{curl}\, \mathbf{h} \cdot \mathrm{curl}\, \mathbf{h}' + \int_{\Gamma_e} \mathbf{n} \times \mathbf{e}_s \cdot \mathbf{h}' = 0 \tag{68}$$

The unknown of this formulation is directly the magnetic field $\mathbf{h}$, which is discretized as:

$$\mathbf{h} = \sum_{k \in \mathbb{E}_c} h_k \mathbf{w}_k(\mathbf{x}) + \sum_{n \in \mathbb{N}_c^C} \phi_n \mathbf{v}_n(\mathbf{x}) + \sum_{n \in \mathbb{C}} I_n \mathbf{c}_n(\mathbf{x}) \tag{69}$$

where $\mathbb{E}_c$ is the set of edges internal to Ω_c, $\mathbb{N}_c^C$ is the set of nodes on Ω_c^C and on its boundary, and $\mathbb{C}$ is the set of cuts required to make Ω_c^C simply connected (see [8,10,23] for details). The key point is that $\mathbf{j}_n = \mathrm{curl}\, \mathbf{c}_n$ is a current density which corresponds to a unit net current in the n^{th} coil, hence $\mathbf{c}_n \in \mathcal{H}_n$. The coefficients I_n represent the currents which flow through the n^{th} coil, which can therefore by imposed strongly. It is found [10] that the corresponding voltage U_n writes:

$$U_n = \int_\Omega \partial_t(\mu \mathbf{h}) \cdot \mathbf{c}_n + \int_{\Omega_c} \rho \,\mathrm{curl}\, \mathbf{h} \cdot \mathrm{curl}\, \mathbf{c}_n = \mathcal{P}[\mathbf{e}, \mathbf{h}'] \tag{70}$$

where the test magnetic field is $\mathbf{h}' = \mathbf{c}_n$.

150 ## 5. Discussion

In this work an original couple of operators $\mathcal{U}_{ij}[\mathbf{e}]$ and $\mathcal{I}_n[\mathbf{h}]$ are introduced. These operators provides a rigorous definition of voltages and currents in terms of the electric and magnetic field only.

The notion of voltage is analysed, and a general physical interpretation is given basing on the electrical power balance of the computational domain. In the static case, the classical definition of voltage (2) is obtained as particular case of (36) by taking as test magnetic field $\mathbf{h}'_\gamma$ the Biôt-Savart field generated by a unit current which flows along the path γ:

$$U_{ij} = \int_{\Omega_c} \mathrm{curl}\, \mathbf{h}'_\gamma \cdot \mathbf{e} + \int_\Omega \mathbf{h}'_\gamma \cdot \partial_t \mathbf{b} = \int_{\Omega_c} \mathbf{j}'_\gamma \cdot \mathbf{e} = \int_\gamma \mathbf{e} \cdot \mathbf{dl} = V_i - V_j \tag{71}$$

In the case of time varying fields, by taking the same test magnetic field $\mathbf{h}'_\gamma$ the integral $\int_\Omega \mathbf{h}'_\gamma \cdot \partial_t \mathbf{b}$ remains, but the voltage can still be expressed as the difference of the scalar potential on the two (isopotential) ports, provided that the hypothesis of theorem 1 are respected:

$$U_{ij} = \int_{\Omega_c} \mathrm{curl}\, \mathbf{h}'_\gamma \cdot \mathbf{e} + \int_\Omega \mathbf{h}'_\gamma \cdot \partial_t \mathbf{b} = V_i - V_j \tag{72}$$

Notice that (72) doesn't contradict the fact that in the case of time varying fields the
155 circulation of $\mathbf{e}$ depends on the particular path γ.

It has to be remarked that not all couples $(\mathbf{a}_1, v_1)$ satisfy the hypothesis of theorem 1. A notable example of gauge where the hypothesis are not satisfied is the so-called temporal gauge [21] (also called Weyl gauge):

$$\mathbf{e}_1 = -\partial_t \mathbf{a}_1^* \quad ; \quad v_1 \equiv 0 \tag{73}$$

This potential $\mathbf{a}_1^*$ is sometimes called modified vector potential [24]. Luckily this is not a problem, because in order to demonstrate the well posedness of the definition of voltage it is enough that a single couple of potentials which satisfies the hypothesis of the theorem exists: then, the uniqueness of voltages is demonstrated. Moreover, potentials are only intermediate actors in the demonstration, and in practice it is not necessary to compute them: it is enough that at least a couple of potentials exists. Notice also that in the case of unbounded domains Helmholtz decomposition ensures by itself the uniqueness of the scalar potential [18], and thus of voltages.

Finally, notice that the theorem 1 requires that the electric field is normal to the ports which connect the computational domain to electric circuits. This is a limitative hypothesis which is required to define voltages, but not for currents. Perhaps it is possible to weak the hypothesis of the theorem so as to define voltages between arbitrary ports (that is, where the electric field is not necessarily normal) basing on the power balance. Also, it can be conjectured that in the case of physical electric fields it is always possible to find at least a couple of potentials $(\mathbf{a}_1, v_1)$ which satisfies the hypothesis of theorem 1.

The findings on existing Finite Element formulations are resumed in Table 1 (the notation may change with respect of the original works). It is found that coupling formulas in [8–10,14,20] are particular cases of usage of the newly defined operators $\mathcal{U}_{ij}[\mathbf{e}]$ and $\mathcal{I}_n[\mathbf{h}]$.

Table 1. Global quantities expressed by using (36) or (40) in various formulations

Formulation	Reference	Global quantity	Test field
Electrokinetic	[8]	I_n	$\mathbf{e}' = -\operatorname{grad} f_n$
Eddy current $(\mathbf{t} - \mathbf{t}_0 - \phi)$	[14,20]	U_n	$\mathbf{h}' = \mathbf{t}_{0,n}$
Eddy current $(\mathbf{a} - v)$	[9]	I_n	$\mathbf{e}' = -\operatorname{grad} v_{0,n}$
Eddy current $(\mathbf{h} - \phi)$	[10]	U_n	$\mathbf{h}' = \mathbf{c}_n$

It is important to observe that even if the analysis of existing formulations has been taken on with the Finite Element method only, these new operators don't rely on any particular numerical method. Hence, in principle they can be used to devise the implementation of coupling between external electrical circuits and computational domains with any other numerical method.

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