Controllability of the linear elasticity as a first-order system using a stabilized space-time mixed formulation

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Abstract

The aim of this paper is to study the boundary controllability of the linear elasticity system as a first-order system in both space and time. Using the observability inequality known for the usual second-order elasticity system, we deduce an equivalent observability inequality for the associated first-order system. Then, the control of minimal L^2-norm can be found as the solution to a space-time mixed formulation. This first-order framework is particularly interesting from a numerical perspective since it is possible to solve the space-time mixed formulation using only piecewise linear C^0-finite elements. Numerical simulations illustrate the theoretical results.

1 Introduction and notations

This work is devoted to the study of the boundary null controllability for the linear elasticity system by means of a first-order equivalent formulation. The main motivation to consider this first order framework is a computational one. If there is a rich literature concerning the numerical approximation of controls for the wave equation (see, for instance, [14, 23, 1, 10, 12, 11] and references therein), there are only a few works studying the linear elasticity case (see, for instance, [9, 19]).

The present work is based on the reformulation of the boundary null controllability of the elastic system as a space-time mixed formulation. This idea was already exploited for the controllability of the wave equation in [7, 8]. Notice that, in order to have a conformal finite element approximation of the solution of this mixed formulation for the wave equation, C^1 finite elements are needed. To avoid this issue, in [16], a first-order formulation was introduced for the numerical approximation of the boundary control of the wave equation. Another possibility is to employ a non-conformal stabilized approximation (see, for instance, [4, 5]).

In the sequel we adapt the framework proposed in [16] to the linear elasticity system. This will allow us to numerical approach the boundary null control of minimal L^2 norm for the elasticity system using a space-time stabilized mixed formulation and P_1 finite elements.

Let $n \in \mathbb{N}^\star$ and let $\Omega \subset \mathbb{R}^n$ be a non-empty bounded domain with a regular enough boundary $\Gamma := \partial\Omega$. For $T > 0$, we define $Q := \Omega \times (0,T)$ and $\Sigma := \Gamma \times (0,T)$. We also set $\Gamma_0 \subset \Gamma$ and $\Sigma_0 := \Gamma_0 \times (0,T)$. We then consider the following linear elasticity system:

$$\boldsymbol{u}'' - \mu\boldsymbol{\Delta u} - (\lambda+\mu)\nabla \operatorname{div} \boldsymbol{u} = \boldsymbol{0} \qquad \text{in } Q, \tag{1a}$$

$$\boldsymbol{u} = \boldsymbol{h} \qquad \text{on } \Sigma_0, \tag{1b}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Sigma \setminus \Sigma_0, \tag{1c}$$

$$(\boldsymbol{u}, \boldsymbol{u}')(\cdot, 0) = (\boldsymbol{u}^0, \boldsymbol{u}^1) \qquad \text{in } \Omega, \tag{1d}$$

where $\boldsymbol{u} = (u_1, \dots, u_n)^\top$ is the state and $\boldsymbol{h} = (h_1, \dots, h_n)^\top$ is a boundary control acting on Σ_0. Here and henceforth, we denote by $\boldsymbol{u}'$ the derivative of $\boldsymbol{u}$ with respect to the time variable t. Moreover, we use bold characters for the vector-valued functions and the associated spaces. We write for instance $\boldsymbol{L}^2(\Omega) := L^2(\Omega;\mathbb{R}^n)$. Equation (1a), often referred to as the Navier-Cauchy equation, models the evolution of the (small) displacement $\boldsymbol{u}$ of an elastic body occupying the domain Ω. The constants $\mu > 0$ and λ satisfying $\lambda + \mu \geq 0$ are called the Lamé's coefficients. They describe the elastic properties of an isotropic material. For more details related to the model, we refer to the textbook [21].

We first recall some facts about the null controllability of system (1) and the control of minimal L^2-norm. If the boundary Γ is regular, C^2 being sufficient, for any $(\boldsymbol{u}^0, \boldsymbol{u}^1) \in \boldsymbol{L}^2(\Omega) \times \boldsymbol{H}^{-1}(\Omega)$ and $\boldsymbol{h} \in \boldsymbol{L}^2(\Sigma_0)$, there exists (see, for instance, [15]) a unique transposition solution $\boldsymbol{u}$ of (1) with regularity

$$\boldsymbol{u} \in C\big([0,T]; \boldsymbol{L}^2(\Omega)\big) \cap C^1\big([0,T]; \boldsymbol{H}^{-1}(\Omega)\big). \tag{2}$$

The null controllability problem for (1) in time T is the following. For each initial datum $(\boldsymbol{u}^0, \boldsymbol{u}^1) \in \boldsymbol{L}^2(\Omega) \times \boldsymbol{H}^{-1}(\Omega)$, find a control $\boldsymbol{h} \in \boldsymbol{L}^2(\Sigma_0)$ such that the corresponding solution $\boldsymbol{u}$ of (1) satisfies

$$(\boldsymbol{u}, \boldsymbol{u}')(\cdot, T) = (\boldsymbol{0}, \boldsymbol{0}) \qquad \text{in } \Omega. \tag{3}$$

It is well-known (see, for instance, [15] or [17]) that, under some geometric conditions on Ω and Γ_0, the system (1) is null-controllable in any large time $T > T^\star = \dfrac{2}{\sqrt{\mu}} R(\boldsymbol{x}_0)$, where $R(\boldsymbol{x}_0)$ is depending on the geometry of the the domain Ω and the control boundary Γ_0. Moreover, as a consequence of the Hilbert Uniqueness Method (HUM) of J.-L. Lions [15], the controllability of (1) is equivalent to an observability inequality for the associated adjoint system

$$\boldsymbol{\varphi}'' - \mu \boldsymbol{\Delta} \boldsymbol{\varphi} - (\lambda + \mu) \nabla \operatorname{div} \boldsymbol{\varphi} = \boldsymbol{0} \qquad \text{in } Q, \tag{4a}$$

$$\boldsymbol{\varphi} = \boldsymbol{0} \qquad \text{on } \Sigma, \tag{4b}$$

$$(\boldsymbol{\varphi}, \boldsymbol{\varphi}')(\cdot, 0) = (\boldsymbol{\varphi}^0, \boldsymbol{\varphi}^1) \qquad \text{in } \Omega. \tag{4c}$$

More exactly, there exists a constant $C_{\text{obs}} > 0$ such that, for any $(\boldsymbol{\varphi}^0, \boldsymbol{\varphi}^1) \in \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)$, the solution $\boldsymbol{\varphi}$ of (4) satisfies

$$\mu \|\boldsymbol{\varphi}^0\|^2_{\boldsymbol{H}_0^1(\Omega)} + (\lambda + \mu) \|\operatorname{div} \boldsymbol{\varphi}^0\|^2_{L^2(\Omega)} + \|\boldsymbol{\varphi}^1\|^2_{\boldsymbol{L}^2(\Omega)} \leq C_{\text{obs}} \int_{\Sigma_0} \left(\mu \left| \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{\nu}} \right|^2 + (\lambda + \mu) |\operatorname{div} \boldsymbol{\varphi}|^2 \right). \tag{5}$$

As we point out in Remark 1, the boundary term appearing in the right-hand side of inequality (5) is well-defined.

We recall that there is an infinite number of controls in time T. Thus, the problem of finding the control of minimal L^2-norm arises naturally. In fact, the control of minimal L^2-norm is unique and can be found (see [15]) as the solution to a minimization problem involving the conjugate functional

$$\mathcal{J}^\star(\boldsymbol{\varphi}^0, \boldsymbol{\varphi}^1) \coloneqq \frac{1}{2} \int_{\Sigma_0} \left| \mu \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{\nu}} + (\lambda + \mu) \operatorname{div} \boldsymbol{\varphi}\, \boldsymbol{\nu} \right|^2 + \big\langle \boldsymbol{u}^0, \boldsymbol{\varphi}^1 \big\rangle_{\boldsymbol{L}^2(\Omega)} - \big\langle \boldsymbol{u}^1, \boldsymbol{\varphi}^0 \big\rangle_{\boldsymbol{H}^{-1}, \boldsymbol{H}_0^1}, \tag{6}$$

where $\boldsymbol{\varphi}$ is the solution of the adjoint system (4) associated with $(\boldsymbol{\varphi}^0, \boldsymbol{\varphi}^1) \in \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)$. We denote by $\langle \cdot, \cdot \rangle_{\boldsymbol{H}^{-1}, \boldsymbol{H}_0^1}$ the duality pairing between $\boldsymbol{H}^{-1}(\Omega)$ and $\boldsymbol{H}_0^1(\Omega)$. For any $\boldsymbol{u}^1 \in \boldsymbol{H}^{-1}(\Omega)$ and $\boldsymbol{\varphi}^0 \in \boldsymbol{H}_0^1(\Omega)$, the pairing can be expressed

$$\big\langle \boldsymbol{u}^1, \boldsymbol{\varphi}^0 \big\rangle_{\boldsymbol{H}^{-1}, \boldsymbol{H}_0^1} = \mu \int_\Omega \boldsymbol{\nabla} (\mathcal{A}_0^{-1} \boldsymbol{u}^1) : \boldsymbol{\nabla} \boldsymbol{\varphi}^0 + (\lambda + \mu) \int_\Omega \operatorname{div} (\mathcal{A}_0^{-1} \boldsymbol{u}_1) \operatorname{div} \boldsymbol{\varphi}^0, \tag{7}$$

where $\boldsymbol{\nabla} \boldsymbol{\varphi}^0$ is the Jacobian matrix of the vector-valued function $\boldsymbol{\varphi}^0$ and $\mathcal{A}_0^{-1} : \boldsymbol{H}^{-1}(\Omega) \to \boldsymbol{H}_0^1(\Omega)$ is the inverse of the operator $\mathcal{A}_0 \coloneqq -\mu \boldsymbol{\Delta} - (\lambda + \mu) \operatorname{div}$. More precisely, we have $\boldsymbol{g} = \mathcal{A}_0^{-1} \boldsymbol{f}$ if $\boldsymbol{g}$ is solution to

$$\begin{aligned} -\mu \boldsymbol{\Delta} \boldsymbol{g} - (\lambda + \mu) \nabla \operatorname{div} \boldsymbol{g} &= \boldsymbol{f} \qquad && \text{in } \Omega, \\ \boldsymbol{g} &= \boldsymbol{0} \qquad && \text{on } \Gamma. \end{aligned}$$

Besides, it is known that the control $\widehat{\boldsymbol{h}}$ of minimal $\boldsymbol{L}^2(\Sigma_0)$-norm for (1) can be written

$$\widehat{\boldsymbol{h}} = \left(\mu \frac{\partial \widehat{\boldsymbol{\varphi}}}{\partial \boldsymbol{\nu}} + (\lambda + \mu) \operatorname{div} \widehat{\boldsymbol{\varphi}}\, \boldsymbol{\nu} \right) \Big|_{\Sigma_0}, \tag{8}$$

where $\widehat{\boldsymbol{\varphi}}$ is the solution of (4) associated with the minimum point $(\widehat{\boldsymbol{\varphi}}^0, \widehat{\boldsymbol{\varphi}}^1) \in \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)$ of the functional $\mathcal{J}^\star$. We mention that the boundedness by below and the coercivity of $\mathcal{J}^\star$ are consequences of the observability inequality (5).

Remark 1. We recall here the following existence, uniqueness and boundary regularity result for the adjoint system. For any $(\boldsymbol{\varphi}^0, \boldsymbol{\varphi}^1) \in \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{f} \in L^1(0,T;\boldsymbol{L}^2(\Omega))$, there exists a unique weak solution

$$\boldsymbol{\varphi} \in C\big([0,T];\boldsymbol{H}_0^1(\Omega)\big) \cap C^1\big([0,T];\boldsymbol{L}^2(\Omega)\big) \tag{9}$$

to the system

$$\boldsymbol{\varphi}'' - \mu\boldsymbol{\Delta}\boldsymbol{\varphi} - (\lambda+\mu)\nabla \operatorname{div}\boldsymbol{\varphi} = \boldsymbol{f} \qquad \text{in } Q, \tag{10a}$$

$$\boldsymbol{\varphi} = \boldsymbol{0} \qquad \text{on } \Sigma, \tag{10b}$$

$$(\boldsymbol{\varphi}, \boldsymbol{\varphi}')(\cdot,0) = (\boldsymbol{\varphi}^0, \boldsymbol{\varphi}^1) \qquad \text{in } \Omega. \tag{10c}$$

Moreover, $\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{\nu}} \in \boldsymbol{L}^2(\Sigma)$, $\operatorname{div}\boldsymbol{\varphi} \in L^2(\Sigma)$ and there exists a positive constant $C = C(\Omega,T)$ such that

$$\int_\Sigma \left(\mu \left| \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{\nu}} \right|^2 + (\lambda+\mu)|\operatorname{div}\boldsymbol{\varphi}|^2 \right) \leq C\Big(\mu\|\boldsymbol{\varphi}^0\|^2_{\boldsymbol{H}_0^1(\Omega)} + (\lambda+\mu)\|\operatorname{div}\boldsymbol{\varphi}^0\|^2_{L^2(\Omega)} + \|\boldsymbol{\varphi}^1\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{f}\|^2_{L^1(0,T;\boldsymbol{L}^2(\Omega))} \Big).$$

In view of this regularity result, the boundary term appearing in (5) makes sense.

As it was pointed out in [8], we can reformulate the minimization of $\mathcal{J}^\star$ in terms of trajectories of the elasticity system. Thus, there is a bijective relation between the trajectories and the initial data. As a consequence, the function $\widehat{\boldsymbol{\varphi}}$ appearing in (8) can be found as the minimum point of the new conjugate functional

$$\widetilde{\mathcal{J}}^\star(\boldsymbol{\varphi}) := \frac{1}{2}\int_{\Sigma_0} \left| \mu \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{\nu}} + (\lambda+\mu)\operatorname{div}\boldsymbol{\varphi}\,\boldsymbol{\nu} \right|^2 + \big\langle \boldsymbol{u}^0, \boldsymbol{\varphi}'(\cdot,0)\big\rangle_{\boldsymbol{L}^2(\Omega)} - \big\langle \boldsymbol{u}^1, \boldsymbol{\varphi}(\cdot,0)\big\rangle_{\boldsymbol{H}^{-1},\boldsymbol{H}_0^1}, \tag{11}$$

where $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$, with

$$\boldsymbol{\Phi} := \Big\{ \boldsymbol{\varphi} \in \boldsymbol{L}^2(Q); \quad \boldsymbol{\varphi} \text{ satisfies (4a)–(4b)}, \ \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{\nu}} \in \boldsymbol{L}^2(\Sigma) \text{ and } \operatorname{div}\boldsymbol{\varphi} \in L^2(\Sigma) \Big\}. \tag{12}$$

The minimization of the functional $\mathcal{J}^\star$ is also exploited for the numerical approximation of the control of minimal L^2-norm (see, for instance, [12, 11, 23] for the wave equation and [9, 19] for the elasticity system).

Following the ideas employed in [16] for the controllability of the wave equation, we write (4) as a first-order hyperbolic system. More precisely, if $\boldsymbol{\varphi}$ is a solution of (4a), we define the new variables

$$\boldsymbol{w} := \boldsymbol{\varphi}' \qquad \text{and} \qquad \boldsymbol{Q} := \boldsymbol{\nabla}\boldsymbol{\varphi}, \tag{13}$$

where $\boldsymbol{\nabla}\boldsymbol{\varphi}$ is the Jacobian matrix of $\boldsymbol{\varphi}$. Then, equation (4a) implies that $(\boldsymbol{w}, \boldsymbol{Q})$ solves the following first-order system,

$$\boldsymbol{w}' - \mu \operatorname{div}\boldsymbol{Q} - (\lambda+\mu)\nabla \operatorname{tr}\boldsymbol{Q} = \boldsymbol{0} \qquad \text{in } Q, \tag{14a}$$

$$\boldsymbol{Q}' - \boldsymbol{\nabla}\boldsymbol{w} = \boldsymbol{0} \qquad \text{in } Q. \tag{14b}$$

Reciprocally, if we are able to establish the well-posedness of (14) in some suitable functional space, by uniqueness, we may recover solutions of (4a). A slightly different reformulation of the elasticity system as a first order system was proposed in [2]. Note that the new variable $\boldsymbol{Q} = (q_{ij})_{1\leq i,j\leq n}$ is a matrix, that $\operatorname{tr}\boldsymbol{Q} := \sum_{i=1}^n q_{ii}$ is the trace of $\boldsymbol{Q}$ and that $\operatorname{div}\boldsymbol{Q}$ is the vector defined by

$$\operatorname{div}\boldsymbol{Q} := \left(\sum_{j=1}^n \frac{\partial q_{ij}}{\partial x_j} \right)_{1\leq i\leq n}. \tag{15}$$

At this point, it is possible to reformulate the minimization problem associated with $\widetilde{\mathcal{J}}^\star$ in terms of the solution to a first-order system. Before doing so, we introduce the initial-boundary value problem associated with (14). First, we must choose the adequate boundary conditions for $(\boldsymbol{w}, \boldsymbol{Q})$. Since the solution $\boldsymbol{\varphi}$ of (4) satisfies the boundary condition $\boldsymbol{\varphi} = \boldsymbol{0}$ on Σ, we also have $\boldsymbol{\varphi}' = \boldsymbol{0}$ on Σ. Thus, we shall impose $\boldsymbol{w} = \boldsymbol{0}$ on Σ in order to be consistent with (13). In fact, this boundary condition is sufficient for our purposes and it is not necessary to impose a boundary condition on $\boldsymbol{Q}$. Regarding the initial

condition, the natural choice is to set $(\boldsymbol{w},\boldsymbol{Q})(\cdot,0)=(\boldsymbol{\varphi}^1,\boldsymbol{\nabla}\boldsymbol{\varphi}^0)$, which means that the initial datum for $(\boldsymbol{w},\boldsymbol{Q})$ should belong to

$$\mathbf{W}\coloneqq \boldsymbol{L}^2(\Omega)\times\boldsymbol{\nabla}\boldsymbol{H}^1_0,\qquad \text{with}\quad \boldsymbol{\nabla}\boldsymbol{H}^1_0\coloneqq\Big\{\boldsymbol{\nabla}\boldsymbol{\varphi};\quad \boldsymbol{\varphi}\in\boldsymbol{H}^1_0(\Omega)\Big\}. \tag{16}$$

The space $\boldsymbol{\nabla}\boldsymbol{H}^1_0$ is a Hilbert space with respect to the inner product

$$\big\langle \boldsymbol{Q},\overline{\boldsymbol{Q}}\big\rangle_{\boldsymbol{\nabla}\boldsymbol{H}^1_0}\coloneqq \mu\big\langle \boldsymbol{Q},\overline{\boldsymbol{Q}}\big\rangle_{\boldsymbol{L}^2(\Omega)}+(\lambda+\mu)\big\langle \operatorname{tr}\boldsymbol{Q},\operatorname{tr}\overline{\boldsymbol{Q}}\big\rangle_{L^2(\Omega)}. \tag{17}$$

Moreover, $\boldsymbol{\nabla}\boldsymbol{H}^1_0$ is a closed subspace of $L^2(\Omega;M_n(\mathbb{R}))$. The Hilbert space $\mathbf{W}$ is endowed with the inner product

$$\big\langle(\boldsymbol{w},\boldsymbol{Q}),(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\big\rangle_{\mathbf{W}}\coloneqq\big\langle\boldsymbol{w},\overline{\boldsymbol{w}}\big\rangle_{\boldsymbol{L}^2(\Omega)}+\big\langle\boldsymbol{Q},\overline{\boldsymbol{Q}}\big\rangle_{\boldsymbol{\nabla}\boldsymbol{H}^1_0}. \tag{18}$$

Moreover, for $\boldsymbol{Q}\in\boldsymbol{\nabla}\boldsymbol{H}^1_0$, we denote by $\boldsymbol{\sigma}(\boldsymbol{Q})$ the following stress tensor

$$\boldsymbol{\sigma}(\boldsymbol{Q})\coloneqq\mu\boldsymbol{Q}+(\lambda+\mu)\operatorname{tr}\boldsymbol{Q}\,\boldsymbol{I}, \tag{19}$$

where $\boldsymbol{I}$ is the identity matrix. Remark that equation (14a) then writes

$$\boldsymbol{w}'-\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{Q})=\boldsymbol{0}\qquad\text{in } Q. \tag{20}$$

We also define

$$\mathbf{V}\coloneqq\boldsymbol{H}^1_0(\Omega)\times\boldsymbol{\nabla}\boldsymbol{H}^2_0,\qquad\text{with}\quad\boldsymbol{\nabla}\boldsymbol{H}^2_0\coloneqq\Big\{\boldsymbol{\nabla}\boldsymbol{\varphi};\quad\boldsymbol{\varphi}\in\boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}^1_0(\Omega)\Big\}. \tag{21}$$

The space $\mathbf{W}$ is the closure of $\mathbf{V}$ with respect to the norm $\|\cdot\|_{\mathbf{W}}$. Besides, for $\boldsymbol{Q}\in\boldsymbol{\nabla}\boldsymbol{H}^2_0$, note that $\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{Q})\in\boldsymbol{L}^2(\Omega)$. With these notations, the first-order initial-boundary value problem associated with (4) is the following. For any $(\boldsymbol{w}^0,\boldsymbol{Q}^0)\in\mathbf{W}$, find $(\boldsymbol{w},\boldsymbol{Q})$ solution of (14) and satisfying the following boundary and initial value conditions

$$\boldsymbol{w}=\boldsymbol{0}\qquad\text{on }\Sigma, \tag{22a}$$

$$(\boldsymbol{w},\boldsymbol{Q})(\cdot,0)=(\boldsymbol{w}^0,\boldsymbol{Q}^0)\qquad\text{in }\Omega. \tag{22b}$$

In what follows, we denote

$$\mathcal{M}(\boldsymbol{w},\boldsymbol{Q})\coloneqq\Big(\boldsymbol{w}'-\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{Q}),\boldsymbol{Q}'-\boldsymbol{\nabla}\boldsymbol{w}\Big). \tag{23}$$

In order to state a suitable well-posedness result for the first-order system (14)–(22), we consider the closed and densely defined operator $\mathcal{A}:\mathbf{V}\to\mathbf{W}$ defined by

$$\mathcal{A}(\boldsymbol{w},\boldsymbol{Q})\coloneqq\Big(\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{Q}),\boldsymbol{\nabla}\boldsymbol{w}\Big),\quad\forall(\boldsymbol{w},\boldsymbol{Q})\in\mathbf{V}. \tag{24}$$

We show in Appendix A that $\mathcal{A}$ generates a C_0-group of contractions which we denote $(\mathbb{T}_t)_{t\in\mathbb{R}}$. Using this semigroup approach (see, for instance, [18] or [22]), we can introduce the notion of mild solution.

Definition 1. Let $(\boldsymbol{w}^0,\boldsymbol{Q}^0)\in\mathbf{W}$ and $(\boldsymbol{f},\boldsymbol{F})\in L^1(0,T;\mathbf{W})$. We say that $(\boldsymbol{w},\boldsymbol{Q})$ is a mild solution of

$$\boldsymbol{w}'-\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{Q})=\boldsymbol{f}\qquad\text{in }Q, \tag{25a}$$

$$\boldsymbol{Q}'-\boldsymbol{\nabla}\boldsymbol{w}=\boldsymbol{F}\qquad\text{in }Q, \tag{25b}$$

$$\boldsymbol{w}=\boldsymbol{0}\qquad\text{on }\Sigma, \tag{25c}$$

$$(\boldsymbol{w},\boldsymbol{Q})(\cdot,0)=(\boldsymbol{w}^0,\boldsymbol{Q}^0)\qquad\text{in }\Omega, \tag{25d}$$

if

$$(\boldsymbol{w},\boldsymbol{Q})(\cdot,t)=\mathbb{T}_t(\boldsymbol{w}^0,\boldsymbol{Q}^0)+\int_0^t\mathbb{T}_{t-s}(\boldsymbol{f},\boldsymbol{F})(\cdot,s)\,\mathrm{d}s,\quad\forall t\in(0,T). \tag{26}$$

As a consequence of well-known facts concerning the theory of semigroups and the definition of mild solutions, we have the following well-posedness statement for (25).

Lemma 1. *Let* $(\boldsymbol{w}^0,\boldsymbol{Q}^0)\in\mathbf{W}$ *and* $(\boldsymbol{f},\boldsymbol{F})\in L^1(0,T;\mathbf{W})$.

i) The first-order system (25) *has a unique mild solution* $(\boldsymbol{w},\boldsymbol{Q}) \in C([0,T];\mathbf{W})$ *and there exists a positive constant* $C = C(\Omega,T)$ *such that*

$$\sup_{t\in(0,T)} \|(\boldsymbol{w},\boldsymbol{Q})(\cdot,t)\|^2_{\mathbf{W}} \leq C\Big(\|(\boldsymbol{w}^0,\boldsymbol{Q}^0)\|^2_{\mathbf{W}} + \|(\boldsymbol{f},\boldsymbol{F})\|^2_{L^1(0,T;\mathbf{W})}\Big). \tag{27}$$

ii) If $(\boldsymbol{w}^0,\boldsymbol{Q}^0) \in \mathbf{V}$ *and* $(\boldsymbol{f},\boldsymbol{F}) \in C^1([0,T];\mathbf{W})$*, then* $(\boldsymbol{w},\boldsymbol{Q}) \in C([0,T];\mathbf{V}) \cap C^1([0,T];\mathbf{W})$ *and the equation* $\mathcal{M}(\boldsymbol{w},\boldsymbol{Q}) = (\boldsymbol{f},\boldsymbol{F})$ *holds in* $\mathbf{W}$*.*

Similarly to the boundary regularity result stated in Remark 1, we have a boundary regularity result for the solutions of (25). Indeed, we prove in Lemma 3 that if $(\boldsymbol{w},\boldsymbol{Q}) \in C([0,T];\mathbf{W})$ is a solution of (25), then $\boldsymbol{Q}\boldsymbol{\nu} \in \boldsymbol{L}^2(\Sigma)$, $\operatorname{tr}\boldsymbol{Q} \in L^2(\Sigma)$ and the inequality

$$\int_\Sigma \Big(\mu|\boldsymbol{Q}\boldsymbol{\nu}|^2 + (\lambda+\mu)|\operatorname{tr}\boldsymbol{Q}|^2\Big) \leq C\Big(\|(\boldsymbol{w}^0,\boldsymbol{Q}^0)\|^2_{\mathbf{W}} + \|(\boldsymbol{f},\boldsymbol{F})\|^2_{L^1(0,T;\mathbf{W})}\Big)$$

holds true for some positive constant $C = C(\Omega,T)$.

We are now in position to restate the minimization problem associated with $\widetilde{\mathcal{J}}^\star$ (cf (11)) in terms of the solution to the first-order system (25). A natural way to proceed is to consider a minimization problem in which the quantities $\boldsymbol{\varphi}'$ and $\boldsymbol{\nabla}\boldsymbol{\varphi}$ are replaced by $\boldsymbol{w}$ and $\boldsymbol{Q}$ respectively. We then define the following spaces of trajectories,

$$\mathbf{Z} \coloneqq \Big\{(\boldsymbol{w},\boldsymbol{Q}) \in C\big([0,T];\mathbf{W}\big) \text{ mild solution of (25) associated with } (\boldsymbol{w}^0,\boldsymbol{Q}^0) \in \mathbf{W},\ (\boldsymbol{f},\boldsymbol{F}) \in L^2(0,T;\mathbf{W})\Big\}, \tag{28}$$

$$\mathbf{Z}_0 \coloneqq \Big\{(\boldsymbol{w},\boldsymbol{Q}) \in C\big([0,T];\mathbf{W}\big) \text{ mild solution of (25) associated with } (\boldsymbol{w}^0,\boldsymbol{Q}^0) \in \mathbf{W},\ (\boldsymbol{f},\boldsymbol{F}) = (\mathbf{0},\mathbf{0})\Big\}. \tag{29}$$

With these definitions, the adequate minimization problem in terms of the first-order system is

$$\min_{(\boldsymbol{w},\boldsymbol{Q})\in\mathbf{Z}_0} \widehat{\mathcal{J}}^\star(\boldsymbol{w},\boldsymbol{Q}), \tag{30}$$

where

$$\widehat{\mathcal{J}}^\star(\boldsymbol{w},\boldsymbol{Q}) \coloneqq \frac{1}{2}\int_{\Sigma_0} \Big|\boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu}\Big|^2 + \big\langle \boldsymbol{u}^0, \boldsymbol{w}(\cdot,0)\big\rangle_{\boldsymbol{L}^2(\Omega)} - \big\langle \boldsymbol{\nabla}(\mathcal{A}_0^{-1}\boldsymbol{u}^1), \boldsymbol{Q}(\cdot,0)\big\rangle_{\boldsymbol{\nabla}\boldsymbol{H}^1_0}. \tag{31}$$

As for the elasticity system, the boundedness by below and the coercivity of $\widehat{\mathcal{J}}^\star$ can be obtained as a consequence of an observability inequality which takes the form

$$\|(\boldsymbol{w},\boldsymbol{Q})(\cdot,0)\|^2_{\mathbf{W}} \leq \mu^{-1} C_{\text{obs}} \int_{\Sigma_0} |\boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu}|^2,$$

for any $(\boldsymbol{w},\boldsymbol{Q}) \in \mathbf{Z}_0$ and under suitable conditions on Ω, Γ_0, (μ,λ) and T.

The remaining part of this work is structured as follows. In Section 2, we prove an a priori estimate and a generalized observability inequality for system (25). The third section rewrites the minimization problem (30) as a mixed formulation. The well-posedness of this mixed formulation is a direct consequence of the estimates proved in Section 2 and provides an easily computable expression for the control of minimal L^2-norm associated with (1). Sections 4 and 5 propose a stabilized version of the mixed formulation introduced in Section 3 and an equivalent reformulation of this stabilized mixed formulation as a dual extremal problem involving only the state variable. Such a reformulation is particularly interesting from the numerical perspective and is exploited in Section 6 to numerically illustrate the controllability for the wave equation and the elasticity system written as first-order systems in both space and time.

2 Boundary regularity and observability

We first derive an identity which is useful to obtain a priori estimates for (25).

Lemma 2. *With Ω of class C^2, let $\boldsymbol{\beta} \in C^1(\overline{\Omega};\mathbb{R}^n)$ be a vector field. For any solution $(\boldsymbol{w},\boldsymbol{Q})$ of (25) associated with $(\boldsymbol{w}^0,\boldsymbol{Q}^0) \in \mathbf{V}$ and $(\boldsymbol{f},\boldsymbol{F}) \in C^1([0,T];\mathbf{W})$, the following identity holds,*

$$\begin{aligned}\frac{1}{2}\int_\Sigma \Big(\mu|\boldsymbol{Q}\boldsymbol{\nu}|^2 + (\lambda+\mu)|\operatorname{tr}\boldsymbol{Q}|^2\Big)(\boldsymbol{\beta}\cdot\boldsymbol{\nu}) &= \Big[\int_\Omega \boldsymbol{w}\cdot(\boldsymbol{Q}\boldsymbol{\beta})\Big]_0^T \\ &+ \frac{1}{2}\iint_Q \Big(|\boldsymbol{w}|^2 - \mu|\boldsymbol{Q}|^2 - (\lambda+\mu)|\operatorname{tr}\boldsymbol{Q}|^2\Big)\operatorname{div}\boldsymbol{\beta} \\ &+ \mu\iint_Q \boldsymbol{Q}:(\boldsymbol{Q}\boldsymbol{\nabla}\boldsymbol{\beta}) + (\lambda+\mu)\iint_Q \operatorname{tr}\boldsymbol{Q}\operatorname{tr}(\boldsymbol{Q}\boldsymbol{\nabla}\boldsymbol{\beta}) \\ &- \iint_Q \boldsymbol{f}\cdot(\boldsymbol{Q}\boldsymbol{\beta}) - \iint_Q \boldsymbol{w}\cdot(\boldsymbol{F}\boldsymbol{\beta}).\end{aligned} \tag{32}$$

Proof. Let $(\boldsymbol{w}^0,\boldsymbol{Q}^0) \in \mathbf{V}$ and $(\boldsymbol{f},\boldsymbol{F}) \in C^1([0,T];\mathbf{W})$. There exists two functions $\boldsymbol{\varphi}^0 \in \boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}^1_0(\Omega)$ and $\boldsymbol{\psi} \in C^1([0,T];\boldsymbol{H}^1_0(\Omega))$ such that $\boldsymbol{Q}^0 = \boldsymbol{\nabla}\boldsymbol{\varphi}^0$ and $\boldsymbol{F} = \boldsymbol{\nabla}\boldsymbol{\psi}$. We start by some properties of $(\boldsymbol{w},\boldsymbol{Q})$ on the boundary Σ. The matrix $\boldsymbol{Q}$ satisfies

$$\boldsymbol{Q}' - \boldsymbol{\nabla}\boldsymbol{w} = \boldsymbol{F} \quad \text{in } Q \qquad \text{and} \qquad \boldsymbol{Q}(\cdot,0) = \boldsymbol{Q}^0 \quad \text{in } \Omega,$$

therefore

$$\boldsymbol{Q}(\boldsymbol{x},t) = \boldsymbol{\nabla}\boldsymbol{\varphi}^0(\boldsymbol{x}) + \int_0^t \boldsymbol{\nabla}\boldsymbol{w}(\boldsymbol{x},\tau)\,\mathrm{d}\tau + \int_0^t \boldsymbol{\nabla}\boldsymbol{\psi}(\boldsymbol{x},\tau)\,\mathrm{d}\tau, \quad \forall(\boldsymbol{x},t)\in Q. \tag{33}$$

Let $1 \le i \le n$. Since $\boldsymbol{\varphi}^0 \in \boldsymbol{H}^1_0(\Omega)$, we have $\nabla\varphi_i^0 = \frac{\partial\varphi_i^0}{\partial\boldsymbol{\nu}}\boldsymbol{\nu}$ on Γ. Similarly, for any $\tau\in[0,T]$, since $\boldsymbol{w}(\cdot,\tau) = \boldsymbol{0}$ on Γ and $\boldsymbol{\psi}(\cdot,\tau)\in\boldsymbol{H}^1_0(\Omega)$, we also have $\nabla w_i(\cdot,\tau) = \frac{\partial w_i}{\partial\boldsymbol{\nu}}(\cdot,\tau)\boldsymbol{\nu}$ and $\nabla\psi_i(\cdot,\tau) = \frac{\partial\psi_i}{\partial\boldsymbol{\nu}}(\cdot,\tau)\boldsymbol{\nu}$ on Γ. Hence, if $\boldsymbol{q}_i$ denotes the i-th line of $\boldsymbol{Q}$, it follows from (33) that $\boldsymbol{q}_i = (\boldsymbol{q}_i\cdot\boldsymbol{\nu})\boldsymbol{\nu}$ on Σ. Thus, we get

$$|\boldsymbol{Q}|^2 = |\boldsymbol{Q}\boldsymbol{\nu}|^2, \quad (\boldsymbol{Q}\boldsymbol{\nu})\cdot\boldsymbol{\nu} = \operatorname{tr}\boldsymbol{Q}, \quad \boldsymbol{Q}\boldsymbol{\beta} = (\boldsymbol{\beta}\cdot\boldsymbol{\nu})\,\boldsymbol{Q}\boldsymbol{\nu} \quad \text{and} \quad (\boldsymbol{Q}\boldsymbol{\beta})\cdot\boldsymbol{\nu} = \operatorname{tr}\boldsymbol{Q}\,(\boldsymbol{\beta}\cdot\boldsymbol{\nu}) \qquad \text{on } \Sigma. \tag{34}$$

Multiplying equation (25a) by $\boldsymbol{Q}\boldsymbol{\beta}$ and integrating over Q, we have

$$\iint_Q \boldsymbol{w}'\cdot(\boldsymbol{Q}\boldsymbol{\beta}) - \mu\iint_Q \operatorname{div}\boldsymbol{Q}\cdot(\boldsymbol{Q}\boldsymbol{\beta}) - (\lambda+\mu)\iint_Q \nabla\operatorname{tr}\boldsymbol{Q}\cdot(\boldsymbol{Q}\boldsymbol{\beta}) = \iint_Q \boldsymbol{f}\cdot(\boldsymbol{Q}\boldsymbol{\beta}). \tag{35}$$

We treat each term one after the other. We recall that there exists $\boldsymbol{\varphi}\in C^1([0,T];\boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}^1_0(\Omega))$ such that $\boldsymbol{Q} = \boldsymbol{\nabla}\boldsymbol{\varphi}$.

- For the first term, by integration by parts, we get

$$\begin{aligned}\iint_Q \boldsymbol{w}'\cdot(\boldsymbol{Q}\boldsymbol{\beta}) &= \Big[\int_\Omega \boldsymbol{w}\cdot(\boldsymbol{Q}\boldsymbol{\beta})\Big]_0^T - \iint_Q \boldsymbol{w}\cdot(\boldsymbol{Q}'\boldsymbol{\beta}) \\ &= \Big[\int_\Omega \boldsymbol{w}\cdot(\boldsymbol{Q}\boldsymbol{\beta})\Big]_0^T - \iint_Q \boldsymbol{w}\cdot(\boldsymbol{\nabla}\boldsymbol{w}\,\boldsymbol{\beta}) - \iint_Q \boldsymbol{w}\cdot(\boldsymbol{F}\boldsymbol{\beta}).\end{aligned}$$

Moreover, since $\boldsymbol{w} = \boldsymbol{0}$ on Σ, we have

$$\iint_Q \boldsymbol{w}\cdot(\boldsymbol{\nabla}\boldsymbol{w}\,\boldsymbol{\beta}) = \frac{1}{2}\iint_Q \nabla|\boldsymbol{w}|^2\cdot\boldsymbol{\beta} = -\frac{1}{2}\iint_Q |\boldsymbol{w}|^2\operatorname{div}\boldsymbol{\beta}.$$

Thus, we get

$$\iint_Q \boldsymbol{w}'\cdot(\boldsymbol{Q}\boldsymbol{\beta}) = \Big[\int_\Omega \boldsymbol{w}\cdot(\boldsymbol{Q}\boldsymbol{\beta})\Big]_0^T + \frac{1}{2}\iint_Q |\boldsymbol{w}|^2\operatorname{div}\boldsymbol{\beta} - \iint_Q \boldsymbol{w}\cdot(\boldsymbol{F}\boldsymbol{\beta}). \tag{36}$$

- For the second term, by integration by parts and since $\boldsymbol{Q} = \boldsymbol{\nabla}\boldsymbol{\varphi}$, we get

$$\begin{aligned}\iint_Q \operatorname{div}\boldsymbol{Q}\cdot(\boldsymbol{Q}\boldsymbol{\beta}) &= \int_\Sigma (\boldsymbol{Q}\boldsymbol{\nu})\cdot(\boldsymbol{Q}\boldsymbol{\beta}) - \iint_Q \boldsymbol{Q}:\boldsymbol{\nabla}(\boldsymbol{Q}\boldsymbol{\beta}) \\ &= \int_\Sigma |\boldsymbol{Q}\boldsymbol{\nu}|^2(\boldsymbol{\beta}\cdot\boldsymbol{\nu}) - \iint_Q \boldsymbol{Q}:(\boldsymbol{\nabla}\boldsymbol{Q}\,\boldsymbol{\beta}) - \iint_Q \boldsymbol{Q}:(\boldsymbol{Q}\boldsymbol{\nabla}\boldsymbol{\beta}).\end{aligned}$$

Moreover, we have

$$\iint_Q \boldsymbol{Q} : (\boldsymbol{\nabla Q}\,\boldsymbol{\beta}) = \frac{1}{2}\iint_Q \nabla|\boldsymbol{Q}|^2\cdot\boldsymbol{\beta} = \frac{1}{2}\int_\Sigma |\boldsymbol{Q}|^2(\boldsymbol{\beta}\cdot\boldsymbol{\nu}) - \frac{1}{2}\iint_Q |\boldsymbol{Q}|^2\operatorname{div}\boldsymbol{\beta}$$
$$= \frac{1}{2}\int_\Sigma |\boldsymbol{Q\nu}|^2(\boldsymbol{\beta}\cdot\boldsymbol{\nu}) - \frac{1}{2}\iint_Q |\boldsymbol{Q}|^2\operatorname{div}\boldsymbol{\beta}.$$

Thus, we get

$$\iint_Q \operatorname{div}\boldsymbol{Q}\cdot(\boldsymbol{Q\beta}) = \frac{1}{2}\int_\Sigma |\boldsymbol{Q\nu}|^2(\boldsymbol{\beta}\cdot\boldsymbol{\nu}) + \frac{1}{2}\iint_Q |\boldsymbol{Q}|^2\operatorname{div}\boldsymbol{\beta} - \iint_Q \boldsymbol{Q} : (\boldsymbol{Q\nabla\beta}). \tag{37}$$

- For the third term, by integration by parts and since $\boldsymbol{Q} = \boldsymbol{\nabla\varphi}$, we get

$$\iint_Q \nabla\operatorname{tr}\boldsymbol{Q}\cdot(\boldsymbol{Q\beta}) = \int_\Sigma \operatorname{tr}\boldsymbol{Q}\,(\boldsymbol{Q\beta})\cdot\boldsymbol{\nu} - \iint_Q \operatorname{tr}\boldsymbol{Q}\operatorname{div}(\boldsymbol{Q\beta})$$
$$= \int_\Sigma |\operatorname{tr}\boldsymbol{Q}|^2(\boldsymbol{\beta}\cdot\boldsymbol{\nu}) - \iint_Q \operatorname{tr}\boldsymbol{Q}\,\nabla\operatorname{tr}\boldsymbol{Q}\cdot\boldsymbol{\beta} - \iint_Q \operatorname{tr}\boldsymbol{Q}\operatorname{tr}(\boldsymbol{Q\nabla\beta}).$$

Moreover, we have

$$\iint_Q \operatorname{tr}\boldsymbol{Q}\,\nabla\operatorname{tr}\boldsymbol{Q}\cdot\boldsymbol{\beta} = \frac{1}{2}\iint_Q \nabla|\operatorname{tr}\boldsymbol{Q}|^2\cdot\boldsymbol{\beta} = \frac{1}{2}\int_\Sigma |\operatorname{tr}\boldsymbol{Q}|^2(\boldsymbol{\beta}\cdot\boldsymbol{\nu}) - \frac{1}{2}\iint_Q |\operatorname{tr}\boldsymbol{Q}|^2\operatorname{div}\boldsymbol{\beta}.$$

Thus, we get

$$\iint_Q \nabla\operatorname{tr}\boldsymbol{Q}\cdot(\boldsymbol{Q\beta}) = \frac{1}{2}\int_\Sigma |\operatorname{tr}\boldsymbol{Q}|^2(\boldsymbol{\beta}\cdot\boldsymbol{\nu}) + \frac{1}{2}\iint_Q |\operatorname{tr}\boldsymbol{Q}|^2\operatorname{div}\boldsymbol{\beta} - \iint_Q \operatorname{tr}\boldsymbol{Q}\operatorname{tr}(\boldsymbol{Q\nabla\beta}). \tag{38}$$

Replacing (36) to (38) in (35), we obtain identity (32). □

We recall that, if $\boldsymbol{Q}\in\boldsymbol{\nabla H}^2_0$, then $\boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu} = \mu\boldsymbol{Q\nu} + (\lambda+\mu)\operatorname{tr}\boldsymbol{Q}\,\boldsymbol{\nu}$ is well-defined as a distribution in $\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ and Green formula is satisfied,

$$\int_\Omega \operatorname{div}\boldsymbol{\sigma}(\boldsymbol{Q})\cdot\overline{\boldsymbol{w}} + \int_\Omega \boldsymbol{\sigma}(\boldsymbol{Q}) : \boldsymbol{\nabla}\overline{\boldsymbol{w}} = \big\langle\boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu},\overline{\boldsymbol{w}}\big\rangle_{\boldsymbol{H}^{-\frac{1}{2}},\boldsymbol{H}^{\frac{1}{2}}},\quad \forall\overline{\boldsymbol{w}}\in\boldsymbol{H}^1(\Omega), \tag{39}$$

where $\langle\cdot,\cdot\rangle_{\boldsymbol{H}^{-\frac{1}{2}},\boldsymbol{H}^{\frac{1}{2}}}$ denotes the duality pairing between $\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ and $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$. The next lemma improves the regularity of $\boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu}$ when $(\boldsymbol{w},\boldsymbol{Q})$ is a solution of (25) belonging to $C([0,T];\mathbf{V})\cap C^1([0,T];\mathbf{W})$ and allows to define $\boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu}$ when $(\boldsymbol{w},\boldsymbol{Q})$ is a mild solution of (25).

Lemma 3. *With Ω of class C^2, let $(\boldsymbol{w}^0,\boldsymbol{Q}^0)\in\mathbf{V}$, $(\boldsymbol{f},\boldsymbol{F})\in C^1([0,T];\mathbf{W})$ and let $(\boldsymbol{w},\boldsymbol{Q})\in C([0,T];\mathbf{V})\cap C^1([0,T];\mathbf{W})$ be the corresponding solution of (25). Then, $\boldsymbol{Q\nu}\in\boldsymbol{L}^2(\Sigma)$, $\operatorname{tr}\boldsymbol{Q}\in L^2(\Sigma)$ and there exists a positive constant $C = C(\Omega,T)$ such that*

$$\int_\Sigma \Big(\mu|\boldsymbol{Q\nu}|^2 + (\lambda+\mu)|\operatorname{tr}\boldsymbol{Q}|^2\Big) \le C\Big(\|(\boldsymbol{w}^0,\boldsymbol{Q}^0)\|^2_{\mathbf{W}} + \|(\boldsymbol{f},\boldsymbol{F})\|^2_{L^1(0,T;\mathbf{W})}\Big). \tag{40}$$

Moreover, we have $\boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu}\in\boldsymbol{L}^2(\Sigma)$ and

$$\int_\Sigma |\boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu}|^2 \le (\lambda+3\mu)\,C\Big(\|(\boldsymbol{w}^0,\boldsymbol{Q}^0)\|^2_{\mathbf{W}} + \|(\boldsymbol{f},\boldsymbol{F})\|^2_{L^1(0,T;\mathbf{W})}\Big). \tag{41}$$

Consequently, there is a unique bounded map

$$\Lambda : \mathbf{W}\times L^1(0,T;\mathbf{W}) \to \boldsymbol{L}^2(\Sigma)$$

such that $\Lambda((\boldsymbol{w}^0,\boldsymbol{Q}^0),(\boldsymbol{f},\boldsymbol{F})) = \boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu}$ when $(\boldsymbol{w},\boldsymbol{Q})\in C([0,T];\mathbf{V})\cap C^1([0,T];\mathbf{W})$. Accordingly, we denote $\boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu} = \Lambda((\boldsymbol{w}^0,\boldsymbol{Q}^0),(\boldsymbol{f},\boldsymbol{F}))$ when $(\boldsymbol{w},\boldsymbol{Q})$ is the mild solution of (25) associated with $(\boldsymbol{w}^0,\boldsymbol{Q}^0)\in\mathbf{W}$ and $(\boldsymbol{f},\boldsymbol{F})\in L^1(0,T;\mathbf{W})$. In that case, estimate (41) still holds.

Proof. Due to the regularity of the boundary Γ, there exists a vector field $\boldsymbol{\beta} \in C^1(\overline{\Omega};\mathbb{R}^n)$ such that $\boldsymbol{\beta} = \boldsymbol{\nu}$ on Γ. Thus, with this choice of $\boldsymbol{\beta}$, identity (32) implies

$$\int_\Sigma \Big(\mu|\boldsymbol{Q\nu}|^2 + (\lambda+\mu)|\operatorname{tr}\boldsymbol{Q}|^2\Big) \le C\left((1+T)\sup_{t\in(0,T)}\|(\boldsymbol{w},\boldsymbol{Q})(\cdot,t)\|^2_{\mathbf{W}} + \|(\boldsymbol{f},\boldsymbol{F})\|^2_{L^1(0,T;\mathbf{W})}\right).$$

We then apply estimate (27) to obtain (40). Besides, from (34), we know that $(\boldsymbol{Q\nu})\cdot\boldsymbol{\nu} = \operatorname{tr}\boldsymbol{Q}$ on Σ. It follows that

$$\begin{aligned}\int_\Sigma |\boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu}|^2 = \int_\Sigma \Big|\mu\boldsymbol{Q\nu} + (\lambda+\mu)\operatorname{tr}\boldsymbol{Q}\,\boldsymbol{\nu}\Big|^2 &= \int_\Sigma \Big(\mu^2|\boldsymbol{Q\nu}|^2 + (\lambda+\mu)^2|\operatorname{tr}\boldsymbol{Q}|^2 + 2\mu(\lambda+\mu)\operatorname{tr}\boldsymbol{Q}\,(\boldsymbol{Q\nu})\cdot\boldsymbol{\nu}\Big)\\ &= \int_\Sigma \Big(\mu^2|\boldsymbol{Q\nu}|^2 + (\lambda+3\mu)(\lambda+\mu)|\operatorname{tr}\boldsymbol{Q}|^2\Big)\\ &\le (\lambda+3\mu)\int_\Sigma \Big(\mu|\boldsymbol{Q\nu}|^2 + (\lambda+\mu)|\operatorname{tr}\boldsymbol{Q}|^2\Big).\end{aligned}$$

□

Remark 2. The above lemma can also be proved using the following Rellich-Nečas identities. If $\boldsymbol{q},\boldsymbol{\beta} \in C^1(\overline{\Omega};\mathbb{R}^n)$ are two vector fields, we can check that

$$\begin{aligned}2\operatorname{div}((\boldsymbol{q}\cdot\boldsymbol{\beta})\boldsymbol{q}) - \operatorname{div}(|\boldsymbol{q}|^2\boldsymbol{\beta}) = 2(\boldsymbol{q}\cdot\boldsymbol{\beta})\operatorname{div}\boldsymbol{q} - |\boldsymbol{q}|^2\operatorname{div}\boldsymbol{\beta} + 2(\boldsymbol{\nabla\beta}\,\boldsymbol{q})\cdot\boldsymbol{q}\\ + 2\Big((\boldsymbol{\nabla q}\,\boldsymbol{q})\cdot\boldsymbol{\beta} - (\boldsymbol{\nabla q}\,\boldsymbol{\beta})\cdot\boldsymbol{q}\Big).\end{aligned}$$

Consequently, for $\boldsymbol{Q} \in C^1(\overline{\Omega};M_n(\mathbb{R}))$, we can adapt the previous identity to get

$$\begin{aligned}2\operatorname{div}(\boldsymbol{Q}^\top(\boldsymbol{Q\beta})) - \operatorname{div}(|\boldsymbol{Q}|^2\boldsymbol{\beta}) = 2(\boldsymbol{Q\beta})\cdot\operatorname{div}\boldsymbol{Q} - |\boldsymbol{Q}|^2\operatorname{div}\boldsymbol{\beta} + 2(\boldsymbol{Q\nabla\beta}) : \boldsymbol{Q}\\ + 2\Big(\boldsymbol{\beta}^\top\boldsymbol{\nabla Q}^\top - \boldsymbol{\nabla Q}\,\boldsymbol{\beta}\Big) : \boldsymbol{Q}.\end{aligned}$$

Similarly, we show that

$$\begin{aligned}2\operatorname{div}((\boldsymbol{Q\beta})\operatorname{tr}\boldsymbol{Q}) - \operatorname{div}(|\operatorname{tr}\boldsymbol{Q}|^2\boldsymbol{\beta}) = 2(\boldsymbol{Q\beta})\cdot\nabla\operatorname{tr}\boldsymbol{Q} - |\operatorname{tr}\boldsymbol{Q}|^2\operatorname{div}\boldsymbol{\beta} + 2\operatorname{tr}(\boldsymbol{Q\nabla\beta})\operatorname{tr}\boldsymbol{Q}\\ + 2\Big(\operatorname{tr}(\boldsymbol{\beta}^\top\boldsymbol{\nabla Q}^\top) - \operatorname{tr}(\boldsymbol{\nabla Q}\,\boldsymbol{\beta})\Big)\operatorname{tr}\boldsymbol{Q}.\end{aligned}$$

Finally, note that if $\boldsymbol{Q} \in \boldsymbol{\nabla H}^2_0$, we have $\boldsymbol{\beta}^\top\boldsymbol{\nabla Q}^\top = \boldsymbol{\nabla Q}\,\boldsymbol{\beta}$.

Lemma 4. *We recall that $\mathbf{Z}_0$ is the set of mild solutions $(\boldsymbol{w},\boldsymbol{Q})$ of (22), with $(\boldsymbol{w}^0,\boldsymbol{Q}^0) \in \mathbf{W}$. For any $T > T^\star$, there exists a constant $C_{\mathrm{obs}} > 0$ such that*

$$\|(\boldsymbol{w}^0,\boldsymbol{Q}^0)\|^2_{\mathbf{W}} \le C_{\mathrm{obs}}\int_{\Sigma_0}\Big(\mu|\boldsymbol{Q\nu}|^2 + (\lambda+\mu)|\operatorname{tr}\boldsymbol{Q}|^2\Big), \quad \forall(\boldsymbol{w},\boldsymbol{Q}) \in \mathbf{Z}_0. \tag{42}$$

The critical time $T^\star > 0$ depends on Ω, Γ_0 and (λ,μ). Besides, we have

$$\|(\boldsymbol{w}^0,\boldsymbol{Q}^0)\|^2_{\mathbf{W}} \le \mu^{-1}C_{\mathrm{obs}}\int_{\Sigma_0}|\boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu}|^2, \quad \forall(\boldsymbol{w},\boldsymbol{Q}) \in \mathbf{Z}_0. \tag{43}$$

Proof. To obtain (42), we use the observability inequality (5) associated with the adjoint system (4). Let $(\boldsymbol{w}^0,\boldsymbol{Q}^0) \in \mathbf{W}$. Then, there exists $\boldsymbol{\varphi}^0 \in \boldsymbol{H}^1_0(\Omega)$ such that $\boldsymbol{Q}^0 = \boldsymbol{\nabla\varphi}^0$. We set $\boldsymbol{\varphi}^1 = \boldsymbol{w}^0$ and we define $(\boldsymbol{w},\boldsymbol{Q}) = (\boldsymbol{\varphi}',\boldsymbol{\nabla\varphi})$, where $\boldsymbol{\varphi}$ is the solution of (4) associated with $(\boldsymbol{\varphi}^0,\boldsymbol{\varphi}^1)$. If we apply the observability inequality (5) to $\boldsymbol{\varphi}$, we get that for $T > T^\star$, the following estimate holds,

$$\|(\boldsymbol{w}^0,\boldsymbol{Q}^0)\|^2_{\mathbf{W}} \le C_{\mathrm{obs}}\int_{\Sigma_0}\Big(\mu|\boldsymbol{Q\nu}|^2 + (\lambda+\mu)|\operatorname{tr}\boldsymbol{Q}|^2\Big).$$

Finally, it is easy to check that $(\boldsymbol{w},\boldsymbol{Q})$ is the mild solution of (22) associated with $(\boldsymbol{w}^0,\boldsymbol{Q}^0)$. Besides, it follows from (34) that

$$\int_{\Sigma_0}|\boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu}|^2 = \int_{\Sigma_0}\Big(\mu^2|\boldsymbol{Q\nu}|^2 + (\lambda+3\mu)(\lambda+\mu)|\operatorname{tr}\boldsymbol{Q}|^2\Big) \ge \mu\int_{\Sigma_0}\Big(\mu|\boldsymbol{Q\nu}|^2 + (\lambda+\mu)|\operatorname{tr}\boldsymbol{Q}|^2\Big).$$

□

Proposition 1. *We recall that* $\mathbf{Z}$ *is the set of mild solutions* $(\boldsymbol{w},\boldsymbol{Q})$ *of* (25)*, with* $(\boldsymbol{w}^0,\boldsymbol{Q}^0)\in\mathbf{W}$ *and* $(\boldsymbol{f},\boldsymbol{F})\in L^2(0,T;\mathbf{W})$*. For any* $T>T^\star$*, there exists a positive constant* $C=C(\Omega,T)$ *such that*

$$\|(\boldsymbol{w}^0,\boldsymbol{Q}^0)\|^2_{\mathbf{W}}\leq C\Big(\|\boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu}\|^2_{\boldsymbol{L}^2(\Sigma_0)}+\|(\boldsymbol{f},\boldsymbol{F})\|^2_{L^1(0,T;\mathbf{W})}\Big),\quad\forall(\boldsymbol{w},\boldsymbol{Q})\in\mathbf{Z}. \tag{44}$$

Proof. We can split a solution $(\boldsymbol{w},\boldsymbol{Q})\in\mathbf{Z}$ as $(\boldsymbol{w},\boldsymbol{Q})=(\boldsymbol{w}_1,\boldsymbol{Q}_1)+(\boldsymbol{w}_2,\boldsymbol{Q}_2)$, where $(\boldsymbol{w}_1,\boldsymbol{Q}_1)$ solves (25) with $(\boldsymbol{f},\boldsymbol{F})=(\boldsymbol{0},\boldsymbol{0})$ and $(\boldsymbol{w}_2,\boldsymbol{Q}_2)$ solves (25) with $(\boldsymbol{w}^0,\boldsymbol{Q}^0)=(\boldsymbol{0},\boldsymbol{0})$. By Lemma 4 and Lemma 3, we have respectively

$$\|(\boldsymbol{w}^0,\boldsymbol{Q}^0)\|^2_{\mathbf{W}}\leq\mu^{-1}C_{\mathrm{obs}}\|\boldsymbol{\sigma}(\boldsymbol{Q}_1)\boldsymbol{\nu}\|^2_{\boldsymbol{L}^2(\Sigma_0)}$$

and

$$\|\boldsymbol{\sigma}(\boldsymbol{Q}_2)\boldsymbol{\nu}\|^2_{\boldsymbol{L}^2(\Sigma_0)}\leq(\lambda+3\mu)\,C\|(\boldsymbol{f},\boldsymbol{F})\|^2_{L^1(0,T;\mathbf{W})}.$$

We then get the result using $\|\boldsymbol{\sigma}(\boldsymbol{Q}_1)\boldsymbol{\nu}\|^2_{\boldsymbol{L}^2(\Sigma_0)}\leq 2\|\boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu}\|^2_{\boldsymbol{L}^2(\Sigma_0)}+2\|\boldsymbol{\sigma}(\boldsymbol{Q}_2)\boldsymbol{\nu}\|^2_{\boldsymbol{L}^2(\Sigma_0)}$. □

Remark 3.

i) Since $(\boldsymbol{w},\boldsymbol{Q})\in\mathbf{Z}$ is a mild solution of (25) for some $(\boldsymbol{w}^0,\boldsymbol{Q}^0)\in\mathbf{W}$ and $(\boldsymbol{f},\boldsymbol{F})\in L^2(0,T;\mathbf{W})$, by Lemma 3, the normal trace $\boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu}\in\boldsymbol{L}^2(\Sigma_0)$ is well-defined.

ii) For any $(\boldsymbol{w},\boldsymbol{Q})\in\mathbf{Z}$, there exists two unique pairs $(\boldsymbol{w}^0,\boldsymbol{Q}^0)\in\mathbf{W}$ and $(\boldsymbol{f},\boldsymbol{F})\in L^2(0,T;\mathbf{W})$ such that $(\boldsymbol{w},\boldsymbol{Q})$ is the corresponding solution of (25). We then denote $\mathcal{M}(\boldsymbol{w},\boldsymbol{Q})=(\boldsymbol{f},\boldsymbol{F})$. With this notation, we have

$$\mathbf{Z}_0=\Big\{(\boldsymbol{w},\boldsymbol{Q})\in\mathbf{Z};\quad\mathcal{M}(\boldsymbol{w},\boldsymbol{Q})=(\boldsymbol{0},\boldsymbol{0})\Big\}.$$

iii) Note that the generalized observability inequality (44) implies

$$\|(\boldsymbol{w}^0,\boldsymbol{Q}^0)\|^2_{\mathbf{W}}\leq C\Big(\|\boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu}\|^2_{\boldsymbol{L}^2(\Sigma_0)}+\|\mathcal{M}(\boldsymbol{w},\boldsymbol{Q})\|^2_{L^2(0,T;\mathbf{W})}\Big),\quad\forall(\boldsymbol{w},\boldsymbol{Q})\in\mathbf{Z}. \tag{45}$$

For any fixed $\eta>0$, we endow $\mathbf{Z}$ with the inner product

$$\big\langle(\boldsymbol{w},\boldsymbol{Q}),(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\big\rangle_{\mathbf{Z}}\coloneqq\int_{\Sigma_0}(\boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu})\cdot(\boldsymbol{\sigma}(\overline{\boldsymbol{Q}})\boldsymbol{\nu})+\eta\int_0^T\big\langle\mathcal{M}(\boldsymbol{w},\boldsymbol{Q}),\mathcal{M}(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\big\rangle_{\mathbf{W}}. \tag{46}$$

We denote $\|\cdot\|_{\mathbf{Z}}$ the associated norm.

Lemma 5. *The space* $\mathbf{Z}$ *endowed with the inner product* $\langle\cdot,\cdot\rangle_{\mathbf{Z}}$ *is a Hilbert space.*

Proof. We check that $\mathbf{Z}$ is closed with respect to the norm $\|\cdot\|_{\mathbf{Z}}$. Let $\{(\boldsymbol{w}_k,\boldsymbol{Q}_k)\}_{k\geq0}\subset\mathbf{Z}$ be a sequence converging to some $(\boldsymbol{w},\boldsymbol{Q})$ for the norm $\|\cdot\|_{\mathbf{Z}}$. Then, $\mathcal{M}(\boldsymbol{w}_k,\boldsymbol{Q}_k)=(\boldsymbol{f}_k,\boldsymbol{F}_k)$ converges to some $(\boldsymbol{f},\boldsymbol{F})\in L^2(0,T;\mathbf{W})$ and the generalized observability inequality (45) implies that $(\boldsymbol{w}_k^0,\boldsymbol{Q}_k^0)$ converges to some $(\boldsymbol{w}^0,\boldsymbol{Q}^0)\in\mathbf{W}$. By the mapping properties of the semigroup which defines the mild solutions of (25), we get that $(\boldsymbol{w}_k,\boldsymbol{Q}_k)$ converges to the mild solution of (25) associated with $(\boldsymbol{w}^0,\boldsymbol{Q}^0)$ and $(\boldsymbol{f},\boldsymbol{F})$. We therefore have $(\boldsymbol{w},\boldsymbol{Q})\in\mathbf{Z}$. □

3 Mixed formulation

Problem (30) is a constrained minimization problem in $\mathbf{Z}$ where the constraint is the equation solved by $(\boldsymbol{w},\boldsymbol{Q})$, i.e. $\mathcal{M}(\boldsymbol{w},\boldsymbol{Q})=(\boldsymbol{0},\boldsymbol{0})$. We take into account this constraint by introducing a Lagrange multiplier $(\boldsymbol{\zeta},\boldsymbol{\Theta})\in\boldsymbol{\Lambda}\coloneqq L^2(0,T;\mathbf{W})$. It leads to the following mixed formulation. Find $\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big)\in\mathbf{Z}\times\boldsymbol{\Lambda}$ solution of

$$a\big((\boldsymbol{w},\boldsymbol{Q}),(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\big)+b\big((\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big)=\ell(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}}),\qquad\forall(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\in\mathbf{Z}, \tag{47a}$$

$$b\big((\boldsymbol{w},\boldsymbol{Q}),(\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}})\big)=0,\qquad\forall(\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}})\in\boldsymbol{\Lambda}, \tag{47b}$$

where

$$a:\mathbf{Z}\times\mathbf{Z}\to\mathbb{R},\qquad a\big((\boldsymbol{w},\boldsymbol{Q}),(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\big)\coloneqq\big\langle\boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu},\boldsymbol{\sigma}(\overline{\boldsymbol{Q}})\boldsymbol{\nu}\big\rangle_{\boldsymbol{L}^2(\Sigma_0)}, \tag{48}$$

$$b:\mathbf{Z}\times\boldsymbol{\Lambda}\to\mathbb{R},\qquad b\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big)\coloneqq\big\langle\mathcal{M}(\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big\rangle_{L^2(0,T;\mathbf{W})}, \tag{49}$$

$$\ell:\mathbf{Z}\to\mathbb{R},\qquad\ell(\boldsymbol{w},\boldsymbol{Q})\coloneqq\big\langle(-\boldsymbol{u}^0,\boldsymbol{\nabla}(-\mathcal{A}_0)^{-1}\boldsymbol{u}^1),(\boldsymbol{w},\boldsymbol{Q})(\cdot,0)\big\rangle_{\mathbf{W}}. \tag{50}$$

Theorem 1. *We have the following.*

i) The mixed formulation (47) *is well-posed.*

ii) Let $\mathcal{L} : \mathbf{Z} \times \mathbf{\Lambda} \to \mathbb{R}$ *be the Lagrangian defined by*

$$\mathcal{L}\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big) \coloneqq \frac{1}{2}a\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{w},\boldsymbol{Q})\big) + b\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big) - \ell(\boldsymbol{w},\boldsymbol{Q}). \tag{51}$$

Then, the unique solution $\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big) \in \mathbf{Z} \times \mathbf{\Lambda}$ *of* (47) *is solution to the saddle-point problem*

$$\sup_{(\boldsymbol{\zeta},\boldsymbol{\Theta})\in\mathbf{\Lambda}} \inf_{(\boldsymbol{w},\boldsymbol{Q})\in\mathbf{Z}} \mathcal{L}\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big).$$

iii) The solution $(\boldsymbol{w},\boldsymbol{Q})$ *of* (47) *is the minimum point of* $\widehat{\mathcal{J}}^\star$ *over* $\mathbf{Z}_0$*. The Lagrange multiplier* $\boldsymbol{\zeta}$ *is the controlled solution of* (1) *in the sense of transposition associated to the control*

$$\boldsymbol{h} = \boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu}\big|_{\Sigma_0}. \tag{52}$$

Remark 4. We can also consider the augmented Lagrangian $\mathcal{L}_r$, which is defined for any $r = (r_1, r_2)$, $r_1, r_2 > 0$, by

$$\mathcal{L}_r\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big) \coloneqq \frac{1}{2}a_r\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{w},\boldsymbol{Q})\big) + b\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big) - \ell(\boldsymbol{w},\boldsymbol{Q}), \tag{53}$$

$$\begin{aligned} a_r\big((\boldsymbol{w},\boldsymbol{Q}),(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\big) \coloneqq a\big((\boldsymbol{w},\boldsymbol{Q}),(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\big) &+ r_1\big\langle \mathcal{M}(\boldsymbol{w},\boldsymbol{Q}), \mathcal{M}(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\big\rangle_{L^2(0,T;\mathbf{W})} \\ &+ r_2\big\langle \boldsymbol{Q}\boldsymbol{\nu}^\perp, \overline{\boldsymbol{Q}}\boldsymbol{\nu}^\perp\big\rangle_{\boldsymbol{L}^2(\Sigma_0)}. \end{aligned} \tag{54}$$

Since $a_r\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{w},\boldsymbol{Q})\big) = a\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{w},\boldsymbol{Q})\big)$ for any $(\boldsymbol{w},\boldsymbol{Q}) \in \mathbf{Z}_0$, the Lagragians $\mathcal{L}$ and $\mathcal{L}_r$ share the same saddle point. Remark that the augmented term associated with r_2 is zero. Nevertheless, we choose to add here this term which will be useful for the numerical implementation. Indeed, we need to impose $\boldsymbol{Q} \in \boldsymbol{\nabla}\boldsymbol{H}^1_0$ for the numerical spaces. In dimension two, $\boldsymbol{\nu}^\perp$ is any unit vector orthogonal to $\boldsymbol{\nu}$, hence if the r_2 is non zero, then this augmentation term impose that the tangent part of $\boldsymbol{Q}$ is close to zero.

Proof. i) In virtue of [3, Theorem 4.2.1], we have to check the following properties.

1) The bilinear forms a and b are continuous. It is obvious from the definition of $\|\cdot\|_{\mathbf{Z}}$.

2) The linear form ℓ is continuous. It is a direct consequence of the generalized observability inequality (45).

3) The form a is coercive on the kernel

$$\mathcal{N}(b) \coloneqq \Big\{(\boldsymbol{w},\boldsymbol{Q}) \in \mathbf{Z};\quad b\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big) = 0,\ \forall (\boldsymbol{\zeta},\boldsymbol{\Theta}) \in \mathbf{\Lambda}\Big\}.$$

It is clear from the definition of a. If $(\boldsymbol{w},\boldsymbol{Q}) \in \mathcal{N}(b) = \mathbf{Z}_0$, then $a\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{w},\boldsymbol{Q})\big) = \|(\boldsymbol{w},\boldsymbol{Q})\|^2_{\mathbf{Z}}$.

4) The form b satisfies an *inf-sup* condition over $\mathbf{Z} \times \mathbf{\Lambda}$. There exists a constant $\delta > 0$ such that

$$\inf_{(\boldsymbol{\zeta},\boldsymbol{\Theta})\in\mathbf{\Lambda}} \sup_{(\boldsymbol{w},\boldsymbol{Q})\in\mathbf{Z}} \frac{b\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big)}{\|(\boldsymbol{w},\boldsymbol{Q})\|_{\mathbf{Z}}\|(\boldsymbol{\zeta},\boldsymbol{\Theta})\|_{\mathbf{\Lambda}}} \geq \delta. \tag{55}$$

We fix $(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\Theta}}) \in \mathbf{\Lambda}$ and we consider the unique $(\widehat{\boldsymbol{w}},\widehat{\boldsymbol{Q}}) \in \mathbf{Z}$ such that $\mathcal{M}(\widehat{\boldsymbol{w}},\widehat{\boldsymbol{Q}}) = (\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\Theta}})$ and $(\widehat{\boldsymbol{w}},\widehat{\boldsymbol{Q}})(\cdot,0) = (\boldsymbol{0},\boldsymbol{0})$. The trace inequality (41) implies

$$\|\boldsymbol{\sigma}(\widehat{\boldsymbol{Q}})\boldsymbol{\nu}\|_{\boldsymbol{L}^2(\Sigma_0)} \leq C\|(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\Theta}})\|_{\mathbf{\Lambda}} = C\|\mathcal{M}(\widehat{\boldsymbol{w}},\widehat{\boldsymbol{Q}})\|_{\mathbf{\Lambda}}. \tag{56}$$

We also have $b\big((\widehat{\boldsymbol{w}},\widehat{\boldsymbol{Q}}),(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\Theta}})\big) = \|(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\Theta}})\|^2_{\mathbf{\Lambda}}$. Therefore,

$$\begin{aligned} \sup_{(\boldsymbol{w},\boldsymbol{Q})\in\mathbf{Z}} \frac{b\big((\boldsymbol{w},\boldsymbol{Q}),(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\Theta}})\big)}{\|(\boldsymbol{w},\boldsymbol{Q})\|_{\mathbf{Z}}\|(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\Theta}})\|_{\mathbf{\Lambda}}} &\geq \frac{\|(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\Theta}})\|_{\mathbf{\Lambda}}}{\|(\widehat{\boldsymbol{w}},\widehat{\boldsymbol{Q}})\|_{\mathbf{Z}}} = \frac{\|\mathcal{M}(\widehat{\boldsymbol{w}},\widehat{\boldsymbol{Q}})\|_{\mathbf{\Lambda}}}{\|(\widehat{\boldsymbol{w}},\widehat{\boldsymbol{Q}})\|_{\mathbf{Z}}} \\ &= \frac{\|\mathcal{M}(\widehat{\boldsymbol{w}},\widehat{\boldsymbol{Q}})\|_{\mathbf{\Lambda}}}{\Big(\|\boldsymbol{\sigma}(\widehat{\boldsymbol{Q}})\boldsymbol{\nu}\|^2_{\boldsymbol{L}^2(\Sigma_0)} + \eta\|\mathcal{M}(\widehat{\boldsymbol{w}},\widehat{\boldsymbol{Q}})\|^2_{\mathbf{\Lambda}}\Big)^{\frac{1}{2}}} \geq \frac{1}{\sqrt{C^2+\eta}}, \end{aligned}$$

where the last inequality is a consequence of (56). Thus, inequality (55) holds with $\delta = (C^2+\eta)^{-\frac{1}{2}}$.

ii) This property is due to the symmetry and positivity of a.
iii) If $\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big) \in \mathbf{Z}\times\boldsymbol{\Lambda}$ solves the mixed formulation (47), then equation (47b) implies $\mathcal{M}(\boldsymbol{w},\boldsymbol{Q}) = (\mathbf{0},\mathbf{0})$. Therefore, $(\boldsymbol{w},\boldsymbol{Q}) \in \mathbf{Z}_0$ and $\mathcal{L}\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big) = \widehat{\mathcal{J}}^\star(\boldsymbol{w},\boldsymbol{Q})$.

Besides, equation (47a) holds for any $(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}}) \in \mathbf{Z}$. In particular, if choosing $\boldsymbol{h}$ given by (52), we have

$$\int_{\Sigma_0} \boldsymbol{h}\cdot(\boldsymbol{\sigma}(\overline{\boldsymbol{Q}})\boldsymbol{\nu}) + \iint_Q \boldsymbol{\zeta}\cdot\overline{\boldsymbol{f}} = \big\langle(-\boldsymbol{u}^0,\boldsymbol{\nabla}(-\mathcal{A}_0)^{-1}\boldsymbol{u}^1),(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})(\cdot,0)\big\rangle_{\mathbf{W}}, \tag{57}$$

for any $(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}}) \in \mathbf{Z}$ such that $\mathcal{M}(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}}) = (\overline{\boldsymbol{f}},\mathbf{0})$, with $\overline{\boldsymbol{f}} \in \boldsymbol{L}^2(Q)$. Let $(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}}) \in \mathbf{Z}$ with $\mathcal{M}(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}}) = (\overline{\boldsymbol{f}},\mathbf{0})$ for some $\overline{\boldsymbol{f}} \in \boldsymbol{L}^2(Q)$. Then, it is easy to check that $(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})$ can be realized as $(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}}) = (\overline{\boldsymbol{\varphi}}',\boldsymbol{\nabla}\overline{\boldsymbol{\varphi}})$, where $\overline{\boldsymbol{\varphi}} \in C([0,T];\boldsymbol{H}^1_0(\Omega)) \cap C^1([0,T];\boldsymbol{L}^2(\Omega))$ is the unique weak solution of

$$\begin{aligned} \overline{\boldsymbol{\varphi}}'' - \mu\boldsymbol{\Delta}\overline{\boldsymbol{\varphi}} - (\lambda+\mu)\nabla\operatorname{div}\overline{\boldsymbol{\varphi}} &= \overline{\boldsymbol{f}} && \text{in } Q,\\ \overline{\boldsymbol{\varphi}} &= \mathbf{0} && \text{on } \Sigma,\\ (\overline{\boldsymbol{\varphi}},\overline{\boldsymbol{\varphi}}')(\cdot,T) &= (\mathbf{0},\mathbf{0}) && \text{in } \Omega. \end{aligned} \tag{58}$$

Hence, the formulation (57) is equivalent to

$$\int_{\Sigma_0} \boldsymbol{h}\cdot\Big(\mu\frac{\partial\overline{\boldsymbol{\varphi}}}{\partial\boldsymbol{\nu}} + (\lambda+\mu)\operatorname{div}\overline{\boldsymbol{\varphi}}\,\boldsymbol{\nu}\Big) + \iint_Q \boldsymbol{\zeta}\cdot\Big(\overline{\boldsymbol{\varphi}}'' - \mu\boldsymbol{\Delta}\overline{\boldsymbol{\varphi}} - (\lambda+\mu)\nabla\operatorname{div}\overline{\boldsymbol{\varphi}}\Big) = - \big\langle\boldsymbol{u}^0,\overline{\boldsymbol{\varphi}}'(\cdot,0)\big\rangle_{\boldsymbol{L}^2(\Omega)} + \big\langle\boldsymbol{u}^1,\overline{\boldsymbol{\varphi}}(\cdot,0)\big\rangle_{\boldsymbol{H}^{-1},\boldsymbol{H}^1_0},$$

for any solution $\overline{\boldsymbol{\varphi}}$ of (58), with $\overline{\boldsymbol{f}} \in \boldsymbol{L}^2(Q)$. It means that $\boldsymbol{\zeta}$ is a transposition solution of (1), so $\boldsymbol{\zeta}$ is the controlled state associated with the initial datum $(\boldsymbol{u}^0,\boldsymbol{u}^1)$ and the control $\boldsymbol{h}$. □

4 Stabilized mixed formulation

In this section, we introduce a stabilized mixed formulation equivalent to (47). This stabilization adds some uniform coercivity with respect to the multiplier variable, which allows to bypass the Babuška-Brezzi *inf-sup* condition. We define the Hilbert space

$$\begin{aligned} \boldsymbol{\Xi} \coloneqq \Big\{&(\boldsymbol{\zeta},\boldsymbol{\Theta}) \in C([0,T];\mathbf{W}) \text{ mild solution of (60) associated with}\\ &(\boldsymbol{\zeta}^0,\boldsymbol{\Theta}^0) \in \mathbf{W},\ \boldsymbol{h} \in \boldsymbol{L}^2(\Sigma_0),\ (\boldsymbol{g},\boldsymbol{G}) \in L^2(0,T;\mathbf{W})\Big\}. \end{aligned} \tag{59}$$

For $(\boldsymbol{\zeta}^0,\boldsymbol{\Theta}^0) \in \mathbf{W}$, $\boldsymbol{h} \in \boldsymbol{L}^2(\Sigma_0)$ and $(\boldsymbol{g},\boldsymbol{G}) \in L^2(0,T;\mathbf{W})$, we say that $(\boldsymbol{\zeta},\boldsymbol{\Theta}) \in C([0,T];\mathbf{W})$ is a mild solution of

$$\begin{aligned} \mathcal{M}(\boldsymbol{\zeta},\boldsymbol{\Theta}) &= (\boldsymbol{g},\boldsymbol{G}) && \text{in } Q,\\ \boldsymbol{\zeta} &= \boldsymbol{h} && \text{on } \Sigma_0,\\ \boldsymbol{\zeta} &= \mathbf{0} && \text{on } \Sigma\setminus\Sigma_0,\\ (\boldsymbol{\zeta},\boldsymbol{\Theta})(\cdot,0) &= (\boldsymbol{\zeta}^0,\boldsymbol{\Theta}^0) && \text{in } \Omega, \end{aligned} \tag{60}$$

if

$$(\boldsymbol{\zeta},\boldsymbol{\Theta})(\cdot,t) = \mathbb{T}_t(\boldsymbol{\zeta}^0,\boldsymbol{\Theta}^0) + \int_0^t \mathbb{T}_{t-s}\mathcal{B}\boldsymbol{h}(\cdot,s)\,\mathrm{d}s + \int_0^t \mathbb{T}_{t-s}(\boldsymbol{g},\boldsymbol{G})(\cdot,s)\,\mathrm{d}s, \quad \forall t\in[0,T], \tag{61}$$

where $\mathcal{B} : L^2(\Gamma_0) \to \mathbf{V}'$ is defined by transposition

$$\big\langle\mathcal{B}\boldsymbol{h},(\boldsymbol{w},\boldsymbol{Q})\big\rangle_{\mathbf{V}',\mathbf{V}} = -\big\langle\boldsymbol{h},\boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu}\big\rangle_{\boldsymbol{L}^2(\Gamma_0)}, \quad \forall(\boldsymbol{w},\boldsymbol{Q}) \in \mathbf{V}.$$

The space $\boldsymbol{\Xi}$ is endowed with the norm given by the following inner product,

$$\begin{aligned} \big\langle(\boldsymbol{\zeta},\boldsymbol{\Theta}),(\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}})\big\rangle_{\boldsymbol{\Xi}} \coloneqq{}& \big\langle\boldsymbol{\zeta},\overline{\boldsymbol{\zeta}}\big\rangle_{\boldsymbol{L}^2(\Sigma_0)} + \big\langle\mathcal{M}(\boldsymbol{\zeta},\boldsymbol{\Theta}),\mathcal{M}(\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}})\big\rangle_{L^2(0,T;\mathbf{W})}\\ &+ \big\langle(\boldsymbol{\zeta},\boldsymbol{\Theta})(\cdot,0),(\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}})(\cdot,0)\big\rangle_{\mathbf{W}} + \big\langle(\boldsymbol{\zeta},\boldsymbol{\Theta})(\cdot,T),(\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}})(\cdot,T)\big\rangle_{\mathbf{W}}. \end{aligned} \tag{62}$$

Let $r = (r_1,r_2)$, $r_1,r_2>0$, and $\alpha = (\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, $\alpha_1 \in (0,1)$, $\alpha_2,\alpha_3,\alpha_4>0$. We want to find $\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big) \in \mathbf{Z}\times\boldsymbol{\Xi}$ such that

$$a_{r,\alpha}\big((\boldsymbol{w},\boldsymbol{Q}),(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\big) + b_\alpha\big((\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big) = \ell(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}}), \qquad \forall(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}}) \in \mathbf{Z}, \tag{63a}$$

$$b_\alpha\big((\boldsymbol{w},\boldsymbol{Q}),(\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}})\big) - c_\alpha\big((\boldsymbol{\zeta},\boldsymbol{\Theta}),(\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}})\big) = d_\alpha(\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}}), \qquad \forall(\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}}) \in \boldsymbol{\Xi}, \tag{63b}$$

where $a_{r,\alpha} : \mathbf{Z} \times \mathbf{Z} \to \mathbb{R}$, $b_\alpha : \mathbf{Z} \times \mathbf{\Xi} \to \mathbb{R}$, $c_\alpha : \mathbf{\Xi} \times \mathbf{\Xi} \to \mathbb{R}$ and $d_\alpha : \mathbf{\Xi} \to \mathbb{R}$ are defined by

$$a_{r,\alpha}\big((\boldsymbol{w},\boldsymbol{Q}),(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\big) \coloneqq a_r\big((\boldsymbol{w},\boldsymbol{Q}),(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\big) - \alpha_1\big\langle \boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu}, \boldsymbol{\sigma}(\overline{\boldsymbol{Q}})\boldsymbol{\nu}\big\rangle_{\boldsymbol{L}^2(\Sigma_0)}, \tag{64}$$

$$b_\alpha\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big) \coloneqq b\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big) + \alpha_1\big\langle \boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu}, \boldsymbol{\zeta}\big\rangle_{\boldsymbol{L}^2(\Sigma_0)}, \tag{65}$$

$$\begin{aligned} c_\alpha\big((\boldsymbol{\zeta},\boldsymbol{\Theta}),(\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}})\big) \coloneqq{}& \alpha_1\big\langle \boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}}\big\rangle_{\boldsymbol{L}^2(\Sigma_0)} + \alpha_2\big\langle \mathcal{M}(\boldsymbol{\zeta},\boldsymbol{\Theta}), \mathcal{M}(\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}})\big\rangle_{L^2(0,T;\mathbf{W})} \\ &+ \alpha_3\big\langle (\boldsymbol{\zeta},\boldsymbol{\Theta})(\cdot,0), (\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}})(\cdot,0)\big\rangle_{\mathbf{W}} + \alpha_4\big\langle (\boldsymbol{\zeta},\boldsymbol{\Theta})(\cdot,T), (\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}})(\cdot,T)\big\rangle_{\mathbf{W}}, \end{aligned} \tag{66}$$

$$d_\alpha(\boldsymbol{\zeta},\boldsymbol{\Theta}) \coloneqq \alpha_3\big\langle (-\boldsymbol{u}^0, \boldsymbol{\nabla}(-\mathcal{A}_0)^{-1}\boldsymbol{u}^1), (\boldsymbol{\zeta},\boldsymbol{\Theta})(\cdot,0)\big\rangle_{\mathbf{W}}. \tag{67}$$

In view of [3, Remark 4.3.1], if $\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big) \in \mathbf{Z}\times\mathbf{\Xi}$ solves (63), then it is solution to the saddle-point problem

$$\inf_{(\boldsymbol{w},\boldsymbol{Q})\in\mathbf{Z}} \sup_{(\boldsymbol{\zeta},\boldsymbol{\Theta})\in\mathbf{\Xi}} \mathcal{L}_{r,\alpha}\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big),$$

with the Lagrangian $\mathcal{L}_{r,\alpha} : \mathbf{Z} \times \mathbf{\Xi} \to \mathbb{R}$ given by

$$\begin{aligned} \mathcal{L}_{r,\alpha}\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big) &= \mathcal{L}_r\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big) - \frac{\alpha_1}{2}\|\boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu} - \boldsymbol{\zeta}\|^2_{\boldsymbol{L}^2(\Sigma_0)} - \frac{\alpha_2}{2}\|\mathcal{M}(\boldsymbol{\zeta},\boldsymbol{\Theta})\|^2_{L^2(0,T;\mathbf{W})} \\ &\quad - \frac{\alpha_3}{2}\|(\boldsymbol{u}^0, -\boldsymbol{\nabla}(-\mathcal{A}_0)^{-1}\boldsymbol{u}^1) - (\boldsymbol{\zeta},\boldsymbol{\Theta})(\cdot,0)\|^2_{\mathbf{W}} - \frac{\alpha_4}{2}\|(\boldsymbol{\zeta},\boldsymbol{\Theta})(\cdot,T)\|^2_{\mathbf{W}} \\ &= \frac{1}{2}a_{r,\alpha}\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{w},\boldsymbol{Q})\big) + b_\alpha\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big) - \frac{1}{2}c_\alpha\big((\boldsymbol{\zeta},\boldsymbol{\Theta}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big) \\ &\quad - \ell(\boldsymbol{w},\boldsymbol{Q}) - d_\alpha(\boldsymbol{\zeta},\boldsymbol{\Theta}), \end{aligned}$$

where $\mathcal{L}_r$ is the augmented Lagrangian defined in (53).

Remark 5. Using integration by parts, we could find an alternative expression for b_α. For $(\boldsymbol{w},\boldsymbol{Q}) \in \mathbf{Z}$ and $(\boldsymbol{\zeta},\boldsymbol{\Theta}) \in \mathbf{\Xi}$, we have

$$\begin{aligned} b_\alpha\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big) = {}& -\big\langle (\boldsymbol{w},\boldsymbol{Q}), \mathcal{M}(\boldsymbol{\zeta},\boldsymbol{\Theta})\big\rangle_{L^2(0,T;\mathbf{W})} - (1-\alpha_1)\big\langle \boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu}, \boldsymbol{\zeta}\big\rangle_{\boldsymbol{L}^2(\Sigma_0)} \\ &- \big\langle (\boldsymbol{w},\boldsymbol{Q})(\cdot,0), (\boldsymbol{\zeta},\boldsymbol{\Theta})(\cdot,0)\big\rangle_{\mathbf{W}} + \big\langle (\boldsymbol{w},\boldsymbol{Q})(\cdot,T), (\boldsymbol{\zeta},\boldsymbol{\Theta})(\cdot,T)\big\rangle_{\mathbf{W}}. \end{aligned}$$

This expression may be useful for the numerical simulations. This idea is similar to the partitioned finite elements methods (see, for instance, [6, 20]).

Theorem 2. *For any $r = (r_1, r_2)$, $r_1, r_2 > 0$, and any $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $\alpha_1 \in (0,1)$, $\alpha_2, \alpha_3, \alpha_4 > 0$, the stabilized mixed formulation* (63) *is well-posed.*

Proof. From [3, Proposition 4.2.1], it is a straightforward consequence of the boundedness of $a_{r,\alpha}$, b_α, c_α, d_α and ℓ, as well as the coercivity of $a_{r,\alpha}$ and c_α on $\mathbf{Z}$ and $\mathbf{\Xi}$ respectively. □

Proposition 2. *The solution of the mixed formulation* (47) *and the solution of the stabilized mixed formulation* (63) *coincide.*

Proof. For any $r = (r_1, r_2)$, $r_1, r_2 > 0$, and any $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $\alpha_1 \in (0,1)$, $\alpha_2, \alpha_3, \alpha_4 > 0$, we check that the saddle point $\big((\boldsymbol{w}_r,\boldsymbol{Q}_r),(\boldsymbol{\zeta}_r,\boldsymbol{\Theta}_r)\big) \in \mathbf{Z}\times\mathbf{\Lambda}$ of $\mathcal{L}_r$ is also a saddle point of $\mathcal{L}_{r,\alpha}$. We observe that, by Theorem 1, $\boldsymbol{\zeta}_r$ is a controlled solution of (1), in the sens of the transposition, with initial datum in $\mathbf{W}$ and boundary control in $\boldsymbol{L}^2(\Sigma_0)$. Thus, $\big((\boldsymbol{w}_r,\boldsymbol{Q}_r),(\boldsymbol{\zeta}_r,\boldsymbol{\Theta}_r)\big) \in \mathbf{Z}\times\mathbf{\Xi}$. Besides, since $\big((\boldsymbol{w}_r,\boldsymbol{Q}_r),(\boldsymbol{\zeta}_r,\boldsymbol{\Theta}_r)\big)$ solves the augmented formulation associated with (47), for any $(\boldsymbol{\zeta},\boldsymbol{\Theta}) \in \mathbf{\Xi}$, we have

$$\mathcal{L}_{r,\alpha}\big((\boldsymbol{w}_r,\boldsymbol{Q}_r),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big) \le \mathcal{L}_r\big((\boldsymbol{w}_r,\boldsymbol{Q}_r),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big) \le \mathcal{L}_r\big((\boldsymbol{w}_r,\boldsymbol{Q}_r),(\boldsymbol{\zeta}_r,\boldsymbol{\Theta}_r)\big) = \mathcal{L}_{r,\alpha}\big((\boldsymbol{w}_r,\boldsymbol{Q}_r),(\boldsymbol{\zeta}_r,\boldsymbol{\Theta}_r)\big).$$

Therefore, $(\boldsymbol{\zeta}_r,\boldsymbol{\Theta}_r)$ maximizes $(\boldsymbol{\zeta},\boldsymbol{\Theta}) \mapsto \mathcal{L}_{r,\alpha}\big((\boldsymbol{w}_r,\boldsymbol{Q}_r),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big)$. Conversely, the functional $F : \mathbf{Z} \to \mathbb{R}$ given by $F(\boldsymbol{w},\boldsymbol{Q}) = \mathcal{L}_{r,\alpha}\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta}_r,\boldsymbol{\Theta}_r)\big)$ admits a unique extremal point due to the ellipticity of $a_{r,\alpha}$. Moreover, for any $(\boldsymbol{w},\boldsymbol{Q}),(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}}) \in \mathbf{Z}$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\varepsilon}F\big((\boldsymbol{w},\boldsymbol{Q}) + \varepsilon(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\big)\Big|_{\varepsilon=0} = {}& a_r\big((\boldsymbol{w},\boldsymbol{Q}),(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\big) + b\big((\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}}),(\boldsymbol{\zeta}_r,\boldsymbol{\Theta}_r)\big) - \ell(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}}) \\ &- \alpha_1\big\langle \boldsymbol{\sigma}(\boldsymbol{Q})\boldsymbol{\nu} - \boldsymbol{\zeta}_r, \boldsymbol{\sigma}(\overline{\boldsymbol{Q}})\boldsymbol{\nu}\big\rangle_{\boldsymbol{L}^2(\Sigma_0)}. \end{aligned}$$

Therefore, from (47), we get

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}F\big((\boldsymbol{w}_r,\boldsymbol{Q}_r)+\varepsilon(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\big)\Big|_{\varepsilon=0}=0,\quad\forall(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\in\mathbf{Z}.$$

Thus, $(\boldsymbol{w}_r,\boldsymbol{Q}_r)$ minimizes $(\boldsymbol{w},\boldsymbol{Q})\mapsto\mathcal{L}_{r,\alpha}\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta}_r,\boldsymbol{\Theta}_r)\big)$. Consequently, the pair $\big((\boldsymbol{w}_r,\boldsymbol{Q}_r),(\boldsymbol{\zeta}_r,\boldsymbol{\Theta}_r)\big)$ is also a saddle point of $\mathcal{L}_{r,\alpha}$. The result follows from the uniqueness of the saddle point. □

5 Dual extremal problem

Here, we derive an extremal problem which is dual to (30) and only involves the variable $(\boldsymbol{\zeta},\boldsymbol{\Theta})$. For $r=(r_1,r_2)$ and $\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, we define the linear operator $\mathcal{A}_{r,\alpha}:\boldsymbol{\Xi}\to\boldsymbol{\Xi}$ in the following way. For any $(\boldsymbol{\zeta},\boldsymbol{\Theta})\in\boldsymbol{\Xi}$, $\mathcal{A}_{r,\alpha}(\boldsymbol{\zeta},\boldsymbol{\Theta})\in\boldsymbol{\Xi}$ is the unique solution of

$$\big\langle\mathcal{A}_{r,\alpha}(\boldsymbol{\zeta},\boldsymbol{\Theta}),(\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}})\big\rangle_{\boldsymbol{\Xi}}=b_\alpha\big((\boldsymbol{w},\boldsymbol{Q}),(\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}})\big)+c_\alpha\big((\boldsymbol{\zeta},\boldsymbol{\Theta}),(\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}})\big),\quad\forall(\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}})\in\boldsymbol{\Xi},\tag{68}$$

where $(\boldsymbol{w},\boldsymbol{Q})\in\mathbf{Z}$ is the unique solution of

$$a_{r,\alpha}\big((\boldsymbol{w},\boldsymbol{Q}),(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\big)=b_\alpha\big((\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big),\quad\forall(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\in\mathbf{Z}.\tag{69}$$

The conditions $r_1,r_2>0$ and $\alpha_1<1$ imply that the augmented bilinear form $a_{r,\alpha}$ is coercive on $\mathbf{Z}$, hence (69) has a unique solution.

Lemma 6. *The operator $\mathcal{A}_{r,\alpha}$ is a strongly elliptic, symmetric isomorphism from $\boldsymbol{\Xi}$ onto $\boldsymbol{\Xi}$.*

Proof. For simplicity, we take $\eta=\frac{r_1}{1-\alpha_1}$ in (46). Since $a_{r,\alpha}$ satisfies the coercivity estimate

$$a_{r,\alpha}\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{w},\boldsymbol{Q})\big)\geq(1-\alpha_1)\|(\boldsymbol{w},\boldsymbol{Q})\|^2_{\mathbf{Z}},\quad\forall(\boldsymbol{w},\boldsymbol{Q})\in\mathbf{Z},$$

then the Lax-Milgram lemma implies that for $(\boldsymbol{\zeta},\boldsymbol{\Theta})\in\boldsymbol{\Xi}$, the corresponding solution $(\boldsymbol{w},\boldsymbol{Q})\in\mathbf{Z}$ of (69) satisfies

$$\|(\boldsymbol{w},\boldsymbol{Q})\|_{\mathbf{Z}}\leq\frac{\|b_\alpha\|}{1-\alpha_1}\|(\boldsymbol{\zeta},\boldsymbol{\Theta})\|_{\boldsymbol{\Xi}}.$$

Therefore, taking $(\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}})=\mathcal{A}_{r,\alpha}(\boldsymbol{\zeta},\boldsymbol{\Theta})$ in (68) leads to

$$\|\mathcal{A}_{r,\alpha}(\boldsymbol{\zeta},\boldsymbol{\Theta})\|_{\boldsymbol{\Xi}}\leq\left(\frac{\|b_\alpha\|^2}{1-\alpha_1}+\|c_\alpha\|\right)\|(\boldsymbol{\zeta},\boldsymbol{\Theta})\|_{\boldsymbol{\Xi}},$$

which gives the boundedness of $\mathcal{A}_{r,\alpha}$. Let $(\boldsymbol{\zeta},\boldsymbol{\Theta}),(\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}})\in\boldsymbol{\Xi}$ and denote respectively $(\boldsymbol{w},\boldsymbol{Q})\in\mathbf{Z}$ and $(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\in\mathbf{Z}$ the corresponding solutions of (69). Then, from (68), we have

$$\big\langle\mathcal{A}_{r,\alpha}(\boldsymbol{\zeta},\boldsymbol{\Theta}),(\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}})\big\rangle_{\boldsymbol{\Xi}}=a_{r,\alpha}\big((\boldsymbol{w},\boldsymbol{Q}),(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\big)+c_\alpha\big((\boldsymbol{\zeta},\boldsymbol{\Theta}),(\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{\Theta}})\big),\tag{70}$$

which implies the positivity and symmetry of $\mathcal{A}_{r,\alpha}$. Besides, from the coercivity of c_α, there exists $C>0$ such that, for any $(\boldsymbol{\zeta},\boldsymbol{\Theta})\in\boldsymbol{\Xi}$,

$$\big\langle\mathcal{A}_{r,\alpha}(\boldsymbol{\zeta},\boldsymbol{\Theta}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big\rangle_{\boldsymbol{\Xi}}\geq c_\alpha\big((\boldsymbol{\zeta},\boldsymbol{\Theta}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big)\geq C\|(\boldsymbol{\zeta},\boldsymbol{\Theta})\|^2_{\boldsymbol{\Xi}}.$$

□

The ellipticity of $\mathcal{A}_{r,\alpha}$ allows to introduce a coercive functional $\widehat{\mathcal{J}}^{\star\star}$ in the next lemma.

Lemma 7. *For any $r=(r_1,r_2)$, $r_1,r_2>0$, let $(\boldsymbol{w}_0,\boldsymbol{Q}_0)\in\mathbf{Z}$ be the unique solution of*

$$a_{r,\alpha}\big((\boldsymbol{w}_0,\boldsymbol{Q}_0),(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\big)=\ell(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}}),\quad\forall(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\in\mathbf{Z}.\tag{71}$$

Let also $\widehat{\mathcal{J}}^{\star\star}:\boldsymbol{\Xi}\to\mathbb{R}$ be the functional defined by

$$\widehat{\mathcal{J}}^{\star\star}(\boldsymbol{\zeta},\boldsymbol{\Theta})=\frac{1}{2}\big\langle\mathcal{A}_{r,\alpha}(\boldsymbol{\zeta},\boldsymbol{\Theta}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big\rangle_{\boldsymbol{\Xi}}-b_\alpha\big((\boldsymbol{w}_0,\boldsymbol{Q}_0),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big)+d_\alpha(\boldsymbol{\zeta},\boldsymbol{\Theta}).$$

Then, the following equality holds,

$$\sup_{(\boldsymbol{\zeta},\boldsymbol{\Theta})\in\boldsymbol{\Xi}}\inf_{(\boldsymbol{w},\boldsymbol{Q})\in\mathbf{Z}}\mathcal{L}_{r,\alpha}\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big)=-\inf_{(\boldsymbol{\zeta},\boldsymbol{\Theta})\in\boldsymbol{\Xi}}\widehat{\mathcal{J}}^{\star\star}(\boldsymbol{\zeta},\boldsymbol{\Theta})+\mathcal{L}_{r,\alpha}\big((\boldsymbol{w}_0,\boldsymbol{Q}_0),(\boldsymbol{0},\boldsymbol{0})\big).$$

Proof. For any $(\boldsymbol{\zeta},\boldsymbol{\Theta})\in\boldsymbol{\Xi}$, let $(\boldsymbol{w}^\star,\boldsymbol{Q}^\star)\in\mathbf{Z}$ be the minimum point of $(\boldsymbol{w},\boldsymbol{Q})\mapsto\mathcal{L}_{r,\alpha}\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big)$. Thus, $(\boldsymbol{w}^\star,\boldsymbol{Q}^\star)$ satisfies the equation

$$a_{r,\alpha}\big((\boldsymbol{w}^\star,\boldsymbol{Q}^\star),(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\big)+b_\alpha\big((\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big)=\ell(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}}),\quad\forall(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\in\mathbf{Z},$$

and can be decomposed as $(\boldsymbol{w}^\star,\boldsymbol{Q}^\star)=(\boldsymbol{w}_0,\boldsymbol{Q}_0)-(\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{Q}})$, where $(\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{Q}})\in\mathbf{Z}$ solves

$$a_{r,\alpha}\big((\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{Q}}),(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\big)=b_\alpha\big((\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big),\quad\forall(\overline{\boldsymbol{w}},\overline{\boldsymbol{Q}})\in\mathbf{Z}.$$

We then have

$$\begin{aligned}\inf_{(\boldsymbol{w},\boldsymbol{Q})\in\mathbf{Z}}\mathcal{L}_{r,\alpha}\big((\boldsymbol{w},\boldsymbol{Q}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big)=\mathcal{L}_{r,\alpha}\big((\boldsymbol{w}^\star,\boldsymbol{Q}^\star),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big)&=\mathcal{L}_{r,\alpha}\big((\boldsymbol{w}_0,\boldsymbol{Q}_0)-(\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{Q}}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big)\\&\coloneqq X_1+X_2+X_3+X_4,\end{aligned}$$

with

$$\begin{cases}X_1=\dfrac{1}{2}a_{r,\alpha}\big((\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{Q}}),(\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{Q}})\big)-b_\alpha\big((\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{Q}}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big)-\dfrac{1}{2}c_\alpha\big((\boldsymbol{\zeta},\boldsymbol{\Theta}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big),\\X_2=b_\alpha\big((\boldsymbol{w}_0,\boldsymbol{Q}_0),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big)-d_\alpha(\boldsymbol{\zeta},\boldsymbol{\Theta}),\\X_3=\dfrac{1}{2}a_{r,\alpha}\big((\boldsymbol{w}_0,\boldsymbol{Q}_0),(\boldsymbol{w}_0,\boldsymbol{Q}_0)\big)-\ell(\boldsymbol{w}_0,\boldsymbol{Q}_0),\\X_4=-a_{r,\alpha}\big((\boldsymbol{w}_0,\boldsymbol{Q}_0),(\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{Q}})\big)+\ell(\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{Q}}).\end{cases}$$

From the definition of $(\boldsymbol{w}_0,\boldsymbol{Q}_0)$, we have $X_4=0$ and $X_3=\mathcal{L}_{r,\alpha}\big((\boldsymbol{w}_0,\boldsymbol{Q}_0),(\mathbf{0},\mathbf{0})\big)$. Finally, the definition of $(\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{Q}})$ implies

$$X_1=-\frac{1}{2}b_\alpha\big((\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{Q}}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big)-\frac{1}{2}c_\alpha\big((\boldsymbol{\zeta},\boldsymbol{\Theta}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big)=-\frac{1}{2}\big\langle\mathcal{A}_{r,\alpha}(\boldsymbol{\zeta},\boldsymbol{\Theta}),(\boldsymbol{\zeta},\boldsymbol{\Theta})\big\rangle_{\boldsymbol{\Xi}}$$

and the result follows. □

From the ellipticity of the operator $\mathcal{A}_{r,\alpha}$, the minimization of $\widehat{\mathcal{J}}^{\star\star}$ in $\mathbf{Z}$ is well-posed. We observe that, in contrast with the initial problem of finding the control of minimal L^2-norm, the minimization of $\widehat{\mathcal{J}}^{\star\star}$ in $\mathbf{Z}$ does not entail any constraint.

Due to the symmetry and ellipticity of the operator $\mathcal{A}_{r,\alpha}$, the *conjugate gradient* method is well-suited for the numerical minimization of $\widehat{\mathcal{J}}^{\star\star}$. The Polak-Ribière version of the conjugate gradient method reads as follows. For a given convergence criterion $\epsilon>0$, we follow the steps in Algorithm 1.

6 Numerical approximation and simulations

In this section, we numerically illustrate the proposed method for the approximation of the control and the controlled solution for the two-dimensional wave equation and for the two-dimensional elasticity system. More precisely, we put in place the procedure described in Section 5 by considering a conformal finite elements approximation of the spaces $\mathbf{Z}$, $\boldsymbol{\Xi}$ and of the operators $\mathcal{A}_{r,\alpha}$, $\mathcal{M}$, together with the inner products of these spaces. In that purpose, we consider uniform triangulations $\mathcal{T}_h$ (structured or not) of the domain Q. For such each triangulation, we denote by $h>0$ the largest diameter of tetrahedra composing $\mathcal{T}_h$. Two examples of domains Q and the associated meshes are displayed in Figure 1.

We define $\boldsymbol{\Xi}_h\subset\boldsymbol{\Xi}$ as follows:

$$\begin{aligned}\boldsymbol{\Xi}_h=\Bigg\{(\boldsymbol{\zeta}_h,\boldsymbol{\Theta}_h)\in\boldsymbol{\Xi};\quad\boldsymbol{\zeta}=\begin{pmatrix}\zeta_{h,1}\\\zeta_{h,2}\end{pmatrix},\ \boldsymbol{\Theta}=\begin{pmatrix}\Theta_{h,11}&\Theta_{h,12}\\\Theta_{h,21}&\Theta_{h,22}\end{pmatrix},\\\zeta_{h,i|K},\ \Theta_{h,ij|K}\in\mathbb{P}_1\text{ for every }K\in\mathcal{T}_h\text{ and }i,j\in\{1,2\}\Bigg\},\end{aligned}\tag{72}$$

with $\mathbb{P}_1$ being the space of polynomials of variables (x,y) and of degree less than 1. Algorithm 1 easily adapts to this finite dimensional framework considering finite elements approximations of operators $\mathcal{A}_{r,\alpha}$, $\mathcal{M}$ and of the inner product over $\boldsymbol{\Xi}$. The implementation of this algorithm was done using FreeFEM [13]. If it is not specified otherwise, we choose the augmentation parameters $r_1=r_2=10^{-1}$ and the stabilization parameters $\alpha_1=\alpha_2=\alpha_3=\alpha_4=10^{-3}$. Remark that the control $\boldsymbol{h}$ can be obtained in two different manners. On the one hand, since $\boldsymbol{\zeta}$ correspond to the controlled state of the system (1),

Algorithm 1 Conjugate algorithm for iteratively solving (70)

1) *Initialization.* We choose an initial candidate $(\boldsymbol{\zeta}_0, \boldsymbol{\Theta}_0) \in \boldsymbol{\Xi}$.
We compute $(\boldsymbol{w}_0, \boldsymbol{Q}_0) \in \mathbf{Z}$ solution of

$$a_{r,\alpha}\big((\boldsymbol{w}_0, \boldsymbol{Q}_0), (\overline{\boldsymbol{w}}, \overline{\boldsymbol{Q}})\big) = b_\alpha\big((\overline{\boldsymbol{w}}, \overline{\boldsymbol{Q}}), (\boldsymbol{\zeta}_0, \boldsymbol{\Theta}_0)\big) - \ell(\overline{\boldsymbol{w}}, \overline{\boldsymbol{Q}}), \quad \forall (\overline{\boldsymbol{w}}, \overline{\boldsymbol{Q}}) \in \mathbf{Z}.$$

We compute $(\boldsymbol{g}_0, \boldsymbol{G}_0) \in \boldsymbol{\Xi}$ solution of

$$\big\langle (\boldsymbol{g}_0, \boldsymbol{G}_0), (\overline{\boldsymbol{\zeta}}, \overline{\boldsymbol{\Theta}})\big\rangle_{\boldsymbol{\Xi}} = b_\alpha\big((\boldsymbol{w}_0, \boldsymbol{Q}_0), (\overline{\boldsymbol{\zeta}}, \overline{\boldsymbol{\Theta}})\big) + c_\alpha\big((\boldsymbol{\zeta}_0, \boldsymbol{\Theta}_0), (\overline{\boldsymbol{\zeta}}, \overline{\boldsymbol{\Theta}})\big) + d_\alpha(\overline{\boldsymbol{\zeta}}, \overline{\boldsymbol{\Theta}}), \quad \forall (\overline{\boldsymbol{\zeta}}, \overline{\boldsymbol{\Theta}}) \in \boldsymbol{\Xi}.$$

We set $(\boldsymbol{v}_0, \boldsymbol{P}_0) = (\boldsymbol{g}_0, \boldsymbol{G}_0)$.

2) *Iteration.* For $n \geq 0$, we compute $(\widetilde{\boldsymbol{w}}_n, \widetilde{\boldsymbol{Q}}_n) \in \mathbf{Z}$ solution of

$$a_{r,\alpha}\big((\widetilde{\boldsymbol{w}}_n, \widetilde{\boldsymbol{Q}}_n), (\overline{\boldsymbol{w}}, \overline{\boldsymbol{Q}})\big) = b_\alpha\big((\overline{\boldsymbol{w}}, \overline{\boldsymbol{Q}}), (\boldsymbol{v}_n, \boldsymbol{P}_n)\big), \quad \forall (\overline{\boldsymbol{w}}, \overline{\boldsymbol{Q}}) \in \mathbf{Z}.$$

We compute $\mathcal{A}_{r,\alpha}(\boldsymbol{v}_n, \boldsymbol{P}_n) \in \boldsymbol{\Xi}$ solution of

$$\big\langle \mathcal{A}_{r,\alpha}(\boldsymbol{v}_n, \boldsymbol{P}_n), (\overline{\boldsymbol{\zeta}}, \overline{\boldsymbol{\Theta}})\big\rangle_{\boldsymbol{\Xi}} = b_\alpha\big((\widetilde{\boldsymbol{w}}_n, \widetilde{\boldsymbol{Q}}_n), (\overline{\boldsymbol{\zeta}}, \overline{\boldsymbol{\Theta}})\big) + c_\alpha\big((\boldsymbol{v}_n, \boldsymbol{P}_n), (\overline{\boldsymbol{\zeta}}, \overline{\boldsymbol{\Theta}})\big), \quad \forall (\overline{\boldsymbol{\zeta}}, \overline{\boldsymbol{\Theta}}) \in \boldsymbol{\Xi}.$$

We compute

$$\alpha_n = \frac{\|(\boldsymbol{g}_n, \boldsymbol{G}_n)\|^2_{\boldsymbol{\Xi}}}{\big\langle \mathcal{A}_{r,\alpha}(\boldsymbol{v}_n, \boldsymbol{P}_n), (\boldsymbol{v}_n, \boldsymbol{P}_n)\big\rangle_{\boldsymbol{\Xi}}}$$

and we update

$$(\boldsymbol{\zeta}_{n+1}, \boldsymbol{\Theta}_{n+1}) = (\boldsymbol{\zeta}_n, \boldsymbol{\Theta}_n) - \alpha_n(\boldsymbol{v}_n, \boldsymbol{P}_n), \qquad (\boldsymbol{w}_{n+1}, \boldsymbol{Q}_{n+1}) = (\boldsymbol{w}_n, \boldsymbol{Q}_n) - \alpha_n(\widetilde{\boldsymbol{w}}_n, \widetilde{\boldsymbol{Q}}_n),$$
$$(\boldsymbol{g}_{n+1}, \boldsymbol{G}_{n+1}) = (\boldsymbol{g}_n, \boldsymbol{G}_n) - \alpha_n \mathcal{A}_{r,\alpha}(\boldsymbol{v}_n, \boldsymbol{P}_n).$$

3) *Convergence test.* If $\|(\boldsymbol{g}_{n+1}, \boldsymbol{G}_{n+1})\|_{\boldsymbol{\Xi}} \leq \epsilon \|(\boldsymbol{g}_0, \boldsymbol{G}_0)\|_{\boldsymbol{\Xi}}$, we stop and set $(\boldsymbol{\zeta}_n, \boldsymbol{\Theta}_n)$ as the approximation of the minimum point of $\widehat{\mathcal{J}}^{\star\star}$. Else, we compute

$$\beta_n = \frac{\big\langle (\boldsymbol{g}_{n+1}, \boldsymbol{G}_{n+1}), (\boldsymbol{g}_{n+1}, \boldsymbol{G}_{n+1}) - (\boldsymbol{g}_n, \boldsymbol{G}_n)\big\rangle_{\boldsymbol{\Xi}}}{\|(\boldsymbol{g}_n, \boldsymbol{G}_n)\|^2_{\boldsymbol{\Xi}}}$$

and we update

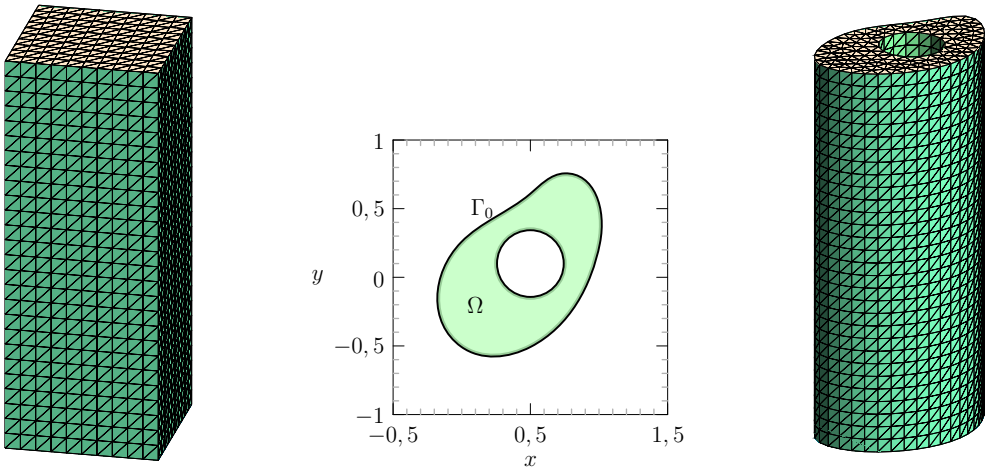
$$(\boldsymbol{v}_{n+1}, \boldsymbol{P}_{n+1}) = (\boldsymbol{g}_{n+1}, \boldsymbol{G}_{n+1}) + \beta_n(\boldsymbol{v}_n, \boldsymbol{P}_n).$$

Do $n \leftarrow n + 1$ and return to step 2).

we have $\boldsymbol{h} = \boldsymbol{\zeta}|_{\Sigma_0}$. On the other hand, we can compute $\boldsymbol{h}$ using the dual variables $(\boldsymbol{w}, \boldsymbol{Q})$ and the relation (52).

As a first step for the validation of the proposed method, we employ Algorithm 1 for the approximation of the boundary control and the corresponding controlled state for a two-dimensional wave equation. Indeed, choosing $\mu = 1$ and $\lambda = -\mu$, system (1) is nothing else than two independent wave equations with constant coefficients in dimension two. The boundary controllability of two-dimensional wave equations was firstly considered from a numerical perspective in [11, 12]. An example of initial data and the corresponding boundary control of minimal L^2-norm are explicitly given in this paper. The numerical experiments corresponding to the domain $\Omega = (0, 1)^2$ and to the initial data in [12, 10] for which the boundary control is explicitly known are presented in Subsection 6.1.

Once the method validated for the boundary controllability of the two-dimensional wave equation, numerical experiments were conducted for the elasticity system in dimension two. More exactly, we set $\mu = \lambda = 1$. We numerically illustrate this situation in Subsection 6.2 for two different geometries of Ω: the unit square and the non-convex domain represented in Figure 1 (b).

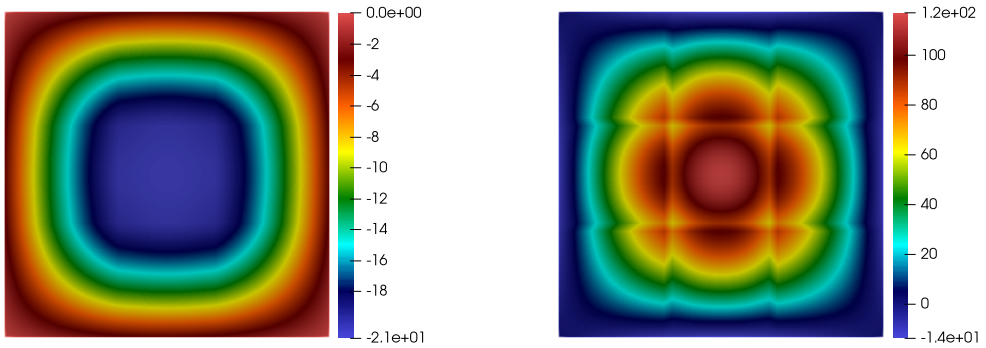


(a) (b) (c)

Figure 1: Domains Q and associated meshes. (a) A structured mesh of $Q = (0,1)^2 \times (0,T)$. (b) An example of non-convex domain Ω and (c) a mesh of $Q = \Omega \times (0,T)$ associated to this domain.

6.1 Two dimensional wave equation

In this subsection, we set $\Omega = (0,1)^2$ the unit square with the control acting on its whole boundary ($\Gamma_0 = \Gamma$). We consider the initial data constructed in [11, 12] using Fourier series. For this initial data, displayed in Figure 2, and for $T = \frac{1}{\sqrt{2}}\left(3 + \frac{3}{4}\right)$ the boundary control can be explicitly computed (see [12] for details).



u^0 u^1

Figure 2: Initial data (u^0, u^1) constructed in [12].

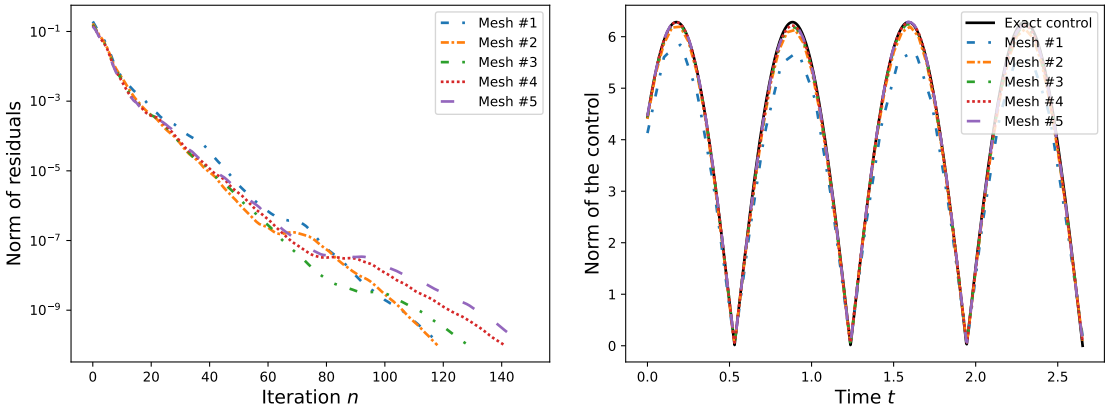
Remark that by taking $\mu = -\lambda$, system (1) is composed by two independent wave equations. Therefore, we can drop the second equation and also $\dim(\boldsymbol{\Xi}_h)$ is smaller.

We consider five levels of meshes described in Table 1. The coarsest of these meshes is depicted in Figure 1 (a).

Mesh number	1	2	3	4	5
Diameter h of elements	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{30}$	$\frac{1}{40}$	$\frac{1}{50}$
Number of nodes	3 267	23 814	76 880	179 867	345 933
Number of tetrahedra	15 600	127 200	426 600	1 017 600	1 980 000

Table 1: Description of five meshes of the domain $Q = (0,1)^2 \times (0,T)$.

In Figure 3 (a), we illustrate the evolution of the norm of the residuals $\|(\boldsymbol{g}_n, \boldsymbol{G}_n)\|_{\boldsymbol{\Xi}}$ appearing in Algorithm 1. Here and henceforth, we fixed the value of the stopping criterion parameter in the conjugate gradient algorithm Algorithm 1 to $\epsilon = 10^{-10}$. In Figure 3 (b), we display the norm of the boundary control obtained from $(\boldsymbol{\zeta}_n, \boldsymbol{\Theta}_n)$ once the conjugate gradient algorithm has converged.



(a) (b)

Figure 3: Results for initial data (u^0, u^1) displayed in Figure 2. (a) Evolution of the norm résiduals $(\boldsymbol{g}_n, \boldsymbol{G}_n)$. (b) Norm L^2 of the control $\boldsymbol{h}(t)$.

In Figure 4 we display the evolution of the norm of the L^2 error between the exact control (explictly known) and the numerical controls computed from dual and primal variables for two different choices of stabilization parameter α_i. Analyzing these results, we observe a convergence of order $h^{0.5}$ for the numerical controls to the explicit control, for the controls obtained from both the primal and dual variables when $\alpha_i = 10^{-1}$. For an augmentation parameter $\alpha_i = 10^{-3}$ we observe a better convergence (of order h) for the control computed from the primal variable. The same order of convergence is observed for large values of h when the control is computed from the dual variable, but the approximation is affected by a numerical locking phenomenon at 0.02 which is probably due to the values of the stabilization parameters which are not large enough. A more involved study will be needed for finding the optimal values of these parameters.

We also remark that the number of iterations needed for the convergence of the conjugate gradient algorithm remains stable with respect to the discretization parameter h. Nevertheless, as expected, for smaller values of augmentation and stabilization parameters a higher number of iterations is need for converge of the conjugate gradient algorithm described in Algorithm 1. In Figure 5, we display the solution $(\boldsymbol{\zeta}_n, \boldsymbol{\Theta}_n)$ obtained once the Algorithm 1 has converged.

Similar results are obtained for different choices of parameter $r = (r_1, r_2)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

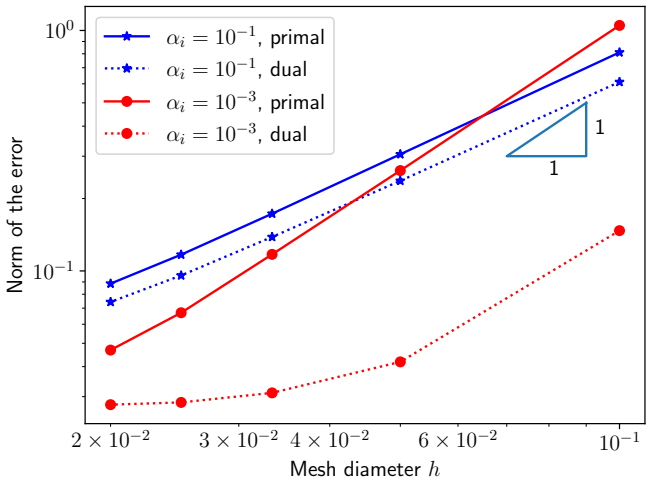
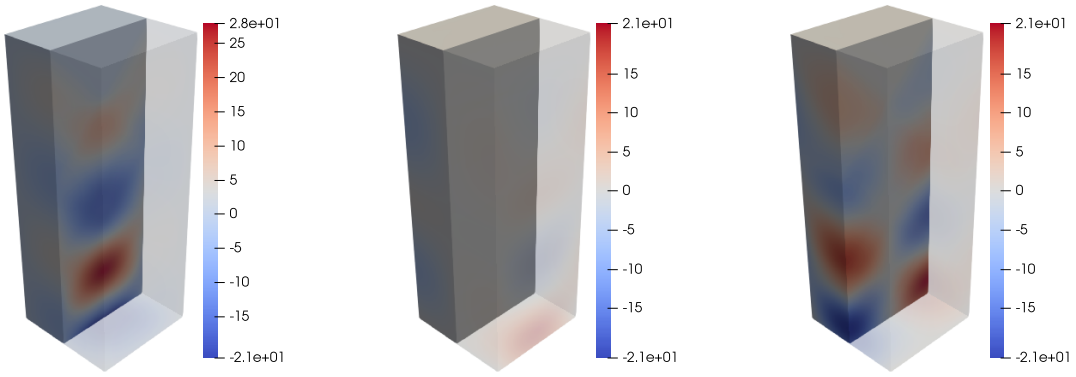


Figure 4: Norm of the error between the exact and numerical controls for different values of h and two different values of α_1.

Another interesting illustration (see Figure 6 (a)) is the evolution with respect to the time of the norm of the primal and dual solutions $(\boldsymbol{\zeta}, \boldsymbol{\Theta})$ and $(\boldsymbol{w}, \boldsymbol{Q})$, respectively. As expected, the energy norm of



(a) (b) (c)

Figure 5: Solution $(\zeta_n, \boldsymbol{\Theta}_n)$ obtained once the conjugate algorithm converged for the mesh 5. (a) ζ_n. (b) $\Theta_{n,1}$. (c) $\Theta_{n,2}$.

the primal variable goes to zero when the time is close to the control time T and the energy norm of the dual variable is conserved in time.

6.2 Elasticity system

In this subsection, we illustrate the approximation of the boundary control for the two-dimensional elasticity system ($\mu = \lambda = 1$) for two different domain geometries. In the first case, we consider Ω the unit square and in the second one Ω is a non convex domain with regular boundary.

6.2.1 Control acting on the whole boundary Γ of $\Omega = (0,1)^2$

For simplicity, we consider here the same geometric situation as in Section 6.1: $\Omega = (0,1)^2$, $T = \frac{1}{\sqrt{2}}\big(3+\frac{3}{4}\big)$ and the initial datum $(\boldsymbol{u}^0, \boldsymbol{u}^1) \in \boldsymbol{L}^2(\Omega) \times \boldsymbol{H}^{-1}(\Omega)$ given by

$$\boldsymbol{u}^0 = (u^0, u^0)^\top, \quad \boldsymbol{u}^1 = (u^1, u^1)^\top, \tag{73}$$

with u^0 and u^1 depicted in Figure 2. The conjugate gradient algorithm converges for every mesh in less than 200 iterations for an norm of the residuals smaller than $\epsilon = 10^{-10}$. The norm $L^2(\Omega)$ of the controls $\boldsymbol{h}(t)$ is depicted in Figure 7 for all the five levels of mesh considered here. In the left column of this figure we display the evolution of the norm of the controls computed as the boundary trace of $\boldsymbol{\zeta}_n$ for $\alpha_1 = 10^{-3}$ (top) and $\alpha_1 = 0.9$ (bottom). The norm of theses controls are very similar for both values of the parameter α_1. On the right side of the figure we depict the norm of the control computed from $\boldsymbol{Q}_n$ using (52) for the same values of α_1 as before. In that case we observe that for $\alpha_1 = 10^{-3}$ the control has some extra oscillations near $t = 0$ and $t = T$. As expected, for values of α_1 close to 1, the controls obtained from the primal variable $\boldsymbol{\zeta}$ are very close to the controls obtained from the dual variable $\boldsymbol{Q}$. Similarly to the wave equation, the controls seem to converge with respect to the discretization parameter h. In Figure 6 (b) we also observe a very similar comportment to the wave equation for the evolution in time of the norms of primal and dual variables.

The six components of the solution $(\boldsymbol{\zeta}_n, \boldsymbol{\Theta}_n)$ obtained once the Algorithm 1 has converged for the finest mesh in Table 1 are graphically represented in Figure 8.

6.2.2 Control acting on a part of the regular boundary of a non-convex Ω

In this subsection we consider the non-convex domain Ω graphically represented in Figure 1 (b). The control acts only on the exterior boundary Γ_0 of Ω.

For this choice of Ω, we consider a control time $T = 3$ and the initial data $(\boldsymbol{u}^0, \boldsymbol{u}^1) \in \boldsymbol{L}^2(\Omega) \times \boldsymbol{H}^{-1}(\Omega)$ given by

$$\boldsymbol{u}^0 = \boldsymbol{u}^1 \text{ solution to } \begin{cases} -\boldsymbol{\Delta} \boldsymbol{u}^0 = 100 & \text{in } \Omega, \\ \boldsymbol{u}^0 = 0 & \text{on } \Gamma. \end{cases} \tag{74}$$

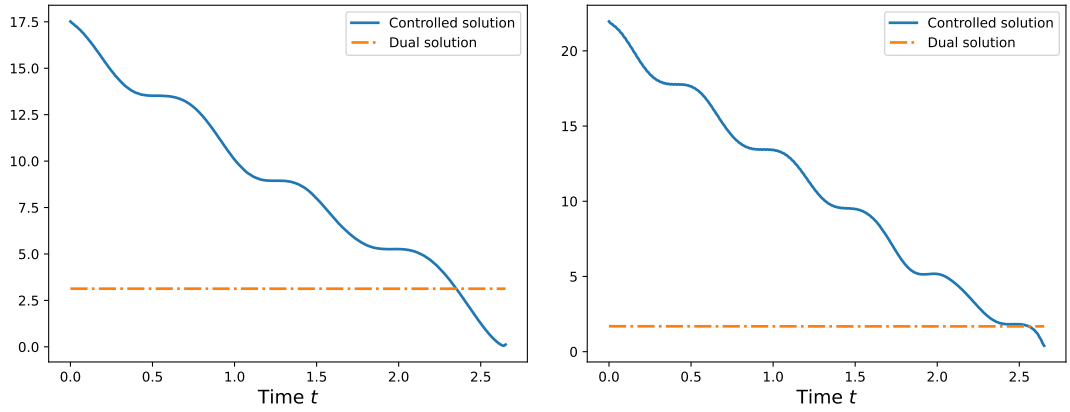


Figure 6: Evolution with respect to the time t of the norms of primal and dual solutions for: (a) the wave equation; (b) the elasticity system and the initial data in Figure 2.

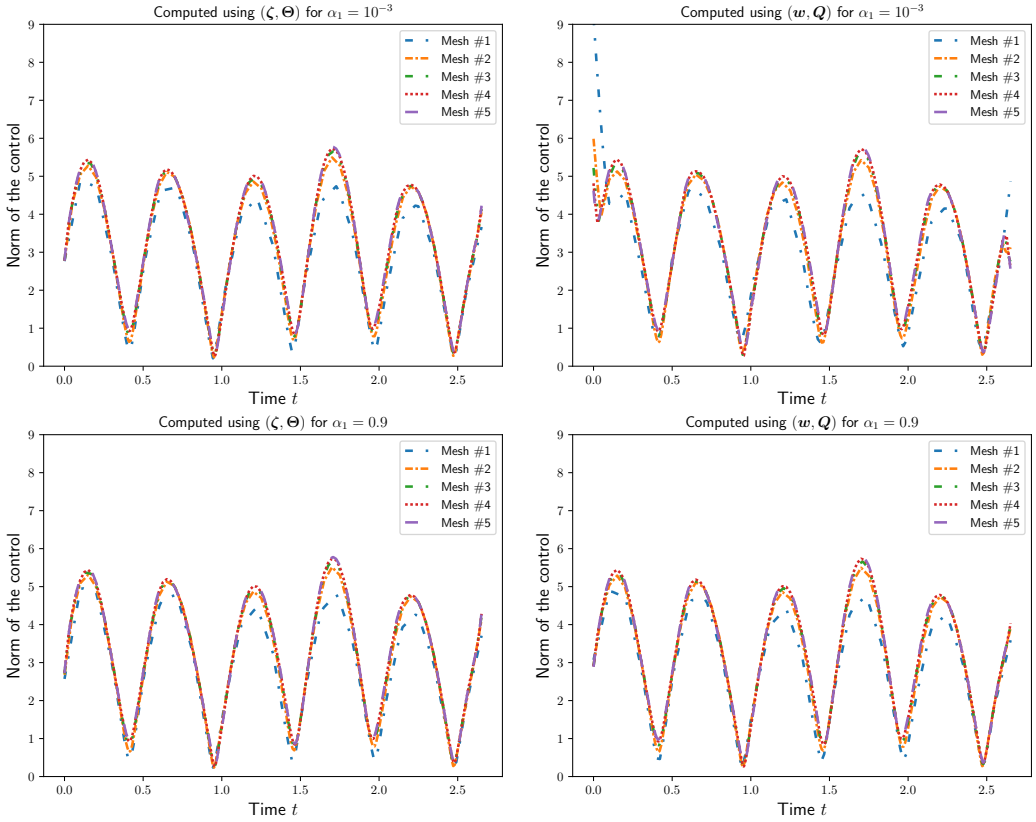


Figure 7: Norm of the control $\boldsymbol{h}$ for the five meshes described in Table 1 computed from $(\boldsymbol{\zeta}_n, \boldsymbol{\Theta}_n)$ (left) and from $(\boldsymbol{w}_n, \boldsymbol{Q}_n)$ (right), for $\alpha_1 = 10^{-3}$ (up) and $\alpha_1 = 9 \times 10^{-1}$ (bottom), respectively.

Mesh number	1	2	3	4	5
Diameter h of elements	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{30}$	$\frac{1}{40}$	$\frac{1}{50}$
Number of nodes	4 557	29 707	99 094	212 234	406 945
Number of tetrahedra	21 510	155 700	515 080	1 185 760	2 303 550

Table 2: Description of five meshes of the domain $Q = \Omega \times (0, T)$ for Ω displayed in Figure 1 (b).

We consider five levels of meshes described in Table 2. The coarsest of these meshes is plotted in Figure 1 (c).

For the set of simulations corresponding to this non-convex domain and initial data given by (74)

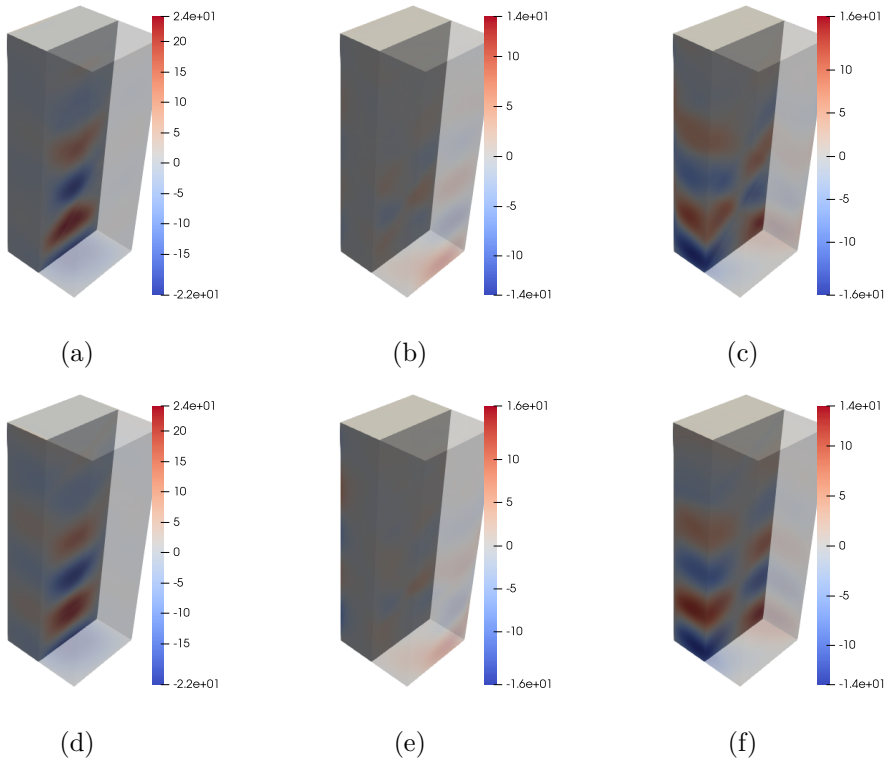


Figure 8: The six components of the solution for the initial datum in Figure 2 and the finest mesh in Table 1. a) $\zeta_{n,1}$. (b) $\Theta_{n,11}$. (c) $\Theta_{n,12}$. (d) $\zeta_{n,2}$. (e) $\Theta_{n,21}$. (f) $\Theta_{n,22}$.

we fixed $\alpha_1 = 0.9$ and we consider $\alpha_2 \in \{10^{-3}, 10^{-2}, 10^{-1}\}$. The parameter α_2 enforce that the primal variable verifies the controlled elasticity system in $L^2(0,T;\mathbf{W})$.

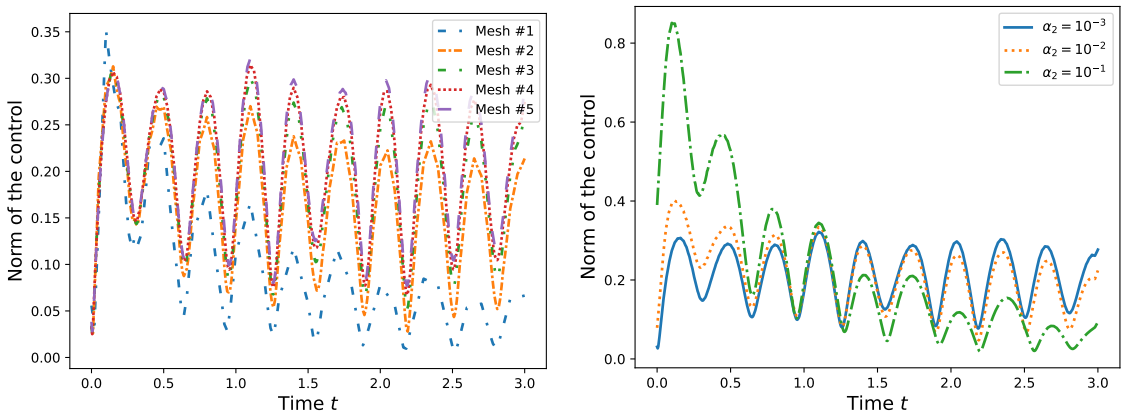


Figure 9: Norm of the control for initial data given by (74). (a) $\alpha_2 = 10^{-3}$ and different meshes. (b) Computation on the mesh ♯5 and different values for α_2.

The norm of the control for $\alpha_2 = 10^{-3}$ and the five meshes described in Table 2 is displayed in Figure 9 (a). As expected we observe the convergence of the controls with respect to the discretization parameter h. In Figure 9 (b) we illustrate the norm of the control computed on the finest mesh in Table 2 for three different values of α_2. As expected, for larger values of α_2 the primal solution $\boldsymbol{\zeta}$ is more regular and the corresponding control is closer to zero near the control time T.

7 Conclusion and perspectives

This paper studied the boundary null controllability of the elasticity system reformulated as a first order in time and space system. We also proposed a constructive method to compute the boundary control of minimal L^2-norm for the elasticity system as the solution of a stabilized mixed formulation. Although the numerical computation of the control remains extremely challenging for the elasticity system, the proposed method was numerically implemented and the obtained results are very promising. The facts that we considered a first order system in space and time combined to the very careful augmentation and stabilization of the mixed formulation were essential for the numerical implementation. More exactly, we are able to successfully solve the mixed formulation by approaching it using only P_1 finite elements and by the means of a conjugate gradient algorithm. Nevertheless, several difficulties remain and in what follows we propose some perspectives to tackle them.

As for all space-time methods, the main numerical difficulty was related to the high dimension of discrete system to solve. Indeed, the time appears as a third dimension for a system being two-dimensional in space. Even using a parallel solver for the linear systems to solve in the gradient descent algorithm, this was not enough for very refined meshes. In a further work, we will investigate the possibility to iteratively solve the mixed formulation characterizing the control and the controlled solution by a domain decomposition technique combined to a conjugate gradient method. This will allow to fully parallelize the method and not only the resolution of the involved linear systems as it is actually the case.

Another delicate question is the choice of the values for the augmentation parameters $(r_i)_{1\leq i\leq 2}$ and for the stabilization parameters $(\alpha_i)_{1\leq i\leq 4}$. For larger values of these parameters we need less iterations for the conjugate gradient algorithm convergence but the errors are bigger. For smaller values of parameters we need a larger number of iterations for the conjugate gradient algorithm and the convergence of control seems better. A more involved numerically study is needed in order to numerically investigate the dependence of the *inf-sup* constant associated to the mixed formulation to these parameters. We will consider this question in a further work.

A Operator $\mathcal{A}$ is the generator of a group of contractions

The following lemma states that the operator $\mathcal{A}$ given by (24) generates a group of contractions.

Lemma 8. *The operator $\mathcal{A} : \mathbf{V} \to \mathbf{W}$ defined by (24) is the infinitesimal generator of a C_0-group of contractions $(\mathbb{T}_t)_{t\in\mathbb{R}}$ on $\mathbf{W}$.*

Proof. It is sufficient to apply the Hille-Yosida theorem to the operators $\mathcal{A}$ and $-\mathcal{A}$. We verify the hypotheses of this theorem for $\mathcal{A}$, the situation being very similar for $-\mathcal{A}$. More precisely, we should check that $(0,+\infty) \subset \rho(\mathcal{A})$ and that, for every $\xi > 0$, the corresponding resolvent operator verifies $\|(\xi\,\mathcal{I}_d - \mathcal{A})^{-1}\| \leq \xi^{-1}$.

Let $\xi > 0$ and $(\boldsymbol{f}, \boldsymbol{F}) \in \mathbf{W}$. In order to show that $\xi \in \rho(\mathcal{A})$, we should find $(\boldsymbol{w}, \boldsymbol{Q}) \in \mathbf{V}$ such that $(\xi\,\mathcal{I}_d - \mathcal{A})(\boldsymbol{w}, \boldsymbol{Q}) = (\boldsymbol{f}, \boldsymbol{F})$. This equation is equivalent to

$$\xi\boldsymbol{w} - \mu\,\mathrm{div}\,\boldsymbol{Q} - (\lambda+\mu)\nabla\,\mathrm{tr}\,\boldsymbol{Q} = \boldsymbol{f}, \tag{75a}$$

$$\xi\boldsymbol{Q} - \boldsymbol{\nabla}\boldsymbol{w} = \boldsymbol{F}. \tag{75b}$$

From (75b), we express $\boldsymbol{Q}$ as a function of $\boldsymbol{w}$. Replacing in (75a), we obtain

$$\xi\boldsymbol{w} - \xi^{-1}\mu\boldsymbol{\Delta}\boldsymbol{w} - \xi^{-1}(\lambda+\mu)\nabla\,\mathrm{div}\,\boldsymbol{w} = \boldsymbol{f} + \xi^{-1}\mu\,\mathrm{div}\,\boldsymbol{F} + \xi^{-1}(\lambda+\mu)\nabla\,\mathrm{tr}\,\boldsymbol{F},$$

which can be written in variational form as: find $\boldsymbol{w} \in \boldsymbol{H}^1_0(\Omega)$ such that, for every $\overline{\boldsymbol{w}} \in \boldsymbol{H}^1_0(\Omega)$,

$$\xi\int_\Omega \boldsymbol{w}\cdot\overline{\boldsymbol{w}} + \xi^{-1}\mu\int_\Omega \boldsymbol{\nabla}\boldsymbol{w} : \boldsymbol{\nabla}\overline{\boldsymbol{w}} + \xi^{-1}(\lambda+\mu)\int_\Omega \mathrm{div}\,\boldsymbol{w}\,\mathrm{div}\,\overline{\boldsymbol{w}} = \int_\Omega \boldsymbol{f}\cdot\overline{\boldsymbol{w}} - \xi^{-1}\mu\int_\Omega \boldsymbol{F} : \boldsymbol{\nabla}\overline{\boldsymbol{w}} - \xi^{-1}(\lambda+\mu)\int_\Omega \mathrm{tr}\,\boldsymbol{F}\,\mathrm{div}\,\overline{\boldsymbol{w}}. \tag{76}$$

Using the Lax-Milgram theorem, the above equation admits a unique weak solution $\boldsymbol{w} \in \boldsymbol{H}^1_0(\Omega)$. Then, we recover $\boldsymbol{Q} = \xi^{-1}(\boldsymbol{\nabla}\boldsymbol{w} + \boldsymbol{F}) \in \boldsymbol{\nabla}\boldsymbol{H}^1_0$. From this expression for $\boldsymbol{Q}$, we deduce

$$\int_\Omega \boldsymbol{Q} : \boldsymbol{\nabla}\overline{\boldsymbol{w}} = \xi^{-1}\int_\Omega (\boldsymbol{\nabla}\boldsymbol{w} + \boldsymbol{F}) : \boldsymbol{\nabla}\overline{\boldsymbol{w}} \qquad \text{and} \qquad \mathrm{tr}\,\boldsymbol{Q} = \xi^{-1}(\mathrm{div}\,\boldsymbol{w} + \mathrm{tr}\,\boldsymbol{F}).$$

Together with (76), it implies that, for every $\overline{\boldsymbol{w}} \in \boldsymbol{H}_0^1(\Omega)$,

$$\mu \int_\Omega \boldsymbol{Q} : \boldsymbol{\nabla}\overline{\boldsymbol{w}} + (\lambda+\mu) \int_\Omega \operatorname{tr}\boldsymbol{Q} \operatorname{div}\overline{\boldsymbol{w}} = \int_\Omega (\boldsymbol{f} - \xi\boldsymbol{w}) \cdot \overline{\boldsymbol{w}}.$$

Let $\boldsymbol{\varphi} \in \boldsymbol{H}_0^1(\Omega)$ such that $\boldsymbol{Q} = \boldsymbol{\nabla}\boldsymbol{\varphi}$, from the previous variational formulation, we see that

$$\boldsymbol{\varphi} = \mathcal{A}_0^{-1}(\boldsymbol{f} - \xi\boldsymbol{w}).$$

Therefore, since $\boldsymbol{f} - \xi\boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$, we have $\boldsymbol{\varphi} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$ and $\boldsymbol{Q} \in \boldsymbol{\nabla}\boldsymbol{H}_0^2$. Finally, $\xi \in \rho(\mathcal{A})$ and we obtain $(0, +\infty) \subset \rho(\mathcal{A})$.

In order to prove the estimate for the norm of the resolvent operator $(\xi\,\mathcal{I}_d - \mathcal{A})^{-1}$, from (75), we easily get

$$\xi \int_\Omega |\boldsymbol{w}|^2 - \mu \int_\Omega \operatorname{div}\boldsymbol{Q} \cdot \boldsymbol{w} - (\lambda+\mu) \int_\Omega \nabla \operatorname{tr}\boldsymbol{Q} \cdot \boldsymbol{w} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w},$$
$$\xi \int_\Omega |\boldsymbol{Q}|^2 - \int_\Omega \boldsymbol{\nabla}\boldsymbol{w} : \boldsymbol{Q} = \int_\Omega \boldsymbol{F} : \boldsymbol{Q}.$$

Integrating by parts and summing the above two relations, we obtain

$$\xi \int_\Omega |\boldsymbol{w}|^2 + \xi \int_\Omega |\boldsymbol{Q}|^2 + (\mu-1) \int_\Omega \boldsymbol{Q} : \boldsymbol{\nabla}\boldsymbol{w} + (\lambda+\mu) \int_\Omega \operatorname{tr}\boldsymbol{Q} \operatorname{div}\boldsymbol{w} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w} + \int_\Omega \boldsymbol{F} : \boldsymbol{Q}. \tag{77}$$

From (75b), we can equivalently write that

$$\boldsymbol{\nabla}\boldsymbol{w} = \xi\boldsymbol{Q} - \boldsymbol{F} \qquad \text{and} \qquad \operatorname{div}\boldsymbol{w} = \xi \operatorname{tr}\boldsymbol{Q} - \operatorname{tr}\boldsymbol{F}.$$

Replacing in (77), it follows that

$$\xi \int_\Omega |\boldsymbol{w}|^2 + \xi \int_\Omega |\boldsymbol{Q}|^2 + (\mu-1) \int_\Omega \boldsymbol{Q} : (\xi\boldsymbol{Q} - \boldsymbol{F}) + (\lambda+\mu) \int_\Omega \operatorname{tr}\boldsymbol{Q}(\xi \operatorname{tr}\boldsymbol{Q} - \operatorname{tr}\boldsymbol{F}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w} + \int_\Omega \boldsymbol{F} : \boldsymbol{Q}.$$

Therefore, we have

$$\xi \int_\Omega |\boldsymbol{w}|^2 + \xi\mu \int_\Omega |\boldsymbol{Q}|^2 + \xi(\lambda+\mu) \int_\Omega |\operatorname{tr}\boldsymbol{Q}|^2 = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w} + \mu \int_\Omega \boldsymbol{F} : \boldsymbol{Q} + (\lambda+\mu) \int_\Omega \operatorname{tr}\boldsymbol{F} \operatorname{tr}\boldsymbol{Q}.$$

Finally, applying the Cauchy-Schwarz inequality, we get that

$$\xi \|(\boldsymbol{w}, \boldsymbol{Q})\|_{\mathbf{W}} \leq \|(\boldsymbol{f}, \boldsymbol{F})\|_{\mathbf{W}} \qquad \text{and} \qquad \|(\xi\,\mathcal{I}_d - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathbf{W})} \leq \xi^{-1},$$

which concludes the proof of the lemma. □

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