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Homogenization of a class of singular elliptic problems in two-component domains

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Abstract. This paper deals with the homogenization of a quasilinear elliptic problem having a singular lower order term and posed in a two-component domain with an ε -periodic imperfect interface. We prescribe a Dirichlet condition on the exterior boundary, while we assume that the continuous heat flux is proportional to the jump of the solution on the interface via a function of order ε^γ .

We prove an homogenization result for $-1 < \gamma < 1$ by means of the periodic unfolding method (see *SIAM J. Math. Anal.* **40** (2008) 1585–1620 and *The Periodic Unfolding Method* (2018) Springer), adapted to two-component domains in (*J. Math. Sci.* **176** (2011) 891–927).

One of the main tools in the homogenization process is a convergence result for a suitable auxiliary linear problem, associated with the weak limit of the sequence $\{u^\varepsilon\}$ of the solutions, as $\varepsilon \rightarrow 0$. More precisely, our result shows that the gradient of u^ε behaves like that of the solution of the auxiliary problem, which allows us to pass to the limit in the quasilinear term, and to study the singular term near its singularity, via an accurate a priori estimate.

Keywords: Two-component domains, Homogenization, Periodic Unfolding Method, Quasilinear elliptic equations, Singular equations

1. Introduction

In this paper we study the asymptotic behavior of a class of quasilinear elliptic problems presenting singular lower order terms and posed in two-component domains.

More precisely, the two-component domain Ω is a bounded open subset of \mathbb{R}^N which is the union of two disjoint open subsets, Ω_1^ε and Ω_2^ε , and their common boundary Γ^ε . The disconnected component Ω_2^ε is the union of the ε -periodic translated sets of εY_2 , where Y_2 is well contained in the reference cell Y and $\Gamma = \partial Y_2$. The connected component Ω_1^ε is obtained by removing from Ω the closure of Ω_2^ε such that the interface $\Gamma^\varepsilon = \partial \Omega_2^\varepsilon$.

We deal with the homogenization, as ε goes to zero, of the following problem:

$$\begin{cases} -\operatorname{div}(A^\varepsilon(x, u^\varepsilon)\nabla u^\varepsilon) = f\zeta(u^\varepsilon) & \text{in } \Omega \setminus \Gamma^\varepsilon, \\ u_1^\varepsilon = 0 & \text{on } \partial\Omega, \\ (A^\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)v^\varepsilon = (A^\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon)v^\varepsilon & \text{on } \Gamma^\varepsilon, \\ (A^\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)v^\varepsilon = -\varepsilon^\gamma h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma^\varepsilon, \end{cases}$$

where v^ε is the unit external normal vector to Ω_1^ε . We prescribe a Dirichlet condition on the exterior boundary, while we assume that the continuous heat flux is proportional to the jump of the solution on the interface by means of a function of order ε^γ .

1 The quasilinear diffusion matrix field is defined by $A^\varepsilon(x, t) = A(\frac{x}{\varepsilon}, t)$, where the matrix field A is 1
 2 uniformly elliptic, bounded, periodic in the first variable and Carathéodory. The nonlinear real function 2
 3 $\zeta(s)$ is nonnegative and singular at $s = 0$, while f is a nonnegative datum whose summability depends 3
 4 on the growth of ζ near its singularity. Concerning the boundary condition, $h^\varepsilon(x) = h(\frac{x}{\varepsilon})$ where h is 4
 5 assumed to be a periodic, nonnegative and bounded function, and $\gamma \in] - 1, 1[$. 5

6 This problem describes the stationary heat diffusion in a two-component composite with an ε -periodic 6
 7 imperfect interface. In particular, quasilinear diffusion terms describe the behavior of materials like 7
 8 glass or wood, in which the heat diffusion depends on the range of the temperature (see for instance 8
 9 [6]). We refer to [20, Section 3] for more details about source terms depending on the solution itself 9
 10 and becoming infinite when the solutions vanish. These kind of source terms can model, for instance, 10
 11 electrical conductors where each point becomes a source of heat as a current flows inside. The boundary 11
 12 condition on Γ^ε models a jump of the solution through a rough interface and we refer to [9] for a physical 12
 13 justification of this model (see also [29]). 13

14 In view of a counterexample suggested by H.C. Hummel in [28], one cannot expect bounded a priori 14
 15 estimates for the solution when $\gamma > 1$. For this case we refer to [23] where different a priori estimates 15
 16 needed. As for our problem, existence and uniqueness results has been proved in [25,26] for $\gamma \leq 1$. 16

17 Here we prove an homogenization result for $\gamma \in] - 1, 1[$ by means of the periodic unfolding method. 17
 18 This method was originally introduced by D. Cioranescu, A. Dalambian and G. Griso in [13] and [14] 18
 19 for fixed domains and extended to perforated ones in [16] and [17]. Then, it has been adapted to two- 19
 20 component domains by P. Donato, K.H. Le Nguyen and R. Tardieu in [22] (see also [21]). 20

21 This work has been partially inspired by the works [20] and [24]. Let us point out the main differences 21
 22 with respect to them and the additional difficulties. In [20] the authors treat the same singularity when 22
 23 A is linear and in a different geometrical framework where the domain has two connected components 23
 24 separated by an oscillating interface. This interface tends to a flat one, so that the integral on Γ^ε goes 24
 25 to the one on Γ , roughly speaking. But here Γ^ε is the union of disconnected sets (of order $\frac{1}{\varepsilon}$) so that its 25
 26 measure goes to infinity, whence the integral on Γ^ε needs particular care. 26

27 Let us mention that in [24] the authors consider the same singular problem with A quasilinear but 27
 28 posed in a periodically perforated domain. Here the holes are replaced by a second material so that we 28
 29 have to treat in addition all the integrals on the second component and the boundary term with jump. 29

30 In Section 2 we present the setting of the problem and we state the main results. 30

31 In Section 3 we give a short presentation of the periodic unfolding method, adapted to two-component 31
 32 domains. 32

33 In Section 4 we prove two a priori estimates uniform with respect to ε and we give an estimate of 33
 34 the integral of the singular term close to the singular set $\{u^\varepsilon = 0\}$, in terms of the quasilinear one (cf. 34
 35 Proposition 4.3). 35

36 In Section 5 we prove a crucial convergence result, Theorem 5.4. It is one of the main tools when 36
 37 proving our homogenization result and it shows that the gradient of u^ε is equivalent to the gradient of 37
 38 the solution of a suitable auxiliary linear problem, associated with a weak cluster point of the sequence 38
 39 $\{u^\varepsilon\}$, as $\varepsilon \rightarrow 0$. This idea was originally introduced in [4] (see also [5]) where some nonlinear problems 39
 40 with quadratic growth are considered. We refer to [10] and [22] for the study of the auxiliary problem, 40
 41 while for the proof of the convergence result we adapt some techniques from [20] and [24]. 41

42 Section 6 is devoted to the proofs of Proposition 2.5 and the homogenization theorem. For these results 42
 43 Theorem 5.4 plays an important role not only in the study of the quasilinear term but also in that of the 43
 44 singular one. Actually, as done in [20] and [24], we split the integral of the singular term into the sum of 44
 45 two integrals: one on the set where the solution is close to the singularity and one where is it far from it, 45
 46 46

which results not singular. Near the singularity, making use of the estimate given in Proposition 4.3, we shift the study of the singular term to that of the quasilinear one, for which we can use Theorem 5.4.

Finally, let us point out that the homogenization result we prove here shows that the conductivity of the first material is the same obtained when there is no material occupying Ω_2^ε . On the other hand, since in the limit problem it appears f instead of $\theta_1 f$ (being θ_1 the proportion of the material occupying Ω_1^ε), one has anyway to take into account even the source term in the second component.

The first paper on this subject is due to [3] for the linear (nonsingular) case by multiple scale method. In [23] and [29] the authors also studied the linear case by using the Tartar method. For similar homogenization elliptic problems we refer to [8,20,24,28] and [30]. In [21,22] the authors study the linear case in presence of linear and nonlinear boundary conditions, respectively.

2. Setting of the problem and main results

Throughout the paper, we use the same notation as in [21,22] for the periodic unfolding method in two-component domains.

The domain. For $N \in \mathbb{N}$, $N \geq 2$, let Ω be a bounded open set in \mathbb{R}^N with a Lipschitz-continuous boundary $\partial\Omega$. Also let $Y \doteq \prod_{i=1}^N [0, l_i[$ be a reference cell, with $l_i > 0$, $i = 1, \dots, N$. We suppose that Y_1 and Y_2 are two disjoint connected open subsets of Y such that $Y_2 \neq \emptyset$, $\overline{Y_2} \subset Y$ and $Y = Y_1 \cup \overline{Y_2}$, with a common boundary $\Gamma = \partial Y_2$ Lipschitz-continuous.

Let $\{\varepsilon\}_{\varepsilon>0}$ be a positive parameter taking values in a sequence converging to zero and set:

- for any $k \in \mathbb{Z}^N$, $k_l = (k_1 l_1, \dots, k_N l_N)$ and

$$Y^k = k_l + Y, \quad Y_i^k = k_l + Y_i, \quad i = 1, 2,$$

- $K_\varepsilon = \{k \in \mathbb{Z}^N \mid \varepsilon \overline{Y_2^k} \subset \Omega\}$ and

$$\Omega_2^\varepsilon = \bigcup_{k \in K_\varepsilon} \varepsilon Y_2^k, \quad \Omega_1^\varepsilon = \Omega \setminus \overline{\Omega_2^\varepsilon}, \quad \Gamma^\varepsilon = \partial \Omega_2^\varepsilon.$$

By construction, Ω results the union of the two disjoint components and their common boundary, i.e. $\Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon$ (see Fig. 1).

Also we introduce the following sets:

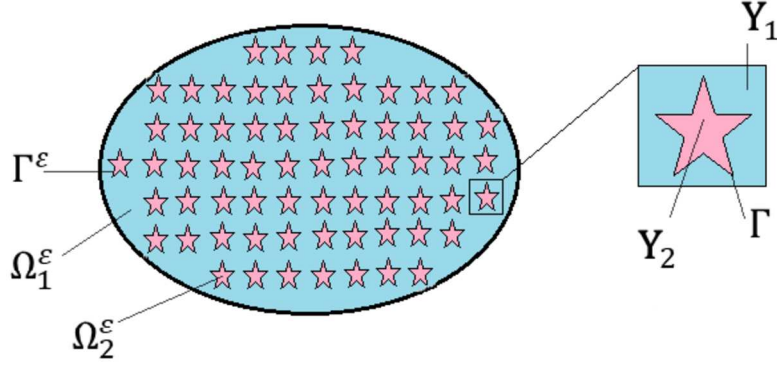
$$\widehat{K}_\varepsilon = \{k \in \mathbb{Z}^N \mid \varepsilon Y^k \subset \Omega\}, \quad \widehat{\Omega}_\varepsilon = \text{interior} \left\{ \bigcup_{k \in \widehat{K}_\varepsilon} \varepsilon (k_l + \overline{Y}) \right\}, \quad \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon,$$

and

$$\widehat{\Omega}_i^\varepsilon = \bigcup_{k \in \widehat{K}_\varepsilon} \varepsilon Y_i^k, \quad \Lambda_i^\varepsilon = \Omega_i^\varepsilon \setminus \widehat{\Omega}_i^\varepsilon, \quad i = 1, 2, \quad \widehat{\Gamma}^\varepsilon = \partial \widehat{\Omega}_2^\varepsilon.$$

Also,

$$\text{there exists } \varepsilon_0 > 0 \text{ such that } \forall \varepsilon \leq \varepsilon_0, \quad \Lambda_2^\varepsilon = \emptyset. \quad (2.1)$$

Fig. 1. The two-component domain Ω and the reference cell Y .

Let us note that here, for the sake of simplicity, we do not allow the second component to meet the boundary of the domain and also we suppose Y_2 connected. Actually the cases where the holes meet the boundary of Ω and Y_2 has a finite number of connected components can be treated as done in [21] and [22], respectively.

In the sequel we denote by

- $\mathcal{M}_\omega(v) \doteq \frac{1}{|\omega|} \int_\omega v \, dx$ the average of any function $v \in L^1(\omega)$, for any open set ω of \mathbb{R}^N ,
- χ_ω the characteristic function of any open set ω of \mathbb{R}^N ,
- \sim the zero extension to the whole of Ω of functions defined on Ω_1^ε or Ω_2^ε ,
- $\theta_i \doteq \frac{|Y_i|}{|Y|}$, $i = 1, 2$,
- c different positive constants independent on ε ,
- $\mathcal{M}(\alpha, \beta, Y)$ the set of matrix fields $A = (a_{i,j})_{1 \leq i,j \leq N} \in (L^\infty(Y))^{N \times N}$ such that

$$(A(x)\lambda, \lambda) \geq \alpha |\lambda|^2 \quad \text{and} \quad |A(x)\lambda| \leq \beta |\lambda|, \quad \forall \lambda \in \mathbb{R}^N \text{ and a.e. in } Y,$$

with $\alpha, \beta \in \mathbb{R}$, $0 < \alpha \leq \beta$,

- $\{e_1, \dots, e_N\}$ the canonical basis of \mathbb{R}^N ,
- $v_i = v|_{\Omega_i^\varepsilon}$ the restriction to Ω_i^ε of functions v defined in Ω , $i = 1, 2$.

Moreover let us recall the classical decomposition for every real function v

$$v = v^+ - v^-, \quad v^+ \doteq \max\{v, 0\} \quad \text{and} \quad v^- \doteq -\min\{v, 0\} \quad \text{a.e. in } \Omega, \quad (2.2)$$

where v^+ and v^- are both nonnegative. In particular, one has

$$\begin{aligned} (v_1 - v_2)(v_1^- - v_2^-) &= (v_1^+ - v_2^+)(v_1^- - v_2^-) - (v_1^- - v_2^-)^2 \\ &= -v_1^+ v_2^- - v_2^+ v_1^- - (v_1^- - v_2^-)^2 \leq 0. \end{aligned} \quad (2.3)$$

The problem. The aim of the paper is to study the asymptotic behavior, as ε goes to zero, of the following problem:

$$\begin{cases} -\operatorname{div}(A^\varepsilon(x, u^\varepsilon)\nabla u^\varepsilon) = f\zeta(u^\varepsilon) & \text{in } \Omega \setminus \Gamma^\varepsilon, \\ u_1^\varepsilon = 0 & \text{on } \partial\Omega, \\ (A^\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)v^\varepsilon = (A^\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon)v^\varepsilon & \text{on } \Gamma^\varepsilon, \\ (A^\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)v^\varepsilon = -\varepsilon^\gamma h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma^\varepsilon, \end{cases} \quad (2.4)$$

where v^ε is the unit external normal vector to Ω_1^ε , and we prescribe a Dirichlet condition on the exterior boundary and a jump of the solution on the interface Γ^ε , in the case $\gamma \in]-1, 1[$.

Assumptions on the data.

H₁) The real $N \times N$ matrix field $A : (y, t) \in Y \times \mathbb{R} \mapsto A(y, t) = (a_{i,j}(y, t))_{i,j=1,\dots,N} \in \mathbb{R}^{N^2}$ satisfies the following conditions:

- $$\left\{ \begin{array}{l} \text{i) } A \text{ is a Carathéodory function s.t.} \\ \quad A(\cdot, t) \text{ is } Y\text{-periodic and in } \mathcal{M}(\alpha, \beta, Y), \text{ for every } t \in \mathbb{R}; \\ \text{ii) there exists a real function } \omega : \mathbb{R} \rightarrow \mathbb{R} \text{ satisfying the following conditions:} \\ \quad - \omega \text{ is continuous and non decreasing, with } \omega(t) > 0 \forall t > 0, \\ \quad - |A(y, t_1) - A(y, t_2)| \leq \omega(|t_1 - t_2|) \text{ for a.e. } y \in Y, \forall t_1 \neq t_2, \\ \quad - \forall s > 0, \quad \lim_{y \rightarrow 0^+} \int_y^s \frac{dt}{\omega(t)} = +\infty. \end{array} \right.$$

H₂) The functions ζ and f verify

- $$\left\{ \begin{array}{l} \text{i) } \zeta : [0, +\infty[\rightarrow [0, +\infty] \text{ is a function such that} \\ \quad \zeta \in C^0([0, +\infty[), 0 \leq \zeta(s) \leq \frac{1}{s^k} \text{ for every } s \in]0, +\infty[, \text{ with } 0 < k \leq 1; \\ \text{ii) } \zeta \text{ is non increasing;} \\ \text{iii) } f \geq 0 \text{ a.e. in } \Omega, f \not\equiv 0, \text{ with } f \in L^l(\Omega), \text{ for } l \geq \frac{2}{1+k} (\geq 1). \end{array} \right.$$

H₃) $-1 < \gamma < 1$, and h is a Y -periodic function in $L^\infty(\Gamma)$ such that

there exists $h_0 \in \mathbb{R} : 0 < h_0 < h(y)$ a.e. on Γ .

Under the above assumptions we set, for every $t \in \mathbb{R}$,

$$A^\varepsilon(x, t) \doteq A\left(\frac{x}{\varepsilon}, t\right) \quad \text{for a.e. } x \in \Omega, \quad h^\varepsilon(x) \doteq h\left(\frac{x}{\varepsilon}\right) \quad \text{for a.e. } x \in \Gamma^\varepsilon. \quad (2.5)$$

The functional framework. We now introduce the functional spaces used in the literature to handle (2.4)-type problems.

Let $V^\varepsilon \doteq \{v \in H^1(\Omega_1^\varepsilon) | v = 0 \text{ on } \partial\Omega\}$ endowed with the norm

$$\|v\|_{V^\varepsilon} = \|\nabla v\|_{L^2(\Omega_1^\varepsilon)}.$$

Remark 2.1. It is known (see for instance [18, Lemma 1], [19]) that a Poincaré inequality in V^ε holds with a constant c_P independent on ε , that is

$$\|v\|_{L^2(\Omega_1^\varepsilon)} \leq c_P \|\nabla v\|_{L^2(\Omega_1^\varepsilon)} \quad \forall v \in V^\varepsilon. \quad (2.6)$$

Consequently, the norm in V^ε is equivalent to that in $H^1(\Omega_1^\varepsilon)$ via a constant independent on ε .

For every $\gamma \in \mathbb{R}$, let H_γ^ε be the space defined by

$$H_\gamma^\varepsilon \doteq \{v \in L^2(\Omega) \mid v_1 \in V^\varepsilon, v_2 \in H^1(\Omega_2^\varepsilon)\},$$

which, after the identification $\nabla v := \widetilde{\nabla}v_1 + \widetilde{\nabla}v_2$, is equipped by the norm

$$\|v\|_{H_\gamma^\varepsilon}^2 \doteq \|\nabla v\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 + \varepsilon^\gamma \|v_1 - v_2\|_{L^2(\Gamma^\varepsilon)}^2.$$

Proposition 2.2 ([22,29]). *Let $\gamma \leq 1$. There exist some positive constants c_1, c_2 and C , independent of ε , such that*

$$\forall v \in H_\gamma^\varepsilon, \quad c_1 \|v\|_{V^\varepsilon \times H^1(\Omega_2^\varepsilon)}^2 \leq \|v\|_{H_\gamma^\varepsilon}^2 \leq c_2 (1 + \varepsilon^{\gamma-1}) \|v\|_{V^\varepsilon \times H^1(\Omega_2^\varepsilon)}^2.$$

In addition, if $v^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon)$ is a bounded sequence in H_γ^ε , then

$$\|v_1^\varepsilon\|_{H^1(\Omega_1^\varepsilon)} \leq C,$$

$$\|v_2^\varepsilon\|_{H^1(\Omega_2^\varepsilon)} \leq C,$$

$$\|v_1^\varepsilon - v_2^\varepsilon\|_{L^2(\Gamma^\varepsilon)} \leq C\varepsilon^{-\frac{\gamma}{2}}.$$

The variational formulation associated with problem (2.4) reads

$$\begin{cases} \text{Find } u^\varepsilon \in H_\gamma^\varepsilon \text{ such that } u^\varepsilon > 0 \text{ a.e. in } \Omega, \\ \int_\Omega f \zeta(u^\varepsilon) \varphi \, dx < +\infty, \quad \text{and} \\ \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \nabla \varphi \, dx + \int_{\Gamma^\varepsilon} \varepsilon^\gamma h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) (\varphi_1 - \varphi_2) \, d\sigma \\ = \int_\Omega f \zeta(u^\varepsilon) \varphi \, dx, \quad \forall \varphi \in H_\gamma^\varepsilon. \end{cases} \quad (2.7)$$

In [25,26] it is proved that, under assumptions $H_1)$ – $H_3)$, problem (2.7) admits a unique solution.

Let us introduce here the homogenized matrix $A^0(t)$, $t \in \mathbb{R}$, corresponding to our case $\gamma \in]-1, 1[$. It is defined by

$$A^0(t)\lambda \doteq \frac{1}{|Y|} \int_{Y_1} A(y, t) (\lambda - \nabla_y \chi_\lambda(y, t)) \, dy \quad \forall \lambda \in \mathbb{R}^N, \quad (2.8)$$

where, for every $\lambda \in \mathbb{R}^N$, $\chi_\lambda(\cdot, t) \in H^1(Y_1)$ are unique solutions of the cell problems

$$\begin{cases} -\operatorname{div}(A(\cdot, t) \nabla_y \chi_\lambda(\cdot, t)) = -\operatorname{div}(A(\cdot, t)\lambda) & \text{in } Y_1, \\ A(\cdot, t) (\lambda - \nabla_y \chi_\lambda(\cdot, t)) \nu_1 = 0 & \text{on } \Gamma, \\ \chi_\lambda(\cdot, t) \text{ Y-periodic and } \mathcal{M}_{Y_1}(\chi_\lambda) = 0. \end{cases} \quad (2.9)$$

The homogenized matrix A^0 is actually the one obtained in the framework of perforated domains. It has been originally introduced in [18] for linear problems with Neumann conditions in perforated domain, successively extended to quasilinear ones in [1] and [2].

We recall (see [7] and [18]) that the matrix A^0 satisfies the following properties:

- i) A^0 is continuous and $A^0(t) \in M\left(\alpha, \frac{\beta^2}{\alpha}, \Omega\right)$ for every $t \in \mathbb{R}$;
- ii) there exists a positive constant C , depending only on α, β, Y and T s.t.

$$|A^0(t_1) - A^0(t_2)| \leq C\omega(|t_1 - t_2|)$$
 for every $t_1, t_2 \in \mathbb{R}$, with $t_1 \neq t_2$, where ω is the function given in H_1).

Remark 2.3. Observe that assumptions H_{1ii} and H_{2ii} are only needed for the uniqueness of the solution of problem (2.7). If they do not hold true, the homogenized problem is still the same but all the convergences remain valid only for a subsequence.

The main results. We now state the main results of this work, which will be proved in Section 6.

Theorem 2.4. Under assumptions H_1 – H_3), let $u^\varepsilon \in H_Y^\varepsilon$ be the unique solution of problem (2.7). Then, there exist a subsequence (still denoted by ε), $u_1 \in H_0^1(\Omega)$, $\widehat{u}_1 \in L^2(\Omega, H_{\text{per}}^1(Y_1))$ with $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$ for almost every $x \in \Omega$ and $\bar{u}_2 \in L^2(\Omega, H^1(Y_2))$ with $\mathcal{M}_\Gamma(\bar{u}_2) = 0$ for almost every $x \in \Omega$ such that

$$\left\{ \begin{array}{ll} \text{i)} & \mathcal{T}_i^\varepsilon(u_i^\varepsilon) \rightarrow u_1 \quad \text{strongly in } L^2(\Omega, H^1(Y_i)), i = 1, 2, \\ \text{ii)} & \mathcal{T}^\varepsilon(\zeta(u^\varepsilon)) \rightarrow \zeta(u_1) \quad \text{a.e. in } \Omega \times Y, \\ \text{iii)} & \widetilde{u}_1^\varepsilon \rightharpoonup \theta_1 u_1 \quad \text{weakly in } L^2(\Omega), \\ \text{iv)} & \mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_1 \quad \text{weakly in } L^2(\Omega \times Y_1) \\ \text{v)} & \mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \rightharpoonup \nabla_y \bar{u}_2 \quad \text{weakly in } L^2(\Omega \times Y_2), \end{array} \right. \quad (2.11)$$

and

$$u_1 \geq 0 \text{ a.e. in } \Omega \quad \text{and} \quad \int_\Omega f \zeta(u_1) \varphi \, dx < +\infty, \quad \forall \varphi \in H_0^1(\Omega). \quad (2.12)$$

Moreover, the pair (u_1, \widehat{u}_1) is the unique solution of the unfolded limit equation

$$\left\{ \begin{array}{l} \forall \varphi \in H_0^1(\Omega) \quad \text{and} \quad \forall \psi \in L^2(\Omega; H_{\text{per}}^1(Y_1)) \\ \int_{\Omega \times Y_1} A(y, u_1)(\nabla u_1 + \nabla_y \widehat{u}_1)(\nabla \varphi + \nabla_y \psi) \, dx \, dy = |Y| \int_\Omega f \zeta(u_1) \varphi \, dx. \end{array} \right. \quad (2.13)$$

Proposition 2.5. Under assumptions H_1 – H_3), let $(u_1, \widehat{u}_1, \bar{u}_2)$ be given by Theorem 2.4. Then

$$\left\{ \begin{array}{l} \widehat{u}_1(y, x) = - \sum_{i=1}^N \chi_{e_i}(y, u_1(x)) \frac{\partial u_1}{\partial x_i}(x) \in L^2(\Omega; H_{\text{per}}^1(Y_1)), \\ \nabla_y \bar{u}_2 \equiv 0 \text{ a.e. in } \Omega \times Y_2, \end{array} \right.$$

where $\chi_{e_i}(\cdot, u_1)$, $i = 1, \dots, N$ are the solutions of the cell problems (2.9), written for $\lambda = e_i$.

Then the homogenization result for problem (2.7) is

Theorem 2.6. *Under assumptions H_1 – H_3), let $u^\varepsilon \in H_Y^\varepsilon$ be the unique solution of problem (2.7) and u_1 given by Theorem 2.4. Then $u_1 > 0$ almost everywhere in Ω and u_1 is the unique solution of the following singular limit problem:*

$$\begin{cases} -\operatorname{div}(A^0(u_1)\nabla u_1) = f\zeta(u_1) & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.14)$$

where the homogenized matrix $A^0(t)$ is given by (2.8) and verifies

$$A^0(u_1)\nabla u_1 = \frac{1}{|Y|} \int_{Y_1} A(y, u_1)(\nabla u_1 + \nabla_y \widehat{u}_1) dy. \quad (2.15)$$

Consequently, convergences (2.11) (with $\nabla_y \bar{u}_2 \equiv 0$) hold for the whole sequence.

3. The periodic unfolding method

In this section we give a short presentation of the periodic unfolding method adapted to two-component domains by P. Donato, K.H. Le Nguyen and R. Tardieu in [22]. This method was originally introduced by D. Cioranescu, A. Damlamian and G. Griso in [13] and [14] for fixed domains and extended to perforated ones in [16] and [17].

To this aim, we recall the unfolding operators $\mathcal{T}_1^\varepsilon$ and $\mathcal{T}_2^\varepsilon$ and the boundary unfolding operator $\mathcal{T}_\varepsilon^b$: $\mathcal{T}_1^\varepsilon$ and $\mathcal{T}_\varepsilon^b$ are exactly those ones introduced in [17] for perforated domains, while $\mathcal{T}_2^\varepsilon$ has been introduced for the two-component domain in [22].

Let $z \in \mathbb{R}^N$, we denote by $[z]_Y$ its integer part such that $z - [z]_Y$ belongs to Y and set $\{z\}_Y \doteq z - [z]_Y$. Then, for every positive ε ,

$$x = \varepsilon \left(\left[\frac{x}{\varepsilon} \right]_Y + \left\{ \frac{x}{\varepsilon} \right\}_Y \right) \quad \forall x \in \mathbb{R}^N.$$

Definition 3.1. For any Lebesgue-measurable function ϕ on Ω_i^ε , the unfolding operators $\mathcal{T}_i^\varepsilon$, $i = 1, 2$, are defined as follows:

$$\mathcal{T}_i^\varepsilon(\phi)(x, y) \doteq \begin{cases} \phi(\varepsilon[\frac{x}{\varepsilon}]_Y + \varepsilon y) & \text{a.e. for } (x, y) \in \widehat{\Omega}_\varepsilon \times Y_i, \\ 0 & \text{a.e. for } (x, y) \in \Lambda_\varepsilon \times Y_i. \end{cases} \quad (3.1)$$

For any Lebesgue-measurable function ϕ on Γ^ε , the boundary unfolding operator $\mathcal{T}_\varepsilon^b$ is defined as follows:

$$\mathcal{T}_\varepsilon^b(\phi)(x, y) \doteq \begin{cases} \phi(\varepsilon[\frac{x}{\varepsilon}]_Y + \varepsilon y) & \text{a.e. for } (x, y) \in \widehat{\Omega}_\varepsilon \times \Gamma, \\ 0 & \text{a.e. for } (x, y) \in \Lambda_\varepsilon \times \Gamma. \end{cases} \quad (3.2)$$

Remark 3.2. If ϕ is defined in Ω , we simply write $\mathcal{T}_i^\varepsilon(\phi)$ instead of $\mathcal{T}_i^\varepsilon(\phi_i)$, $i = 1, 2$, for the sake of simplicity. Also we define $\mathcal{T}^\varepsilon(\phi)$ as follows:

$$\mathcal{T}^\varepsilon(\phi)(x, y) \doteq \begin{cases} \mathcal{T}_1^\varepsilon(\phi) & \text{in } \Omega \times Y_1, \\ \mathcal{T}_2^\varepsilon(\phi) & \text{in } \Omega \times Y_2. \end{cases}$$

We now recall the main properties of the unfolding operators.

Proposition 3.3 ([17,22]). *Let $p \in [1, +\infty[$ and $i = 1, 2$.*

1. $\mathcal{T}_i^\varepsilon$ is a linear and continuous operator from $L^p(\Omega_i^\varepsilon)$ to $L^p(\Omega \times Y)$.
2. $\mathcal{T}_i^\varepsilon(\phi\psi) = \mathcal{T}_i^\varepsilon(\phi)\mathcal{T}_i^\varepsilon(\psi)$ for every $\phi, \psi \in L^p(\Omega_i^\varepsilon)$.
3. Let $\phi \in L^p(Y_i)$ be a Y -periodic function and set $\phi_\varepsilon(x) = \phi(\frac{x}{\varepsilon})$. Then

$$\mathcal{T}_i^\varepsilon(\phi_\varepsilon) \rightarrow \phi \quad \text{strongly in } L^p(\Omega \times Y_i).$$

4. For all $\phi \in L^1(\Omega_i^\varepsilon)$, one has

$$\frac{1}{|Y|} \int_{\Omega \times Y_i} \mathcal{T}_i^\varepsilon(\phi)(x, y) dx dy = \int_{\widehat{\Omega}_i^\varepsilon} \phi(x) dx = \int_{\Omega_i^\varepsilon} \phi(x) dx - \int_{\Lambda_i^\varepsilon} \phi(x) dx.$$

5. For all $\phi \in L^1(\Gamma^\varepsilon)$, one has

$$\int_{\Gamma^\varepsilon} \phi(x) d\sigma_x = \frac{1}{\varepsilon|Y|} \int_{\Omega \times \Gamma} \mathcal{T}_\varepsilon^b(\phi)(x, y) dx d\sigma_y.$$

6. $\|\mathcal{T}_i^\varepsilon(\phi)\|_{L^p(\Omega \times Y_i)} \leq |Y|^{\frac{1}{p}} \|\phi\|_{L^p(\Omega_i^\varepsilon)}$ for every $\phi \in L^p(\Omega_i^\varepsilon)$.
7. $\|\mathcal{T}_\varepsilon^b(\phi)\|_{L^p(\Omega \times \Gamma)} \leq \varepsilon^{\frac{1}{p}} |Y|^{\frac{1}{p}} \|\phi\|_{L^p(\Gamma^\varepsilon)}$ for every $\phi \in L^p(\Gamma^\varepsilon)$.
8. For $\phi \in L^p(\Omega)$, $\mathcal{T}_i^\varepsilon(\phi) \rightarrow \phi$ strongly in $L^p(\Omega \times Y_i)$.
9. Let $\{\phi_\varepsilon\}$ be a sequence in $L^p(\Omega)$ such that $\phi_\varepsilon \rightarrow \phi$ strongly in $L^p(\Omega)$. Then

$$\mathcal{T}_i^\varepsilon(\phi_\varepsilon) \rightarrow \phi \quad \text{strongly in } L^p(\Omega \times Y_i).$$

10. Let $\{\phi_\varepsilon\}$ be a sequence in $L^p(\Omega_i^\varepsilon)$ such that $\|\phi_\varepsilon\|_{L^p(\Omega_i^\varepsilon)} \leq c$.

If $\mathcal{T}_i^\varepsilon(\phi_\varepsilon) \rightarrow \widehat{\phi}$ weakly in $L^p(\Omega \times Y_i)$, then

$$\widetilde{\phi}_\varepsilon \rightarrow \theta_i \mathcal{M}_{Y_i}(\widehat{\phi}) \quad \text{weakly in } L^p(\Omega).$$

11. If $\phi \in W^{1,p}(\Omega_i^\varepsilon)$, then $\nabla_y[\mathcal{T}_i^\varepsilon(\phi)] = \varepsilon \mathcal{T}_i^\varepsilon(\nabla \phi)$ and $\mathcal{T}_i^\varepsilon(\phi) \in L^p(\Omega, W^{1,p}(Y_i))$.
12. If $\phi \in L^s(\Gamma^\varepsilon)$ for $s \in [1, +\infty[$, then

$$\|\mathcal{T}_\varepsilon^b(\phi)\|_{L^s(\Omega \times \Gamma)} \leq |Y|^{\frac{1}{s}} \varepsilon^{\frac{1}{s}} \|\phi\|_{L^s(\Gamma^\varepsilon)}.$$

We state below the main propositions proved in [12] and [22] concerning the jump on the interface and some convergence results, under the same notations as in [21].

Lemma 3.4 ([22]). *Let $\varphi \in \mathcal{D}(\Omega)$ and $u^\varepsilon \in H_\gamma^\varepsilon$. For ε small enough one has*

$$\varepsilon \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) \varphi \, d\sigma_x = \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y) (\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon)) \mathcal{T}_1^\varepsilon(\varphi) \, dx \, d\sigma_y.$$

Theorem 3.5 ([12]). [17,21,22] *Let $\gamma \in \mathbb{R}$ and u^ε be a bounded sequence in H_γ^ε . Then,*

$$\begin{aligned} \|\mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon)\|_{L^2(\Omega \times Y_1)} &\leq c, \\ \|\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon)\|_{L^2(\Omega \times Y_2)} &\leq c, \\ \|\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon)\|_{L^2(\Omega \times \Gamma)} &\leq c\varepsilon^{\frac{1-\gamma}{2}}, \end{aligned} \tag{3.3}$$

and there exist a subsequence (still denoted by ε), $u_1 \in H_0^1(\Omega)$ and $\widehat{u}_1 \in L^2(\Omega, H_{\text{per}}^1(Y_1))$ with $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$ for almost every $x \in \Omega$ such that

$$\begin{cases} \mathcal{T}_1^\varepsilon(u_1^\varepsilon) \rightarrow u_1 & \text{strongly in } L^2(\Omega, H^1(Y_1)), \\ \mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon) \rightarrow \nabla u_1 + \nabla_y \widehat{u}_1 & \text{weakly in } L^2(\Omega \times Y_1). \end{cases} \tag{3.4}$$

Moreover, if $\gamma \leq 1$, there exist a subsequence (still denoted by ε), $u_2 \in L^2(\Omega)$ and $\bar{u}_2 \in L^2(\Omega, H^1(Y_2))$ with $\mathcal{M}_\Gamma(\bar{u}_2) = 0$ for almost every $x \in \Omega$ such that

$$\begin{cases} \mathcal{T}_2^\varepsilon(u_2^\varepsilon) \rightharpoonup u_2 & \text{weakly in } L^2(\Omega, H^1(Y_2)), \\ \mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \rightharpoonup \nabla_y \bar{u}_2 & \text{weakly in } L^2(\Omega \times Y_2). \end{cases} \tag{3.5}$$

Furthermore, if $\gamma < 1$, then $u_1 = u_2$ and also

$$\mathcal{T}_2^\varepsilon(u_2^\varepsilon) \rightarrow u_1 \text{ strongly in } L^2(\Omega, H^1(Y_2)). \tag{3.6}$$

4. A priori estimates

In this section we prove two uniform a priori estimates (with respect to ε) for a solution of problem (2.7). In addition, we recall Proposition 4.3 which gives a bound to the integral of the singular term close to the singularity.

These estimates allow us to prove the first part of Theorem 2.4 at the end of this section.

Proposition 4.1. *Under assumptions H_1 – H_3 , let $u^\varepsilon \in H_\gamma^\varepsilon$ be the solution of problem (2.7). The following a priori estimate holds true:*

$$\|\nabla u^\varepsilon\|_{L^2(\Omega \setminus \Gamma^\varepsilon)} \leq C_1 \|f\|_{L^{\frac{2}{1+k}}(\Omega)}, \tag{4.1}$$

where C_1 depends on α , c_p and c_1 .

Also

$$\|u_1^\varepsilon - u_2^\varepsilon\|_{L^2(\Gamma^\varepsilon)} \leq \varepsilon^{-\frac{\gamma}{2}} C_2 \|f\|_{L^{\frac{2}{1+k}}(\Omega)}, \quad (4.2)$$

where C_2 depends on α , h_0 , c_p , c_1 .

Proof. Let $u^\varepsilon \in H_\gamma^\varepsilon$ be the solution of problem (2.7) and let us choose u^ε as test function. By H_1)- H_3) and applying the Young inequality with exponents $\frac{2}{1-k}$ and $\frac{2}{1+k}$, we get for every $\eta > 0$

$$\begin{aligned} & \alpha \|\nabla u^\varepsilon\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 + \varepsilon^\gamma h_0 \int_{\Gamma^\varepsilon} (u_1^\varepsilon - u_2^\varepsilon)^2 d\sigma \\ & \leq \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \nabla u^\varepsilon dx + \int_{\Gamma^\varepsilon} \varepsilon^\gamma h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon)^2 d\sigma \\ & = \int_{\Omega} f \zeta(u^\varepsilon) u^\varepsilon dx \leq \int_{\Omega} f u^{1-k} dx \leq \eta \|u^\varepsilon\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 + c(\eta) \|f\|_{L^{\frac{2}{1+k}}(\Omega)}^{\frac{2}{1+k}}. \end{aligned}$$

In view of Remark 2.1, the above inequality leads to

$$\alpha \|\nabla u^\varepsilon\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 + \varepsilon^\gamma h_0 \int_{\Gamma^\varepsilon} (u_1^\varepsilon - u_2^\varepsilon)^2 d\sigma \leq c(\eta, c_p) \|u^\varepsilon\|_{V^\varepsilon \times H^1(\Omega_2^\varepsilon)}^2 + c(\eta) \|f\|_{L^{\frac{2}{1+k}}(\Omega)}^{\frac{2}{1+k}}.$$

Thanks to Proposition 2.2 we get

$$\alpha \|\nabla u^\varepsilon\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 + \varepsilon^\gamma h_0 \int_{\Gamma^\varepsilon} (u_1^\varepsilon - u_2^\varepsilon)^2 d\sigma \leq c(\eta, c_p, c_1) \|u^\varepsilon\|_{H_\gamma^\varepsilon}^2 + c(\eta) \|f\|_{L^{\frac{2}{1+k}}(\Omega)}^{\frac{2}{1+k}}.$$

So that,

$$(\alpha - c(\eta, c_p, c_1)) \|\nabla u^\varepsilon\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 + \varepsilon^\gamma (h_0 - c(\eta, c_p, c_1)) \|u_1^\varepsilon - u_2^\varepsilon\|_{L^2(\Gamma^\varepsilon)}^2 \leq c(\eta) \|f\|_{L^{\frac{2}{1+k}}(\Omega)}^{\frac{2}{1+k}}.$$

Whence, choosing η sufficiently small so that $\alpha - c(\eta, c_p, c_1) \geq 0$ and $h_0 - c(\eta, c_p, c_1) \geq 0$, we deduce the result from the previous estimate. \square

Proposition 4.2. Under assumptions H_1)- H_3), let $u^\varepsilon \in H_\gamma^\varepsilon$ be the solution of problem (2.7). Then

$$\|f \zeta(u^\varepsilon) \varphi\|_{L^1(\Omega)} \leq c,$$

for every nonnegative $\varphi \in H_\gamma^\varepsilon$ with c depending on α , β , c_p , c_1 , $\|\nabla \varphi\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}$ and $\|f\|_{L^1(\Omega)}$.

Proof. Let $u^\varepsilon \in H_\gamma^\varepsilon$ be the solution of problem (2.7) and let us choose a nonnegative $\varphi \in H_0^1(\Omega)$ as test function. Since φ has no jump on Γ^ε , the boundary term vanishes. Hence, by using the Hölder inequality and nonnegativity of f , ζ and φ , we have

$$0 \leq \int_{\Omega} f \zeta(u^\varepsilon) \varphi dx = \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \nabla \varphi dx \leq \beta \|\nabla u^\varepsilon\|_{L^2(\Omega \setminus \Gamma^\varepsilon)} \|\nabla \varphi\|_{L^2(\Omega)},$$

which implies the result for $\varphi \in H_0^1(\Omega)$, via the estimate (4.1). If now $\varphi = (\varphi_1^\varepsilon, \varphi_2^\varepsilon) \in H_\gamma^\varepsilon$ is nonnegative, since Γ_ε is Lipschitz continuous, there exist nonnegative ψ_1 and ψ_2 in $H_0^1(\Omega)$ such that $\varphi = (\varphi_1^\varepsilon, \varphi_2^\varepsilon) = (\psi_1|_{\Omega_1^\varepsilon}, \psi_2|_{\Omega_2^\varepsilon})$. Then

$$\begin{aligned} 0 &\leq \int_{\Omega} f \zeta(u^\varepsilon) \varphi \, dx = \int_{\Omega_1^\varepsilon} f \zeta(u^\varepsilon) \varphi_1^\varepsilon \, dx + \int_{\Omega_2^\varepsilon} f \zeta(u^\varepsilon) \varphi_2^\varepsilon \, dx \\ &\leq \int_{\Omega} f \zeta(u^\varepsilon) \psi_1 \, dx + \int_{\Omega} f \zeta(u^\varepsilon) \psi_2 \, dx \leq c. \end{aligned}$$

□

We also recall the following result from [26] (written for $h = \varepsilon^\gamma h^\varepsilon$ and $\lambda = 0$). It gives an estimate of the integral of the singular term close to the singular set $\{u^\varepsilon = 0\}$.

Proposition 4.3 ([26]). *Under assumptions H_1 – H_3), let $u^\varepsilon \in H_\gamma^\varepsilon$ be the solution of problem (2.7) and δ a fixed positive real number. Then*

$$\begin{aligned} \int_{\{0 < u^\varepsilon \leq \delta\}} f \zeta(u^\varepsilon) \varphi \, dx &\leq \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \nabla \varphi Z_\delta(u^\varepsilon) \, dx \\ &\quad + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) (Z_\delta(u_1^\varepsilon) \varphi_1 - Z_\delta(u_2^\varepsilon) \varphi_2) \, d\sigma, \end{aligned}$$

for every $\varphi \in H_\gamma^\varepsilon$, $\varphi \geq 0$, where Z_δ is an auxiliary function defined by

$$Z_\delta(s) = \begin{cases} 1, & \text{if } 0 \leq s \leq \delta, \\ -\frac{s}{\delta} + 2, & \text{if } \delta \leq s \leq 2\delta, \\ 0, & \text{if } s \geq 2\delta. \end{cases} \quad (4.3)$$

Proposition 4.4. *Under assumptions H_1 – H_3), let $u^\varepsilon \in H_\gamma^\varepsilon$ be the unique solution of problem (2.7). Then, there exist a subsequence (still denoted by ε), $u_1 \in H_0^1(\Omega)$, $\widehat{u}_1 \in L^2(\Omega, H_{\text{per}}^1(Y_1))$ with $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$ for almost every $x \in \Omega$ and $\overline{u}_2 \in L^2(\Omega, H^1(Y_2))$ with $\mathcal{M}_\Gamma(\overline{u}_2) = 0$ for almost every $x \in \Omega$ such that (2.11) and (2.12) hold.*

Proof. Let u^ε be the solution of problem (2.7). Proposition 4.1 allows us to apply Theorem 3.5. It provides the existence of $u_1 \in H_0^1(\Omega)$, $\widehat{u}_1 \in L^2(\Omega, H_{\text{per}}^1(Y_1))$ with $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$ for almost every $x \in \Omega$ and $\overline{u}_2 \in L^2(\Omega, H^1(Y_2))$ with $\mathcal{M}_\Gamma(\overline{u}_2) = 0$ for almost every $x \in \Omega$ such that, up to a subsequence, one has convergences (2.11)_{i,iv,v}, since $\gamma < 1$.

Now observe that, by construction, for every $x \in \Omega$ there exists $\varepsilon_x > 0$ such that

$$x \in \widehat{\Omega}_\varepsilon, \quad \forall \varepsilon \leq \varepsilon_x.$$

Also, by definition of unfolding one has, for $i = 1, 2$

$$\mathcal{T}_i^\varepsilon(\zeta(\phi))(x, y) = \begin{cases} \zeta(\mathcal{T}_i^\varepsilon(\phi)(x, y)) & \text{a.e. in } \widehat{\Omega}_\varepsilon \times Y_i, \\ 0 & \text{a.e. in } \Lambda_\varepsilon \times Y_i. \end{cases} \quad (4.4)$$

Consequently, for almost every $(x, y) \in \Omega \times Y$, there exists $\varepsilon_x > 0$ such that

$$\mathcal{T}^\varepsilon(\zeta(u^\varepsilon)) = \zeta(\mathcal{T}^\varepsilon(u^\varepsilon)), \quad \forall \varepsilon \leq \varepsilon_x. \quad (4.5)$$

On the other hand, using the continuity of ζ and (2.11)_i, we have

$$\zeta(\mathcal{T}^\varepsilon(u^\varepsilon)) \rightarrow \zeta(u_1) \quad \text{a.e. in } \Omega \times Y. \quad (4.6)$$

This, together with (4.5), gives the convergence (2.11)_{ii}.

Also, from Proposition 2.2 and convergence (2.11)_i, Proposition 3.3₁₀ gives

$$\tilde{u}_1^\varepsilon \rightharpoonup \theta_1 \mathcal{M}_{Y_1}(u_1) \quad \text{weakly in } L^2(\Omega),$$

which reads as (2.11)_{iii} being u_1 independent on y .

To show that u_1 is nonnegative almost everywhere in Ω , we note that every solution u^ε is positive almost everywhere in Ω . Then, the definition of the unfolding operator implies $\mathcal{T}^\varepsilon(u^\varepsilon) \geq 0$ almost everywhere in $\Omega \times Y$ so that, in view of (2.11)_i,

$$u_1 \geq 0 \quad \text{a.e. in } \Omega.$$

It remains to prove the second condition in (2.12). Let us choose first a nonnegative $\varphi \in H_0^1(\Omega)$. Propositions 3.3_{2,4} and 4.2, for the subsequence mentioned before, lead to

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}^\varepsilon(f) \mathcal{T}^\varepsilon(\zeta(u^\varepsilon)) \mathcal{T}^\varepsilon(\varphi) \, dx \, dy \leq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} f \zeta(u^\varepsilon) \varphi \, dx < +\infty. \quad (4.7)$$

Now, from Proposition 3.3₈, $\mathcal{T}^\varepsilon(f)$ and $\mathcal{T}^\varepsilon(\varphi)$ converge to f and φ , respectively, almost everywhere in $\Omega \times Y$, up to a subsequence. Thus, by (2.11)_{ii},

$$\mathcal{T}^\varepsilon(f) \mathcal{T}^\varepsilon(\zeta(u^\varepsilon)) \mathcal{T}^\varepsilon(\varphi) \rightarrow f \zeta(u_1) \varphi \quad \text{a.e. in } \Omega \times Y.$$

Since $\mathcal{T}^\varepsilon(f)$, $\mathcal{T}^\varepsilon(\zeta(u^\varepsilon))$ and $\mathcal{T}^\varepsilon(\varphi)$ are nonnegative functions, we can use Fatou's lemma and (4.7) to obtain

$$\frac{1}{|Y|} \int_{\Omega \times Y} f \zeta(u_1) \varphi \, dx \, dy \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}^\varepsilon(f) \mathcal{T}^\varepsilon(\zeta(u^\varepsilon)) \mathcal{T}^\varepsilon(\varphi) \, dx \, dy < +\infty.$$

Being the functions f and u_1 independent on y , this implies in particular that

$$\int_{\Omega} f \zeta(u_1) \varphi \, dx < +\infty$$

and ends the proof for $\varphi \geq 0$. For φ with any sign, it suffices to decompose φ as $\varphi^+ - \varphi^-$. \square

5. A convergence result for an auxiliary problem

This section is devoted to the study of a suitable auxiliary problem and a crucial convergence result, which is one of the main tools needed for proving Theorem 2.6. In the same spirit of what we have done in [24] in the case of a periodically perforated domain, we first consider the auxiliary linear problem (5.2) below and state existence and homogenization results for it. Then, we prove the convergence result given in Theorem 5.4. It shows that the gradient of u^ε is equivalent to the gradient of the solution of the auxiliary linear problem, associated with a weak cluster point of the sequence $\{u^\varepsilon\}$, as $\varepsilon \rightarrow 0$.

We refer to [10] and [22] for the study of the auxiliary problem. For the proof of the convergence result we adapt some techniques from [20] and [24], which inspired this work.

First of all, as in [10] we introduce a linear operator \mathcal{L}_ε from $H^{-1}(\Omega)$ to $(V^\varepsilon)'$ verifying the following assumption:

H₄) If $\{\varphi^\varepsilon\}$ is a sequence such that

$$\|\varphi^\varepsilon\|_{V^\varepsilon} \leq c \quad \text{and} \quad \tilde{\varphi}^\varepsilon \rightharpoonup \theta_1 \varphi_0 \quad \text{weakly in } L^2(\Omega), \quad (5.1)$$

then

$$\lim_{\varepsilon \rightarrow 0} \langle \mathcal{L}_\varepsilon(Z), \varphi^\varepsilon \rangle_{(V^\varepsilon)', V^\varepsilon} = \langle Z, \varphi_0 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

Let us point out that assumption H₄) is satisfied, for example, by the adjoint of the linear operator P_1^ε introduced by Cioranescu and Saint Jean Paulin in [18]. Then, see [11, Remark 3.1], if H₄) holds, one has $\varphi_0 \in H_0^1(\Omega)$.

5.1. The auxiliary problem

The suitable auxiliary problem we are interested in is the following one:

$$\begin{cases} -\operatorname{div}(A^\varepsilon(x, u_1^\varepsilon) \nabla v_1^\varepsilon) = \mathcal{L}_\varepsilon(Z) & \text{in } \Omega_1^\varepsilon, \\ -\operatorname{div}(A^\varepsilon(x, u_2^\varepsilon) \nabla v_2^\varepsilon) = 0 & \text{in } \Omega_2^\varepsilon, \\ v_1^\varepsilon = 0 & \text{on } \partial\Omega, \\ (A^\varepsilon(x, u_1^\varepsilon) \nabla v_1^\varepsilon) v^\varepsilon = (A^\varepsilon(x, u_2^\varepsilon) \nabla v_2^\varepsilon) v^\varepsilon & \text{on } \Gamma^\varepsilon, \\ (A^\varepsilon(x, u_1^\varepsilon) \nabla v_1^\varepsilon) v^\varepsilon = -\varepsilon^\gamma h^\varepsilon(v_1^\varepsilon - v_2^\varepsilon) & \text{on } \Gamma^\varepsilon, \end{cases} \quad (5.2)$$

under assumptions H₁), H₃)–H₄), and $Z \in H^{-1}(\Omega)$.

The variational formulation of problem (5.2) is

$$\begin{cases} \text{Find } v^\varepsilon \in H_\gamma^\varepsilon \text{ such that} \\ \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla v^\varepsilon \nabla \varphi \, dx + \int_{\Gamma^\varepsilon} \varepsilon^\gamma h^\varepsilon(v_1^\varepsilon - v_2^\varepsilon)(\varphi_1 - \varphi_2) \, d\sigma \\ = \langle \mathcal{L}_\varepsilon(Z), \varphi_1 \rangle_{(V^\varepsilon)', V^\varepsilon}, \quad \forall \varphi \in H_\gamma^\varepsilon. \end{cases} \quad (5.3)$$

The existence and the uniqueness of a solution of problem (5.3) is a straightforward consequence of the Lax-Milgram theorem.

Observe now that in view of Proposition 4.1 and (2.11)_{iii}, the sequence $\{u^\varepsilon\}$ satisfies conditions (5.1), i.e.

$$\|u_1^\varepsilon\|_{V^\varepsilon} \leq c \quad \text{and} \quad \tilde{u}_1^\varepsilon \rightharpoonup \theta_1 u_1 \quad \text{weakly in } L^2(\Omega). \quad (5.4)$$

The homogenization result below extends Theorem 3.3 of [22] to the case where the matrix field depends on x . We omit its proof, which follows along the line of [22, Theorem 3.3]. The terms containing the functional Z and the quasilinearity of A can be treated as in [10] and [7], respectively.

Theorem 5.1. *Under assumptions $H_1), H_3)-H_4)$ and $Z \in H^{-1}(\Omega)$, let $v^\varepsilon \in H_\gamma^\varepsilon$ be the unique solution of problem (5.3). Then, there exist $v_1 \in H_0^1(\Omega)$ and $\widehat{v}_1 \in L^2(\Omega, H_{\text{per}}^1(Y_1))$ with $\mathcal{M}_\Gamma(\widehat{v}_1) = 0$ such that*

$$\begin{cases} i) \tilde{v}_i^\varepsilon \rightharpoonup \theta_i v_1 & \text{weakly in } L^2(\Omega), \quad i = 1, 2, \\ ii) \mathcal{T}_i^\varepsilon(v_i^\varepsilon) \rightarrow v_1 & \text{strongly in } L^2(\Omega; H^1(Y_i)), \quad i = 1, 2 \\ iii) \mathcal{T}_1^\varepsilon(\nabla v_1^\varepsilon) \rightharpoonup \nabla v_1 + \nabla_y \widehat{v}_1 & \text{weakly in } L^2(\Omega \times Y_1), \\ iv) \mathcal{T}_2^\varepsilon(\nabla v_2^\varepsilon) \rightharpoonup 0 & \text{weakly in } L^2(\Omega \times Y_2), \end{cases} \quad (5.5)$$

and the pair (v_1, \widehat{v}_1) is the unique solution of the limit equation

$$\begin{cases} \forall \varphi \in H_0^1(\Omega) \quad \text{and} \quad \forall \psi \in L^2(\Omega; H_{\text{per}}^1(Y_1)), \\ \int_{\Omega \times Y_1} A(y, u_1)(\nabla v_1 + \nabla_y \widehat{v}_1)(\nabla \varphi + \nabla_y \psi) \, dx \, dy = |Y| \langle Z, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \end{cases}$$

Finally, v_1 is the unique solution of the following limit problem:

$$\begin{cases} -\operatorname{div}(A^0(u_1)\nabla v_1) = Z & \text{in } \Omega, \\ v_1 = 0 & \text{on } \partial\Omega, \end{cases} \quad (5.6)$$

where the homogenized matrix A^0 is given by (2.8).

Let us consider problem (5.2) for $Z = -\operatorname{div}(A^0(u_1)\nabla u_1)$. Its variational formulation is

$$\begin{aligned} & \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla v^\varepsilon \nabla \varphi \, dx + \int_{\Gamma^\varepsilon} \varepsilon^\gamma h^\varepsilon(v_1^\varepsilon - v_2^\varepsilon)(\varphi_1 - \varphi_2) \, d\sigma \\ & = \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_1)\nabla u_1)), \varphi_1 \rangle_{(V^\varepsilon)^\gamma, V^\varepsilon}, \quad \forall \varphi \in H_\gamma^\varepsilon, \end{aligned} \quad (5.7)$$

which admits a unique solution $v^\varepsilon \in H_\gamma^\varepsilon$. Theorem 5.1 (written for $Z = -\operatorname{div}(A^0(u_1)\nabla u_1)$) gives

$$\begin{cases} -\operatorname{div}(A^0(u_1)\nabla v_1) = -\operatorname{div}(A^0(u_1)\nabla u_1) & \text{in } \Omega, \\ v_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Thanks to the assumptions on A , the uniqueness of this problem ensures that

$$v_1 = u_1. \quad (5.8)$$

Moreover, the same arguments used to prove (2.11)_{iii} give

$$\|v^\varepsilon\|_{H_Y^\varepsilon} \leq c \quad \text{and} \quad \widetilde{v}_1^\varepsilon \rightharpoonup \theta_1 u_1 \quad \text{weakly in } L^2(\Omega). \quad (5.9)$$

As in [20] we define now the following auxiliary functions u_m :

$$\forall m \in \mathbb{N}, m \geq 1, \quad u_m \doteq T_m(u_1),$$

where T_m is the usual truncation function at level m , so that

$$0 \leq u_m \leq u_1 \quad \text{and} \quad u_m \rightarrow u_1 \quad \text{strongly in } H_0^1(\Omega), \quad \text{as } m \rightarrow +\infty. \quad (5.10)$$

Then, we denote $v_\varepsilon^m \in H_0^1(\Omega)$ the solution of the following problem:

$$\begin{cases} -\operatorname{div}(A^\varepsilon(x, u_1^\varepsilon) \nabla v_{\varepsilon,1}^m) = \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_1) \nabla u_m)) & \text{in } \Omega_1^\varepsilon, \\ -\operatorname{div}(A^\varepsilon(x, u_2^\varepsilon) \nabla v_{\varepsilon,2}^m) = 0 & \text{in } \Omega_2^\varepsilon, \\ v_\varepsilon^m = 0 & \text{on } \partial\Omega. \end{cases}$$

Its variational formulation is

$$\int_{\Omega} A^\varepsilon(x, u^\varepsilon) \nabla v_\varepsilon^m \nabla \varphi \, dx = \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_1) \nabla u_m)), \varphi_1 \rangle_{(V^\varepsilon)', V^\varepsilon}, \quad \forall \varphi \in H_Y^\varepsilon. \quad (5.11)$$

Again, the existence and the uniqueness of a solution of problem (5.11) is straightforward proved by using the Lax-Milgram theorem.

Remark 5.2. Let us notice that $v_\varepsilon^m \in H_0^1(\Omega)$, and consequently $v_{\varepsilon,1}^m = v_{\varepsilon,2}^m$ on Γ^ε , for every ε .

Thus, in particular, Theorem 5.1 applies to this case with no jump (for $Z = -\operatorname{div}(A^0(u_1) \nabla u_m)$) and, again by uniqueness, we obtain

$$\|v_\varepsilon^m\|_{H_Y^\varepsilon} \leq c \quad \text{and} \quad \widetilde{v}_{\varepsilon,1}^m \rightharpoonup \theta_1 u_m \quad \text{weakly in } L^2(\Omega). \quad (5.12)$$

Also the following convergence holds true:

$$\mathcal{T}^\varepsilon(v_\varepsilon^m) \rightarrow u_m \quad \text{strongly in } L^2(\Omega; H^1(Y)). \quad (5.13)$$

Moreover, by classical results from [31], we have that for every fixed m

$$\|v_\varepsilon^m\|_{L^\infty(\Omega)} \leq c_m, \quad \text{for every } \varepsilon. \quad (5.14)$$

Remark 5.3. The sequence $\{(v_{\varepsilon,1}^m)^-\}$ satisfies conditions (5.1). Indeed, in view of the estimate in (5.12) and convergence (5.13), one has

$$\|(v_{\varepsilon,1}^m)^-\|_{V_\varepsilon} \leq \|v_{\varepsilon,1}^m\|_{V_\varepsilon} \leq c,$$

1 and

$$2 \quad \mathcal{T}_1^\varepsilon((v_{\varepsilon,1}^m)^-) \rightarrow u_m^- = 0 \quad \text{strongly in } L^2(\Omega \times Y_1), \quad 3$$

4 since $u_m \geq 0$ by construction. Also, by using Proposition 3.3₁₀, one has

$$5 \quad \widetilde{(v_{\varepsilon,1}^m)^-} \rightharpoonup \theta_1 \mathcal{M}_{Y_1}(u_m^-) = 0 \quad \text{weakly in } L^2(\Omega). \quad 6$$

7 Let us also point out that, from (5.13), one has even

$$8 \quad \mathcal{T}_2^\varepsilon((v_{\varepsilon,2}^m)^-) \rightarrow u_m^- = 0 \quad \text{strongly in } L^2(\Omega \times Y_2). \quad 9$$

10 This is needed in the next subsection.

11 5.2. A convergence result

12 We are now able to prove the main result of this section. Here we adapt the arguments we used in [24] for the quasilinear singular case in periodically perforated domains to the two-component case, where the holes are replaced by the second component. This is why we analyze in detail only the terms differing from the previous work, namely the boundary term and the quasilinear diffusion term in the second component.

13 The proof follows the same steps introduced in that of [20, Theorem 8.5], which concerns the singular case when A is linear and the domain is made up of two connected components separated by an oscillating interface.

14 **Theorem 5.4.** *Under assumptions $H_1)$ - $H_4)$, let u^ε and v^ε be solutions of problems (2.7) and (5.7), respectively. Then, up to a subsequence,*

$$15 \quad \lim_{\varepsilon \rightarrow 0} \|\nabla u^\varepsilon - \nabla v^\varepsilon\|_{L^2(\Omega \setminus \Gamma^\varepsilon)} = 0. \quad 16$$

17 **Proof.** The proof is done in 3 steps.

18 **Step 1.** Let us prove that

$$19 \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Gamma^\varepsilon} |\nabla (v_\varepsilon^m)^-|^2 dx = 0 \quad \forall m \geq 1. \quad 20 \quad (5.15)$$

21 Let us choose $-(v_\varepsilon^m)^- \in H_\gamma^\varepsilon$ as test function in (5.11). From the ellipticity of A , taking into account that $\nabla v_\varepsilon^m = \nabla (v_\varepsilon^m)^+ - \nabla (v_\varepsilon^m)^-$, and (2.3) we get

$$22 \quad 0 \leq \alpha \|\nabla (v_\varepsilon^m)^-\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 \leq - \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla v_\varepsilon^m \nabla (v_\varepsilon^m)^- dx$$

$$23 \quad = \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_1) \nabla u_m)), -(v_{\varepsilon,1}^m)^- \rangle_{(V^\varepsilon)', V^\varepsilon}.$$

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From Remark 5.3 and H₄), we obtain

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_1)\nabla u_m)), -(v_{\varepsilon,1}^m)^- \rangle_{(V^\varepsilon)', V^\varepsilon} \\ &= \langle -\operatorname{div}(A^0(u_1)\nabla u_m), 0 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0, \end{aligned}$$

which concludes the first step.

Step 2. In this step we show that

$$\lim_{m \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Gamma^\varepsilon} |\nabla(u^\varepsilon - v_\varepsilon^m)|^2 dx = 0. \quad (5.16)$$

To do that, let us choose $u^\varepsilon - v_\varepsilon^m \in H_\gamma^\varepsilon$ as test function in (2.7) and (5.11). By subtraction, H₁)_i and the nonnegativity of the boundary term, one has

$$\begin{aligned} & \alpha \|\nabla(u^\varepsilon - v_\varepsilon^m)\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 \\ & \leq \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla(u^\varepsilon - v_\varepsilon^m) \nabla(u^\varepsilon - v_\varepsilon^m) dx + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon)^2 d\sigma \\ & = \int_{\Omega} f\zeta(u^\varepsilon)(u^\varepsilon - v_\varepsilon^m) dx - \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_1)\nabla u_m)), (u^\varepsilon - v_\varepsilon^m)_1 \rangle_{V_\varepsilon', V_\varepsilon}. \end{aligned} \quad (5.17)$$

Let us prove that, as $\varepsilon \rightarrow 0$, we have

$$\begin{aligned} \alpha \limsup_{\varepsilon \rightarrow 0} \|\nabla(u^\varepsilon - v_\varepsilon^m)\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 & \leq \int_{\Omega} f\zeta(u_1)(u_1 - u_m) \chi_{\{u_1 > 0\}} dx \\ & \quad - \langle (-\operatorname{div}(A^0(u_1)\nabla u_m)), u_1 - u_m \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \end{aligned} \quad (5.18)$$

so that we obtain the result (5.16) as $m \rightarrow +\infty$, via convergence (5.10) and the Lebesgue theorem.

Concerning the second term in the right-hand side of the previous inequality, let us observe that $u^\varepsilon - v_\varepsilon^m$ satisfies (5.1), in view of (5.4) and (5.12). So that, by H₄)

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_1)\nabla u_m)), (u^\varepsilon - v_\varepsilon^m)_1 \rangle_{V_\varepsilon', V_\varepsilon} \\ &= \langle -\operatorname{div}(A^0(u_1)\nabla u_m), u_1 - u_m \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \end{aligned} \quad (5.19)$$

Let $\delta > 0$. We split the integral of the singular term in (5.17) in two terms as follows:

$$\begin{aligned} \int_{\Omega} f\zeta(u^\varepsilon)(u^\varepsilon - v_\varepsilon^m) dx &= \int_{\Omega} f\zeta(u^\varepsilon)(u^\varepsilon - (v_\varepsilon^m)^+) dx + \int_{\Omega} f\zeta(u^\varepsilon)(v_\varepsilon^m)^- dx \\ &\leq \sum_{i=1}^2 (I_{\varepsilon,i}^\delta + J_{\varepsilon,i}^\delta) + K_\varepsilon, \quad i = 1, 2, \end{aligned} \quad (5.20)$$

where, for $i = 1, 2$,

$$I_{\varepsilon,i}^{\delta} \doteq \int_{\Omega_{\varepsilon} \cap \{0 < u_i^{\varepsilon} \leq \delta\}} f \zeta(u_i^{\varepsilon}) u_i^{\varepsilon} dx, \quad J_{\varepsilon,i}^{\delta} \doteq \int_{\Omega_{\varepsilon} \cap \{u_i^{\varepsilon} > \delta\}} f \zeta(u_i^{\varepsilon}) (u_i^{\varepsilon} - (v_{\varepsilon,i}^m)^+) dx, \quad \text{and}$$

$$K_{\varepsilon} \doteq \int_{\Omega} f \zeta(u^{\varepsilon}) (v_{\varepsilon}^m)^- dx.$$

The terms corresponding to $i = 1$ are exactly the ones considered in [24, Theorem 5.5] for the case of periodically perforated domains. It is easy to check that the additional terms $I_{\varepsilon,2}^{\delta}$ and $J_{\varepsilon,2}^{\delta}$ can be treated exactly in the same way. Indeed, all the convergences used there for the first component hold true also here in the second component, thanks to (2.1) and convergence (3.6).

Hence,

$$\begin{cases} \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} (I_{\varepsilon,1}^{\delta} + I_{\varepsilon,2}^{\delta}) = 0, \\ \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} (J_{\varepsilon,1}^{\delta} + J_{\varepsilon,2}^{\delta}) = (\theta_1 + \theta_2) \int_{\Omega} f \zeta(u_1) (u_1 - u_m) \chi_{\{u_1 > 0\}} dx \\ \quad = \int_{\Omega} f \zeta(u_1) (u_1 - u_m) \chi_{\{u_1 > 0\}} dx. \end{cases} \quad (5.21)$$

On the contrary, the term K_{ε} needs to be treated specifically, since its computation gives rise to a different boundary term. To this aim, let us observe that

$$\int_{\Omega \cap \{u^{\varepsilon} > \delta\}} f \zeta(u^{\varepsilon}) (v_{\varepsilon}^m)^- \chi_{\{u_1 = \delta\}} dx = 0,$$

for every $\delta \in \mathbb{R}^+ \setminus D$, where D is a countable set of values (see for instance [20,27] and [24]).

Hence, for $\delta_0 \in \mathbb{R}^+ \setminus D$, we can write

$$K_{\varepsilon} = \int_{\Omega \cap \{0 < u^{\varepsilon} \leq \delta_0\}} f \zeta(u^{\varepsilon}) (v_{\varepsilon}^m)^- dx + \int_{\Omega \cap \{u^{\varepsilon} > \delta_0\}} f \zeta(u^{\varepsilon}) (v_{\varepsilon}^m)^- \chi_{\{u_1 \neq \delta_0\}} dx \doteq A_{\varepsilon} + B_{\varepsilon}. \quad (5.22)$$

From Proposition 4.3 written with $\delta = \delta_0$ we get

$$\begin{aligned} 0 &\leq A_{\varepsilon} \\ &\leq \int_{\Omega \setminus \Gamma^{\varepsilon}} A^{\varepsilon}(x, u^{\varepsilon}) \nabla u^{\varepsilon} \nabla (v_{\varepsilon}^m)^- Z_{\delta_0}(u^{\varepsilon}) dx \\ &\quad + \varepsilon^{\gamma} \int_{\Gamma^{\varepsilon}} h^{\varepsilon}(u_1^{\varepsilon} - u_2^{\varepsilon}) (Z_{\delta_0}(u_1^{\varepsilon}) (v_{\varepsilon,1}^m)^- - Z_{\delta_0}(u_2^{\varepsilon}) (v_{\varepsilon,2}^m)^-) d\sigma. \end{aligned}$$

The first term in the right-hand side of the previous inequality goes to zero as ε goes to zero, because of $H_1)_i$, the Hölder inequality, (4.1) and (5.15). On the other hand, since $v_{\varepsilon}^m \in H_0^1(\Omega)$, also $(v_{\varepsilon}^m)^-$ belongs to $H_0^1(\Omega)$, so that $(v_{\varepsilon,1}^m)^- = (v_{\varepsilon,2}^m)^-$ on Γ^{ε} and, since Z_{δ} is nonincreasing (see definition (4.3)), we have

$$\begin{aligned} \varepsilon^{\gamma} \int_{\Gamma^{\varepsilon}} h^{\varepsilon}(u_1^{\varepsilon} - u_2^{\varepsilon}) (Z_{\delta_0}(u_1^{\varepsilon}) (v_{\varepsilon,1}^m)^- - Z_{\delta_0}(u_2^{\varepsilon}) (v_{\varepsilon,2}^m)^-) d\sigma \\ = \varepsilon^{\gamma} \int_{\Gamma^{\varepsilon}} h^{\varepsilon}(v_{\varepsilon,1}^m)^- (u_1^{\varepsilon} - u_2^{\varepsilon}) (Z_{\delta_0}(u_1^{\varepsilon}) - Z_{\delta_0}(u_2^{\varepsilon})) d\sigma \leq 0. \end{aligned}$$

Thus we deduce that

$$\lim_{\varepsilon \rightarrow 0} A_\varepsilon = 0. \quad (5.23)$$

As done for the previous terms, in order to handle B_ε we split it as follows:

$$B_\varepsilon = B_{\varepsilon,1} + B_{\varepsilon,2} \quad \text{with } B_{\varepsilon,i} \doteq \int_{\Omega_i^\varepsilon \cap \{u_i^\varepsilon > \delta_0\}} f \zeta(u_i^\varepsilon) (v_{\varepsilon,i}^m)^- \chi_{\{u_1 \neq \delta_0\}} dx, \quad i = 1, 2.$$

The same arguments used to prove that $\lim_{\varepsilon \rightarrow 0} B_{\varepsilon,1} = 0$ in the proof of [24, Theorem 5.5] apply also to the second component. So, using again (2.1) and convergence (3.6), we get

$$\lim_{\varepsilon \rightarrow 0} B_{\varepsilon,2} = 0.$$

This implies, together with (5.22)–(5.23),

$$\lim_{\varepsilon \rightarrow 0} K_\varepsilon = 0. \quad (5.24)$$

By collecting (5.17), (5.19), (5.20), (5.21), (5.24), we finally obtain the validity of (5.18).

Step 3. In this last step, we show that

$$\lim_{m \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Gamma^\varepsilon} |\nabla(v_\varepsilon^m - v^\varepsilon)|^2 dx = 0. \quad (5.25)$$

We take $v_\varepsilon^m - v^\varepsilon$ as test function in the variational formulations (5.11) and (5.7). By subtraction, we have

$$\begin{aligned} & \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla(v_\varepsilon^m - v^\varepsilon) \nabla(v_\varepsilon^m - v^\varepsilon) dx - \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(v_1^\varepsilon - v_2^\varepsilon) [(v_\varepsilon^m - v^\varepsilon)_1 - (v_\varepsilon^m - v^\varepsilon)_2] d\sigma \\ &= \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_1) \nabla(u_m - u_1))), (v_\varepsilon^m - v^\varepsilon)_1 \rangle_{V_\varepsilon', V_\varepsilon}. \end{aligned}$$

In view of Remark 5.2, it results

$$-\varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(v_1^\varepsilon - v_2^\varepsilon) [(v_\varepsilon^m - v^\varepsilon)_1 - (v_\varepsilon^m - v^\varepsilon)_2] d\sigma = \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(v_1^\varepsilon - v_2^\varepsilon)^2 d\sigma \geq 0.$$

Consequently, by passing to the limit on ε , for H_1) and H_4) (whose assumptions are satisfied both by v_ε and v_ε^m thanks to (5.9) and (5.12)), we get

$$\begin{aligned} 0 &\leq \alpha \lim_{\varepsilon \rightarrow 0} \|\nabla(v_\varepsilon^m - v^\varepsilon)\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 \\ &\leq \lim_{\varepsilon \rightarrow 0} \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_1) \nabla(u_m - u_1))), (v_\varepsilon^m - v^\varepsilon)_1 \rangle_{V_\varepsilon', V_\varepsilon} \\ &= \langle -\operatorname{div}(A^0(u_1) \nabla(u_m - u_1)), u_m - u_1 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \end{aligned}$$

This together with (5.10) gives

$$\begin{aligned} 0 &\leq \lim_{m \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \alpha \|\nabla(v_\varepsilon^m - v^\varepsilon)\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 \\ &\leq \lim_{m \rightarrow +\infty} \int_{\Omega} A^0(u_1) \nabla(u_m - u_1) \nabla(u_m - u_1) dx = 0. \end{aligned}$$

At last, coupling (5.16) and (5.25) we obtain the desired result. \square

6. Proof of the homogenization result

In this last section we prove the second part of Theorem 2.4, Proposition 2.5 and Theorem 2.6.

First, we treat Proposition 2.5 which is a consequence of the convergence result given in Theorem 5.4.

Proof of Proposition 2.5. Let v^ε be the solution of problem (5.7). The homogenization result given in Theorem 5.1 (written for $Z = -\operatorname{div}(A^0(u_1)\nabla u_1)$ and $v_1 = u_1$, see (5.8)) gives, for the subsequence verifying (2.4),

$$\begin{cases} \mathcal{T}_1^\varepsilon(\nabla v_1^\varepsilon) \rightharpoonup \nabla u_1 + \nabla_y \widehat{v}_1 & \text{weakly in } L^2(\Omega \times Y_1), \\ \mathcal{T}_2^\varepsilon(\nabla v_2^\varepsilon) \rightharpoonup 0 & \text{weakly in } L^2(\Omega \times Y_2), \end{cases} \quad (6.1)$$

where \widehat{v}_1 is a function in $L^2(\Omega; H_{\text{per}}^1(Y_1))$ with $\mathcal{M}_\Gamma(\widehat{v}_1) = 0$, such that

$$\widehat{v}_1(x, y) = - \sum_{i=1}^N \chi_{e_i}(y, u_1(x)) \frac{\partial u_1}{\partial x_i}(x).$$

Now let us observe that, from Theorem 5.4 and Proposition 3.3₉, we get

$$\mathcal{T}^\varepsilon(\nabla u^\varepsilon - \nabla v^\varepsilon) \rightarrow 0 \quad \text{strongly in } L^2(\Omega \times Y).$$

This, together with (2.11)_{iv,v} and (6.1), leads to

$$\begin{cases} \nabla_y \widehat{v}_1 = \nabla_y \widehat{u}_1 & \text{a.e. in } \Omega \times Y_1, \\ \nabla_y \widehat{u}_2 \equiv 0 & \text{a.e. in } \Omega \times Y_2, \end{cases} \quad (6.2)$$

which implies $\widehat{v}_1 = \widehat{u}_1 + w(x)$, for some function w only depending on x .

Since $\mathcal{M}_\Gamma(\widehat{v}_1) = \mathcal{M}_\Gamma(\widehat{u}_1) = 0$ and

$$\mathcal{M}_\Gamma(\widehat{v}_1) = \mathcal{M}_\Gamma(\widehat{u}_1) + \mathcal{M}_\Gamma(w),$$

we derive $w = 0$ and

$$\widehat{v}_1 = \widehat{u}_1.$$

1 Whence we obtain the convergence 1

$$2 \quad \mathcal{T}_1^\varepsilon(\nabla v_1^\varepsilon) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_1 \quad \text{weakly in } L^2(\Omega \times Y_1), \quad (6.3) \quad 2$$

3 and the claimed expression of \widehat{u}_1 . \square 3

4 We are now able to prove the homogenization theorems. 4

5 **Proof of Theorem 2.4.** Let us first remark that, under our assumptions, Proposition 4.4 holds true. In 5

$$6 \quad \mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega \times Y_2). \quad (6.4) \quad 6$$

7 Notice that the function $u_1 \in H_0^1(\Omega)$ is nonnegative due to (2.12). Moreover, conditions (5.4) ensures 7

8 the validity of H_4) and, as a consequence, Theorem 5.4 holds true. 8

9 Now, to identify the limit problem satisfied by (u_1, \widehat{u}_1) , we take $\varphi, \phi \in \mathcal{D}(\Omega)$ and $\xi \in H_{\text{per}}^1(Y)$, and 9

10 use 10

$$11 \quad \psi_\varepsilon(x) = \varphi(x) + \varepsilon \phi(x) \xi \left(\frac{x}{\varepsilon} \right) \in H_\gamma^\varepsilon \quad 11$$

12 as test function in (2.7). We obtain 12

$$13 \quad \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \nabla \psi_\varepsilon \, dx + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) (\psi_{\varepsilon,1} - \psi_{\varepsilon,2}) \, d\sigma = \int_{\Omega} f \zeta(u^\varepsilon) \psi_\varepsilon \, dx. \quad (6.5) \quad 13$$

14 Let us consider the solution v^ε of problem (5.3) and write 14

$$15 \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \nabla \psi_\varepsilon \, dx = \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla v^\varepsilon \nabla \psi_\varepsilon \, dx \quad 15$$

$$16 \quad + \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla (u^\varepsilon - v^\varepsilon) \nabla \psi_\varepsilon \, dx. \quad (6.6) \quad 16$$

17 By using assumption H_1), the Hölder inequality and Theorem 5.4, one has 17

$$18 \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla (u^\varepsilon - v^\varepsilon) \nabla \psi_\varepsilon \, dx = 0, \quad (6.7) \quad 18$$

19 taking into account that the norm of ψ_ε is bounded in H_γ^ε . 19

20 By Theorem 5.1 (written for $Z = -\text{div}(A^0(u_1) \nabla u_1)$), we obtain 20

$$21 \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla v^\varepsilon \nabla \psi_\varepsilon \, dx \quad 21$$

$$22 \quad = \frac{1}{|Y|} \int_{\Omega \times Y_1} A(y, u_1) (\nabla u_1 + \nabla_y \widehat{u}_1) (\nabla \varphi + \phi \nabla_y \xi) \, dx \, dy, \quad (6.8) \quad 22$$

23 in view of (6.3). 23

Concerning the boundary term in (6.5), by unfolding, using Lemma 3.4 and (3.3) we get

$$\begin{aligned}
& \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon)(\psi_{\varepsilon,1} - \psi_{\varepsilon,2}) d\sigma \\
&= \varepsilon^{\gamma+1} \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon)(\phi_1 \xi_1 - \phi_2 \xi_2) d\sigma \\
&= \frac{\varepsilon^\gamma}{|Y|} \int_{\Omega \times \Gamma} h(y) [\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon)] [\xi_1(y) \mathcal{T}_1^\varepsilon(\varphi_1) - \xi_2(y) \mathcal{T}_2^\varepsilon(\varphi_2)] dx d\sigma_y \\
&\leq c \varepsilon^\gamma \|\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon)\|_{L^2(\Omega \times \Gamma)} \|\mathcal{T}_1^\varepsilon(\varphi_1) [\xi_1(y) - \xi_2(y)]\|_{L^2(\Omega \times \Gamma)} \leq c \varepsilon^\gamma \varepsilon^{\frac{1-\gamma}{2}} = c \varepsilon^{\frac{1+\gamma}{2}}.
\end{aligned}$$

Since $-1 < \gamma < 1$, passing to the limit as $\varepsilon \rightarrow 0$, we obtain

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon)(\psi_{\varepsilon,1} - \psi_{\varepsilon,2}) d\sigma = 0. \quad (6.9)$$

In order to pass to the limit in the singular term of (6.5), we define

$$\mu_\varepsilon(x) \doteq \varepsilon \phi(x) \xi\left(\frac{x}{\varepsilon}\right), \quad \text{that is } \psi_\varepsilon = \varphi + \mu_\varepsilon. \quad (6.10)$$

Then one has

$$\mathcal{T}^\varepsilon(\mu_\varepsilon) = \varepsilon \mathcal{T}^\varepsilon(\phi) \xi \quad \text{and} \quad \nabla \mu_\varepsilon = \varepsilon \nabla \phi \xi\left(\frac{\cdot}{\varepsilon}\right) + \phi \nabla_y \xi\left(\frac{\cdot}{\varepsilon}\right),$$

and

$$\begin{cases} \text{i) } \mathcal{T}^\varepsilon(\mu_\varepsilon) \rightarrow 0 & \text{strongly in } L^2(\Omega \times Y), \\ \text{ii) } \mathcal{T}^\varepsilon(\nabla \mu_\varepsilon) \rightarrow \phi \nabla_y \xi & \text{strongly in } L^2(\Omega \times Y). \end{cases} \quad (6.11)$$

From now on, without loss of generality we assume $\varphi \geq 0$ and $\mu_\varepsilon \geq 0$ in (6.10). Indeed we can decompose these functions in their positive and negative parts as in (2.2). Now let us split the singular integral into two terms: one near the singularity and one far from it. For every positive δ we write

$$0 \leq \int_{\Omega} f \zeta(u^\varepsilon) \psi_\varepsilon dx = \int_{\{0 < u^\varepsilon \leq \delta\}} f \zeta(u^\varepsilon) \psi_\varepsilon dx + \int_{\{u^\varepsilon > \delta\}} f \zeta(u^\varepsilon) \psi_\varepsilon dx \doteq I_\varepsilon^\delta + J_\varepsilon^\delta. \quad (6.12)$$

In view of Proposition 4.3 written for $\varphi, \mu_\varepsilon \geq 0$ we have

$$\begin{aligned}
0 &\leq \limsup_{\varepsilon \rightarrow 0} I_\varepsilon^\delta \\
&\leq \limsup_{\varepsilon \rightarrow 0} \left[\int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \nabla \psi_\varepsilon Z_\delta(u^\varepsilon) dx \right. \\
&\quad \left. + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) [Z_\delta(u_1^\varepsilon) \psi_{\varepsilon,1} - Z_\delta(u_2^\varepsilon) \psi_{\varepsilon,2}] d\sigma \right]. \quad (6.13)
\end{aligned}$$

Let us first show that

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) [Z_\delta(u_1^\varepsilon) \psi_{\varepsilon,1} - Z_\delta(u_2^\varepsilon) \psi_{\varepsilon,2}] d\sigma \leq 0, \quad (6.14)$$

so that it results

$$0 \leq \limsup_{\varepsilon \rightarrow 0} I_\varepsilon^\delta \leq \limsup_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \nabla \psi_\varepsilon Z_\delta(u^\varepsilon) dx. \quad (6.15)$$

Taking into account the decomposition given in (6.10), and the fact that since $\varphi, \phi \in \mathcal{D}(\Omega)$ then $\varphi_1 = \varphi_2$ and $\phi_1 = \phi_2$ on Γ^ε , we have

$$\begin{aligned} & \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) \{ \varphi_1 [Z_\delta(u_1^\varepsilon) - Z_\delta(u_2^\varepsilon)] + \varepsilon \phi_1 [Z_\delta(u_1^\varepsilon) \xi_1 - Z_\delta(u_2^\varepsilon) \xi_2] \} d\sigma \\ &= \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) \varphi_1 [(Z_\delta(u_1^\varepsilon) - Z_\delta(u_2^\varepsilon))] d\sigma \\ & \quad + \varepsilon^{\gamma+1} \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) \phi_1 [(Z_\delta(u_1^\varepsilon) \xi_1 - Z_\delta(u_2^\varepsilon) \xi_2)] d\sigma \\ & \leq \varepsilon^{\gamma+1} \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) \phi_1 [(Z_\delta(u_1^\varepsilon) \xi_1 - Z_\delta(u_2^\varepsilon) \xi_2)] d\sigma, \end{aligned}$$

where we used H_3), $\varphi_1 \geq 0$ and the growth condition of Z_δ .

From the properties of the unfolding operators (Lemma 3.4 and estimate (3.3)), the fact that $Z_\delta \leq 1$ by definition and $\gamma \in]-1, 1[$, we get

$$\begin{aligned} & \varepsilon^{\gamma+1} \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) \phi_1 [(Z_\delta(u_1^\varepsilon) \xi_1 - Z_\delta(u_2^\varepsilon) \xi_2)] d\sigma \\ & \leq c \varepsilon^\gamma \|\mathcal{T}_1^\varepsilon(\phi_1) [\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon)]\|_{L^2(\Omega \times \Gamma)} \|\xi_1 (Z_\delta(\mathcal{T}_1^\varepsilon(u_1^\varepsilon)) - \xi_2 Z_\delta(\mathcal{T}_2^\varepsilon(u_2^\varepsilon)))\|_{L^2(\Omega \times \Gamma)} \\ & \leq c \varepsilon^\gamma \varepsilon^{\frac{1-\gamma}{2}} = c \varepsilon^{\frac{\gamma+1}{2}} \rightarrow 0, \end{aligned}$$

which implies (6.14).

Now, we write

$$\begin{aligned} \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \nabla \psi_\varepsilon Z_\delta(u^\varepsilon) dx &= \int_{\Omega_1^\varepsilon} A^\varepsilon(x, u_1^\varepsilon) \nabla u_1^\varepsilon \nabla \psi_{\varepsilon,1} Z_\delta(u_1^\varepsilon) dx \\ & \quad + \int_{\Omega_2^\varepsilon} A^\varepsilon(x, u_2^\varepsilon) \nabla u_2^\varepsilon \nabla \psi_{\varepsilon,2} Z_\delta(u_2^\varepsilon) dx. \end{aligned}$$

The arguments used in Theorem 2.8 of [24] to handle the term related to the first component apply also to the second one. So that, in view of (6.4) and (6.15), we obtain

$$0 \leq \limsup_{\varepsilon \rightarrow 0} I_\varepsilon^\delta \leq \frac{1}{|Y|} \int_{\Omega \times Y_1} A(y, u_1) (\nabla u_1 + \nabla_y \widehat{u}_1) (\nabla \varphi + \phi \nabla_y \xi) \chi_{\{u_1=0\}} dx dy = 0, \quad (6.16)$$

since, due to the expression of \widehat{u}_1 given by Proposition 2.5, the functions ∇u_1 and $\nabla_y \widehat{u}_1$ vanish where u_1 is equal to 0.

In order to study the limit behavior of J_ε^δ (defined in (6.12)), once again we split it as follows:

$$J_\varepsilon^\delta = \int_{\{u_1^\varepsilon > \delta\}} f \zeta(u_1^\varepsilon) \psi_{\varepsilon,1} dx + \int_{\{u_2^\varepsilon > \delta\}} f \zeta(u_2^\varepsilon) \psi_{\varepsilon,2} dx.$$

The integral on the first component is the same considered in the proof of [24, Theorem 2.8]. By using those same computations also on the second component, we get

$$\lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} J_\varepsilon^\delta = (\theta_1 + \theta_2) \int_{\Omega} f \zeta(u_1) \varphi \chi_{\{u_1 > 0\}} dx = \int_{\Omega} f \zeta(u_1) \varphi \chi_{\{u_1 > 0\}} dx. \quad (6.17)$$

Then, when passing to the limit in (6.5), we combine (6.6)–(6.9), (6.12), (6.16), (6.17) and have

$$\int_{\Omega \times Y_1} A(y, u_1) (\nabla u_1 + \nabla_y \widehat{u}_1) (\nabla \varphi + \phi \nabla_y \xi) dx dy = |Y| \int_{\Omega} f \zeta(u_1) \varphi \chi_{\{u_1 > 0\}} dx,$$

for every $\varphi, \phi \in \mathcal{D}(\Omega)$ and $\xi \in H_{\text{per}}^1(Y)$. By density we obtain

$$\int_{\Omega \times Y_1} A(y, u_1) (\nabla u_1 + \nabla_y \widehat{u}_1) (\nabla \varphi + \nabla_y \psi) dx dy = |Y| \int_{\Omega} f \zeta(u_1) \varphi \chi_{\{u_1 > 0\}} dx \quad (6.18)$$

for every $\varphi \in H_0^1(\Omega)$ and $\psi \in L^2(\Omega; H_{\text{per}}^1(Y))$. \square

Proof of Theorem 2.6. From the expression of \widehat{u}_1 given in Proposition 2.5, a standard computation shows that u_1 is a solution of the following problem:

$$\begin{cases} -\operatorname{div}(A^0(u_1) \nabla u_1) = f \zeta(u_1) \chi_{\{u_1 > 0\}} & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega. \end{cases} \quad (6.19)$$

Also, since the conditions given in (2.10) are satisfied by A^0 , Theorem 6 from [25] shows that u_1 is the unique solution of this problem. This implies the uniqueness of \widehat{u}_1 under the condition $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$, in view of Proposition 2.5. Hence convergence (2.11) holds for the whole sequence, as well as (6.4).

We lastly prove that $u_1 > 0$ almost everywhere in Ω . From the strong maximum principle, by contradiction, we derive $u_1 \equiv 0$ in Ω . This imply $\int_{\Omega} f \zeta(u_1) \chi_{\{u_1 > 0\}} \varphi dx = 0$ for every $\varphi \in \mathcal{D}(\Omega)$. This means $f \equiv 0$ on Ω which contradicts assumption $H_2)_{\text{iii}}$. Consequently $u_1 > 0$ almost everywhere in Ω and $\chi_{\{u_1 > 0\}} \equiv 1$. Then u_1 satisfies the limit equation (2.14). \square

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