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# Homogenization of a class of singular elliptic problems in two-component domains

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1           the auxiliary one, associated with a weak cluster point of the sequence  
 2            $\{u^\varepsilon\}$ , as  $\varepsilon \rightarrow 0$ . This allows us not only to pass to the limit in the quasi-  
 3           linear term, but also to study the singular term near its singularity,  
 4           via an accurate a priori estimate.

## 5   1   Introduction

In this paper we study the asymptotic behavior of a class of quasilinear elliptic problems presenting singular lower order terms and posed in two-component domains.

More precisely, the two-component domain  $\Omega$  is a bounded open subset of  $\mathbb{R}^N$  which is the union of two disjoint open subsets,  $\Omega_1^\varepsilon$  and  $\Omega_2^\varepsilon$ , and their common boundary  $\Gamma^\varepsilon$ . The disconnected component  $\Omega_2^\varepsilon$  is the union of the  $\varepsilon$ -periodic translated sets of  $\varepsilon Y_2$ , where  $Y_2$  is well contained in the reference cell  $Y$  and  $\Gamma = \partial Y_2$ . The connected component  $\Omega_1^\varepsilon$  is obtained by removing from  $\Omega$  the closure of  $\Omega_2^\varepsilon$  such that the interface  $\Gamma^\varepsilon = \partial\Omega_2^\varepsilon$ .

We deal with the homogenization, as  $\varepsilon$  goes to zero, of the following problem:

$$\left\{ \begin{array}{ll} -\operatorname{div}(A^\varepsilon(x, u^\varepsilon)\nabla u^\varepsilon) = f\zeta(u^\varepsilon) & \text{in } \Omega \setminus \Gamma^\varepsilon, \\ u_1^\varepsilon = 0 & \text{on } \partial\Omega, \\ (A^\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)\nu^\varepsilon = (A^\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon)\nu^\varepsilon & \text{on } \Gamma^\varepsilon, \\ (A^\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)\nu^\varepsilon = -\varepsilon^\gamma h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma^\varepsilon, \end{array} \right.$$

6   where  $\nu^\varepsilon$  is the unit external normal vector to  $\Omega_1^\varepsilon$ . We prescribe a Dirichlet  
 7   condition on the exterior boundary, while we assume that the continuous  
 8   heat flux is proportional to the jump of the solution on the interface by

1 means of a function of order  $\varepsilon^\gamma$ .

2 The quasilinear diffusion matrix field is defined by  $A^\varepsilon(x, t) = A\left(\frac{x}{\varepsilon}, t\right)$ , where  
 3 the matrix field  $A$  is uniformly elliptic, bounded, periodic in the first variable  
 4 and Carathéodory. The nonlinear real function  $\zeta(s)$  is nonnegative and sin-  
 5 gular at  $s = 0$ , while  $f$  is a nonnegative datum whose summability depends  
 6 on the growth of  $\zeta$  near its singularity. Concerning the boundary condition,  
 7  $h^\varepsilon(x) = h\left(\frac{x}{\varepsilon}\right)$  where  $h$  is assumed to be a periodic, nonnegative and bounded  
 8 function, and  $\gamma \in ] - 1, 1[$ .

9 This problem describes the stationary heat diffusion in a two-component  
 10 composite with an  $\varepsilon$ -periodic imperfect interface. In particular, quasilinear  
 11 diffusion terms describe the behavior of materials like glass or wood, in which  
 12 the heat diffusion depends on the range of the temperature (see for instance  
 13 [6]). We refer to [19, Section 3] for more details about source terms depend-  
 14 ing on the solution itself and becoming infinite when the solutions vanish.  
 15 These kind of source terms can model, for instance, electrical conductors  
 16 where each point becomes a source of heat as a current flows inside. The  
 17 boundary condition on  $\Gamma^\varepsilon$  models a jump of the solution through a rough  
 18 interface and we refer to [9] for a physical justification of this model (see also  
 19 [30]).

20 In view of a counterexample suggested by H.C. Hummel in [29], one cannot  
 21 expect bounded a priori estimates for the solution when  $\gamma > 1$ . For this  
 22 case we refer to [23] where different a priori estimates needed. As for our  
 23 problem, existence and uniqueness results has been proved in [26]-[27] for  
 24  $\gamma \leq 1$ .

25 Here we prove an homogenization result for  $\gamma \in ] - 1, 1[$  by means of the

1 periodic unfolding method. This method was originally introduced by D.  
2 Cioranescu, A. Damlamian and G. Griso in [13] and [14] for fixed domains  
3 and extended to perforated ones in [15] and [16]. Then, it has been adapted  
4 to two-component domains by P. Donato, K.H. Le Nguyen and R. Tardieu  
5 in [22] (see also [21]).

6 This work has essentially been inspired by the works [19] and [25]. Let us  
7 point out the main differences with respect to them and the additional dif-  
8 ficulties. In [19] the authors treat the same singularity when  $A$  is linear  
9 and the domain has two connected components separated by an oscillating  
10 interface. This interface tends to a flat one, so that the integral on  $\Gamma^\varepsilon$  goes  
11 to the one on  $\Gamma$ , roughly speaking. But here  $\Gamma^\varepsilon$  is the union of disconnected  
12 sets (of order  $\frac{1}{\varepsilon}$ ) so that its measures goes to infinity, whence the integral on  
13  $\Gamma^\varepsilon$  needs particular care.

14 Let us mention that in [25] the authors consider the same singular problem  
15 with  $A$  quasilinear but posed in a periodically perforated domain. Here the  
16 holes are replaced by a second material so that we have to treat in addition  
17 all the integrals on the second component and the boundary term with jump.  
18 In Section 2 we present the setting of the problem and we state the main  
19 results.

20 In Section 3 we give a short presentation of the periodic unfolding method,  
21 adapted to two-component domains.

22 In Section 4 we prove two a priori estimates uniform with respect to  $\varepsilon$  and  
23 we give an estimate of the integral of the singular term close to the singular  
24 set  $\{u^\varepsilon = 0\}$ , in terms of the quasilinear one (cf. Proposition 4.3).

1 In Section 5 we prove a crucial convergence result, Theorem 5.4. It is one  
2 of the main tools when proving our homogenization result and it shows that  
3 the gradient of  $u^\varepsilon$  is equivalent to the gradient of the solution of a suitable  
4 auxiliary linear problem, associated with a weak cluster point of the sequence  
5  $\{u^\varepsilon\}$ , as  $\varepsilon \rightarrow 0$ . This idea was originally introduced in [4] (see also [5]) where  
6 some nonlinear problems with quadratic growth are considered. We refer to  
7 [10] and [22] for the study of the auxiliary problem, while for the proof of  
8 the convergence result we adapt some techniques from [19] and [25].

9 Section 6 is devoted to the proofs of Proposition 2.5 and the homogenization  
10 theorem. For these results Theorem 5.4 plays an important role not only  
11 in the study of the quasilinear term but also in that of the singular one.  
12 Actually, as done in [19] and [25], we split the integral of the singular term  
13 into the sum of two integrals: one on the set where the solution is close to  
14 the singularity and one where it is far from it, which results not singular.  
15 Near the singularity, making use of the estimate given in Proposition 4.3, we  
16 shift the study of the singular term to that of the quasilinear one, for which  
17 we can use Theorem 5.4.

18 Finally, let us point out that the homogenization result we prove here shows  
19 that the conductivity of the first material is the same obtained when there is  
20 no material occupying  $\Omega_2^\varepsilon$ . On the other hand, since in the limit problem it  
21 appears  $f$  instead of  $\theta_1 f$  (being  $\theta_1$  the proportion of the material occupying  
22  $\Omega_1^\varepsilon$ ), one has anyway to take into account even the source term in the second  
23 component.

24 The first paper on this subject is due to [3] for the linear (nonsingular)

1 case by multiple scale method. In [23] and [30] the authors also studied the  
 2 linear case by using the Tartar method. For similar homogenization elliptic  
 3 problems we refer to [8], [19], [25], [29] and [31]. In [22] - [21] the authors  
 4 study the linear case in presence of linear and nonlinear boundary conditions,  
 5 respectively.

## 6 **2 Setting of the problem and main results**

7 Throughout the paper, we use the same notation as in [21]-[22] for the peri-  
 8 odic unfolding method in two-component domains.

### 9 **The domain**

10 For  $N \in \mathbb{N}, N \geq 2$ , let  $\Omega$  be a bounded open set in  $\mathbb{R}^N$  with a Lipschitz-  
 11 continuous boundary  $\partial\Omega$ . Also let  $Y \doteq \prod_{i=1}^N [0, l_i[$  be a reference cell, with  
 12  $l_i > 0, i = 1, \dots, N$ . We suppose that  $Y_1$  and  $Y_2$  are two disjoint connected  
 13 open subsets of  $Y$  such that  $Y_2 \neq \emptyset, \overline{Y_2} \subset Y$  and  $Y = Y_1 \cup \overline{Y_2}$ , with a common  
 14 boundary  $\Gamma = \partial Y_2$  Lipschitz-continuous.  
 15 Let  $\{\varepsilon\}_{\varepsilon > 0}$  be a positive parameter taking values in a sequence converging  
 16 to zero and set:

- for any  $k \in \mathbb{Z}^N, k_l = (k_1 l_1, \dots, k_N l_N)$  and

$$Y^k = k_l + Y, \quad Y_i^k = k_l + Y_i, \quad i = 1, 2,$$

-  $K_\varepsilon = \{k \in \mathbb{Z}^N \mid \varepsilon \overline{Y_2^k} \subset \Omega\}$  and

$$\Omega_2^\varepsilon = \bigcup_{k \in K_\varepsilon} \varepsilon Y_2^k, \quad \Omega_1^\varepsilon = \Omega \setminus \overline{\Omega_2^\varepsilon}, \quad \Gamma^\varepsilon = \partial \Omega_2^\varepsilon.$$

By construction,  $\Omega$  results the union of the two disjoint components and their common boundary, i.e.  $\Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon$  (see Figure 1).

Also we introduce the following sets:

$$\widehat{K}_\varepsilon = \{k \in \mathbb{Z}^N \mid \varepsilon Y^k \subset \Omega\}, \quad \widehat{\Omega}_\varepsilon = \text{interior} \left\{ \bigcup_{k \in \widehat{K}_\varepsilon} \varepsilon(k_l + \overline{Y}) \right\}, \quad \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon,$$

and

$$\widehat{\Omega}_i^\varepsilon = \bigcup_{k \in \widehat{K}_\varepsilon} \varepsilon Y_i^k, \quad \Lambda_i^\varepsilon = \Omega_i^\varepsilon \setminus \widehat{\Omega}_i^\varepsilon, \quad i = 1, 2, \quad \widehat{\Gamma}^\varepsilon = \partial \widehat{\Omega}_2^\varepsilon.$$

1 Also,

$$2 \text{ there exists } \varepsilon_0 > 0 \text{ such that } \forall \varepsilon \leq \varepsilon_0, \Lambda_2^\varepsilon = \emptyset. \quad (2.1)$$

3 Let us note that here, for the sake of simplicity, we do not allow the second  
 4 component to meet the boundary of the domain and also we suppose  $Y_2$   
 5 connected. Actually the cases where the holes meet the boundary of  $\Omega$  and  
 6  $Y_2$  has a finite number of connected components can be treated as done in  
 [21] and [22], respectively.



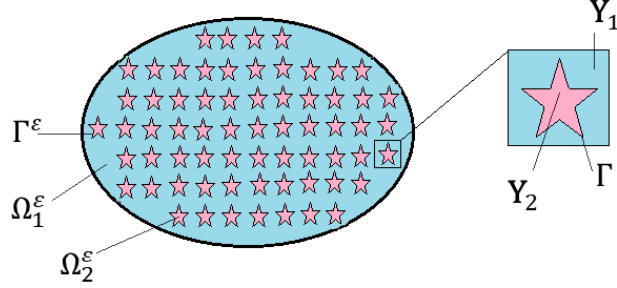


Figure 1: The two-component domain  $\Omega$  and the reference cell  $Y$

1 In the sequel we denote by

2 •  $\mathcal{M}_\omega(v) \doteq \frac{1}{|\omega|} \int_\omega v dx$  the average of any function  $v \in L^1(\omega)$ , for any  
 3 open set  $\omega$  of  $\mathbb{R}^N$ ,

4 •  $\chi_\omega$  the characteristic function of any open set  $\omega$  of  $\mathbb{R}^N$ ,

5 •  $\sim$  the zero extension to the whole of  $\Omega$  of functions defined on  $\Omega_1^\varepsilon$  or  
 6  $\Omega_2^\varepsilon$ ,

7 •  $\theta_i \doteq \frac{|Y_i|}{|Y|}$ ,  $i = 1, 2$ ,

8 •  $c$  different positive constants independent on  $\varepsilon$ ,

9 •  $\mathcal{M}(\alpha, \beta, Y)$  the set of matrix fields  $A = (a_{i,j})_{1 \leq i,j \leq N} \in (L^\infty(Y))^{N \times N}$   
 10 such that

11  $(A(x)\lambda, \lambda) \geq \alpha|\lambda|^2$  and  $|A(x)\lambda| \leq \beta|\lambda|$ ,  $\forall \lambda \in \mathbb{R}^N$  and a.e. in  $Y$ ,

12 with  $\alpha, \beta \in \mathbb{R}$ ,  $0 < \alpha \leq \beta$ ,

13 •  $\{e_1, \dots, e_N\}$  the canonical basis of  $\mathbb{R}^N$ ,

14 •  $v_i = v|_{\Omega_i^\varepsilon}$  the restriction to  $\Omega_i^\varepsilon$  of functions  $v$  defined in  $\Omega$ ,  $i = 1, 2$ .

1 Moreover let us recall the classical decomposition for every real function  $v$

$$v = v^+ - v^-, \quad v^+ \doteq \max\{v, 0\} \quad \text{and} \quad v^- \doteq -\min\{v, 0\} \quad \text{a.e. in } \Omega, \quad (2.2)$$

2 where  $v^+$  and  $v^-$  are both nonnegative. In particular, one has

$$\begin{aligned} (v_1 - v_2)(v_1^- - v_2^-) &= (v_1^+ - v_2^+)(v_1^- - v_2^-) - (v_1^- - v_2^-)^2 \\ &= -v_1^+ v_2^- - v_2^+ v_1^- - (v_1^- - v_2^-)^2 \leq 0. \end{aligned} \quad (2.3)$$

### 3 **The problem**

4 The aim of the paper is to study the asymptotic behavior, as  $\varepsilon$  goes to zero,  
5 of the following problem:

$$\left\{ \begin{array}{ll} -\operatorname{div}(A^\varepsilon(x, u^\varepsilon)\nabla u^\varepsilon) = f\zeta(u^\varepsilon) & \text{in } \Omega \setminus \Gamma^\varepsilon, \\ u_1^\varepsilon = 0 & \text{on } \partial\Omega, \\ (A^\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)\nu^\varepsilon = (A^\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon)\nu^\varepsilon & \text{on } \Gamma^\varepsilon, \\ (A^\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)\nu^\varepsilon = -\varepsilon^\gamma h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma^\varepsilon, \end{array} \right. \quad (2.4)$$

6 where  $\nu^\varepsilon$  is the unit external normal vector to  $\Omega_1^\varepsilon$ , and we prescribe a Dirich-  
7 let condition on the exterior boundary and a jump of the solution on the  
8 interface  $\Gamma^\varepsilon$ , in the case  $\gamma \in ]-1, 1[$ .

1 **Assumptions on the data**

H<sub>1</sub>) The real  $N \times N$  matrix field  $A : (y, t) \in Y \times \mathbb{R} \mapsto A(y, t) = (a_{i,j}(y, t))_{i,j=1,\dots,N} \in \mathbb{R}^{N^2}$  satisfies the following conditions:

- $$\left\{ \begin{array}{l} \text{i) } A \text{ is a Carathéodory function s.t.} \\ \quad A(\cdot, t) \text{ is } Y\text{-periodic and in } \mathcal{M}(\alpha, \beta, Y), \text{ for every } t \in \mathbb{R}; \\ \text{ii) there exists a real function } \omega : \mathbb{R} \rightarrow \mathbb{R} \text{ satisfying the following conditions:} \\ \quad - \omega \text{ is continuous and non decreasing, with } \omega(t) > 0 \forall t > 0, \\ \quad - |A(y, t_1) - A(y, t_2)| \leq \omega(|t_1 - t_2|) \text{ for a.e. } y \in Y, \forall t_1 \neq t_2, \\ \quad - \forall s > 0, \lim_{y \rightarrow 0^+} \int_y^s \frac{dt}{\omega(t)} = +\infty. \end{array} \right.$$

H<sub>2</sub>) The functions  $\zeta$  and  $f$  verify

- $$\left\{ \begin{array}{l} \text{i) } \zeta : [0, +\infty[ \rightarrow [0, +\infty] \text{ is a function such that} \\ \quad \zeta \in \mathcal{C}^0([0, +\infty[), 0 \leq \zeta(s) \leq \frac{1}{s^k} \text{ for every } s \in ]0, +\infty[, \text{ with } 0 < k \leq 1; \\ \text{ii) } \zeta \text{ is non increasing;} \end{array} \right.$$

iii)  $f \geq 0$  a.e. in  $\Omega$ ,  $f \not\equiv 0$ , with  $f \in L^l(\Omega)$ , for  $l \geq \frac{2}{1+k} (\geq 1)$ .

H<sub>3</sub>)  $-1 < \gamma < 1$ , and  $h$  is a  $Y$ -periodic function in  $L^\infty(\Gamma)$  such that

there exists  $h_0 \in \mathbb{R} : 0 < h_0 < h(y)$  a.e. on  $\Gamma$ .

2

◇

1 Under the above assumptions we set, for every  $t \in \mathbb{R}$ ,

$$A^\varepsilon(x, t) \doteq A\left(\frac{x}{\varepsilon}, t\right) \text{ for a.e. } x \in \Omega, \quad h^\varepsilon(x) \doteq h\left(\frac{x}{\varepsilon}\right) \text{ for a.e. } x \in \Gamma^\varepsilon. \quad (2.5)$$

2 **The functional framework**

We now introduce the functional spaces used to handle (2.4)-type problems.

Let  $V^\varepsilon \doteq \{v \in H^1(\Omega_1^\varepsilon) \mid v = 0 \text{ on } \partial\Omega\}$  endowed with the norm

$$\|v\|_{V^\varepsilon} = \|\nabla v\|_{L^2(\Omega_1^\varepsilon)}.$$

3 **REMARK 2.1.** It is known (see for instance [17, Lemma 1], [18]) that a  
 4 Poincaré inequality in  $V^\varepsilon$  holds with a constant  $c_P$  independent on  $\varepsilon$ , that  
 5 is

$$\|v\|_{L^2(\Omega_1^\varepsilon)} \leq c_P \|\nabla v\|_{L^2(\Omega_1^\varepsilon)} \quad \forall v \in V^\varepsilon. \quad (2.6)$$

6 Consequently, the norm in  $V^\varepsilon$  is equivalent to that in  $H^1(\Omega_1^\varepsilon)$  via a constant  
 7 independent on  $\varepsilon$ .

For every  $\gamma \in \mathbb{R}$ , let  $H_\gamma^\varepsilon$  be the space defined by

$$H_\gamma^\varepsilon \doteq \{v \in L^2(\Omega) \mid v_1 \in V^\varepsilon, v_2 \in H^1(\Omega_2^\varepsilon)\},$$

which, after the identification  $\nabla v := \widetilde{\nabla}v_1 + \widetilde{\nabla}v_2$ , is equipped by the norm

$$\|v\|_{H_\gamma^\varepsilon}^2 \doteq \|\nabla v\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 + \varepsilon^\gamma \|v_1 - v_2\|_{L^2(\Gamma^\varepsilon)}^2.$$

**PROPOSITION 2.2** ([30][22]). *Let  $\gamma \leq 1$ . There exist some positive constants  $c_1, c_2$  and  $C$ , independent of  $\varepsilon$ , such that*

$$\forall v \in H_\gamma^\varepsilon, \quad c_1 \|v\|_{V^\varepsilon \times H^1(\Omega_2^\varepsilon)}^2 \leq \|v\|_{H_\gamma^\varepsilon}^2 \leq c_2(1 + \varepsilon^{\gamma-1}) \|v\|_{V^\varepsilon \times H^1(\Omega_2^\varepsilon)}^2.$$

*In addition, if  $v^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon)$  is a bounded sequence in  $H_\gamma^\varepsilon$ , then*

$$\|v_1^\varepsilon\|_{H^1(\Omega_1^\varepsilon)} \leq C,$$

$$\|v_2^\varepsilon\|_{H^1(\Omega_2^\varepsilon)} \leq C,$$

$$\|v_1^\varepsilon - v_2^\varepsilon\|_{L^2(\Gamma^\varepsilon)} \leq C\varepsilon^{-\frac{\gamma}{2}}.$$

- 1 The variational formulation associated with problem (2.4) reads

$$\left\{ \begin{array}{l} \text{Find } u^\varepsilon \in H_\gamma^\varepsilon \text{ such that } u^\varepsilon > 0 \text{ a.e. in } \Omega, \\ \int_{\Omega} f\zeta(u^\varepsilon)\varphi dx < +\infty, \text{ and} \\ \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \nabla \varphi dx + \int_{\Gamma^\varepsilon} \varepsilon^\gamma h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon)(\varphi_1 - \varphi_2) d\sigma \\ = \int_{\Omega} f\zeta(u^\varepsilon)\varphi dx, \quad \forall \varphi \in H_\gamma^\varepsilon. \end{array} \right. \quad (2.7)$$

- 2 In [26]-[27] it is proved that, under assumptions H<sub>1</sub>)- H<sub>3</sub>), problem (2.7)  
 3 admits a unique solution.  
 4 Let us introduce here the homogenized matrix  $A^0(t)$ ,  $t \in \mathbb{R}$ , corresponding

1 to our case  $\gamma \in ]-1, 1[$ . It is defined by

$$A^0(t)\lambda \doteq \frac{1}{|Y|} \int_{Y_1} A(y, t)(\lambda - \nabla_y \chi_\lambda(y, t)) dy \quad \forall \lambda \in \mathbb{R}^N, \quad (2.8)$$

2 where, for every  $\lambda \in \mathbb{R}^N$ ,  $\chi_\lambda(\cdot, t) \in H^1(Y_1)$  are unique solutions of the cell  
 3 problems

$$\begin{cases} -\operatorname{div}(A(\cdot, t)\nabla_y \chi_\lambda(\cdot, t)) = -\operatorname{div}(A(\cdot, t)\lambda) & \text{in } Y_1, \\ A(\cdot, t)(\lambda - \nabla_y \chi_\lambda(\cdot, t))\nu_1 = 0 & \text{on } \Gamma, \\ \chi_\lambda(\cdot, t) \text{ Y-periodic and } \mathcal{M}_{Y_1}(\chi_\lambda) = 0. \end{cases} \quad (2.9)$$

4 The homogenized matrix  $A^0$  is actually the one obtained in the framework of  
 5 perforated domains. It has been originally introduced in [17] for linear prob-  
 6 lems with Neumann conditions in perforated domain, successively extended  
 7 to quasilinear ones in [1], [2].

8 We recall (see [7] and [17]) that the matrix  $A^0$  satisfies the following prop-  
 9 erties:

i)  $A^0$  is continuous and  $A^0(t) \in M\left(\alpha, \frac{\beta^2}{\alpha}, \Omega\right)$  for every  $t \in \mathbb{R}$ ;

ii) there exists a positive constant  $C$ , depending only on  $\alpha, \beta, Y$  and  $T$  s.t.

$$|A^0(t_1) - A^0(t_2)| \leq C\omega(|t_1 - t_2|)$$

for every  $t_1, t_2 \in \mathbb{R}$ , with  $t_1 \neq t_2$ , where  $\omega$  is the function given in  $H_1$ ).  
 (2.10)

10 **REMARK 2.3.** Observe that assumptions  $H_1)_{ii}$  and  $H_2)_{ii}$  are only needed  
 11 for the uniqueness of the solution of problem (2.7). If they do not hold true,

1 the homogenized problem is the one we obtain here but all the convergences  
 2 remain valid only for a subsequence.

### 3 The main results

4 We now state some main results of this work, which will be proved in the  
 5 next sections.

6 **PROPOSITION 2.4.** *Under assumptions  $H_1$ -  $H_3$ ), let  $u^\varepsilon \in H_\gamma^\varepsilon$  be the unique  
 7 solution of problem (2.7). Then, there exist a subsequence (still denoted by  
 8  $\varepsilon$ ),  $u_1 \in H_0^1(\Omega)$ ,  $\widehat{u}_1 \in L^2(\Omega, H_{per}^1(Y_1))$  with  $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$  for almost every  
 9  $x \in \Omega$  and  $\bar{u}_2 \in L^2(\Omega, H^1(Y_2))$  with  $\mathcal{M}_\Gamma(\bar{u}_2) = 0$  for almost every  $x \in \Omega$   
 10 such that*

$$\left\{ \begin{array}{ll} i) \mathcal{T}_i^\varepsilon(u_i^\varepsilon) \rightarrow u_i & \text{strongly in } L^2(\Omega, H^1(Y_i)), \quad i = 1, 2, \\ ii) \mathcal{T}^\varepsilon(\zeta(u^\varepsilon)) \rightarrow \zeta(u_1) & \text{a.e. in } \Omega \times Y, \\ iii) \widetilde{u}_1^\varepsilon \rightharpoonup \theta_1 u_1 & \text{weakly in } L^2(\Omega), \\ iv) \mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_1 & \text{weakly in } L^2(\Omega \times Y_1) \\ v) \mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \rightharpoonup \nabla_y \bar{u}_2 & \text{weakly in } L^2(\Omega \times Y_2). \end{array} \right. \quad (2.11)$$

11 Moreover,

$$u_1 \geq 0 \text{ a.e. in } \Omega \quad \text{and} \quad \int_\Omega f\zeta(u_1)\varphi dx < +\infty, \quad \forall \varphi \in H_0^1(\Omega). \quad (2.12)$$

12 Its proof is given at the end of Section 4.

**PROPOSITION 2.5.** *Under assumptions  $H_1$ -  $H_3$ ), let  $(u_1, \widehat{u}_1, \bar{u}_2)$  be given by Proposition 2.4. Then*

$$\begin{cases} \widehat{u}_1(y, x) = - \sum_{i=1}^N \chi_{e_i}(y, u_1(x)) \frac{\partial u_1}{\partial x_i}(x) \in L^2(\Omega; H_{per}^1(Y_1)), \\ \nabla_y \bar{u}_2 \equiv 0 \quad \text{a.e. in } \Omega \times Y_2, \end{cases}$$

1 where  $\chi_{e_i}(\cdot, u_1)$ ,  $i = 1, \dots, N$  are the solutions of the cell problems (2.9),  
 2 written for  $\lambda = e_i$ .

3 Then the homogenization result for problem (2.7) is

4 **THEOREM 2.6.** *Under assumptions  $H_1$ -  $H_3$ ), let  $u^\varepsilon \in H_\gamma^\varepsilon$  be the unique  
 5 solution of problem (2.7) and  $(u_1, \widehat{u}_1, \bar{u}_2)$  be given by Propositions 2.4 and  
 6 2.5. Then the pair  $(u_1, \widehat{u}_1)$  is the unique solution of the limit equation*

$$\begin{cases} \forall \varphi \in H_0^1(\Omega) \text{ and } \forall \psi \in L^2(\Omega; H_{per}^1(Y_1)) \\ \int_{\Omega \times Y_1} A(y, u_1)(\nabla u_1 + \nabla_y \widehat{u}_1)(\nabla \varphi + \nabla_y \psi) dx dy = |Y| \int_{\Omega} f \zeta(u_1) \varphi dx. \end{cases} \quad (2.13)$$

7 Finally,  $u_1 > 0$  almost everywhere in  $\Omega$  and  $u_1$  is the unique solution of the  
 8 following singular limit problem:

$$\begin{cases} -\operatorname{div}(A^0(u_1) \nabla u_1) = f \zeta(u_1) & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.14)$$



1 where the homogenized matrix  $A^0(t)$  is given by (2.8) and verifies

$$A^0(u_1)\nabla u_1 = \frac{1}{|Y|} \int_{Y_1} A(y, u_1)(\nabla u_1 + \nabla_y \hat{u}_1) dy. \quad (2.15)$$

2 Consequently, convergences (2.11) (together with  $\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \rightharpoonup 0$  weakly in

3  $L^2(\Omega \times Y_2)$ ) hold for the whole sequence.

4 These two last results are proved in Section 6.

### 5 **3 The periodic unfolding method**

In this section we give a short presentation of the periodic unfolding method adapted to two-component domains by P. Donato, K.H. Le Nguyen and R. Tardieu in [22]. This method was originally introduced by D. Cioranescu, A. Dalmlamian and G. Griso in [13] and [14] for fixed domains and extended to perforated ones in [15] and [16].

To this aim, we recall the unfolding operators  $\mathcal{T}_1^\varepsilon$  and  $\mathcal{T}_2^\varepsilon$  and the boundary unfolding operator  $\mathcal{T}_\varepsilon^b$ :  $\mathcal{T}_1^\varepsilon$  and  $\mathcal{T}_\varepsilon^b$  are exactly those ones introduced in [16] for perforated domains, while  $\mathcal{T}_2^\varepsilon$  has been introduced later in [22].

Let  $z \in \mathbb{R}^N$ , we denote by  $[z]_Y$  its integer part such that  $z - [z]_Y$  belongs to  $Y$  and set  $\{z\}_Y \doteq z - [z]_Y$ . Then, for every positive  $\varepsilon$ ,

$$x = \varepsilon \left( \left[ \frac{x}{\varepsilon} \right]_Y + \left\{ \frac{x}{\varepsilon} \right\}_Y \right) \quad \forall x \in \mathbb{R}^N.$$

- 1 **DEFINITION 3.1.** For any Lebesgue-measurable function  $\phi$  on  $\Omega_i^\varepsilon$ , the un-  
 2 folding operators  $\mathcal{T}_i^\varepsilon$ ,  $i = 1, 2$ , are defined as follows:

$$\mathcal{T}_i^\varepsilon(\phi)(x, y) \doteq \begin{cases} \phi\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{a.e. for } (x, y) \in \widehat{\Omega}_\varepsilon \times Y_i, \\ 0 & \text{a.e. for } (x, y) \in \Lambda_\varepsilon \times Y_i. \end{cases} \quad (3.1)$$

- 3 For any Lebesgue-measurable function  $\phi$  on  $\Gamma^\varepsilon$ , the boundary unfolding op-  
 4 erator  $\mathcal{T}_\varepsilon^b$  is defined as follows:

$$\mathcal{T}_\varepsilon^b(\phi)(x, y) \doteq \begin{cases} \phi\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{a.e. for } (x, y) \in \widehat{\Omega}_\varepsilon \times \Gamma, \\ 0 & \text{a.e. for } (x, y) \in \Lambda_\varepsilon \times \Gamma. \end{cases} \quad (3.2)$$

**REMARK 3.2.** If  $\phi$  is defined in  $\Omega$ , we simply write  $\mathcal{T}_i^\varepsilon(\phi)$  instead of  $\mathcal{T}_i^\varepsilon(\phi_i)$ ,  
 $i = 1, 2$ , for the sake of semplicity. Also we define  $\mathcal{T}^\varepsilon(\varphi)$  as follows:

$$\mathcal{T}^\varepsilon(\phi)(x, y) \doteq \begin{cases} \mathcal{T}_1^\varepsilon(\phi) & \text{in } \Omega \times Y_1, \\ \mathcal{T}_2^\varepsilon(\phi) & \text{in } \Omega \times Y_2. \end{cases}$$

- 5 We now recall the main properties of the unfolding operators.

6 **PROPOSITION 3.3** ([16][22]). *Let  $p \in [1, +\infty[$  and  $i = 1, 2$ .*

- 7 1.  $\mathcal{T}_i^\varepsilon$  is a linear and continuous operator from  $L^p(\Omega_i^\varepsilon)$  to  $L^p(\Omega \times Y)$ .  
 8 2.  $\mathcal{T}_i^\varepsilon(\phi\psi) = \mathcal{T}_i^\varepsilon(\phi)\mathcal{T}_i^\varepsilon(\psi)$  for every  $\phi, \psi \in L^p(\Omega_i^\varepsilon)$ .  
 3. Let  $\phi \in L^p(Y_i)$  be a  $Y$ -periodic function and set  $\phi_\varepsilon(x) = \phi\left(\frac{x}{\varepsilon}\right)$ . Then

$$\mathcal{T}_i^\varepsilon(\phi_\varepsilon) \rightarrow \phi \quad \text{strongly in } L^p(\Omega \times Y_i).$$

4. For all  $\phi \in L^1(\Omega_i^\varepsilon)$ , one has

$$\frac{1}{|Y|} \int_{\Omega \times Y_i} \mathcal{T}_i^\varepsilon(\phi)(x, y) dx dy = \int_{\widehat{\Omega}_i^\varepsilon} \phi(x) dx = \int_{\Omega_i^\varepsilon} \phi(x) dx - \int_{\Lambda_i^\varepsilon} \phi(x) dx.$$

5. For all  $\phi \in L^1(\Gamma^\varepsilon)$ , one has

$$\int_{\Gamma^\varepsilon} \phi(x) d\sigma_x = \frac{1}{\varepsilon|Y|} \int_{\Omega \times \Gamma} \mathcal{T}_\varepsilon^b(\phi)(x, y) dx d\sigma_y.$$

1 6.  $\|\mathcal{T}_i^\varepsilon(\phi)\|_{L^p(\Omega \times Y_i)} \leq |Y|^{\frac{1}{p}} \|\phi\|_{L^p(\Omega_i^\varepsilon)}$  for every  $\phi \in L^p(\Omega_i^\varepsilon)$ .

2 7.  $\|\mathcal{T}_\varepsilon^b(\phi)\|_{L^p(\Omega \times \Gamma)} \leq \varepsilon^{\frac{1}{p}} |Y|^{\frac{1}{p}} \|\phi\|_{L^p(\Gamma^\varepsilon)}$  for every  $\phi \in L^p(\Gamma^\varepsilon)$ .

3 8. For  $\phi \in L^p(\Omega)$ ,  $\mathcal{T}_i^\varepsilon(\phi) \rightarrow \phi$  strongly in  $L^p(\Omega \times Y_i)$ .

9. Let  $\{\phi_\varepsilon\}$  be a sequence in  $L^p(\Omega)$  such that  $\phi_\varepsilon \rightarrow \phi$  strongly in  $L^p(\Omega)$ .

Then

$$\mathcal{T}_i^\varepsilon(\phi_\varepsilon) \rightarrow \phi \text{ strongly in } L^p(\Omega \times Y_i).$$

10. Let  $\{\phi_\varepsilon\}$  be a sequence in  $L^p(\Omega_i^\varepsilon)$  such that  $\|\phi_\varepsilon\|_{L^p(\Omega_i^\varepsilon)} \leq c$ .

If  $\mathcal{T}_i^\varepsilon(\phi_\varepsilon) \rightharpoonup \widehat{\phi}$  weakly in  $L^p(\Omega \times Y_i)$ , then

$$\widetilde{\phi}_\varepsilon \rightharpoonup \theta_i \mathcal{M}_{Y_i}(\widehat{\phi}) \text{ weakly in } L^p(\Omega).$$

4 11. If  $\phi \in W^{1,p}(\Omega_i^\varepsilon)$ , then  $\nabla_y[\mathcal{T}_i^\varepsilon(\phi)] = \varepsilon \mathcal{T}_i^\varepsilon(\nabla \phi)$  and  $\mathcal{T}_i^\varepsilon(\phi) \in L^p(\Omega, W^{1,p}(Y_i))$ .

12. If  $\phi \in L^s(\Gamma^\varepsilon)$  for  $s \in [1, +\infty[$ , then

$$\|\mathcal{T}_\varepsilon^b(\phi)\|_{L^s(\Omega \times \Gamma)} \leq |Y|^{\frac{1}{s}} \varepsilon^{\frac{1}{s}} \|\phi\|_{L^s(\Gamma^\varepsilon)}.$$

1 We state below the main propositions proved in [12] and [22] concerning  
 2 the jump on the interface and some convergence results, under the same  
 3 notations as in [21].

**LEMMA 3.4.** [22] *Let  $\varphi \in \mathcal{D}(\Omega)$  and  $u^\varepsilon \in H_\gamma^\varepsilon$ . For  $\varepsilon$  small enough one has*

$$\varepsilon \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) \varphi d\sigma_x = \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y) (\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon)) \mathcal{T}_1^\varepsilon(\varphi) dx d\sigma_y.$$

4 **THEOREM 3.5.** [12][16][21][22] *Let  $\gamma \in \mathbb{R}$  and  $u^\varepsilon$  be a bounded sequence in*  
 5  *$H_\gamma^\varepsilon$ . Then,*

$$\begin{aligned} \|\mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon)\|_{L^2(\Omega \times Y_1)} &\leq c, \\ \|\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon)\|_{L^2(\Omega \times Y_2)} &\leq c, \\ \|\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon)\|_{L^2(\Omega \times \Gamma)} &\leq c\varepsilon^{\frac{1-\gamma}{2}}, \end{aligned} \tag{3.3}$$

6 *and there exist a subsequence (still denoted by  $\varepsilon$ ),  $u_1 \in H_0^1(\Omega)$  and  $\widehat{u}_1 \in$*   
 7  *$L^2(\Omega, H_{per}^1(Y_1))$  with  $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$  for almost every  $x \in \Omega$  such that*

$$\begin{cases} \mathcal{T}_1^\varepsilon(u_1^\varepsilon) \rightarrow u_1 & \text{strongly in } L^2(\Omega, H^1(Y_1)), \\ \mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_1 & \text{weakly in } L^2(\Omega \times Y_1). \end{cases} \tag{3.4}$$

8 *Moreover, if  $\gamma \leq 1$ , there exist a subsequence (still denoted by  $\varepsilon$ ),  $u_2 \in L^2(\Omega)$*   
 9 *and  $\bar{u}_2 \in L^2(\Omega, H^1(Y_2))$  with  $\mathcal{M}_\Gamma(\bar{u}_2) = 0$  for almost every  $x \in \Omega$  such that*

$$\begin{cases} \mathcal{T}_2^\varepsilon(u_2^\varepsilon) \rightharpoonup u_2 & \text{weakly in } L^2(\Omega, H^1(Y_2)), \\ \mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \rightharpoonup \nabla_y \bar{u}_2 & \text{weakly in } L^2(\Omega \times Y_2). \end{cases} \tag{3.5}$$

10 *Furthermore, if  $\gamma < 1$ , then  $u_1 = u_2$  and also*

$$\mathcal{T}_2^\varepsilon(u_2^\varepsilon) \rightarrow u_1 \text{ strongly in } L^2(\Omega, H^1(Y_2)). \tag{3.6}$$

## 1 4 A priori estimates

2 In this section we prove two a priori estimates, uniform with respect to  $\varepsilon$ ,  
 3 for a solution of problem (2.7). In addition, we recall Proposition 4.3 which  
 4 gives a bound to the integral of the singular term close to the singularity.

5 These estimates allow us to prove Proposition 2.4 at the end of this section.

6 **PROPOSITION 4.1.** *Under assumptions  $H_1)$ -  $H_3)$ , let  $u^\varepsilon \in H_\gamma^\varepsilon$  be the solu-*  
 7 *tion of problem (2.7). The following a priori estimate holds true:*

$$\|\nabla u^\varepsilon\|_{L^2(\Omega \setminus \Gamma^\varepsilon)} \leq C_1 \|f\|_{L^{\frac{2}{1+k}}(\Omega)}, \quad (4.1)$$

8 where  $C_1$  depends on  $\alpha$ ,  $c_p$  and  $c_1$ .

9 Also

$$\|u_1^\varepsilon - u_2^\varepsilon\|_{L^2(\Gamma^\varepsilon)} \leq \varepsilon^{-\frac{\gamma}{2}} C_2 \|f\|_{L^{\frac{2}{1+k}}(\Omega)}, \quad (4.2)$$

10 where  $C_2$  depends on  $\alpha$ ,  $h_0$ ,  $c_p$ ,  $c_1$ .

*Proof.* Let  $u^\varepsilon \in H_\gamma^\varepsilon$  be the solution of problem (2.7) and let us choose  $u^\varepsilon$  as  
 test function. By  $H_1)$ - $H_3)$  and applying the Young inequality with exponents  
 $\frac{2}{1-k}$  and  $\frac{2}{1+k}$ , we get for every  $\eta > 0$

$$\begin{aligned} & \alpha \|\nabla u^\varepsilon\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 + \varepsilon^\gamma h_o \int_{\Gamma^\varepsilon} (u_1^\varepsilon - u_2^\varepsilon)^2 d\sigma \\ & \leq \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \nabla u^\varepsilon dx + \int_{\Gamma^\varepsilon} \varepsilon^\gamma h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon)^2 d\sigma \\ & = \int_{\Omega} f \zeta(u^\varepsilon) u^\varepsilon dx \leq \int_{\Omega} f u^{1-k} dx \leq \eta \|u^\varepsilon\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 + c(\eta) \|f\|_{L^{\frac{2}{1+k}}(\Omega)}^2. \end{aligned}$$

In view of Remark 2.1, the above inequality leads to

$$\alpha \|\nabla u^\varepsilon\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 + \varepsilon^\gamma h_o \int_{\Gamma^\varepsilon} (u_1^\varepsilon - u_2^\varepsilon)^2 d\sigma \leq c(\eta, c_p) \|u^\varepsilon\|_{V^\varepsilon \times H^1(\Omega_2^\varepsilon)}^2 + c(\eta) \|f\|_{L^{\frac{2}{1+k}}(\Omega)}^{\frac{2}{1+k}}.$$

Thanks to Proposition 2.2 we get

$$\alpha \|\nabla u^\varepsilon\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 + \varepsilon^\gamma h_o \int_{\Gamma^\varepsilon} (u_1^\varepsilon - u_2^\varepsilon)^2 d\sigma \leq c(\eta, c_p, c_1) \|u^\varepsilon\|_{H_\gamma^\varepsilon}^2 + c(\eta) \|f\|_{L^{\frac{2}{1+k}}(\Omega)}^{\frac{2}{1+k}}.$$

So that,

$$(\alpha - c(\eta, c_p, c_1)) \|\nabla u^\varepsilon\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 + \varepsilon^\gamma (h_o - c(\eta, c_p, c_1)) \|u_1^\varepsilon - u_2^\varepsilon\|_{L^2(\Gamma^\varepsilon)}^2 \leq c(\eta) \|f\|_{L^{\frac{2}{1+k}}(\Omega)}^{\frac{2}{1+k}}.$$

- 1 From this estimate we deduce the result. □

**PROPOSITION 4.2.** *Under assumptions  $H_1$ )- $H_3$ ), let  $u^\varepsilon \in H_\gamma^\varepsilon$  be the solution of problem (2.7). Then*

$$\|f\zeta(u^\varepsilon)\varphi\|_{L^1(\Omega)} \leq c,$$

- 2 *for every nonnegative  $\varphi \in H_\gamma^\varepsilon$  with  $c$  depending on  $\alpha, \beta, c_p, c_1, \|\nabla\varphi\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}$*   
 3 *and  $\|f\|_{L^1(\Omega)}$ .*

*Proof.* Let  $u^\varepsilon \in H_\gamma^\varepsilon$  be the solution of problem (2.7) and let us choose a nonnegative  $\varphi \in H_0^1(\Omega)$  as test function. Since  $\varphi$  has no jump on  $\Gamma^\varepsilon$ , the boundary term vanishes. Hence, by using the Hölder inequality and nonnegativity of  $f, \zeta$  and  $\varphi$ , we have

$$0 \leq \int_{\Omega} f\zeta(u^\varepsilon)\varphi dx = \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \nabla \varphi dx \leq \beta \|\nabla u^\varepsilon\|_{L^2(\Omega \setminus \Gamma^\varepsilon)} \|\nabla \varphi\|_{L^2(\Omega)},$$

which implies the result for  $\varphi \in H_0^1(\Omega)$ , via the estimate (4.1). If now  $\varphi = (\varphi_1^\varepsilon, \varphi_2^\varepsilon) \in H_\gamma^\varepsilon$  is nonnegative, since  $\Gamma_\varepsilon$  is Lipschitz continuous, there exist nonnegative  $\psi_1$  and  $\psi_2$  in  $H_0^1(\Omega)$  such that  $\varphi = (\varphi_1^\varepsilon, \varphi_2^\varepsilon) = (\psi_1|_{\Omega_1^\varepsilon}, \psi_2|_{\Omega_2^\varepsilon})$ . Then

$$\begin{aligned} 0 \leq \int_{\Omega} f\zeta(u^\varepsilon)\varphi dx &= \int_{\Omega_1^\varepsilon} f\zeta(u^\varepsilon)\varphi_1^\varepsilon dx + \int_{\Omega_2^\varepsilon} f\zeta(u^\varepsilon)\varphi_2^\varepsilon dx \\ &\leq \int_{\Omega} f\zeta(u^\varepsilon)\psi_1 dx + \int_{\Omega} f\zeta(u^\varepsilon)\psi_2 dx \leq c. \end{aligned}$$

1

□

- 2 We also recall the following result from [26] (written for  $h = \varepsilon^\gamma h^\varepsilon$  and  $\lambda = 0$ ).  
 3 It gives an estimate of the integral of the singular term close to the singular  
 4 set  $\{u^\varepsilon = 0\}$ .

**PROPOSITION 4.3.** [26] *Under assumptions  $H_1$ )- $H_3$ ), let  $u^\varepsilon \in H_\gamma^\varepsilon$  be the solution of problem (2.7) and  $\delta$  a fixed positive real number. Then*

$$\begin{aligned} \int_{\{0 < u^\varepsilon \leq \delta\}} f\zeta(u^\varepsilon)\varphi dx &\leq \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon)\nabla u^\varepsilon \nabla \varphi Z_\delta(u^\varepsilon) dx \\ &\quad + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon)(Z_\delta(u_1^\varepsilon)\varphi_1 - Z_\delta(u_2^\varepsilon)\varphi_2) d\sigma, \end{aligned}$$

- 5 for every  $\varphi \in H_\gamma^\varepsilon$ ,  $\varphi \geq 0$ , where  $Z_\delta$  is an auxiliary function defined by

$$Z_\delta(s) = \begin{cases} 1, & \text{if } 0 \leq s \leq \delta, \\ -\frac{s}{\delta} + 2, & \text{if } \delta \leq s \leq 2\delta, \\ 0, & \text{if } s \geq 2\delta. \end{cases} \quad (4.3)$$

**Proof of Proposition 2.4.** Let  $u^\varepsilon$  be the solution of problem (2.7). Proposition 4.1 allows us to apply Theorem 3.5. It provides the existence of  $u_1 \in H_0^1(\Omega)$ ,  $\widehat{u}_1 \in L^2(\Omega, H_{per}^1(Y_1))$  with  $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$  for almost every  $x \in \Omega$  and  $\bar{u}_2 \in L^2(\Omega, H^1(Y_2))$  with  $\mathcal{M}_\Gamma(\bar{u}_2) = 0$  for almost every  $x \in \Omega$  such that, up to a subsequence, one has convergences (2.11)<sub>i,iv,v</sub>, since  $\gamma < 1$ .

Now observe that, by construction, for every  $x \in \Omega$  there exists  $\varepsilon_x > 0$  such that

$$x \in \widehat{\Omega}_\varepsilon, \quad \forall \varepsilon \leq \varepsilon_x.$$

1 Also, by definition of unfolding one has, for  $i = 1, 2$

$$\mathcal{T}_i^\varepsilon(\zeta(\phi))(x, y) = \begin{cases} \zeta(\mathcal{T}_i^\varepsilon(\phi)(x, y)) & \text{a.e. in } \widehat{\Omega}_\varepsilon \times Y_i, \\ 0 & \text{a.e. in } \Lambda_\varepsilon \times Y_i. \end{cases} \quad (4.4)$$

2 Consequently, for almost every  $(x, y) \in \Omega \times Y$ , there exists  $\varepsilon_x > 0$  such that

$$\mathcal{T}^\varepsilon(\zeta(u^\varepsilon)) = \zeta(\mathcal{T}^\varepsilon(u^\varepsilon)), \quad \forall \varepsilon \leq \varepsilon_x. \quad (4.5)$$

3 On the other hand, using the continuity of  $\zeta$  and (2.11)<sub>i</sub>, we have

$$\zeta(\mathcal{T}^\varepsilon(u^\varepsilon)) \rightarrow \zeta(u_1) \quad \text{a.e. in } \Omega \times Y. \quad (4.6)$$

This, together with (4.5), gives the convergence (2.11)<sub>ii</sub>.

Also, from Proposition 2.2 and convergence (2.11)<sub>i</sub>, Proposition 3.3<sub>10</sub> gives

$$\widetilde{u}_1^\varepsilon \rightharpoonup \theta_1 \mathcal{M}_{Y_1}(u_1) \quad \text{weakly in } L^2(\Omega),$$



which reads as (2.11)<sub>iii</sub> being  $u_1$  independent on  $y$ .

To show that  $u_1$  is nonnegative almost everywhere in  $\Omega$ , we note that every solution  $u^\varepsilon$  is positive almost everywhere in  $\Omega$ . Then, the definition of the unfolding operator implies  $\mathcal{T}^\varepsilon(u^\varepsilon) \geq 0$  almost everywhere in  $\Omega \times Y$  so that, in view of (2.11)<sub>i</sub>,

$$u_1 \geq 0 \quad \text{a.e. in } \Omega.$$

- 1 It remains to prove the second condition in (2.12). Let us choose first a  
 2 nonnegative  $\varphi \in H_0^1(\Omega)$ . Propositions 3.3<sub>2,4</sub> and 4.2, for the subsequence  
 3 mentioned before, lead to

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}^\varepsilon(f) \mathcal{T}^\varepsilon(\zeta(u^\varepsilon)) \mathcal{T}^\varepsilon(\varphi) dx dy \leq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} f \zeta(u^\varepsilon) \varphi dx < +\infty. \quad (4.7)$$

Now, from Proposition 3.3<sub>8</sub>,  $\mathcal{T}^\varepsilon(f)$  and  $\mathcal{T}^\varepsilon(\varphi)$  converge to  $f$  and  $\varphi$ , respectively, almost everywhere in  $\Omega \times Y$ , up to a subsequence. Thus, by (2.11)<sub>ii</sub>,

$$\mathcal{T}^\varepsilon(f) \mathcal{T}^\varepsilon(\zeta(u^\varepsilon)) \mathcal{T}^\varepsilon(\varphi) \rightarrow f \zeta(u_1) \varphi \quad \text{a.e. in } \Omega \times Y.$$

Since  $\mathcal{T}^\varepsilon(f)$ ,  $\mathcal{T}^\varepsilon(\zeta(u^\varepsilon))$  and  $\mathcal{T}^\varepsilon(\varphi)$  are nonnegative functions, we can use Fatou's lemma and (4.7) to obtain

$$\frac{1}{|Y|} \int_{\Omega \times Y} f \zeta(u_1) \varphi dx dy \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}^\varepsilon(f) \mathcal{T}^\varepsilon(\zeta(u^\varepsilon)) \mathcal{T}^\varepsilon(\varphi) dx dy < +\infty.$$

Being the functions  $f$  and  $u_1$  independent on  $y$ , this implies in particular that

$$\int_{\Omega} f \zeta(u_1) \varphi dx < +\infty$$

1 and ends the proof for  $\varphi \geq 0$ . For  $\varphi$  with any sign, it suffices to decompose  
 2  $\varphi$  as  $\varphi^+ - \varphi^-$ . □

### 3 **5 A convergence result for an auxiliary problem**

4 This section is devoted to the study of a suitable auxiliary problem and a  
 5 crucial convergence result, which is one of the main tools needed for proving  
 6 Theorem 2.6. In the same spirit of what we have done in [25] in order to  
 7 homogenize a similar singular problem posed in a periodically perforated  
 8 domain, we first consider the auxiliary linear problem (5.2) below and state  
 9 existence and homogenization results for it. Then, we prove the convergence  
 10 result given in Theorem 5.4. It shows that the gradient of  $u^\varepsilon$  is equivalent to  
 11 the gradient of the solution of the auxiliary linear problem, associated with  
 12 a weak cluster point of the sequence  $\{u^\varepsilon\}$ , as  $\varepsilon \rightarrow 0$ .

13 We refer to [10] and [22] for the study of the auxiliary problem. For the  
 14 proof of the convergence result we adapt some techniques from [19] and [25],  
 15 which inspired this work.

16 First of all, as in [10] we introduce a linear operator  $\mathcal{L}_\varepsilon$  from  $H^{-1}(\Omega)$  to  
 17  $(V^\varepsilon)'$  verifying the following assumption:

18 H<sub>4</sub>) If  $\{\varphi^\varepsilon\}$  is a sequence such that

$$\|\varphi^\varepsilon\|_{V^\varepsilon} \leq c \quad \text{and} \quad \widetilde{\varphi^\varepsilon} \rightharpoonup \theta_1 \varphi_0 \text{ weakly in } L^2(\Omega), \quad (5.1)$$

then

$$\lim_{\varepsilon \rightarrow 0} \langle \mathcal{L}_\varepsilon(Z), \varphi^\varepsilon \rangle_{(V^\varepsilon)', V^\varepsilon} = \langle Z, \varphi_0 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

1 Let us point out that assumption  $H_4$ ) is satisfied, for example, by the adjoint  
 2 of the linear operator  $P_1^\varepsilon$  introduced by Cioranescu and Saint Jean Paulin in  
 3 [17]. Then, see [11, Remark 3.1], if  $H_4$ ) holds, one has  $\varphi_0 \in H_0^1(\Omega)$ .

#### 4 **5.1 The auxiliary problem**

5 The suitable auxiliary problem we are interested in is the following one:

$$\left\{ \begin{array}{ll} -\operatorname{div}(A^\varepsilon(x, u_1^\varepsilon)\nabla v_1^\varepsilon) = \mathcal{L}_\varepsilon(Z) & \text{in } \Omega_1^\varepsilon, \\ -\operatorname{div}(A^\varepsilon(x, u_2^\varepsilon)\nabla v_2^\varepsilon) = 0 & \text{in } \Omega_2^\varepsilon, \\ v_1^\varepsilon = 0 & \text{on } \partial\Omega, \\ (A^\varepsilon(x, u_1^\varepsilon)\nabla v_1^\varepsilon)\nu^\varepsilon = (A^\varepsilon(x, u_2^\varepsilon)\nabla v_2^\varepsilon)\nu^\varepsilon & \text{on } \Gamma^\varepsilon, \\ (A^\varepsilon(x, u_1^\varepsilon)\nabla v_1^\varepsilon)\nu^\varepsilon = -\varepsilon^\gamma h^\varepsilon(v_1^\varepsilon - v_2^\varepsilon) & \text{on } \Gamma^\varepsilon, \end{array} \right. \quad (5.2)$$

6 under assumptions  $H_1)$ ,  $H_3)$ - $H_4)$ , and  $Z \in H^{-1}(\Omega)$ .

7 The variational formulation of problem (5.2) is

$$\left\{ \begin{array}{l} \text{Find } v^\varepsilon \in H_\gamma^\varepsilon \text{ such that} \\ \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon)\nabla v^\varepsilon \nabla \varphi dx + \int_{\Gamma^\varepsilon} \varepsilon^\gamma h^\varepsilon(v_1^\varepsilon - v_2^\varepsilon)(\varphi_1 - \varphi_2) d\sigma \\ = \langle \mathcal{L}_\varepsilon(Z), \varphi_1 \rangle_{(V^\varepsilon)', V^\varepsilon}, \quad \forall \varphi \in H_\gamma^\varepsilon. \end{array} \right. \quad (5.3)$$

8 The existence and the uniqueness of a solution of problem (5.3) is a straight-  
 9 forward consequence of the Lax-Milgram theorem.

10 Observe now that in view of Proposition 4.1 and (2.11)*iii*, the sequence  $\{u^\varepsilon\}$

1 satisfies conditions (5.1), i.e.

$$\|u_1^\varepsilon\|_{V^\varepsilon} \leq c \quad \text{and} \quad \widetilde{u}_1^\varepsilon \rightharpoonup \theta_1 u_1 \text{ weakly in } L^2(\Omega). \quad (5.4)$$

2 The homogenization result below extends Theorem 3.3 of [22] to the case  
 3 where the matrix field depends on  $x$ . We omit its proof, which follows along  
 4 the line of [22, Theorem 3.3]. The terms containing the functional  $Z$  and the  
 5 quasilinearity of  $A$  can be treated as in [10] and [7], respectively.

6 **THEOREM 5.1.** *Under assumptions  $H_1)$ ,  $H_3)$ - $H_4)$  and  $Z \in H^{-1}(\Omega)$ , let  
 7  $v^\varepsilon \in H_\gamma^\varepsilon$  be the unique solution of problem (5.3). Then, there exist  $v_1 \in$   
 8  $H_0^1(\Omega)$  and  $\widehat{v}_1 \in L^2(\Omega, H_{per}^1(Y_1))$  with  $\mathcal{M}_\Gamma(\widehat{v}_1) = 0$  such that*

$$\left\{ \begin{array}{ll} i) \widetilde{v}_i^\varepsilon \rightharpoonup \theta_i v_1 & \text{weakly in } L^2(\Omega), \quad i = 1, 2, \\ ii) \mathcal{T}_i^\varepsilon(v_i^\varepsilon) \rightarrow v_1 & \text{strongly in } L^2(\Omega; H^1(Y_i)), \quad i = 1, 2 \\ iii) \mathcal{T}_1^\varepsilon(\nabla v_1^\varepsilon) \rightharpoonup \nabla v_1 + \nabla_y \widehat{v}_1 & \text{weakly in } L^2(\Omega \times Y_1), \\ iv) \mathcal{T}_2^\varepsilon(\nabla v_2^\varepsilon) \rightharpoonup 0 & \text{weakly in } L^2(\Omega \times Y_2), \end{array} \right. \quad (5.5)$$

and the pair  $(v_1, \widehat{v}_1)$  is the unique solution of the limit equation

$$\left\{ \begin{array}{l} \forall \varphi \in H_0^1(\Omega) \text{ and } \forall \psi \in L^2(\Omega; H_{per}^1(Y_1)), \\ \int_{\Omega \times Y_1} A(y, u_1)(\nabla v_1 + \nabla_y \widehat{v}_1)(\nabla \varphi + \nabla_y \psi) dx dy = |Y| \langle Z, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \end{array} \right.$$

1 Finally,  $v_1$  is the unique solution of the following limit problem:

$$\begin{cases} -\operatorname{div}(A^0(u_1)\nabla v_1) = Z & \text{in } \Omega, \\ v_1 = 0 & \text{on } \partial\Omega, \end{cases} \quad (5.6)$$

2 where the homogenized matrix  $A^0$  is given by (2.8).

3 Let us consider problem (5.2) for  $Z = -\operatorname{div}(A^0(u_1)\nabla u_1)$ . Its variational  
4 formulation is

$$\begin{aligned} \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla v^\varepsilon \nabla \varphi dx + \int_{\Gamma^\varepsilon} \varepsilon^\gamma h^\varepsilon (v_1^\varepsilon - v_2^\varepsilon) (\varphi_1 - \varphi_2) d\sigma \\ = \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_1)\nabla u_1)), \varphi_1 \rangle_{(V^\varepsilon)', V^\varepsilon}, \quad \forall \varphi \in H_\gamma^\varepsilon, \end{aligned} \quad (5.7)$$

which admits a unique solution  $v^\varepsilon \in H_\gamma^\varepsilon$ . Theorem 5.1 (written for  $Z = -\operatorname{div}(A^0(u_1)\nabla u_1)$ ) gives

$$\begin{cases} -\operatorname{div}(A^0(u_1)\nabla v_1) = -\operatorname{div}(A^0(u_1)\nabla u_1) & \text{in } \Omega, \\ v_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

5 Thanks to the assumptions on  $A$ , the uniqueness of this problem ensures  
6 that

$$v_1 = u_1. \quad (5.8)$$

7 Moreover, the same arguments used to prove (2.11)<sub>iii</sub> give

$$\|v^\varepsilon\|_{H_\gamma^\varepsilon} \leq c \quad \text{and} \quad \tilde{v}_1^\varepsilon \rightharpoonup \theta_1 u_1 \quad \text{weakly in } L^2(\Omega). \quad (5.9)$$

As in [19] we define now the following auxiliary functions  $u_m$ :

$$\forall m \in \mathbb{N}, m \geq 1, \quad u_m \doteq T_m(u_1),$$

1 where  $T_m$  is the usual truncation function at level  $m$ , so that

$$0 \leq u_m \leq u_1 \quad \text{and} \quad u_m \rightarrow u_1 \quad \text{strongly in } H_0^1(\Omega), \quad \text{as } m \rightarrow +\infty. \quad (5.10)$$

Then, we denote  $v_\varepsilon^m \in H_0^1(\Omega)$  the solution of the following problem:

$$\begin{cases} -\operatorname{div}(A^\varepsilon(x, u_1^\varepsilon) \nabla v_{\varepsilon,1}^m) = \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_1) \nabla u_m)) & \text{in } \Omega_1^\varepsilon, \\ -\operatorname{div}(A^\varepsilon(x, u_2^\varepsilon) \nabla v_{\varepsilon,2}^m) = 0 & \text{in } \Omega_1^\varepsilon, \\ v_\varepsilon^m = 0 & \text{on } \partial\Omega. \end{cases}$$

2 Its variational formulation is

$$\int_{\Omega} A^\varepsilon(x, u^\varepsilon) \nabla v_\varepsilon^m \nabla \varphi dx = \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_1) \nabla u_m)), \varphi_1 \rangle_{(V^\varepsilon)', V^\varepsilon}, \quad \forall \varphi \in H_\gamma^\varepsilon. \quad (5.11)$$

3 Again, the existence and the uniqueness of a solution of problem (5.11) is  
4 straightforward proved by using the Lax-Milgram theorem.

5 **REMARK 5.2.** Let us notice that  $v_\varepsilon^m \in H_0^1(\Omega)$ , and consequently  $v_{\varepsilon,1}^m = v_{\varepsilon,2}^m$   
6 on  $\Gamma^\varepsilon$ , for every  $\varepsilon$ .

7 Thus, in particular, Theorem 5.1 applies to this case with no jump (for  
8  $Z = -\operatorname{div}(A^0(u_1) \nabla u_m)$ ) and, again by uniqueness, we obtain

$$\|v_\varepsilon^m\|_{H_\gamma^\varepsilon} \leq c \quad \text{and} \quad \widetilde{v_{\varepsilon,1}^m} \rightharpoonup \theta_1 u_m \quad \text{weakly in } L^2(\Omega). \quad (5.12)$$

1 Also the following convergence holds true:

$$\mathcal{T}^\varepsilon(v_\varepsilon^m) \rightarrow u_m \text{ strongly in } L^2(\Omega; H^1(Y)). \quad (5.13)$$

2 Moreover, by classical results from [32], we have that for every fixed  $m$

$$\|v_\varepsilon^m\|_{L^\infty(\Omega)} \leq c_m, \text{ for every } \varepsilon. \quad (5.14)$$

**REMARK 5.3.** The sequence  $\{(v_{\varepsilon,1}^m)^-\}$  satisfies conditions (5.1). Indeed, in view of the estimate in (5.12) and convergence (5.13), one has

$$\|(v_{\varepsilon,1}^m)^-\|_{V_\varepsilon} \leq \|v_{\varepsilon,1}^m\|_{V_\varepsilon} \leq c,$$

and

$$\mathcal{T}_1^\varepsilon((v_{\varepsilon,1}^m)^-) \rightarrow u_m^- = 0 \text{ strongly in } L^2(\Omega \times Y_1),$$

since  $u_m \geq 0$  by construction. Also, by using Proposition 3.3<sub>10</sub>, one has

$$\widetilde{(v_{\varepsilon,1}^m)^-} \rightharpoonup \theta_1 \mathcal{M}_{Y_1}(u_m^-) = 0 \text{ weakly in } L^2(\Omega).$$

Let us also point out that, from (5.13), one has even

$$\mathcal{T}_2^\varepsilon((v_{\varepsilon,2}^m)^-) \rightarrow u_m^- = 0 \text{ strongly in } L^2(\Omega \times Y_2).$$

3 This is needed in the next subsection.

1 **5.2 A convergence result**

2 We are now able to prove the main result of this section. Here we adapt  
 3 the arguments we used in [25] for the quasilinear singular case in period-  
 4 ically perforated domains to the two-component case, where the holes are  
 5 replaced by the second component. This is why we analyze in detail only the  
 6 terms differing from the previous work, namely the boundary term and the  
 7 quasilinear diffusion term in the second component.

8 The proof follows the same steps introduced in that of [19, Theorem 8.5],  
 9 which concerns the singular case when  $A$  is linear and the domain is made  
 10 up of two connected components separated by an oscillating interface.

11

**THEOREM 5.4.** *Under assumptions  $H_1)$ -  $H_4)$ , let  $u^\varepsilon$  and  $v^\varepsilon$  be solutions of problems (2.7) and (5.7), respectively. Then, up to a subsequence,*

$$\lim_{\varepsilon \rightarrow 0} \|\nabla u^\varepsilon - \nabla v^\varepsilon\|_{L^2(\Omega \setminus \Gamma^\varepsilon)} = 0.$$

12 *Proof.* The proof is done in 3 steps.

13 **Step 1.** Let us prove that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Gamma^\varepsilon} |\nabla (v_\varepsilon^m)^-|^2 dx = 0 \quad \forall m \geq 1. \quad (5.15)$$

Let us choose  $-(v_\varepsilon^m)^- \in H_\gamma^\varepsilon$  as test function in (5.11). From the ellipticity



of  $A$ , taking into account that  $\nabla v_\varepsilon^m = \nabla(v_\varepsilon^m)^+ - \nabla(v_\varepsilon^m)^-$ , and (2.3) we get

$$\begin{aligned} 0 &\leq \alpha \|\nabla(v_\varepsilon^m)^-\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 \leq - \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla v_\varepsilon^m \nabla(v_\varepsilon^m)^- dx \\ &= \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_1) \nabla u_m)), -(v_{\varepsilon,1}^m)^- \rangle_{(V^\varepsilon)', V^\varepsilon}. \end{aligned}$$

From Remark 5.3 and  $H_4$ ), we obtain

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_1) \nabla u_m)), -(v_{\varepsilon,1}^m)^- \rangle_{(V^\varepsilon)', V^\varepsilon} \\ &= \langle -\operatorname{div}(A^0(u_1) \nabla u_m), 0 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0, \end{aligned}$$

- 1 which concludes the first step.
- 2 **Step 2.** In this step we show that

$$\lim_{m \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Gamma^\varepsilon} |\nabla(u^\varepsilon - v_\varepsilon^m)|^2 dx = 0. \quad (5.16)$$

- 3 To do that, let us choose  $u^\varepsilon - v_\varepsilon^m \in H_\gamma^\varepsilon$  as test function in (2.7) and (5.11).
- 4 By subtraction,  $H_1)_i$  and the nonnegativity of the boundary term, one has

$$\begin{aligned} &\alpha \|\nabla(u^\varepsilon - v_\varepsilon^m)\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 \\ &\leq \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla(u^\varepsilon - v_\varepsilon^m) \nabla(u^\varepsilon - v_\varepsilon^m) dx + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon)^2 d\sigma \\ &= \int_{\Omega} f\zeta(u^\varepsilon)(u^\varepsilon - v_\varepsilon^m) dx - \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_1) \nabla u_m)), (u^\varepsilon - v_\varepsilon^m)_1 \rangle_{V_\varepsilon', V_\varepsilon}. \end{aligned} \quad (5.17)$$

1 Let us prove that, as  $\varepsilon \rightarrow 0$ , we have

$$\begin{aligned} \alpha \limsup_{\varepsilon \rightarrow 0} \|\nabla(u^\varepsilon - v_\varepsilon^m)\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 &\leq \int_{\Omega} f\zeta(u_1)(u_1 - u_m)\chi_{\{u_1 > 0\}} dx \\ &- \langle (-\operatorname{div}(A^0(u_1)\nabla u_m)), u_1 - u_m \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \end{aligned} \quad (5.18)$$

2 so that we obtain the result (5.16) as  $m \rightarrow +\infty$ , via convergence (5.10) and  
3 the Lebesgue theorem.

4 Concerning the second term in the right-hand side of the previous inequality,  
5 let us observe that  $u^\varepsilon - v_\varepsilon^m$  satisfies (5.1), in view of (5.4) and (5.12). So  
6 that, by  $H_4$ )

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_1)\nabla u_m)), (u^\varepsilon - v_\varepsilon^m)_1 \rangle_{V'_\varepsilon, V_\varepsilon} \\ = \langle -\operatorname{div}(A^0(u_1)\nabla u_m), u_1 - u_m \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \end{aligned} \quad (5.19)$$

7 Let  $\delta > 0$ . We split the integral of the singular term in (5.17) in two terms  
8 as follows:

$$\begin{aligned} \int_{\Omega} f\zeta(u^\varepsilon)(u^\varepsilon - v_\varepsilon^m) dx &= \int_{\Omega} f\zeta(u^\varepsilon)(u^\varepsilon - (v_\varepsilon^m)^+) dx + \int_{\Omega} f\zeta(u^\varepsilon)(v_\varepsilon^m)^- dx \\ &\leq \sum_{i=1}^2 (I_{\varepsilon,i}^\delta + J_{\varepsilon,i}^\delta) + K_\varepsilon, \quad i = 1, 2, \end{aligned} \quad (5.20)$$

where, for  $i = 1, 2$ ,

$$I_{\varepsilon,i}^\delta \doteq \int_{\Omega_i^\varepsilon \cap \{0 < u_i^\varepsilon \leq \delta\}} f\zeta(u_i^\varepsilon)u_i^\varepsilon dx, \quad J_{\varepsilon,i}^\delta \doteq \int_{\Omega_i^\varepsilon \cap \{u_i^\varepsilon > \delta\}} f\zeta(u_i^\varepsilon)(u_i^\varepsilon - (v_{\varepsilon,i}^m)^+) dx,$$

$$\text{and } K_\varepsilon \doteq \int_{\Omega} f\zeta(u^\varepsilon)(v_\varepsilon^m)^- dx.$$

- 1 The terms corresponding to  $i = 1$  are exactly the ones considered in [25,  
 2 Theorem 5.5] for the case of periodically perforated domains. It is easy to  
 3 check that the additional terms  $I_{\varepsilon,2}^\delta$  and  $J_{\varepsilon,2}^\delta$  can be treated exactly in the  
 4 same way. Indeed, all the convergences used there for the first component  
 5 hold true also here in the second component, thanks to (2.1) and convergence  
 6 (3.6).  
 7 Hence,

$$\left\{ \begin{array}{l} \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} (I_{\varepsilon,1}^\delta + I_{\varepsilon,2}^\delta) = 0, \\ \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} (J_{\varepsilon,1}^\delta + J_{\varepsilon,2}^\delta) = (\theta_1 + \theta_2) \int_{\Omega} f\zeta(u_1)(u_1 - u_m)\chi_{\{u_1 > 0\}} dx \\ \qquad \qquad \qquad = \int_{\Omega} f\zeta(u_1)(u_1 - u_m)\chi_{\{u_1 > 0\}} dx. \end{array} \right. \quad (5.21)$$

On the contrary, the term  $K_\varepsilon$  needs to be treated specifically, since its computation gives rise to a different boundary term. To this aim, let us observe that

$$\int_{\Omega \cap \{u^\varepsilon > \delta\}} f\zeta(u^\varepsilon)(v_\varepsilon^m)^- \chi_{\{u_1 = \delta\}} dx = 0,$$

- 8 for every  $\delta \in \mathbb{R}^+ \setminus D$ , where  $D$  is a countable set of values (see for instance  
 9 [28], [19] and [25]).  
 10 Hence, for  $\delta_0 \in \mathbb{R}^+ \setminus D$ , we can write

$$K_\varepsilon = \int_{\Omega \cap \{0 < u^\varepsilon \leq \delta_0\}} f\zeta(u^\varepsilon)(v_\varepsilon^m)^- dx + \int_{\Omega \cap \{u^\varepsilon > \delta_0\}} f\zeta(u^\varepsilon)(v_\varepsilon^m)^- \chi_{\{u_1 \neq \delta_0\}} dx \doteq A_\varepsilon + B_\varepsilon. \quad (5.22)$$

From Proposition 4.3 written with  $\delta = \delta_0$  we get

$$0 \leq A_\varepsilon \leq \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \nabla (v_\varepsilon^m)^- Z_{\delta_0}(u^\varepsilon) dx \\ + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) (Z_{\delta_0}(u_1^\varepsilon)(v_{\varepsilon,1}^m)^- - Z_{\delta_0}(u_2^\varepsilon)(v_{\varepsilon,2}^m)^-) d\sigma.$$

The first term in the right-hand side of the previous inequality goes to zero as  $\varepsilon$  goes to zero, because of  $H_1)_i$ , the Hölder inequality, (4.1) and (5.15). On the other hand, since  $v_\varepsilon^m \in H_0^1(\Omega)$ , also  $(v_\varepsilon^m)^-$  belongs to  $H_0^1(\Omega)$ , so that  $(v_{\varepsilon,1}^m)^- = (v_{\varepsilon,2}^m)^-$  on  $\Gamma^\varepsilon$  and, since  $Z_\delta$  is nonincreasing (see definition (4.3)), we have

$$\varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) (Z_{\delta_0}(u_1^\varepsilon)(v_{\varepsilon,1}^m)^- - Z_{\delta_0}(u_2^\varepsilon)(v_{\varepsilon,2}^m)^-) d\sigma \\ = \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(v_{\varepsilon,1}^m)^- (u_1^\varepsilon - u_2^\varepsilon) (Z_{\delta_0}(u_1^\varepsilon) - Z_{\delta_0}(u_2^\varepsilon)) d\sigma \leq 0.$$

1 Thus we deduce that

$$\lim_{\varepsilon \rightarrow 0} A_\varepsilon = 0. \quad (5.23)$$

As done for the previous terms, in order to handle  $B_\varepsilon$  we split it as follows:

$$B_\varepsilon = B_{\varepsilon,1} + B_{\varepsilon,2} \text{ with } B_{\varepsilon,i} \doteq \int_{\Omega_i^\varepsilon \cap \{u_i^\varepsilon > \delta_0\}} f\zeta(u_i^\varepsilon)(v_{\varepsilon,i}^m)^- \chi_{\{u_1 \neq \delta_0\}} dx, i = 1, 2.$$

The same arguments used to prove that  $\lim_{\varepsilon \rightarrow 0} B_{\varepsilon,1} = 0$  in the proof of [25, Theorem 5.5] apply also to the second component. So, using again (2.1) and convergence (3.6), we get

$$\lim_{\varepsilon \rightarrow 0} B_{\varepsilon,2} = 0.$$

1 This implies, together with (5.22)-(5.23),

$$\lim_{\varepsilon \rightarrow 0} K_\varepsilon = 0. \quad (5.24)$$

2 By collecting (5.17), (5.19), (5.20), (5.21), (5.24), we finally obtain the va-  
 3 lidity of (5.18).

4 **Step 3.** In this last step, we show that

$$\lim_{m \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Gamma^\varepsilon} |\nabla(v_\varepsilon^m - v^\varepsilon)|^2 dx = 0. \quad (5.25)$$

We take  $v_\varepsilon^m - v^\varepsilon$  as test function in the variational formulations (5.11) and (5.7). By subtraction, we have

$$\begin{aligned} & \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla(v_\varepsilon^m - v^\varepsilon) \nabla(v_\varepsilon^m - v^\varepsilon) dx - \varepsilon^\gamma \int_{\Gamma} h^\varepsilon(v_1^\varepsilon - v_2^\varepsilon) [(v_\varepsilon^m - v^\varepsilon)_1 - (v_\varepsilon^m - v^\varepsilon)_2] d\sigma \\ &= \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_1) \nabla(u_m - u_1))), (v_\varepsilon^m - v^\varepsilon)_1 \rangle_{V'_\varepsilon, V_\varepsilon}. \end{aligned}$$

In view of Remark 5.2, it results

$$-\varepsilon^\gamma \int_{\Gamma} h^\varepsilon(v_1^\varepsilon - v_2^\varepsilon) [(v_\varepsilon^m - v^\varepsilon)_1 - (v_\varepsilon^m - v^\varepsilon)_2] d\sigma = \varepsilon^\gamma \int_{\Gamma} h^\varepsilon(v_1^\varepsilon - v_2^\varepsilon)^2 d\sigma \geq 0.$$

Consequently, by passing to the limit on  $\varepsilon$ , for  $H_1$ ) and  $H_4$ ) (whose assump-

tions are satisfied both by  $v_\varepsilon$  and  $v_\varepsilon^m$  thanks to (5.9) and (5.12)), we get

$$\begin{aligned} 0 &\leq \alpha \lim_{\varepsilon \rightarrow 0} \|\nabla(v_\varepsilon^m - v^\varepsilon)\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 \\ &\leq \lim_{\varepsilon \rightarrow 0} \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_1)\nabla(u_m - u_1))), (v_\varepsilon^m - v^\varepsilon)_1 \rangle_{V'_\varepsilon, V_\varepsilon} \\ &= \langle -\operatorname{div}(A^0(u_1)\nabla(u_m - u_1)), u_m - u_1 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \end{aligned}$$

This together with (5.10) gives

$$\begin{aligned} 0 &\leq \lim_{m \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \alpha \|\nabla(v_\varepsilon^m - v^\varepsilon)\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 \\ &\leq \lim_{m \rightarrow +\infty} \int_{\Omega} A^0(u_1)\nabla(u_m - u_1)\nabla(u_m - u_1)dx = 0. \end{aligned}$$

1 At last, coupling (5.16) and (5.25) we obtain the desired result. □

## 2 **6 Proofs of Proposition 2.5 and Theorem 2.6**

3 In this last section we prove Proposition 2.5 and Theorem 2.6.

4 First, we treat Proposition 2.5 which is a consequence of the convergence  
5 result given in Theorem 5.4.

6 **Proof of Proposition 2.5.** Let  $v^\varepsilon$  be the solution of problem (5.7). The ho-  
7 mogenization result given in Theorem 5.1 (written for  $Z = -\operatorname{div}(A^0(u_1)\nabla u_1)$   
8 and  $v_1 = u_1$ , see (5.8)) gives, for the subsequence verifying (2.4),

$$\begin{cases} \mathcal{T}_1^\varepsilon(\nabla v_1^\varepsilon) \rightharpoonup \nabla u_1 + \nabla_y \widehat{v}_1 & \text{weakly in } L^2(\Omega \times Y_1), \\ \mathcal{T}_2^\varepsilon(\nabla v_2^\varepsilon) \rightharpoonup 0 & \text{weakly in } L^2(\Omega \times Y_2), \end{cases} \quad (6.1)$$

where  $\widehat{v}_1$  is a function in  $L^2(\Omega; H_{per}^1(Y_1))$  with  $\mathcal{M}_\Gamma(\widehat{v}_1) = 0$ , such that

$$\widehat{v}_1(x, y) = - \sum_{i=1}^N \chi_{e_i}(y, u_1(x)) \frac{\partial u_1}{\partial x_i}(x).$$

Now let us observe that, from Theorem 5.4 and Proposition 3.39, we get

$$\mathcal{T}^\varepsilon(\nabla u^\varepsilon - \nabla v^\varepsilon) \rightarrow 0 \quad \text{strongly in } L^2(\Omega \times Y).$$

- 1 This, together with (2.11)<sub>*iv,v*</sub> and (6.1), leads to

$$\begin{cases} \nabla_y \widehat{v}_1 = \nabla_y \widehat{u}_1 & \text{a.e. in } \Omega \times Y_1, \\ \nabla_y \widehat{u}_2 \equiv 0 & \text{a.e. in } \Omega \times Y_2, \end{cases} \quad (6.2)$$

which implies  $\widehat{v}_1 = \widehat{u}_1 + w(x)$ , for some function  $w$  only depending on  $x$ .

Since  $\mathcal{M}_\Gamma(\widehat{v}_1) = \mathcal{M}_\Gamma(\widehat{u}_1) = 0$  and

$$\mathcal{M}_\Gamma(\widehat{v}_1) = \mathcal{M}_\Gamma(\widehat{u}_1) + \mathcal{M}_\Gamma(w),$$

we derive  $w = 0$  and

$$\widehat{v}_1 = \widehat{u}_1.$$

- 2 Whence we obtain the convergence

$$\mathcal{T}_1^\varepsilon(\nabla v_1^\varepsilon) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_1 \quad \text{weakly in } L^2(\Omega \times Y_1), \quad (6.3)$$

- 3 and the claimed expression of  $\widehat{u}_1$ . □

- 4 We are now able to prove the homogenization theorem.

- 1 **Proof of Theorem 2.6.** Let us first remark that, under our assumptions,  
 2 convergences (2.11) hold true for a subsequence of  $u^\varepsilon$  (still denoted by  $\varepsilon$ ),  
 3 where  $\widehat{u}_1 \in L^2(\Omega; H_{per}^1(Y_1))$  is given by Proposition 2.5. Let us highlight  
 4 also that

$$\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \rightharpoonup 0 \text{ weakly in } L^2(\Omega \times Y_2), \quad (6.4)$$

by collecting (2.11)<sub>iv</sub> and (6.2)<sub>ii</sub>.

Notice that, due to (2.12), the function  $u_1 \in H_0^1(\Omega)$  is nonnegative. Also, conditions (5.4) ensures the validity of  $H_4$ ) and, as a consequence, Theorem 5.4 holds true.

Now, to identify the limit problem satisfied by  $(u_1, \widehat{u}_1)$ , we take  $\varphi, \phi \in \mathcal{D}(\Omega)$  and  $\xi \in H_{per}^1(Y)$ , and use

$$\psi_\varepsilon(x) = \varphi(x) + \varepsilon \phi(x) \xi \left( \frac{x}{\varepsilon} \right) \in H_\gamma^\varepsilon$$

- 5 as test function in (2.7). We obtain

$$\int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \nabla \psi_\varepsilon dx + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) (\psi_{\varepsilon,1} - \psi_{\varepsilon,2}) d\sigma = \int_{\Omega} f \zeta(u^\varepsilon) \psi_\varepsilon dx. \quad (6.5)$$

- 6 Let us consider the solution  $v^\varepsilon$  of problem (5.3) and write

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \nabla \psi_\varepsilon dx &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla v^\varepsilon \nabla \psi_\varepsilon dx \\ &+ \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla (u^\varepsilon - v^\varepsilon) \nabla \psi_\varepsilon dx. \end{aligned} \quad (6.6)$$



- 1 By using assumption  $H_1$ ), the Hölder inequality and Theorem 5.4, one has

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla(u^\varepsilon - v^\varepsilon) \nabla \psi_\varepsilon dx = 0, \quad (6.7)$$

- 2 taking into account that the norm of  $\psi_\varepsilon$  is bounded in  $H_\gamma^\varepsilon$ .  
 3 By Theorem 5.1 (written for  $Z = -\operatorname{div}(A^0(u_1) \nabla u_1)$ ), we obtain

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla v^\varepsilon \nabla \psi_\varepsilon dx \\ &= \frac{1}{|Y|} \int_{\Omega \times Y_1} A(y, u_1) (\nabla u_1 + \nabla_y \hat{u}_1) (\nabla \varphi + \phi \nabla_y \xi) dx dy, \end{aligned} \quad (6.8)$$

in view of (6.3).

Concerning the boundary term in (6.5), by unfolding, using Lemma 3.4 and (3.3) we get

$$\begin{aligned} & \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) (\psi_{\varepsilon,1} - \psi_{\varepsilon,2}) d\sigma = \varepsilon^{\gamma+1} \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) (\phi_1 \xi_1 - \phi_2 \xi_2) d\sigma \\ &= \frac{\varepsilon^\gamma}{|Y|} \int_{\Omega \times \Gamma} h(y) [\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon)] [\xi_1(y) \mathcal{T}_1^\varepsilon(\varphi_1) - \xi_2(y) \mathcal{T}_2^\varepsilon(\varphi_2)] dx d\sigma_y \\ &\leq c \varepsilon^\gamma \|\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon)\|_{L^2(\Omega \times \Gamma)} \|\mathcal{T}_1^\varepsilon(\varphi_1) [\xi_1(y) - \xi_2(y)]\|_{L^2(\Omega \times \Gamma)} \\ &\leq c \varepsilon^\gamma \varepsilon^{\frac{1-\gamma}{2}} = c \varepsilon^{\frac{1+\gamma}{2}}. \end{aligned}$$

- 4 Since  $-1 < \gamma < 1$ , passing to the limit as  $\varepsilon \rightarrow 0$ , we obtain

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) (\psi_{\varepsilon,1} - \psi_{\varepsilon,2}) d\sigma = 0. \quad (6.9)$$

1 In order to pass to the limit in the singular term of (6.5), we define

$$\mu_\varepsilon(x) \doteq \varepsilon\phi(x)\xi\left(\frac{x}{\varepsilon}\right), \quad \text{that is } \psi_\varepsilon = \varphi + \mu_\varepsilon. \quad (6.10)$$

Then one has

$$\mathcal{T}^\varepsilon(\mu_\varepsilon) = \varepsilon\mathcal{T}^\varepsilon(\phi)\xi \quad \text{and} \quad \nabla\mu_\varepsilon = \varepsilon\nabla\phi\xi\left(\frac{\cdot}{\varepsilon}\right) + \phi\nabla_y\xi\left(\frac{\cdot}{\varepsilon}\right),$$

2 and

$$\begin{cases} \text{i) } \mathcal{T}^\varepsilon(\mu_\varepsilon) \rightarrow 0 & \text{strongly in } L^2(\Omega \times Y), \\ \text{ii) } \mathcal{T}^\varepsilon(\nabla\mu_\varepsilon) \rightarrow \phi\nabla_y\xi & \text{strongly in } L^2(\Omega \times Y). \end{cases} \quad (6.11)$$

3 From now on, without loss of generality we assume  $\varphi \geq 0$  and  $\mu_\varepsilon \geq 0$   
 4 in (6.10). Indeed we can decompose these functions in their positive and  
 5 negative parts as in (2.2). Now let us split the singular integral into two  
 6 terms: one near the singularity and one far from it. For every positive  $\delta$  we  
 7 write

$$0 \leq \int_\Omega f\zeta(u^\varepsilon)\psi_\varepsilon dx = \int_{\{0 < u^\varepsilon \leq \delta\}} f\zeta(u^\varepsilon)\psi_\varepsilon dx + \int_{\{u^\varepsilon > \delta\}} f\zeta(u^\varepsilon)\psi_\varepsilon dx \doteq I_\varepsilon^\delta + J_\varepsilon^\delta. \quad (6.12)$$

8 In view of Proposition 4.3 written for  $\varphi, \mu_\varepsilon \geq 0$  we have

$$\begin{aligned} 0 &\leq \limsup_{\varepsilon \rightarrow 0} I_\varepsilon^\delta \\ &\leq \limsup_{\varepsilon \rightarrow 0} \left[ \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \nabla \psi_\varepsilon Z_\delta(u^\varepsilon) dx \right. \\ &\quad \left. + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) [Z_\delta(u_1^\varepsilon)\psi_{\varepsilon,1} - Z_\delta(u_2^\varepsilon)\psi_{\varepsilon,2}] d\sigma \right]. \end{aligned} \quad (6.13)$$

1 Let us first show that

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) [Z_\delta(u_1^\varepsilon)\psi_{\varepsilon,1} - Z_\delta(u_2^\varepsilon)\psi_{\varepsilon,2}] d\sigma \leq 0, \quad (6.14)$$

2 so that it results

$$0 \leq \limsup_{\varepsilon \rightarrow 0} I_\varepsilon^\delta \leq \limsup_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \nabla \psi_\varepsilon Z_\delta(u^\varepsilon) dx. \quad (6.15)$$

Taking into account the decomposition given in (6.10), and the fact that since  $\varphi, \phi \in \mathcal{D}(\Omega)$  then  $\varphi_1 = \varphi_2$  and  $\phi_1 = \phi_2$  on  $\Gamma^\varepsilon$ , we have

$$\begin{aligned} & \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) \{ \varphi_1 [Z_\delta(u_1^\varepsilon) - Z_\delta(u_2^\varepsilon)] + \varepsilon \phi_1 [Z_\delta(u_1^\varepsilon)\xi_1 - Z_\delta(u_2^\varepsilon)\xi_2] \} d\sigma \\ &= \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) \varphi_1 [(Z_\delta(u_1^\varepsilon) - Z_\delta(u_2^\varepsilon))] d\sigma \\ &+ \varepsilon^{\gamma+1} \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) \phi_1 [(Z_\delta(u_1^\varepsilon)\xi_1 - Z_\delta(u_2^\varepsilon)\xi_2)] d\sigma \\ &\leq \varepsilon^{\gamma+1} \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) \phi_1 [(Z_\delta(u_1^\varepsilon)\xi_1 - Z_\delta(u_2^\varepsilon)\xi_2)] d\sigma, \end{aligned}$$

where we used H<sub>3</sub>),  $\varphi_1 \geq 0$  and the growth condition of  $Z_\delta$ .

From the properties of the unfolding operators (Lemma 3.4 and estimate

(3.3)), the fact that  $Z_\delta \leq 1$  by definition and  $\gamma \in ]-1, 1[$ , we get

$$\begin{aligned}
 & \varepsilon^{\gamma+1} \int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) \phi_1 [(Z_\delta(u_1^\varepsilon) \xi_1 - Z_\delta(u_2^\varepsilon) \xi_2)] d\sigma \\
 & \leq c \varepsilon^\gamma \|\mathcal{T}_1^\varepsilon(\phi_1) [\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon)]\|_{L^2(\Omega \times \Gamma)} \|\xi_1 (Z_\delta(\mathcal{T}_1^\varepsilon(u_1^\varepsilon)) - \xi_2 Z_\delta(\mathcal{T}_2^\varepsilon(u_2^\varepsilon)))\|_{L^2(\Omega \times \Gamma)} \\
 & \leq c \varepsilon^\gamma \varepsilon^{\frac{1-\gamma}{2}} = c \varepsilon^{\frac{\gamma+1}{2}} \rightarrow 0,
 \end{aligned}$$

which implies (6.14).

Now, as for (6.15), we write

$$\begin{aligned}
 \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \nabla \psi_\varepsilon Z_\delta(u^\varepsilon) dx &= \int_{\Omega_1^\varepsilon} A^\varepsilon(x, u_1^\varepsilon) \nabla u_1^\varepsilon \nabla \psi_{\varepsilon,1} Z_\delta(u_1^\varepsilon) dx \\
 &+ \int_{\Omega_2^\varepsilon} A^\varepsilon(x, u_2^\varepsilon) \nabla u_2^\varepsilon \nabla \psi_{\varepsilon,2} Z_\delta(u_2^\varepsilon) dx.
 \end{aligned}$$

- 1 The arguments used in Theorem 2.8 of [25] to handle the term related to the
- 2 first component apply also to the second one. So that, in view of (6.4), we
- 3 obtain

$$0 \leq \limsup_{\varepsilon \rightarrow 0} I_\varepsilon^\delta \leq \frac{1}{|Y|} \int_{\Omega \times Y_1} A(y, u_1) (\nabla u_1 + \nabla_y \hat{u}_1) (\nabla \varphi + \phi \nabla_y \xi) \chi_{\{u_1=0\}} dx dy = 0, \tag{6.16}$$

since, due to the expression of  $\hat{u}_1$  given by Proposition 2.5, the functions  $\nabla u_1$  and  $\nabla_y \hat{u}_1$  vanish where  $u_1$  is equal to 0.

In order to study the limit behavior of  $J_\varepsilon^\delta$  (defined in (6.12)), once again we split it as follows:

$$J_\varepsilon^\delta = \int_{\{u_1^\varepsilon > \delta\}} f\zeta(u_1^\varepsilon) \psi_{\varepsilon,1} dx + \int_{\{u_2^\varepsilon > \delta\}} f\zeta(u_2^\varepsilon) \psi_{\varepsilon,2} dx.$$

- 1 The integral on the first component is the same considered in the proof of  
 2 [25, Theorem 2.8]. By using those same computations also on the second  
 3 component, we get

$$\lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} J_\varepsilon^\delta = (\theta_1 + \theta_2) \int_{\Omega} f\zeta(u_1)\varphi\chi_{\{u_1>0\}} dx = \int_{\Omega} f\zeta(u_1)\varphi\chi_{\{u_1>0\}} dx. \quad (6.17)$$

Then, when passing to the limit in (6.5), we combine (6.6)-(6.9),(6.12),(6.16),  
 (6.17) and have

$$\int_{\Omega \times Y_1} A(y, u_1)(\nabla u_1 + \nabla_y \widehat{u}_1)(\nabla \varphi + \phi \nabla_y \xi) dx dy = |Y| \int_{\Omega} f\zeta(u_1)\varphi\chi_{\{u_1>0\}} dx,$$

- 4 for every  $\varphi, \phi \in \mathcal{D}(\Omega)$  and  $\xi \in H_{per}^1(Y)$ . By density we obtain

$$\int_{\Omega \times Y_1} A(y, u_1)(\nabla u_1 + \nabla_y \widehat{u}_1)(\nabla \varphi + \nabla_y \psi) dx dy = |Y| \int_{\Omega} f\zeta(u_1)\varphi\chi_{\{u_1>0\}} dx \quad (6.18)$$

- 5 for every  $\varphi \in H_0^1(\Omega)$  and  $\psi \in L^2(\Omega; H_{per}^1(Y))$ .  
 6 Moreover formula (2.15) is a known consequence of the expression of  $\widehat{u}_1$  given  
 7 in Proposition 2.5 and, consequently, a standard computation shows that  $u_1$   
 8 is a solution of the following problem:

$$\begin{cases} -div(A^0(u_1)\nabla u_1) = f\zeta(u_1)\chi_{\{u_1>0\}} & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega. \end{cases} \quad (6.19)$$

- 9 Also  $u_1$  results the unique solution of this problem. It is showed by applying  
 10 Theorem 6 from [27] since the conditions given in (2.10) are satisfied by  $A^0$ .  
 11 This implies the uniqueness of  $\widehat{u}_1$  under the condition  $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$ , in view

1 of Proposition 2.5. This provides the validity of convergence (2.11) for the  
 2 whole sequence, as well as (6.4).

3 We lastly prove that  $u_1 > 0$  almost everywhere in  $\Omega$ . From the strong  
 4 maximum principle, by contradiction, we derive  $u_1 \equiv 0$  in  $\Omega$ . This imply  
 5  $\int_{\Omega} f\zeta(u_1)\chi_{\{u_1>0\}}\varphi dx = 0$  for every  $\varphi \in \mathcal{D}(\Omega)$ . This means  $f \equiv 0$  on  $\Omega$  which  
 6 contradicts assumption H<sub>2</sub>)*iii*. Consequently  $u_1 > 0$  almost everywhere in  $\Omega$   
 7 and  $\chi_{\{u_1>0\}} \equiv 1$ . Then  $u_1$  satisfies the limit equation (2.14).  $\square$

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