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# Statistical Identification of the Different Phases of Stock Market Bubbles

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## Abstract

Eugene Fama, Economics Nobel Prize winner, said in 2016 that people have not come up with ways of statistically identifying bubbles. This paper presents a nonparametric change-point test to identify the different phases of stock bubbles, and we derive its asymptotic distribution under the null hypothesis. Using simulation, we obtain the corresponding critical value. In the empirical analysis, we employ this test and apply the binary segmentation method to the 1990s Nasdaq bubble and obtain the same result as Phillips et al. (2011). We also apply this test to the S&P 500 index, the Shanghai stock index, and the Nikkei 225 index, and successfully identify the bubbles' different phases in each stock market.

*Keywords:* Change-point analysis, Weak dependence assumption, Phases of stock market bubbles

*JEL classification:* C12, G10, G15

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# How to Identify the Different Phases of Stock Market Bubbles Statistically?

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## Abstract

Eugene Fama once mentioned in 2016 that people have not come up with ways of identifying bubbles statistically. This paper presents the nonparametric change-point method to identify different stages of stock bubbles, and we derive its asymptotic distribution under the null hypothesis. By simulation, we obtain the corresponding critical value. In the empirical analysis, we employ this test and binary segmentation method to the 1990s Nasdaq bubble and get the same result as Phillips et al. (2011). We also apply this test to the S&P 500 index, the Shanghai stock index, the Nikkei 225 index, the FTSE 100 index, and the CAC 40 index respectively, and successfully identify the bubbles' different phases in each stock market.

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## 1. Introduction

As theories of bubble develop in various ways, scholars began to pay attention to the bubble test. Most existing bubble tests are aimed at testing rational bubbles due to their relatively intuitive statistical characteristics (LeRoy & Porter, 1981; Shiller, 1981; Hamilton & Whiteman, 1985; Froot & Obstfeld, 1991; Evans, 1991; West, 1987; Diba & Grossman, 1988; Homm & Breitung, 2012; Koustas & Serletis, 2005). Moreover, the bubble tests are mainly based on the fractionally integration method (Cuñado et al., 2005; Sibbertsen & Kruse, 2009) and unit root test (Kim, 2000; Demetrescu et al., 2008).

However, academics and practitioners in finance are still no happy with these developments of bubble tests. Eugene Fama mentioned in 2016 that "Statistically, people have not come up with ways of identifying bubbles."<sup>1</sup>

Phillips et al. (2011) proposed the sup augmented Dickey-Fuller method (SADF) to detect the start date and end date of an explosive bubble, which proved to be superior to previous tests to find a single explosive bubble. Further, Phillips et al. (2015) extended the methodology to the generalized sup augmented Dickey-Fuller method (GSADF) for multiple explosive bubbles. Harvey et al. (2016) pointed out that non-stationary volatility can have a serious impact on the performance of the Dickey-Fuller statistics. Several studies provide tests to detect explosive bubbles under possible heteroscedastic volatility. Homm & Breitung (2012) investigated several tests to detect rational bubbles using the CUSUM procedure. Astill et al. (2021) suggested modified CUSUM statistics to account for heteroscedasticity. Harvey et al. (2017, 2020) utilized a BIC-based approach to improve bubble dating accuracy in comparison to the SADF and GSADF methods.

Motivated by the change-point analysis in Horváth et al. (2017), this paper employs change-point analysis to identify various structural breaks of the stock market prices. Based on weak-dependence assumptions, we apply the nonparametric change-point analysis to the general bubble model, which is more suitable for financial data. Further, we will verify the

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<sup>1</sup>See Chicago Booth Review , June 30, 2016, available at <http://review.chicagobooth.edu/economics/2016/video/are-markets-efficient>.

accuracy of this nonparametric change-point test in the international stock markets.

This paper is organized as follows. Section 2 illustrates the nonparametric change-point test with weak-dependence assumptions and derive the asymptotic distribution under the null hypothesis of “No Change-point” (or “No Bubble”). Section 3 exhibits empirical applications of the nonparametric change-point test for stock market bubbles identification. Section 4 makes conclusions.

## 2. Basic Model and Statistical Test

The model in Phillips & Yu (2011) and Phillips et al. (2015) describes a period of mildly explosive bubble, which has been summarized by Linton (2019) as follows:

$$P_i = \begin{cases} P_{i-1} + \varepsilon_i, & 1 \leq i \leq \tau^b \\ \delta_n P_{i-1} + \varepsilon_i, & \tau^b \leq i \leq \tau^e \\ P_{i-1} + \varepsilon_i, & i \geq \tau^e \end{cases}, \quad (1)$$

where  $P_i$  is the log of stock price,  $\delta_n = 1 + cn^{-\alpha}$  for  $c > 0$ ,  $0 < \alpha < 1$ ,  $n$  is the fixed sample size;  $\tau^b$  represents the start date of mildly explosive bubble, and  $\tau^e$  represents the crash time of mildly explosive bubble.

Our testing model is based on the first-order difference of stock price:

$$R_i = P_i - P_{i-1} = \begin{cases} \varepsilon_i, & 1 \leq i \leq \tau^b \\ BB_i + \varepsilon_i, & \tau^b \leq i \leq \tau^e \\ \varepsilon_i, & i \geq \tau^e \end{cases}, \quad (2)$$

where  $P_i$  is the stock price,  $BB_i$  indicates the stock bubble, and  $[\tau^b, \tau^e]$  signifies the period of the stock bubble.

There are two weak dependence assumptions of  $\varepsilon_i$ , which relaxes the *i.i.d* assumption.

**Assumption 2.1.** *There is a sequence of standard Wiener Processes  $\{W_n^{(1)}(t), 0 \leq t \leq 1\}$*

and  $E[\varepsilon_i] = 0$ ,  $E[\varepsilon_i^2] = \sigma^2$  for  $i = 1, 2, \dots, n$  such that

$$\max_{0 < t \leq \frac{1}{2}} \frac{1}{[tn]^\gamma} \left| \sum_{i=1}^{[tn]} \varepsilon_i - \sigma \sqrt{n} W_n^{(1)}(t) \right| = O_P(1)$$

where  $\gamma < \frac{1}{2}$ .

**Assumption 2.2.** *There is a sequence of standard Wiener Processes  $\{W_n^{(2)}(1-t), 0 \leq t \leq 1\}$ , which is independent of  $\{W_n^{(1)}(t), 0 \leq t \leq 1\}$ , and  $E[\varepsilon_i] = 0$ ,  $E[\varepsilon_i^2] = \sigma^2$  for  $i = 1, 2, \dots, n$  such that*

$$\max_{\frac{1}{2} \leq t < 1} \frac{1}{n - [tn]^\gamma} \left| \sum_{i=[tn]+1}^n \varepsilon_i - \sigma \sqrt{n} W_n^{(2)}(1-t) \right| = O_P(1)$$

where  $\gamma < \frac{1}{2}$ .

These two assumptions are more widely applicable to the most financial time series, such as the AR model, the ARMA model, and the GARCH model (see Horváth et al., 2020).

Now, we translate the basic model into the change-point problem:

$$R_i = \begin{cases} \varepsilon_i, & 1 \leq i \leq k^* \\ BB_i + \varepsilon_i, & k^* + 1 \leq i \leq n \end{cases}, \quad (3)$$

where  $k^*$  signifies the time of change. Due to our test is the nonstationary test for the fixed sample testing procedure, it can not only be used to identify the beginning of the bubble, but also to identify the end of the bubble.

Now, we test the null hypothesis of “No Change-point” (or “No Bubble”)

$$H_0 : BB_i = 0$$

against the alternative hypothesis of “One Change-point” (or “One Bubble”)

$$H_A : BB_i \neq 0$$

Let we start with the detector

$$T_n = \sqrt{\frac{n}{k(n-k)}} \left[ S(k) - \frac{k}{n} S(n) \right] \quad (4)$$

where  $S(k) = \sum_{1 \leq i \leq k} R_i$ . In the continuous situation,  $T_n$  can be expressed as  $T_n(t) = \frac{1}{\sqrt{t(1-t)}\sqrt{n}} [S(\lfloor t(n+1) \rfloor) - tS(n)]$ .

We will investigate the asymptotic behavior of  $T_n(t)$ . Based on the weak-dependence assumptions, we construct a sequence of stochastic processes  $\{B_n(t), 0 \leq t \leq 1\}$ :

$$B_n(t) = \begin{cases} W_n^{(1)}(t) - t \left[ W_n^{(1)}(\frac{1}{2}) + W_n^{(2)}(\frac{1}{2}) \right], & 0 \leq t \leq \frac{1}{2} \\ (1-t) \left[ W_n^{(1)}(\frac{1}{2}) + W_n^{(2)}(\frac{1}{2}) \right] - W_n^{(2)}(1-t), & \frac{1}{2} \leq t \leq 1 \end{cases} \quad (5)$$

It is easy to check that for each  $n$ ,  $\{B_n(t), 0 \leq t \leq 1\}$  is the Gaussian process with  $E[B_n(t)] = 0$  and  $E[B_n(t)B_n(s)] = \min\{t, s\} - st$ . Hence the distribution of  $B_n(t)$  does not depend on  $n$ ; it is a Brownian bridge for any  $n$ . The next theorem is used to derive the asymptotic distribution of the standardized  $T_n(t)$ .

**Theorem 2.1.** *Under the “No Change-point” null hypothesis, Assumptions 2.1 and 2.2, we have with some  $\alpha > 0$*

$$\frac{1}{\sigma} \sup_{0 < t < 1} |T_n(t)| = \sup_{1/(n+1) \leq t \leq 1-1/(n+1)} \frac{|B_n(t)|}{\sqrt{t(1-t)}} + o_P((\log n)^{-\alpha}),$$

where  $\{B_n(t), 0 \leq t \leq 1\}$  is the Brownian bridge of equation (5).

Theorem 2.1 can be used to derive the asymptotic distribution of  $\sup_{0 < t < 1} |T_n(t)|$ . We note that according to the law of iterated logarithm,

$$\sup_{1/(n+1) \leq t \leq 1-1/(n+1)} \frac{|B_n(t)|}{\sqrt{t(1-t)}} \xrightarrow{P} \infty,$$

but with non-standard normalization  $\sup_{1/(n+1) \leq t \leq 1-1/(n+1)} \frac{|B_n(t)|}{\sqrt{t(1-t)}}$  has a limit distribution.

Let

$$a(x) = (2 \log x)^{1/2}$$

and

$$b(x) = 2 \log x + \frac{1}{2} \log \log x - \frac{1}{2} \log \pi.$$

**Theorem 2.2.** *Under the “No Change-point” null hypothesis, Assumptions 2.1 and 2.2, we have for any  $x$*

$$\lim_{n \rightarrow \infty} P \left\{ a(\log n) \frac{1}{\sigma} \sup_{0 < t < 1} |T_n(t)| \leq x + b(\log n) \right\} = \exp(-2e^{-x}).$$

The application of Theorem 2.2 requires the estimation of  $\sigma$ . In this paper we use

$$\hat{\sigma}_n^2 = \frac{1}{n} \min_{1 < k < n} \left\{ \sum_{i=1}^k (R_i - \hat{R}_k)^2 + \sum_{i=k+1}^n (R_i - \tilde{R}_k)^2 \right\} \quad (6)$$

where  $\hat{R}_k = \frac{1}{k} \sum_{i=1}^k R_i$  and  $\tilde{R}_k = \frac{1}{n-k} \sum_{i=k+1}^n R_i$ . We note that  $\hat{\sigma}_n^2$  is an asymptotically consistent estimator for the variance under the null hypothesis as well as under the one change-point alternative hypothesis. We need an additional condition:

**Assumption 2.3.**

$$\left| \frac{1}{n} \sum_{i=1}^n (R_i - E[R_i])^2 - \sigma^2 \right| = o_P((\log \log n)^{-1/2})$$

**Theorem 2.3.** *Under the “No Change-point” null hypothesis, Assumptions 2.1, 2.2, and 2.3, we have*

$$|\hat{\sigma}_n^2 - \sigma^2| = o_P((\log \log n)^{-1/2})$$

and

$$\lim_{n \rightarrow \infty} P \left\{ a(\log n) \frac{1}{\hat{\sigma}_n} \sup_{0 < t < 1} |T_n(t)| \leq x + b(\log n) \right\} = \exp(-2e^{-x})$$

for all  $x$ .



Based on Theorems 2.1 - 2.3, the critical region can be constructed by

$$\lim_{n \rightarrow \infty} P \left\{ \sup_{0 < t < 1} \frac{1}{\hat{\sigma}_n} |T_n(t)| > c_n(\alpha) \right\} = \lim_{n \rightarrow \infty} P \left\{ \sup_{1/(n+1) \leq t \leq 1-1/(n+1)} \frac{|B(t)|}{\sqrt{t(1-t)}} > c_n(\alpha) \right\} = \alpha \quad (7)$$

for  $0 < \alpha < 1$ , where  $c_n(\alpha)$  denotes the critical value at the  $\alpha$  significance level. Performing Monte Carlo simulations, we get the critical values shown in Table 1.

Table 1: The critical values  $c_n(\alpha)$

Simulation Times	$1 - \alpha$	n=20	n=50	n=100	n=200	n=500	n=1000
5000	0.90	2.5527	2.7171	2.8441	2.8925	2.9570	3.0208
	0.95	2.8014	2.9913	3.0883	3.1211	3.2174	3.2671
	0.99	3.3677	3.5347	3.6038	3.5868	3.6938	3.7523
10000	0.90	2.5489	2.7104	2.8364	2.8934	2.9721	3.0234
	0.95	2.8038	2.9766	3.0839	3.1337	3.2383	3.2856
	0.99	3.3891	3.4956	3.6038	3.6186	3.7121	3.7792

### 3. Empirical Analysis

We employ the augmented nonparametric change-point test, which is applying the binary segmentation method (Vostrikova, 1981; Horváth et al., 2017) to detect the multiple change-points. The augmented nonparametric change-point test can identify different phases of a bubble (in most case, we can identify three or more change times in the bubble period: the starting date, the crash date, the end date, and others). The main idea of the augmented nonparametric change-point test is: in the first step, we test the whole sample and find the estimator of change time  $k^*$ , if rejected  $H_0 : BB_i = 0$  under the time  $k^*$ , then denote  $k^*$  as the first change time; in the second step, we divide the sample into two subsamples, the first subsample covers the interval from time 1 to time  $k^* - 1$ , and the second subsample includes the range from time  $k^* + 1$  to time n, then apply the nonparametric change-point test to these two subsamples respectively and find the second or even third estimator of change time. Repeat the second step until we identify all stages of a bubble.

### 3.1. The 1990s Nasdaq

Like Phillips et al. (2011), we try to identify the bubble in the 1990s Nasdaq with the nonparametric change-point test. Figure B.1 plots the Nasdaq Composite price index from April 1976 to June 2005, containing 351 observations. Table 2 reports the change-point test results for sample and/or subsample periods with binary segmentation method<sup>2</sup>. Interestingly, the start point of the 1990s Nasdaq bubble in our test is January 1995, which is six months earlier than the occurrence date in Phillips et al. (2011), July 1995. Moreover, both the crash point and end point are the same as the conclusions in Phillips et al. (2011) (details can be seen in Table 4). Different from Phillips et al. (2011), we have also successfully identified two rounds of explosive growth in August 1998 and September 1999.

As Phillips et al. (2011), we choose another sample period in the 1990s Nasdaq (from June 1979 to June 2005 with 313 observations) for the robust test, and results are shown in Figure B.2 and Table 3. Compared to test period one, the bubble test result with test period two is roughly the same, except that there was an additional round of explosive rise in March 1997. More encouragingly, all the start date, explosive rising date, crash date, and end date are robust in our nonparametric change-point test.

Table 2: Augmented nonparametric change-point test for the Nasdaq Composite index between April 1976 and June 2005

No.	sample Period	Change-point Date	Statistic
1	Apr.1976 - Jun.2005	Feb.2000	3.3023**
2	Apr.1976 - Jan.2000	Sept.1999	9.0169***
3	Apr.1976 - Aug.1999	Aug.1998	6.8360***
4	Apr.1976 - Jul.1998	Jan.1995	4.6382***
5	Mar.2000 - Jun.2005	Mar.2001	3.3483**

Note: \*\*\*,\*\*,\* indicate the significance level of 1%, 5%, 10%, respectively.

Harvey et al. (2016) pointed out that the limit results of Phillips et al. (2011) do not hold under heteroscedasticity. Harvey et al. (2016) proposed a new test to cover possible heteroscedasticity and suggested the wild bootstrap to get critical values for their tests. Our

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<sup>2</sup>we only report the subsample periods with change-point, instead of all subsample periods with binary segmentation method

Table 3: Augmented nonparametric change-point test for the Nasdaq Composite index between June 1979 and June 2005

No.	Subsample Period	Change-point Date	Statistic
1	Jun.1979 - Jun.2005	Feb.2000	3.2001**
2	Jun.1979 - Jan.2000	Sept.1999	8.3624***
3	Jun.1979 - Aug.1999	Aug.1998	6.2986***
4	Jun.1979 - Jul.1998	Mar.1997	4.2037***
5	Jun.1979 - Feb.1997	Jan.1995	3.4854**
6	Mar.2000 - Jun.2005	Mar.2001	3.3483**

Note: \*\*\*, \*\*, \* indicate the significance level of 1%, 5%, 10%, respectively.

Table 4: Comparison of the Phillips ADF test and the Nonparametric Change-point test

Test Period		Bubble Test	Start Date	End Date
Apr.1976 - Jun.2005		Phillips ADF test	Jul.1995	Mar.2001
		Nonparametric Change-point test	Jan.1995	Mar.2001
Jun.1979 - Jun.2005		Phillips ADF test	Jul.1995	Sept.2000
		Nonparametric Change-point test	Jan.1995	Mar.2001

Note: The Phillips ADF test results from Phillips et al. (2011).

results are derived under homoscedasticity. However, the method of Górecki et al. (2017) can be used to extend our results to cover the case of heteroscedastic observations.

### 3.2. Further Applications to Other Stock Indices

In this section, we apply the nonparametric change-point test to other stock market indices. We check for bubbles in the S&P 500 between August 1970 and December 2003 (401 observations in total). Figure B.3 and Table 5 show that the S&P 500 bubble we identified is also related to the Dot-com bubble. Similarly to the Nasdaq bubble in the 1990s we identified above, the beginning point of the S&P 500 bubble is December 1994, which is only one month earlier than the beginning point of the Nasdaq bubble. Since the Federal Reserve raised interest rates during 1994-1995, the S&P 500 had been rising. From 1999 to early 2000, the economy began to lose control. The Dot-com bubble started to burst in March 2000, but at this time, the market was still optimistic, and some investors believed it was a short-term correction in the stock market. The S&P 500 bubble started to burst in August 2000 and finished in September 2002.

Figure B.4 and Table 6 display the Shanghai stock index bubble which corresponds to the 2008 global financial crisis. Since the outbreak of the Sub-prime Mortgage Crisis in 2007,

Table 5: Augmented nonparametric change-point test for the S&P 500 index

No.	Subsample Period	Change-point Date	Statistic
1	Aug.1970 - Dec.2003	Aug.2000	3.0867*
2	Aug.1970 - Jul.2000	Dec.1994	4.6196***
3	Sept.2000 - Dec.2003	Sept.2002	2.8732*

Note: \*\*\*, \*\*, \* indicate the significance level of 1%, 5%, 10%, respectively.

banks and other financial institutions are facing enormous liquidity risks. Even if the central banks of various countries inject large amounts of funds many times, they can not prevent the financial crisis. For the Shanghai stock index, the bubble started in October 2006 (it may be interrelated to the share-trading reform in the Chinese stock market: a large number of funds poured into the stock market, coupled with the large-scale issuance of open-end funds and the expectation of RMB appreciation), and came to collapse in October 2007 due to the global financial crisis, and finally ended after one year (October 2008).

Table 6: Augmented nonparametric change-point test for the Shanghai stock index

No.	Subsample Period	Change-point Date	Statistic
1	Jan.1995 - Dec.2009	Oct.2007	3.0406*
2	Jan.1995 - Sept.2007	Oct.2006	8.7906***
3	Nov.2007 - Dec.2009	Oct.2008	2.7443*

Note: \*\*\*, \*\*, \* indicate the significance level of 1%, 5%, 10%, respectively.

In addition, we identify the Nikkei bubble in the Japanese stock market (see Figure B.5 and Table 7), which began in September 1982 and crashed in December 1989 according to our statistical analysis. Under the favorable conditions of both the domestic and international economic environment, the Japanese stock market began to recover slowly in 1985 and gradually entered the surge stage in January 1986. In December 1989, the Nikkei 225 reached its peak, and stock prices plunged immediately. The testing result demonstrates that the Nikkei bubble ended in September 1990.

Our method is also applicable to European stock markets, such as the United Kingdom and France. Figure B.6 and Table 8 show the test results of the Financial Times Stock Exchange 100 index (FTSE 100 index) for the period April 2002 to August 2009. We find that the UK bubble started in October 2002 and crashed in November 2007. Figure B.7 and Table 9 report the French stock bubble based on the Cotation Assistée en Continu 40

Table 7: Augmented nonparametric change-point test for the Nikkei 225 index

No.	Subsample Period	Change-point Date	Statistic
1	Jan.1970 - Jan.1995	Dec.1989	3.5239**
2	Jan.1970 - Nov.1989	Jan.1986	5.8222***
3	Jan.1970 - Dec.1985	Sept.1982	3.1194*
4	Jan.1990 - Jan.1995	Sept.1990	3.1470**

Note: \*\*\*, \*\*, \* indicate the significance level of 1%, 5%, 10%, respectively.

index (CAC 40 index) from March 1990 to May 2004. The results show that our method can statistically identify the start date (October 2007), the crash date (August 2000), and the end date (September 2002) for the French stock bubble.

Table 8: Augmented nonparametric change-point test for the FTSE 100 index

No.	Subsample Period	Change-point Date	Statistic
1	Apr.2002-Aug.2009	Oct.2002	3.1085**
2	Nov.2002-Aug.2009	Nov.2007	3.0391*

Note: \*\*\*, \*\*, \* indicate the significance level of 1%, 5%, 10%, respectively.

Table 9: Augmented nonparametric change-point test for the CAC 40 index

No.	Subsample Period	Change-point Date	Statistic
1	Mar.1990 - May.2004	Aug.2000	3.0176*
2	Mar.1990 - Jul.2000	Sept.1998	3.7419***
3	Mar.1990 - Aug.1998	Jul.1998	3.4920**
4	Mar.1990 - Jun.1998	Oct.1997	4.1014***
5	Nov.1997 - May.2004	Sept.2002	2.7498*

Note: \*\*\*, \*\*, \* indicate the significance level of 1%, 5%, 10%, respectively.

#### 4. Conclusion

Given the fact that financial data are serially dependent, instead of independently and identically distributed, this paper presents the nonparametric change-point method for bubble testing and successfully applies it to identify different stages of stock bubbles. We introduce a new nonparametric test and we derive its asymptotic distribution under the null hypothesis of “No Bubble”. By simulation, we obtain the corresponding critical value. To detect the different phases of the bubble, we employ the binary segmentation method.

In the empirical analysis, by using the same test periods in the 1990s Nasdaq, we get the same collapsing date and end date as Phillips et al. (2011), but the beginning date in our

bubble test is six months earlier than that in Phillips et al. (2011). We apply the same test to the S&P 500 index, the Shanghai stock index, the Nikkei 225 index, the FTSE 100 index, and the CAC 40 index respectively, and successfully find out the start date, the crash date, and the end date of bubbles in each stock market.

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## Appendix A. Proof of Theorem 2.1

*Proof.* We recall the definition of  $\{B_n(t), 0 \leq t \leq 1\}$  in equation (5). Using Assumption 2.1 and 2.2, we get that

$$\sup_{1/(n+1) \leq t \leq 1-1/(n+1)} \left| T_n(t) - \frac{\sigma B_n(t)}{\sqrt{t(1-t)}} \right| = O_P(1) \quad (\text{A.1})$$

and

$$\sup_{(\log n)^4/n \leq t \leq 1-(\log n)^4/n} \left| T_n(t) - \frac{\sigma B_n(t)}{\sqrt{t(1-t)}} \right| = O_P \left( (\log n)^{-4(\frac{1}{2}-\gamma)} \right). \quad (\text{A.2})$$

Lemma 1.3 on page 257 in Csörgő & Horváth (1993) yields

$$\frac{1}{(2 \log \log n)^{1/2}} \sup_{1/(n+1) \leq t \leq 1-1/(n+1)} \frac{|B_n(t)|}{\sqrt{t(1-t)}} \xrightarrow{P} 1, \quad (\text{A.3})$$

$$\sup_{1/(n+1) \leq t \leq (\log n)^4/n} \frac{|B_n(t)|}{\sqrt{t(1-t)}} = O_P \left( (\log \log \log n)^{1/2} \right), \quad (\text{A.4})$$

and

$$\sup_{1-(\log n)^4/n \leq t \leq 1-1/(n+1)} \frac{|B_n(t)|}{\sqrt{t(1-t)}} = O_P \left( (\log \log \log n)^{1/2} \right). \quad (\text{A.5})$$

Putting together equations (A.3)-(A.5), we conclude

$$P \left\{ \sup_{1/(n+1) \leq t \leq 1-1/(n+1)} \frac{|B_n(t)|}{\sqrt{t(1-t)}} = \sup_{(\log n)^4/n \leq t \leq 1-(\log n)^4/n} \frac{|B_n(t)|}{\sqrt{t(1-t)}} \right\} \rightarrow 1, \text{ as } n \rightarrow \infty. \quad (\text{A.6})$$

Now the weighted approximation in equation (A.1) with equations (A.3)-(A.5) yields

$$\frac{1}{(2 \log \log n)^{1/2}} \sup_{1/(n+1) \leq t \leq 1-1/(n+1)} |T_n(t)| \xrightarrow{P} \sigma, \quad (\text{A.7})$$

$$\sup_{1/(n+1) \leq t \leq (\log n)^4/n} |T_n(t)| = O_P \left( (\log \log \log n)^{1/2} \right), \quad (\text{A.8})$$

and

$$\sup_{1-(\log n)^4/n \leq t \leq 1-1/(n+1)} |T_n(t)| = O_P \left( (\log \log \log n)^{1/2} \right). \quad (\text{A.9})$$

By the definition of  $T_n(t)$  and Assumptions 2.1 and 2.2, we obtain

$$\sup_{0 < t \leq 1/(n+1)} |T_n(t)| = O_P \left( n^{-1/2} \right) \quad (\text{A.10})$$

and

$$\sup_{1-1/(n+1) \leq t < 1} |T_n(t)| = O_P \left( n^{-1/2} \right) \quad (\text{A.11})$$

As in equation (A.6), we obtain from equations (A.7)-(A.11) that

$$P \left\{ \sup_{0 < t < 1} |T_n(t)| = \sup_{(\log n)^4/n \leq t \leq 1-(\log n)^4/n} |T_n(t)| \right\} \rightarrow 1, \text{ as } n \rightarrow \infty. \quad (\text{A.12})$$

The result in Theorem 2.1 follows from equations (A.2), (A.6), and (A.12).  $\square$

## Appendix B. Proof of Theorem 2.2

*Proof.* We note that the distribution of  $\{B_n(t), 0 \leq t \leq 1\}$  does not depend on  $n$ ; it is a Brownian bridge for each  $n$ . If  $\{B(t), 0 \leq t \leq 1\}$  is a Brownian bridge, Lemma 1.3 on page 257 in Csörgő & Horváth (1993) gives

$$\lim_{n \rightarrow \infty} P \left\{ a(\log n) \sup_{1/(n+1) \leq t \leq 1-1/(n+1)} \frac{|B_n(t)|}{\sqrt{t(1-t)}} \leq x + b(\log n) \right\} = \exp(-2e^{-x}) \quad (\text{B.1})$$

for all  $x$ . Hence Theorem 2.1 implies Theorem 2.2.  $\square$

Appendix B.1. Proof of Theorem 2.3

*Proof.* We have that

$$\begin{aligned}
& \sum_{i=1}^k (R_i - \hat{R}_k)^2 + \sum_{i=k+1}^n (R_i - \tilde{R}_k)^2 \\
&= \sum_{i=1}^n (R_i - E[R_i])^2 + 2 \sum_{i=1}^k (R_i - E[R_i])(E[R_i] - \hat{R}_k) + \sum_{i=1}^k (E[R_i] - \hat{R}_k)^2 \\
&+ 2 \sum_{i=k+1}^n (R_i - E[R_i])(E[R_i] - \tilde{R}_k) + \sum_{i=k+1}^n (E[R_i] - \tilde{R}_k)^2.
\end{aligned}$$

Regardless if we have the null or the alternative hypothesis, according to Assumption 2.3:

$$\left| \frac{1}{n} \sum_{i=1}^n (R_i - E[R_i])^2 - \sigma^2 \right| = o_P((\log \log n)^{-1/2}).$$

Under the null hypothesis,  $E[R_i] = \mu$  does not depend on  $i$  and therefore

$$2 \sum_{i=1}^k (R_i - \mu)(\mu - \hat{R}_k) + k(\mu - \hat{R}_k)^2 = -\frac{1}{k} \left( \sum_{i=1}^k (R_i - \mu) \right)^2,$$

and according to the proof of Theorem 2.1 and Theorem 2.2, we have that

$$\max_{1 \leq k \leq n} \left( \frac{1}{\sqrt{k}} \sum_{i=1}^k (R_i - \mu) \right)^2 = O_P(\log \log n).$$

Similar arguments give

$$\max_{1 \leq k < n} \left| 2 \sum_{i=k+1}^n (R_i - \mu)(\mu - \tilde{R}_k) + \sum_{i=k+1}^n (\mu - \tilde{R}_k)^2 \right| = O_P(\log \log n).$$

Hence the first part of Theorem 2.3 is proven. The second half follows from Theorem 2.2 and Slutsky's lemma.  $\square$

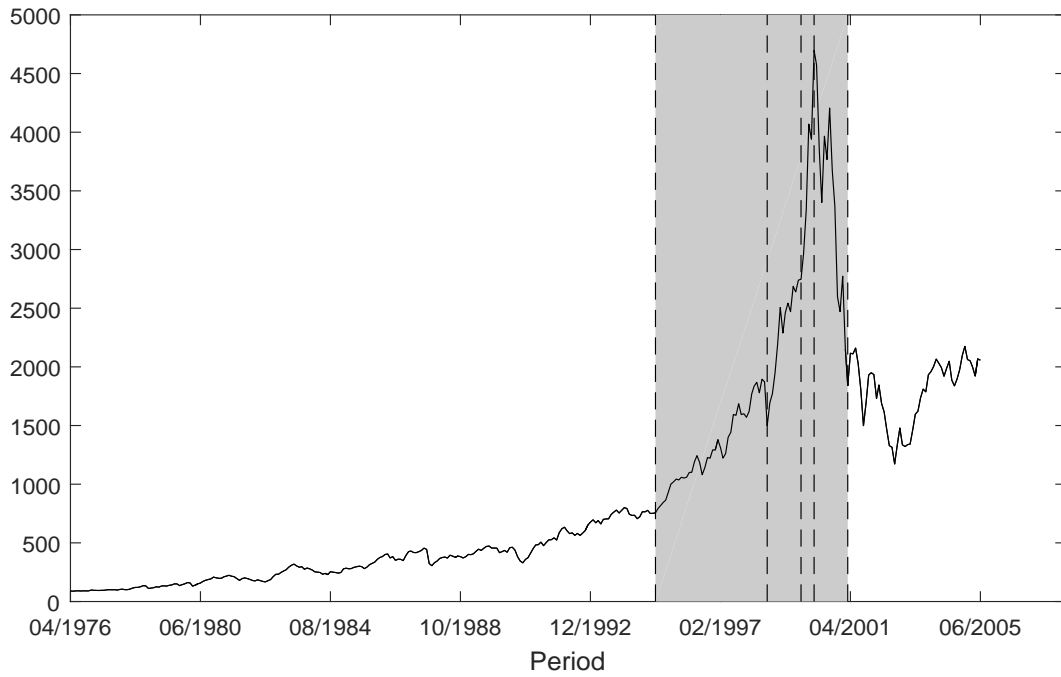


Figure B.1: Change-point detection in the Nasdaq Composite index between April 1976 and June 2005 (gray area refers to the detecting bubble period)

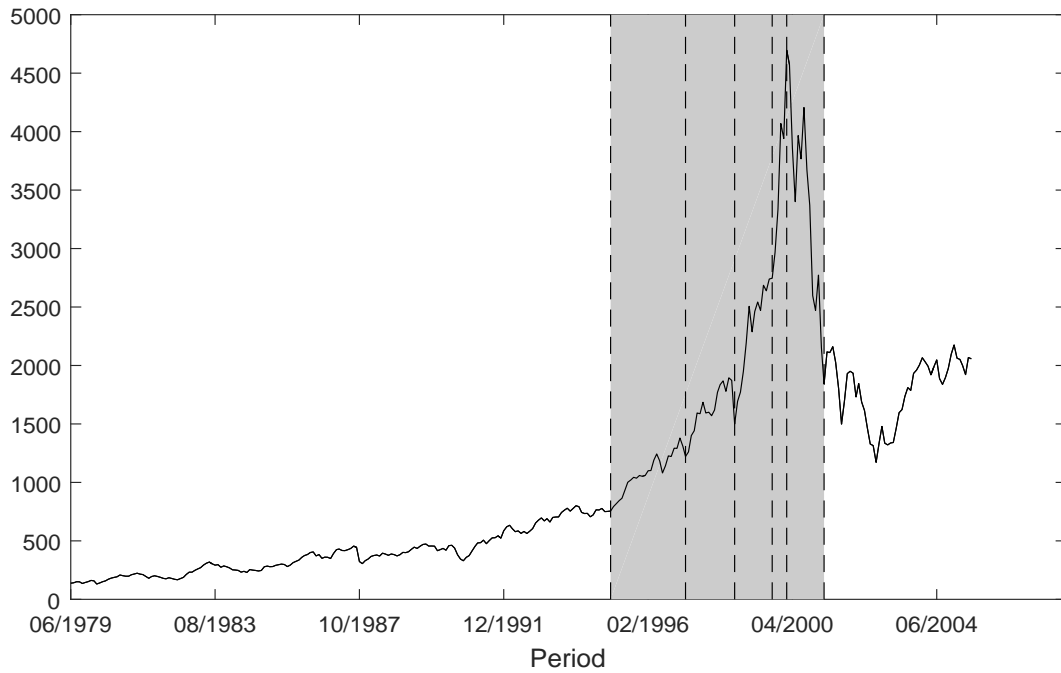


Figure B.2: Change-point detection in the Nasdaq Composite index between June 1979 and June 2005 (gray area refers to the detecting bubble period)

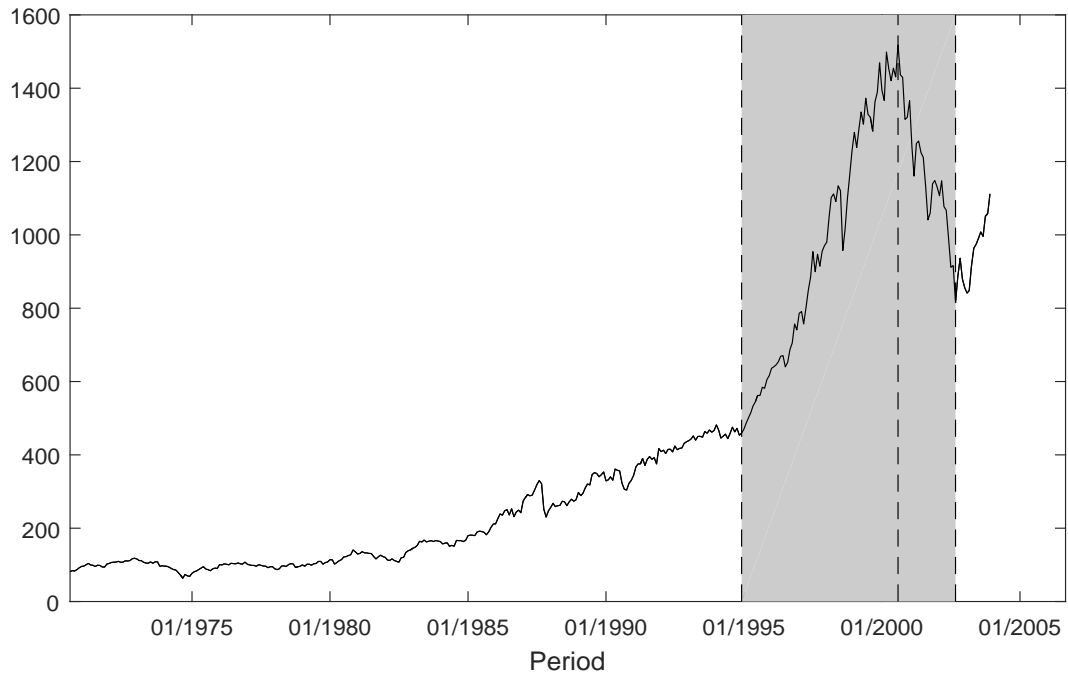


Figure B.3: Change-point detection in the S&P 500 index between August 1970 and December 2003 (gray area refers to the detecting bubble period)

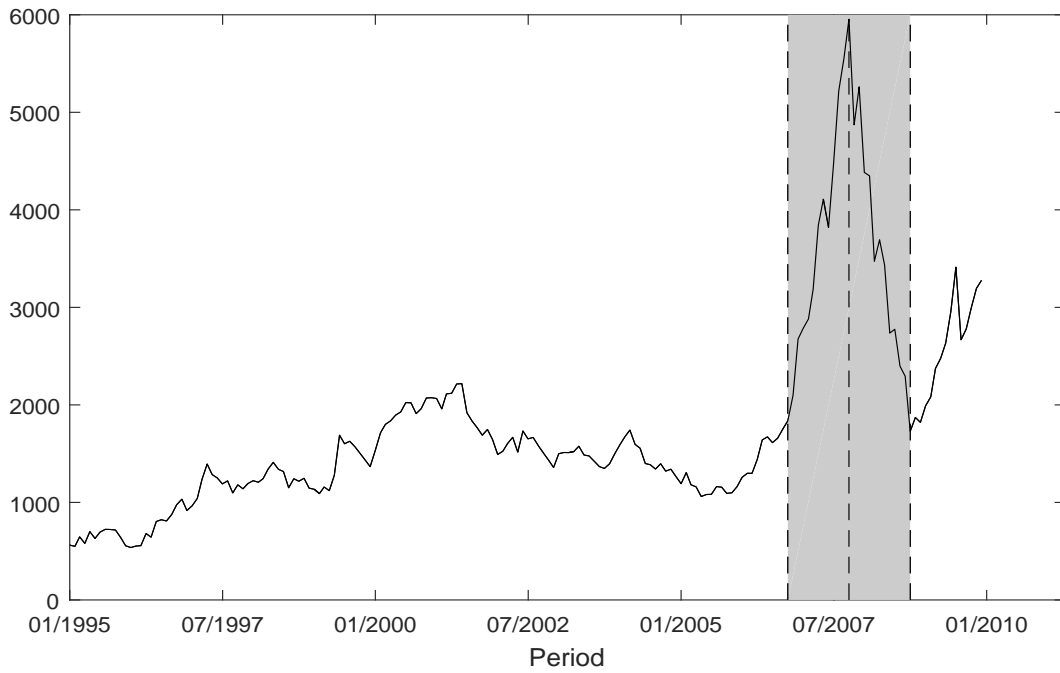


Figure B.4: Change-point detection in the Shanghai stock index between January 1995 and December 2009 (gray area refers to the detecting bubble period)

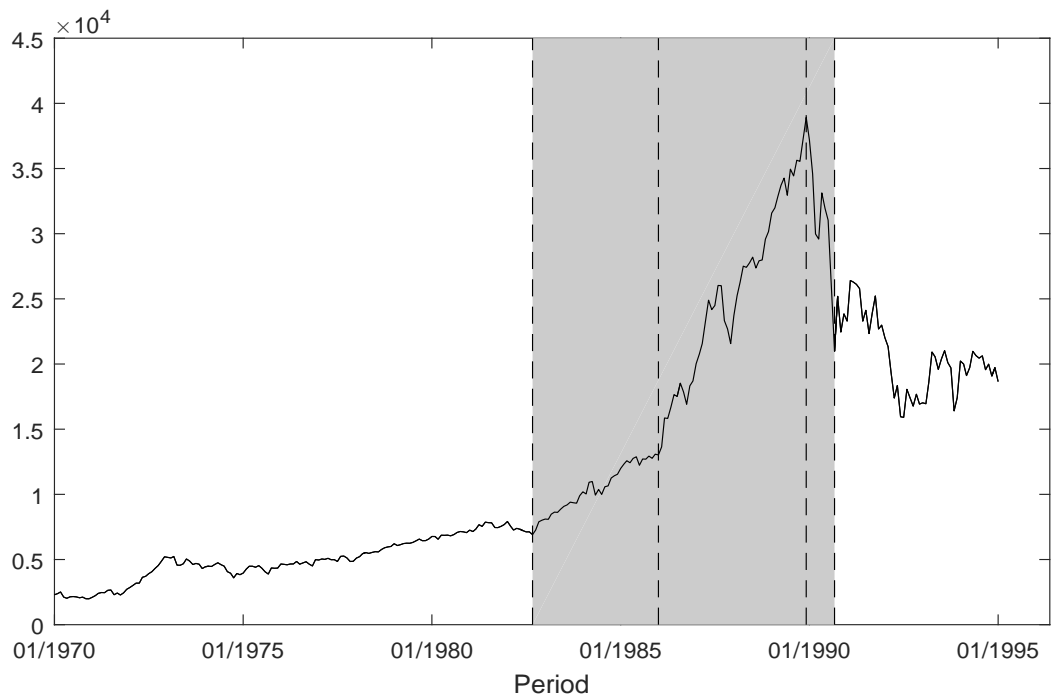


Figure B.5: Change-point detection in the Nikkei 225 index between January 1970 and January 1995 (gray area refers to the detecting bubble period)

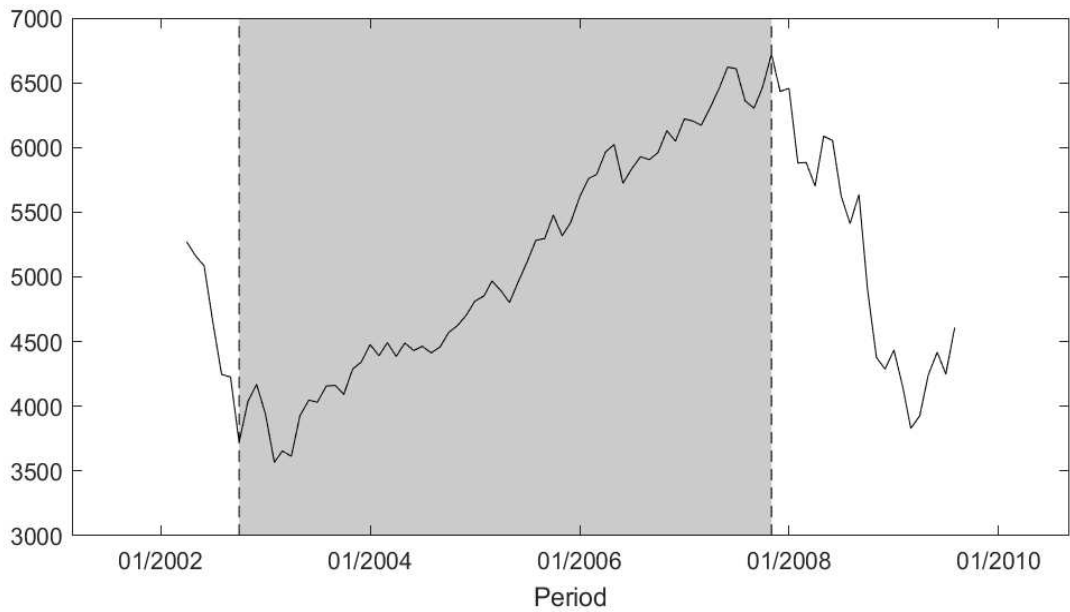


Figure B.6: Change-point detection in the FTSE 100 index between April 2002 and August 2009 (gray area refers to the detecting bubble period)

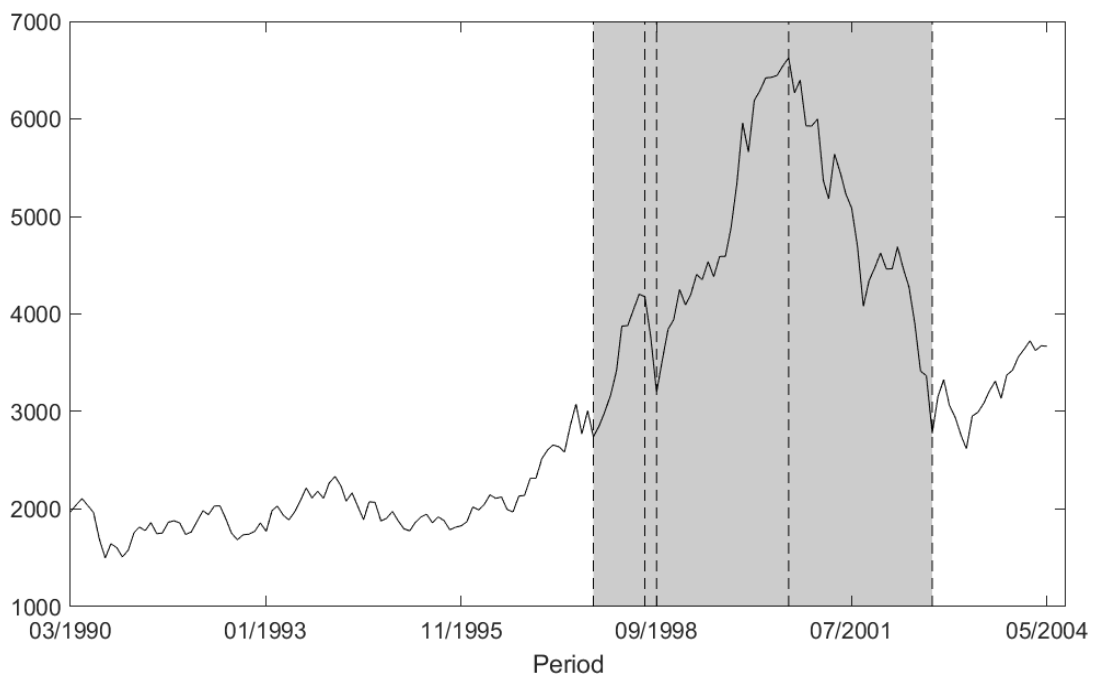


Figure B.7: Change-point detection in the CAC 40 index between March 1990 and May 2004 (gray area refers to the detecting bubble period)