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LONG-TIME BEHAVIOR OF SSEP WITH SLOW BOUNDARY

LINJIE ZHAO

ABSTRACT. We consider the symmetric simple exclusion process with slow boundary first introduced in [Baldasso *et al.*, Journal of Statistical Physics, 167(5), 2017]. We prove a law of large number for the empirical measure of the process under a longer time scaling instead of the usual diffusive time scaling.

1. INTRODUCTION

Interacting particle systems in contact with reservoirs have been investigated in various literature [5, 7, 8]. We study in this article the symmetric simple exclusion process (SSEP) on a line segment $\{1, \dots, N-1\}$ with slow boundary, which is first introduced by Baldasso *et al.* [1]. Here, N is the scaling parameter. There is at most one particle per site. In the bulk, a particle jumps to one of its neighbors at rate one provided the target site is empty. Fix parameters $c > 0, \theta \geq 0$ and $\alpha, \beta \in (0, 1)$. At the boundary site 1 (resp. $N-1$), a particle is created at rate $c\alpha N^{-\theta}$ (resp. $c\beta N^{-\theta}$) if site 1 (resp. $N-1$) is empty, and a particle is destroyed at rate $c(1-\alpha)N^{-\theta}$ (resp. $c(1-\beta)N^{-\theta}$) if site 1 (resp. $N-1$) is occupied. Therefore, the particle density of the left (resp. right) reservoir is α (resp. β), and the interaction strength between the bulk and the reservoirs is $cN^{-\theta}$. The hydrodynamic equation of the model turns out to be the heat equation with Dirichlet boundary conditions if $\theta < 1$, with Robin boundary conditions if $\theta = 1$ and with Neumann boundary conditions if $\theta > 1$. We refer the readers to [1] for more background of the model.

The hydrodynamic limit of the model is considered under the diffusive time scaling, i.e., with time speeded up by N^2 and space divided by N . The aim of this article is to consider the behavior of the process under a longer time scaling $N^{2+\gamma}$, $\gamma > 0$. Since the process is irreducible, it has a unique invariant measure. Under the invariant measure, the empirical measure converges in probability to the stationary solution of the corresponding hydrodynamic equations as $N \rightarrow \infty$. This is called hydrostatic limit [1, 13]. The hydrostatic limit could be formally interpreted as taking $\gamma = \infty$. For $0 < \gamma < \infty$, it is natural to expect that the limit of the empirical measure should coincide with the hydrostatic limit. This is indeed true for $\theta \leq 1$, since the stationary solution of the corresponding hydrodynamic equation is unique in this case. For $\theta > 1$, since the stationary solution is not unique, three regimes appear depending on whether $\gamma < \theta - 1$, $\gamma = \theta - 1$ or $\gamma > \theta - 1$. See Theorem 2.3 for details.

Despite the simple structure of the model, it has attracted a lot of attention since then. The equilibrium/non-equilibrium fluctuations from the hydrodynamic limit are considered in [9, 11]. The large deviation of the SSEP with slow boundary are investigated in [2, 6, 10].

Key words and phrases. Exclusion process; slow boundary; empirical measure; law of large numbers.

The paper is organized as follows. In Section 2 we define the model rigorously via its infinitesimal generator, review the hydrodynamic/hydrostatic limit already proven in [1, 13], and state the main result of the article. In Section 3 we introduce the notation of Dirichlet forms and prove the so-called replacement lemmas under a longer time scaling. The estimates involving the Dirichlet forms are mostly borrowed from [1]. The proof of Theorem 2.3 is presented in Section 4. In the last Section 5 we explain how to adapt our results to mixed slow boundaries, i.e., when the strengths of interactions with left and right reservoirs are different.

2. NOTATION AND RESULTS

2.1. The model. The state space of the process $(\eta_t)_{t \geq 0}$ is $\Omega_N := \{0, 1\}^{I_N}$, where $I_N := \{1, \dots, N-1\}$. Here, N is the scaling parameter. For a configuration $\eta \in \Omega_N$, $\eta(x) = 1$ if and only if there is a particle at site x . Fix parameters $c > 0$, $\theta \geq 0$ and $\alpha, \beta \in (0, 1)$. The parameter θ denotes the strength of interaction with reservoirs and α, β are the particle densities of reservoirs. The generator \mathcal{L}_N of the process $(\eta_t)_{t \geq 0}$ is given by

$$\mathcal{L}_N = \mathcal{L}_{N,0} + \mathcal{L}_{N,L}^\alpha + \mathcal{L}_{N,R}^\beta.$$

Above, the generator $\mathcal{L}_{N,0}$ of the bulk dynamics acting on functions $f : \Omega_N \rightarrow \mathbb{R}$ is

$$(\mathcal{L}_{N,0}f)(\eta) = \sum_{x=1}^{N-2} [f(\eta^{x,x+1}) - f(\eta)],$$

where $\eta^{x,y}$ is the configuration obtained from η by exchanging the values of $\eta(x)$ and $\eta(y)$, i.e., $\eta^{x,y}(x) = \eta(y)$, $\eta^{x,y}(y) = \eta(x)$ and $\eta^{x,y}(z) = \eta(z)$ for $z \neq x, y$. The generators $\mathcal{L}_{N,L}^\alpha$ and $\mathcal{L}_{N,R}^\beta$ correspond to the boundary effects, and are given by

$$(\mathcal{L}_{N,L}^\alpha f)(\eta) := cN^{-\theta} r_{\alpha,L}(\eta) [f(\eta^1) - f(\eta)], \quad (\mathcal{L}_{N,R}^\beta f)(\eta) := cN^{-\theta} r_{\beta,R}(\eta) [f(\eta^{N-1}) - f(\eta)],$$

where η^x is the configuration obtained from η by flipping the value of $\eta(x)$, i.e., $\eta^x(x) = 1 - \eta(x)$ and $\eta^x(z) = \eta(z)$ for $z \neq x$, and

$$r_{\alpha,L}(\eta) = \alpha(1 - \eta(1)) + (1 - \alpha)\eta(1), \quad r_{\beta,R}(\eta) = \beta(1 - \eta(N-1)) + (1 - \beta)\eta(N-1).$$

Denote by μ_N the initial measure of the process. For any positive integer k , let $C^k[0, 1]$ be the family of functions on $[0, 1]$ such that the m -th derivative is uniformly continuous in $(0, 1)$ for any $m \leq k$.

2.2. Hydrodynamic limit. It has been proven in [1] that phase transitions occur for the SSEP with slow boundary, depending on whether $\theta < 1$, $\theta = 1$ or $\theta > 1$. To state the hydrodynamic limit, we impose the following assumptions on the initial measure μ_N : there exists a measurable initial density profile $\rho_0 : [0, 1] \rightarrow [0, 1]$ such that for any $G \in C[0, 1]$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{x=1}^{N-1} \eta(x) G\left(\frac{x}{N}\right) = \int_0^1 \rho_0(u) G(u) du$$

in probability with respect to μ_N . The following result characterize the macroscopic density profile under the diffusive time scaling.

Theorem 2.1 (Cf. [1, Theorem 2.8]). *For any $t \geq 0$ and for any $G \in C[0, 1]$,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{x=1}^{N-1} \eta_{tN^2}(x) G\left(\frac{x}{N}\right) = \int_0^1 \rho(t, u) G(u) du$$

in probability, where

(i) *if $0 \leq \theta < 1$, then $\rho(t, u)$ is the unique weak solution to the heat equation with Dirichlet boundary*

$$\begin{cases} \partial_t \rho(t, u) = \Delta \rho(t, u), & u \in (0, 1), \quad t \geq 0 \\ \rho(t, 0) = \alpha, \rho(t, 1) = \beta, & t \geq 0, \\ \rho(0, u) = \rho_0(u), & u \in [0, 1] \end{cases} \quad (2.1)$$

(ii) *if $\theta = 1$, then $\rho(t, u)$ is the unique weak solution to the heat equation with Robin boundary*

$$\begin{cases} \partial_t \rho(t, u) = \Delta \rho(t, u), & u \in (0, 1), \quad t \geq 0 \\ \partial_u \rho(t, 0) = c(\rho(t, 0) - \alpha), & t \geq 0, \\ \partial_u \rho(t, 1) = c(\beta - \rho(t, 1)), & t \geq 0 \\ \rho(0, u) = \rho_0(u), & u \in [0, 1] \end{cases} \quad (2.2)$$

(iii) *if $\theta > 1$, then $\rho(t, u)$ is the unique weak solution to the heat equation with Neumann boundary*

$$\begin{cases} \partial_t \rho(t, u) = \Delta \rho(t, u), & u \in (0, 1), \quad t \geq 0 \\ \partial_u \rho(t, 0) = 0, \partial_u \rho(t, 1) = 0, & t \geq 0 \\ \rho(0, u) = \rho_0(u), & u \in [0, 1] \end{cases} \quad (2.3)$$

We refer the readers to [1] for rigorous definitions of weak solutions to the above PDEs.

2.3. Hydrostatic limit. Since the process $(\eta_t)_{t \geq 0}$ is irreducible, it has a unique invariant measure denoted by μ_N^{ss} . The following result characterize the macroscopic density profile under the invariant measure μ_N^{ss} .

Theorem 2.2 (Cf. [1, Theorem 2.2] and [13]). *For any $G \in C[0, 1]$,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{x=1}^{N-1} \eta(x) G\left(\frac{x}{N}\right) = \int_0^1 \bar{\rho}(u) G(u) du$$

in probability with respect to μ_N^{ss} , where

$$\bar{\rho}(u) = \begin{cases} (\beta - \alpha)u + \alpha, & \text{if } \theta \in [0, 1) \\ \frac{c(\beta - \alpha)}{2+c}u + \alpha + \frac{\beta - \alpha}{2+c}, & \text{if } \theta = 1 \\ \frac{\beta + \alpha}{2}, & \text{if } \theta \in (1, \infty) \end{cases}$$

We say $\rho : [0, 1] \rightarrow [0, 1]$ is a stationary solution to (2.1) if

$$\Delta \rho(u) = 0, \quad u \in (0, 1), \quad \rho(0) = \alpha, \quad \rho(1) = \beta.$$

Stationary solutions to (2.2) and (2.3) could be defined in the same way. Note that the hydrostatic limits $\bar{\rho}$ are stationary solutions to the corresponding hydrodynamic equations as stated in Theorem 2.1. We underline that the stationary solution to (2.1) and (2.2) is unique, while the stationary solution to (2.3) is not unique. Indeed, any constant function is a stationary solution to (2.3). The above theorem tells us that the correct choice is $(\alpha + \beta)/2$.

2.4. Long-time limit. In this subsection, we state the main result of the article. Fix $\gamma > 0$. We shall consider the process speeded up by $N^{2+\gamma}$. Denote by $(\eta_t^N)_{t \geq 0}$ the process with generator $N^{2+\gamma} \mathcal{L}_N$. Then $(\eta_t^N)_{t \geq 0}$ and $(\eta_{tN^{2+\gamma}})_{t \geq 0}$ have the same distribution. We are interested in the long time behavior of the empirical measure π_t^N of the process defined as

$$\pi_t^N(du) = \frac{1}{N} \sum_{x=1}^{N-1} \eta_t^N(x) \delta_{x/N}(du),$$

where $\delta_{x/N}(du)$ is the Dirac measure on the point x/N . Whence, π_t^N is a random measure on $[0, 1]$ with total mass bounded by one. With this notation, for any $G \in C[0, 1]$,

$$\langle \pi_t^N, G \rangle = \frac{1}{N} \sum_{x=1}^{N-1} \eta_t^N(x) G\left(\frac{x}{N}\right).$$

For $\theta > 1$ and $0 < \gamma \leq \theta - 1$, we assume that the average number of particles converges in the following sense: there exists $m_0 \in [0, 1]$ such that

$$\lim_{N \rightarrow \infty} E_{\mu_N} \left[\left| \frac{1}{N-1} \sum_{x=1}^{N-1} \eta(x) - m_0 \right| \right] = 0. \quad (2.4)$$

We underline that we impose no restrictions on the initial measure μ_N in the rest of the cases.

Denote by $\mathbb{P}_{\mu_N}^N$ the probability measure on $D([0, \infty), \Omega_N)$ associated to the process $(\eta_t^N)_{t \geq 0}$ and the initial measure μ_N , and by $\mathbb{E}_{\mu_N}^N$ the corresponding expectation.

We now state the law of large numbers for the empirical measure π_t^N .

Theorem 2.3. *For any $t > 0$ and for any $G \in C[0, 1]$,*

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N \left[\left| \int_0^t \left\{ \langle \pi_s^N, G \rangle - \int_0^1 \rho_{\theta, \gamma}(s, u) G(u) du \right\} ds \right| \right] = 0,$$

where

$$\rho_{\theta, \gamma}(t, u) = \begin{cases} (\beta - \alpha)u + \alpha, & \text{if } 0 \leq \theta < 1 \\ \frac{c(\beta - \alpha)}{2+c}u + \alpha + \frac{\beta - \alpha}{2+c}, & \text{if } \theta = 1, \\ m_0, & \text{if } \theta > 1, 0 < \gamma < \theta - 1, \\ \frac{\beta + \alpha}{2}, & \text{if } \theta > 1, \gamma > \theta - 1, \\ \frac{\beta + \alpha}{2} + \left(m_0 - \frac{\alpha + \beta}{2} \right) e^{-2ct}, & \text{if } \theta > 1, \gamma = \theta - 1. \end{cases} \quad (2.5)$$

Remark 2.4. *Note that $\rho_{\theta, \infty} = \bar{\rho}$. This is not surprising since the hydrostatic limit stated in Theorem 2.2 could be formally interpreted as taking $\gamma = \infty$. Compared with Theorem 2.2, the above theorem states that phase transition occurs even in the supercritical case $\theta > 1$.*

Remark 2.5. *The result should also hold if the density reservoirs vary slowly with time, i.e., if replacing α (resp. β) with some smooth function $\rho_-(t) : \mathbb{R}_+ \rightarrow (0, 1)$ (resp. $\rho_+(t) : \mathbb{R}_+ \rightarrow (0, 1)$). This is called quasi-static hydrodynamic limit [3, 4]. We underline that the case $\theta = 0$ is covered by [4, Theorem 2.1/3.1].*

3. PRELIMINARY RESULTS

In this section, we introduce the notion of Dirichlet forms and prove several replacement lemmas in different regimes, which are crucial in the proof of Theorem 2.3.

3.1. Dirichlet form. For a probability measure μ on Ω_N and a function $g : \Omega_N \rightarrow \mathbb{R}$, the Dirichlet forms corresponding to the bulk/boundary dynamics are defined as

$$\begin{aligned} D_{N,0}(g, \mu) &:= \frac{1}{2} \sum_{x=1}^{N-2} \sum_{\eta \in \Omega_N} (g(\eta^{x,x+1}) - g(\eta))^2 \mu(\eta), \\ D_{N,L}^\alpha(g, \mu) &:= \frac{1}{2} \sum_{\eta \in \Omega_N} cN^{-\theta} r_{\alpha,L}(\eta) (g(\eta^1) - g(\eta))^2 \mu(\eta), \\ D_{N,R}^\beta(g, \mu) &:= \frac{1}{2} \sum_{\eta \in \Omega_N} cN^{-\theta} r_{\beta,R}(\eta) (g(\eta^{N-1}) - g(\eta))^2 \mu(\eta). \end{aligned}$$

For any two functions $f, g : \Omega_N \rightarrow \mathbb{R}$, denote

$$\langle f, g \rangle_\mu = \sum_{\eta \in \Omega_N} f(\eta)g(\eta)\mu(\eta).$$

For any density profile $\lambda : [0, 1] \rightarrow [0, 1]$, let $\nu_{\lambda(\cdot)}^N$ be the product measure on Ω_N with marginals given by

$$\nu_{\lambda(\cdot)}^N\{\eta : \eta(x) = 1\} = \lambda\left(\frac{x}{N}\right), \quad x \in I_N.$$

In particular, if $\lambda(\cdot) \equiv \rho$ for some $\rho \in [0, 1]$, we simply write ν_ρ^N .

The following lemma compares $\langle \mathcal{L}_N g, g \rangle_\mu$ with $D_N(g, \mu)$.

Lemma 3.1. (i) Let $\lambda : [0, 1] \rightarrow (0, 1)$ be a smooth density profile such that there exists a neighborhood of 0 where $\gamma(\cdot) = \alpha$, and a neighborhood of 1 where $\gamma(\cdot) = \beta$. Let f be a $\nu_{\lambda(\cdot)}^N$ -density,

$$f \geq 0, \quad \sum_{\eta \in \Omega_N} f(\eta)\nu_{\lambda(\cdot)}^N(\eta) = 1.$$

Then

$$\langle \mathcal{L}_N \sqrt{f}, \sqrt{f} \rangle_{\nu_{\lambda(\cdot)}^N} = -(1/2)D_{N,0}(\sqrt{f}, \nu_{\lambda(\cdot)}^N) - D_{N,L}^\alpha(\sqrt{f}, \nu_{\lambda(\cdot)}^N) - D_{N,R}^\beta(\sqrt{f}, \nu_{\lambda(\cdot)}^N) + \mathcal{O}(N^{-1}),$$

where $|\mathcal{O}(N^{-1})| \leq CN^{-1}$ for some finite constant C .

(ii) Let $\rho \in (0, 1)$ be a constant. Let f be a ν_ρ^N -density. Then

$$\langle \mathcal{L}_N \sqrt{f}, \sqrt{f} \rangle_{\nu_\rho^N} = -D_{N,0}(\sqrt{f}, \nu_\rho^N) - D_{N,L}^\alpha(\sqrt{f}, \nu_\rho^N) - D_{N,R}^\beta(\sqrt{f}, \nu_\rho^N) + \mathcal{O}(N^{-\theta}).$$

The first statement (i) is a direct consequence of [1, Lemma 5.1 (ii) and Lemma 5.2]. The second statement (ii) follows directly from [1, Lemma 5.1 (i) and Corollary 5.3]. For this reason, we omit the proof here.

The following lemma bound the occupation variables at the boundary sites by the corresponding Dirichlet forms.

Lemma 3.2. *Let $\lambda : [0, 1] \rightarrow (0, 1)$ be a smooth density profile such that there exists a neighborhood of 0 where $\gamma(\cdot) = \alpha$, and a neighborhood of 1 where $\gamma(\cdot) = \beta$. Let f be a $\nu_{\lambda(\cdot)}^N$ -density. Then there exists a finite constant C_α such that for any $B > 0$,*

$$\left| \sum_{\eta \in \Omega_N} (\eta(1) - \alpha) f(\eta) \nu_{\lambda(\cdot)}^N(\eta) \right| \leq C_\alpha B + C_\alpha N^\theta B^{-1} D_{N,L}^\alpha(\sqrt{f}, \nu_{\lambda(\cdot)}^N).$$

The same result holds at the right boundary: there exists a finite constant C_β such that for any $B > 0$,

$$\left| \sum_{\eta \in \Omega_N} (\eta(N-1) - \beta) f(\eta) \nu_{\lambda(\cdot)}^N(\eta) \right| \leq C_\beta B + C_\beta N^\theta B^{-1} D_{N,R}^\beta(\sqrt{f}, \nu_{\lambda(\cdot)}^N).$$

The above lemma is a direct consequence of [1, Lemmas 5.6 and 5.7]. For that reason, we omit the proof here.

3.2. Replacement lemmas. In this subsection, we prove several replacement lemmas under the longer time scaling $N^{2+\gamma}$, $\gamma > 0$.

The following lemma states that in the subcritical regime $0 \leq \theta < 1$, we could replace the occupation variable at the boundary sites with the corresponding particle density of reservoirs.

Lemma 3.3 (Replacement lemma for the case $0 \leq \theta < 1$). *Suppose $0 \leq \theta < 1$. Then for any $t > 0$,*

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N \left[\left| \int_0^t (\eta_s^N(1) - \alpha) ds \right| \right] = 0.$$

The same result holds with $\eta_s^N(1)$ replaced with $\eta_s^N(N-1)$ and α with β .

Proof. Let $\lambda : [0, 1] \rightarrow (0, 1)$ be a smooth density profile such that there exists a neighborhood of 0 where $\gamma(\cdot) = \alpha$, and a neighborhood of 1 where $\gamma(\cdot) = \beta$. By the relative entropy inequality (cf. [12, A.1.8]), for any $A > 0$, the expectation in the lemma could be bounded from above by

$$\frac{H(\mu_N | \nu_{\lambda(\cdot)}^N)}{AN} + \frac{1}{AN} \log \mathbb{E}_{\nu_{\lambda(\cdot)}^N}^N \left[\exp \left\{ AN \left| \int_0^t (\eta_s^N(1) - \alpha) ds \right| \right\} \right], \quad (3.1)$$

where for any probability measures μ, ν on Ω_N such that μ is absolutely continuous with respect to ν , $H(\mu | \nu)$ is the relative entropy of μ with respect to ν defined as

$$H(\mu | \nu) = \sum_{\eta \in \Omega_N} \mu(\eta) \log \frac{\mu(\eta)}{\nu(\eta)}.$$

It is not hard to prove that $H(\mu_N | \nu_{\lambda(\cdot)}^N) \leq C_\lambda N$ for some finite constant C_λ . Therefore, the first term in (3.1) is bounded by C_λ/A . In the sequel, we shall take $A = A(N) \rightarrow \infty$ as $N \rightarrow \infty$. Whence the first term in (3.1) vanishes in the limit. Since

$$\lim_{N \rightarrow \infty} r_N^{-1} \log(a_N + b_N) = \max \left\{ \lim_{N \rightarrow \infty} r_N^{-1} \log a_N, \lim_{N \rightarrow \infty} r_N^{-1} \log b_N \right\}$$

for any positive sequences $\{a_N\}_{N \geq 1}$, $\{b_N\}_{N \geq 1}$ and $\{r_N\}_{N \geq 1}$ such that $\lim_{N \rightarrow \infty} r_N = \infty$, we could remove the modulus inside the exponential for the second term in (3.1). By the

Feynman-Kac formula (cf. [1, Lemma 7.3]), the second term in (3.1) is bounded by

$$t \sup_{f: \nu_{\gamma(\cdot)}^N\text{-density}} \left\{ \sum_{\eta \in \Omega_N} (\eta(1) - \alpha) f(\eta) \nu_{\lambda(\cdot)}^N(\eta) + \frac{N^{1+\gamma}}{A} \langle \mathcal{L}_N \sqrt{f}, \sqrt{f} \rangle_{\nu_{\lambda(\cdot)}^N} \right\}. \quad (3.2)$$

By Lemma 3.1 (i),

$$\langle \mathcal{L}_N \sqrt{f}, \sqrt{f} \rangle_{\nu_{\lambda(\cdot)}^N} \leq -D_{N,L}^\alpha(\sqrt{f}, \nu_{\lambda(\cdot)}^N) + \mathcal{O}(N^{-1}).$$

Together with Lemma 3.2, for any $B > 0$, we may bound (3.2) by

$$t \sup_{f: \nu_{\lambda(\cdot)}^N\text{-density}} \left\{ C_\alpha B + C_\alpha N^\theta B^{-1} D_{N,L}^\alpha(\sqrt{f}, \nu_{\lambda(\cdot)}^N) - \frac{N^{1+\gamma}}{A} D_{N,L}^\alpha(\sqrt{f}, \nu_{\lambda(\cdot)}^N) + \mathcal{O}(N^\gamma/A) \right\}$$

for some finite constant C_α . Taking $B = C_\alpha A N^{\theta-1-\gamma}$ and $A = N^\gamma \log N$, the above term is bounded by $t(C_\alpha^2 N^{\theta-1} \log N + \mathcal{O}(1/\log N))$, which converges to zero as $N \rightarrow \infty$ since $\theta < 1$. This concludes the proof. \square

Let $m^N(\eta)$ be the average number of particles in the system

$$m^N(\eta) = \frac{1}{N-1} \sum_{x=1}^{N-1} \eta(x).$$

Denote $m_t^N = m^N(\eta_t^N)$. The next result states that in the supercritical regime $\theta > 1$, we could replace the occupation variables at the boundary sites with the average number of particles in the system.

Lemma 3.4 (Replacement lemma for the case $\theta > 1$). *Suppose $\theta > 1$. Then for any $t > 0$,*

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N \left[\left| \int_0^t (\eta_s^N(1) - m_s^N) ds \right| \right] = 0.$$

The same result holds with $\eta_s^N(1)$ replaced with $\eta_s^N(N-1)$

Proof. The proof is similar to that of Lemma 3.3, and we only sketch the proof here. Fix a constant $\rho \in (0, 1)$. By the relative entropy inequality (cf. [12, A.1.8]), for any $A > 0$, the expectation in the lemma could be bounded from above by

$$\frac{H(\mu_N | \nu_\rho^N)}{AN} + \frac{1}{AN} \log \mathbb{E}_{\nu_\rho^N}^N \left[\exp \left\{ AN \left| \int_0^t (\eta_s^N(1) - m_s^N) ds \right| \right\} \right]. \quad (3.3)$$

As in Lemma 3.3, the first term is bounded by C/A for some finite constant C . By the Feynman-Kac formula (cf. [1, Lemma 7.3]), the second term in (3.3) is bounded by

$$t \sup_{f: \nu_\rho^N\text{-density}} \left\{ \sum_{\eta \in \Omega_N} (\eta(1) - m^N(\eta)) f(\eta) \nu_\rho^N(\eta) + \frac{N^{1+\gamma}}{A} \langle \mathcal{L}_N \sqrt{f}, \sqrt{f} \rangle_{\nu_\rho^N} \right\}. \quad (3.4)$$

We may rewrite $\eta(1) - m^N(\eta)$ as a telescope sum

$$\frac{1}{N-1} \sum_{x=1}^{N-1} \sum_{y=1}^{x-1} (\eta(y) - \eta(y+1)).$$

Making the change of variables $\eta \mapsto \eta^{y,y+1}$,

$$\sum_{\eta \in \Omega_N} (\eta(1) - m^N(\eta)) f(\eta) \nu_\rho^N(\eta) = \frac{1}{2(N-1)} \sum_{x=1}^{N-1} \sum_{y=1}^{x-1} \sum_{\eta \in \Omega_N} (\eta(y) - \eta(y+1)) (f(\eta) - f(\eta^{y,y+1})) \nu_\rho^N(\eta).$$

By Cauchy-Schwarz inequality, for any $B > 0$, we may bound the last term by

$$\begin{aligned} & \frac{B}{4(N-1)} \sum_{x=1}^{N-1} \sum_{y=1}^{x-1} \sum_{\eta \in \Omega_N} (\sqrt{f}(\eta) - \sqrt{f}(\eta^{y,y+1}))^2 \nu_\rho^N(\eta) \\ & \quad + \frac{1}{4B(N-1)} \sum_{x=1}^{N-1} \sum_{y=1}^{x-1} \sum_{\eta \in \Omega_N} (\sqrt{f}(\eta) + \sqrt{f}(\eta^{y,y+1}))^2 \nu_\rho^N(\eta) \\ & \leq \frac{B}{2} D_{N,0}(\sqrt{f}, \nu_\rho^N) + \frac{N}{B}. \end{aligned}$$

The last inequality follows from the basic inequality $(a+b)^2 \leq 2(a^2+b^2)$ and the fact that f is a density with respect to ν_ρ^N . By Lemma 3.1 (ii),

$$\langle \mathcal{L}_N \sqrt{f}, \sqrt{f} \rangle_{\nu_\rho^N} \leq -D_{N,0}(\sqrt{f}, \nu_\rho^N) + \mathcal{O}(N^{-\theta}).$$

Whence, (3.4) is bounded by

$$t \sup_{f: \nu_\rho^N\text{-density}} \left\{ \frac{B}{2} D_{N,0}(\sqrt{f}, \nu_\rho^N) + \frac{N}{B} - \frac{N^{1+\gamma}}{A} D_{N,0}(\sqrt{f}, \nu_\rho^N) + \mathcal{O}(N^{1+\gamma-\theta}/A) \right\}$$

Taking $B = 2N^{1+\gamma}/A$ and $A = N^\gamma/(\log N)$, the above term is bounded by $1/(2 \log N) + \mathcal{O}(N^{1-\theta} \log N)$, which converges to zero as $N \rightarrow \infty$ since $\theta > 1$. This concludes the proof. \square

The next lemma concerns about the long time behavior of the average number of particles in the supercritical case.

Lemma 3.5 (Replacement lemma for the average particle number.). *Suppose $\theta > 1$. Recall m_0 defined in (2.4) is the average number of particles at the initial time. For any $t > 0$,*

(i) *if $0 \leq \gamma < \theta - 1$, then*

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N \left[\left| \int_0^t (m_s^N - m_0) ds \right| \right] = 0,$$

(ii) *if $\gamma = \theta - 1$, then*

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N \left[\left| \int_0^t (m_s^N - m_s) ds \right| \right] = 0,$$

where

$$m_s = \frac{\alpha + \beta}{2} + \left(m_0 - \frac{\alpha + \beta}{2} \right) e^{-2cs}.$$

(iii) *if $\gamma > \theta - 1$, then*

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N \left[\left| \int_0^t (m_s^N - (\alpha + \beta)/2) ds \right| \right] = 0.$$

Proof. The statement (ii) is a direct consequence of [13, Proposition 4.5]. For the rest of the statements, consider the martingale \mathbf{m}_t^N defined as

$$\mathbf{m}_t^N := m_t^N - m_0^N - \int_0^t N^{2+\gamma} \mathcal{L}_N m_s^N ds, \quad (3.5)$$

whose quadratic variation at time t is given by

$$\int_0^t \{N^{2+\gamma} \mathcal{L}_N (m_s^N)^2 - 2m_s^N N^{2+\gamma} \mathcal{L}_N m_s^N\} ds.$$

A simple calculation shows that the quadratic variation of \mathbf{m}_t^N is bounded by $CN^{\gamma-\theta}$ for some finite constant C , and that the integral term in (3.5) equals

$$\frac{cN^{2+\gamma-\theta}}{N-1} \int_0^t (\alpha - \eta_s^N(1) + \beta - \eta_s^N(N-1)) ds.$$

If $0 \leq \gamma < \theta - 1$, by Doob's inequality, for any $T > 0$,

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N \left[\sup_{0 \leq t \leq T} (\mathbf{m}_t^N)^2 \right] = 0.$$

The integral term in (3.5) is of order $N^{1+\gamma-\theta}$, which converges to zero as $N \rightarrow \infty$ uniformly in a bounded time interval. Therefore,

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N \left[\sup_{0 \leq t \leq T} |m_t^N - m_0| \right] = 0$$

This proves the first statement (i).

If $\gamma > \theta - 1$, divided by $N^{1+\gamma-\theta}$ in (3.5), we have

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N \left[\sup_{0 \leq t \leq T} (N^{\theta-\gamma-1} \mathbf{m}_t^N)^2 \right] = 0.$$

Since $m_t^N \leq 1$, by (3.5),

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N \left[\left| \int_0^t (\alpha - \eta_s^N(1) + \beta - \eta_s^N(N-1)) ds \right| \right] = 0.$$

By Lemma 3.4, we could replace $\eta_s^N(1)$ and $\eta_s^N(N-1)$ in the time integral with m_s^N . This concludes the proof. \square

4. PROOF OF THEOREM 2.3

In this section, we prove Theorem 2.3 depending on whether $0 \leq \theta < 1$, $\theta = 1$ or $\theta > 1$. For $H \in C^2[0, 1]$, consider the martingale defined as

$$M_t^N(H) = \langle \pi_t^N, H \rangle - \langle \pi_0^N, H \rangle - \int_0^t N^{2+\gamma} \mathcal{L}_N \langle \pi_s^N, H \rangle ds, \quad (4.1)$$

whose quadratic variation at time t is given by

$$\int_0^t \{N^{2+\gamma} \mathcal{L}_N \langle \pi_s^N, H \rangle^2 - 2\langle \pi_s^N, H \rangle N^{2+\gamma} \mathcal{L}_N \langle \pi_s^N, H \rangle\} ds.$$

Direct calculations show that the quadratic variation of $M_t^N(H)$ is bounded by $C_H(N^{\gamma-1} + N^{\gamma-\theta})$ for some finite constant C_H . Therefore, for any $T > 0$,

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N \left[\sup_{0 \leq t \leq T} (N^{-\gamma} M_t^N(H))^2 \right] = 0.$$

Since there is at most one particle per site, $|\langle \pi_t^N, H \rangle| \leq \|H\|_\infty$ uniformly in t , where $\|H\|_\infty := \max_{u \in [0,1]} |H(u)|$ is the uniform norm. Divided by N^γ in (4.1),

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N \left[\left| \int_0^t N^2 \mathcal{L}_N \langle \pi_s^N, H \rangle ds \right| \right] = 0.$$

Direct calculations yield that

$$\begin{aligned} N^2 \mathcal{L}_N \langle \pi_s^N, H \rangle &= \langle \pi_s^N, H'' \rangle - \eta_s^N (N-1) H'(1) + \eta_s^N (1) H'(0) \\ &+ cN^{1-\theta} (\alpha - \eta_s^N(1)) H(0) + cN^{1-\theta} (\beta - \eta_s^N(N-1)) H(1) + \mathcal{O}(N^{-\theta} + N^{-1}). \end{aligned}$$

Whence,

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N \left[\left| \int_0^t \left\{ \langle \pi_s^N, H'' \rangle - \eta_s^N (N-1) H'(1) + \eta_s^N (1) H'(0) \right. \right. \right. \\ \left. \left. \left. + cN^{1-\theta} (\alpha - \eta_s^N(1)) H(0) + cN^{1-\theta} (\beta - \eta_s^N(N-1)) H(1) \right\} ds \right| \right] = 0. \end{aligned} \quad (4.2)$$

4.1. The case $0 \leq \theta < 1$. In this subsection, we prove Theorem 2.3 for the case $0 \leq \theta < 1$. Fix $G \in C[0,1]$. The main technique here is to find an appropriate function H such that $H'' = G$ on $(0,1)$ and that $H(0) = H(1) = 0$. With such a function H , the second line in (4.2) vanishes and the result follows by the corresponding replacement lemmas.

Proof of Theorem 2.3 in the case $0 \leq \theta < 1$. For $G \in C[0,1]$, let

$$H(u) = H_G(u) = \int_0^u \int_0^v G(w) dw dv + u \int_0^1 (v-1)G(v) dv. \quad (4.3)$$

It is easy to check that $H \in C^2[0,1]$, $H'' = G$ on $(0,1)$, and that

$$H(0) = H(1) = 0, \quad H'(0) = \int_0^1 (u-1)G(u) du, \quad H'(1) = \int_0^1 uG(u) du.$$

Substituting the function H into (4.2),

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N \left[\left| \int_0^t \left\{ \langle \pi_s^N, G \rangle - \int_0^1 [(\eta_s^N(N-1) - \eta_s^N(1))u + \eta_s^N(1)] G(u) du \right\} ds \right| \right] = 0.$$

By Lemma 3.3, we could replace $\eta_s^N(1)$ (resp. $\eta_s^N(N-1)$) with α (resp. β). This concludes the proof for the case $0 \leq \theta < 1$. \square

4.2. The case $\theta = 1$. In this subsection, we prove Theorem 2.3 for the case $\theta = 1$. Fix $G \in C[0,1]$. We need to find an appropriate function H such that $H'' = G$ on $(0,1)$ and that the coefficients of $\eta(1)$ and $\eta(N-1)$ vanish in (4.2). Note that in this case we do not need any replacement lemma.

Proof of Theorem 2.3 in the case $\theta = 1$. For $G \in C[0, 1]$, let

$$H(u) = \int_0^1 \left(\frac{v}{2+c} - \frac{1+c}{c(2+c)} \right) G(v) dv + u \int_0^1 \left(\frac{cv}{2+c} - \frac{1+c}{2+c} \right) G(v) dv + \int_0^u \int_0^v G(w) dw dv.$$

It is easy to check that $H \in C^2[0, 1]$, $H'' = G$ on $(0, 1)$, and that

$$\begin{aligned} H(0) &= \int_0^1 \left(\frac{v}{2+c} - \frac{1+c}{c(2+c)} \right) G(v) dv, & H(1) &= \int_0^1 \left(-\frac{v}{2+c} - \frac{1}{c(2+c)} \right) G(v) dv, \\ H'(0) &= \int_0^1 \left(\frac{cv}{2+c} - \frac{1+c}{2+c} \right) G(v) dv, & H'(1) &= \int_0^1 \left(\frac{cv}{2+c} + \frac{1}{2+c} \right) G(v) dv. \end{aligned}$$

Taking the function H into (4.2) and calculating the coefficients of $\eta(1)$ and $\eta(N-1)$, we have

$$-H'(1) - cH(1) = 0, \quad H'(0) - cH(0) = 0.$$

The constant term in (4.2) is given by

$$c\alpha H(0) + c\beta H(1) = \int_0^1 \left(\frac{c(\alpha - \beta)}{2+c} v - \alpha - \frac{\beta - \alpha}{2+c} \right) G(v) dv.$$

Whence, the integrand in (4.2) is equal to

$$\langle \pi_s^N, G \rangle - \int_0^1 \left(\frac{c(\beta - \alpha)}{2+c} v + \alpha + \frac{\beta - \alpha}{2+c} \right) G(v) dv.$$

By (4.2), the time integral of the above term converges in $L^1(\mathbb{P}_{\mu_N}^N)$ to zero as $N \rightarrow \infty$. This concludes the proof for the case $\theta = 1$. \square

4.3. The case $\theta > 1$. In this subsection, we prove Theorem 2.3 for the case $\theta > 1$. In this case, the last line in (4.2) converges to zero as $N \rightarrow \infty$. Whence, we do not need special properties of the function H .

Proof of Theorem 2.3 in the case $\theta > 1$. For $G \in C[0, 1]$, let

$$H(u) = \int_0^u \int_0^v G(w) dw dv.$$

Taking the function H into (4.2) and by Lemma 3.4,

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N \left[\left| \int_0^t \left\{ \langle \pi_s^N, G \rangle - m_s^N (H'(1) - H'(0)) \right\} ds \right| \right] = 0.$$

It is easy to check

$$H'(1) - H'(0) = \int_0^1 G(u) du.$$

Therefore,

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N \left[\left| \int_0^t \left\{ \langle \pi_s^N, G \rangle - m_s^N \int_0^1 G(u) du \right\} ds \right| \right] = 0.$$

By Lemma 3.5, we conclude the proof for the case $\theta > 1$. \square

5. MIXED SLOW BOUNDARIES

In this section, we discuss the extension of Theorem 2.3 to mixed slow boundaries, where the strengths of interactions with left/right reservoirs are different. To be precise, for non-negative parameters $\theta_1 \neq \theta_2$, we consider the process with generator

$$\mathcal{L}_N = \mathcal{L}_{N,0} + \mathcal{L}_{N,L}^{\alpha,\theta_1} + \mathcal{L}_{N,R}^{\beta,\theta_2},$$

where $\mathcal{L}_{N,0}$ corresponds to the buck dynamics which is defined in Section 2, and

$$\begin{aligned} (\mathcal{L}_{N,L}^{\alpha,\theta_1} f)(\eta) &:= cN^{-\theta_1} r_{\alpha,L}(\eta) [f(\eta^1) - f(\eta)], \\ (\mathcal{L}_{N,R}^{\beta,\theta_2} f)(\eta) &:= cN^{-\theta_2} r_{\beta,R}(\eta) [f(\eta^{N-1}) - f(\eta)]. \end{aligned}$$

Since the proof of Theorem 2.3 adapts directly to mixed slow boundaries, we shall state most of the results without proof. Roughly speaking, if the stationary solution of the hydrodynamic equation is unique, then under the longer time scale $N^{2+\gamma}$, $\gamma > 0$, the limiting density profile is the corresponding stationary solution. If the stationary solutions are not unique, a correct one has to be chosen. The later case happens if $\theta_1, \theta_2 > 1$, which is more interesting. Note that in Theorem 2.3, the parameter θ lies in five regimes.

Case 1. If θ_1 and θ_2 lie in the same regime as stated in Theorem 2.3, then the limiting density profile, denoted by $\rho_{\theta,\gamma}$, is given by (2.5).

Case 2. If θ_1 and θ_2 are in different regimes, and the stationary solution is unique, then $\rho_{\theta,\gamma}$ is given by the corresponding stationary solution. Without loss of generality, we may assume $\theta_1 < \theta_2$. Then there exist three cases:

(2.1) If $0 \leq \theta_1 < 1$ and $\theta_2 = 1$, then

$$\rho_{\theta,\gamma}(t, u) = \frac{c(\beta - \alpha)}{1 + c} u + \alpha,$$

which is the unique stationary solution to

$$\Delta\rho(u) = 0, \quad u \in (0, 1), \quad \rho(0) = \alpha, \quad \partial_u \rho(1) = c(\beta - \rho(1)).$$

(2.2) If $0 \leq \theta_1 < 1$ and $\theta_2 > 1$, then

$$\rho_{\theta,\gamma}(t, u) = \alpha,$$

which is the unique stationary solution to

$$\Delta\rho(u) = 0, \quad u \in (0, 1), \quad \rho(0) = \alpha, \quad \partial_u \rho(1) = 0.$$

(2.3) If $\theta_1 = 1$ and $\theta_2 > 1$, then

$$\rho_{\theta,\gamma}(t, u) = \alpha,$$

which is the unique stationary solution to

$$\Delta\rho(u) = 0, \quad u \in (0, 1), \quad \partial_u \rho(0) = c(\rho(0) - \alpha), \quad \partial_u \rho(1) = 0.$$

Case 3. It remains to consider $\theta_1, \theta_2 > 1$. Let us assume $\theta_1 < \theta_2$ as in **Case 2**. Then we have

$$\rho_{\theta,\gamma}(t, u) = \begin{cases} \alpha & \text{if } \theta_1 - 1 < \gamma \leq \theta_2 - 1, \\ \alpha + (m_0 - \alpha)e^{-ct} & \text{if } \theta_1 - 1 = \gamma < \theta_2 - 1. \end{cases} \quad (5.1)$$

Next we only explain how to adapt the proof of Theorem 2.3 to **Case 3**, since this is the most interesting case.

Proof of Eq. (5.1). Since $\theta_1, \theta_2 > 1$, it is easy to check that Lemma 3.4 is still valid. Repeating the proof presented in Subsection 4.3, we only need to replace the average number of particles m_s^N in the system by a proper quantity, which is exactly $\rho_{\theta,\gamma}$. As in the proof of Lemma 3.5, in this case the martingale \mathbf{m}_t^N is given by

$$\mathbf{m}_t^N = m_t^N - m_0^N - \int_0^t \left\{ \frac{cN^{2+\gamma-\theta_1}}{N-1} (\alpha - \eta_s^N(1)) + \frac{cN^{2+\gamma-\theta_2}}{N-1} (\beta - \eta_s^N(N-1)) \right\} ds \quad (5.2)$$

and the quadratic variation of \mathbf{m}_t^N is bounded by $CN^{\gamma-\theta_1}$ since we assume $\theta_1 < \theta_2$.

If $\theta_1 - 1 < \gamma \leq \theta_2 - 1$, divide the martingale \mathbf{m}_t^N in (5.2) by $N^{1+\gamma-\theta_1}$, and let $N \rightarrow \infty$. It follows immediately that we may replace $\eta_s^N(1)$, whence m_s^N according to Lemma 3.4, by α .

If $\theta_1 - 1 = \gamma < \theta_2 - 1$, let $N \rightarrow \infty$ in (5.2). Then any limit m_t of m_t^N satisfies

$$0 = m_t - m_0 - \int_0^t c(\alpha - m_s) ds$$

where we use Lemma 3.4 again to replace $\eta_s^N(1)$ by m_s^N . m_t has an explicit solution

$$m_t = \alpha + (m_0 - \alpha)e^{-ct}.$$

This concludes the proof. \square

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