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| D T Vo. The oscillating random walk on  $\mathbb{Z}$ . 2022. hal-03510517

**HAL Id: hal-03510517**

**<https://hal.science/hal-03510517v1>**

Preprint submitted on 4 Jan 2022

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# The oscillating random walk on $\mathbb{Z}$

January 4, 2022

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## Abstract

The paper is concerned with a new approach for the recurrence property of the oscillating process on  $\mathbb{Z}$  in Kemperman's sense. In the case when the random walk is ascending on  $\mathbb{Z}^-$  and descending on  $\mathbb{Z}^+$ , we determine the invariant measure of the embedded process of successive crossing times and then prove a necessary and sufficient condition for recurrence. Finally, we make use of this result to show that the general oscillating process is recurrent under some Hölder-typed moment assumptions.

Keywords: random walks, irreducible class, invariant measure

## 1 Introduction and notations

### 1.1 Introduction

In parallel with many studies of classical stochastic processes, oscillating random walks, which was introduced systematically by KEMPERMAN [8], have been found to be good models with several applications, see [7] for instance. This paper deals with the homogeneous Markov chain  $\mathcal{X}^{(\alpha)} = (X_n^{(\alpha)})_{n \geq 0}$  indexed by a parameter  $\alpha \in [0, 1]$  such that  $X_0^{(\alpha)} = x_0$  with some fixed  $x_0 \in \mathbb{Z}$  and for  $n \geq 1$ ,

$$X_{n+1}^{(\alpha)} := X_n^{(\alpha)} + \left( \xi_{n+1} \mathbb{1}_{\{X_n^{(\alpha)} \leq -1\}} + \eta_{n+1} \mathbb{1}_{\{X_n^{(\alpha)} = 0\}} + \xi'_{n+1} \mathbb{1}_{\{X_n^{(\alpha)} \geq 1\}} \right), \quad (1.1)$$

where

- the  $\xi_n, n \geq 1$ , have common distribution  $\mu$ ,
- the  $\xi'_n, n \geq 1$ , have common distribution  $\mu'$ ,

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- the  $\eta_n, n \geq 1$ , have common distribution given linearly by

$$\mathbb{P}[\eta_n = y] := \alpha\mu(y) + (1 - \alpha)\mu'(y) \text{ for any } y \in \mathbb{Z},$$

- $(\xi_n, \xi'_n, \eta_n)_{n \geq 1}$  is a sequence of independent and identically distributed (abbreviate i.i.d.) random variables.

When we want to emphasize the dependence of  $\mathcal{X}^{(\alpha)}$  on the distributions  $\mu$  and  $\mu'$ , the process  $(X_n^{(\alpha)})_{n \geq 0}$  is denoted by  $\mathcal{X}^{(\alpha)}(\mu, \mu')$  instead  $\mathcal{X}^{(\alpha)}$ .

It is merely to say that the excursion will be directed by  $\mu$  (resp.  $\mu'$ ) as long as the process stays on the negative (resp. positive) side and therefore, this present model is often called the *oscillating random walk* with respect to (w.r.t.) zero level. The choice of zero is arbitrary and can be replaced by any fixed level. In case of  $\alpha \in \{0; 1\}$ , we use the terminology “*crossing*” to mean the point 0 belongs to only one of the two half lines and it belongs to both if  $0 < \alpha < 1$ . We are mainly interested in recurrence of this process on its essential (i.e. maximal irreducible) classes.

The case  $\mu = \mu'$  is well-treated by MIJATOVIĆ and VYSOTSKY [11], except providing detailed illustrations of the trajectory. The highlight result is an invariant measure (and a probability) constructed in a probabilistic manner and under the additional assumption of (topological) recurrence of the chain, it is up to a multiplicative constant finite invariant measure. For the sake of completeness, we will study all irreducible classes of  $\mathcal{X}^{(\alpha)}$  from the simple case when  $\xi_n \geq 0$  and  $\xi'_n \leq 0$  to the general one. Appropriately refining the formula in [11], we obtain the exact invariant measure and then apply the idea of KNIGHT [10] to get its discrete version, see Section 2.

Section 3 is devoted to such an important sub-process of  $\mathcal{X}^{(0)}$  evolving within a definite state space, whose elements are recorded at corresponding successive crossing times. A particular interest will be put on the structure of (essential) irreducible classes, especially on  $\mathcal{N}$ , the set of isolated states which are impossible to reach from any opposite states in a single step. Theorem 3.3 stipulates some mild conditions for  $\mathcal{N}$  to be empty and also yields an expression for it. In analogy to the approach for reflected random walks, we finally compute the invariant measure for the sub-process based on some arguments developed in the previous work of PEIGNÉ and WOESS [13].

The recurrence of the general oscillating random walk  $\mathcal{X}^{(\alpha)}$  is dealt with in Section 4. Some powerful tools such as the KEMPERMAN’s criterion [8] (a divergent series represented in term of renewal functions) or the integral criterion of ROGOZIN and FOSS [15] (a transformation established with the help of Wiener-Hopf factorization) are mentioned for reference. We also furnish a new approach coming from the fact that the recurrence of the crossing sub-process implies to the recurrence of the full process. To do this, we first

show that the process of crossing is (positive) recurrent if the tail distribution condition

$$\sum_{n=0}^{+\infty} \mathbb{P}(\xi_1 > n) \mathbb{P}(\xi'_1 < -n) < +\infty$$

holds when  $\xi_n \geq 0, \xi'_n \leq 0$ . Moreover, it can be attained under the hypothesis  $\mathbb{E}[\xi_1^p] < +\infty$  and  $\mathbb{E}[(-\xi'_1)^q] < +\infty$ , where  $p, q \in ]0, 1[$  satisfying  $p + q = 1$ . When jumps are generalized on  $\mathbb{Z}$ , there may have different possibilities, for instance, both  $\xi_n$  and  $\xi'_n$  are either drifted (positive and negative, respectively) or centered as well as mixed and by Theorem 4.8, we will address suitable conditions to each corresponding situation.

## 1.2 Notations

Throughout this paper, we fix some frequently used notations

- $S_\mu$  (resp.  $S'_\mu$ ) : the support of  $\mu$  (resp.  $\mu'$ ).
- $D$  (resp.  $D'$ ) : the maximum of  $\mu$  (resp. the minimum of  $\mu'$ ).  
We adhere to the convention that  $D = +\infty$  (resp.  $D' = -\infty$ ) when  $S_\mu$  (resp.  $S_{\mu'}$ ) is unbounded from above (resp. from below).
- $d$  (resp.  $d'$ ) : the greatest common divisor of  $S_\mu$  (resp.  $S'_\mu$ ).
- $\mathbb{Z}^+/\mathbb{Z}^-$  : the set of positive/negative integers (and  $\mathbb{Z}_0^+/\mathbb{Z}_0^-$  if 0 is included).
- $r_x$ : the remainder of  $x$  in the Euclidean division by  $\delta$  (i.e.  $0 \leq r_x < \delta$ ).

Let us end this paragraph dedicated to notations by reminding that for any fixed  $0 \leq \alpha \leq 1$ , the chain (1.1) is denoted by  $\mathcal{X}^{(\alpha)}(\mu, \mu')$  (and simply  $\mathcal{X}^{(\alpha)}$  when there is no ambiguity on the choices of  $\mu$  and  $\mu'$ ).

## 2 Irreducible classes and invariant measure of $\mathcal{X}^{(0)}(\mu, \mu')$

It is easy to check that if  $\mu = \mu'$  (in this case  $\mathcal{X}^{(0)}(\mu, \mu')$  becomes an ordinary random walk on  $\mathbb{Z}$  with the unique jump measure  $\mu$ ) then  $d = d'$  and the irreducible classes of  $\mathcal{X}^{(0)}(\mu, \mu')$  are the sets  $r + d\mathbb{Z}$  with  $0 \leq r < d$ . In this section, we describe the essential classes of  $\mathcal{X}^{(0)}(\mu, \mu')$  when  $\mu \neq \mu'$ .

### 2.1 The chain $\mathcal{X}^{(0)}(\mu, \mu')$ when $S_\mu \subset \mathbb{Z}^+$ and $S_{\mu'} \subset \mathbb{Z}^-$

For any  $x \in \mathbb{Z}$ , let  $\mathcal{I}(x)$  be the irreducible class of  $x$ . It holds  $\mathcal{I}(0) \subset \{D', \dots, D - 1\}$ . Furthermore, for any starting point  $x$ , after finitely many steps, the chain  $\mathcal{X}^{(0)}$  stays for ever a.s. in the subset  $\{D', \dots, D - 1\}$ .

**Theorem 2.1.** We suppose that  $S_\mu \subset \mathbb{Z}^+$  and  $S_{\mu'} \subset \mathbb{Z}^-$ .

- Assume first  $D$  and  $D'$  are finite. If  $d \wedge d' = \delta$ , then
- i) there exist  $\delta$  irreducible essential classes

$$\{D', \dots, D-1\} \cap (r + \delta\mathbb{Z}) \quad \text{with } 0 \leq r < \delta;$$

(in particular the irreducible class of 0 equals  $\mathcal{I}(0) = \{D', \dots, D-1\} \cap \delta\mathbb{Z}$ );

- ii) if  $x \geq D$  or  $x < D'$  then  $x$  is transient and, after finitely many steps, reaches  $\mathbb{P}$ -a.s. the essential class  $\{D', \dots, D-1\} \cap (r_x + \delta\mathbb{Z})$ .

- If  $D' = -\infty$  and  $D$  is finite then

- i) there exist  $\delta$  irreducible essential classes

$$]-\infty, D-1] \cap (r + \delta\mathbb{Z}) \quad \text{with } 0 \leq r < \delta;$$

- ii) the  $x \geq D$  are all transient and, after finitely many steps, reaches  $\mathbb{P}$ -a.s. the essential class  $]-\infty, \dots, D-1] \cap (r_x + \delta\mathbb{Z})$ .

(similar dual statement follows when  $D'$  is finite and  $D = +\infty$ ).

- If  $D = +\infty$  and  $D' = -\infty$  then there exist  $\delta$  irreducible essential classes, which are all essential:

$$r + \delta\mathbb{Z} \quad \text{with } 0 \leq r < \delta.$$

*Proof.* The proof of KEMPERMAN based on the theory of semi-groups of  $\mathbb{Z}$  entirely solved for the chain  $\mathcal{X}^{(\alpha)}$ ,  $0 \leq \alpha \leq 1$  providing that  $D = -D' = +\infty$  (see Remark 2.3). Back to the current model, we will prove by induction, but let us first fix some notations.

Let  $T$  be a finite subset of  $S_\mu \cup S_{\mu'}$  s.t.  $T \cap S_\mu \neq \emptyset$  and  $T \cap S_{\mu'} \neq \emptyset$ ; without loss of generality, we assume  $0 \notin T$ .

For any  $x \in \mathbb{Z}$ , we denote by  $\mathcal{O}_T(x)$  the “orbit of  $x$  under  $T$ ”, that is the set of sequences  $\mathbf{x} = (x_i)_{i \geq 0}$  defined by induction as follows:  $x_0 = x$  and, for any  $i \geq 1$ ,

- if  $x_i \leq -1$  then  $x_{i+1} = x_i + s$  for some  $s \in T \cap S_\mu$ ;
- if  $x_i \geq 0$  then  $x_{i+1} = x_i + s'$  for some  $s' \in T \cap S_{\mu'}$ .

Notice that all the  $x_i$  but finitely many do belong to  $\{\min(S_{\mu'}), \dots, \max(S_\mu) - 1\}$ .

For any  $x, y \in \mathbb{Z}$ , we write  $x \xrightarrow{T} y$  if there exists  $\mathbf{x} \in \mathcal{O}_T(x)$  s.t.  $x_0 = x$  and  $x_n = y$  for some  $n \geq 0$ . When there exists no such sequence  $\mathbf{x}$ , we write  $x \not\xrightarrow{T} y$ . The notation  $x \overset{T}{\leftrightarrow} y$  means  $x \xrightarrow{T} y$  and  $y \xrightarrow{T} x$ .

The relation  $\overset{T}{\leftrightarrow}$  is an equivalence relation on  $\mathbb{Z}$  whose classes are called  $T$ -irreducible classes. The  $T$ -irreducible class of  $x$  is denoted  $\mathcal{I}_T(x)$ .

The relation  $\xrightarrow{T}$  induces a partial order relation on  $\mathcal{I}_T$ , denoted again  $\xrightarrow{T}$ . The maximal irreducible classes for this relation are called  $T$ -essential; a non essential irreducible class is said  $T$ -transient.

We now describe  $\mathcal{I}_T$ , by induction on the cardinality of  $T$ .

**Step 1- Case when  $T = \{s, s'\}$  with  $s \in \mathbb{Z}^+$  and  $s' \in \mathbb{Z}^-$ .**

• We assume first  $s \wedge s' = 1, x = 0$  and prove that  $\{s', \dots, s-1\}$  is the unique  $T$ -essential class. Furthermore,  $\mathcal{I}_T(x)$  equals  $\{x\}$  and is  $T$ -transient when  $x \geq s$  or  $x < s'$ .

Indeed, for any  $\omega = (\omega_i)_{i \geq 0}$  in  $\mathcal{O}_T(0)$ , the  $\omega_i$  all belong to  $\mathbb{N}s + \mathbb{N}s'$  and to  $\{s', \dots, s-1\}$ . Hence, there exist  $j > i \geq 1$  such that  $\omega_j = \omega_i$ . Since  $\omega_i \neq \omega_{i+1}, \omega_{j-1} \neq \omega_j$  and  $\omega_i \xrightarrow{T} \omega_j = \omega_i$ , there exists  $k, \ell \geq 1$  such that  $ks + \ell s' = 0$ . The condition  $s \wedge s' = 1$  yields  $k = 0 \pmod{s'}$  and  $\ell = 0 \pmod{s}$ , hence  $k \geq |s'|$  and  $\ell \geq s$ . Consequently, the sub-orbit  $\{\omega_i, \omega_{i+1}, \dots, \omega_{j-1}\}$  contains at least  $s + |s'|$  elements; since it is included in  $\{s', \dots, s-1\}$ , it holds in fact  $\{\omega_i, \omega_{i+1}, \dots, \omega_{j-1}\} = \{s', \dots, s-1\}$ . This proves that  $x \xrightarrow{T} y$  for any  $x, y \in \{s', \dots, s-1\}$ , hence  $\{s', \dots, s-1\} \subset \mathcal{I}_T(0)$ . Eventually,  $\{s', \dots, s-1\} = \mathcal{I}_T(0)$  since the elements of all the orbits of 0 remain in  $\{s', \dots, s-1\}$ . This also implies that  $\{s', \dots, s-1\}$  is  $T$ -essential.

As a consequence of this argument, the orbit  $\mathcal{O}_T(0)$  contains a unique sequence  $\omega$ , which is periodic with period  $\omega_0, \omega_1, \dots, \omega_{s+|s'|-1}$  where  $\omega_i \in \{s', \dots, s-1\}$  and  $\omega_i \neq \omega_j$  for any  $0 \leq i < j < s + |s'|$ . We write for short  $\omega = \overline{\omega_0, \dots, \omega_{s+|s'|-1}}$  and emphasize that  $\{\omega_0, \dots, \omega_{s+|s'|-1}\} = \{s', \dots, s-1\}$ .

Now, if  $x \geq s$  or  $x < s'$  and  $\mathbf{x} = (x_i)_{i \geq 0} \in \mathcal{O}_T(x)$ , then  $x_i \neq x$  for any  $i \geq 1$ ; in other words,  $\mathcal{I}_T(x) = \{x\}$  and  $x$  is  $T$ -transient.

• Assume now  $s \wedge s' = \delta \geq 2$ . Then, the  $T$ -essential classes are  $\{s', \dots, s-1\} \cap (x + \delta\mathbb{Z})$ . Furthermore, if  $x \geq s$  or  $x < s'$ , then  $\mathcal{I}(x)$  equals  $\{x\}$  and is  $T$ -transient.

In this case, the set  $\mathcal{O}_T(0)$  still contains a unique sequence  $\omega = \overline{\omega_0, \dots, \omega_{k-1}}$  with  $k = \frac{s+|s'|}{\delta}$  and  $\{\omega_0, \dots, \omega_{k-1}\} = \{s', \dots, s-1\} \cap \delta\mathbb{Z}$ . As a direct consequence, for any  $x \in \mathbb{Z}$ , the set  $\mathcal{O}_T(x)$  is included in  $x + \delta\mathbb{Z}$  and also contains a unique sequence  $\mathbf{x}$ , which is ultimately periodic with period  $r_x + \omega_0, \dots, r_x + \omega_{k-1}$ . The description of the  $T$ -irreducible classes follows immediately.

**Step 2- Case when  $T \subset \mathbb{Z}$  is finite and satisfies  $T \cap \mathbb{Z}^- \neq \emptyset$  and  $T \cap \mathbb{Z}^+ \neq \emptyset$ .**

The proof is made by induction, from  $T$  to  $\mathbf{T} := T \cup \{t'\}$  with  $t' \in \mathbb{Z}^-$ ; the case when  $\mathbf{T} := T \cup \{t\}$  with  $t \in \mathbb{Z}^+$  is studied in the same way. By Step 1, the property is true for  $T = \{s, s'\}, s > 0$  and  $s' < 0$ .

Hypothesis of induction: Let  $T$  be a set of non-zero integers s.t.  $T \cap \mathbb{Z}^- \neq \emptyset$  and  $T \cap \mathbb{Z}^+ \neq \emptyset$ .

We set  $\delta_T := \gcd(T)$  and denote  $s$  (resp.  $s'$ ) the largest (resp. smallest) element of  $T$ .

We assume that

- the  $T$ -essential classes are  $\{s', \dots, s-1\} \cap (r + \delta_T\mathbb{Z})$  with  $0 \leq r < \delta_T$ ;
- when  $x \geq s$  or  $x < s'$ , then  $\mathcal{I}_T(x) = \{x\}$  and  $x$  is  $T$ -transient; furthermore, for any  $\mathbf{x} = (x_i)_{i \geq 0} \in \mathcal{O}_T(x)$ , all  $x_i$  but finitely many belong to the  $T$ -essential class  $\{s', \dots, s-1\} \cap (x + \delta_T\mathbb{Z})$ .

Conclusion: For  $t' \in \mathbb{Z}^-$  then the same property holds for  $\mathbf{T} = T \cup \{t'\}$ . In other words, setting  $\delta_{\mathbf{T}} := \gcd(\mathbf{T})$  and  $m_{\mathbf{T}} := \min(s', t')$ ,

- the  $\mathbf{T}$ -essential classes are  $\{m_{\mathbf{T}}, \dots, s-1\} \cap (r + \delta_{\mathbf{T}}\mathbb{Z})$  with  $0 \leq r < \delta_{\mathbf{T}}$ ;
- if  $x \geq s$  or  $x < m_{\mathbf{T}}$ , then  $\mathcal{I}_{\mathbf{T}}(x) = \{x\}$  and  $x$  is  $\mathbf{T}$ -transient; furthermore, for any  $\mathbf{x} = (x_i)_{i \geq 0} \in \mathcal{O}_{\mathbf{T}}(x)$ , all  $x_i$  but finitely many belong to the  $\mathbf{T}$ -essential class  $\{m_{\mathbf{T}}, \dots, s-1\} \cap (x + \delta_{\mathbf{T}}\mathbb{Z})$ .

The same argument as in Step 1 works to deduce the case  $\delta_{\mathbf{T}} \geq 2$  from the case  $\delta_{\mathbf{T}} = 1$ ; thus, we only consider the case  $\delta_{\mathbf{T}} = 1$ .

Notice that  $\{m_{\mathbf{T}}, \dots, s-1\}$  is absorbing; in other words, for any  $x \in \mathbb{Z}$  and any  $\mathbf{x} = (x_i)_{i \geq 0} \in \mathcal{O}_{\mathbf{T}}(\mathbf{x})$ , all  $x_i$  but finitely many belong to this set. In particular,  $x$  is transient when  $x \geq s$  or  $x < m_{\mathbf{T}}$ . It thus remains to check that  $\{m_{\mathbf{T}}, \dots, s-1\}$  is irreducible.

Let us first prove that

$$\{s', \dots, s-1\} \xrightarrow{\mathbf{T}} \{s', \dots, s-1\} \quad (2.1)$$

For any  $x \in \{s', \dots, s-1\}$ , we choose  $\ell_x \geq 0$  s.t.  $x + t' + \ell_x s \in \{0, \dots, s-1\}$  and notice that  $x \xrightarrow{\mathbf{T}} x + t' + \ell_x s$ . Now, for any  $\mathbf{y} = (y_i)_{i \geq 0} \in \mathcal{O}_T(x + t' + \ell_x s)$ , all  $y_i$  but finitely many belong to the  $T$ -essential class  $\mathcal{I}_T(x + t') = \{s, \dots, s'-1\} \cap (x + t' + \delta_T\mathbb{Z})$ ; thus,  $\{x\} \xrightarrow{\mathbf{T}} \{s', \dots, s-1\} \cap (x + t' + \delta_T\mathbb{Z})$ . Reiterating the argument, we get, for  $k \geq 1$

$$\{x\} \xrightarrow{\mathbf{T}} \{s', \dots, s-1\} \cap (x + kt' + \delta_T\mathbb{Z}). \quad (2.2)$$

Since  $\gcd(t', \delta_T) = \delta_{\mathbf{T}} = 1$ , the class of  $t' \pmod{\delta_T}$  generates  $\mathbb{Z}/\delta_T\mathbb{Z}$ , so

$$\bigcup_{0 \leq k < \delta_T} \{s', \dots, s-1\} \cap (x + kt' + \delta_T\mathbb{Z}) = \{s', \dots, s-1\}.$$

This yields immediately  $\{x\} \xrightarrow{\mathbf{T}} \{s', \dots, s-1\}$ : indeed, for any  $y \in \{s', \dots, s-1\}$  there exists  $k_y \geq 1$  s.t.  $y \in \{s', \dots, s-1\} \cap (x + k_y t' + \delta_T\mathbb{Z})$  and (2.2) readily implies  $x \xrightarrow{\mathbf{T}} y$ . This holds for any  $x \in \{s', \dots, s-1\}$  and proves (2.1).

This implies  $\{m_{\mathbf{T}}, \dots, s-1\} \xrightarrow{\mathbf{T}} \{m_{\mathbf{T}}, \dots, s-1\}$  when  $s' < t'$ . It remains to consider the case when  $t' < s'$ ; we fix  $x \in \{t', \dots, s'-1\}$ .

The same argument as above proves that  $\{x\} \xrightarrow{\mathbf{T}} \{s', \dots, s-1\}$ .

Conversely, we decompose  $x$  as  $x = t' + k_x s + r_x$  with  $k_x \geq 0$  and  $0 \leq r_x < s$ . For any  $y \in \{s', \dots, s-1\}$ , property (2.1) yields  $y \xrightarrow{\mathbf{T}} r_x$  since  $0 \leq r_x < s$ ; now, immediately,  $r_x \xrightarrow{\mathbf{T}} t' + k_x s + r_x = x$  so that  $y \xrightarrow{\mathbf{T}} x$  as expected.

### Step 3 -Proof of Theorem 2.1

When  $D$  and  $D'$  are finite, we set  $T = S_{\mu} \cup S_{\mu'}$  and apply Step 2.

If  $D' = -\infty$  and  $D$  is finite, we set  $T = T_{s'} = S_\mu \cup (S_{\mu'} \cap \{s', \dots, -1\})$  with  $s' \leq -1$ , apply Step 2 then let  $s' \rightarrow -\infty$ . The two other cases are treated in the same way. ■

It is worth remarking that one may extend the above to adapt for  $\mathcal{X}^{(\alpha)}$ , that is to say, the chain (starting at any initial point) will be absorbed after finitely many steps by the essential class

$$\begin{cases} \{D', \dots, D-1\} & \text{if } \alpha = 0, \\ \{D', \dots, D\} & \text{if } 0 < \alpha < 1, \\ \{D'+1, \dots, D\} & \text{if } \alpha = 1. \end{cases}$$

under the assumption of bounded jumps. The remaining statements of the theorem are proved in an analogous way.

## 2.2 The chain $\mathcal{X}^{(0)}$ in the general case

We start by considering the irreducible classes of  $\mathcal{X}^{(0)}$  on the additional assumptions that  $S_\mu \cap \mathbb{Z}^- \neq \emptyset$  and  $S_{\mu'} \cap \mathbb{Z}^+ \neq \emptyset$ . Intuitively, the oscillating random walk  $\mathcal{X}^{(0)}$  starting from 0 can visit arbitrarily large integers and so, it is quite natural to think that  $\mathcal{I}(0)$  contains the whole line  $\mathbb{Z}$  in this case, under the hypothesis  $d \wedge d' = 1$ . In fact, it is false and depends deeply on the structure of  $S_\mu$  and  $S'_{\mu'}$ . The following theorem, which is based on the ideas developed in Step 2 above, clarifies this point.

**Theorem 2.2.** *We write  $S_\mu^+$  (resp.  $S_{\mu'}^+$ ) and  $S_\mu^-$  (resp.  $S_{\mu'}^-$ ) as the positive and negative components of  $S_\mu$  (resp.  $S'_{\mu'}$ ), respectively. Suppose that these subsets are all non-empty. If  $d \wedge d' = \delta$  and  $d \neq d'$  then*

- Case when  $D < d'$

i) if  $x \in \{D, \dots, d'-1\} + d'\mathbb{Z}_0^+$  then  $x$  is transient and its irreducible class is

$$\mathcal{I}(x) = (x + d'\mathbb{Z}) \cap [D, +\infty[;$$

ii) otherwise,  $x$  is essential and its essential class is given by

$$\mathcal{I}(x) = (r_x + \delta\mathbb{Z}) \setminus (\{D, \dots, d'-1\} + d'\mathbb{Z}_0^+).$$

- Case when  $D' > -d$

i) if  $x \in \{-d, \dots, D'-1\} + d\mathbb{Z}_0^-$  then  $x$  is transient and its irreducible class is

$$\mathcal{I}(x) = (x + d\mathbb{Z}) \cap ]-\infty, D'-1];$$



ii) otherwise,  $x$  is essential and its essential class is given by

$$\mathcal{I}(x) = (r_x + \delta\mathbb{Z}) \setminus (\{-d, \dots, D' - 1\} + d\mathbb{Z}_0^-).$$

• Case when  $D \geq d'$  and  $D' \leq -d$

There is/are  $\delta$  irreducible class(es), which is/are all essential:

$$r + \delta\mathbb{Z} \text{ with } 0 \leq r < \delta.$$

*Proof.* Before delving into details, we shall pay more attention to the fact that these two conditions cannot be attained at the same time due to  $-D < d' < D' < d$  and we thus arrive at a contradiction. In particular, if  $D < d'$  then  $-D' \geq d$  and vice versa.

• Case when  $D' < d$

i) The converse is easily done since  $x \xleftrightarrow{S_\mu \cup S_{\mu'}} x + kd'$  for any  $k \in \mathbb{Z}$  satisfying  $x + kd' \geq D$ . Indeed, the assumption of  $S_{\mu'}$  leads to the semi-group generated by  $S_{\mu'}$ , says  $T_{\mu'}$ , is equal to  $d'\mathbb{Z}$ . One can write  $kd' = \sum s'_i$  as the finite sum of elements in  $S_{\mu'}$ . Selecting first the positive elements (if any) and then the negative ones, it immediately follows that  $(x + d'\mathbb{Z}) \cap [D, +\infty[ \subset C(x)$ . Now, we will show by contraposition that  $C(x) \subset (x + d'\mathbb{Z}) \cap [D, +\infty[$ . Suppose  $z \in ]-\infty, D - 1] \cap C(x)$ . Let  $\tau$  be the last time entering to the set  $\{0, \dots, D - 1\}$  of  $\mathcal{X}^{(0)}$  before visiting  $x$  for the first time. Since  $z \xrightarrow{S_\mu \cup S_{\mu'}} x$ , we have  $\mathbb{P}_z[\tau < +\infty] = 1$  and then put  $X_\tau^{(0)} = y \in \{0, \dots, D - 1\}$ . Observe that  $x - y \in \{1, \dots, d' - 1\} + d'\mathbb{Z}_0^+$ , i.e.  $x - y \notin d'\mathbb{N}$  and thus,  $z \not\xrightarrow{S_\mu \cup S_{\mu'}} x$  (contradiction). When  $z \geq D$ , the crossing process starting at  $z$  is directed only by  $S_{\mu'}$  and therefore,  $z \in x + d'\mathbb{Z}$ .

ii) A reasoning similar to the above yields that

$$y \xleftrightarrow{S_\mu \cup S_{\mu'}} y + kd' \text{ if } y \in \{0, \dots, D - 1\} + d'\mathbb{Z}_0^+,$$

where  $k \in \mathbb{Z}$  s.t.  $y + kd' \geq 0$ .

Moreover, by Theorem 2.1

$$\{D', \dots, D - 1\} \cap (r_x + \delta\mathbb{Z}) \xrightarrow{S_\mu^+ \cup S_{\mu'}^-} \{D', \dots, D - 1\} \cap (r_x + \delta\mathbb{Z}). \quad (2.3)$$

Now, we fixed  $t \in ]-\infty, D'[\cap (r_x + \delta\mathbb{Z})$ , then there is some  $z_t \in \{D', \dots, -1\} \cap (r_x + \delta\mathbb{Z})$  s.t.  $z_t \equiv t \pmod{d}$  due to  $-D' \geq d$ . Combining (2.3) with the fact that  $T_\mu = d\mathbb{Z}$ , we get

$$\{t\} \xleftrightarrow{S_\mu \cup S_{\mu'}} \{D', \dots, D - 1\} \cap (r_x + \delta\mathbb{Z}).$$

This property holds for every choice of  $t$ , so

$$\{-\infty, D'[\cap(r_x + \delta\mathbb{Z})\} \xleftrightarrow{S_\mu \cup S_{\mu'}} \{D', \dots, D-1\} \cap(r_x + \delta\mathbb{Z})$$

which achieves the proof.

- We finish by mentioning that, suitably modified, the above argument applies to the other cases. ■

**Remark 2.3.** *Suppose that  $S_\mu$  is unbounded from above ( $D = +\infty$ ) and  $S_{\mu'}$  is unbounded from below ( $D' = -\infty$ ). In such situation, all states connect and it is in fact possible that the process  $\mathcal{X}^{(0)}$  with a positive probability will never reach a given neighborhood of 0 due to infinitely many extremely large jumps across 0. Consequently, there exists (with a positive probability) an orbit between every two points, which has no intermediary state belonging to a given finite set  $F$ , even if  $0 \notin F$  (see [8]).*

### 2.3 On the invariant measure for $\mathcal{X}^{(0)}$ when $S_\mu \subset \mathbb{Z}^+$ and $S_{\mu'} \subset \mathbb{Z}^-$

The concept of invariant measure plays a crucial role in the study in the long-time behaviour and asymptotic properties of a Markov chain. To adapt for the current situation, a proper adjustment to the invariant measure in [11] is indispensable and what we found is the following

**Lemma 2.4.** *Assume  $S_\mu \subset \mathbb{Z}^+$  and  $S_{\mu'} \subset \mathbb{Z}^-$ . The measure  $\lambda$  on  $\mathbb{R}$  given by*

$$\lambda(dx) = (\mathbb{1}_{]-\infty, 0[}(x) \mathbb{P}[\xi'_1 < x] + \mathbb{1}_{]0, +\infty[}(x) \mathbb{P}[\xi_1 > x]) dx,$$

where  $dx$  is Lebesgue measure, is invariant for  $\mathcal{X}^{(0)}$ .

*Proof.* For any  $a \geq 0$ , it follows without difficulty that

$$\begin{aligned} \lambda P]a, +\infty[ &= \lim_{N \rightarrow +\infty} \int_{-N}^0 \mathbb{P}[\xi'_1 < x] \mathbb{P}[x + \xi_1 > a] dx + \lim_{N \rightarrow +\infty} \int_0^N \mathbb{P}[\xi_1 > x] \mathbb{P}[x + \xi'_1 > a] dx \\ &= \lim_{N \rightarrow +\infty} \int_{-N}^0 \mathbb{P}[\xi'_1 < x] \mathbb{P}[\xi_1 > a - x] dx + \lim_{N \rightarrow +\infty} \int_0^N \mathbb{P}[\xi_1 > x] \mathbb{P}[\xi'_1 > a - x] dx \\ &= \lim_{N \rightarrow +\infty} \int_a^{N+a} \mathbb{P}[\xi'_1 < a - y] \mathbb{P}[\xi_1 > y] dy + \lim_{N \rightarrow +\infty} \int_a^N \mathbb{P}[\xi_1 > y] \mathbb{P}[\xi'_1 > a - y] dy \\ &= \lim_{N \rightarrow +\infty} \int_a^N \mathbb{P}[\xi_1 > y] dy + \lim_{N \rightarrow +\infty} \int_N^{N+a} \mathbb{P}[\xi'_1 < a - y] \mathbb{P}[\xi_1 > y] dy \\ &= \lambda]a, +\infty[ \end{aligned}$$

$$\text{since } \int_N^{N+a} \mathbb{P}[\xi'_1 < a - y] \mathbb{P}[\xi_1 > y] dy \leq \int_0^a \mathbb{P}[\xi_1 > z + N] dz \rightarrow 0 \text{ as } N \rightarrow +\infty.$$

The same computation yields  $\lambda P] - \infty, -a[ = \lambda] - \infty, -a[$  and thus, the proof is complete.  $\blacksquare$

As a direct consequence, we obtain the following result.

**Corollary 2.5.** *Assume  $S_\mu \subset \mathbb{Z}^+$  and  $S_{\mu'} \subset \mathbb{Z}^-$ . Then  $\mathcal{X}^{(0)}$  is positive recurrent on each essential class iff both measures  $\mu$  and  $\mu'$  have finite first moment.*

**Lemma 2.6.** *Assume  $S_\mu \subset \mathbb{Z}^+$  and  $S_{\mu'} \subset \mathbb{Z}^-$ . The discrete measure  $\nu$  on  $\mathbb{Z}$  given by*

$$\nu(m) := \begin{cases} \mu'] - \infty, m] & \text{if } m \leq -1 \\ \mu[m + 1, +\infty[ & \text{if } m \geq 0. \end{cases} \quad (2.4)$$

*is invariant for the homogeneous random walk  $\mathcal{X}^{(0)}$ . Moreover, for arbitrary  $x_0 \in \mathbb{Z}$ , it induces a corresponding invariant measure  $\nu_{x_0}$  on  $\mathcal{I}(r_{x_0})$  by its restriction on this essential class (in other words, if  $A \subset \mathcal{I}(r_{x_0})$  then  $\nu_{x_0}(A) = \sum_{x \in A} \nu(x)$ ).*

*Proof.* At first, we briefly outline the idea of the result. Let  $X$  be a measurable space that is compatible with the Borel  $\sigma$ -algebra  $\mathcal{B}$ . Suppose we have successfully founded an invariant measure  $\lambda$  on  $(X, \mathcal{B})$  with the corresponding transition operator  $P$ . Let  $\sim$  be an equivalence relation on  $X$  and denote by  $\tilde{X} := X_{/\sim}$  the quotient space of  $X$  (whose equivalence classes belong to  $\mathcal{B}$ ) under this relation. We assume  $\tilde{X}$  is countable and holds for any elements  $C_i, C_j \in \tilde{X}$ ,

$$P(x, C_j) = P(y, C_j) \text{ for any } x, y \in C_i. \quad (*)$$

Then the kernel  $P$  induces a Markov transition  $\tilde{P}$  on  $\tilde{X}$  s.t.  $\tilde{P}(C_i, C_j) := P(x, C_j)$  with  $x \in C_i$ ; furthermore, the measure  $\tilde{\lambda}$  on  $\tilde{X}$  defined by  $\tilde{\lambda}(C_i) := \lambda(C_i)$  is  $\tilde{P}$ -invariant. Indeed, for any  $C' \in \tilde{X}$

$$\begin{aligned} \tilde{\lambda}\tilde{P}(C') &= \sum_C \tilde{\lambda}(C)\tilde{P}(C, C') \\ &= \sum_C \int_C P(x, C')\lambda(dx) \\ &= \int_X P(x, C')\lambda(dx) \\ &= \tilde{\lambda}(C'). \end{aligned}$$

Let us now explain how to apply this general principle to get the exact formula of the invariant measure for the oscillating random walk. Consider the following equivalence

relation

$$x \sim y \iff \exists n \in \mathbb{Z} \text{ s.t. } x, y \in [n, n+1[$$

which apparently satisfies the condition (\*). Taking  $X = \mathbb{R}$  and  $C_j := [j, j+1[$  for any  $j \in \mathbb{Z}$ , one admits  $\nu(m) := \lambda(C_m)$  (compare (2.4)) as the discrete invariant measure of  $\mathcal{X}^{(0)}$ . ■

**Remark 2.7.** *When the state 0 is supposed to be merged to the negative side ( $\alpha = 1$ ), we replace  $C_j$  by  $C_j^* = ]j-1, j]$  and the resulting invariant measure  $\nu^*$  is given by*

$$\nu^*(m) := \begin{cases} \mu' ]-\infty, m-1] & \text{if } m \leq 0 \\ \mu ]m, +\infty[ & \text{if } m \geq 1. \end{cases}$$

### 3 On the crossing sub-process of $\mathcal{X}^{(0)}$ when $S_\mu \subset \mathbb{Z}^+$ and $S_{\mu'} \subset \mathbb{Z}^-$

In this section, we would like to study the recurrence and the invariant measure of the embedding process of  $\mathcal{X}^{(0)}$ , which contains only states at the crossing times. For convenience, let us define the random variables  $(S_n)_{n \geq 0}$  and  $(S'_n)_{n \geq 0}$ , the simple random walks associated with laws  $\mu$  and  $\mu'$  respectively by  $S_0 = S'_0 = 0$  and for  $n \geq 1$ ,

$$S_n = \xi_1 + \xi_2 + \cdots + \xi_n,$$

and

$$S'_n = \xi'_1 + \xi'_2 + \cdots + \xi'_n.$$

Denote by  $\mu^{*n}$  the  $n$ -fold convolution of  $\mu$  with itself (also the distribution of  $S_n$ ) and  $\mathcal{U} = \sum_{n \geq 0} \mu^{*n}$  its potential kernel; similarly for  $\mu'^{*n}$  and  $\mathcal{U}'$ . Now we consider the sequence of crossing times  $\mathbf{C} = (C_k)_{k \geq 0}$  at which the process changes its sign whenever crossing 0. Assume that  $C_0 = 0$  and we designate  $C_k$  as the time of  $k^{\text{th}}$ - crossing given by

$$C_{k+1} := \inf \{ n > C_k : X_{C_k}^{(0)} + (\xi_{C_k+1} + \xi_{C_k+2} + \cdots + \xi_n) \geq 0 \text{ if } X_{C_k}^{(0)} \leq -1 \quad (3.1) \\ \text{or } X_{C_k}^{(0)} + (\xi'_{C_k+1} + \xi'_{C_k+2} + \cdots + \xi'_n) < 0 \text{ if } X_{C_k}^{(0)} \geq 0 \}$$

This forms a sequence of stopping times with respect to the filtration  $\mathbb{F} := (F_n)_{n \geq 0}$  where  $F_n := \sigma(\xi_k, \xi'_k \mid k \leq n)$ . By the law of large number, one gets  $S_n \rightarrow +\infty$  and  $S'_n \rightarrow -\infty$   $\mathbb{P}$ -almost surely and thus,  $\mathbb{P}_x[C_k < +\infty] = 1$  for all  $x \in \mathbb{Z}$ .

**Lemma 3.1.** *Assume  $S_\mu \subset \mathbb{Z}^+$  and  $S_{\mu'} \subset \mathbb{Z}^-$ . The sub-process  $(X_{C_k}^{(0)})_{k \geq 0}$  is a (time-*

homogeneous) Markov chain on  $\mathbb{Z}$  with its transition kernel determined by

$$C(x, y) = \begin{cases} \sum_{t=0}^{-x-1} \mu(y-x-t)\mathcal{U}(t) & \text{if } x < 0 \text{ and } y \geq 0, \\ \sum_{t=-x}^0 \mu'(y-x-t)\mathcal{U}'(t) & \text{if } x \geq 0 \text{ and } y < 0, \\ 0 & \text{otherwise.} \end{cases} \quad (3.2)$$

The process  $\mathcal{X}_{\mathbf{C}}^{(0)} := (X_{C_k}^{(0)})_{k \geq 0}$  is called the **crossing sub-process** of  $\mathcal{X}^{(0)}$ .

*Proof.* The Markov property is obvious.

If  $x < 0$  and  $y \geq 0$  (similar to  $x \geq 0, y < 0$ ) then we have

$$\begin{aligned} C(x, y) &= \mathbb{P}[X_{C_1} = y \mid X_0 = x] \\ &= \sum_{n=1}^{+\infty} \underbrace{\mathbb{P}[x + S_{n-1} \leq -1, x + S_n = y]}_{\text{since } (S_n)_{n \geq 1} \text{ is increasing}} \\ &= \sum_{n=1}^{+\infty} \sum_{t=0}^{-x-1} \mathbb{P}[S_{n-1} = t] \mathbb{P}[\xi_n = y - x - t] \\ &= \sum_{t=0}^{-x-1} \mathbb{P}[\xi_1 = y - x - t] \sum_{n=1}^{+\infty} \mathbb{P}[S_{n-1} = t] \\ &= \sum_{t=0}^{-x-1} \mu(y - x - t)\mathcal{U}(t). \end{aligned}$$

■

### 3.1 Irreducible classes of $\mathcal{X}_{\mathbf{C}}^{(0)}$

In case of reflected random walk, it is well-known in [13] that the full process and its process of reflections possess the common essential classes. Since the reflected random walk is regarded as the anti-symmetric case of our general model in which we identify the points themselves and their mirror images relative to 0, it comes naturally a question whether this phenomenon possibly occurs. There is no solid information to give an exact answer other than the intuitive relationship  $\mathcal{I}_{\mathbf{C}}(x) \subset \mathcal{I}(x)$ , where  $\mathcal{I}_{\mathbf{C}}(x)$  represents the irreducible class of  $x$  with respect to the crossing sub-process  $\mathcal{X}_{\mathbf{C}}^{(0)}$  starting at any given  $x \in \mathbb{Z}$ . Thus it is reasonable to attempt, using the below construction, to gain an understanding of the structure of  $\mathcal{I}_{\mathbf{C}}$ .

**Lemma 3.2.** *Assume  $S_\mu \subset \mathbb{Z}^+$  and  $S_{\mu'} \subset \mathbb{Z}^-$ , and, for any fixed  $0 \leq r < \delta$ , let us*

decompose  $\mathcal{I}(r)$  into  $\mathcal{I}^+(r) \cup \mathcal{I}^-(r)$  where  $\mathcal{I}^+(r) := \mathcal{I}(r) \cap \mathbb{Z}_0^+$  and  $\mathcal{I}^-(r) := \mathcal{I}(r) \cap \mathbb{Z}^-$ . Set

$$\mathcal{I}_{\mathbf{C}}^+(r) := \{y \in \mathcal{I}^+(r) : (y - S_{\mu}) \cap \mathcal{I}^-(r) \neq \emptyset\}, \quad (3.3)$$

and

$$\mathcal{I}_{\mathbf{C}}^-(r) := \{y \in \mathcal{I}^-(r) : (y - S_{\mu'}) \cap \mathcal{I}^+(r) \neq \emptyset\}. \quad (3.4)$$

Then  $\mathcal{I}_{\mathbf{C}}(r) = \mathcal{I}_{\mathbf{C}}^+(r) \cup \mathcal{I}_{\mathbf{C}}^-(r)$  is an essential class of the crossing sub-process  $\mathcal{X}_{\mathbf{C}}^{(0)}$ . Furthermore, all the  $X_{C_k}^{(0)}$  but finitely many belong to  $\mathcal{I}_{\mathbf{C}}(r_{x_0})$   $\mathbb{P}$ -a.s for any initial point  $x_0 \in \mathbb{Z}$ .

*Proof.* For any  $x, y \in \mathcal{I}_{\mathbf{C}}(r)$ , we write  $x \rightsquigarrow y$  to indicate that the crossing process  $\mathcal{X}^{(0)}$  starting at  $x$ , reaches  $y$  (with a positive probability) at certain crossing time  $C_k$ . Equivalently, there is such  $z \in \mathcal{I}(r)$  and  $n \in \mathbb{N}$  that  $x \xrightarrow{n} z \xrightarrow{1} y$ <sup>2</sup>, where  $y$  and  $z$  have the opposite signs.

The attractive property is immediate from the definition, so it remains to check that  $\mathcal{I}_{\mathbf{C}}(r)$  is an irreducible class for  $\mathcal{X}_{\mathbf{C}}^{(0)}$ , i.e.  $x \rightsquigarrow y$  for any given  $x, y \in \mathcal{I}_{\mathbf{C}}(r)$ . Without loss of generality, we suppose  $x, y \in \mathcal{I}_{\mathbf{C}}^+(r)$ . There exists some  $s \in S_{\mu}$  and  $n \geq 0$  s.t.  $y - s \in \mathcal{I}^-(r)$  and  $p^{(n)}(x, y - s) > 0$ . Then in a single step from  $y - s$ , the crossing process  $\mathcal{X}^{(0)}$  reaches  $y$  with the probability  $\mu(s) > 0$  at a crossing time as desired.  $\blacksquare$

**Theorem 3.3.** *Let  $(a_i)_{i \geq 1}$  and  $(b_j)_{j \geq 1}$  be strictly increasing sequences of positive integers. Set  $S_{\mu} = (a_i)_{i \geq 1}$  and  $S_{\mu'} = (-b_j)_{j \geq 1}$ . For any  $0 \leq r < \delta$ , the structure of  $\mathcal{I}_{\mathbf{C}}(r)$  and its complement will be completely revealed in view of the following conditions:*

(i).  $D = +\infty \implies \mathcal{I}_{\mathbf{C}}^-(r) = \mathcal{I}^-(r)$ .

(ii).  $D' = -\infty \implies \mathcal{I}_{\mathbf{C}}^+(r) = \mathcal{I}^+(r)$ .

(iii). If  $-D' < +\infty$  and  $D = +\infty$  then

$$\mathcal{I}_{\mathbf{C}}^+(r) = \mathcal{I}^+(r) \iff \sup_{k \geq 1} \{a_k - a_{k-1}\} \leq -D' \text{ with } a_0 = 0.$$

(iv). If  $D' = -\infty$  and  $D < +\infty$  then

$$\mathcal{I}_{\mathbf{C}}^-(r) = \mathcal{I}^-(r) \iff \sup_{\ell \geq 1} \{b_{\ell} - b_{\ell-1}\} \leq D \text{ with } b_0 = 0.$$

(v). If  $D < +\infty$  and  $-D' < +\infty$  then

$$\mathcal{I}_{\mathbf{C}}^+(r) = \mathcal{I}^+(r) \iff \max_{1 \leq k \leq m} \{a_k - a_{k-1}\} \leq -D',$$

---

<sup>2</sup>The notation means that  $p^{(n)}(x, z) > 0$  and  $p(z, y) > 0$ .

and

$$\mathcal{I}_{\mathbb{C}}^-(r) = \mathcal{I}^-(r) \iff \max_{1 \leq \ell \leq n} \{b_\ell - b_{\ell-1}\} \leq D,$$

where  $a_0 = b_0 = 0$ ;  $D = a_m$  and  $D' = -b_n$  for some  $m, n \geq 1$ .

(vi). Set  $I^+ := \{k \geq 1 \mid a_k - a_{k-1} > -D'\}$  and  $I^- := \{\ell \geq 1 \mid b_\ell - b_{\ell-1} > D\}$  in case (v) is violated. For every choice of  $k \in I^+$  and  $\ell \in I^-$ , we define

$$\mathcal{N}_k^+(r) := \left\{ a_{k-1} + r + \delta s : 0 \leq s \leq \frac{a_k - a_{k-1} + D'}{\delta} - 1 \right\},$$

and

$$\mathcal{N}_\ell^-(r) := \left\{ -b_{\ell-1} + r + \delta s : \frac{b_{\ell-1} - b_\ell + D}{\delta} \leq s \leq -1 \right\}.$$

Then  $\mathcal{N}(r) := \bigcup_{(k,\ell) \in I^+ \times I^-} \mathcal{N}_k^+(r) \cup \mathcal{N}_\ell^-(r)$  is the set of non-crossing points.

*Proof.* (i)-(ii). By definition.

(iii)-(iv). Note that (3.3) and (3.4) can be rewritten as

$$\mathcal{I}_{\mathbb{C}}^+(r) = \mathcal{I}^+(r) \cap \bigcup_{k \geq 1} A_k \text{ with } A_k := \{a_k + D' + r, \dots, a_k + r - \delta\}$$

and

$$\mathcal{I}_{\mathbb{C}}^-(r) = \mathcal{I}^-(r) \cap \bigcup_{\ell \geq 1} B_\ell \text{ with } B_\ell := \{-b_\ell + r, \dots, -b_\ell + D + r - \delta\}.$$

To cover  $\mathcal{I}^+(r)$  by countably many same length sub-intervals, it requires  $r \in A_1$  and further, no point in the form  $r + \delta\mathbb{Z}$  stays inside the gap between  $A_k$  and  $A_{k+1}$  since  $\{a_k + D' + r\}_{k \geq 1}$  is a strictly increasing sequence. More precisely,

$$\mathcal{I}_{\mathbb{C}}^+(r) = \mathcal{I}^+(r) \iff \begin{cases} a_1 + D' + r \leq r \\ a_{k+1} + D' + r \leq a_k + r, \forall k \geq 1 \end{cases} \iff \begin{cases} a_1 \leq -D' \\ a_{k+1} - a_k \leq -D', \forall k \geq 1 \end{cases}$$

Identically, we also infer

$$\mathcal{I}_{\mathbb{C}}^-(r) = \mathcal{I}^-(r) \iff \begin{cases} -b_1 + D + r - \delta \geq r - \delta \\ -b_\ell + r \leq -b_{\ell+1} + D + r, \forall \ell \geq 1 \end{cases} \iff \begin{cases} b_1 \leq D \\ b_{\ell+1} - b_\ell \leq D, \forall \ell \geq 1. \end{cases}$$

(v). This is a direct consequence of (iii) and (iv).

(vi). A straightforward argument yields

$$(\mathcal{N}_k^+(r) - S_\mu) \cap \mathcal{I}^-(r) = (\mathcal{N}_\ell^-(r) - S_{\mu'}) \cap \mathcal{I}^+(r) = \emptyset$$

for every pair  $(k, \ell) \in I^+ \times I^-$  and it is enough to end the proof. ■

**Example 3.4.** We simply deal with the case when  $D, -D' < +\infty$  by, for instance, taking  $S_\mu := \{2, 4, 10\}$  and  $S'_\mu := \{-4, -1\}$ . Obviously,  $D = 10, D' = -4, d = 2, d' = 1$  and the chain  $\mathcal{X}^{(0)}$  has the unique essential class  $\mathcal{I}(0) := \{-4, \dots, 9\}$ . An easy verification on (v) would lead to  $\mathcal{I}_\mathbb{C}^-(0) = \mathcal{I}^-(0)$  while  $\mathcal{I}_\mathbb{C}^+(0) \subsetneq \mathcal{I}^+(0)$  and its complement  $\mathcal{N}_3^+ = \{4, 5\}$  according to (vi). Hence, equality is achieved only by replacing  $S_\mu$  by  $\{2, 6, 10\}$ .

**Remark 3.5.** We do emphasize that it fails to let  $\mathcal{I}_\mathbb{C}^+(r) \subsetneq \mathcal{I}^+(r)$  and  $\mathcal{I}_\mathbb{C}^-(r) \subsetneq \mathcal{I}^-(r)$  simultaneously occur due to (v). In other words,  $\mathcal{I}_\mathbb{C}(r)$  always has at least one side which coincides with  $\mathcal{I}(r)$ . The case when  $S_\mu \subset \mathbb{Z}$  and  $S_{\mu'} \subset \mathbb{Z}$  is much complicated since the behaviour of the chain now is significantly affected by many factors. However, in connection with Theorem 2.2 and the above theorem, one may derive some properties of  $\mathcal{I}_\mathbb{C}$ , for instance,  $\mathcal{I}_\mathbb{C}(r) = \mathcal{I}(r)$  if and only if  $D = -D' = +\infty$ .

### 3.2 Invariant measure for $\mathcal{X}_\mathbb{C}^{(0)}$

Use of the explicit formula (2.4) of  $\nu$  enables us to derive the invariant measure of the sub-process  $\mathcal{X}_\mathbb{C}^{(0)}$ . In particular

**Theorem 3.6.** Assume  $S_\mu \subset \mathbb{Z}^+$  and  $S_{\mu'} \subset \mathbb{Z}^-$ . Let  $\rho$  be the measure on  $\mathbb{Z}$  defined by

$$\rho(n) := \begin{cases} \sum_{k=1}^{+\infty} \mu(k) \mu'[n-k+1, n] & \text{if } n \leq -1, \\ \sum_{k=1}^{+\infty} \mu'(-k) \mu[n+1, n+k] & \text{if } n \geq 0. \end{cases} \quad (3.5)$$

Then, for any  $x_0 \in \mathbb{Z}$ , the restriction  $\rho_{x_0}$  of  $\rho$  to  $\mathcal{I}_\mathbb{C}(r_{x_0})$  is an invariant measure for  $\mathcal{X}_\mathbb{C}^{(0)}$ .

*Proof.* Consider the signed measures  $\mathcal{A}$  and  $\mathcal{A}'$  defined by  $\mathcal{A}(m) = \delta_0(m) - \mu(m)$  if  $m \geq 0$  and  $\mathcal{A}'(m) = \delta_0(m) - \mu'(m)$  if  $m \leq 0$ . It is easily seen that

$$\mathcal{A} * \mathcal{U} = \mathcal{U} * \mathcal{A} = \delta_0 \text{ and } \mathcal{A}' * \mathcal{U}' = \mathcal{U}' * \mathcal{A}' = \delta_0. \quad (3.6)$$

In addition, we also have

$$\rho(n) := \begin{cases} \sum_{k=0}^{+\infty} \mathcal{A}(k) \nu(n-k) & \text{if } n \leq -1, \\ \sum_{k=0}^{+\infty} \mathcal{A}'(-k) \nu(n+k) & \text{if } n \geq 0. \end{cases} \quad (3.7)$$



Indeed, if  $n \geq 0$  then we get

$$\begin{aligned}
\sum_{k=0}^{+\infty} \mathcal{A}'(-k) \nu(n+k) &= \sum_{k=0}^{+\infty} (\delta_0(-k) - \mu'(-k)) \nu(n+k) \\
&= (1 - \mu'(0)) \nu(n) - \sum_{k=1}^{+\infty} \mu'(-k) \nu(n+k) \\
&= \sum_{k=1}^{+\infty} \mu'(-k) (\mu[n+1, +\infty[-\mu[n+k+1, +\infty[) \\
&= \sum_{k=1}^{+\infty} \mu'(-k) \mu[n+1, n+k].
\end{aligned}$$

Conversely,  $\nu$  can be represented in terms of  $\rho$  and  $\mathcal{U}'$  directly from (3.6) and (3.7), that is, for  $n \geq 0$

$$\begin{aligned}
\sum_{k=0}^{+\infty} \mathcal{U}'(-k) \rho(n+k) &= \sum_{k=0}^{+\infty} \mathcal{U}'(-k) \sum_{\ell=0}^{+\infty} \mathcal{A}'(-\ell) \nu(n+k+\ell) \\
&= \sum_{k=0}^{+\infty} \mathcal{U}'(-k) \sum_{s=k}^{+\infty} \mathcal{A}'(k-s) \nu(n+s) \\
&= \sum_{s=0}^{+\infty} \nu(n+s) \underbrace{\sum_{k=0}^s \mathcal{U}'(-k) \mathcal{A}'(-s+k)}_{\delta_0(-s)} \\
&= \nu(n)
\end{aligned}$$

and the same property holds for  $n \leq -1$ . Briefly, one may write

$$\nu(n) := \mathbb{1}_{]-\infty, 0[}(n) \left( \sum_{k=0}^{+\infty} \mathcal{U}(k) \rho(n-k) \right) + \mathbb{1}_{[0, +\infty[}(n) \left( \sum_{k=0}^{+\infty} \mathcal{U}'(-k) \rho(n+k) \right). \quad (3.8)$$

We claim that

$$\nu_{x_0}(n) := \mathbb{E}_{\rho_{x_0}} \left( \sum_{j=0}^{C_1-1} \mathbb{1}_n(X_j^{(0)}) \right), \quad \text{if } n \in \mathcal{I}(r_{x_0}) \quad (3.9)$$

where  $\mathbb{E}_{\rho_{x_0}}(\cdot) = \sum_{w \in \mathcal{I}_{\mathbf{C}}(r_0)} \rho_{x_0}(w) \mathbb{E}_w(\cdot)$  indicates the expectation governed by  $\rho_{x_0}$ .

If  $n \in \mathcal{I}^-(r_{x_0})$  then the crossing process can reach  $n$  before the first crossing time if and only if  $X_0^{(0)} = \omega \in \mathcal{I}_{\mathbf{C}}^-(r_{x_0})$  and  $\omega \leq n$ . In other words, there is  $k \in \mathbb{Z}_0^+$  and  $i \geq 1$  s.t.

$S_i = k$  and  $\omega = n - k$ . Since  $\sum_{j=0}^{C_1-1} \mathbb{1}_n(X_j^{(0)}) = \sum_{j=0}^{+\infty} \mathbb{1}_n(X_j^{(0)}) \mathbb{1}_{\{C_1 > j\}}$ ,

$$\begin{aligned}
\mathbb{E}_{\rho_{x_0}} \left( \sum_{j=0}^{C_1-1} \mathbb{1}_n(X_j^{(0)}) \right) &= \sum_{k=0}^{+\infty} \rho_{x_0}(n-k) \mathbb{E}_{n-k} \left( \sum_{j=0}^{+\infty} \mathbb{1}_n(X_j^{(0)}) \mathbb{1}_{\{C_1 > j\}} \right) \\
&= \sum_{k=0}^{+\infty} \rho_{x_0}(n-k) \sum_{j=0}^{+\infty} \mathbb{P}_{n-k}[X_j^{(0)} = n, C_1 > j] \\
&= \sum_{k=0}^{+\infty} \rho_{x_0}(n-k) \sum_{j=0}^{+\infty} \mathbb{P}[S_j = k] \\
&= \sum_{k=0}^{+\infty} \rho_{x_0}(n-k) \mathcal{U}(k) \\
&= \nu_{x_0}(n).
\end{aligned}$$

Hence, (3.9) is true for every  $n \in \mathcal{I}(r_{x_0})$  and yields that

$$\sum_{m \in \mathcal{I}(r_{x_0})} \nu_{x_0}(m) p(m, n) = \mathbb{E}_{\rho_{x_0}} \left( \sum_{j=1}^{C_1} \mathbb{1}_n(X_j^{(0)}) \right).$$

Since  $\nu_{x_0}$  is invariant on  $\mathcal{I}(r_{x_0})$ , i.e.  $\nu_{x_0} = \nu_{x_0} P$ , we again apply (3.9) and simplify as

$$\mathbb{E}_{\rho_{x_0}} \left( \mathbb{1}_n(X_0^{(0)}) \right) = \mathbb{E}_{\rho_{x_0}} \left( \mathbb{1}_n(X_{C_1}^{(0)}) \right).$$

The left hand side is  $\rho_{x_0}(n)$  and the right hand side is the sum  $\sum_{m \in \mathcal{I}_{\mathbf{C}}(r_{x_0})} \rho_{x_0}(m) C(m, n)$ ,

which prove that  $\rho_{x_0}$  is an invariant measure for  $\mathcal{X}_{\mathbf{C}}^{(0)}$  on  $\mathcal{I}_{\mathbf{C}}(r_{x_0})$ . ■

## 4 Criteria for the recurrence of $\mathcal{X}^{(\alpha)}$ , $0 \leq \alpha \leq 1$

**From now on, we assume  $d = d' = 1$  and keep in mind that the recurrence of  $\mathcal{X}^{(\alpha)}$  always means the recurrence of the state 0 since  $\mathcal{X}^{(\alpha)}$  has an unique irreducible class in this case.**

## 4.1 Classical approach

In this subsection, we consider the first passage of  $(S_n)_{n \geq 0}$  to the subset  $]0, +\infty[$  and of  $(S'_n)_{n \geq 0}$  to the subset  $] - \infty, 0[$ , namely

$$\ell_+ := \inf\{k > 0 : S_k > 0\} \text{ and } \ell'_- := \inf\{k > 0 : S'_k < 0\}$$

(with the convention  $\inf \emptyset = +\infty$ ). The random variables  $\ell_+$  and  $\ell'_-$  are stopping times with respect to the canonical filtration  $(\sigma(\xi_k, \xi'_k, 1 \leq k \leq n))_{n \geq 1}$ . In the sequel, we only consider cases when these random variables are  $P$ -as finite i.e. equivalently when  $\mathbb{P}[\limsup S_n = +\infty] = \mathbb{P}[\liminf S'_n = -\infty] = 1$ ; hence, the random variables  $S_{\ell_+}$  and  $S'_{\ell'_-}$  are well defined in these cases and **we denote by  $\mu_+$  and  $\mu'_-$  their respective distributions**

Let us define also, for  $h \geq 1$ , the renewal functions associated with the ladder heights  $S_{\ell_+}$  and  $S_{\ell'_-}$ , respectively by

$$C(h) := \sum_{n=1}^{+\infty} \mathbb{P}[S_n = h, \min_{1 \leq i \leq n} S_i > 0],$$

$$C'(-h) := \sum_{n=1}^{+\infty} \mathbb{P}[S'_n = -h, \min_{1 \leq j \leq n} S'_j > 0].$$

We now get a glimpse of the following well-known criterion of recurrence of  $\mathcal{X}^{(\alpha)}$

**Theorem 4.1.** (Kemperman) *The general oscillating random walk  $\mathcal{X}^{(\alpha)}$  is recurrent if and only if*

$$\sum_{h=1}^{+\infty} C(h) C'(-h) = +\infty. \quad (4.1)$$

However, it is quite theoretical and difficult to check in several cases. Next, we will take into consideration an equivalent condition to (4.1), which has the additional advantage of being easily computable.

**Theorem 4.2.** (Rogozin-Foss) *If for some  $\epsilon > 0$*

$$\int_{-\epsilon}^{\epsilon} \frac{1}{|1 - \mathbb{E}[e^{itS_{\ell_+}}]| |1 - \mathbb{E}[e^{itS'_{\ell'_-}}]|} dt < +\infty, \quad (4.2)$$

*then the original crossing process is transient.*

*If, in addition,  $\operatorname{Re}((1 - \mathbb{E}[e^{itS_{\ell_+}}])(1 - \mathbb{E}[e^{itS'_{\ell'_-}}])) \geq 0$  for  $|t| < \epsilon$  for some  $\epsilon > 0$  and the*

below condition holds

$$\int_{-\epsilon}^{\epsilon} \operatorname{Re} \left( \frac{1}{(1 - \mathbb{E}[e^{itS_{\ell_+}}]) (1 - \mathbb{E}[e^{itS'_{\ell'_-}])} \right) dt = +\infty, \quad (4.3)$$

then  $\mathcal{X}^{(\alpha)}$  is recurrent.

We also refer to the recent paper [2], Proposition 4.4 for a luminous proof.

In the simple case when  $\mu = \mu'$ , in other words when  $\mathcal{X}^{(\alpha)}$  is an homogeneous classical random walk on  $\mathbb{Z}$ , these conditions turn into the above conditions turn into

$$\int_{\epsilon}^{\epsilon} |1 - \hat{\mu}(t)|^{-1} dt < +\infty, \quad (4.4)$$

and

$$\int_{\epsilon}^{\epsilon} \operatorname{Re} \left( \frac{1}{1 - \hat{\mu}(t)} \right) dt = +\infty, \quad (4.5)$$

In fact, (4.5) is a necessary and sufficient condition for  $(S_n)_{n \geq 0}$  be recurrent; see KESTEN, SPITZER [9], [14] and [2] for proofs and comments. In [2], Proposition 2.2, the reader will find an explicit and simple relation between the integral of the function  $\operatorname{Re} \left( \frac{1}{1 - \hat{\mu}(t)} \right)$  and the Green function of the random walk  $(S_n)_n$  which enlightens the above statement.

In the next subsection we develop another approach to identify quite general conditions which ensure that  $\mathcal{X}^{(\alpha)}$  is recurrent. We first consider the case when  $S_{\mu} \subset \mathbb{Z}^+$  and  $S_{\mu'} \subset \mathbb{Z}^-$  then the general case, replacing the couple  $(\mu, \mu')$  by  $(\mu_+, \mu'_-)$ .

## 4.2 Tail condition criterion for the recurrence of $\mathcal{X}^{(0)}$ when $S_{\mu} \subset \mathbb{Z}^+$ and $S_{\mu'} \subset \mathbb{Z}^-$

An easy observation gives that the crossing sub-process  $\mathcal{X}_{\mathbf{C}}^{(0)}$  is positive recurrent on  $\mathcal{I}_{\mathbf{C}}(0)$  (equivalently,  $\rho(\mathcal{I}_{\mathbf{C}}(0)) < +\infty$ ) when  $D$  and  $D'$  are both finite. Thus, it is reasonable to study the recurrence of  $\mathcal{X}_{\mathbf{C}}^{(0)}$  in the non-trivial cases.

**Proposition 4.3.** *Assume that  $S_{\mu} \subset \mathbb{Z}^+$  and  $S_{\mu'} \subset \mathbb{Z}^-$ . Then the total mass of  $\rho$  on  $\mathcal{I}_{\mathbf{C}}(0)$  is finite if and only if*

$$\sum_{n=0}^{+\infty} H(n) H'(-n) < +\infty, \quad (4.6)$$

where  $H(n) = \mu]n, +\infty[$  and  $H'(-n) = \mu'] - \infty, -n[$  respectively stands for the tail distributions of  $\mu$  and  $\mu'$ .

In this case, the Markov chain  $\mathcal{X}_{\mathbf{C}}^{(0)}$  is positive recurrent and  $\mathcal{X}^{(0)}(\mu, \mu')$  is recurrent on its essential class.

*Proof.* We compute  $\rho(\mathbb{Z}^+)$  by substituting the formula (3.5)

$$\begin{aligned}
\sum_{n=0}^{+\infty} \rho(n) &= \sum_{n=0}^{+\infty} \sum_{k=1}^{+\infty} \mu'(-k) \mu[n+1, n+k] \\
&= \sum_{k=1}^{+\infty} \mu'(-k) \sum_{n=0}^{+\infty} [H(n) - H(n+k)] \\
&= \sum_{k=1}^{+\infty} \mu'(-k) \left[ \sum_{n=0}^{k-1} H(n) - \lim_{N \rightarrow +\infty} (H(N+1) + \dots + H(N+k)) \right] \\
&= \sum_{n=0}^{+\infty} H(n) \sum_{k=n+1}^{+\infty} \mu'(-k) \\
&= \sum_{n=0}^{+\infty} H(n) H'(-n).
\end{aligned}$$

One also obtains  $\sum_{n=-\infty}^{-1} \rho(n) = \sum_{n=0}^{+\infty} H(n) H'(-n)$  which immediately implies (4.6).  $\blacksquare$

Apparently, (4.6) holds providing that the first moment of either  $\xi_n$  or  $-\xi'_n$  is finite. This assumption can be sharpened by constraining finite Hölder moments as below

**Corollary 4.4.** *Assume that  $S_\mu \subset \mathbb{Z}^+$ ,  $S_{\mu'} \subset \mathbb{Z}^-$  and  $\mathbb{E}[\xi_1^p], \mathbb{E}[(-\xi'_1)^q] < +\infty$  with  $p, q \in ]0, 1[$  satisfying  $p + q = 1$ . Then (4.6) holds and the Markov chain  $\mathcal{X}^{(0)}$  is recurrent on its unique essential class (and positive recurrent when  $\mathbb{E}[\xi_1], \mathbb{E}[-\xi'_1] < +\infty$  by Corollary 2.5).*

*Proof.* The formula  $\mathbb{E}[X^k] := k \int_0^{+\infty} t^{k-1} \mathbb{P}[X \geq t] dt$  yields

$$k \sum_{n=0}^{+\infty} (n+1)^{k-1} \mathbb{P}[X \geq n+1] \leq \mathbb{E}[X^k] \quad (4.7)$$

so that, by the Markov's inequality for  $H^p(n)$  and  $H^q(n)$ , we get

$$\begin{aligned}
\sum_{n=0}^{+\infty} H(n) H'(-n) &= \sum_{n=0}^{+\infty} H^q(n) H'^p(-n) [H^p(n) H'^q(-n)] \\
&\leq \mathbb{E}[\xi_1^p]^p \mathbb{E}[(-\xi'_1)^q]^q \sum_{n=0}^{+\infty} \left[ (n+1)^{-q^2} H^q(n) (n+1)^{-p^2} H'^p(-n) \right].
\end{aligned}$$

The product inside the bracket can be transformed into sum by using the *Young's inequality* and then together with (4.7), it yields

$$\begin{aligned} \sum_{n=0}^{+\infty} H(n) H'(-n) &\leq \mathbb{E}[\xi_1^p]^p \mathbb{E}[(-\xi_1')^q]^q \left( q \sum_{n=0}^{+\infty} (n+1)^{-q} H(n) + p \sum_{n=0}^{+\infty} (n+1)^{-p} H'(-n) \right) \\ &\leq \mathbb{E}[\xi_1^p]^p \mathbb{E}[(-\xi_1')^q]^q \left( \frac{q}{p} \mathbb{E}[\xi_1^p] + \frac{p}{q} \mathbb{E}[(-\xi_1')^q] \right) < +\infty. \end{aligned}$$

■

**Remark 4.5.** *The condition  $\mathbb{E}[\xi_1^p], \mathbb{E}[(-\xi_1')^q] < +\infty$  with  $p+q = 1$  is a sufficient condition for the recurrence of  $\mathcal{X}^{(0)}$ . Notice that it is not far to be sharp. We refer to Proposition 5.12 in [13] for an example in the case of the reflected random walk on  $\mathbb{N}$ , which corresponds to the antisymmetric case, i.e.  $S = -S'$  (with  $p = q = 1/2$  there). The reader can find other examples in [15] Theorem 2 in the case when  $S$  and  $S'$  are stable random walks on  $\mathbb{Z}$  but  $S \neq -S'$ .*

### 4.3 Recurrence of $\mathcal{X}^{(0)}$ in the general case

To treat this model, let us first introduce the basic decomposition of  $\xi_n$  and  $\xi'_n$ , namely

$$\xi_n = \xi_n^+ - \xi_n^- \quad \text{and} \quad \xi'_n = \xi_n'^+ - \xi_n'^-,$$

where  $\xi_n^\pm = \max\{\pm\xi_n, 0\}$  and  $\xi_n'^\pm = \max\{\pm\xi'_n, 0\}$ .

Consider the following assumptions

- (H)  $(\mathbb{E}[\xi_1^-] < \mathbb{E}[\xi_1^+] \leq +\infty)$  or  $(\mathbb{E}[\xi_1^-] = \mathbb{E}[\xi_1^+] < +\infty)$ ;
- (H')  $(\mathbb{E}[(\xi_1')^+] < \mathbb{E}[(\xi_1')^-] \leq +\infty)$  or  $(\mathbb{E}[(\xi_1')^-] = \mathbb{E}[(\xi_1')^+] < +\infty)$ .

**Lemma 4.6.** *If H (resp. H') is satisfied then  $\limsup S_n = +\infty$  (resp.  $\liminf S'_n = -\infty$ ) almost surely.*

Hence, when both H and H' hold, the random variables  $\ell_+$  and  $\ell'_-$  are  $\mathbb{P}$ -a.s. finite; more generally, there are infinitely many  $\mathbb{P}$ -a.s. finite crossing times in this case. Let us introduce the ladder times  $(t_k)_{k \geq 0}$  defined recursively by:  $t_0 = 0$  and, for  $k \geq 1$ ,

$$t_{k+1} := \begin{cases} \inf\{n > t_k \mid \xi_{t_k+1} + \dots + \xi_n > 0\} & \text{if } X_{t_k}^{(0)} \leq -1, \\ \inf\{n > t_k \mid \xi'_{t_k+1} + \dots + \xi'_n < 0\} & \text{if } X_{t_k}^{(0)} \geq 0. \end{cases} \quad (4.8)$$

Notice that, in the first line, the random variable  $t_{k+1}$  is an ascending ladder epoch of  $S$ , while, in the second line, it is a descending ladder epoch of  $S'$ . These random times are

$\mathbb{P}$ -a.s. finite and the increments  $(t_{k+1} - t_k)_{k \geq 0}$  form a sequence of independent random variables with laws

$$\mathcal{L}(t_{k+1} - t_k \mid X_{t_k}^{(0)} < 0) = \mathcal{L}(t_1 \mid X_0^{(0)} = x) \quad \text{and} \quad \mathcal{L}(t_{k+1} - t_k \mid X_{t_k}^{(0)} \geq 0) = \mathcal{L}(t_1 \mid X_0^{(0)} = y)$$

for any  $x < 0 \leq y$ . Similarly,

$$\mathcal{L}(S_{t_{k+1}} - S_{t_k} \mid X_{t_k}^{(0)} < 0) = \mathcal{L}(S_{t_1} \mid X_0^{(0)} = x) = \mu_+,$$

and

$$\mathcal{L}(S'_{t_{k+1}} - S'_{t_k} \mid X_{t_k}^{(0)} \geq 0) = \mathcal{L}(S'_{t_1} \mid X_0^{(0)} = y) = \mu'_-.$$

It is perhaps worth remarking that, by setting  $Y_k := S_{t_k} - S_{t_{k-1}}$  when  $X_{t_k}^{(0)} < 0$  and  $Y'_k := S'_{t_k} - S'_{t_{k-1}}$  when  $X_{t_k}^{(0)} \geq 0$ , the sub-process  $(X_{t_k}^{(0)})_{k \geq 0}$  turns out to be an crossing process associated with the distributions  $\mu_+$  and  $\mu'_-$  of  $Y_k$  and  $Y'_k$  respectively. In other words, the process  $(X_{t_k}^{(0)})_{k \geq 0}$  has the same distribution as  $\mathcal{X}^{(0)}(\mu_+, \mu'_-)$ .

**Lemma 4.7.** *Assume that both  $\mathbf{H}$  and  $\mathbf{H}'$  hold. Then, the oscillating process  $\mathcal{X}^{(0)}(\mu, \mu')$  is recurrent if and only if the oscillating process  $\mathcal{X}^{(0)}(\mu_+, \mu'_-)$  is recurrent.*

A proof of this statement for the process  $\mathcal{X}^{(1)}$  appears in the recent paper [2] of J. BREMONT, lemma 4.2 (ii). For the sake of completeness, we detail the argument below, introducing the first return time at 0 of  $\mathcal{X}^{(\alpha)}$ ,  $0 \leq \alpha \leq 1$ , which will be useful latter on.

*Proof.* For any  $0 \leq \alpha \leq 1$ , let  $\tau^{(\alpha)}$  be the first return time at 0 of  $\mathcal{X}^{(\alpha)}$  given by

$$\tau^{(\alpha)} := \inf\{n \geq 1 : X_n^{(\alpha)} = 0\}.$$

In the present proof, we only consider the case when  $\alpha = 0$ .

Starting at 0, we know that  $\tau^{(0)} < +\infty$  almost surely and since  $X_{\tau^{(0)}}^{(0)} = 0$ , there are only two possibilities: if  $X_{\tau^{(0)}-1}^{(0)} \geq 1$  then  $\tau^{(0)}$  must be a ladder time while the case  $X_{\tau^{(0)}-1}^{(0)} \leq -1$  will imply the existence of some  $k \geq 1$  s.t.  $\tau^{(0)} = C_k$ , the  $k^{\text{th}}$ -crossing time of  $\mathcal{X}^{(0)}$  and of course that  $\tau^{(0)}$  is also a ladder time. This means  $(X_{t_k}^{(0)})_{k \geq 0}$  is recurrent and so does  $\mathcal{X}^{(0)}(\mu_+, \mu'_-)$ . The converse is obvious.  $\blacksquare$

It is easily seen that  $\mathcal{X}^{(0)}$  and  $(X_{t_k}^{(0)})_{k \geq 0}$  admit a common crossing sub-process since there is at most a crossing moment between two consecutive ladder times  $t_k$  and  $t_{k+1}$  happening when  $X_{t_k}^{(0)} < 0$  and  $X_{t_{k+1}}^{(0)} \geq 0$  or vice versa. Therefore, we can take advantage of Corollary 4.4 (applied to the process  $\mathcal{X}^{(0)}(\mu_+, \mu'_-)$ ) to deduce the recurrence of  $(X_{t_k}^{(0)})_{k \geq 0}$  and finally that of  $\mathcal{X}^{(0)}$  by Lemma 4.7.

**Theorem 4.8.** *Let  $p, q \in ]0, 1[$  s.t.  $p + q = 1$ . Then each of the following is sufficient for the oscillating process  $\mathcal{X}^{(0)}(\mu, \mu')$  to be recurrent on its essential class*

- (a)  $(\mathbb{E}[\xi_1^-] < \mathbb{E}[\xi_1^+], \mathbb{E}[(\xi_1^+)^p] < +\infty)$  and  $(\mathbb{E}[(\xi_1')^+] < \mathbb{E}[(\xi_1')^-], \mathbb{E}[(\xi_1'^-)^q] < +\infty)$ ;
- (b)  $(\mathbb{E}[\xi_1^-] = \mathbb{E}[\xi_1^+], \mathbb{E}[(\xi_1^+)^{1+p}] < +\infty)$  and  $(\mathbb{E}[(\xi_1')^+] = \mathbb{E}[(\xi_1')^-], \mathbb{E}[(\xi_1'^-)^{1+q}] < +\infty)$ ;
- (c)  $(\mathbb{E}[\xi_1^-] < \mathbb{E}[\xi_1^+], \mathbb{E}[(\xi_1^+)^p] < +\infty)$  and  $(\mathbb{E}[(\xi_1')^+] = \mathbb{E}[(\xi_1')^-], \mathbb{E}[(\xi_1'^-)^{1+q}] < +\infty)$ .

*The similar condition holds when swapping the roles of  $\xi_1$  and  $\xi_1'$ .*

*Proof.* As mentioned above, it remains to check that  $\mathbb{E}[(Y_n)^p] < +\infty$  and  $\mathbb{E}[(-Y_n')^q] < +\infty$ . The set of conditions (a) means that the chain moves with positive drift on the left and negative drift on the right while (b) represents the center case which was already done by CHOW and LAI (see [3]). The others are partly mixed from both of (a) and (b), so we will leave the proof only for the first case.

Notice that  $0 < p < 1$  and by the Wald's identity, we obtain

$$\begin{aligned} \mathbb{E}[(\xi_1 + \dots + \xi_{t_1})^p \mid X_0^{(0)} = x < 0] &\leq \mathbb{E}[(\xi_1^+ + \dots + \xi_{t_1}^+)^p \mid X_0^{(0)} = x < 0] \\ &\leq \mathbb{E}[(\xi_1^+)^p + \dots + (\xi_{t_1}^+)^p \mid X_0^{(0)} = x < 0] \\ &= \mathbb{E}[(\xi_1^+)^p] \mathbb{E}[t_1 \mid X_0^{(0)} = x < 0] < +\infty \end{aligned}$$

due to  $\mathbb{E}[t_1 \mid X_0^{(0)} = x < 0] < +\infty$ . Indeed, if  $\mathbb{E}[\xi_1^+] < +\infty$  then  $\mathbb{E}[\xi_1] > 0$  and  $\mathbb{E}[|\xi_1|] < +\infty$  and the FELLER's result tells us that  $\mathbb{E}[t_1 \mid X_0^{(0)} = x < 0] < +\infty$  (see [5]). On the other hands, if  $\mathbb{E}[\xi_1^+] = +\infty$  then there is  $L > 0$  s.t.  $\xi_n^{(L)} = \min\{\xi_n, L\}$  (which has finite first moment) satisfies  $\mathbb{E}[\xi_n^{(L)}] = \mathbb{E}[\xi_1^{(L)}] > 0$ . The first ascending ladder time  $t_1^{(L)}$  associated with  $S_n^{(M)} = \xi_1^{(L)} + \dots + \xi_n^{(L)}$  has finite expectation by what we just said. Therefore,  $t_1$  is integrable since  $t_1 \leq t_1^{(L)}$ . The argument showing  $\mathbb{E}[(-\xi_1' - \dots - \xi_{t_1}')^q \mid X_0^{(0)} = x \geq 0] < +\infty$  goes exactly the same line. ■

**Remark 4.9.** *As a direct consequence of the above proof and Corollary 2.5, we deduce that when  $\mathbb{E}[\xi_1^-] < \mathbb{E}[\xi_1^+] < +\infty$  and  $\mathbb{E}[(\xi_1')^+] < \mathbb{E}[(\xi_1')^-] < +\infty$  then the process  $\mathcal{X}^{(0)}$  is positive recurrent. Indeed, in this case,  $\mathbb{E}[\tau^{(0)}] < +\infty$  and the result follows by a classical theorem of induced processes.*

#### 4.4 Recurrence of $\mathcal{X}^{(\alpha)}$ with $0 \leq \alpha \leq 1$

We end this section by proving our main result

**Corollary 4.10.** *If at least one of the assumptions of Theorem 4.8 is satisfied, then the general oscillating process  $\mathcal{X}^{(\alpha)}(\mu, \mu')$  is recurrent.*



*Proof.* It is clear that  $\mathcal{X}^{(0)}$  and  $\mathcal{X}^{(1)}$  (suitably modified) are recurrent. Now, we have

$$\begin{aligned}
\mathbb{P}_0[\tau^{(\alpha)} = n] &= \mathbb{P}_0[X_1^{(\alpha)} \neq 0, X_2^{(\alpha)} \neq 0, \dots, X_{n-1}^{(\alpha)} \neq 0, X_n^{(\alpha)} = 0] \\
&= \sum_{k=1}^{+\infty} \mathbb{P}_0[X_1^{(\alpha)} = k] \mathbb{P}_k[X_1^{(\alpha)} \neq 0, X_2^{(\alpha)} \neq 0, \dots, X_{n-1}^{(\alpha)} \neq 0, X_n^{(\alpha)} = 0] \\
&+ \sum_{k=-\infty}^{-1} \mathbb{P}_0[X_1^{(\alpha)} = k] \mathbb{P}_k[X_1^{(\alpha)} \neq 0, X_2^{(\alpha)} \neq 0, \dots, X_{n-1}^{(\alpha)} \neq 0, X_n^{(\alpha)} = 0] \\
&= \alpha \underbrace{\sum_{k \neq 0} \mu(k) \mathbb{P}_k[\tau^{(1)} = n - 1]}_{\mathbb{P}_0[\tau^{(1)} = n]} + (1 - \alpha) \underbrace{\sum_{k \neq 0} \mu'(k) \mathbb{P}_k[\tau^{(0)} = n - 1]}_{\mathbb{P}_0[\tau^{(0)} = n]}.
\end{aligned}$$

Summing over  $n \geq 1$  in both sides, it readily implies  $\mathbb{P}_0[\tau^{(\alpha)} < +\infty] = 1$  as expected. ■

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