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► **To cite this version:**

Cyrille Kenne, Pascal Zongo, René Dorville, Gisèle Mophou. Optimal control of a coupled degenerate population dynamics model with unknown birth rates. *Nonlinear Studies*, 2021, 28 (4), pp.1225-1252. hal-03510055

**HAL Id: hal-03510055**

**<https://hal.science/hal-03510055>**

Submitted on 4 Jan 2022

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# Optimal control of a coupled degenerate population dynamics model with unknown birth rates

Cyrille Kenne\*    Pascal Zongo†    René Dorville‡    Gisèle Mophou §

July 29, 2021

## Abstract

We consider a coupled model of degenerate population dynamics with age dependence and spatial structure but with unknown data on the birth rate of individuals. Using the notion of low-regret, we prove that we can bring the state of the second population in a given region to a desired state by acting on the first population via a localized control over a part of habitat. We provide the optimality systems that characterize the adapted low-regret control and prove its convergence to a so-called no-regret control that we characterize.

Mathematics Subject Classification. 49J20, 92D25, 93C41

**Key-words** : Degenerate population dynamics model, incomplete data, optimal control, No-regret control, Low-regret control.

## 1 Introduction

In this paper we consider a coupled degenerate population dynamics model with missing data. This model can be used to describe the dispersion of a gene in two given populations which are in interaction ( e.g cancer cells and healthy cells), see [2] for instance. More precisely, let  $T > 0$ ,  $A > 0$ ,  $U = (0, T) \times (0, A)$ ,  $Q = U \times (0, 1)$ ,  $Q_A = (0, A) \times (0, 1)$ ,  $Q_T = (0, T) \times (0, 1)$ ,  $\Sigma = U \times \{0, 1\}$ ,  $\omega$  and  $\mathcal{O}$  are nonempty subsets of the spatial domain  $(0, 1)$ . We also set  $Q_\omega = U \times \omega$ . Then, we consider the following model of population dynamics:

$$\left\{ \begin{array}{ll} y_t + y_a - (k(x)y_x)_x + \mu_1(a)y = f + v\chi_{Q_\omega} & \text{in } Q, \\ z_t + z_a - (k(x)z_x)_x + \mu_2(a)z = \kappa y\chi_{\mathcal{O}} & \text{in } Q, \\ y = z = 0 & \text{on } \Sigma, \\ y(0, a, x) = y_0(a, x), \quad z(0, a, x) = z_0(a, x) & \text{in } Q_A, \\ y(t, 0, x) = \int_0^A g_1(a)y(t, a, x) da, \quad z(t, 0, x) = \int_0^A g_2(a)z(t, a, x) da & \text{in } Q_T, \end{array} \right. \quad (1)$$

where  $y(t, a, x)$  and  $z(t, a, x)$  denote respectively the distributions of individuals of age  $a$  at time  $t$  having a gene type  $x$ . The terms  $-\mu_2(a)z(t, a, x)$  and  $-\mu_1(a)y(t, a, x)$  describe the natural mortality of individuals of age  $a$  at time  $t$  and of gene type  $x$  of the population  $z$  and  $y$  respectively. The term  $\int_0^A g_i(a)W(t, a, x) da$ , is the flux of new born individuals while the functions  $g_i(a)$ ,  $i = 1, 2$  are the

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age-specific fertility rates,  $i = 1, W = y$  or  $i = 2, W = z$ .  $A$  is the maximal life expectancy. The function  $f \in L^2(Q)$  represents the external supply of population. We have denoted by  $k$  the diffusion coefficient, by  $v$  the control. The functions  $y_0, z_0 \in L^2(Q_A)$  are the initials conditions,  $\chi_X$  denote the characteristic function of the set  $X$ . Here, we don't have the distribution of newborns in the two populations, these measurements are expressed here by the unknown variables  $g_1$  and  $g_2$ .

We assume as in [1] that the dispersal coefficient  $k$  verifies

$$\begin{cases} k \in \mathcal{C}([0, 1]) \cap \mathcal{C}^1((0, 1]), k > 0 \text{ in } (0, 1] \text{ and } k(0) = 0, \\ \exists \gamma \in [0, 1) : xk'(x) \leq \gamma k(x), x \in [0, 1], \end{cases} \quad (2)$$

the death rates  $\mu_i, i = 1, 2$  are such that

$$\mu_i \in L^\infty(0, A), \mu_i \geq 0 \text{ a.e. } (0, A), \quad (3)$$

and the fertility rates  $g_i, i = 1, 2$  are unknown and verify

$$g_i \in L^\infty(0, A), g_i \geq 0 \text{ a.e in } [0, A]. \quad (4)$$

For the sake of simplicity, we replace  $y$  by  $y/\kappa$ ,  $f$  by  $f/\kappa$ ,  $v$  by  $v/\kappa$ , and  $y_0$  by  $y_0/\kappa$  and we will consider from now on, the following system:

$$\left\{ \begin{array}{ll} y_t + y_a - (k(x)y_x)_x + \mu_1(a)y = f + v\chi_\omega & \text{in } Q, \\ z_t + z_a - (k(x)z_x)_x + \mu_2(a)z = y\chi_\mathcal{O} & \text{in } Q, \\ y = z = 0 & \text{on } \Sigma, \\ y(0, a, x) = y_0(a, x), \quad z(0, a, x) = z_0(a, x) & \text{in } Q_A, \\ y(t, 0, x) = \int_0^A g_1(a)y(t, a, x) da, \quad z(t, 0, x) = \int_0^A g_2(a)z(t, a, x) da & \text{in } Q_T. \end{array} \right. \quad (5)$$

As in [5, 15], we denote by  $H_k^1(0, 1)$  and  $H_k^2(0, 1)$  the weighted Sobolev spaces respectively given by

$$\begin{cases} H_k^1(0, 1) := \{u \in L^2(0, 1) : u \text{ is abs. cont. in } [0, 1] : \sqrt{k}u_x \in L^2(\Omega), u(0) = u(1) = 0\}, \\ H_k^2(0, 1) := \{u \in H_k^1(0, 1) : (ku_x)_x \in H^1(0, 1)\}, \end{cases} \quad (6)$$

and endowed respectively with the norms,

$$\begin{cases} \|u\|_{H_k^1(0,1)}^2 := \|u\|_{L^2(0,1)}^2 + \|\sqrt{k(x)}u_x\|_{L^2(0,1)}^2, & u \in H_k^1(0, 1), \\ \|u\|_{H_k^2(0,1)}^2 := \|u\|_{H_k^1(0,1)}^2 + \|(k(x)u_x)_x\|_{L^2(0,1)}^2, & u \in H_k^2(0, 1). \end{cases} \quad (7)$$

Throughout the rest of the paper, we use  $\|\cdot\|_\infty$  to denote the  $L^\infty$ -norm in  $(0, A)$ .

**Remark 1** Under the assumptions (2), we have

1.  $H_k^1(0, 1)$  is compactly embedded in  $L^2(0, 1)$  (see [3]).
2. The trace at  $x = 0$  of a function of  $H_k^1(0, 1)$  exists.

Set

$$E = [\mathcal{C}([0, T]; L^2(Q_A)) \cap \mathcal{C}([0, A]; L^2(Q_T)) \cap L^2(U, H_k^1(0, 1))]^2, \quad (8)$$

$$W_k(T, A) = \{\rho \in L^2(U; H_k^1(0, 1)); \rho_t + \rho_a \in L^2(U; H_k^{-1}(0, 1))\}. \quad (9)$$

Then we have that

$$W_k(T, A) \subset \mathcal{C}([0, T], L^2(Q_A)) \cap \mathcal{C}([0, A], L^2(Q_T)). \quad (10)$$

Under the assumption on the data one can prove as in [15, 5, 2] that the problem (5) has a unique solution  $(y, z)$  in  $E$ . Moreover, there exists constant  $C = C(T, A, \|\mu_1\|_\infty, \|\mu_2\|_\infty, \|g_1\|_\infty, \|g_2\|_\infty) > 0$ , such that

$$\begin{aligned} & \|y(T)\|_{L^2(Q_A)}^2 + \|y(A)\|_{L^2(Q_T)}^2 + \|z(T)\|_{L^2(Q_A)}^2 + \|z(A)\|_{L^2(Q_T)}^2 + \\ & \left( \|\sqrt{k}y_x\|_{L^2(Q)}^2 + \|y\|_{L^2(Q)}^2 \right) + \left( \|\sqrt{k}z_x\|_{L^2(Q)}^2 + \|z\|_{L^2(Q)}^2 \right) \leq \\ & C \left( \|v\|_{L^2(Q_\omega)}^2 + \|z_0\|_{L^2(Q_A)}^2 + \|y_0\|_{L^2(Q_A)}^2 \right). \end{aligned} \quad (11)$$

The purpose of this paper is to bring the distribution of cancer cells  $z$  in a given region  $\mathcal{O} \subset (0, 1)$  to a desired distribution  $z_d$  by acting on the system via a control  $v$  over a part  $\omega$  of the domain. Such a control may correspond to a removal on the subdomain  $\omega$ . We define the cost function

$$J(v, g_1, g_2) = \|z(v, g_1, g_2) - z_d\|_{L^2(U \times \mathcal{O})}^2 + N \|v\|_{L^2(Q_\omega)}^2, \quad (12)$$

where the term  $z_d \in L^2(U \times \mathcal{O})$  is the desired state, and  $N > 0$  is the cost coefficient and are given.

We point out that the objective is not to determine the unknown birth rates  $g_i$ ,  $i = 1, 2$ , but to find an optimal control  $v$  solution to the following optimization problem:

$$\inf_{v \in L^2(Q_\omega)} \sup_{(g_1, g_2) \in [L^2(0, A)]^2} (J(v, g_1, g_2) - J(0, g_1, g_2)). \quad (13)$$

Problem (13) is called the no-regret control problem. The notions of no-regret control and low-regret control were introduced by Lions [8] in order to control a phenomenon described by a parabolic equation with missing initial condition.

The most challenging point is to prove that this family of controls (called low-regret controls) converge towards the no-regret control. In [13], Nakoulima et al. applied this notion to linear evolution equations with incomplete data. They proved that the low-regret controls converge to the no-regret control for which they obtained a singular optimality system. In the nonlinear case, this notion was also considered by Nakoulima et al. [12] to control (on the whole domain) a nonlinear system with incomplete data. They observe on the one hand that the no-regret control is typically not easy to characterize and, on the other hand that the low-regret cost function may not be convex. So, by adapting this cost, they proved that the adapted low-regret control converge towards a no-regret control. Finally they characterized this no-regret by a singular optimality system. Velin[14], studied systems governed by quasilinear equations with unknown boundary condition and a control acting on the whole domain. After established some regularity results, he proved by proceeding as in [12] that the adapted low-regret control converge towards a no-regret control characterized by a singular optimality system. Note that in the above papers, the convergence of the low-regret control towards the no-regret control is obtained by controlling on the whole domain. Recently, Kenne et al. [7] used the notion of no-regret and low-regret to control a model describing the dynamics of population with age dependence and spatial structure with missing birth rate by acting on a part of the domain. They proved that they can bring the state of the system to a desired state by acting on the system via a localized distributed control. The results of this latter paper are extended here to a degenerated coupled model with two missing data. We also refer to [10, 6, 11] for more literature.

In this paper we are interested in the optimization problem (13), where the cost function is given by (12). Since the application from the space of the unknown birth rates to space of the state variables is nonlinear, we linearize the cost and prove the existence of a no-regret control associated to this new cost. Then we consider an adapted low-regret control problem obtained by relaxing the no-regret problem. We prove the existence and uniqueness of the adapted low-regret control and show its convergence toward the no-regret control. Finally we show that the optimality systems of the adapted low-regret control converge to optimality systems which characterize the no-regret control.

The rest of this paper is structured as follows. In Section 2, we give some regularity results. We study the low-regret and no-regret control and their characterizations in Section 3. A conclusion is given in Section 4. The last section takes the form of an appendix wherein we will give the proofs of some basic tools.

## 2 Preliminary results

To solve the optimization problem (13), we need some preliminary results. Set  $\Omega = (0, 1)$  and consider the following model of population dynamics:

$$\left\{ \begin{array}{ll} y_t + y_a - (k(x)y_x)_x + \mu_1(a)y = h & \text{in } Q, \\ z_t + z_a - (k(x)z_x)_x + \mu_2(a)z = y\chi_{\mathcal{O}} & \text{in } Q, \\ y(t, a, 1) = y(t, a, 0) = 0 & \text{on } U, \\ z(t, a, 1) = z(t, a, 0) = 0 & \text{on } U, \\ y(0, a, x) = y_0(a, x), \quad z(0, a, x) = z_0(a, x) & \text{in } Q_A, \\ y(t, 0, x) = b_1(t, x), \quad z(t, 0, x) = b_2(t, x) & \text{in } Q_T. \end{array} \right. \quad (14)$$

Let us state the following well posedness result.

**Proposition 2.1** *Assume that (3) holds true. Let  $h \in L^2(Q)$ ,  $(b_1, b_2) \in L^2(Q_T)^2$  and  $(y_0, z_0) \in L^2(Q_A)^2$ . Let*

$$E = [\mathcal{C}([0, T]; L^2(Q_A)) \cap \mathcal{C}([0, A]; L^2(Q_T)) \cap L^2(U, H_k^1(\Omega))]^2.$$

*Then the problem (14) has a unique solution  $(y, z)$  in  $E$ .*

*Moreover, there exists constant  $C = C(T, A, \|\mu_1\|_\infty, \|\mu_2\|_\infty) > 0$ , such that*

$$\begin{aligned} & \|y(T)\|_{L^2(Q_A)}^2 + \|y(A)\|_{L^2(Q_T)}^2 + \|z(T)\|_{L^2(Q_A)}^2 + \|z(A)\|_{L^2(Q_T)}^2 + \\ & \left( \|\sqrt{k}y_x\|_{L^2(Q)}^2 + \|y\|_{L^2(Q)}^2 \right) + \left( \|\sqrt{k}z_x\|_{L^2(Q)}^2 + \|z\|_{L^2(Q)}^2 \right) \leq \\ & C \left( \|h\|_{L^2(Q)}^2 + \|z_0\|_{L^2(Q_A)}^2 + \|y_0\|_{L^2(Q_A)}^2 + \|b_1\|_{L^2(Q_T)}^2 + \|b_2\|_{L^2(Q_T)}^2 \right). \end{aligned} \quad (15)$$

**Proof.** For the existence, we refer to [15, 5, 2]. To prove (15), we make a change of variable. For  $r \in \mathbb{R}$ , we set

$$\tilde{y}(t, a, x) = e^{-rt}y(t, a, x), \quad \tilde{z}(t, a, x) = e^{-rt}z(t, a, x),$$

where  $(y, z)$  is the solution of (5). Then  $(\tilde{y}, \tilde{z})$  is the solution of

$$\left\{ \begin{array}{ll} \tilde{y}_t + \tilde{y}_a - (k(x)\tilde{y}_x)_x + \tilde{\mu}_1(a)\tilde{y} = e^{-rt}h & \text{in } Q, \\ \tilde{z}_t + \tilde{z}_a - (k(x)\tilde{z}_x)_x + \tilde{\mu}_2(a)\tilde{z} = \tilde{y}\chi_{\mathcal{O}} & \text{in } Q, \\ \tilde{y}(t, a, 1) = \tilde{y}(t, a, 0) = 0 & \text{on } U, \\ \tilde{z}(t, a, 1) = \tilde{z}(t, a, 0) = 0 & \text{on } U, \\ \tilde{y}(0, a, x) = y_0(a, x), \quad \tilde{z}(0, a, x) = z_0(a, x) & \text{in } Q_A, \\ \tilde{y}(a, 0, x) = e^{-rt}b_1(t, x), \quad \tilde{z}(a, 0, x) = e^{-rt}b_2(t, x) & \text{in } Q_T. \end{array} \right. \quad (16)$$

where in view of (3),  $\tilde{\mu}_i(a) = r + \mu_i(a)$ ,  $i = 1, 2$ .

Now, let  $(t, a) \in U$ . If we multiply the first and the second equation in (16) respectively by  $\tilde{y}(t, a)$  and  $\tilde{z}(t, a)$ , and integrate by parts over  $\Omega$ , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\tilde{y}(t, a)\|_{L^2(\Omega)}^2 + \frac{1}{2} \frac{d}{da} \|\tilde{y}(t, a)\|_{L^2(\Omega)}^2 + \left\| \sqrt{k}\tilde{y}_x(t, a) \right\|_{L^2(\Omega)}^2 + \int_{\Omega} \tilde{\mu}_1(a)\tilde{y}^2(t, a, x) dx \\ & = \int_{\Omega} e^{-rt}h(t, a, x)\tilde{y}(t, a, x) dx, \\ & \frac{1}{2} \frac{d}{dt} \|\tilde{z}(t, a)\|_{L^2(\Omega)}^2 + \frac{1}{2} \frac{d}{da} \|\tilde{z}(t, a)\|_{L^2(\Omega)}^2 + \left\| \sqrt{k}\tilde{z}_x(t, a) \right\|_{L^2(\Omega)}^2 + \int_{\Omega} \tilde{\mu}_2(a)\tilde{z}^2(t, a, x) dx \\ & = \int_{\mathcal{O}} \tilde{y}(t, a, x)\tilde{z}(t, a, x) dx. \end{aligned}$$

since  $\tilde{\mu}_i(a) = r + \mu_i(a)$ ,  $i = 1, 2$ , using Young and Cauchy inequalities,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\tilde{y}(t, a)\|_{L^2(\Omega)}^2 + \frac{1}{2} \frac{d}{da} \|\tilde{y}(t, a)\|_{L^2(\Omega)}^2 + \left\| \sqrt{k}\tilde{y}_x(t, a) \right\|_{L^2(\Omega)}^2 + \\ & \left( r - \|\mu_1\|_\infty - \frac{1}{2} \right) \|\tilde{y}(t, a)\|_{L^2(\Omega)}^2 \leq \frac{1}{2} \|e^{-rt}h(t, a)\|_{L^2(\Omega)}^2, \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\tilde{z}(t, a)\|_{L^2(\Omega)}^2 + \frac{1}{2} \frac{d}{da} \|\tilde{z}(t, a)\|_{L^2(\Omega)}^2 + \left\| \sqrt{k} \tilde{z}_x(t, a) \right\|_{L^2(\Omega)}^2 + \\ & \left( r - \|\mu_2\|_\infty - \frac{1}{2} \right) \|\tilde{z}(t, a)\|_{L^2(\Omega)}^2 \leq \frac{1}{2} \|\tilde{y}(t, a)\|_{L^2(\Omega)}^2. \end{aligned}$$

choose  $r = \max(\|\mu_1\|_\infty, \|\mu_2\|_\infty) + \frac{3}{2}$ , then it follows that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\tilde{y}(t, a)\|_{L^2(\Omega)}^2 + \frac{1}{2} \frac{d}{da} \|\tilde{y}(t, a)\|_{L^2(\Omega)}^2 + \left\| \sqrt{k} \tilde{y}_x(t, a) \right\|_{L^2(\Omega)}^2 + \|\tilde{y}(t, a)\|_{L^2(\Omega)}^2 \\ & \leq \frac{1}{2} \|h(t, a)\|_{L^2(\Omega)}^2 \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\tilde{z}(t, a)\|_{L^2(\Omega)}^2 + \frac{1}{2} \frac{d}{da} \|\tilde{z}(t, a)\|_{L^2(\Omega)}^2 + \left\| \sqrt{k} \tilde{z}_x(t, a) \right\|_{L^2(\Omega)}^2 + \|\tilde{z}(t, a)\|_{L^2(\Omega)}^2 \\ & \leq \frac{1}{2} \|\tilde{y}(t, a)\|_{L^2(\Omega)}^2, \end{aligned}$$

which after an integration by part over  $(0, T) \times (0, A)$  gives

$$\begin{aligned} & \frac{1}{2} \|\tilde{y}(T)\|_{L^2(Q_A)}^2 + \frac{1}{2} \|\tilde{y}(A)\|_{L^2(Q_T)}^2 + \left\| \sqrt{k} \tilde{y}_x \right\|_{L^2(Q)}^2 + \|\tilde{y}\|_{L^2(Q)}^2 \\ & \leq \frac{1}{2} \|h\|_{L^2(Q)}^2 + \frac{1}{2} \|y_0\|_{L^2(Q_A)}^2 + \frac{1}{2} \|b_1\|_{L^2(Q_T)}^2 \end{aligned} \quad (17)$$

and

$$\begin{aligned} & \frac{1}{2} \|\tilde{z}(T)\|_{L^2(Q_A)}^2 + \frac{1}{2} \|\tilde{z}(A)\|_{L^2(Q_T)}^2 + \left\| \sqrt{k} \tilde{z}_x \right\|_{L^2(Q)}^2 + \|\tilde{z}\|_{L^2(Q)}^2 \\ & \leq \frac{1}{2} \|\tilde{y}\|_{L^2(Q)}^2 + \frac{1}{2} \|z_0\|_{L^2(Q_A)}^2 + \frac{1}{2} \|b_2\|_{L^2(Q_T)}^2. \end{aligned} \quad (18)$$

By combining (17) and (18), and replacing  $\tilde{y}$  and  $\tilde{z}$  by  $e^{-rt}y$  and  $e^{-rt}z$  respectively, we deduce (15) with  $C = e^{rT}$ , and where  $r = \max(\|\mu_1\|_\infty, \|\mu_2\|_\infty) + \frac{3}{2}$ . This completes the proof.  $\blacksquare$

For the rest of the section and for the sequel, we denote by  $C(X)$  a positive constant whose value varies from a line to another but depends on  $X$ . Let us adopt the following notation

$$\begin{cases} L_i &= \partial_t + \partial_a - (k(x)\partial_x)_x + \mu_i, \\ L_i^* &= -\partial_t - \partial_a - (k(x)\partial_x)_x + \mu_i. \end{cases} \quad (19)$$

The following two propositions are proved as in [7, Propositions 2.1 and 2.2] using a change of variable on the states and energy estimate.

**Proposition 2.2** *Let  $(y(v, g_1, g_2), z(v, g_1, g_2))$  be a solution of (5). Then the application  $(v, g_1, g_2) \mapsto (y(v, g_1, g_2), z(v, g_1, g_2))$  from  $L^2(Q_\omega) \times L^2(0, A) \times L^2(0, A)$  onto  $[L^2(Q)]^2$  is continuous.*

**Proposition 2.3** *Let  $\lambda > 0$ . Let  $g_1, g_2, h_1, h_2 \in L^2(0, A)$  and  $v, w \in L^2(Q_\omega)$ . Let also  $(y, z) = (y(v, g_1, g_2), z(v, g_1, g_2))$  be a solution of (5). Set  $\bar{y}_\lambda = \frac{y(v + \lambda w, g_1 + \lambda h_1, g_2 + \lambda h_2) - y(v, g_1, g_2)}{\lambda}$ , and  $\bar{z}_\lambda = \frac{z(v + \lambda w, g_1 + \lambda h_1, g_2 + \lambda h_2) - z(v, g_1, g_2)}{\lambda}$ . Then as  $\lambda \rightarrow 0$ ,  $(\bar{y}_\lambda, \bar{z}_\lambda)$  converges strongly in  $[L^2(Q)]^2$  to  $(\bar{y}, \bar{z})$ , where  $(\bar{y}, \bar{z})$  is solution of*

$$\begin{cases} L_1 \bar{y} &= w \chi_\omega & \text{in } Q, \\ L_2 \bar{z} &= \bar{y} \chi_\mathcal{O} & \text{in } Q, \\ \bar{y} = \bar{z} &= 0 & \text{on } \Sigma, \\ \bar{y}(0, \cdot, \cdot) = 0, \quad \bar{z}(0, \cdot, \cdot) &= 0 & \text{in } Q_A, \\ \bar{y}(\cdot, 0, \cdot) = \delta_3, \quad \bar{z}(\cdot, 0, \cdot) &= \delta_4 & \text{in } Q_T, \end{cases} \quad (20)$$

with

$$\delta_3 = \int_0^A g_1(a) \bar{y} \, da + \int_0^A h_1(a) y(t, a, x; v, g_1, g_2) \, da,$$

and

$$\delta_4 = \int_0^A g_2(a) \bar{z} \, da + \int_0^A h_2(a) z(t, a, x; v, g_1, g_2) \, da.$$

**Remark 2** Note that using the definition of the cost function given by (12) and Proposition 2.3, we have that

$$\lim_{\lambda \rightarrow 0} \frac{J(v + \lambda w, g_1 + \lambda h_1, g_2 + \lambda h_2) - J(v, g_1, g_2)}{\lambda} = 2N \int_{Q_\omega} v w \, dt \, da \, dx + 2 \int_U \int_{\mathcal{O}} \bar{z} (z(v, g_1, g_2) - z_d) \, dt \, da \, dx$$

where  $(\bar{y}, \bar{z})$  satisfies (20).

### 3 Resolution of the optimization problem (13)

This section is devoted to the optimization problem (13). As the low-regret and no-regret notion introduced by Lions [8] uses the decomposition of the solution of (5) on the form  $y(v, g_1, g_2) = y(v, 0, 0) + \varphi(g_1, g_2)$  where  $y(v, 0, 0)$  is solution of (5) with  $g_1 = g_2 = 0$  and  $\varphi(g_1, g_2)$  is a function depending of  $g_1$  and  $g_2$ , this decomposition is no longer valid because the map  $(g_1, g_2) \mapsto y(v, g_1, g_2)$  from  $[L^2(0, A)]^2$  to  $L^2(U; H_k^1(\Omega))$  is non-linear. Thus using the regularity results of  $(y, z)$  given in Proposition 2.2 and Proposition 2.3, we replace the cost function defined in (12) by its linearized form with respect to  $g_1$  and  $g_2$ . Thus, we consider as in [9] the new cost-function

$$J_1(v, g_1, g_2) = J(v, 0, 0) + \frac{\partial J}{\partial g_1}(v, 0, 0)(g_1) + \frac{\partial J}{\partial g_2}(v, 0, 0)(g_2). \quad (21)$$

Then, we consider the following new optimization problem:

$$\inf_{v \in L^2(Q_\omega)} \sup_{(g_1, g_2) \in [L^2(0, A)]^2} (J_1(v, g_1, g_2) - J_1(0, g_1, g_2)). \quad (22)$$

The solution of (22) if it exists is called no-regret control for the non-linear problem. Let  $(y(v, 0, 0), z(v, 0, 0)) \in [L^2(U; H_k^1(\Omega))]^2$  be the solution of

$$\begin{cases} L_1 y(v, 0, 0) & = f + v \chi_\omega & \text{in } Q, \\ L_2 z(v, 0, 0) & = y(v, 0, 0) \chi_{\mathcal{O}} & \text{in } Q, \\ y = z & = 0 & \text{on } \Sigma, \\ y(0, a, x) = y_0(a, x), \quad z(0, a, x) & = z_0(a, x) & \text{in } Q_A, \\ y(t, 0, x) = 0, \quad z(t, 0, x) & = 0 & \text{in } Q_T. \end{cases} \quad (23)$$

Then we have the following result.

**Proposition 3.1** For any  $(v, g_1, g_2) \in L^2(Q_\omega) \times [L^2(0, A)]^2$ , the following equality holds:

$$\begin{aligned} J_1(v, g_1, g_2) &= J(v, 0, 0) + 2 \int_U \int_{\mathcal{O}} \left( \frac{\partial z}{\partial g_1}(v, 0, 0)(g_1) \right) (z(v, 0, 0) - z_d) \, dt \, da \, dx \\ &\quad + 2 \int_U \int_{\mathcal{O}} \left( \frac{\partial z}{\partial g_2}(v, 0, 0)(g_2) \right) (z(v, 0, 0) - z_d) \, dt \, da \, dx \end{aligned} \quad (24)$$

where  $J$  is the cost function defined in (12).

**Proof.** Observing on the one hand

$$\begin{aligned}
J(v, tg_1, 0) &= \|z(v, tg_1, 0) - z_d\|_{L^2(U \times \mathcal{O})}^2 + N\|v\|_{L^2(Q_\omega)}^2 \\
&= J(v, 0, 0) + \|z(v, tg_1, 0) - z(v, 0, 0)\|_{L^2(U \times \mathcal{O})}^2 \\
&+ 2 \int_U \int_{\mathcal{O}} (z(v, tg_1, 0) - z(v, 0, 0))(z(v, 0, 0) - z_d) dt da dx,
\end{aligned}$$

and on the other hand

$$\frac{\partial J}{\partial g_1}(v, 0, 0)(g_1) = \lim_{t \rightarrow 0} \frac{J(v, tg_1, 0) - J(v, 0, 0)}{t},$$

using Proposition 2.3 and Remark 2, we obtain that

$$\begin{aligned}
\frac{\partial J}{\partial g_1}(v, 0, 0)(g_1) &= \lim_{t \rightarrow 0} \frac{J(v, tg_1, 0) - J(v, 0, 0)}{t} \\
&= 2 \int_U \int_{\mathcal{O}} \left( \frac{\partial z}{\partial g_1}(v, 0, 0)(g_1) \right) (z(v, 0, 0) - z_d) dt da dx.
\end{aligned}$$

Analogously,

$$\frac{\partial J}{\partial g_2}(v, 0, 0)(g_2) = 2 \int_U \int_{\mathcal{O}} \left( \frac{\partial z}{\partial g_2}(v, 0, 0)(g_2) \right) (z(v, 0, 0) - z_d) dt da dx.$$

Using these two identities in (21) yields (24). ■

**Proposition 3.2** For any  $(v, g_1, g_2) \in L^2(Q_\omega) \times [L^2(0, A)]^2$ , we have

$$\begin{aligned}
J_1(v, g_1, g_2) - J_1(0, g_1, g_2) &= J(v, 0, 0) - J(0, 0, 0) + 2 \int_0^A S_1(a; v) g_1(a) da \\
&+ 2 \int_0^A S_2(a; v) g_2(a) da,
\end{aligned} \tag{25}$$

where for any  $a \in (0, A)$ ,

$$S_1(a; v) = \int_{Q_T} [y(t, a, x; v, 0, 0) \xi_1(v)(t, 0, x) - y(t, a, x; 0, 0, 0) \xi_1(0)(t, 0, x)] dt dx, \tag{26}$$

$$S_2(a; v) = \int_{Q_T} [z(t, a, x; v, 0, 0) \eta_2(v)(t, 0, x) - z(t, a, x; 0, 0, 0) \eta_2(0)(t, 0, x)] dt dx, \tag{27}$$

where  $(\xi_1(v), \eta_1(v))$  is solution to

$$\begin{cases}
L_1^* \xi_1(v) &= \eta_1(v) \chi_{\mathcal{O}} & \text{in } Q, \\
L_2^* \eta_1(v) &= (z(v, 0, 0) - z_d) \chi_{\mathcal{O}} & \text{in } Q, \\
\xi_1(v) = \eta_1(v) &= 0 & \text{on } \Sigma, \\
\xi_1(v)(T, \cdot, \cdot) = 0, \quad \eta_1(v)(T, \cdot, \cdot) &= 0 & \text{in } Q_A, \\
\xi_1(v)(\cdot, A, \cdot) = 0, \quad \eta_1(v)(\cdot, A, \cdot) &= 0 & \text{in } Q_T,
\end{cases} \tag{28}$$

and  $(\xi_2(v), \eta_2(v))$  is solution to

$$\begin{cases}
L_1^* \xi_2(v) &= \eta_2(v) \chi_{\mathcal{O}} & \text{in } Q, \\
L_2^* \eta_2(v) &= (z(v, 0, 0) - z_d) \chi_{\mathcal{O}} & \text{in } Q, \\
\xi_2(v) = \eta_2(v) &= 0 & \text{on } \Sigma, \\
\xi_2(v)(T, \cdot, \cdot) = 0, \quad \eta_2(v)(T, \cdot, \cdot) &= 0 & \text{in } Q_A, \\
\xi_2(v)(\cdot, A, \cdot) = 0, \quad \eta_2(v)(\cdot, A, \cdot) &= 0 & \text{in } Q_T.
\end{cases} \tag{29}$$



**Proof.** From(21),

$$J_1(0, g_1, g_2) = J(0, 0, 0) + \frac{\partial J}{\partial g_1}(0, 0, 0)(g_1) + \frac{\partial J}{\partial g_2}(0, 0, 0)(g_2).$$

In view of (24), we have

$$\begin{aligned} J_1(v, g_1, g_2) - J_1(0, g_1, g_2) &= J(v, 0, 0) - J(0, 0, 0) \\ &+ 2 \int_U \int_{\mathcal{O}} \left( \frac{\partial z}{\partial g_1}(v, 0, 0)(g_1) \right) (z(v, 0, 0) - z_a) dt da dx \\ &+ 2 \int_U \int_{\mathcal{O}} \left( \frac{\partial z}{\partial g_2}(v, 0, 0)(g_2) \right) (z(v, 0, 0) - z_a) dt da dx \\ &- 2 \int_U \int_{\mathcal{O}} \left( \frac{\partial z}{\partial g_1}(0, 0, 0)(g_1) \right) (z(0, 0, 0) - z_a) dt da dx \\ &- 2 \int_U \int_{\mathcal{O}} \left( \frac{\partial z}{\partial g_2}(0, 0, 0)(g_2) \right) (z(0, 0, 0) - z_a) dt da dx. \end{aligned} \quad (30)$$

From Proposition 2.3, we have that  $(\bar{y}(g_1), \bar{z}(g_1)) = \left( \frac{\partial y}{\partial g_1}(v, 0, 0)(g_1), \frac{\partial z}{\partial g_1}(v, 0, 0)(g_1) \right)$  is solution to

$$\begin{cases} L_1 \bar{y}(g_1) &= 0 & \text{in } Q, \\ L_2 \bar{z}(g_1) &= \bar{y}(g_1) \chi_{\mathcal{O}} & \text{in } Q, \\ \bar{y}(g_1) = \bar{z}(g_1) &= 0 & \text{on } \Sigma, \\ \bar{y}(g_1)(0, \cdot, \cdot) = 0, \quad \bar{z}(g_1)(0, \cdot, \cdot) &= 0 & \text{in } Q_A, \\ \bar{y}(g_1)(\cdot, 0, \cdot) = \eta_{11}, \quad \bar{z}(g_1)(\cdot, 0, \cdot) &= 0 & \text{in } Q_T, \end{cases} \quad (31)$$

where

$$\eta_{11} = \int_0^A g_1(a) y(t, a, x; v, 0, 0) da.$$

So, if we multiply the first and the second equations of (31) respectively by  $\xi_1(v)$  and  $\eta_1(v)$  solutions of (28), and integrate by parts over  $Q$ , we get

$$\begin{aligned} &\int_U \int_{\mathcal{O}} \left( \frac{\partial y}{\partial g_1}(v, 0, 0)(g_1) \right) \eta_1(v) dt da dx \\ &= \int_Q g_1(a) y(t, a, x; v, 0, 0) \xi_1(v)(t, 0, x) dt da dx, \end{aligned} \quad (32)$$

and

$$\begin{aligned} &\int_U \int_{\mathcal{O}} \left( \frac{\partial z}{\partial g_1}(v, 0, 0)(g_1) \right) (z(v, 0, 0) - z_a) dt da dx \\ &= \int_U \int_{\mathcal{O}} \left( \frac{\partial y}{\partial g_1}(v, 0, 0)(g_1) \right) \eta_1(v) dt da dx. \end{aligned} \quad (33)$$

Then, combining (32) and (33), we obtain

$$\begin{aligned} &\int_U \int_{\mathcal{O}} \left( \frac{\partial z}{\partial g_1}(v, 0, 0)(g_1) \right) (z(v, 0, 0) - z_a) dt da dx \\ &= \int_Q g_1(a) y(t, a, x; v, 0, 0) \xi_1(v)(t, 0, x) dt da dx, \end{aligned} \quad (34)$$

from which, we deduce that

$$\begin{aligned} &\int_U \int_{\mathcal{O}} \left( \frac{\partial z}{\partial g_1}(0, 0, 0)(g_1) \right) (z(0, 0, 0) - z_a) dt da dx \\ &= \int_Q g_1(a) y(t, a, x; 0, 0, 0) \xi_1(0)(t, 0, x) dt da dx. \end{aligned} \quad (35)$$

Using again Proposition 2.3, we have that  $(\tilde{y}(g_2), \tilde{z}(g_2)) = \left( \frac{\partial y}{\partial g_2}(v, 0, 0)(g_2), \frac{\partial z}{\partial g_2}(v, 0, 0)(g_2) \right)$  is solution to

$$\begin{cases} L_1 \tilde{y}(g_2) & = 0 & \text{in } Q, \\ L_2 \tilde{z}(g_2) & = \tilde{y}(g_2) \chi_{\mathcal{O}} & \text{in } Q, \\ \tilde{y}(g_2) = \tilde{z}(g_2) & = 0 & \text{on } \Sigma, \\ \tilde{y}(g_2)(0, \cdot, \cdot) = 0, \quad \tilde{z}(g_2)(0, \cdot, \cdot) & = 0 & \text{in } Q_A, \\ \tilde{y}(g_2)(\cdot, 0, \cdot) = 0, \quad \tilde{z}(g_2)(\cdot, 0, \cdot) & = \eta_{12} & \text{in } Q_T, \end{cases} \quad (36)$$

where

$$\eta_{12} = \int_0^A g_2(a) z(t, a, x; v, 0, 0) da.$$

So, if we multiply the first and the second equations of (36) respectively by  $\xi_2(v)$  and  $\eta_2(v)$  solutions of (29) and integrate by parts over  $Q$ , we get

$$\int_U \int_{\mathcal{O}} \left( \frac{\partial y}{\partial g_2}(v, 0, 0)(g_2) \right) \eta_2(v) dt da dx = 0 \quad (37)$$

and

$$\begin{aligned} & - \int_Q g_2(a) z(t, a, x; v, 0, 0) \xi_2(v)(t, 0, x) dt da dx \\ & + \int_U \int_{\mathcal{O}} \left( \frac{\partial z}{\partial g_2}(v, 0, 0)(g_2) \right) (z(v, 0, 0) - z_d) dt da dx \\ & = \int_U \int_{\mathcal{O}} \left( \frac{\partial y}{\partial g_2}(v, 0, 0)(g_2) \right) \eta_2(v) dt da dx. \end{aligned} \quad (38)$$

Also, by combining (37) and (38), we are lead to

$$\begin{aligned} & \int_U \int_{\mathcal{O}} \left( \frac{\partial z}{\partial g_2}(v, 0, 0)(g_2) \right) (z(v, 0, 0) - z_d) dt da dx \\ & = \int_Q g_2(a) z(t, a, x; v, 0, 0) \xi_2(v)(t, 0, x) dt da dx, \end{aligned} \quad (39)$$

from which we deduce that

$$\int_U \int_{\mathcal{O}} \left( \frac{\partial z}{\partial g_2}(0, 0, 0)(g_2) \right) (z(0, 0, 0) - z_d) dt da dx \quad (40)$$

$$= \int_Q g_2(a) z(t, a, x; 0, 0, 0) \xi_2(0)(t, 0, x) dt da dx. \quad (41)$$

Using (30), (34), (35), (39) and (40), (25) follows, with (26) and (27). This completes the proof.  $\blacksquare$

Proceeding as for the the proof of Proposition 2.2, we have the following result.

**Lemma 3.1** *Let  $(\xi_1(v), \eta_1(v))$  and  $(\xi_2(v), \eta_2(v))$  be respectively solutions of problems (28) and (29). Then the applications  $v \mapsto (\xi_1(v), \eta_1(v))$  and  $v \mapsto (\xi_2(v), \eta_2(v))$  are continuous from  $L^2(Q_\omega)$  onto  $[L^2(Q)]^2$ . Moreover the applications  $v \mapsto (\xi_1(v)(\cdot, 0, \cdot), \eta_1(v)(\cdot, 0, \cdot))$  and  $v \mapsto (\xi_2(v)(\cdot, 0, \cdot), \eta_2(v)(\cdot, 0, \cdot))$  are also continuous from  $L^2(Q_\omega)$  onto  $[L^2(Q_T)]^2$ .*

**Proposition 3.3** *Let  $S_1(\cdot; v)$  and  $S_2(\cdot; v)$  be the functions defined respectively in (26) and (27). Then the maps  $v \mapsto S_1(\cdot; v)$  and  $v \mapsto S_2(\cdot; v)$  are continuous from  $L^2(Q_\omega)$  onto  $L^2(0, A)$ .*

**Proof.** We write the proof for  $S_1(\cdot; v)$ , the proof for  $S_2(\cdot; v)$  follows by using the same arguments. Let  $v_1, v_2 \in L^2(Q_\omega)$ . Then in view of (26),

$$\begin{aligned} S_1(a; v_1) - S_1(a; v_2) & = \int_{Q_T} (y(t, a, x; v_1, 0, 0) - y(t, a, x; v_2, 0, 0)) \xi_1(v_1)(t, 0, x) dt dx \\ & + \int_{Q_T} y(t, a, x; v_2, 0, 0) (\xi_1(v_1)(t, 0, x) - \xi_1(v_2)(t, 0, x)) dt dx. \end{aligned}$$

Using the Cauchy Schwartz inequality, we have

$$|S_1(a; v_1) - S_1(a; v_2)| \leq \|y(\cdot, a, \cdot; v_1, 0, 0) - y(\cdot, a, \cdot; v_2, 0, 0)\|_{L^2(Q_T)} \|\xi_1(v_1)(\cdot, 0, \cdot)\|_{L^2(Q_T)} \\ + \|y(\cdot, a, \cdot; v_2, 0, 0)\|_{L^2(Q_T)} \|\xi(v_1)(\cdot, 0, \cdot) - \xi(v_2)(\cdot, 0, \cdot)\|_{L^2(Q_T)}.$$

Hence,

$$\int_0^A |S_1(a; v_1) - S_1(a; v_2)|^2 da \leq \\ 2\|y(\cdot, a, \cdot; v_1, 0, 0) - y(\cdot, a, \cdot; v_2, 0, 0)\|_{L^2(Q)}^2 \|\xi(v_1)(\cdot, 0, \cdot)\|_{L^2(Q_T)}^2 + \\ 2\|y(\cdot, a, \cdot; v_2, 0, 0)\|_{L^2(Q)}^2 \|\xi(v_1)(\cdot, 0, \cdot) - \xi(v_2)(\cdot, 0, \cdot)\|_{L^2(Q_T)}^2.$$

It then follows from Proposition 2.2 and Lemma 3.1 that  $S_1(\cdot, v_1) \rightarrow S_1(\cdot, v_2)$  as  $v_1 \rightarrow v_2$ . This completes the proof. ■

The following Lemma will be useful to prove the existence of the no-regret and low-regret controls.

**Lemma 3.2** *Let  $S_1(\cdot, v)$  and  $S_2(\cdot, v)$  be respectively defined as in (26) and (27) for any  $a \in L^2(0, A)$ . For any  $\gamma > 0$ , we consider the sequences  $(y^\gamma, z^\gamma) = (y(t, a, x; u^\gamma, 0, 0), z(t, a, x; u^\gamma, 0, 0))$ ,  $(\xi_1(u^\gamma), \eta_1(u^\gamma))$  and  $(\xi_2(u^\gamma), \eta_2(u^\gamma))$ , respectively, solutions of (23), (28) and (29) with  $v = u^\gamma$ . Assume that there exists  $C_1, C_2 > 0$  independent of  $\gamma$  such that*

$$\|S_1(\cdot, u^\gamma)\|_{L^2(0, A)} < C_1 \quad \text{and} \quad \|S_2(\cdot, u^\gamma)\|_{L^2(0, A)} < C_2$$

Assume also that  $(\hat{y}, \hat{z}) = (y(t, a, x; \hat{u}, 0, 0), y(t, a, x; \hat{u}, 0, 0)) \in [L^2(U; H_k^1(\Omega))]^2$  solution of (23), and  $\hat{u} \in L^2(Q_\omega)$ ,  $\hat{\xi}_1(\cdot, 0, \cdot)$ ,  $\hat{\xi}_2(\cdot, 0, \cdot)$ ,  $\hat{\eta}_1(\cdot, 0, \cdot)$ ,  $\hat{\eta}_2(\cdot, 0, \cdot) \in L^2(Q_T)$  are such that

$$u^\gamma \rightharpoonup \hat{u} \text{ weakly in } L^2(Q_\omega), \quad (42a)$$

$$(y^\gamma, z^\gamma) \rightharpoonup (\hat{y}, \hat{z}) \text{ weakly in } [L^2(U, H_k^1(\Omega))]^2, \quad (42b)$$

$$\xi_1(u^\gamma)(\cdot, 0, \cdot) \rightharpoonup \hat{\xi}_1(\cdot, 0, \cdot) \text{ weakly in } L^2(Q_T), \quad (42c)$$

$$\eta_2(u^\gamma)(\cdot, 0, \cdot) \rightharpoonup \hat{\eta}_2(\cdot, 0, \cdot) \text{ weakly in } L^2(Q_T). \quad (42d)$$

Then we have

$$S_1(a; u^\gamma) \rightharpoonup S_1(a; \hat{u}) \text{ weakly in } \mathcal{D}'(0, A),$$

and

$$S_2(a; u^\gamma) \rightharpoonup S_2(a; \hat{u}) \text{ weakly in } \mathcal{D}'(0, A).$$

**Proof.** Let  $\mathcal{D}((0, A))$  be the set of  $C^\infty$  function with compact support on  $(0, A)$ . Set for any  $\phi \in \mathcal{D}((0, A))$

$$\tilde{y}^\gamma(t, x) = \int_0^A y(t, a, x; u^\gamma, 0, 0) \phi(a) da, \quad (t, x) \in Q_T, \quad (43a)$$

$$\tilde{z}^\gamma(t, x) = \int_0^A z(t, a, x; u^\gamma, 0, 0) \phi(a) da, \quad (t, x) \in Q_T. \quad (43b)$$

Then, in view of (42b), there exist two constants  $C_1, C_2 > 0$  independent of  $\gamma$  such that

$$\|\tilde{y}^\gamma\|_{L^2(Q_T)} \leq \|y^\gamma\|_{L^2(Q)} \|\phi\|_{L^2(0, A)} \leq C_1, \\ \|\tilde{z}^\gamma\|_{L^2(Q_T)} \leq \|z^\gamma\|_{L^2(Q)} \|\phi\|_{L^2(0, A)} \leq C_2.$$

Consequently, there exist  $\tilde{y}, \tilde{z} \in L^2(Q_T)$  such that

$$\tilde{y}^\gamma \rightharpoonup \tilde{y} \text{ weakly in } L^2(Q_T), \quad (44a)$$

$$\tilde{z}^\gamma \rightharpoonup \tilde{z} \text{ weakly in } L^2(Q_T). \quad (44b)$$

Moreover, using (43a) and (42b), we deduce that

$$\lim_{\gamma \rightarrow 0} \int_{Q_T} \tilde{y}^\gamma(t, x) \psi(t, x) dt dx = \int_{Q_T} \int_0^A y(t, a, x; \hat{u}, 0) \phi(a) \psi(t, x) dx dt da, \quad \forall \psi \in \mathcal{D}(Q_T).$$

This means that

$$\tilde{y}^\gamma \rightharpoonup \int_0^A y(t, a, x; \hat{u}, 0) \phi(a) da \text{ weakly in } \mathcal{D}'(Q_T).$$

It follows from (43a) and the uniqueness of the limit that,

$$\tilde{y}(t, x) = \int_0^A y(t, a, x; \hat{u}, 0) \phi(a) da, \quad (t, x) \in Q_T. \quad (45)$$

Arguing as the same for  $\tilde{z}^\gamma$  while using (43b) and (42b), we obtain that

$$\tilde{z}(t, x) = \int_0^A z(t, a, x; \hat{u}, 0) \phi(a) da, \quad (t, x) \in Q_T. \quad (46)$$

Because  $(y^\gamma, z^\gamma) = (y(t, a, x; u^\gamma, 0, 0), z(t, a, x; u^\gamma, 0, 0))$  solves (23) with  $v = u^\gamma$ , we have that  $(\tilde{y}^\gamma, \tilde{z}^\gamma)$  solves

$$\begin{cases} \tilde{y}_t^\gamma - (k(x)\tilde{y}_x^\gamma)_x & = k_1^\gamma & \text{in } Q_T, \\ \tilde{z}_t^\gamma - (k(x)\tilde{z}_x^\gamma)_x & = k_2^\gamma & \text{in } Q_T, \\ \tilde{y}^\gamma = \tilde{z}^\gamma & = 0 & \text{on } \Sigma, \\ \tilde{y}^\gamma(0) = \int_0^A y^0(a, x) \phi(a) da, \quad \tilde{z}^\gamma(0) & = \int_0^A z^0(a, x) \phi(a) da & \text{in } \Omega, \end{cases}$$

where

$$k_1^\gamma(t, x) = \int_0^A (f + u^\gamma \chi_\omega) \phi da - \int_0^A \mu_1(a) y^\gamma \phi da - \int_0^A \frac{\partial y^\gamma}{\partial a} \phi da,$$

and

$$k_2^\gamma(t, x) = \int_0^A y^\gamma \chi_\phi \phi da - \int_0^A \mu_2(a) z^\gamma \phi da - \int_0^A \frac{\partial z^\gamma}{\partial a} \phi da.$$

Consequently, in view of (42a) and (42b), there exist two positive constants  $C_1$  and  $C_2$  independent of  $\gamma$  such that

$$\begin{aligned} \|k_1^\gamma\|_{L^2(Q_T)} &\leq \left( 2\|f\|_{L^2(Q)}^2 + 2\|u^\gamma\|_{L^2(Q_\omega)}^2 + \|\mu_1\|_{L^\infty(0,A)}^2 \|y^\gamma\|_{L^2(Q)}^2 \right)^{1/2} \|\phi\|_{L^2(0,A)} + \\ \|y^\gamma\|_{L^2(Q)} \left\| \frac{\partial \phi}{\partial a} \right\|_{L^2(0,A)} &\leq C_1, \end{aligned}$$

and

$$\begin{aligned} \|k_2^\gamma\|_{L^2(Q_T)} &\leq \left( 2\|y^\gamma\|_{L^2(Q)}^2 + \|\mu_2\|_{L^\infty(0,A)}^2 \|z^\gamma\|_{L^2(Q)}^2 \right)^{1/2} \|\phi\|_{L^2(0,A)} + \\ \|z^\gamma\|_{L^2(Q)} \left\| \frac{\partial \phi}{\partial a} \right\|_{L^2(0,A)} &\leq C_2. \end{aligned}$$

It then follows that there is  $C_1, C_2 > 0$ , independent of  $\gamma$ , such that

$$\begin{cases} \|\tilde{y}^\gamma\|_{L^2((0,T);H_k^1(\Omega))} &\leq C_1, \\ \|\tilde{y}_t^\gamma\|_{L^2((0,T);H_k^{-1}(\Omega))} &\leq C_1, \end{cases}$$

and

$$\begin{cases} \|\tilde{z}^\gamma\|_{L^2((0,T);H_k^1(\Omega))} &\leq C_2, \\ \|\tilde{z}_t^\gamma\|_{L^2((0,T);H_k^{-1}(\Omega))} &\leq C_2. \end{cases}$$

Therefore, it follows from Remark 1 and Aubin-Lions's Lemma that

$$\tilde{y}^\gamma \rightarrow \tilde{y} \text{ strongly in } L^2(Q_T) \quad \text{and} \quad \tilde{z}^\gamma \rightarrow \tilde{z} \text{ strongly in } L^2(Q_T) \quad (47)$$

where for  $(t, x) \in Q_T$

$$\tilde{y}(t, x) = \int_0^A y(t, a, x; \hat{u}, 0) \phi(a) da \quad \text{and} \quad \tilde{z}(t, x) = \int_0^A z(t, a, x; \hat{u}, 0) \phi(a) da,$$

because of (45) and (46).

Now in view of (26)

$$S_1(a; u^\gamma) = \int_{Q_T} [y(t, a, x; u^\gamma, 0, 0) \xi_1(u^\gamma)(t, 0, x) - y(t, a, x; 0, 0, 0) \xi_1(0)(t, 0, x)] dt dx.$$

Therefore using (43a),

$$\begin{aligned} \int_0^A S_1(a; u^\gamma) \phi(a) da &= \int_{Q_T} \int_0^A (y(t, a, x; u^\gamma, 0, 0) \phi(a) da) \xi_1(u^\gamma)(t, 0, x) dt da dx \\ &\quad - \int_Q y(t, a, x; 0, 0, 0) \xi_1(0)(t, 0, x) \phi(a) dt da dx \\ &= \int_{Q_T} \tilde{y}^\gamma(t, x) \xi_1(u^\gamma)(t, 0, x) dt da dx \\ &\quad - \int_Q y(t, a, x; 0, 0, 0) \xi_1(0)(t, 0, x) \phi(a) dt da dx, \quad \forall \phi \in \mathcal{D}(0, A) \end{aligned}$$

Passing this latter identity to the limit while using (47), (45) and (42c), we obtain

$$\begin{aligned} \int_0^A S_1(a; u^\gamma) \phi(a) da &\rightarrow \int_{Q_T} \tilde{y}(t, x) \xi_1(\hat{u})(t, 0, x) dt da dx \\ &\quad - \int_Q y(t, a, x; 0, 0, 0) \xi_1(0)(t, 0, x) \phi(a) dt da dx \\ &= \int_{Q_T} \int_0^A (y(t, a, x; \hat{u}, 0, 0) \phi(a) da) \xi_1(\hat{u})(t, 0, x) dt da dx \\ &\quad - \int_Q y(t, a, x; 0, 0, 0) \xi_1(0)(t, 0, x) \phi(a) dt da dx \quad \forall \phi \in \mathcal{D}(0, A), \end{aligned}$$

which in view of (26), proves

$$S_1(a; u^\gamma) \rightharpoonup S_1(a; \hat{u}) \text{ weakly in } \mathcal{D}'(0, A).$$

Proceeding as above while using (43b), (47), (46) and (42d), we obtain that

$$S_2(a; u^\gamma) \rightharpoonup S_2(a; \hat{u}) \text{ weakly in } \mathcal{D}'(0, A).$$

This completes the proof.  $\blacksquare$

From now on, we denote by  $\mathcal{D}(\Theta)$  the set of  $\mathcal{C}^\infty$  function with compact support on  $\Theta$  and by  $\mathcal{D}'(\Theta)$ , its dual.

### 3.1 Existence of No-regret control and Low-regret control

In view of (25), the optimization problem (22) is equivalent to the following problem:

$$\inf_{v \in L^2(Q_\omega)} \sup_{(g_1, g_2) \in [L^2(0, A)]^2} \left[ J(v, 0, 0) - J(0, 0, 0) + 2 \int_0^A S_1(a; v) g_1(a) da + 2 \int_0^A S_2(a; v) g_2(a) da \right]. \quad (48)$$

As  $\int_0^A S_i(a; v)g_i(a)da$  is either equal to 0 or  $+\infty$ ,  $i = 1, 2$ , we look for the control  $v$  in the set,

$$\mathcal{M} = \left\{ v \in L^2(Q_\omega); \int_0^A S_i(a; v)g_i(a)da = 0, \quad \forall g_i \in L^2(0, A), i = 1, 2 \right\}. \quad (49)$$

**Lemma 3.3** *The problem (48) has a solution  $\tilde{u}$  in  $\mathcal{M}$ .*

**Proof.** It is clear that the set  $\mathcal{M}$  is strongly closed in  $L^2(Q_\omega)$ . On the one hand the application  $v \mapsto J(v, 0, 0) - J(0, 0, 0)$  is coercive on  $L^2(Q_\omega)$ , bounded below by  $-J(0, 0, 0)$ , and continuous because of Proposition 2.2. On the other hand, since the applications  $v \mapsto S_i(\cdot; v)$ ,  $i = 1, 2$  are continuous on  $L^2(Q_\omega)$ , then using minimizing sequences and Lemma 3.2 there exists a no-regret control  $\tilde{u}$  in  $\mathcal{M}$  satisfying (48). ■

In order to characterize the no regret control  $\tilde{u}$ , we consider for any  $\gamma > 0$ , *low-regret-control problem*:

$$\inf_{v \in L^2(Q_\omega)} \sup_{(g_1, g_2) \in [L^2(0, A)]^2} \left[ J(v, 0, 0) - J(0, 0, 0) + 2 \int_0^A S_1(a; v)g_1(a)da + 2 \int_0^A S_2(a; v)g_2(a)da - \gamma \|g_1\|_{L^2(0, A)}^2 - \gamma \|g_2\|_{L^2(0, A)}^2 \right]. \quad (50)$$

Using Fenchel-Legendre transform (see [4]), we obtain that,

$$\begin{aligned} & \sup_{(g_1, g_2) \in [L^2(0, A)]^2} \left[ J(v, 0, 0) - J(0, 0, 0) + 2 \int_0^A S_1(a; v)g_1(a)da + 2 \int_0^A S_2(a; v)g_2(a)da - \gamma \|g_1\|_{L^2(0, A)}^2 - \gamma \|g_2\|_{L^2(0, A)}^2 \right] \\ &= J(v, 0, 0) - J(0, 0, 0) + 2\gamma \sup_{g_1 \in L^2(0, A)} \left[ \int_0^A \frac{S_1(a; v)}{\gamma} g_1(a)da - \frac{1}{2} \|g_1\|_{L^2(0, A)}^2 \right] \\ & \quad + 2\gamma \sup_{g_2 \in L^2(0, A)} \left[ \int_0^A \frac{S_2(a; v)}{\gamma} g_2(a)da - \frac{1}{2} \|g_2\|_{L^2(0, A)}^2 \right] \\ &= J(v, 0, 0) - J(0, 0, 0) + \frac{1}{\gamma} \|S_1(\cdot; v)\|_{L^2(0, A)}^2 + \frac{1}{\gamma} \|S_2(\cdot; v)\|_{L^2(0, A)}^2, \end{aligned}$$

and (50) is reduced to

$$\inf_{v \in L^2(Q_\omega)} \mathcal{J}^\gamma(v), \quad (51)$$

where

$$\mathcal{J}^\gamma(v) = J(v, 0, 0) - J(0, 0, 0) + \frac{1}{\gamma} \|S_1(\cdot; v)\|_{L^2(0, A)}^2 + \frac{1}{\gamma} \|S_2(\cdot; v)\|_{L^2(0, A)}^2. \quad (52)$$

**Remark 3** *As  $\mathcal{J}^\gamma(v) \geq -J(0, 0, 0)$  and  $\mathcal{J}^\gamma(0) = 0$ . Using minimizing sequences, Proposition 2.2, Proposition 3.3 and Lemma 3.2, we can prove as for Lemma 3.3 that problem (51) has at least one solution  $u_\gamma \in L^2(Q_\omega)$ . But, as the applications  $v \mapsto S_i(\cdot; v)$ ,  $i = 1, 2$  from  $L^2(Q_\omega)$  to  $L^2(0, A)$  are not necessarily strictly convex, the uniqueness of  $u_\gamma \in L^2(Q_\omega)$ , solution of (51) is not guaranteed. So, we are not sure that control low-regret-control  $u_\gamma$  will converge to a no-regret control  $\tilde{u} \in \mathcal{M}$ . Therefore, in order to have a low-regret control which will converge in  $\mathcal{M}$ , we adapt the cost function  $\mathcal{J}^\gamma$  to a no-regret control  $\tilde{u}$ .*

### 3.2 Existence of the adapted low-regret control

Let  $\tilde{u}$  be a no-regret optimal control. For any  $\gamma > 0$ , we define the adapted cost function  $\tilde{\mathcal{J}}^\gamma$  by:

$$\begin{aligned} v \mapsto \tilde{\mathcal{J}}^\gamma(v) &= J(v, 0, 0) - J(0, 0, 0) + \|v - \tilde{u}\|_{L^2(Q_\omega)}^2 \\ &\quad + \frac{1}{\gamma} \|S_1(\cdot; v)\|_{L^2(0, A)}^2 + \frac{1}{\gamma} \|S_2(\cdot; v)\|_{L^2(0, A)}^2. \end{aligned} \quad (53)$$

Then, we consider the following optimal control problem:

$$\inf_{v \in L^2(Q_\omega)} \tilde{\mathcal{J}}^\gamma(v). \quad (54)$$

**Proposition 3.4** *Let  $\gamma > 0$ . Then problem (54) has at least a solution  $\tilde{u}_\gamma$  in  $L^2(Q_\omega)$ .*

**Proof.** We have  $\tilde{\mathcal{J}}^\gamma(v) \geq -J(0, 0, 0)$  and  $\tilde{\mathcal{J}}^\gamma(0) = \|\tilde{u}\|_{L^2(Q_\omega)}^2$ . Using minimizing sequences, Proposition 2.2, Proposition 3.3, Lemma 3.2 and the fact that  $\lim_{\|v\|_{L^2(Q_\omega)} \rightarrow +\infty} \tilde{\mathcal{J}}^\gamma(v) = +\infty$ , we prove as for

Lemma 3.3 that problem (54) has at least a solution  $\tilde{u}_\gamma$  in  $L^2(Q_\omega)$ . ■

**Proposition 3.5** *Let  $\tilde{u}_\gamma \in L^2(Q_\omega)$  be a solution of (54). Then there exist  $(\tilde{p}_\gamma^1, \tilde{p}_\gamma^2) = (p_1(\tilde{u}_\gamma), p_2(\tilde{u}_\gamma)) \in [L^2(U; H_k^1(\Omega))]^2$ ,  $(\tilde{q}_\gamma^1, \tilde{q}_\gamma^2) = (q_1(\tilde{u}_\gamma), q_2(\tilde{u}_\gamma)) \in [L^2(U; H_k^1(\Omega))]^2$  and  $(\tilde{\varphi}_\gamma^1, \tilde{\varphi}_\gamma^2) = (\varphi_1(\tilde{u}_\gamma), \varphi_2(\tilde{u}_\gamma)) \in [L^2(U; H_k^1(\Omega))]^2$  such that  $\{(\tilde{y}_\gamma, \tilde{z}_\gamma), (\tilde{\xi}_\gamma^1, \tilde{\eta}_\gamma^1), (\tilde{\xi}_\gamma^2, \tilde{\eta}_\gamma^2), (\tilde{p}_\gamma^1, \tilde{p}_\gamma^2), (\tilde{q}_\gamma^1, \tilde{q}_\gamma^2), (\tilde{\varphi}_\gamma^1, \tilde{\varphi}_\gamma^2)\}$  is a solution of the systems:*

$$\begin{cases} L_1 \tilde{y}_\gamma &= f + \tilde{u}_\gamma \chi_\omega & \text{in } Q, \\ L_2 \tilde{z}_\gamma &= \tilde{y}_\gamma \chi_\mathcal{O} & \text{in } Q, \\ \tilde{y}_\gamma = \tilde{z}_\gamma &= 0 & \text{on } \Sigma, \\ \tilde{y}_\gamma(0, a, x) = y_0(a, x), \quad \tilde{z}_\gamma(0, a, x) &= z_0(a, x) & \text{in } Q_A, \\ \tilde{y}_\gamma(t, 0, x) = 0, \quad \tilde{z}_\gamma(t, 0, x) &= 0 & \text{in } Q_T, \end{cases} \quad (55)$$

$$\begin{cases} L_1^* \tilde{\xi}_\gamma^1 &= \tilde{\eta}_\gamma^1 \chi_\mathcal{O} & \text{in } Q, \\ L_2^* \tilde{\eta}_\gamma^1 &= (\tilde{z}_\gamma - z_d) \chi_\mathcal{O} & \text{in } Q, \\ \tilde{\xi}_\gamma^1 = \tilde{\eta}_\gamma^1 &= 0 & \text{on } \Sigma, \\ \tilde{\xi}_\gamma^1(T, \cdot, \cdot) = 0, \quad \tilde{\eta}_\gamma^1(T, \cdot, \cdot) &= 0 & \text{in } Q_A, \\ \tilde{\xi}_\gamma^1(\cdot, A, \cdot) = 0, \quad \tilde{\eta}_\gamma^1(\cdot, A, \cdot) &= 0 & \text{in } Q_T, \end{cases} \quad (56)$$

$$\begin{cases} L_1^* \tilde{\xi}_\gamma^2 &= \tilde{\eta}_\gamma^2 \chi_\mathcal{O} & \text{in } Q, \\ L_2^* \tilde{\eta}_\gamma^2 &= (\tilde{z}_\gamma - z_d) \chi_\mathcal{O} & \text{in } Q, \\ \tilde{\xi}_\gamma^2 = \tilde{\eta}_\gamma^2 &= 0 & \text{on } \Sigma, \\ \tilde{\xi}_\gamma^2(T, \cdot, \cdot) = 0, \quad \tilde{\eta}_\gamma^2(T, \cdot, \cdot) &= 0 & \text{in } Q_A, \\ \tilde{\xi}_\gamma^2(\cdot, A, \cdot) = 0, \quad \tilde{\eta}_\gamma^2(\cdot, A, \cdot) &= 0 & \text{in } Q_T, \end{cases} \quad (57)$$

$$\begin{cases} L_1^* \tilde{q}_\gamma^1 &= \tilde{q}_\gamma^2 \chi_\mathcal{O} & \text{in } Q, \\ L_2^* \tilde{q}_\gamma^2 &= (\tilde{z}_\gamma - z_d) \chi_\mathcal{O} + \varrho_\gamma & \text{in } Q, \\ \tilde{q}_\gamma^1 = \tilde{q}_\gamma^2 &= 0 & \text{on } \Sigma, \\ \tilde{q}_\gamma^1(T, \cdot, \cdot) = 0, \quad \tilde{q}_\gamma^2(T, \cdot, \cdot) &= 0 & \text{in } Q_A, \\ \tilde{q}_\gamma^1(\cdot, A, \cdot) = 0, \quad \tilde{q}_\gamma^2(\cdot, A, \cdot) &= 0 & \text{in } Q_T, \end{cases} \quad (58)$$

$$\begin{cases} L_1 \tilde{p}_\gamma^1 &= 0 & \text{in } Q, \\ L_2 \tilde{p}_\gamma^2 &= \tilde{p}_\gamma^1 \chi_\mathcal{O} & \text{in } Q, \\ \tilde{p}_\gamma^1 = \tilde{p}_\gamma^2 &= 0 & \text{on } \Sigma, \\ \tilde{p}_\gamma^1(0, \cdot, \cdot) = 0, \quad \tilde{p}_\gamma^2(0, \cdot, \cdot) &= 0 & \text{in } Q_A, \\ \tilde{p}_\gamma^1(\cdot, 0, \cdot) = \frac{1}{\sqrt{\gamma}} \int_0^A z(\tilde{u}_\gamma, 0, 0) S_1(a, \tilde{u}_\gamma) da, \quad \tilde{p}_\gamma^2(\cdot, 0, \cdot) &= 0 & \text{in } Q_T, \end{cases} \quad (59)$$

$$\begin{cases} L_1 \tilde{\varphi}_\gamma^1 & = 0 & \text{in } Q, \\ L_2 \tilde{\varphi}_\gamma^2 & = \tilde{\varphi}_\gamma^1 \chi_{\mathcal{O}} & \text{in } Q, \\ \tilde{\varphi}_\gamma^1 = \tilde{\varphi}_\gamma^2 & = 0 & \text{on } \Sigma, \\ \tilde{\varphi}_\gamma^1(0, \cdot, \cdot) = 0, \quad \tilde{\varphi}_\gamma^2(0, \cdot, \cdot) & = 0 & \text{in } Q_A, \\ \tilde{\varphi}_\gamma^1(\cdot, 0, \cdot) = 0, \quad \tilde{\varphi}_\gamma^2(\cdot, 0, \cdot) & = \frac{1}{\sqrt{\gamma}} \int_0^A z(\tilde{u}_\gamma, 0, 0) S_2(a, \tilde{u}_\gamma) da & \text{in } Q_T, \end{cases} \quad (60)$$

and

$$(N+1)\tilde{u}_\gamma - \tilde{u} + \tilde{q}_\gamma^1 = 0 \quad \text{in } Q_\omega, \quad (61)$$

where  $\varrho_\gamma = \frac{1}{\sqrt{\gamma}} \tilde{p}_\gamma^2 \chi_{\mathcal{O}} + \frac{1}{\sqrt{\gamma}} \tilde{\varphi}_\gamma^2 \chi_{\mathcal{O}} + \frac{1}{\gamma} \xi_1(\tilde{u}_\gamma)(t, 0, x) S_1(a; \tilde{u}_\gamma) + \frac{1}{\gamma} \eta_2(\tilde{u}_\gamma)(t, 0, x) S_2(a, \tilde{u}_\gamma)$ ,  $(\tilde{y}_\gamma, \tilde{z}_\gamma) = (y(\tilde{u}_\gamma, 0, 0), z(\tilde{u}_\gamma, 0, 0))$ , and  $(\tilde{\xi}_\gamma^i, \tilde{\eta}_\gamma^i) = (\xi_i(\tilde{u}_\gamma), \eta_i(\tilde{u}_\gamma))$ ,  $i = 1, 2$ .

**Proof.** We write the Euler-Lagrange optimality condition that characterizes  $\tilde{u}_\gamma$ :

$$\lim_{\lambda \rightarrow 0} \frac{\tilde{\mathcal{J}}^\gamma(\tilde{u}_\gamma + \lambda w) - \tilde{\mathcal{J}}^\gamma(\tilde{u}_\gamma)}{\lambda} = 0, \quad \forall w \in L^2(Q_\omega).$$

Using Proposition 2.3 and Proposition 3.2, we obtain after some calculations

$$\begin{aligned} 0 &= \int_U \int_{\mathcal{O}} \frac{\partial z}{\partial v}(\tilde{u}_\gamma, 0, 0)(w) (z(\tilde{u}_\gamma, 0, 0) - z_d) dt da dx + \int_{Q_\omega} (\tilde{u}_\gamma - \tilde{u}) w dt da dx \\ &+ \int_{Q_\omega} N \tilde{u}_\gamma w dt da dx + \frac{1}{\gamma} \int_Q \frac{\partial z}{\partial v}(\tilde{u}_\gamma, 0, 0)(w) \xi_1(\tilde{u}_\gamma)(t, 0, x) S_1(a, \tilde{u}_\gamma) dt da dx \\ &+ \frac{1}{\gamma} \int_Q \frac{\partial \xi_1}{\partial v}(\tilde{u}_\gamma)(w)(t, 0, x) z(\tilde{u}_\gamma, 0, 0) S_1(a; \tilde{u}_\gamma) dt da dx \\ &+ \frac{1}{\gamma} \int_Q \frac{\partial z}{\partial v}(\tilde{u}_\gamma, 0, 0)(w) \eta_2(\tilde{u}_\gamma)(t, 0, x) S_2(a, \tilde{u}_\gamma) dt da dx \\ &+ \frac{1}{\gamma} \int_Q \frac{\partial \eta_2}{\partial v}(\tilde{u}_\gamma)(w)(t, 0, x) z(\tilde{u}_\gamma, 0, 0) S_2(a; \tilde{u}_\gamma) dt da dx, \quad \forall w \in L^2(Q_\omega), \end{aligned} \quad (62)$$

where

$$\begin{aligned} (\bar{y}(w), \bar{z}(w)) &= \left( \frac{\partial y}{\partial v}(\tilde{u}_\gamma, 0, 0)(w), \frac{\partial z}{\partial v}(\tilde{u}_\gamma, 0, 0)(w) \right), \quad (\bar{\xi}_1(w), \bar{\eta}_1(w)) = \left( \frac{\partial \xi_1}{\partial v}(\tilde{u}_\gamma)(w), \frac{\partial \eta_1}{\partial v}(\tilde{u}_\gamma)(w) \right) \text{ and} \\ (\bar{\xi}_2(w), \bar{\eta}_2(w)) &= \left( \frac{\partial \xi_2}{\partial v}(\tilde{u}_\gamma)(w), \frac{\partial \eta_2}{\partial v}(\tilde{u}_\gamma)(w) \right) \text{ are respectively solutions to} \end{aligned}$$

$$\begin{cases} L_1 \bar{y}(w) & = w \chi_\omega & \text{in } Q, \\ L_2 \bar{z}(w) & = \bar{y}(w) \chi_{\mathcal{O}} & \text{in } Q, \\ \bar{y}(w) = \bar{z}(w) & = 0 & \text{on } \Sigma, \\ \bar{y}(w)(0, \cdot, \cdot) = 0, \quad \bar{z}(w)(0, \cdot, \cdot) & = 0 & \text{in } Q_A, \\ \bar{y}(w)(\cdot, 0, \cdot) = 0 \quad \bar{z}(w)(\cdot, 0, \cdot) & = 0 & \text{in } Q_T, \end{cases} \quad (63)$$

$$\begin{cases} L_1^* \bar{\xi}_1(w) & = \bar{\eta}_1(w) \chi_{\mathcal{O}} & \text{in } Q, \\ L_2^* \bar{\eta}_1(w) & = \bar{z}(w) \chi_{\mathcal{O}} & \text{in } Q, \\ \bar{\xi}_1(w) = \bar{\eta}_1(w) & = 0 & \text{on } \Sigma, \\ \bar{\xi}_1(w)(T, \cdot, \cdot) = 0, \quad \bar{\eta}_1(w)(T, \cdot, \cdot) & = 0 & \text{in } Q_A, \\ \bar{\xi}_1(w)(\cdot, A, \cdot) = 0, \quad \bar{\eta}_1(w)(\cdot, A, \cdot) & = 0 & \text{in } Q_T, \end{cases} \quad (64)$$

and

$$\begin{cases} L_1^* \bar{\xi}_2(w) & = \bar{\eta}_2(w) \chi_{\mathcal{O}} & \text{in } Q, \\ L_2^* \bar{\eta}_2(w) & = \bar{z}(w) \chi_{\mathcal{O}} & \text{in } Q, \\ \bar{\xi}_2(w) = \bar{\eta}_2(w) & = 0 & \text{on } \Sigma, \\ \bar{\xi}_2(w)(T, \cdot, \cdot) = 0, \quad \bar{\eta}_2(w)(T, \cdot, \cdot) & = 0 & \text{in } Q_A, \\ \bar{\xi}_2(w)(\cdot, A, \cdot) = 0, \quad \bar{\eta}_2(w)(\cdot, A, \cdot) & = 0 & \text{in } Q_T. \end{cases} \quad (65)$$



To interpret (62), we use the adjoint states  $(\tilde{p}_\gamma^1, \tilde{p}_\gamma^2)$ ,  $(\tilde{q}_\gamma^1, \tilde{q}_\gamma^2)$ ,  $(\tilde{\varphi}_\gamma^1, \tilde{\varphi}_\gamma^2)$ . So if we multiply the first and the second equation of (63) respectively by  $\tilde{q}_\gamma^1$  and  $\tilde{q}_\gamma^2$  solutions of (58) and we combine the result, we multiply the first and the second equation of (64) respectively by  $\frac{1}{\sqrt{\gamma}}\tilde{p}_\gamma^1$  and  $\frac{1}{\sqrt{\gamma}}\tilde{p}_\gamma^2$  solutions of (59), we multiply the first and the second equation of (65) respectively by  $\frac{1}{\sqrt{\gamma}}\tilde{\varphi}_\gamma^1$  and  $\frac{1}{\sqrt{\gamma}}\tilde{\varphi}_\gamma^2$  solutions of (60), then integrate by parts over  $Q$  and combining the result, we obtain respectively

$$\int_Q L_1^* \tilde{q}_\gamma^1 \bar{y}(w) dt da dx + \int_Q L_2^* \tilde{q}_\gamma^2 \bar{z}(w) dt da dx = \int_{Q_\omega} \tilde{q}_\gamma^1 w dt da dx + \int_U \int_{\mathcal{O}} \bar{y}(w) \tilde{q}_\gamma^2 dt da dx,$$

$$\begin{aligned} & \frac{1}{\sqrt{\gamma}} \int_{Q_T} \bar{\xi}_1(w)(t, 0, x) \tilde{p}_\gamma^1(t, 0, x) dt dx + \frac{1}{\sqrt{\gamma}} \int_{Q_T} \bar{\eta}_1(w)(t, 0, x) \tilde{p}_\gamma^2(t, 0, x) dt dx \\ & + \frac{1}{\sqrt{\gamma}} \int_Q L_2 \tilde{p}_\gamma^2 \bar{\eta}_1(w) = \frac{1}{\sqrt{\gamma}} \int_U \int_{\mathcal{O}} \bar{\eta}_1(w) \tilde{p}_\gamma^1 dt da dx + \frac{1}{\sqrt{\gamma}} \int_U \int_{\mathcal{O}} \bar{z}(w) \tilde{p}_\gamma^2 dt da dx, \end{aligned}$$

$$\begin{aligned} & \frac{1}{\sqrt{\gamma}} \int_{Q_T} \bar{\eta}_2(w)(t, 0, x) \tilde{\varphi}_\gamma^2(t, 0, x) dt dx + \frac{1}{\sqrt{\gamma}} \int_Q L_2 \tilde{\varphi}_\gamma^2 \bar{\eta}_2(w) dt da dx \\ & = \frac{1}{\sqrt{\gamma}} \int_U \int_{\mathcal{O}} \bar{\eta}_2(w) \tilde{\varphi}_\gamma^1 dt da dx + \frac{1}{\sqrt{\gamma}} \int_U \int_{\mathcal{O}} \bar{z}(w) \tilde{\varphi}_\gamma^2 dt da dx. \end{aligned}$$

By combining the three latter equalities, we obtain

$$\int_{Q_\omega} ((N+1)\tilde{u}_\gamma - \tilde{u} + \tilde{q}_\gamma^1) w dt da dx = 0, \quad \forall w \in L^2(Q_\omega),$$

which implies that

$$(N+1)\tilde{u}_\gamma - \tilde{u} + \tilde{q}_\gamma^1 = 0 \text{ in } Q_\omega. \quad (66)$$

This completes the proof. ■

### 3.3 Characterization of the no-regret control

In this section, we are about to characterize the no-regret control. Before to do so, let us state the following result which gives the estimations needed for the characterization.

**Proposition 3.6** *Let  $\tilde{u}_\gamma \in L^2(Q_\omega)$  be a solution of (54). Let also  $(\tilde{y}_\gamma, \tilde{z}_\gamma)$ ,  $(\tilde{\xi}_\gamma^1, \tilde{\eta}_\gamma^1)$ ,  $(\tilde{\xi}_\gamma^2, \tilde{\eta}_\gamma^2)$ ,  $(\tilde{p}_\gamma^1, \tilde{p}_\gamma^2)$ ,*

$(\tilde{q}_\gamma^1, \tilde{q}_\gamma^2)$ ,  $(\tilde{\varphi}_\gamma^1, \tilde{\varphi}_\gamma^2)$  be such that (55)-(61) hold true. Then we have the following estimations: for  $i = 1, 2$

$$\|\tilde{u}_\gamma\|_{L^2(Q_\omega)} \leq C(N, C_0), \quad (67)$$

$$\frac{1}{\sqrt{\gamma}} \|S_i(\cdot; \tilde{u}_\gamma)\|_{L^2(0,A)} \leq C_0, \quad (68)$$

$$\|S_i(\cdot; \tilde{u}_\gamma)\|_{L^2(0,A)} \leq \sqrt{\gamma} C_0, \quad (69)$$

$$\|\tilde{y}_\gamma\|_{L^2(U; H_k^1(\Omega))} \leq C(N, T, C_0), \quad (70)$$

$$\|\tilde{z}_\gamma\|_{L^2(U; H_k^1(\Omega))} \leq C(N, T, C_0), \quad (71)$$

$$\|\tilde{\xi}_\gamma^i\|_{L^2(U; H_k^1(\Omega))} \leq C(T, C_0), \quad (72)$$

$$\|\tilde{\eta}_\gamma^i\|_{L^2(U; H_k^1(\Omega))} \leq C(T, C_0), \quad (73)$$

$$\|\tilde{\eta}_\gamma^2(\cdot, 0, \cdot)\|_{L^2(Q_T)} \leq C(T, C_0), \quad (74)$$

$$\|\tilde{\xi}_\gamma^1(\cdot, 0, \cdot)\|_{L^2(Q_T)} \leq C(T, C_0), \quad (75)$$

$$\|\tilde{p}_\gamma^1(\cdot, 0, \cdot)\|_{L^2(Q_T)} \leq C_0, \quad (76)$$

$$\|\tilde{\varphi}_\gamma^2(\cdot, 0, \cdot)\|_{L^2(Q_T)} \leq C_0, \quad (77)$$

$$\|\tilde{p}_\gamma^i\|_{L^2(U; H_k^1(\Omega))} \leq C(T, C_0), \quad (78)$$

$$\|\tilde{\varphi}_\gamma^i\|_{L^2(U; H_k^1(\Omega))} \leq C(T, C_0), \quad (79)$$

$$\|\tilde{q}_\gamma^i\|_{L^2(U; H_k^1(\Omega))} \leq C(N, T, C_0), \quad (80)$$

where  $C_0 = C(\|\tilde{u}\|_{L^2(Q_\omega)}, \|y_0\|_{L^2(Q_A)}, \|z_0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)})$ .

**Proof.** We proceed in three steps.

**Step 1.** We prove the estimations (67)-(72).

As  $\tilde{u}_\gamma$  is solution of (54), we can write

$$\tilde{\mathcal{J}}^\gamma(\tilde{u}_\gamma) \leq \tilde{\mathcal{J}}^\gamma(0) = \|\tilde{u}\|_{L^2(Q_\omega)}. \quad (81)$$

It then follows from the definition of  $\tilde{\mathcal{J}}^\gamma$  and  $J$  given respectively by (53) and (12) that,

$$\begin{aligned} & \|\tilde{z}_\gamma - z_d\|_{L^2(Q)}^2 + N\|\tilde{u}_\gamma\|_{L^2(Q_\omega)}^2 + \|\tilde{u}_\gamma - \tilde{u}\|_{L^2(Q)}^2 + \frac{1}{\gamma}\|S_1(\cdot; \tilde{u}_\gamma)\|_{L^2(0,A)}^2 \\ & + \frac{1}{\gamma}\|S_2(\cdot; \tilde{u}_\gamma)\|_{L^2(0,A)}^2 \leq \|\tilde{u}\|_{L^2(Q_\omega)}^2 + \|z(0, 0, 0) - z_d\|_{L^2(U \times \mathcal{O})}^2. \end{aligned}$$

Hence we deduce (67), (68), (69) and

$$\|\tilde{z}_\gamma - z_d\|_{L^2(U \times \mathcal{O})} \leq C(\|\tilde{u}\|_{L^2(Q_\omega)}, \|y_0\|_{L^2(Q_A)}, \|z_0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}). \quad (82)$$

Observing that  $(\tilde{y}_\gamma, \tilde{z}_\gamma)$ ,  $(\tilde{\xi}_\gamma^1, \tilde{\eta}_\gamma^1)$ , and  $(\tilde{\xi}_\gamma^2, \tilde{\eta}_\gamma^2)$  are respectively solution of (55), (56) and (57), proceeding as in Proposition 2.1, we obtain that

$$\begin{aligned} & \|\tilde{y}_\gamma\|_{L^2(U; H_k^1(\Omega))} \leq C(\|y_0\|_{L^2(Q_A)} + \|z_0\|_{L^2(Q_A)} + \|f\|_{L^2(Q)} + \|\tilde{u}_\gamma\|_{L^2(Q_\omega)}), \\ & \|\tilde{z}_\gamma\|_{L^2(U; H_k^1(\Omega))} \leq C(\|y_0\|_{L^2(Q_A)} + \|z_0\|_{L^2(Q_A)} + \|f\|_{L^2(Q)} + \|\tilde{u}_\gamma\|_{L^2(Q_\omega)}), \\ & \|\tilde{\xi}_\gamma^1(\cdot, 0, \cdot)\|_{L^2(Q_T)} + \|\tilde{\xi}_\gamma^1\|_{L^2(U; H_k^1(\Omega))} + \|\tilde{\eta}_\gamma^1\|_{L^2(U; H_k^1(\Omega))} \leq C\|\tilde{z}_\gamma - z_d\|_{L^2(U \times \mathcal{O})}, \end{aligned}$$

and

$$\|\tilde{\xi}_\gamma^2\|_{L^2(U; H_k^1(\Omega))} + \|\tilde{\eta}_\gamma^2(\cdot, 0, \cdot)\|_{L^2(Q_T)} + \|\tilde{\eta}_\gamma^2\|_{L^2(U; H_k^1(\Omega))} \leq C\|\tilde{z}_\gamma - z_d\|_{L^2(U \times \mathcal{O})},$$

from which we respectively, deduce (70)-(75) because of (67) and (82).

**Step 2.** We prove the estimations (76) and (77).

We observe that for  $i = 1, 2$

$$\left| \frac{1}{\sqrt{\gamma}} \int_0^A z(t, a, x; \tilde{u}_\gamma, 0, 0) S_i(a; \tilde{u}_\gamma) da \right| \leq \frac{1}{\sqrt{\gamma}} \|S_i(\cdot; \tilde{u}_\gamma)\|_{L^2(0, A)} \left( \int_0^A z(t, a, x; \tilde{u}_\gamma, 0, 0)^2 da \right)^{1/2}.$$

So using (68) and (70), we deduce

$$\int_{Q_T} \left| \frac{1}{\sqrt{\gamma}} \int_0^A z(t, a, x; \tilde{u}_\gamma, 0) S_i(a; \tilde{u}_\gamma) da \right|^2 dt dx \leq \frac{1}{\gamma} \|S_i(\cdot; \tilde{u}_\gamma)\|_{L^2(0, A)}^2 \|\tilde{z}_\gamma\|_{L^2(Q)}^2 \leq C,$$

where  $C = C(\|\tilde{u}\|_{L^2(Q_\omega)}, \|y_0\|_{L^2(Q_A)}, \|z_0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}) > 0$ . This means

$$\|\tilde{p}_\gamma^1(\cdot, 0, \cdot)\|_{L^2(Q_T)} \leq C(\|\tilde{u}\|_{L^2(Q_\omega)}, \|y_0\|_{L^2(Q_A)}, \|z_0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}),$$

and

$$\|\tilde{\varphi}_\gamma^2(\cdot, 0, \cdot)\|_{L^2(Q_T)} \leq C(\|\tilde{u}\|_{L^2(Q_\omega)}, \|y_0\|_{L^2(Q_A)}, \|z_0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}).$$

Since  $(\tilde{p}_\gamma^1, \tilde{p}_\gamma^2)$ ,  $(\tilde{\varphi}_\gamma^1, \tilde{\varphi}_\gamma^2)$  are respectively solution of (59) and (60), proceeding as in Proposition 2.1 while using (76) and (77), we obtain

$$\|\tilde{p}_\gamma^i\|_{L^2(U; H_k^1(\Omega))} \leq C(\|\tilde{u}\|_{L^2(Q_\omega)}, \|y_0\|_{L^2(Q_A)}, \|z_0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}),$$

and

$$\|\tilde{\varphi}_\gamma^i\|_{L^2(U; H_k^1(\Omega))} \leq C(\|\tilde{u}\|_{L^2(Q_\omega)}, \|y_0\|_{L^2(Q_A)}, \|z_0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}).$$

**Step 3.** We prove (80).

We observe that  $(\hat{q}_\gamma^1, \hat{q}_\gamma^2)$ , solution of (58), can be decomposed as  $(\hat{q}_\gamma^1, \hat{q}_\gamma^2) = (\bar{q}_\gamma^1 + \hat{q}_\gamma^1, \bar{q}_\gamma^2 + \hat{q}_\gamma^2)$  where  $(\bar{q}_\gamma^1, \bar{q}_\gamma^2)$  is solution to

$$\begin{cases} L_1^* \bar{q}_\gamma^1 & = \bar{q}_\gamma^2 \chi_{\mathcal{O}} & \text{in } Q, \\ L_2^* \bar{q}_\gamma^2 & = (\tilde{z}_\gamma - z_d) \chi_{\mathcal{O}} & \text{in } Q, \\ \bar{q}_\gamma^1 = \bar{q}_\gamma^2 & = 0 & \text{on } \Sigma, \\ \bar{q}_\gamma^1(T, \cdot, \cdot) = 0, \quad \bar{q}_\gamma^2(T, \cdot, \cdot) & = 0 & \text{in } Q_A, \\ \bar{q}_\gamma^1(\cdot, A, \cdot) = 0, \quad \bar{q}_\gamma^2(\cdot, A, \cdot) & = 0 & \text{in } Q_T, \end{cases} \quad (83)$$

and  $(\hat{q}_\gamma^1, \hat{q}_\gamma^2)$  is solution to

$$\begin{cases} L_1^* \hat{q}_\gamma^1 & = \bar{q}_\gamma^2 \chi_{\mathcal{O}} & \text{in } Q, \\ L_2^* \hat{q}_\gamma^2 & = \varrho_\gamma & \text{in } Q, \\ \hat{q}_\gamma^1 = \hat{q}_\gamma^2 & = 0 & \text{on } \Sigma, \\ \hat{q}_\gamma^1(T, \cdot, \cdot) = 0, \quad \hat{q}_\gamma^2(T, \cdot, \cdot) & = 0 & \text{in } Q_A, \\ \hat{q}_\gamma^1(\cdot, A, \cdot) = 0, \quad \hat{q}_\gamma^2(\cdot, A, \cdot) & = 0 & \text{in } Q_T, \end{cases} \quad (84)$$

where  $\varrho_\gamma = \frac{1}{\sqrt{\gamma}} \tilde{p}_\gamma^2 \chi_{\mathcal{O}} + \frac{1}{\sqrt{\gamma}} \tilde{\varphi}_\gamma^2 \chi_{\mathcal{O}} + \frac{1}{\gamma} \xi_1(\tilde{u}_\gamma)(t, 0, x) S_1(a; \tilde{u}_\gamma) + \frac{1}{\gamma} \eta_2(\tilde{u}_\gamma)(t, 0, x) S_2(a, \tilde{u}_\gamma)$ . Proceeding as in Proposition 2.1, while using (82), we obtain

$$\|\bar{q}_\gamma^1\|_{L^2(U; H_k^1(\Omega))} + \|\bar{q}_\gamma^2\|_{L^2(U; H_k^1(\Omega))} \leq C, \quad (85)$$

where  $C = C(\|\tilde{u}\|_{L^2(Q_\omega)}, \|y_0\|_{L^2(Q_A)}, \|z_0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)})$ .  
Combining (62), (66) and (66), we obtain for all  $w \in L^2(Q_\omega)$

$$\begin{aligned} & \int_Q \bar{z}(w) \left( \frac{1}{\sqrt{\gamma}} \tilde{p}_\gamma^2 \chi_{\mathcal{O}} + \frac{1}{\sqrt{\gamma}} \tilde{\varphi}_\gamma^2 \chi_{\mathcal{O}} + \frac{1}{\gamma} \xi_1(\tilde{u}_\gamma)(0) S_1(a; \tilde{u}_\gamma) + \frac{1}{\gamma} \eta_2(\tilde{u}_\gamma)(0) S_2(a, \tilde{u}_\gamma) \right) dt da dx \\ & + \int_U \int_{\mathcal{O}} \bar{z}(w) (\tilde{y}_\gamma - z_d) dt da dx + \int_{Q_\omega} N \tilde{u}_\gamma w dt da dx + \int_{Q_\omega} (\tilde{u}_\gamma - \tilde{u}) w dt da dx = 0. \end{aligned} \quad (86)$$

Set

$$\mathcal{E} = \{ \bar{z}(w), \quad w \in L^2(Q_\omega) \}. \quad (87)$$

Then  $\mathcal{E} \subset L^2(Q)$ . We define on  $\mathcal{E} \times \mathcal{E}$  the inner product:

$$\langle \bar{z}(v), \bar{z}(w) \rangle_{\mathcal{E}} = \int_{Q_\omega} vw \, dt da dx + \int_Q \bar{z}(v) \bar{z}(w) dt da dx, \quad \forall \bar{z}(v), \bar{z}(w) \in \mathcal{E}. \quad (88)$$

Then  $\mathcal{E}$  endowed with the norm

$$\|\bar{z}(w)\|_{\mathcal{E}}^2 = \|w\|_{L^2(Q_\omega)}^2 + \|\bar{z}(w)\|_{L^2(Q)}^2, \quad \forall \bar{z}(w) \in \mathcal{E} \quad (89)$$

is a Hilbert space. We set

$$T_\gamma(\tilde{u}_\gamma) = \frac{1}{\sqrt{\gamma}} \tilde{p}_\gamma^2 \chi_{\mathcal{O}} + \frac{1}{\sqrt{\gamma}} \tilde{\varphi}_\gamma^2 \chi_{\mathcal{O}} + \frac{1}{\gamma} \xi_1(\tilde{u}_\gamma)(0) S_1(a; \tilde{u}_\gamma) + \frac{1}{\gamma} \eta_2(\tilde{u}_\gamma)(0) S_2(a, \tilde{u}_\gamma).$$

Then, in view of (86), we have for any  $w \in L^2(Q_\omega)$ ,

$$\begin{aligned} \int_Q T_\gamma(\tilde{u}_\gamma) \bar{z}(w) dt da dx &= - \int_U \int_{\mathcal{O}} \bar{z}(w) (\tilde{z}_\gamma - z_d) dt da dx \\ &\quad - \int_{Q_\omega} N \tilde{u}_\gamma w dt da dx - \int_{Q_\omega} (\tilde{u}_\gamma - \tilde{u}) w dt da dx. \end{aligned} \quad (90)$$

Using the Cauchy Schwarz inequality, we have

$$\begin{aligned} & \left| - \int_U \int_{\mathcal{O}} \bar{z}(w) (\tilde{z}_\gamma - z_d) dt da dx - \int_{Q_\omega} ((N+1)\tilde{u}_\gamma - \tilde{u}) w dt da dx \right| \leq \\ & \|\tilde{z}_\gamma - z_d\|_{L^2(U \times \mathcal{O})} \|\bar{z}(w)\|_{L^2(Q)} + (N+1) \|\tilde{u}_\gamma\|_{L^2(Q_\omega)} \|w\|_{L^2(Q_\omega)} + \\ & \|\tilde{u}\|_{L^2(Q_\omega)} \|w\|_{L^2(Q_\omega)}. \end{aligned}$$

Therefore, using (82) and (67),

$$\begin{aligned} & \left| - \int_U \int_{\mathcal{O}} \bar{z}(w) (\tilde{z}_\gamma - z_d) dt da dx - \int_{Q_\omega} ((N+1)\tilde{u}_\gamma - \tilde{u}) w dt da dx \right| \leq \\ & \left( \|\tilde{z}_\gamma - z_d\|_{L^2(U \times \mathcal{O})}^2 + [(N+1)\|\tilde{u}_\gamma\|_{L^2(Q_\omega)} + \|\tilde{u}\|_{L^2(Q_\omega)}]^2 \right)^{1/2} \|\bar{z}(w)\|_{\mathcal{E}} \leq \\ & C \|\bar{z}(w)\|_{\mathcal{E}}, \end{aligned}$$

where  $C = C(N, T, \|\tilde{u}\|_{L^2(Q_\omega)}, \|y_0\|_{L^2(Q_A)}, \|z_0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}) > 0$ . It then follows from (90)

$$\left| \int_Q T_\gamma(\tilde{u}_\gamma) \bar{y}(w) dt da dx \right| \leq C \|\bar{z}(w)\|_{\mathcal{E}}.$$

Consequently,

$$\begin{aligned} \|T_\gamma(\tilde{u}_\gamma)\|_{\mathcal{E}'} &= \left\| \frac{1}{\sqrt{\gamma}} \tilde{p}_\gamma^2 \chi_{\mathcal{O}} + \frac{1}{\sqrt{\gamma}} \tilde{\varphi}_\gamma^2 \chi_{\mathcal{O}} + \frac{1}{\gamma} \xi_1(\tilde{u}_\gamma)(0) S_1(a; \tilde{u}_\gamma) + \frac{1}{\gamma} \eta_2(\tilde{u}_\gamma)(0) S_2(a, \tilde{u}_\gamma) \right\|_{\mathcal{E}'} \\ &\leq C. \end{aligned}$$

In particular,

$$\left\| \frac{1}{\sqrt{\gamma}} \tilde{p}_\gamma^2 \chi_{\mathcal{O}} + \frac{1}{\sqrt{\gamma}} \tilde{\varphi}_\gamma^2 \chi_{\mathcal{O}} + \frac{1}{\gamma} \xi_1(\tilde{u}_\gamma)(0) S_1(a; \tilde{u}_\gamma) + \frac{1}{\gamma} \eta_2(\tilde{u}_\gamma)(0) S_2(a, \tilde{u}_\gamma) \right\|_{L^2(Q)} \leq C,$$

where  $C = C(N, T, \|\tilde{u}\|_{L^2(Q_\omega)}, \|y_0\|_{L^2(Q_A)}, \|z_0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}) > 0$ .

Now, proceeding as in Proposition 2.1, while using the latter inequality, we obtain that

$$\|\tilde{q}_\gamma^2\|_{L^2(U; H_k^1(\Omega))} \leq C(N, T, \|\tilde{u}\|_{L^2(Q_\omega)}, \|y_0\|_{L^2(Q_A)}, \|z_0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}).$$

Finally from (85) and the latter inequality, we deduce (80). ■

**Proposition 3.7** *The adapted low-regret optimal control  $\tilde{u}_\gamma$  converges in  $L^2(Q_\omega)$  to the no-regret control  $\tilde{u} \in \mathcal{M}$ .*

**Proof.** In view of (67)-(75), for  $i = 1, 2$ , there exists a subsequence of  $(\tilde{u}_\gamma, \tilde{y}_\gamma, \tilde{z}_\gamma, \tilde{\xi}_\gamma^i, S_i(\cdot, \tilde{u}_\gamma), \tilde{\eta}_\gamma^i)$  still denoted by  $(\tilde{u}_\gamma, \tilde{y}_\gamma, \tilde{z}_\gamma, \tilde{\xi}_\gamma^i, S_i(\cdot, \tilde{u}_\gamma), \tilde{\eta}_\gamma^i)$  and  $\hat{u} \in L^2(Q_\omega)$ ,  $\tilde{y}, \tilde{z} \in L^2(U, H_k^1(\Omega))$ ,  $\tilde{\xi}^i, \tilde{\eta}^i \in L^2(U, H_k^1(\Omega))$ ,  $\alpha_i \in L^2(0, A)$ ,  $\tau_i \in L^2(Q_T)$  such that

$$\tilde{u}_\gamma \rightharpoonup \hat{u} \text{ weakly in } L^2(Q_\omega), \quad (91)$$

$$\frac{1}{\sqrt{\gamma}} S_i(\cdot, \tilde{u}_\gamma) \rightharpoonup \alpha_i \text{ weakly in } L^2(0, A), \quad S_i(\cdot, \tilde{u}_\gamma) \rightarrow 0 \text{ strongly in } L^2(0, A), \quad (92)$$

$$\tilde{y}_\gamma \rightharpoonup \tilde{y}, \quad \tilde{z}_\gamma \rightharpoonup \tilde{z}, \quad \tilde{\xi}_\gamma^i \rightharpoonup \tilde{\xi}^i, \quad \tilde{\eta}_\gamma^i \rightharpoonup \tilde{\eta}^i \text{ weakly in } L^2(U; H_k^1(\Omega)), \quad (93)$$

$$\tilde{\eta}_\gamma^2(\cdot, 0, \cdot) \rightharpoonup \tau_2, \quad \tilde{\xi}_\gamma^1(\cdot, 0, \cdot) \rightharpoonup \tau_1 \text{ weakly in } L^2(Q_T). \quad (94)$$

Let  $\varphi, \phi \in \mathcal{D}(Q)$ , by multiplying the first equation and the second equation of system (55) respectively by  $\varphi$  and  $\phi$ , and integrating by parts over  $Q$ , we obtain

$$\int_Q \tilde{y}_\gamma L_1^* \varphi \, dt \, da \, dx = \int_Q f \varphi \, dt \, da \, dx + \int_{Q_\omega} \tilde{u}_\gamma \varphi \, dt \, da \, dx, \quad \forall \varphi \in \mathcal{D}(Q),$$

and

$$\int_Q \tilde{z}_\gamma L_2^* \phi \, dt \, da \, dx = \int_{Q_{\mathcal{O}}} \tilde{y}_\gamma \phi \, dt \, da \, dx, \quad \forall \phi \in \mathcal{D}(Q).$$

By taking the limit as  $\gamma \rightarrow 0$  in these latter identities, while using (91) and (93), we obtain

$$\int_Q \tilde{y} L_1^* \varphi \, dt \, da \, dx = \int_Q f \varphi \, dt \, da \, dx + \int_{Q_\omega} \hat{u} \varphi \, dt \, da \, dx, \quad \forall \varphi \in \mathcal{D}(Q),$$

and

$$\int_Q \tilde{z} L_2^* \phi \, dt \, da \, dx = \int_{Q_{\mathcal{O}}} \tilde{y} \phi \, dt \, da \, dx, \quad \forall \phi \in \mathcal{D}(Q).$$

An integration by parts gives

$$\int_Q L_1^* \tilde{y}_\gamma \varphi \, dt \, da \, dx = \int_Q f \varphi \, dt \, da \, dx + \int_{Q_\omega} \tilde{u}_\gamma \varphi \, dt \, da \, dx, \quad \forall \varphi \in \mathcal{D}(Q),$$

and

$$\int_Q L_2^* \tilde{z}_\gamma \phi dt da dx = \int_{Q_\circ} \tilde{y}_\gamma \phi dt da dx, \quad \forall \phi \in \mathcal{D}(Q).$$

Hence

$$L_1 \tilde{y} = f + \hat{u} \chi_\omega \quad \text{in} \quad Q, \quad (95)$$

and

$$L_2 \tilde{z} = \tilde{y} \chi_\circ \quad \text{in} \quad Q. \quad (96)$$

Now, we have  $\tilde{y}, \tilde{z} \in L^2(U, H_k^1(\Omega))$  and using (95) and (96),  $\tilde{y}_t + \tilde{y}_a, \tilde{z}_t + \tilde{z}_a \in L^2(U, H_k^{-1}(\Omega))$ . Hence  $\tilde{y}, \tilde{z} \in W_k(0, T)$  and according to Remark 1,  $\tilde{y}(0, \cdot, \cdot), \tilde{z}(0, \cdot, \cdot), \tilde{y}(T, \cdot, \cdot)$  and  $\tilde{z}(T, \cdot, \cdot)$  exist and belong to  $L^2(Q_A)$ . Also,  $\tilde{y}(\cdot, 0, \cdot), \tilde{z}(\cdot, 0, \cdot), \tilde{y}(\cdot, A, \cdot)$  and  $\tilde{z}(\cdot, A, \cdot)$  exist and belong to  $L^2(Q_T)$ . Moreover  $\tilde{y}, \tilde{z} \in L^2(U, H_k^1(\Omega))$  implies that

$$\tilde{y} = \tilde{z} = 0 \quad \text{on} \quad \Sigma. \quad (97)$$

Let  $\varphi, \phi \in \mathcal{C}(\bar{Q})$  such that  $\varphi = \phi = 0$  on  $\Sigma$ ,  $\varphi(t, A, 1) = \phi(t, A, 0) = 0$  in  $Q_T$ , and  $\phi(T, a, x) = 0$  in  $Q_A$ . If we multiply the first equation and the second equation in (55) respectively by  $\varphi$  and  $\phi$ , and we integrate by parts over  $Q$ , we obtain

$$-\int_{Q_A} y_0(x, a) \varphi(0, a, x) da dx + \int_Q \tilde{y}_\gamma L_1^* \varphi dt da dx = \int_Q f \varphi dt da dx + \int_{Q_\omega} \tilde{u}_\gamma \varphi dt da dx,$$

and

$$\int_{Q_A} z_0(x, a) \phi(0, a, x) da dx + \int_Q \tilde{z}_\gamma L_2^* \phi dt da dx = \int_{Q_\circ} \tilde{y}_\gamma \phi dt da dx.$$

By taking the limit as  $\gamma \rightarrow 0$  in these latter identities, while using (91) and (93), we obtain

$$-\int_{Q_A} y_0(x, a) \varphi(0, a, x) da dx + \int_Q \tilde{y} L_1^* \varphi dt da dx = \int_Q f \varphi dt da dx + \int_{Q_\omega} \hat{u} \varphi dt da dx,$$

and

$$\int_{Q_A} z_0(x, a) \phi(0, a, x) da dx + \int_Q \tilde{z} L_2^* \phi dt da dx = \int_U \int_\circ \tilde{y} \phi dt da dx.$$

An integration by parts gives

$$\begin{aligned} & \int_{Q_A} (\tilde{y}(0, a, x) - y_0(x, a)) \varphi(0, a, x) da dx + \int_{Q_T} \tilde{y}(t, 0, x) \varphi(t, 0, x) dt dx \\ & + \int_Q \varphi L_1 \tilde{y} dt da dx = \int_Q f \varphi dt da dx + \int_{Q_\omega} \hat{u} \varphi dt da dx, \end{aligned}$$

and

$$\begin{aligned} & \int_{Q_A} (\tilde{z}(0, a, x) - z_0(x, a)) \phi(0, a, x) da dx + \int_{Q_T} \tilde{z}(t, a, x) \phi(t, 0, x) dt dx \\ & + \int_Q \phi L_2 \tilde{z} dt da dx = \int_U \int_\circ \tilde{y} \phi dt da dx. \end{aligned}$$

Taking (95) and (96) into account, we are lead to

$$\int_{Q_A} (\tilde{y}(0, a, x) - y_0(x, a)) \varphi(0, a, x) da dx + \int_{Q_T} \tilde{y}(t, 0, x) \varphi(t, 0, x) dt dx = 0, \quad (98)$$

and

$$\int_{Q_A} (\tilde{z}(0, a, x) - z_0(x, a)) \phi(0, a, x) da dx + \int_{Q_T} \tilde{z}(t, a, x) \phi(t, 0, x) dt dx = 0. \quad (99)$$

Now on one of the hand, if we take in addition in (98),  $\varphi(\cdot, 0, \cdot) = 0$  in  $Q_T$ , then it follows that

$$\tilde{y}(0, \cdot, \cdot) = y_0 \quad \text{in} \quad Q_A. \quad (100)$$

Taking (100) in account, (99) yields

$$\tilde{y}(\cdot, 0, \cdot) = 0 \quad \text{in} \quad Q_A. \quad (101)$$

On the other hand, if we choose in addition in (99),  $\phi$  such that  $\phi(0, \cdot, \cdot) = 0$  in  $Q_A$ , we obtain

$$\tilde{z}(\cdot, 0, \cdot) = 0 \quad \text{in} \quad Q_A. \quad (102)$$

Moreover taking (102) into account, (99) gives

$$\tilde{z}(0, \cdot, \cdot) = z_0 \quad \text{in} \quad Q_A. \quad (103)$$

Therefore by (95)-(97) and (100)-(103), it follows that  $(\tilde{y}, \tilde{z})$  is solution to

$$\begin{cases} L_1 \tilde{y} & = f + \hat{u} \chi_{\omega} & \text{in } Q, \\ L_2 \tilde{z} & = \tilde{y} \chi_{\mathcal{O}} & \text{in } Q, \\ \tilde{y} = \tilde{z} & = 0 & \text{on } \Sigma, \\ \tilde{y}(0, a, x) = y_0(a, x), \quad \tilde{z}(0, a, x) & = z_0(a, x) & \text{in } Q_A, \\ \tilde{y}(t, 0, x) = 0, \quad \tilde{z}(t, 0, x) & = 0 & \text{in } Q_T. \end{cases} \quad (104)$$

By applying the same arguments as above for  $(\tilde{\xi}_\gamma^1, \tilde{\eta}_\gamma^1)$ ,  $(\tilde{\xi}_\gamma^2, \tilde{\eta}_\gamma^2)$ , and using the convergences (93)-(94), we infer that  $(\tilde{\xi}^1, \tilde{\eta}^1)$ ,  $(\tilde{\xi}^2, \tilde{\eta}^2)$  are respectively solutions to

$$\begin{cases} L_1^* \tilde{\xi}^1 & = \tilde{\eta}^1 \chi_{\mathcal{O}} & \text{in } Q, \\ L_2^* \tilde{\eta}^1 & = (\tilde{z} - z_d) \chi_{\mathcal{O}} & \text{in } Q, \\ \tilde{\xi}^1 = \tilde{\eta}^1 & = 0 & \text{on } \Sigma, \\ \tilde{\xi}^1(T, \cdot, \cdot) = 0, \quad \tilde{\eta}^1(T, \cdot, \cdot) & = 0 & \text{in } Q_A, \\ \tilde{\xi}^1(\cdot, A, \cdot) = 0, \quad \tilde{\eta}^1(\cdot, A, \cdot) & = 0 & \text{in } Q_T, \end{cases} \quad (105)$$

and

$$\begin{cases} L_1^* \tilde{\xi}^2 & = \tilde{\eta}^2 \chi_{\mathcal{O}} & \text{in } Q, \\ L_2^* \tilde{\eta}^2 & = (\tilde{z} - z_d) \chi_{\mathcal{O}} & \text{in } Q, \\ \tilde{\xi}^2 = \tilde{\eta}^2 & = 0 & \text{on } \Sigma, \\ \tilde{\xi}^2(T, \cdot, \cdot) = 0, \quad \tilde{\eta}^2(T, \cdot, \cdot) & = 0 & \text{in } Q_A, \\ \tilde{\xi}^2(\cdot, A, \cdot) = 0, \quad \tilde{\eta}^2(\cdot, A, \cdot) & = 0 & \text{in } Q_T, \end{cases} \quad (106)$$

In addition, we get also

$$\tilde{\xi}^1(\cdot, 0, \cdot) = \tau_1, \quad (107)$$

and

$$\tilde{\eta}^2(\cdot, 0, \cdot) = \tau_2. \quad (108)$$

Now, using (91), (92), (94), and (69), we have from Lemma 3.2 that

$$S_i(\cdot, u^\gamma) \rightharpoonup S_i(\cdot, \hat{u}) \text{ weakly in } \mathcal{D}'(0, A).$$

Hence, using (92) again and thanks to the uniqueness of the limit, we obtain

$$S_i(\cdot, \tilde{u}_\gamma) \rightarrow S_i(\cdot, \hat{u}) = 0 \text{ strongly in } L^2(0, A).$$

Consequently,

$$\int_0^A S_i(a; \tilde{u}_\gamma) g(a) da \rightarrow \int_0^A S_i(a; \hat{u}_\gamma) g(a) da = 0.$$

Thus  $\hat{u} \in \mathcal{M}$  and we also have  $\|S_i(\cdot; \hat{u})\|_{L^2(0,A)} = 0$ . Since  $\tilde{u}$  is a No-regret control and  $\hat{u} \in \mathcal{M}$ , it follows from (48) that

$$J(\tilde{u}, 0, 0) - J(0, 0, 0) \leq J(\hat{u}, 0, 0) - J(0, 0, 0), \quad (109)$$

Observing that  $\tilde{u}_\gamma$  solves the problem  $\inf_{v \in L^2(Q_\omega)} \tilde{\mathcal{J}}^\gamma(v)$ , we have

$$\tilde{\mathcal{J}}^\gamma(\tilde{u}_\gamma) \leq \tilde{\mathcal{J}}^\gamma(\tilde{u}) = J(\tilde{u}, 0, 0) - J(0, 0, 0), \quad (110)$$

which, in view of the definition of  $\tilde{\mathcal{J}}^\gamma$  given by (53), implies that

$$J(\tilde{u}_\gamma, 0, 0) - J(0, 0, 0) + \|\tilde{u}_\gamma - \tilde{u}\|_{L^2(Q_\omega)}^2 \leq \tilde{\mathcal{J}}^\gamma(\tilde{u}_\gamma) \leq \tilde{\mathcal{J}}^\gamma(\tilde{u}) = J(\tilde{u}, 0, 0) - J(0, 0, 0).$$

Using the convexity and lower semi-continuity of  $J$  on  $L^2(Q_\omega)$ , (91) and (92), we obtain

$$J(\hat{u}, 0, 0) - J(0, 0, 0) + \|\hat{u} - \tilde{u}\|_{L^2(Q_\omega)}^2 \leq \liminf_{\gamma \rightarrow 0} \tilde{\mathcal{J}}^\gamma(\tilde{u}_\gamma) \leq J(\tilde{u}, 0, 0) - J(0, 0, 0), \quad (111)$$

which combining with (109) gives

$$\|\hat{u} - \tilde{u}\|_{L^2(Q_\omega)}^2 \leq 0.$$

Hence,

$$\hat{u} = \tilde{u} \text{ in } Q_\omega. \quad (112)$$

Thus the adapted low-regret controls converge in  $L^2(Q_\omega)$  to the no-regret control. Moreover from (112) and (104), it follows that  $(\tilde{y}, \tilde{z}) = (y(\tilde{u}, 0, 0), z(\tilde{u}, 0, 0)) \in [L^2(U; H_k^1(\Omega))]^2$  is the unique solution of

$$\begin{cases} L_1 \tilde{y} & = f + \tilde{u} \chi_\omega & \text{in } Q, \\ L_2 \tilde{z} & = \tilde{y} \chi_\mathcal{O} & \text{in } Q, \\ \tilde{y} = \tilde{z} & = 0 & \text{on } \Sigma, \\ \tilde{y}(0, a, x) = y_0(a, x), \quad \tilde{z}(0, a, x) & = z_0(a, x) & \text{in } Q_A, \\ \tilde{y}(t, 0, x) = 0, \quad \tilde{z}(t, 0, x) & = 0 & \text{in } Q_T. \end{cases} \quad (113)$$

This completes the proof.  $\blacksquare$

**Proposition 3.8** *The No-regret control  $\tilde{u} \in \mathcal{M}$  is characterized by the functions  $\tilde{u}$ ,  $(\tilde{y}, \tilde{z})$ ,  $(\tilde{\xi}^1, \tilde{\eta}^1)$ ,  $(\tilde{\xi}^2, \tilde{\eta}^2)$ ,  $(\tilde{p}^1, \tilde{p}^2)$ ,  $(\tilde{q}^1, \tilde{q}^2)$  and  $(\tilde{\varphi}^1, \tilde{\varphi}^2)$  which are unique solutions of the optimality system:*

$$\begin{cases} L_1 \tilde{y} & = f + \tilde{u} \chi_\omega & \text{in } Q, \\ L_2 \tilde{z} & = \tilde{y} \chi_\mathcal{O} & \text{in } Q, \\ \tilde{y} = \tilde{z} & = 0 & \text{on } \Sigma, \\ \tilde{y}(0, a, x) = y_0(a, x), \quad \tilde{z}(0, a, x) & = z_0(a, x) & \text{in } Q_A, \\ \tilde{y}(t, 0, x) = 0, \quad \tilde{z}(t, 0, x) & = 0 & \text{in } Q_T. \end{cases} \quad (114)$$

$$\begin{cases} L_1^* \tilde{\xi}^1 & = \tilde{\eta}^1 \chi_\mathcal{O} & \text{in } Q, \\ L_2^* \tilde{\eta}^1 & = (\tilde{z} - z_a) \chi_\mathcal{O} & \text{in } Q, \\ \tilde{\xi}^1 = \tilde{\eta}^1 & = 0 & \text{on } \Sigma, \\ \tilde{\xi}^1(T, \cdot, \cdot) = 0, \quad \tilde{\eta}^1(T, \cdot, \cdot) & = 0 & \text{in } Q_A, \\ \tilde{\xi}^1(\cdot, A, \cdot) = 0, \quad \tilde{\eta}^1(\cdot, A, \cdot) & = 0 & \text{in } Q_T, \end{cases} \quad (115)$$

$$\begin{cases} L_1^* \tilde{\xi}^2 & = \tilde{\eta}^2 \chi_\mathcal{O} & \text{in } Q, \\ L_2^* \tilde{\eta}^2 & = (\tilde{z} - z_a) \chi_\mathcal{O} & \text{in } Q, \\ \tilde{\xi}^2 = \tilde{\eta}^2 & = 0 & \text{on } \Sigma, \\ \tilde{\xi}^2(T, \cdot, \cdot) = 0, \quad \tilde{\eta}^2(T, \cdot, \cdot) & = 0 & \text{in } Q_A, \\ \tilde{\xi}^2(\cdot, A, \cdot) = 0, \quad \tilde{\eta}^2(\cdot, A, \cdot) & = 0 & \text{in } Q_T, \end{cases} \quad (116)$$



$$\begin{cases} L_1^* \tilde{q}^1 & = \tilde{q}^2 \chi_{\mathcal{O}} & \text{in } Q, \\ L_2^* \tilde{q}^2 & = (\tilde{z} - z_a) \chi_{\mathcal{O}} + \lambda_1 & \text{in } Q, \\ \tilde{q}^1 = \tilde{q}^2 & = 0 & \text{on } \Sigma, \\ \tilde{q}^1(T, \cdot, \cdot) = 0, \quad \tilde{q}^2(T, \cdot, \cdot) & = 0 & \text{in } Q_A, \\ \tilde{q}^1(\cdot, A, \cdot) = 0, \quad \tilde{q}^2(\cdot, A, \cdot) & = 0 & \text{in } Q_T, \end{cases} \quad (117)$$

$$\begin{cases} L_1 \tilde{p}^1 & = 0 & \text{in } Q, \\ L_2 \tilde{p}^2 & = \tilde{p}^1 \chi_{\mathcal{O}} & \text{in } Q, \\ \tilde{p}^1 = \tilde{p}^2 & = 0 & \text{on } \Sigma, \\ \tilde{p}^1(0, \cdot, \cdot) = 0, \quad \tilde{p}^2(0, \cdot, \cdot) & = 0 & \text{in } Q_A, \\ \tilde{p}^1(\cdot, 0, \cdot) = \lambda_2, \quad \tilde{p}^2(\cdot, 0, \cdot) & = 0 & \text{in } Q_T, \end{cases} \quad (118)$$

$$\begin{cases} L_1 \tilde{\varphi}^1 & = 0 & \text{in } Q, \\ L_2 \tilde{\varphi}^2 & = \tilde{\varphi}^1 \chi_{\mathcal{O}} & \text{in } Q, \\ \tilde{\varphi}^1 = \tilde{\varphi}^2 & = 0 & \text{on } \Sigma, \\ \tilde{\varphi}^1(0, \cdot, \cdot) = 0, \quad \tilde{\varphi}^2(0, \cdot, \cdot) & = 0 & \text{in } Q_A, \\ \tilde{\varphi}^1(\cdot, 0, \cdot) = 0, \quad \tilde{\varphi}^2(\cdot, 0, \cdot) & = \lambda_3 & \text{in } Q_T, \end{cases} \quad (119)$$

and

$$N\tilde{u} + \tilde{q}^1 = 0, \quad (120)$$

where

$$\begin{aligned} \lambda_1 &= \lim_{\gamma \rightarrow 0} \left( \frac{1}{\sqrt{\gamma}} \tilde{p}_\gamma^2 \chi_{\mathcal{O}} + \frac{1}{\sqrt{\gamma}} \tilde{\varphi}_\gamma^2 \chi_{\mathcal{O}} + \frac{1}{\gamma} \xi_1(\tilde{u}_\gamma)(t, 0, x) S_1(a; \tilde{u}_\gamma) + \frac{1}{\gamma} \eta_2(\tilde{u}_\gamma)(t, 0, x) S_2(a, \tilde{u}_\gamma) \right) \\ \lambda_2 &= \lim_{\gamma \rightarrow 0} \frac{1}{\sqrt{\gamma}} \int_0^A z(\tilde{u}_\gamma, 0, 0) S_1(a, \tilde{u}_\gamma) da, \\ \lambda_3 &= \lim_{\gamma \rightarrow 0} \frac{1}{\sqrt{\gamma}} \int_0^A z(\tilde{u}_\gamma, 0, 0) S_2(a, \tilde{u}_\gamma) da. \end{aligned}$$

**Proof.** We have already proved (114)-(116) (see proof Proposition 3.7).

From (76), (77), (78), (79) and (80), we have there exist  $\lambda_2, \lambda_3 \in L^2(Q_T)$ ,  $\lambda_1 \in L^2(Q)$ ,  $\tilde{p}^1, \tilde{p}^2 \in L^2(U; H_k^1(\Omega))$ ,  $\tilde{\varphi}^1, \tilde{\varphi}^2 \in L^2(U; H_k^1(\Omega))$  and  $\tilde{q}^1, \tilde{q}^2 \in L^2(U; H_k^1(\Omega))$  such that the following convergences hold in the weak sense:

$$\tilde{p}_\gamma^1(\cdot, 0, \cdot) \rightharpoonup \lambda_2, \tilde{\varphi}_\gamma^2(\cdot, 0, \cdot) \rightharpoonup \lambda_3 \text{ in } L^2(Q_T) \text{ and } \varrho_\gamma \rightharpoonup \lambda_1 \text{ in } L^2(Q), \quad (121)$$

$$\tilde{p}_\gamma^1 \rightharpoonup \tilde{p}^1, \tilde{p}_\gamma^2 \rightharpoonup \tilde{p}^2, \tilde{q}_\gamma^1 \rightharpoonup \tilde{q}^1, \tilde{\varphi}_\gamma^1 \rightharpoonup \tilde{\varphi}^1, \tilde{q}_\gamma^2 \rightharpoonup \tilde{q}^2, \tilde{\varphi}_\gamma^2 \rightharpoonup \tilde{\varphi}^2 \text{ in } L^2(U; H_k^1(\Omega)). \quad (122)$$

Then, proceeding as for  $\tilde{y}_\gamma$  in the proof of Proposition 3.7, while using (121)-(122), we prove (117), (118) and (119). To obtain (120), we pass to the limit in (66) while using (91), (112) and (122). This completes the proof. ■

## 4 Conclusion

We used the notion of no-regret and low-regret to control a coupled model describing the dynamics of a degenerate population with age dependence and spatial structure with missing birth rates. The control considered acts on a part of the domain  $(0, 1)$ , the observation is done on another part of the same domain. Then we introduce an appropriate Hilbert space and apply the Aubin-Lions Lemma to an appropriate auxiliary problem to prove that the adapted low-regret control converges towards a no-regret control that we characterize. So theoretically, this means that by acting on the small domain  $\omega$ , we brought the state of infected in the subspace  $\mathcal{O}$  to a desired state in order to reduce the disease. Unfortunately, because of the singularities which appear in the optimality system, we were not able to perform numerical experiments.

## Funding

This work was funded by a grant from the African Institute for Mathematical Sciences, [www.nexteinstein.org](http://www.nexteinstein.org), with financial support from the Government of Canada, provided through Global Affairs Canada, [www.international.gc.ca](http://www.international.gc.ca), and the International Development Research Centre, [www.idrc.ca](http://www.idrc.ca).

## Acknowledgments

The authors are very grateful to the editors and the anonymous referees for their valuable comments and suggestions, which helped us to improve the presentation of this work significantly.

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