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## Frugal Gaussian clustering of huge imbalanced datasets through a bin-marginal approach

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## Plan

## (4) Experiments

(5) Discussion

## Motivation: huge and imbalanced data sets

- huge in the sense tall data
$\hookrightarrow$ number of observations (high dimension setting out of scope)
$\hookrightarrow$ out of computer limits
$\hookrightarrow$ or within computer limits but with frugal resource consumption (green computing)
- discover new information
$\hookrightarrow$ more and more clusters: not the focus of this talk
$\hookrightarrow$ reveal (valuable) tiny clusters: imbalanced data sets a few abnormal objects have to be recognized among a large amount of normal ones
credit card fraud detection [Chan and Stolfo 1998)], cancer recognition [Yu et al. 2012], fraudulent calls [Fawcett and Provost 1997]


## Approaches

- supervised approach (classification) with imbalanced data sets
$\hookrightarrow$ create artificial balanced data sets:
oversampling the minority class [Chawla et al. 2002],
undersampling the majority class [Tahir et al. 2009]
$\hookrightarrow$ labeling could be difficult when sample size is very large
- unsupervised approach (clustering) with sample size is very large
$\hookrightarrow$ subsampling [Fraley and Raftery 2002, Xia et al. 2019]
$\hookrightarrow$ difficult to detect very tiny clusters
$\hookrightarrow$ computer science solutions
powerful computers or distributed architectures (MAP-reduce, ...)
$\hookrightarrow$ not frugal
- our aim : clustering of huge and imbalanced datasets under memory contraints


## Another way for data reduction

- unsupervised approach (clustering)
$\hookrightarrow$ from raw to binned data

(a) Raw data

(b) Binned data


## Our bin-marginal approach in a nutshell

Frugal unsupervised MBC (D-dimensional Gaussian mixtures) using marginal binned data:

1. from raw to binned data
$\hookrightarrow$ particular version of the EM algorithm [McLachlan and Jones 1998;
Cadez et al. 2002]
$\hookrightarrow$ another dimensionality pb
2. from binned data to (1D-)marginal counts
$\hookrightarrow$ need to design a new EM algorithm but computationally intractable
3. optimization of a composite likelihood (CL) [Lindsay 1988; Whitaker et al. 2020] instead of the full one
$\hookrightarrow$ for diagonal GMM

- CL + GMM + 2D-bin [Ranalli and Rocci 2016] new in our approach: harder data reduction (1D-bin)


## Plan

4. Experiments
(5) Discussion

## Model Based Clustering with finite GMM

Observations $\mathbf{x}=\left\{\boldsymbol{x}_{i} \in \mathbb{R}^{D}, i=1, \ldots, n\right\}$ are i.i.d. according to a $D$-dimensional Gaussian mixture with $K$ components:

$$
\begin{gathered}
f(\boldsymbol{x} ; \boldsymbol{\psi})=\sum_{k=1}^{K} \pi_{k} \phi\left(\boldsymbol{x} ; \boldsymbol{\mu}_{k}, \boldsymbol{\Sigma}_{k}\right) \\
\sum_{k} \pi_{k}=1, \quad \pi_{k}>0 \quad(k=1, \ldots, K)
\end{gathered}
$$

where $\boldsymbol{\psi}=\left(\pi_{1}, \ldots, \pi_{K}, \boldsymbol{\mu}_{1}, \ldots, \boldsymbol{\mu}_{K}, \boldsymbol{\Sigma}_{1}, \ldots, \boldsymbol{\Sigma}_{K}\right)$ and $\phi($.$) is the$ $D$-variate Gaussian density

## Binned data

unobservable or too many raw data $\boldsymbol{x}_{i}$
$\hookrightarrow$ vector of binned data $\boldsymbol{n}=\left(n_{1}, \ldots, n_{B}\right)$

- the original sample space is divided into a partition $\left\{\mathcal{B}_{b} \subset \mathbb{R}^{d}, b=1, \ldots, B\right\}$
- $n_{b}=\#\left\{\boldsymbol{x}_{i} \in \mathcal{B}_{b}\right\}$
$\boldsymbol{n}$ arises from a multinomial model with pmf [Cadez et al. 2002] ${ }^{1}$

$$
p(\boldsymbol{n} ; \psi) \propto \prod_{b=1}^{B}\left(\sum_{k=1}^{K} \pi_{k} \int_{\mathcal{B}_{b}} \phi\left(\boldsymbol{x} ; \boldsymbol{\mu}_{k}, \boldsymbol{\Sigma}_{k}\right) d \boldsymbol{x}\right)^{n_{b}} .
$$

- trick for sample size reduction: select $B \ll n$

[^0]
## Curse of dimensionality for binned data

- in our case: Cartesian grid $G=G_{1} \times \ldots \times G_{D}$ where $G_{d}$ is a univariate grid with $R_{d}+2$ cut points
$\hookrightarrow B=\prod_{d=1}^{D}\left(R_{d}+1\right)$ bins, representing the grid's coarseness
- works well if $B \ll n$ and univariate context
- when $D$ increases
the number of non-empty bins depends exponentially on the dimension $D$
$\hookrightarrow$ impossible to obtain a manageable amount of binned data
$\hookrightarrow$ several $D$-dimensional numerical integrations.
$\hookrightarrow$ vanishes any kind of gain



## Marginal binned data

- work with the 1-D binned data on each direction separately
- marginal counts: $\boldsymbol{m}=\left\{\boldsymbol{m}_{1}, \ldots, \boldsymbol{m}_{D}\right\}$
for each direction $d=1, \ldots, D, \boldsymbol{m}_{d}=\left(m_{d 1}, \ldots, m_{d B_{d}}\right)$, component $m_{d b_{d}}$ is the count of observations $x_{i d}$ in the $b_{d}$-th bin of the $d$-th dimension

$\hookrightarrow$ store $\sum_{d=1}^{D} B_{d}$ values instead of $\prod_{d=1}^{D} B_{d}$


## Bin-marginal model

- bin-marginal pdf

$$
p_{m}(\boldsymbol{m} ; \psi)=\sum_{\boldsymbol{n}^{\prime} \in \mathcal{F}_{\boldsymbol{m}}} p\left(\boldsymbol{n}^{\prime} ; \boldsymbol{\psi}\right)
$$

where $\mathcal{F}_{\boldsymbol{m}}$ is the set of tables $\boldsymbol{n}^{\prime}$ sharing the same marginals $\boldsymbol{m}$.

- issues
$\hookrightarrow$ identifiability
$\hookrightarrow$ mathematical complexity of the likelihood
$\hookrightarrow$ optimization of the likelihood


## Identifiability

- GMM identifiable up to a label permutation [rakowitz and Spragings 1968] (raw data)
- as so far, no reference for the binned case


## Proposition ( Full binned Gaussian diagonal mixtures - ABK 2021)

Under hypothesis of diagonal covariance matrices, binned D-variate mixtures of at most $K_{\text {max }}$ components are identifiable if $R_{d}>4 K_{\max }-3, d=1, \ldots, D$.

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Under hypothesis of diagonal covariance matrices, binned D-variate mixtures of at most $K_{\text {max }}$ components are identifiable if $R_{d}>4 K_{\text {max }}-3, d=1, \ldots, D$.

- the proof relies on an existing result


## Proposition (11.5-Valiant 2012)

Given the linear combination of $K$ univariate Gaussian densities $f(x)=\sum_{k=1}^{K} \pi_{k} \phi\left(x ; \mu_{k}, \sigma_{k}^{2}\right)$, such that either $\mu_{i} \neq \mu_{j}$ or $\sigma_{i}^{2} \neq \sigma_{j}^{2}$ for $i \neq j$ and for all $k \pi_{k} \neq 0$, the number of solutions to $f(x)=0$ is at most $2(K-1)$.

## Identifiability

- GMM identifiable up to a label permutation [Yakowitz and Spragings 1968] (raw data)
- as so far, no reference for the binned case

Proposition (Full binned Gaussian diagonal mixtures - ABK 2021)
Under hypothesis of diagonal covariance matrices, binned D-variate mixtures of at most $K_{\text {max }}$ components are identifiable if $R_{d}>4 K_{\max }-3, d=1, \ldots, D$.

## Proposition (Marginal-binned Gaussian diagonal mixtures - ABK 2021)

Bin-marginal $D$-variate mixtures of at most $K_{\text {max }}$ components are identifiable if binned $D$-variate mixtures are identifiable.
So, under diagonal covariance matrices hypothesis, identifiability is achieved if $R_{d}>4 K_{\max }-3, d=1, \ldots, D$.

## Plan

4. Experiments
(5) Discussion

## EM algorithm for bin-marginal model

- complete log-likelihood

$$
\ell^{c}(\psi ; \mathbf{x}, \mathbf{z})=\sum_{k=1}^{K} \sum_{i=1}^{n} z_{i k} \log \left(\pi_{k} \phi\left(\boldsymbol{x}_{i}, \boldsymbol{\mu}_{k}, \boldsymbol{\Sigma}_{k}\right)\right)
$$

where $\mathbf{z}$ gathers all $z_{i k}=\mathbb{1}_{\text {observation }} i$ in cluster $k$

## EM algorithm for bin-marginal model

## E-step

- expectation respectively to $p\left(\boldsymbol{x}, \boldsymbol{z} \mid \boldsymbol{m} ; \psi^{(j)}\right)$

$$
\begin{aligned}
Q_{m}\left(\boldsymbol{\psi}, \boldsymbol{\psi}^{(j)}\right) & =\mathbb{E}_{\boldsymbol{\psi}(j)}\left[\ell^{c}(\boldsymbol{\psi} ; \mathbf{X}, \mathbf{Z}) \mid \boldsymbol{m}\right] \\
& =\sum_{n \in \mathcal{F}_{m}} \alpha^{(j)}(\boldsymbol{n}) \sum_{k=1}^{K} \sum_{b=1}^{B} n_{b} \int_{\mathcal{B}_{b}} \tau_{k}^{(j)}(\boldsymbol{x}) \log \left[\pi_{k} \phi\left(\boldsymbol{x} ; \boldsymbol{\mu}_{k}, \boldsymbol{\Sigma}_{k}\right)\right] d \boldsymbol{x} \\
-\alpha^{(j)}(\boldsymbol{n})= & \frac{p\left(\boldsymbol{n} ; \boldsymbol{\psi}^{(j)}\right.}{\sum_{\boldsymbol{n}^{\prime} \in \mathcal{F}_{m}} p\left(\boldsymbol{n}^{\prime} ; \boldsymbol{\psi}^{(j)}\right)} \text { and } \tau_{k}^{(j)}(.)=\frac{\pi_{k}^{(j)} \phi\left(\cdot ; \cdot \boldsymbol{\mu}_{k}^{(j)}, \boldsymbol{\Sigma}_{k}^{(j)}\right)}{f\left(\cdot ; \boldsymbol{\psi}^{(j)}\right)} .
\end{aligned}
$$

## M-step

- $\pi_{k}^{(j+1)}=\frac{1}{n} \sum_{\boldsymbol{n} \in \mathcal{F}_{m}} \alpha^{(j)}(\boldsymbol{n}) \sum_{b=1}^{B} n_{b} \int_{\mathcal{B}_{b}} \tau_{k}^{(j)}(\boldsymbol{x}) d \boldsymbol{x}$
$\hookrightarrow$ both steps involve the computation of all crossed tables $\left(\mathcal{F}_{\boldsymbol{m}}\right)$ : intractable
$\hookrightarrow$ alternative: use of marginal composite likelihood


## Marginal Composite Likelihood

Let $\mathbf{x}$ be a $D$-dimensional sample with $n$ observations
$\boldsymbol{x}_{i}=\left(x_{i 1}, \ldots, x_{i D}\right), i=1, \ldots, n$, generated by a GMM with parameter $\psi$

- pseudo-likelihood only relying on the likelihood of the marginals $L_{d}\left(\psi_{d} ; \mathbf{x}_{d}\right)$
$\hookrightarrow \mathbf{x}_{d}=\left(x_{1 d}, \ldots, x_{n d}\right)$ the component $d$ of the dataset
$\hookrightarrow$ with parameter $\psi_{d}=\left(\pi_{1}, \ldots, \pi_{K}, \mu_{1 d}, \ldots, \mu_{K d}, \sigma_{1 d}^{2}, \ldots, \sigma_{K d}^{2}\right)$

$$
\tilde{L}(\psi ; \mathbf{x})=\prod_{d=1}^{D} L_{d}\left(\psi_{d} ; \mathbf{x}_{d}\right)
$$

- the estimator $\tilde{\psi}$ maximizing $\tilde{L}(\boldsymbol{\psi} ; \mathbf{x})$ is consistent and asymptotically normal [Molenberghs and Verbeke 2005]
- $\hookrightarrow$ EM algorithm with CL for HMM [Gao and Song 2011]
$\hookrightarrow$ CL on bivariate-binned data [Ranalli and Rocci 2016]


## Bin-marginal Composite Likelihood (bmCL)

- our proposal: combine memory reduction (bin-marginal)

$$
\log p_{m}(\boldsymbol{m} ; \boldsymbol{\psi})=\log \sum_{\boldsymbol{n}^{\prime} \in \mathcal{F}_{\boldsymbol{m}}} p\left(\boldsymbol{n}^{\prime} ; \boldsymbol{\psi}\right)
$$

and computational advantages of 1D-marginal CL
$\hookrightarrow$ we aim at maximizing the bin-marginal composite log-lik.:

$$
\begin{aligned}
\tilde{\ell}_{m}(\boldsymbol{\psi} ; \boldsymbol{m}) & =\sum_{d=1}^{D} \ell_{d}\left(\boldsymbol{\psi}_{d} ; \boldsymbol{m}_{d}\right) \\
& =\sum_{d=1}^{D} \sum_{b_{d}=1}^{B_{d}} m_{d b_{d}} \log \left(\int_{\mathcal{B}_{b_{d}}^{d}} f_{d}\left(x_{d} ; \boldsymbol{\psi}_{d}\right) d x_{d}\right)
\end{aligned}
$$

$\hookrightarrow$ diagonal mixtures only...
$\hookrightarrow$ identifiability?

## Bin-marginal CL: generic identifiability

A case of non identifiability

- blue mixture:

$$
\begin{aligned}
& 0.5 \mathcal{N}\left(\binom{\mu_{1}}{\mu_{2}},\left(\begin{array}{cc}
v_{1} & 0 \\
0 & v_{2}
\end{array}\right)\right)+ \\
& 0.5 \mathcal{N}\left(\binom{\nu_{1}}{\nu_{2}},\left(\begin{array}{cc}
w_{1} & 0 \\
0 & w_{2}
\end{array}\right)\right)
\end{aligned}
$$

- red mixture:

$$
\begin{aligned}
& 0.5 \mathcal{N}\left(\binom{\mu_{1}}{\nu_{2}},\left(\begin{array}{cc}
v_{1} & 0 \\
0 & w_{2}
\end{array}\right)\right)+ \\
& 0.5 \mathcal{N}\left(\binom{\nu_{1}}{\mu_{2}},\left(\begin{array}{cc}
w_{1} & 0 \\
0 & v_{2}
\end{array}\right)\right) \\
& \hookrightarrow \mathbb{E}_{\psi^{*}}\left[\tilde{\ell}_{m}\left(\psi^{*} ; \boldsymbol{M}\right)\right]=\mathbb{E}_{\psi^{*}}\left[\tilde{\ell}_{m}(\boldsymbol{\psi} ; \boldsymbol{M})\right]
\end{aligned}
$$



Identifiability except on the set of null measure composed by mixtures having two equal proportions with two components sharing the same projection
$\hookrightarrow$ generic identifiability, then consistency [Whitaker et al. 2020]

## A naive EM algorithm for bin-marginal CL

on each direction $d$ : work with $\boldsymbol{m}_{d}$

- associate the missing vectors $\left(\mathbf{x}_{d}, \mathbf{z}_{d}\right)$, where $\mathbf{z}_{d}$ is $n \times K$ indicator membership matrix for $\mathbf{x}_{d}$.
- run 1D EM algorithm separately
- how two conciliate the partitions from each direction?
$\hookrightarrow$ use the same $\pi_{1}, \ldots, \pi_{K}$ on each direction, in a global EM


## EM algorithm for bin-marginal CL (bmCL)

With $\psi_{d}=\left(\pi_{1}, \ldots, \pi_{K}, \mu_{1 d}, \ldots, \mu_{K d}, \sigma_{1 d}^{2}, \ldots, \sigma_{K d}^{2}\right)$

- bmCL E-step

$$
\tilde{Q}_{m}\left(\boldsymbol{\psi}, \boldsymbol{\psi}^{(j)}\right)=\sum_{d=1}^{D} \int_{\mathcal{X}_{d} \times \mathcal{Z}_{d}} \ell_{d}^{c}\left(\boldsymbol{\psi}_{d} ; \boldsymbol{x}_{d}, \boldsymbol{z}_{d}\right) f\left(\boldsymbol{x}_{d}, \boldsymbol{z}_{d} \mid \boldsymbol{m}_{d} ; \boldsymbol{\psi}_{d}^{(j)}\right) d \boldsymbol{x}_{d} d \boldsymbol{z}_{d}
$$

- bmCL M-step straightforward

$$
\begin{gathered}
\tau_{k d}^{(j)}(.)=\frac{\pi_{k}^{(j)} \phi\left(. ; \mu_{k d}^{(j)}, \sigma_{k d}^{2(j)}\right)}{f\left(. ; \psi_{d}^{(j)}\right)} \\
\pi_{k}^{(j+1)}=\frac{\sum_{d=1}^{D} \sum_{b_{d}=1}^{B_{d}} m_{d b_{d}} \int_{\mathcal{B}_{b_{d}}^{d}} \tau_{k d}^{(j)}\left(x_{d}\right) d x_{d}}{D n} ; \mu_{k d}^{(j+1)}=\frac{\sum_{b_{d}=1}^{B_{d}} m_{d b_{d}} \int_{\mathcal{B}_{b_{d}}^{d}} x_{d} \tau_{k d}^{(j)}\left(x_{d}\right) d x_{d}}{\sum_{b_{d}=1}^{B_{d}} m_{d b_{d}} \int_{\mathcal{B}_{b_{d} d}^{d}} \tau_{k d}^{(j)}\left(x_{d}\right) d x_{d}}
\end{gathered}
$$

- estimated partition: $\widehat{\boldsymbol{z}}=\arg \max _{k} \frac{\widehat{\pi}_{k} \phi\left(: ; \widehat{\widehat{N}}_{\kappa}, \hat{\Sigma}_{k}^{2}\right)}{f(; \cdot, \hat{\psi})}$


## Plan

(5) Discussion

## Numerical experiment on simulated data

- ability to recognize the minority class
- comparison with two competitors (estimation with Rmixmod)
$\hookrightarrow$ classic estimation with the full dataset
$\hookrightarrow$ a subsampling strategy
- clustering quality measured by the ARI score and time, under same memory constraints:
$\hookrightarrow$ bin marginal: grid coarseness $\mathrm{R} \hookrightarrow$ 2R memory space
$\hookrightarrow$ subsampling: 100 different subsamples of size $2 R$
$\hookrightarrow R=50,100,200$


## Experimental settings: 1M obs from 3D 2-classes mixtures

| Scenario | Separation | Imbalance | Small class proportion $\left(\pi_{1}\right)$ | Means |
| :--- | :---: | :---: | :---: | :---: |
| HH |  | High | $10^{-4}$ | $\mu_{1}=(-4,-4,-4)$ |
| HM | High | Medium | $10^{-3}$ | $\mu_{2}=(4,4,4)$ |
| HL |  | Low | $10^{-2}$ |  |
| MH |  | High | $10^{-4}$ | $\mu_{1}=(-3,-3,-3)$ |
| MM | Medium | Medium | $10^{-3}$ | $\mu_{2}=(3,3,3)$ |
| ML |  | Low | $10^{-2}$ |  |
| LH |  | High | $10^{-4}$ | $\mu_{1}=(-2,-2,-2)$ |
| LM | Low | Medium | $10^{-3}$ | $\mu_{2}=(2,2,2)$ |
| LL |  | Low | $10^{-2}$ | $\mu_{1}=(-1,-1,-1)$ |
| VH |  | High | $10^{-4}$ | $\mu_{2}=(1,1,1)$ |
| VM | Very low | Medium | $10^{-3}$ |  |
| VL |  | Low | $10^{-2}$ | $\mu_{1}=(-1,-1,-4)$ |
|  |  |  | $10^{-4}$ | $\mu_{2}=(1,1,4)$ |
| 1HH | One separated | High | Medium | $10^{-3}$ |
| lHM | component | Low | $10^{-2}$ |  |
| lHL |  |  |  |  |

20 replications of each scenario

## Results


(a) HH

(d) MH

(g) LH

(j) VH


(o) 1HL

## Results

## subsampling failures

- probability of failure $\nearrow$ if separation $\nearrow$ and if imbalance ratio $\searrow$
- astonishing... but
- if subsampling does not fail, it works badly



## Results

time vs grid/subsample size equal memory occupancy

-     - subsampling EM (red) - bin-marginal CL-EM (black)
- expected CL-EM time after
optimization in language C++ (blue)
- full dataset (dotted line)
- remarkable improvement relatively to full data set



## Real imbalanced datasets

- image segmentation, fraud detection, hazardous asteroid detection
- three variables

| Dataset | $n$ | $D$ | Small class proportion |
| :--- | :---: | :---: | :---: |
| Cell-1 | 101,430 | 3 | unknown |
| Cell-2 | 65,536 | 3 | unknown |
| Cell-3 | 685,020 | 3 | unknown |
| Comet | $1,083,681$ | 3 | unknown |
| Asteroids | 932,341 | 3 | 0.002 |
| Credit card | 284,807 | 3 | 0.0014 |

## Results: Image segmentation Comet ( $\mathrm{R}=400, \mathrm{~K}=3$ )


(a)


## Results: Image segmentation Cell-1 (R=20, K=4)


(a)


(b)


## Results: Image segmentation Cell-2 (R=20, K=4)


(a)


## Results: Image segmentation Cell-3 (R=20, K=4)


(a)


(b)


## Results: credit card and asteroids




- two known clusters
- Clustering for $K=2,3,4$, then clusters of highest proportions are grouped (mixture of mixtures)
- despite the loss of information, binned method seems to behave better than full GMM and subsampling
- the tiny class contains the abnormal objects (no false negative) but low ARI (false positive)


## Plan

3 Estimation
(4) Experiments
(5) Discussion

## Sum-up

clustering of huge and imbalanced datasets under memory contraints:

- bin marginal composite likelihood (bmCL) approach allows to answer:
$\hookrightarrow$ memory requirements
$\hookrightarrow$ tractability of EM algorithm
$\hookrightarrow$ recovery of tiny classes
$\hookrightarrow$ not very sensitive to grid coarseness
- subsampling
$\hookrightarrow$ easy to implement
$\hookrightarrow$ pb to recover tiny clusters
$\hookrightarrow$ high variability
$\hookrightarrow$ number of subsamples?


## Discussion

- bmCL clearly outperforms subsampling under same memory constraint, and is frugal compared to full sample but
$\hookrightarrow$ generates a lot of missing data
$\hookrightarrow$ prone to slow convergence, open algorithmic question
$\hookrightarrow$ hybrid method bmCL / subsampling?
- preliminary study, seminal for further researches
$\hookrightarrow$ how to deal with frugality while increasing number of clusters
$\hookrightarrow$ strategy when many (tiny) clusters
$\hookrightarrow$ grid definition as a model choice?
$\hookrightarrow$ specific criterion for selecting the number of clusters and grid definition


# Thank you for your attention! 

## Identifiability (main steps)

- work with binned univariate mixtures of at most $K_{\max }$ components: pmf reduces to

$$
\forall \boldsymbol{\psi}, \boldsymbol{\psi}^{*} \in \psi: \quad p(\boldsymbol{n} ; \boldsymbol{\psi})=p\left(\boldsymbol{n} ; \boldsymbol{\psi}^{*}\right) \forall G, \boldsymbol{n} \Rightarrow \boldsymbol{\psi}=\boldsymbol{\psi}^{*}
$$

- if $G$ has $R$ cut points, $\left(a_{1}, \ldots, a_{R}\right)$ then it is needed to prove that the system has only the trivial solution $\psi=\psi^{*}$ at a up to a relabeling whatever the grid is

$$
\left\{\begin{array}{l}
\pi \sum_{k=1}^{K} \Phi\left(\frac{a_{1}-\mu_{k}}{\sigma_{k}}\right)=\sum_{k=1}^{K^{*}} \pi^{*} \Phi\left(\frac{a_{1}-\mu_{k}^{*}}{\sigma_{k}^{*}}\right) \\
\pi \sum_{k=1}^{K} \Phi\left(\frac{a_{2}-\mu_{k}}{\sigma_{k}}\right)=\sum_{k=1}^{K^{*}} \pi^{*} \Phi\left(\frac{a_{2}-\mu_{k}^{*}}{\sigma_{k}^{*}}\right) \\
\vdots \\
\pi \sum_{k=1}^{K} \Phi\left(\frac{a_{R}-\mu_{k}}{\sigma_{k}}\right)=\sum_{k=1}^{K^{*}} \pi^{*} \Phi\left(\frac{a_{R}-\mu_{k}^{*}}{\sigma_{k}^{*}}\right)
\end{array}\right.
$$

- deduce with [Prop. 11.5-Valiant 2012] that binned univariate mixtures of at most $K_{\text {max }}$ Gaussian distributions are identifiable if the binning grid has $R>4 K_{\max }-3$ cut points.
- induction for D-variate mixtures


## EM algorithm for bin-marginal data

- complete log-likelihood

$$
\ell^{c}(\psi ; \mathbf{x}, \mathbf{z})=\sum_{k=1}^{K} \sum_{i=1}^{n} z_{i k} \log \left(\pi_{k} \phi\left(\boldsymbol{x}_{i}, \boldsymbol{\mu}_{k}, \boldsymbol{\Sigma}_{k}\right)\right)
$$

where $z_{i k}=\mathbb{1}_{\text {observation } i}$ in cluster $k$

- E-step

$$
\begin{aligned}
Q_{m}\left(\boldsymbol{\psi}, \boldsymbol{\psi}^{(j-1)}\right) & =\mathbb{E}_{\boldsymbol{\psi}^{(j-1)}}\left[\ell^{C}(\boldsymbol{\psi} ; \mathbf{X}, \mathbf{Z}) \mid \boldsymbol{m}\right] \\
& =\sum_{\boldsymbol{n} \in \mathcal{F}_{\boldsymbol{m}}} p\left(\boldsymbol{n} \mid \boldsymbol{m} ; \boldsymbol{\psi}^{(j-1)}\right) \mathbb{E}_{\boldsymbol{\psi}^{(j-1)}}\left[\ell^{c}(\boldsymbol{\psi} ; \mathbf{X}, \mathbf{Z}) \mid \boldsymbol{n}\right] \\
& =\sum_{\boldsymbol{n} \in \mathcal{F}_{\boldsymbol{m}}} \alpha^{(j-1)}(\boldsymbol{n}) \mathbb{E}_{\boldsymbol{\psi}^{(j-1)}}\left[\ell^{c}(\boldsymbol{\psi} ; \mathbf{X}, \mathbf{Z}) \mid \boldsymbol{n}\right] \\
& =\sum_{\boldsymbol{n} \in \mathcal{F}_{\boldsymbol{m}}} \alpha^{(j-1)}(\boldsymbol{n}) \sum_{k=1}^{K} \sum_{b=1}^{B} n_{b} \int_{\mathcal{B}_{b}} \tau_{k}^{(j-1)}(\boldsymbol{x}) \\
& \times \log \left[\pi_{k} \phi\left(\boldsymbol{x} ; \boldsymbol{\mu}_{k}, \boldsymbol{\Sigma}_{k}\right)\right] d \boldsymbol{x}
\end{aligned}
$$


[^0]:    ${ }^{1}$ also provide an estimate of $\psi$ with a binned version of EM

