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Frugal Gaussian clustering of huge imbalanced datasets through a bin-marginal approach

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Séminaire de statistique du laboratoire de mathématiques d'Avignon
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Outline

1 Introduction

2 Model

3 Estimation

4 Experiments

5 Discussion

Motivation: huge and imbalanced data sets

- ▶ huge in the sense tall data
 - ↔ number of observations (high dimension setting out of scope)
 - ↔ out of computer limits
 - ↔ or within computer limits but with frugal resource consumption (*green computing*)
- ▶ discover new information
 - ↔ more and more clusters: not the focus of this talk
 - ↔ reveal (valuable) tiny clusters: imbalanced data sets
a few *abnormal* objects have to be recognized among a large amount of *normal* ones
credit card fraud detection [*Chan and Stolfo 1998*], cancer recognition [*Yu et al. 2012*], fraudulent calls [*Fawcett and Provost 1997*]

Approaches

- ▶ supervised approach (classification) with imbalanced data sets
 - ↪ create **artificial balanced** data sets:
 - oversampling the minority class [*Chawla et al. 2002*],
 - undersampling the majority class [*Tahir et al. 2009*]
 - ↪ labeling could be difficult when sample size is very large

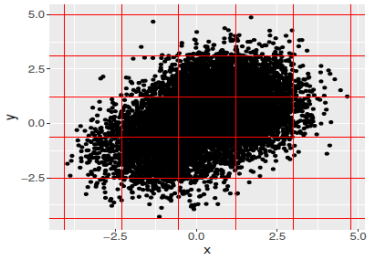
- ▶ unsupervised approach (clustering) with very large sample size
 - ↪ **subsampling** [*Fraley and Raftery 2002, Xia et al. 2019*]
 - ↪ difficult to detect very tiny clusters

 - ↪ **computer science solutions**
 - powerful computers or distributed architectures (MAP-reduce, ...)
 - ↪ not frugal

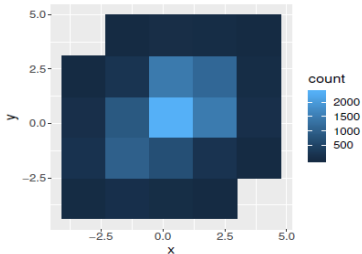
our aim: clustering of huge and imbalanced datasets under memory constraints

Another way for data reduction

- ▶ unsupervised approach (clustering)
 - ↳ from **raw** to **binned** data



(a) Raw data



(b) Binned data

Our bin-marginal approach in a nutshell

Frugal unsupervised D-dim. GMM using **marginal binned data**:

1. from raw to binned data

- ↪ particular version of the EM algorithm [*McLachlan and Jones 1998; Cadez et al. 2002*]
- ↪ but we will be face to another dimensionality problem. . .

2. from binned data to (1D-)marginal counts

- ↪ need to design a **new EM** algorithm but computationally **intractable**. . .

3. optimization of a composite likelihood (CL) [*Lindsay 1988; Whitaker et al. 2020*] instead of the full one

- ↪ restriction for **diagonal** GMM

already exists: CL + GMM + 2D-bin [*Ranalli and Rocci 2016*]

novelty in our approach: harder data reduction (1D-bin)

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Model Based Clustering with finite GMM

Observations $\mathbf{x} = \{\mathbf{x}_i \in \mathbb{R}^D, i = 1, \dots, n\}$ are i.i.d. according to a D -dimensional Gaussian mixture model (GMM) with K components:

$$f(\mathbf{x}; \boldsymbol{\psi}) = \sum_{k=1}^K \pi_k \phi(\mathbf{x}; \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$$

$$\sum_k \pi_k = 1, \quad \pi_k > 0 \quad (k = 1, \dots, K)$$

where $\boldsymbol{\psi} = (\pi_1, \dots, \pi_K, \boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_K, \boldsymbol{\Sigma}_1, \dots, \boldsymbol{\Sigma}_K)$ and $\phi(\cdot)$ is the D -variate Gaussian density

Binned data

unobservable or too many **raw** data \mathbf{x}_i

↪ vector of **binned** data $\mathbf{n} = (n_1, \dots, n_B)$

- ▶ the original sample space is divided into a partition $\{\mathcal{B}_b \subset \mathbb{R}^d, b = 1, \dots, B\}$
- ▶ $n_b = \#\{\mathbf{x}_i \in \mathcal{B}_b\}$

\mathbf{n} arises from a multinomial model with pmf [Cadez et al. 2002]¹

$$p(\mathbf{n}; \psi) \propto \prod_{b=1}^B \left(\sum_{k=1}^K \pi_k \int_{\mathcal{B}_b} \phi(\mathbf{x}; \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) d\mathbf{x} \right)^{n_b}.$$

- ▶ trick for **sample size reduction**: select $B \ll n$

¹also provide an estimate of ψ with a binned version of EM

Curse of dimensionality for binned data

- ▶ in our case: **Cartesian grid** $G = G_1 \times \dots \times G_D$ where G_d is a univariate grid with $R_d + 2$ cut points
 $\hookrightarrow B = \prod_{d=1}^D (R_d + 1)$ **bins**, representing the grid's **coarseness**
- ▶ works well if $B \ll n$ and univariate context

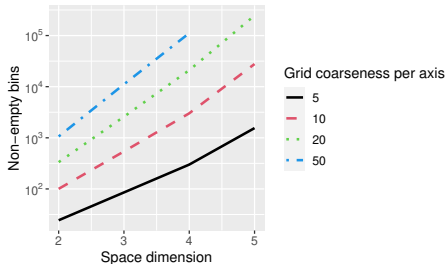
- ▶ when D increases

the number of non-empty bins depends exponentially on the dimension D

\hookrightarrow impossible to obtain a **manageable** amount of binned data

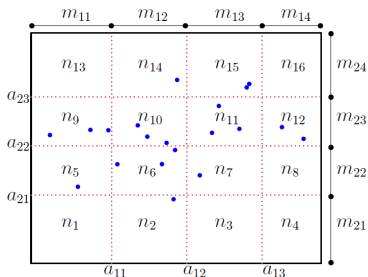
\hookrightarrow several D -dimensional **numerical integrations**.

\hookrightarrow vanishes any kind of gain



Marginal binned data

- ▶ work with the 1-D binned data on each direction **separately**
- ▶ **marginal counts**: $\mathbf{m} = \{\mathbf{m}_1, \dots, \mathbf{m}_D\}$
for each direction $d = 1, \dots, D$, $\mathbf{m}_d = (m_{d1}, \dots, m_{dB_d})$,
component m_{db_d} is the count of observations x_{id} in the b_d -th bin
of the d -th dimension



store $\sum_{d=1}^D B_d$ values instead of $\prod_{d=1}^D B_d$

Bin-marginal model

► bin-marginal pdf

$$p_m(\mathbf{m}; \psi) = \sum_{\mathbf{n}' \in \mathcal{F}_m} p(\mathbf{n}'; \psi),$$

where \mathcal{F}_m is the set of tables \mathbf{n}' sharing the same marginals \mathbf{m} .

► issues

- ↔ identifiability
- ↔ mathematical complexity of the likelihood
- ↔ optimization of the likelihood

Identifiability

- ▶ GMM identifiable up to a label permutation [Yakowitz and Spragings 1968] (raw data)
- ▶ as so far, no reference for the binned case

Proposition (Full binned diagonal GMM - ABK 2021)

Under hypothesis of *diagonal covariance* matrices, *binned D -variate mixtures of at most K_{max} components are identifiable if*

$$R_d > 4K_{max} - 3, d = 1, \dots, D.$$

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- ▶ the proof relies on an existing result

Proposition (11.5 - Valiant 2012)

Given the linear combination of K univariate Gaussian densities $f(x) = \sum_{k=1}^K \pi_k \phi(x; \mu_k, \sigma_k^2)$, such that either $\mu_i \neq \mu_j$ or $\sigma_i^2 \neq \sigma_j^2$ for $i \neq j$ and for all k $\pi_k \neq 0$, the number of solutions to $f(x) = 0$ is at most $2(K - 1)$.

Identifiability

- ▶ GMM identifiable up to a label permutation [*Yakowitz and Spragings 1968*] (raw data)
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Proposition (Full binned diagonal GMM - ABK 2021)

Under hypothesis of *diagonal covariance* matrices, *binned D -variate mixtures* of at most K_{max} components are identifiable if $R_d > 4K_{max} - 3$, $d = 1, \dots, D$.

Proposition (Marginal-binned diag. GMM - ABK 2021)

Bin-marginal D -variate mixtures of at most K_{max} components are identifiable if *binned D -variate mixtures* are identifiable.
So, under diagonal covariance matrices hypothesis, identifiability is achieved if $R_d > 4K_{max} - 3$, $d = 1, \dots, D$.

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EM algorithm for bin-marginal model

► complete log-likelihood

$$\ell^c(\boldsymbol{\psi}; \mathbf{x}, \mathbf{z}) = \sum_{k=1}^K \sum_{i=1}^n z_{ik} \log(\pi_k \phi(\mathbf{x}_i, \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k))$$

where \mathbf{z} gathers all $z_{ik} = \mathbb{1}_{\text{observation } i \text{ in cluster } k}$

EM algorithm for bin-marginal model

E-step

- ▶ expectation respectively to $p(\mathbf{x}, \mathbf{z} | \mathbf{m}; \psi^{(j)})$

$$\begin{aligned} Q_m(\psi, \psi^{(j)}) &= \mathbb{E}_{\psi^{(j)}}[\ell^c(\psi; \mathbf{X}, \mathbf{Z}) | \mathbf{m}] \\ &= \sum_{\mathbf{n} \in \mathcal{F}_m} \alpha^{(j)}(\mathbf{n}) \sum_{k=1}^K \sum_{b=1}^B n_b \int_{\mathcal{B}_b} \tau_k^{(j)}(\mathbf{x}) \log[\pi_k \phi(\mathbf{x}; \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)] d\mathbf{x} \end{aligned}$$

- ▶ $\alpha^{(j)}(\mathbf{n}) = \frac{p(\mathbf{n}; \psi^{(j)})}{\sum_{\mathbf{n}' \in \mathcal{F}_m} p(\mathbf{n}'; \psi^{(j)})}$ and $\tau_k^{(j)}(\cdot) = \frac{\pi_k^{(j)} \phi(\cdot; \boldsymbol{\mu}_k^{(j)}, \boldsymbol{\Sigma}_k^{(j)})}{f(\cdot; \psi^{(j)})}$.

M-step

- ▶ $\pi_k^{(j+1)} = \frac{1}{n} \sum_{\mathbf{n} \in \mathcal{F}_m} \alpha^{(j)}(\mathbf{n}) \sum_{b=1}^B n_b \int_{\mathcal{B}_b} \tau_k^{(j)}(\mathbf{x}) d\mathbf{x}$

both steps involve **intractable** computation of all **crossed** tables \mathcal{F}_m
alternative: use of **marginal composite likelihood**

Marginal Composite Likelihood

Let \mathbf{x} be a D -dimensional sample with n observations

$\mathbf{x}_i = (x_{i1}, \dots, x_{iD})$, $i = 1, \dots, n$, generated by a GMM with parameter ψ

- ▶ **pseudo-likelihood** only relying on the **likelihood of the marginals**

$L_d(\psi_d; \mathbf{x}_d)$

↪ $\mathbf{x}_d = (x_{1d}, \dots, x_{nd})$ the component d of the dataset

↪ with parameter $\psi_d = (\pi_1, \dots, \pi_K, \mu_{1d}, \dots, \mu_{Kd}, \sigma_{1d}^2, \dots, \sigma_{Kd}^2)$

$$\tilde{L}(\psi; \mathbf{x}) = \prod_{d=1}^D L_d(\psi_d; \mathbf{x}_d)$$

- ▶ the estimator $\tilde{\psi}$ maximizing $\tilde{L}(\psi; \mathbf{x})$ is consistent and asymptotically normal [*Molenberghs and Verbeke 2005*]
- ▶ ↪ EM algorithm with CL for HMM [*Gao and Song 2011*]
- ▶ ↪ CL on bivariate-binned data [*Ranalli and Rocci 2016*]

Bin-marginal Composite Likelihood (bmCL)

- ▶ **our proposal**: combine memory reduction (bin-marginal)

$$\log p_m(\mathbf{m}; \psi) = \log \sum_{\mathbf{n}' \in \mathcal{F}_m} p(\mathbf{n}'; \psi)$$

and computational advantages of 1D-marginal CL

↔ we aim at maximizing the **bin-marginal composite** log-lik.:

$$\begin{aligned} \tilde{\ell}_m(\psi; \mathbf{m}) &= \sum_{d=1}^D \ell_d(\psi_d; \mathbf{m}_d) \\ &= \sum_{d=1}^D \sum_{b_d=1}^{B_d} m_{db_d} \log \left(\int_{\mathcal{B}_{b_d}^d} f_d(x_d; \psi_d) dx_d \right). \end{aligned}$$

↔ diagonal mixtures only...

what about identifiability again?

Bin-marginal CL: generic identifiability

A case of non identifiability

▶ blue mixture:

$$0.5\mathcal{N}\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}\right) +$$

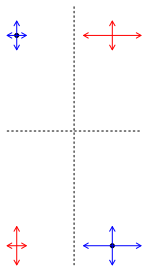
$$0.5\mathcal{N}\left(\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}, \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}\right)$$

▶ red mixture:

$$0.5\mathcal{N}\left(\begin{pmatrix} \mu_1 \\ \nu_2 \end{pmatrix}, \begin{pmatrix} v_1 & 0 \\ 0 & w_2 \end{pmatrix}\right) +$$

$$0.5\mathcal{N}\left(\begin{pmatrix} \nu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} w_1 & 0 \\ 0 & v_2 \end{pmatrix}\right)$$

$$\hookrightarrow \mathbb{E}_{\psi^*}[\tilde{\ell}_m(\psi^*; \mathbf{M})] = \mathbb{E}_{\psi^*}[\tilde{\ell}_m(\psi; \mathbf{M})]$$



Identifiability **except on** the set of **null measure** composed by mixtures having **two equal proportions** with two components sharing the **same projection**

generic identifiability, then consistency [Whitaker et al. 2020]

A naive EM algorithm for bin-marginal CL

on each direction d : work with \mathbf{m}_d

- ▶ associate the **missing** vectors $(\mathbf{x}_d, \mathbf{z}_d)$, where \mathbf{z}_d is $n \times K$ indicator membership matrix for \mathbf{x}_d .
- ▶ run 1D EM algorithm separately
- ▶ how to conciliate the partitions from each direction ?
 - ↪ use the same π_1, \dots, π_K on each direction, in a global EM

formalize more this idea now with a **unique** EM algorithm...

EM algorithm for bin-marginal CL (bmCL)

With $\psi_d = (\pi_1, \dots, \pi_K, \mu_{1d}, \dots, \mu_{Kd}, \sigma_{1d}^2, \dots, \sigma_{Kd}^2)$

► **bmCL E-step**

$$\tilde{Q}_m(\psi, \psi^{(j)}) = \sum_{d=1}^D \int_{\mathcal{X}_d \times \mathcal{Z}_d} \ell_d^c(\psi_d; \mathbf{x}_d, \mathbf{z}_d) f(\mathbf{x}_d, \mathbf{z}_d | \mathbf{m}_d; \psi_d^{(j)}) d\mathbf{x}_d d\mathbf{z}_d.$$

► **bmCL M-step** straightforward

$$\tau_{kd}^{(j)}(\cdot) = \frac{\pi_k^{(j)} \phi(\cdot; \mu_{kd}^{(j)}, \sigma_{kd}^{2(j)})}{f(\cdot; \psi_d^{(j)})}$$

$$\pi_k^{(j+1)} = \frac{\sum_{d=1}^D \sum_{b_d=1}^{B_d} m_{db_d} \int_{\mathcal{B}_{b_d}^d} \tau_{kd}^{(j)}(x_d) dx_d}{Dn}; \quad \mu_{kd}^{(j+1)} = \frac{\sum_{b_d=1}^{B_d} m_{db_d} \int_{\mathcal{B}_{b_d}^d} x_d \tau_{kd}^{(j)}(x_d) dx_d}{\sum_{b_d=1}^{B_d} m_{db_d} \int_{\mathcal{B}_{b_d}^d} \tau_{kd}^{(j)}(x_d) dx_d}$$

► **final** estimated partition: $\hat{\mathbf{z}} = \arg \max_k \frac{\hat{\pi}_k \phi(\cdot; \hat{\mu}_k, \hat{\Sigma}_k^2)}{f(\cdot; \hat{\psi})}$

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Numerical experiment on simulated data

- ▶ ability to recognize the minority class
- ▶ comparison with two competitors (estimation with `Rmixmod`)
 - ↔ classic estimation with the full dataset
 - ↔ a subsampling strategy
- ▶ clustering quality measured by the ARI score and time, under same memory constraints:
 - ↔ bin marginal: grid coarseness R ↔ $2R$ memory space
 - ↔ subsampling: 100 different subsamples of size $2R$
 - ↔ $R=50, 100, 200$

Experimental settings: 1M obs from 3D 2-classes mixtures

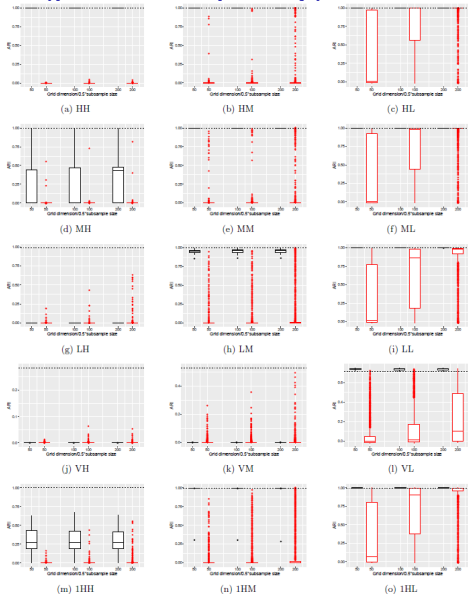
Scenario	Separation	Imbalance	Small class proportion (π_1)	Means
HH	High	High	10^{-4}	$\mu_1 = (-4, -4, -4)$ $\mu_2 = (4, 4, 4)$
HM		Medium	10^{-3}	
HL		Low	10^{-2}	
MH	Medium	High	10^{-4}	$\mu_1 = (-3, -3, -3)$ $\mu_2 = (3, 3, 3)$
MM		Medium	10^{-3}	
ML		Low	10^{-2}	
LH	Low	High	10^{-4}	$\mu_1 = (-2, -2, -2)$ $\mu_2 = (2, 2, 2)$
LM		Medium	10^{-3}	
LL		Low	10^{-2}	
VH	Very low	High	10^{-4}	$\mu_1 = (-1, -1, -1)$ $\mu_2 = (1, 1, 1)$
VM		Medium	10^{-3}	
VL		Low	10^{-2}	
1HH	One separated component	High	10^{-4}	$\mu_1 = (-1, -1, -4)$ $\mu_2 = (1, 1, 4)$
1HM		Medium	10^{-3}	
1HL		Low	10^{-2}	

20 replications of each scenario

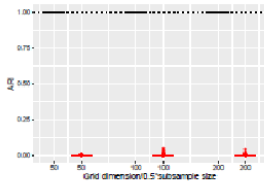
Results (partition quality)

quality vs memory

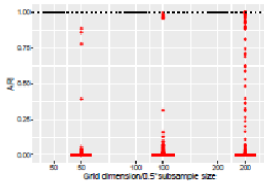
- ▶ bmCL (black) mostly outperforms **subsampling** (red), even with coarser grid,
- ▶ some difficulties only with very little separation and small proportion
- ▶ in general, bmCL approaches full data set results (dotted), with drastically less amount of memory



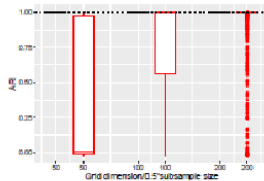
A zoom on some (partition quality) results...



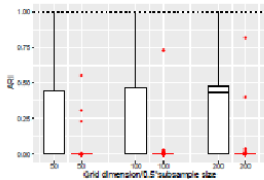
(a) HH



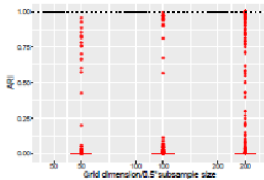
(b) HM



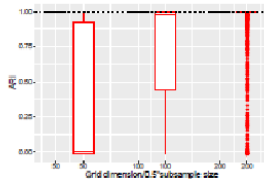
(c) HL



(d) MH



(e) MM

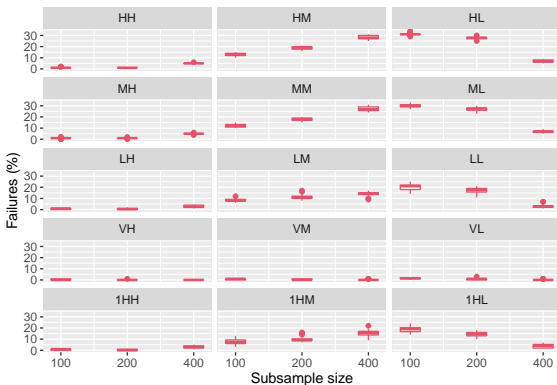


(f) ML

Results (subsampling failures)

subsampling failures

- ▶ probability of failure ↗ if separation ↗ and if imbalance ratio ↘
- ▶ astonishing... but
- ▶ if subsampling does not fail, it works badly

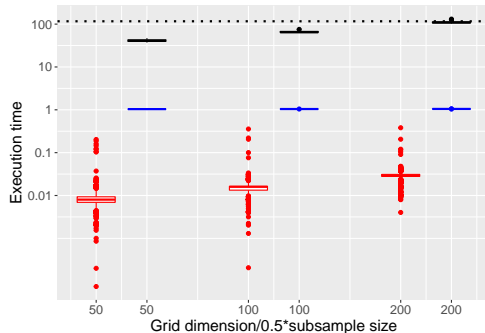


Results (computation time)

time vs grid/subsample size

equal memory occupancy

- ▶ - **subsampling** EM (red)
 - bin-marginal CL-EM (black)
 - expected CL-EM time **after optimization in language C++** (blue)
 - full dataset (dotted line)
-
- ▶ remarkable improvement relatively to full data set



Real imbalanced datasets

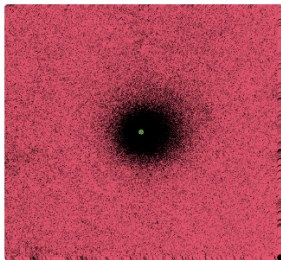
- ▶ image segmentation, fraud detection, hazardous asteroid detection
- ▶ three variables

Dataset	n	D	Small class proportion
Cell-1	101,430	3	unknown
Cell-2	65,536	3	unknown
Cell-3	685,020	3	unknown
Comet	1,083,681	3	unknown
Asteroids	932,341	3	0.002
Credit card	284,807	3	0.0014

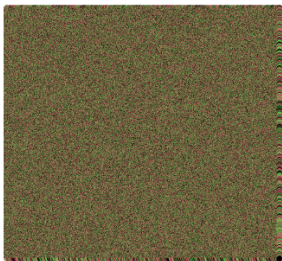
Results: Image segmentation Comet (R=400, K=3)



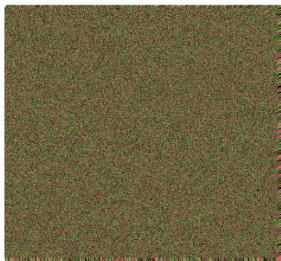
(a)



(b)

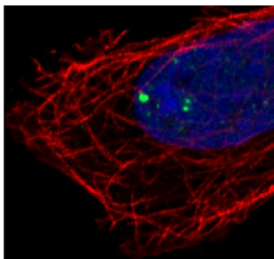


(c)

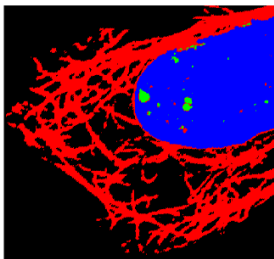


(d)

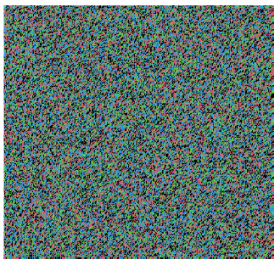
Results: Image segmentation Cell-1 (R=20, K=4)



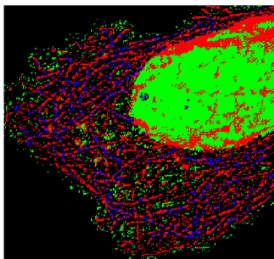
(a)



(b)

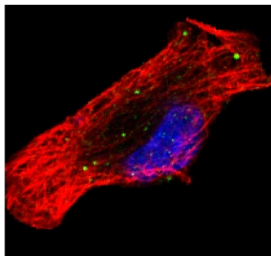


(c)

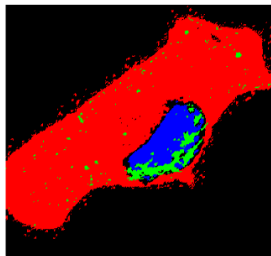


(d)

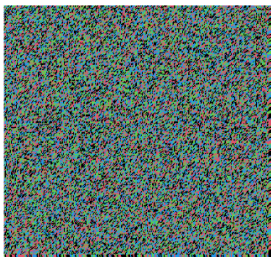
Results: Image segmentation Cell-2 (R=20, K=4)



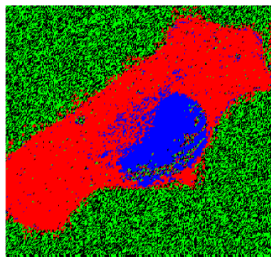
(a)



(b)



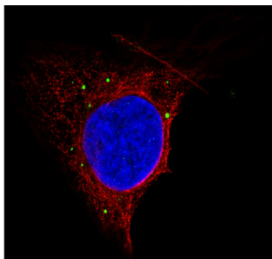
(c)



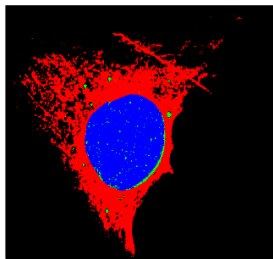
(d)

Figure 7: Cell-2 segmentation: a) Original image; b) Worst and best segmentation obtained with bin-

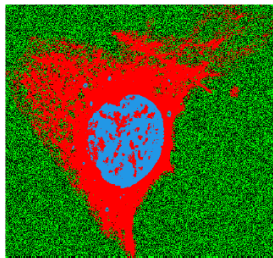
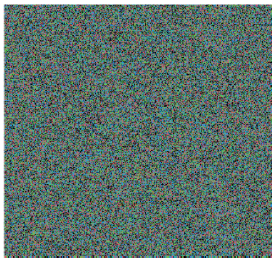
Results: Image segmentation Cell-3 (R=20, K=4)



(a)

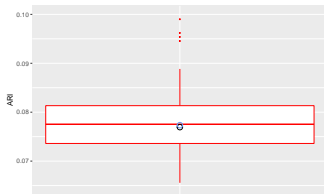


(b)

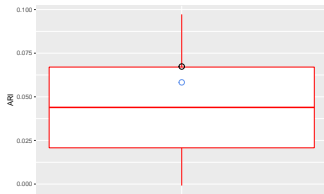


Results: asteroids and credit card

asteroids



credit card



subsampled EM (red boxplots), bin-marginal CL-EM (black circle) and full dataset EM (blue circle)

- ▶ two known clusters
- ▶ ARI very low for all methods, included the full dataset one, but it is not the concern of this experiment
- ▶ despite the loss of information, binned method behave similarly than full dataset and subsampling
- ▶ subsampling has high variability (dependency to the drawn subsample)

Outline

1 Introduction

2 Model

3 Estimation

4 Experiments

5 Discussion

Sum-up

clustering of huge and imbalanced datasets under memory constraints:

- ▶ bin marginal composite likelihood (bmCL) approach allows to answer:
 - ↔ memory requirements
 - ↔ tractability of EM algorithm
 - ↔ recovery of tiny classes
 - ↔ not very sensitive to grid coarseness
- ▶ subsampling
 - ↔ easy to implement
 - ↔ pb to recover tiny clusters
 - ↔ high variability
 - ↔ number of subsamples (in clustering, no information on the tiny cluster)?

Discussion

- ▶ bmCL clearly outperforms subsampling under same memory constraint, and is frugal compared to full sample but
 - ↔ generates a lot of missing data
 - ↔ prone to **slow** convergence, open algorithmic question
 - ↔ hybrid method bmCL / subsampling?
- ▶ preliminary study, seminal for further researches
 - ↔ how to deal with frugality while **increasing** number of clusters
 - ↔ strategy when many (tiny) clusters
 - ↔ **grid** definition as a **model choice**?
 - ↔ specific criterion for selecting the number of clusters and grid definition (remind: likelihood value is intractable)

Thank you for your attention!

Identifiability (main steps)

- ▶ work with binned univariate mixtures of at most K_{max} components: pmf reduces to

$$\forall \psi, \psi^* \in \Psi : p(\mathbf{n}; \psi) = p(\mathbf{n}; \psi^*) \quad \forall G, \mathbf{n} \Rightarrow \psi = \psi^*$$

- ▶ if G has R cut points, (a_1, \dots, a_R) then it is needed to prove that the system has only the trivial solution $\psi = \psi^*$ at a up to a relabeling whatever the grid is

$$\left\{ \begin{array}{l} \pi \sum_{k=1}^K \Phi\left(\frac{a_1 - \mu_k}{\sigma_k}\right) = \sum_{k=1}^{K^*} \pi^* \Phi\left(\frac{a_1 - \mu_k^*}{\sigma_k^*}\right) \\ \pi \sum_{k=1}^K \Phi\left(\frac{a_2 - \mu_k}{\sigma_k}\right) = \sum_{k=1}^{K^*} \pi^* \Phi\left(\frac{a_2 - \mu_k^*}{\sigma_k^*}\right) \\ \vdots \\ \pi \sum_{k=1}^K \Phi\left(\frac{a_R - \mu_k}{\sigma_k}\right) = \sum_{k=1}^{K^*} \pi^* \Phi\left(\frac{a_R - \mu_k^*}{\sigma_k^*}\right) \end{array} \right.$$

- ▶ deduce with [Prop. 11.5 - Valiant 2012] that binned univariate mixtures of at most K_{max} Gaussian distributions are identifiable if the binning grid has $R > 4K_{max} - 3$ cut points.
- ▶ induction for D-variate mixtures

EM algorithm for bin-marginal data

- ▶ complete log-likelihood

$$\ell^c(\boldsymbol{\psi}; \mathbf{x}, \mathbf{z}) = \sum_{k=1}^K \sum_{i=1}^n z_{ik} \log(\pi_k \phi(\mathbf{x}_i, \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k))$$

where $z_{ik} = \mathbb{1}_{\text{observation } i \text{ in cluster } k}$

- ▶ E-step

$$\begin{aligned} Q_m(\boldsymbol{\psi}, \boldsymbol{\psi}^{(j-1)}) &= \mathbb{E}_{\boldsymbol{\psi}^{(j-1)}}[\ell^c(\boldsymbol{\psi}; \mathbf{X}, \mathbf{Z}) | \mathbf{m}] \\ &= \sum_{\mathbf{n} \in \mathcal{F}_m} \rho(\mathbf{n} | \mathbf{m}; \boldsymbol{\psi}^{(j-1)}) \mathbb{E}_{\boldsymbol{\psi}^{(j-1)}}[\ell^c(\boldsymbol{\psi}; \mathbf{X}, \mathbf{Z}) | \mathbf{n}] \\ &= \sum_{\mathbf{n} \in \mathcal{F}_m} \alpha^{(j-1)}(\mathbf{n}) \mathbb{E}_{\boldsymbol{\psi}^{(j-1)}}[\ell^c(\boldsymbol{\psi}; \mathbf{X}, \mathbf{Z}) | \mathbf{n}] \\ &= \sum_{\mathbf{n} \in \mathcal{F}_m} \alpha^{(j-1)}(\mathbf{n}) \sum_{k=1}^K \sum_{b=1}^B n_b \int_{\mathcal{B}_b} \tau_k^{(j-1)}(\mathbf{x}) \\ &\quad \times \log[\pi_k \phi(\mathbf{x}; \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)] d\mathbf{x} \end{aligned}$$