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## Impact of Missing Data on Mixtures and Clustering

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# Impact of Missing Data on Mixtures and Clustering

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Mixtures, Hidden Markov Models, Clustering (MHC2021)

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## Take home message

Missing data may change preconceptions

- **Mixtures:** EM has unexpected behaviour concerning degeneracy dynamic
- **Clustering:** the missing data pattern may convey some information on partition

These topics are in the research agenda of Statisticians since:

The larger the datasets, the more missing data may appear. . .



# Outline

- 1 Introduction
- 2 Impact of MAR data on the EM algorithm
- 3 Impact of MNAR data on clustering
- 4 Concluding remarks

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# Outline

## 1 Introduction



## Missing data: notations

- $y = \{y_1 | \dots | y_n\}^T$ : **full dataset** with  $n$  individuals
- $y_i = (y_{i1}, \dots, y_{id}) \in \mathcal{Y} = \mathbb{R}^d$  with individual  $i \in \{1, \dots, n\}$
- $c = \{c_1 | \dots | c_n\}^T \in \{0, 1\}^{n \times d}$ : **pattern of missing data** for the full dataset
- $c_i = (c_{i1}, \dots, c_{id}) \in \{0, 1\}^d$ : pattern of missing data for individual  $i \in \{1, \dots, n\}$

$$c_{ij} = 1 \Leftrightarrow y_{ij} \text{ is missing}$$

- $y_i^{\text{obs}}$ : the observed variables values for indiv.  $i$  (and  $y^{\text{obs}} = \{y_1^{\text{obs}} | \dots | y_n^{\text{obs}}\}^T$ )
- $y_i^{\text{mis}}$ : the missing variables values for individual  $i$  (and  $y^{\text{mis}} = \{y_1^{\text{mis}} | \dots | y_n^{\text{mis}}\}^T$ )



## Typology of the missingness mechanisms

- Missing completely at random (**MCAR**):

$$p(c|y; \psi) = p(c; \psi) \quad \forall y$$

- Missing at random (**MAR**):

$$p(c|y; \psi) = p(c|y^{\text{obs}}; \psi) \quad \forall y^{\text{mis}}$$

- Missing not at random (**MNAR**): the mechanism is not MCAR nor MAR



## Ignorable vs. non ignorable model

A missing mechanism is ignorable if likelihoods can be decomposed as

$$L(\theta, \psi; \underbrace{y^{\text{obs}}, c}_{\text{observed data}}) = L(\psi; c | y^{\text{obs}}) \times L(\theta; y^{\text{obs}})$$

Some simple algebra show that this occurs when missing mechanism is not MNAR

### Inference of $\theta$

“If the missing mechanism is **ignorable** then likelihood-based inferences for  $\theta$  from  $L(\theta; y^{\text{obs}})$  will be the same as likelihood based inference for  $\theta$  from  $L(\theta, \psi; y^{\text{obs}}, c)$ .”

[Little and Rubin, 2002 Section 6.2]

- M(C)AR is ignorable
- MNAR is not ignorable





## Mixture model and clustering

- **Partition ( $K$  clusters):**  $z = (z_1 | \dots | z_n)^T \in \{0, 1\}^{n \times K}$  where
  - $z_i = (z_{i1}, \dots, z_{iK}) \in \{0, 1\}^K$
  - $z_{ik} = 1$  if  $y_i$  belongs to cluster  $k$ ,  $z_{ik} = 0$  otherwise
- **Mixture model:**  $y_1, \dots, y_n$  are i.i.d. from the  $d$ -variate Gaussian mixture

$$f(y_i; \pi, \theta) = \sum_{k=1}^K \pi_k f_k(y_i; \theta_k)$$

- $\pi_k = p(z_{ik} = 1)$ ,  $\pi = (\pi_1, \dots, \pi_K)$
- $f_k(\cdot; \theta_k) = \phi(\cdot; \mu_k, \Sigma_k)$  is the  $d$ -variate **Gaussian distribution** with mean vector  $\mu_k$  and covariance matrix  $\Sigma_k$
- $\theta = (\theta_1, \dots, \theta_K)$  is the whole mixture parameter
- **Clustering:** MAP principle from the mixture output to estimate the partition



# Outline

- 2 Impact of MAR data on the EM algorithm
  - Gaussian mixture degeneracy *without* missing data
  - Gaussian mixture degeneracy *with* missing data



## Degeneracy genesis: unbounded likelihood

- $d$ -variate Gaussian mixture

$$f(y_i; \theta) = \sum_{k=1}^K \pi_k \underbrace{\frac{1}{(2\pi)^{d/2} |\Sigma_k|^{1/2}} \exp\left(-\frac{1}{2}(y_i - \mu_k)^T \Sigma_k^{-1} (y_i - \mu_k)\right)}_{\phi(y_i; \mu_k, \Sigma_k)}$$

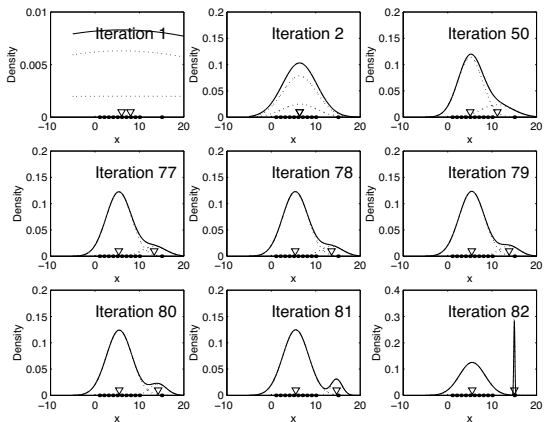
- Sampling:  $y_1, \dots, y_n \stackrel{i.i.d.}{\sim} p(\cdot; \theta)$  without any missing data ( $y^{\text{obs}} = y$ )
- Likelihood:  $\ell(\theta; y) = \ln L(\theta; y) = \sum_{i=1}^n \ln f(y_i; \theta)$

$$\text{particular center } \mu_2 = y_i \quad \Rightarrow \quad \lim_{|\Sigma_2| \rightarrow 0} \ell(\theta; y) = +\infty$$

[Kiefer and Wolfowitz, 1956] [Day, 1969]

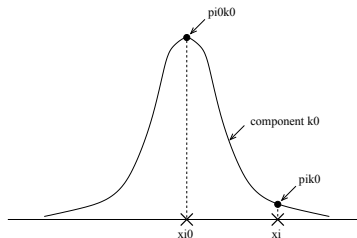


## EM behaviour: illustration



- degeneracy may occur even when starting from large variances
- convergence can be slow when far from the degenerate limit
- convergence extremely fast near degeneracy

## EM behaviour: results



$$u_0 = \left[ \frac{1}{p_{i_0 k_0}}, \{p_{i k_0}\}_{i \neq i_0} \right]$$

degeneracy of component  $k_0$  at  $y_{i_0}$

$$\Leftrightarrow \|u_0\| \rightarrow 0$$

[Biernacki and Chrétien, 2003]

[Ingrassia and Rocci, 2009]

### Proposition 1: Existence of a basin of attraction

$\exists \epsilon > 0$  s.t. if  $\|u_0\| \leq \epsilon$  then  $\|u_0^+\| = o(\|u_0\|)$  with probability 1.

### Proposition 2: Speed towards degeneracy is exponential

$\exists \epsilon > 0, \alpha > 0$  and  $\beta > 0$  s.t. if  $\|u_0\| \leq \epsilon$  then, with probability 1,  
 $|\Sigma_{k_0}^+| \leq \alpha / |\Sigma_{k_0}| \cdot \exp(-\beta / |\Sigma_{k_0}|)$ .



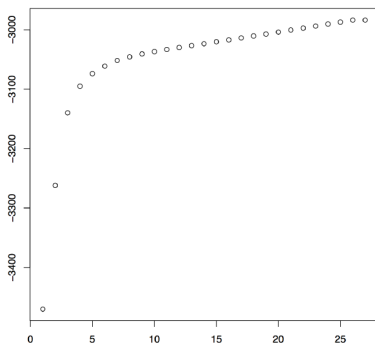
## Outline

- 2 Impact of MAR data on the EM algorithm
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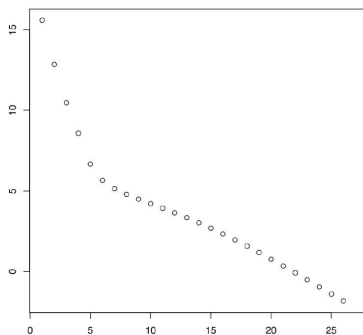
## EM behaviour illustration

- Breast cancer tissue of the UCI database repository: 106 units, 9 variables.
- 10% of missing data randomly generated
- $K = 4$  clusters

Log-likelihood according to the number of iterations



Decrease of the log-determinant of the degenerated component



## Detail from the illustration

	1	2	3	4	5	6	7	8	9
1	211.00		0.09	30.75	151.98	4.94	14.27	27.24	217.13
2	196.86	0.02	0.09	28.59	82.06	2.87	7.97	27.66	200.75
3	144.00	0.12	0.05	19.65	70.43	3.58		7.57	160.37
4	172.52	0.13	0.04		192.22	5.12	19.32	32.19	174.93
5	121.00	0.17	0.09	24.44	144.47	5.91	22.02	10.59	141.77
6	223.00	0.12	0.08	33.10	197.01	5.95	30.45	12.96	252.48
7		0.17	0.23	34.22	94.35	2.76	31.28	13.88	180.61
8	303.00	0.06	0.04	22.57		4.54	21.83	5.72	321.65
9	250.00	0.09	0.09	29.64	180.76	6.10	26.14	13.96	280.12
10	391.00	0.06	0.01	35.78		7.41	22.13	28.11	400.99
11	176.00	0.09	0.08	20.59	79.71		18.23	9.58	191.99
12	145.00		0.11	21.22	82.46	3.89	20.30	6.17	162.51
13	124.13	0.13	0.11	20.59			18.46	9.12	134.89
14	103.00	0.16	0.29	23.75	78.26	3.29	22.32	8.12	124.98

Table : Data belonging to the degenerated component.

- Cvg. towards a degenerated component (no plateau of the log-likelihood)
- Degeneracy relatively slow: log-likelihood linear according to the nb of it.
- Number of points of the degenerated solution greater than the space dimension  $d$  (but the number of complete points lower than  $d$ )





## Intermediate conclusion on missing data

### Like the complete data $y$ case

- Likelihood is unbounded
- EM can be attracted by degenerate solutions

### Unlike the complete data $y$ case

- Risk to consider a degenerated solution as valid
- Risk of losing a lot of time in useless iterations

**Statisticians should be aware of such dangerous EM behaviour. . .**

. . . since missing data are more and more frequent



## Understanding degeneracy speed on a toy example

- Univariate framework, no mixture, only one observed data:  $y$
- Maximum likelihood estimator (**Unbounded likelihood!**):  $\mu = y, \sigma^2 = 0$
- Suppose equivalently that  $n - 1$  data are unobserved (unchanged likelihood)
- Here is one iteration of a (useless) EM algorithm (it.  $q$ )

$$\mu^{(q+1)} = \frac{(n-1)\mu^{(q)} + y}{n} \quad \text{and} \quad \sigma^{2(q+1)} = \frac{(n-1)\sigma^{2(q)} + (y - \mu^{(q+1)})^2}{n}$$

Linear grow of the log-likelihood (have a look also when  $n$  increases!)

$$\ell(\theta^{(q)}; x) \sim -0.5q \log \frac{n-1}{n}$$

Geometrical convergence rate towards 0 for the variance

$$\sigma^{2(q)} \sim \sigma^{2(0)} \left( \frac{n-1}{n} \right)^q$$

## Influence of the missing data rate

% missing data	0	5	10	15	20	25	30
% deg.	16	4	12	11	46	51	100
Average nb of iterations before deg.	2	13	13	82	304	138	215

**Table :** Frequency and speed of degeneracy (deg.) according to the rate of missing data on the breast cancer data set.

When the rate of missing data increases:

- The rate of degeneracy increases
- The number of iterations before degeneracy seems to (globally) increase

Again, statisticians should be aware of such dangerous EM behaviour. . .  
 . . . since missing data are more and more frequent



# Outline

- 3 Impact of MNAR data on clustering
  - A model-based MNAR clustering approach
  - Medical study illustration



## Proposed MNAR models in clustering

### Question we address now

Since MNAR is not ignorable, which distribution  $p(c|y, z; \psi)$  to propose?

### Hypothesis 1: conditional independence

$$p(c_i | y_i, z_{ik} = 1; \psi) = \prod_{j=1}^d p(c_{ij} | y_i, z_{ik} = 1; \psi)$$

### Hypothesis 2: linear function within canonical link functions $\rho$

$$p(c_{ij} = 1 | y_i, z_{ik} = 1; \psi) = \rho(\alpha_{kj} + \beta_{kj} y_{ij})$$

- $\psi = (\alpha, \beta)$  where  $\alpha = (\alpha_{11}, \dots, \alpha_{1d}, \dots, \alpha_{K1}, \dots, \alpha_{Kd})^T \in \mathbb{R}^{Kd}$  and  $\beta = (\beta_{11}, \dots, \beta_{1d}, \dots, \beta_{K1}, \dots, \beta_{Kd})^T \in \mathbb{R}^{Kd}$
- $\rho$  is the cdf of any continuous distribution (logit, probit)

## A by-product zoology of MNAR models

	Effect on the variable $j$		Effect on the class membership $k$		Nb parameters
	Depends on $j$	Depends on $k$	Depends on $j$	Depends on $k$	Continuous
MNAR $z^j y^k$	✓	✓	✓	✓	$2Kd$
MNAR $yz^j$	✓	✗	✓	✓	$(K + 1)d$
MNAR $y^k z$	✓	✓	✗	✓	$K(d + 1)$
MNAR $yz$	✓	✗	✗	✓	$(K + d)$
MNAR $y$	✓	✗	✗	✗	$d$
MNAR $y^k$	✓	✓	✗	✗	$Kd$
MNAR $z$	✗	✗	✗	✓	$K$
MNAR $z^j$	✗	✗	✓	✓	$Kd$

Remarks:

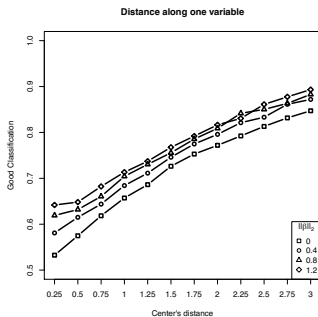
- MNAR $z^j y^k$  is the most complex model
- MNAR $z$ , MNAR $z^j$ : the only effect of missingness is on the class membership,  $\psi = (\alpha_{11}, \dots, \alpha_{1d}, \dots, \alpha_{K1}, \dots, \alpha_{Kd})^T$ ,  $p(c_{ij} = 1 \mid y_i, z_{ik} = 1; \psi) = \rho(\alpha_{kj})$
- MCAR is a specific and simple case



## MNARz analysis: pattern $c$ gives information on partition $z$ !

Draw Bayes error of a MNARz model with two components and 20% of missing data

$$\pi_k = 0.5, \|\mu_2 - \mu_1\| \text{ varies}, \Sigma_1 = \Sigma_2 = \mathbf{I}, |\alpha_2 - \alpha_1| \text{ varies}$$



Both  $\mu_k$  and  $\alpha_k$  act on the Bayes error



## Reinterpretation of the MNAR<sub>z</sub> and MNAR<sub>z</sub><sup>j</sup> models as MAR

Commonly used in Machine Learning [Jones, 1996], [Little and Rubin, 2002], [Josse *et al.*, 2019]

Mixture model for  $y^{\text{obs}}$  and Bernoulli distribution for  $C$   
 $\Leftrightarrow$  MAR mixture model for  $\tilde{y}^{\text{obs}} = (y^{\text{obs}}|c)$

For example,

$$y^{\text{obs}} = \begin{pmatrix} ? & 2.6 & 5 \\ \text{blue} & 1.9 & 4 \\ \text{red} & 2.3 & ? \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then  $\tilde{y}^{\text{obs}}$  is expressed as

$$\tilde{y}^{\text{obs}} = \begin{pmatrix} ? & 2.6 & 5 & 1 & 0 & 0 \\ \text{blue} & 1.9 & 4 & 0 & 0 & 0 \\ \text{red} & 2.3 & ? & 0 & 0 & 1 \end{pmatrix}.$$

### Proposition 3: in terms of maximum likelihood

The maximum likelihood estimate associated to the dataset  $\tilde{y}^{\text{obs}}$  under MAR model is the one associated to the dataset  $y^{\text{obs}}$  under MNAR<sub>z</sub> or MNAR<sub>z</sub><sup>j</sup> models.





## Identifiability

Previous works: [Teicher, 1963] (without NA), [Miao et al., 2016] (for MNAR data)

### Proposition 4

Assume that

- 1 The marginal mixture  $\sum_{k=1}^K \pi_k f_k(y_i; \theta_k)$  is identifiable
- 2 There exists a total ordering  $\preceq$  of  $\mathcal{F}_j \times \mathcal{R}$ , for  $j \in \{1, \dots, d\}$  fixed, where  $\mathcal{F}_j = \{f_{1j}, \dots, f_{Kj}\}$  and  $\mathcal{R} = \{\rho_1, \dots, \rho_K\} = \{\rho(\cdot; \psi_1), \dots, \rho(\cdot; \psi_K)\}$ . The total ordering is s.t.  $\forall k < \ell, F_k = \rho_k f_{kj} \preceq F_\ell = \rho_\ell f_{\ell j}$  implies

$$\lim_{u \rightarrow +\infty} \frac{\rho_\ell(u) f_{\ell j}(u)}{\rho_k(u) f_{kj}(u)} = 0$$

Then the mixture model with one of the MNAR\* mechanisms is identifiable up to label swapping

All MNAR models are identifiable (or at least generically identifiable) for probit/logit

## Estimation procedure overview

- Use EM or Stochastic EM (SEM) algorithms
- MNAR<sub>z</sub> and MNAR<sub>z</sub><sup>j</sup>: EM and SEM are very simple
- MNAR<sub>y</sub>\*: the SE step requires a within Gibbs loop, sometimes involving itself a Sampling Importance Resampling (SIR)

	EM		SEM	
MNAR <sub>z</sub> MNAR <sub>z</sub> <sup>j</sup>	✓		✓	
	Probit	Logit	Probit	Logit
MNAR <sub>y</sub> *	no closed form	no closed form, optim. pb	✓	require algorithms as SIR (costly)

## Model selection

Can select between MCAR and MNAR\* with any information criterion (BIC, ICL)

Even if the missing mechanism is ignorable for MCAR. . .

. . . need to model  $c$  to compare a MCAR and a MNAR model

### CAUTION

- It is just a selection between several proposed MNAR models
- It is not deciding if missingness procedure is “genererically” MNAR or not



## Outline

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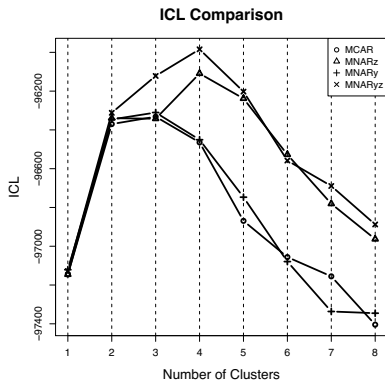


## Hospital data description

- Number of patients:  $n = 5\,146$
- Number of features:  $d = 7$ 
  - Age
  - Size
  - Weight
  - Cardiac frequency
  - Hemoglobin concentration
  - Temperature
  - Minimum Diastolic and Systolic Blood Pressure
- Percentage of missing data: 6.4%

Doctors are convinced that their missing data are MNAR

## ICL comparison



- MCAR, MNARy and MNARz are equivalent until  $K = 3$
- MNARz and MNARyz clearly indicate presence of an additional cluster ( $K = 4$ )

It seems to be an illustration of the effect of  $c$  through MNARz and MNARyz

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# Outline

## 4 Concluding remarks



## To conclude

### Summary

- Statisticians should properly consider the potential missing data impact both from an **algorithmic** and from a **modeling** point of view
- EM: be careful about degeneracy which seems to exacerbated/masqued
- MNAR: interest of the simple but meaningful model MNARz, link with usual methods

### Ongoing works

- EM: propose mechanism to identify/discard degenerate runs
- MNAR: extend to categorical, count and mixed data

Thanks!