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Dealing with missing data in model-based clustering through a MNAR model

Christophe Biernacki, Claire Boyer, Gilles Celeux, Julie Josse, Fabien Laporte,
Matthieu Marbac Lourdelle, Aude Sportisse

THE 14TH PROFESSOR ALEKSANDER ZELIAS INTERNATIONAL CONFERENCE ON
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Take home message

- 1 The missing data **pattern** may convey some information on clustering
- 2 **Embed the missingness mechanism** directly within the clustering modeling step

Outline

1 Introduction

2 A model-based MNAR clustering approach

3 Identifiability

4 Inference procedures

5 Medical study illustration

6 Concluding remarks

Missing data: an inevitable event

The larger the datasets, the more missing data may appear. . .

Two traditional solutions (for obtaining a filled dataset)

- **Discard** individuals with missing data: more variance or a biased subset
- **Impute** missing data: possible bias and underestimation of the variability

General guidelines

- Obtaining a complete dataset is **not** the final goal
- Missing data management should **take into account the initial analysis target**

Our analysis target: **model-based clustering**

Embed missing data management into this paradigm. . .

Missing data: notations

- $Y = \{y_1 | \dots | y_n\}^T$: **full dataset** with n individuals
- $y_i = (y_{i1}, \dots, y_{id}) \in \mathcal{Y}$, depending on the data type: individual $i \in \{1, \dots, n\}$
 - **continuous data**: $\mathcal{Y} = \mathbb{R}^d$
 - **categorical data**: $\mathcal{Y} = \{0, 1\}^{\ell_1} \times \dots \times \{0, 1\}^{\ell_d}$ where ℓ_j is the number of levels for $y_{ij} = (y_{ij}^1, \dots, y_{ij}^{\ell_j})$, where $y_{ij}^\ell = 1$ if y_{ij} takes the level ℓ , 0 otherwise.
 - **mixed data**: combination of continuous and categorical data.
- $C = \{c_1 | \dots | c_n\}^T \in \{0, 1\}^{n \times d}$: **pattern of missing data** for the full dataset
- $c_i = (c_{i1}, \dots, c_{id}) \in \{0, 1\}^d$: pattern of missing data for individual $i \in \{1, \dots, n\}$

$$c_{ij} = 1 \Leftrightarrow y_{ij} \text{ is missing}$$

- y_i^{obs} : the observed variables values for individual i
- y_i^{mis} : the missing variables values for individual i

Missing data: typology of the missing mechanisms

- Missing completely at random (MCAR):

$$\mathbb{P}(c|y; \psi) = \mathbb{P}(c; \psi) \quad \forall y$$

- Missing at random (MAR):

$$\mathbb{P}(c|y; \psi) = \mathbb{P}(c|y^{\text{obs}}; \psi) \quad \forall y^{\text{mis}}$$

- Missing not at random (MNAR): the mechanism is not MCAR nor MAR

Example of MNAR data

The probability to have a missing value on income depends on the value of income (rich people less inclined to reveal their income).

Ignorable vs. non ignorable model

A missing mechanism is ignorable if likelihoods can be decomposed as

$$L(\theta, \psi; \underbrace{y^{\text{obs}}, c}_{\text{observed data}}) = L(\psi; c | y^{\text{obs}}) \times L(\theta; y^{\text{obs}})$$

Some simple algebra show that this occurs when missing mechanism is not MNAR

Inference of θ

“If the missing mechanism is **ignorable** then likelihood-based inferences for θ from $L(\theta; y^{\text{obs}})$ will be the same as likelihood based inference for θ from $L(\theta, \psi; y^{\text{obs}}, c)$.”
([Little and Rubin, 2002] Section 6.2)

- M(C)AR is ignorable
- MNAR is not ignorable

Clustering: model-based approach

- **Partition with K clusters:** $Z = (z_1 | \dots | z_n)^T \in \{0, 1\}^{n \times K}$ where
 - $z_i = (z_{i1}, \dots, z_{iK}) \in \{0, 1\}^K$
 - $z_{ik} = 1$ if y_i belongs to cluster k , $z_{ik} = 0$ otherwise
- **Mixture model:** y_1, \dots, y_n are i.i.d. from the mixture

$$f(y_i; \pi, \theta) = \sum_{k=1}^K \pi_k f_k(y_i; \theta_k)$$

- $\pi_k = P(z_{ik} = 1)$, $\pi = (\pi_1, \dots, \pi_K)$
- $f_k(\cdot; \theta_k)$: pdf of the k -th component parametrized by θ_k , $\theta = (\theta_1, \dots, \theta_K)$
 - **continuous data:** $f_k(\cdot; \theta_k) = \phi(\cdot; \mu_k, \Sigma_k)$ is the d -variate **Gaussian distribution** with mean vector μ_k and covariance matrix Σ_k
 - **categorical data:** the features are independent conditionally to the group membership i.e.

$$f_k(\cdot; \theta_k) = \prod_{j=1}^d f_{kj}(\cdot; \theta_{kj}),$$
 where $f_{kj} = \prod_{\ell=1}^{\ell_j} (\theta_{kj}^\ell)^{y_{ij}^\ell}$ is the **multinomial distribution** with $\theta_{kj} = (\theta_{kj}^\ell = \mathbb{P}(y_{ij}^\ell = 1 | z_{ik} = 1))_{\ell=1, \dots, \ell_j}$
 - **mixed data:** the features are independent conditionally to the group membership, $f_k(\cdot; \theta_k)$ is the product of univariate Gaussian and multinomial distributions
 - Can also be extended to other cases (count data with Poisson distributions for instance)

Question we address in this work

Which distribution $\mathbb{P}(c|y, z; \psi)$ to propose in this clustering context?

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Proposed zoology of MNAR models in clustering

$$\mathbb{P}(c_i | y_i, z_{ik} = 1; \psi) = \prod_{j=1}^d \mathbb{P}(c_{ij} | y_i, z_{ik} = 1; \psi)$$

- **MNAR $_{y^k z^j}$** , with $\psi = (\alpha, \beta)$ where $\alpha = (\alpha_{11}, \dots, \alpha_{1d}, \dots, \alpha_{K1}, \dots, \alpha_{Kd})^T \in \mathbb{R}^{Kd}$ and $\beta = (\beta_{11}, \dots, \beta_{1d}, \dots, \beta_{K1}, \dots, \beta_{Kd})^T \in \mathbb{R}^{Kd}$

$$\mathbb{P}(c_{ij} = 1 | y_i, z_{ik} = 1; \psi) = \rho(\alpha_{kj} + \beta_{kj} y_{ij}),$$

with ρ the cdf of any continuous distribution (logit, probit)

- **MNAR $_{yz}$** , **MNAR $_{y^k z}$** , **MNAR $_{yz^j}$**

- **MNAR $_y$** , **MNAR $_{y^k}$**

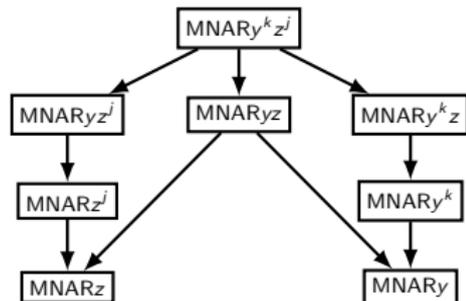
$$\psi = (\beta_{11}, \dots, \beta_{1d}, \dots, \beta_{K1}, \dots, \beta_{Kd})^T$$

$$\mathbb{P}(c_{ij} = 1 | y_i, z_{ik} = 1; \psi) = \rho(\beta_{kj} y_{ij})$$

- **MNAR $_z$** , **MNAR $_z^j$**

$$\psi = (\alpha_{11}, \dots, \alpha_{1d}, \dots, \alpha_{K1}, \dots, \alpha_{Kd})^T$$

$$\mathbb{P}(c_{ij} = 1 | y_i, z_{ik} = 1; \psi) = \rho(\alpha_{kj})$$



Overview of the proposed MNAR models

	Effect on the variable j		Effect on the class membership k		Nb parameters	
	Depends on j	Depends on k	Depends on j	Depends on k	Continuous	Categorical
MNAR $z^j y^k$	✓	✓	✓	✓	$2Kd$	$K(d + \sum_{j=1}^d (\ell_j - 1))$
MNAR $y^j z$	✓	✗	✓	✓	$(K + 1)d$	$Kd + \sum_{j=1}^d (\ell_j - 1)$
MNAR $y^k z$	✓	✓	✗	✓	$K(d + 1)$	$K(1 + \sum_{j=1}^d (\ell_j - 1))$
MNAR yz	✓	✗	✗	✓	$(K + d)$	$K + \sum_{j=1}^d (\ell_j - 1)$
MNAR y	✓	✗	✗	✗	d	$\sum_{j=1}^d (\ell_j - 1)$
MNAR y^k	✓	✓	✗	✗	Kd	$K \sum_{j=1}^d (\ell_j - 1)$
MNAR z	✗	✗	✗	✓	K	K
MNAR z^j	✗	✗	✓	✓	Kd	Kd

Terminology in the sequel:

- MNAR z , MNAR z^j : the only effect of missingness is on the class membership
- MNAR y^* : all the other models which considers the effect of the missingness depending on the variable
- MNAR $*$: all the models

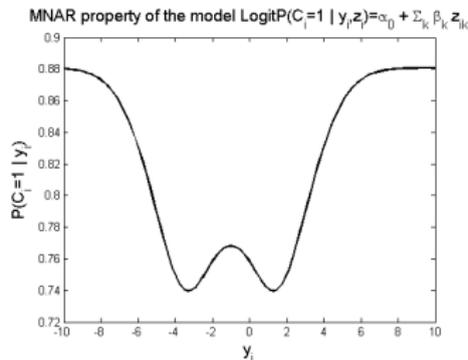
MNARz analysis: it depends on y through z !

$$P(c_{ij} = 1 | y_i; \theta, \psi) = \sum_{k=1}^K P(c_{ij} = 1 | y_i, z_{ik} = 1; \psi) P(z_{ik} = 1 | y_i; \theta)$$

Example of a univariate Gaussian model with the three components

$$0.2N(\cdot; 0, 1) + 0.3N(\cdot; 1, 2) + 0.5N(\cdot; 2, 3)$$

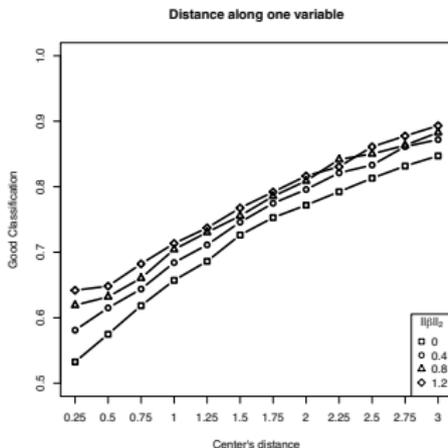
and with parameters of the logit expression: $\alpha_0 = 1, \beta_1 = 1, \beta_2 = -1, \beta_3 = 1$



MNARz analysis: pattern c gives information on partition z !

Draw Bayes error of a MNARz model with two components and 20% of missing data

$$\pi_k = 0.5, \|\mu_2 - \mu_1\| \text{ varies}, \Sigma_1 = \Sigma_2 = I, |\beta_2 - \beta_1| \text{ varies}$$



Both μ_k and β_k act on the Bayes error

Reinterpretation of the MNAR_z and MNAR_z^j models as MAR

Commonly used in Machine Learning

[Jones, 1996, Little and Rubin, 2002, Josse et al., 2019]

Mixture model for Y^{obs} and Bernoulli distribution for C
⇔ MAR mixture model for $\tilde{Y}^{\text{obs}} = (Y^{\text{obs}}|C)$

For example,

$$Y^{\text{obs}} = \begin{pmatrix} ? & 2.6 & 5 \\ \text{blue} & 1.9 & 4 \\ \text{red} & 2.3 & ? \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then \tilde{Y}^{obs} is expressed as

$$\tilde{Y}^{\text{obs}} = \begin{pmatrix} ? & 2.6 & 5 & 1 & 0 & 0 \\ \text{blue} & 1.9 & 4 & 0 & 0 & 0 \\ \text{red} & 2.3 & ? & 0 & 0 & 1 \end{pmatrix}.$$

Proposition 1: in terms of maximum likelihood

The maximum likelihood estimate associated to the dataset \tilde{Y}^{obs} under MAR model is the one associated to the dataset Y^{obs} under MNAR_z or MNAR_z^j models.
⇒ can be extended to other estimation strategies

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Continuous and count data (1)

Previous works: [Teicher, 1963] (without NA), [Miao et al., 2016] (for MNAR data)

Identifiability for a mixture model with MNAR data

⇔ Mixture/MNAR parameters are uniquely determined from available information

Proposition 2: identifiability for continuous and count data

Assume that

- 1 The marginal mixture $\sum_{k=1}^K \pi_k f_k(y_i; \theta_k)$ is identifiable
- 2 There exists a total ordering \preceq of $\mathcal{F}_j \times \mathcal{R}$, for $j \in \{1, \dots, d\}$ fixed, where $\mathcal{F}_j = \{f_{1j}, \dots, f_{Kj}\}$ and $\mathcal{R} = \{\rho_1, \dots, \rho_K\} = \{\rho(\cdot; \psi_1), \dots, \rho(\cdot; \psi_K)\}$. The total ordering is s.t. $\forall k < \ell, F_k = \rho_k f_{kj} \preceq F_\ell = \rho_\ell f_{\ell j}$ implies

$$\lim_{u \rightarrow +\infty} \frac{\rho_\ell(u) f_{\ell j}(u)}{\rho_k(u) f_{kj}(u)} = 0$$

Then the mixture model with one of the MNAR* mechanisms is identifiable up to label swapping

Continuous and count data (2)

Is the total ordering checked for classical distributions ?

f_k	Gaussian		Poisson	
ρ_k	Probit	Logit	Probit	Logit
MNAR $z^j y^k$				
MNAR $y^k z$	✓	generic idenfiability	✓	generic idenfiability
MNAR y^k				
MNAR $y z^j$				
MNAR $y z$				
MNAR y	✓	✓	✓	✓
MNAR z				
MNAR z^j				

Generic identifiability: all not-identifiable parameter choices lie within a proper subvariety, and thus form a set of Lebesgue zero measure

Categorical data

Previous work: [Allman et al., 2009] (without NA)

Recall that for categorical data: conditional independence of the features given the group membership holds i.e. $f_k(\cdot; \theta_k) = \prod_{j=1}^d f_{kj}(\cdot; \theta_{kj})$

Proposition 3: identifiability for categorical data

Assume that $d \geq 2 \lceil \log_2 K \rceil + 1$ and $f_k(\cdot; \theta_k) = \prod_{j=1}^d f_{kj}(\cdot; \theta_{kj})$

- ✓ Then the mixture model with **MNAR_z** or **MNAR_z^j** mechanism is identifiable up to label swapping
- ✗ The mixture model with one of the **MNAR_y*** mechanisms is not identifiable

Mixed data

$f_k(\cdot; \theta_k) = \prod_{j=1}^d f_{kj}(\cdot; \theta_{kj})$, thus identifiability of mixed data directly follows from Proposition 2 for continuous variables and from Proposition 3 for categorical variables

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EM algorithm: looks simple

The expected complete log-likelihood knowing the observed data and a current value of the parameters can be decomposed into two parts

$$\begin{aligned} Q(\theta, \psi, \pi; \theta^r, \psi^r, \pi^r) &= \mathbb{E}[\ell_{\text{comp}}(\theta, \psi, \pi; y, z, c) | y_i^{\text{obs}}, c_i; \theta^r, \psi^r, \pi^r] \\ &= Q_y(\theta, \pi; \theta^r, \psi^r, \pi^r) + Q_c(\psi; \theta^r, \psi^r, \pi^r) \end{aligned}$$

$$Q_y(\theta, \pi; \theta^r, \psi^r, \pi^r) = \sum_{i=1}^n \sum_{k=1}^K (\tau_{ik})^r \log(\pi_k) + \sum_{i=1}^n \sum_{k=1}^K (\tau_{ik})^r E_{iy}^f(\theta)$$

$$Q_c(\psi; \theta^r, \psi^r, \pi^r) = \sum_{i=1}^n \sum_{k=1}^K (\tau_{ik})^r E_{ic}^f(\psi)$$

where for $i = 1, \dots, n$ and $k = 1, \dots, K$,

$$E_{iy}^f(\theta) = \mathbb{E} \left[\log(f_k(y_i; \theta_k)) \mid y_i^{\text{obs}}, z_{ik} = 1, c_i; \theta^r, \psi^r \right]$$

$$E_{ic}^f(\psi) = \mathbb{E} \left[\log(\mathbb{P}(c_i \mid y_i, z_{ik} = 1; \psi)) \mid y_i^{\text{obs}}, z_{ik} = 1, c_i; \theta^r, \psi^r \right]$$

$$(\tau_{ik})^r = \mathbb{P}(z_{ik} = 1 \mid y_i^{\text{obs}}, c_i; \theta^r, \psi^r, \pi^r) \propto \pi_k^r f_k(y_i^{\text{obs}}; \theta_k^r) \mathbb{P}(c_i \mid y_i^{\text{obs}}, z_{ik} = 1; \psi^r)$$

EM algorithm for MNARz and MNARz^j

MNARz, MNARz^j: needs some computations but still simple.

$$\mathbb{P}(c_{ij} = 1 \mid y_i, z_{ik} = 1; \psi) = \rho(\alpha_{kj}) \quad (\text{independent of } y) \quad (\Delta)$$

■ **Gaussian data**: $(y_i \mid z_{ik} = 1; \theta^r) \sim \mathcal{N}(\mu_k, \Sigma_k)$

■ $\mathbb{P}(y_i^{\text{mis}} \mid y_i^{\text{obs}}, z_{ik} = 1, c_i; \theta^r, \psi^r) = \mathbb{P}(y_i^{\text{mis}} \mid y_i^{\text{obs}}, z_{ik} = 1; \theta^r)$ using (Δ) and

$$\left(y_i^{\text{mis}} \mid y_i^{\text{obs}}, z_{ik} = 1; \theta^r \right) \sim \mathcal{N} \left((\tilde{\mu}_{ik}^{\text{mis}})^r, (\tilde{\Sigma}_{ik}^{\text{mis}})^r \right)$$

where $(\tilde{\mu}_{ik}^{\text{mis}})^r$ and $(\tilde{\Sigma}_{ik}^{\text{mis}})^r$ only depend on μ_k^r, Σ_k^r and y_i^{obs}

$\Rightarrow E_{iy}^r(\theta) = \mathbb{E} \left[\log(f_k(y_i; \theta_k)) \mid y_i^{\text{obs}}, z_{ik} = 1, ; \theta^r \right]$ easy to compute (classical formulae)

■ Using (Δ)

$$E_{ic}^r(\psi) = \log(\mathbb{P}(c_i \mid z_{ik} = 1; \psi)) = \sum_{j=1}^d c_{ij} \log \rho(\alpha_{kj}) + (1 - c_{ij}) \log(1 - \rho(\alpha_{kj}))$$

■ Using (Δ)

$$(\tau_{ik})^r \propto \pi_k^r \phi(y_i^{\text{obs}}; (\mu_{ik}^{\text{obs}})^r, (\Sigma_{ik}^{\text{obs,obs}})^r) \prod_{j=1}^d \rho(\alpha_{kj}^r)^{c_{ij}} (1 - \rho(\alpha_{kj}^r))^{1-c_{ij}}$$

EM algorithm for MNAR_z and MNAR_z^j

Recall that: $(y_i^{\text{mis}} \mid y_i^{\text{obs}}, z_{ik} = 1; \theta^r) \sim \mathcal{N}((\tilde{\mu}_{ik}^{\text{mis}})^r, (\tilde{\Sigma}_{ik}^{\text{mis}})^r)$.

- **E-step**: for $k = 1, \dots, K$ and $i = 1, \dots, n$, **compute** $(\tilde{\mu}_{ik}^{\text{mis}})^r, (\tilde{\Sigma}_{ik}^{\text{mis}})^r, (\tau_{ik})^r$ and

$$(\tilde{y}_{i,k})^r = (y_i^{\text{obs}}, (\tilde{\mu}_{ik}^{\text{mis}})^r) \quad \tilde{\Sigma}_{ik}^r = \begin{pmatrix} 0_{i}^{\text{obs,obs}} & 0_{i}^{\text{obs,mis}} \\ 0_{i}^{\text{mis,obs}} & (\tilde{\Sigma}_{ik}^{\text{mis}})^r \end{pmatrix}$$

- **M-step**: for $k = 1, \dots, K$, **compute**

$$\pi_k^{r+1} = \frac{1}{n} \sum_{i=1}^n (\tau_{ik})^r \quad \mu_k^{r+1} = \frac{\sum_{i=1}^n (\tau_{ik})^r (\tilde{y}_{k,i})^r}{\sum_{i=1}^n (\tau_{ik})^r}$$

$$\Sigma_k^{r+1} = \frac{\sum_{i=1}^n [(\tau_{ik})^r ((\tilde{y}_{i,k})^r - \mu_k^{r+1})((\tilde{y}_{i,k})^r - \mu_k^{r+1})^T + \tilde{\Sigma}_{ik}^r]}{\sum_{i=1}^n (\tau_{ik})^r}$$

For ψ^{r+1} : maximization of $Q_c(\psi; \theta^r, \psi^r, \pi^r)$ over ψ with a **Newton-Raphson algorithm** (classical procedure for link functions of interest)

An EM algorithm can also be **easily derived for categorical data**

Not EM algorithm for MNARy*

MNARy* : needs approximations

$$\mathbb{P}(c_{ij} = 1 \mid y_i, z_{ik} = 1; \psi) = \rho(\alpha_{kj} + \beta_{kj}y_{ij}) \quad (\text{not independent of } y)$$

■ **Gaussian data** :

- $(y_i^{\text{mis}} \mid y_i^{\text{obs}}, z_{ik} = 1, c_i)$:
 - ✗ not classical if ρ is **Logit**, ✓ truncated Gaussian distribution if ρ is **Probit**
 - No closed form of $E_{ic}^r(\psi)$ neither for Probit nor for Logit:

$$E_{ic}^r(\psi) = \sum_{j=1}^d c_{ij} \int_{y_{ij}^{\text{mis}}} \log(\rho(\alpha_{kj} + \beta_{kj}y_{ij}^{\text{mis}})) \frac{\rho(\alpha_{kj}^r + \beta_{kj}^r y_{ij}^{\text{mis}})^{c_{ij}} \mathbb{P}(y_{ij}^{\text{mis}} \mid y_i^{\text{obs}}, z_{ik} = 1; \theta^r)}{\int_{y_{ij}^{\text{mis}}} \rho(\alpha_{kj}^r + \beta_{kj}^r x)^{c_{ij}} \mathbb{P}(x \mid y_i^{\text{obs}}, z_{ik} = 1; \theta^r) dx} dy_{ij}^{\text{mis}} + (1 - c_{ij}) \log(1 - \rho(\alpha_{kj} + \beta_{kj}y_{ij}^{\text{obs}}))$$

- ✗ not concave function if ρ is **Logit**
- No closed form of $(\tau_{ik})^r$ neither for Probit nor for Logit

- In the Gaussian case, there is **no closed form** [Pirjol, 2013]
- **SEM easier?** random drawing instead of expectation

SEM algorithm for MNAR_y*

- **SE-step**: draw the missing data $((y_i^{\text{mis}})^{r+1}, z_i^{r+1}) \sim (\cdot | y_i^{\text{obs}}, c_i; \theta^r, \psi^r, \pi^r)$

Use of **One-Gibbs sampling**:

- $(y_i^{\text{mis}})^{r+1} \sim (\cdot | y_i^{\text{obs}}, z_i^r, c_i; \theta^r, \psi^r)$:
 - ✗ not classical if ρ is **Logit**, ✓ truncated Gaussian distribution if ρ is **Probit**
- $z_i^{r+1} \sim (\cdot | y_i^{r+1}, c_i; \theta^r, \psi^r, \pi^r)$: draw the membership k of z_i^{r+1} from the **multinomial distribution** with probabilities

$$\mathbb{P}(z_{ik} = 1 | y_i^{r+1}, c_i; \theta^r, \psi^r, \pi^r) = \frac{\mathbb{P}(c_i | y_i^{r+1}, z_{ik} = 1; \psi^r) \mathbb{P}(y_i^{r+1} | z_{ik} = 1; \theta^r) \pi_k^r}{\sum_{h=1}^K \mathbb{P}(c_i | y_i^{r+1}, z_{ih} = 1; \psi^r) \mathbb{P}(y_i^{r+1} | z_{ih} = 1; \theta^r) \pi_h^r}$$

Let $Y^{r+1} = (y_1^{r+1} | \dots | y_n^{r+1})$, $Z^{r+1} = (z_1^{r+1} | \dots | z_n^{r+1})$ be the imputed matrix and the partition

- **M-step**: for $k = 1, \dots, K$, compute
 - π_k^{r+1} with the proportion of rows of Y^{r+1} belonging to class k
 - $\mu_k^{r+1}, \Sigma_k^{r+1}$ with the mean and covariance matrix of rows of Y^{r+1} belonging to class k
 - ψ^{r+1} with a Newton-Raphson algorithm

Summary for algorithms

	EM			SEM		
	Gaussian		Categorical	Gaussian		Categorical
MNAR _z MNAR _{z^j}	✓		✓	✓		✓
	Probit	Logit		Probit	Logit	
MNAR _{y*}	no closed form	no closed form, optim. pb	not ident.	✓	require algorithms as SIR (costly)	not ident.

What about model selection?

Can select between MCAR and MNAR* with any information criterion (BIC, ICL)

Even if the missing mechanism is ignorable for MCAR. . .

. . . need to model c to compare a MCAR and a MNAR model

CAUTION

- It is just a selection between several proposed MNAR models
- It is not deciding if missingness procedure is “genererically” MNAR or not

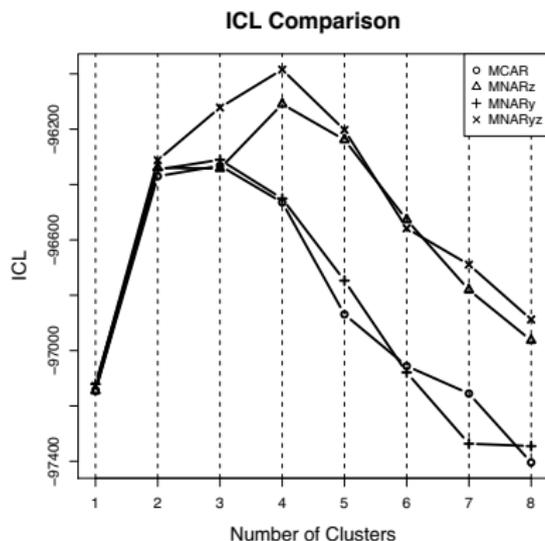
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Hospital Data: continuous features case

- Number of patients: $n = 5\,146$
- Number of features: $d = 7$
 - Age
 - Size
 - Weight
 - Cardiac frequency
 - Hemoglobin concentration
 - Temperature
 - Minimum Diastolic and Systolic Blood Pressure
- Percentage of missing data: 6.4%

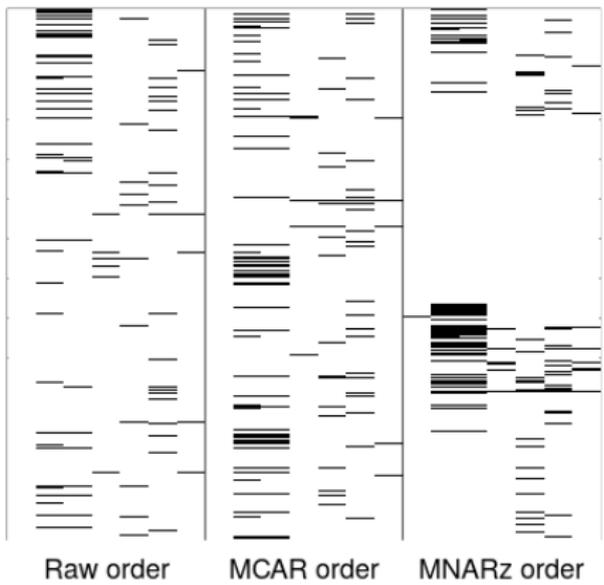
ICL comparison



- MCAR, MNARy and MNARz are equivalent until $K = 3$
- MNARz and MNARyz clearly indicate presence of an additional cluster ($K = 4$)

It seems to be an illustration of the effect of c through MNARz and MNARyz

Missing Pattern



It seems that MNARz modelling leads to a missing free cluster

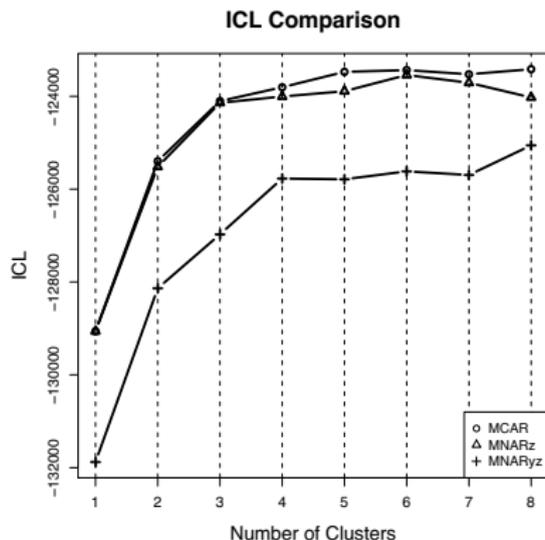
Hospital data: mixed features case

- Number of patients: $n = 5\,146$
- Number of features: $d = 15$ (7 continuous and 8 categorical)
- Percentage of missing data: $\sim 4\%$

Model

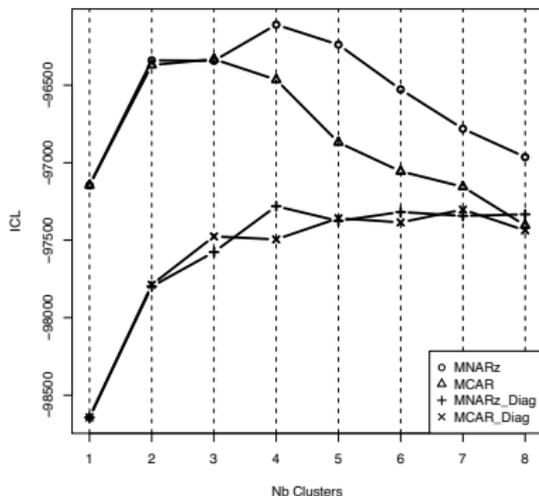
Conditional independence of the variables knowing the cluster

ICL comparison



- MCAR and MNARz are equivalent
- MNARyz seems really inappropriate
- Seems to miss the previous latent structure: [requires a specific exploration...](#)

Exploration 1: local independence is not relevant for this data set



Not accounting for possible conditional dependencies between the continuous variables is inappropriate for this dataset.

Exploration 2: mixture model bias vs missing model bias

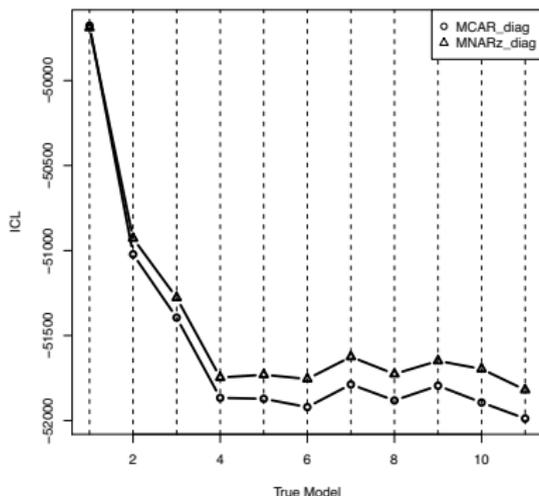
A simulated data set with the following parameters:

- Mixture model: **varying from diagonal to non diagonal hypotheses**
 - 2 clusters, dimension 7
 - 5 000 individuals
 - $\pi_1 = 0.3$, $\pi_2 = 0.7$, $\mu_1 = (0, 0, 0, 0, 0, 0, 0)$, $\mu_2 = (2, 2, 2, 2, 2, 2, 2)$
 - Covariance matrices: $r \in \{1, \dots, 10, \infty\}$ ($r = \infty$ is the diagonal case)

$$\Sigma_k^{(r)} = \begin{pmatrix} 1 & 0.5^r & 0.25^r & \dots & \cdot & \cdot & \cdot \\ 0.5^r & 1 & 0.5^r & \dots & \cdot & \cdot & \cdot \\ 0.25^r & 0.5^r & 1 & \dots & \cdot & \cdot & \cdot \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \cdot & \cdot & \cdot & \dots & 0.25^r & 0.5^r & 1 \end{pmatrix}$$

- Missingness model: **MNAR hypothesis**
 - Proportions of missingness: 0.001 for cluster 1 and 0.06 for cluster 2

Results from exploration 2



- The mixture model bias can not be compensated by the (unbiased) missing mechanism modeling
- It illustrates again the fact that information (on the latent partition) conveys by data is much more important than information conveys by the pattern

Outline

- 1 Introduction
- 2 A model-based MNAR clustering approach
- 3 Identifiability
- 4 Inference procedures
- 5 Medical study illustration
- 6 Concluding remarks**

Summary

- Interest to put a model on c
- Interest of the simple but meaningful model MNAR $_z$
- Link between our models and usual methods

Ongoing works

- Deeper analysis of the previous results with doctors. . .
- Implement the proposed models/algo. in the Mixmod software^a
- Address the **trade-off between biased mixture model and biased missingness mechanism** in particular for the mixed data case

^a<http://www.mixmod.org>

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