Traitement statistique des données manquantes-Part II Numerical and non-numerical data

Christophe Biernacki

Traitement statistique des données manquantes

Atelier Statistique de la SFdS – 10 et 11 mars 2021

–

Part II
Numerical and non-numerical data

C. Biernacki

Laboratoire P. Painlevé, UMR CNRS 8524 & Université de Lille & Inria

Merci à : F. Antonazzo, C. Boyer, G. Celeux, Q. Grimonprez, J. Jacques, J. Josse, C. Keribin, V. Kubicki, F. Laporte, M. Marbac, A. Sportisse, J. Vandaele, V. Vandewalle

Outline

Part I argued in favor of modeling to deal with missing data

Mixture models are a generic/flexible modeling for addressing many classical statistical purposes

Parametric mixture model

- Unknown true distribution[1]:

$$\boldsymbol{x} = (\boldsymbol{x}^o, \boldsymbol{x}^m) = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \overset{iid}{\sim} \mathrm{p}(\cdot)$$

- Parametric mixture assumption:

$$\mathrm{p}(\boldsymbol{x}_1) = \mathrm{p}(\boldsymbol{x}_1; \boldsymbol{\theta}) = \sum_{k=1}^{K} \underbrace{\pi_k}_{\text{proportion}} \underbrace{\mathrm{p}(\boldsymbol{x}_1; \boldsymbol{\alpha}_k)}_{\text{component}}$$

- Mixture parameter:

$$\boldsymbol{\theta} = (\boldsymbol{\pi}, \boldsymbol{\alpha}) \text{ with } \boldsymbol{\pi} = (\pi_1, \ldots, \pi_K) \text{ and } \boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_K)$$

- Model: it includes both the family $\mathrm{p}(\cdot; \boldsymbol{\alpha}_k)$ and the number of components K

$$\mathbf{m} = \{\mathrm{p}(\boldsymbol{x}_1; \boldsymbol{\theta}) : \boldsymbol{\theta} \in \Theta\}$$

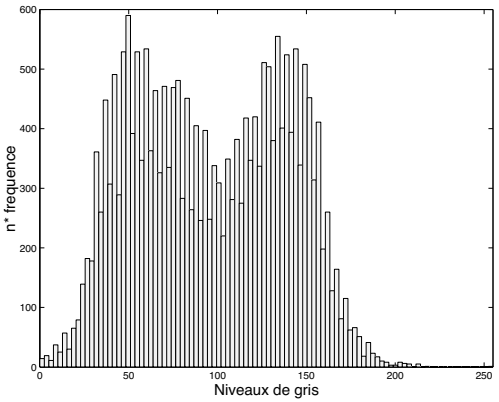
The number of free *continuous* parameters is given by

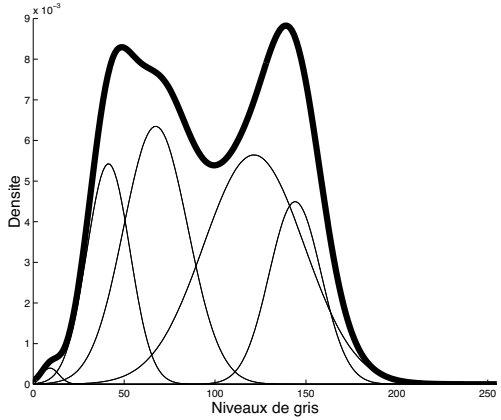
$$\nu = \dim(\Theta)$$

[1] See Part I for for precise definition of $\boldsymbol{x}^o$ and $\boldsymbol{x}^m$.

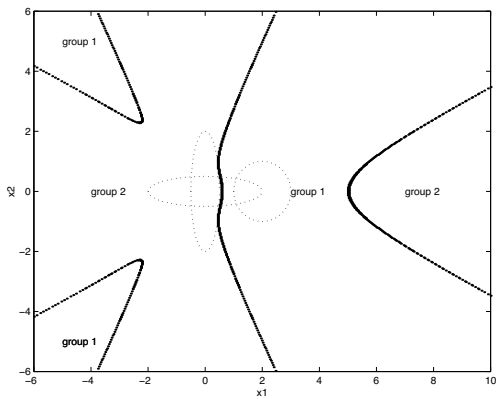
Illustration of mixture models flexibility (1/2)

- Mixture models: extremely flexible family of distributions





- Mixture of mixture models: flexibility for groups also



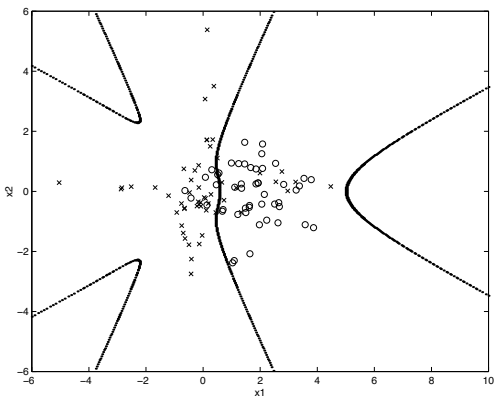
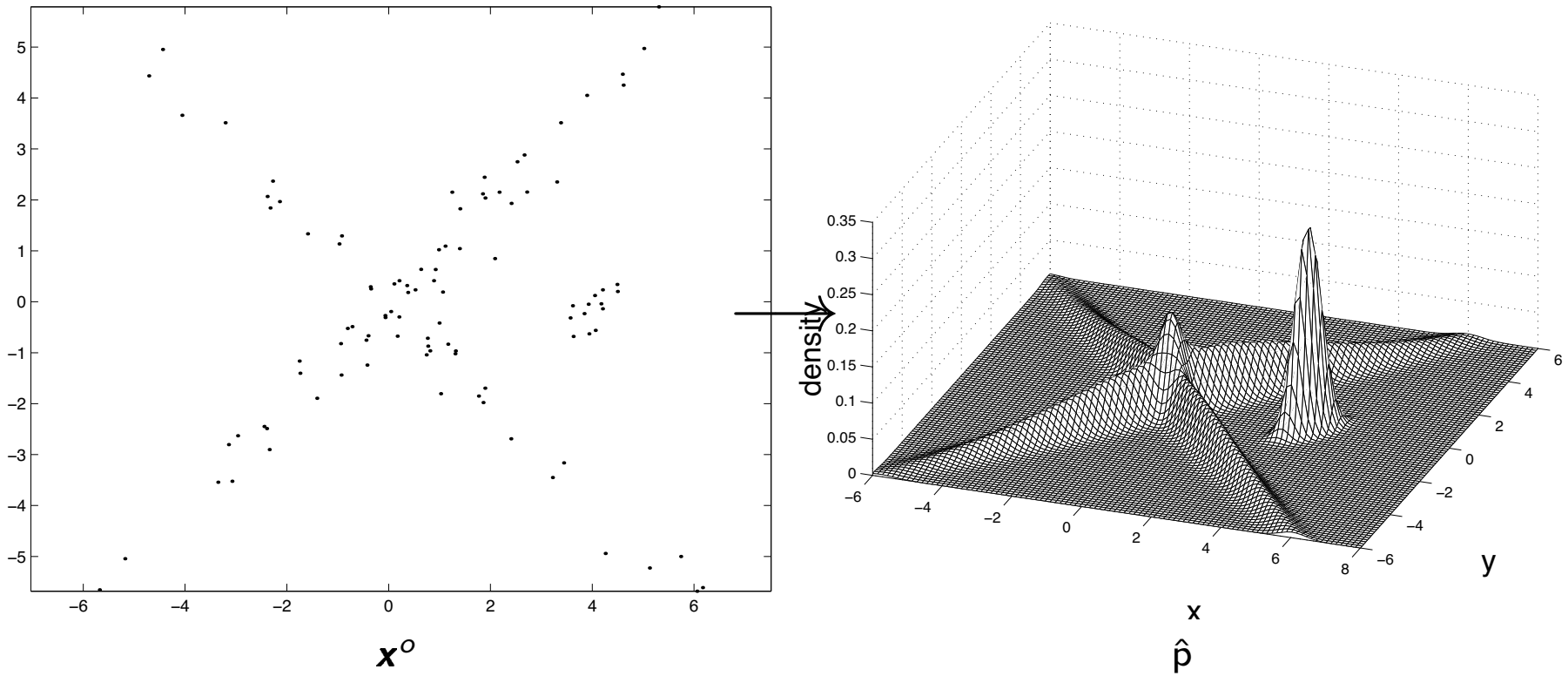


Illustration of mixture models flexibility (2/2)



In many situations, it can be theoretically proved that if K is large enough then a mixture distribution can approximate any distribution!

Sampling assumptions as a missing variable formulation

- A mixture model can be expressed through a binary latent variable:

$$\boldsymbol{z} = (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n)$$

where $\boldsymbol{z}_i = (z_{i1}, \ldots, z_{iK})$ with $z_{ik} \in \{0, 1\}$ and $\sum_{k=1}^{K} z_{ik} = 1$

- Generative process:

$$\begin{aligned} \boldsymbol{z}_1, \ldots, \boldsymbol{z}_n &\overset{iid}{\sim} \text{Mult}_K(1, \boldsymbol{\pi}) \\ \boldsymbol{x}_{i\,|z_{ik}=1} &\overset{ind}{\sim} \text{p}(\cdot; \boldsymbol{\alpha}_k) \end{aligned}$$

- Joint and marginal (or mixture) distributions:

$$\text{p}(\boldsymbol{x}_1, \boldsymbol{z}_1) = \prod_{k=1}^{K} [\pi_k \text{p}(\mathbf{x}_1; \boldsymbol{\alpha}_k)]^{z_{1k}} \quad , \quad \text{p}(\mathbf{x}_1) = \sum_{k=1}^{K} \pi_k \text{p}(\mathbf{x}_1; \boldsymbol{\alpha}_k)$$

With this latent variable formulation, many applications are possible...

The mixture model answer for imputation

Straightforward consequence of the fact that $\mathrm{p}(\cdot;\boldsymbol{\theta})$ is flexible enough to approximate any p

- Single imputation: Straightforward, for instance by the mode[2]

$$\hat{\boldsymbol{x}}^m = \arg\max_{\boldsymbol{x}^m} \mathrm{p}(\boldsymbol{x}^m|\boldsymbol{x}^o;\boldsymbol{\theta})$$

- Multiple imputation: draw multiple values of $\boldsymbol{x}^m$ from the distribution $\mathrm{p}(\boldsymbol{x}^m|\boldsymbol{x}^o;\boldsymbol{\theta})$

[2] Other possibilities, depending on the data type: mean, *etc.*

The mixture model answer for supervised classification (1/2)

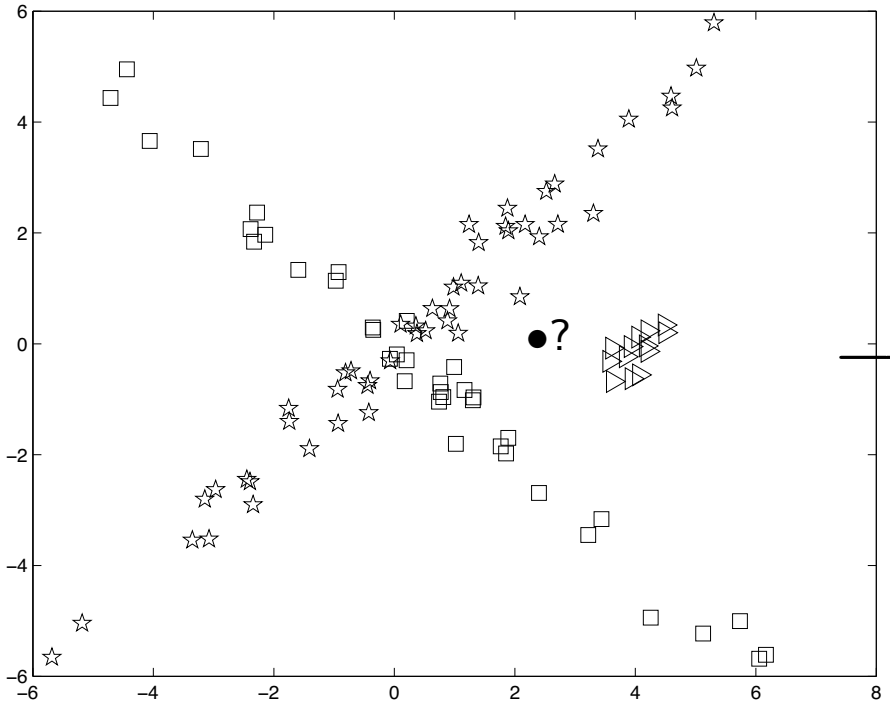
Aim: estimation of an allocation rule $r(\cdot;\boldsymbol{\theta})$ from $(\boldsymbol{x}^o, \boldsymbol{z}^o)$

$$\begin{array}{rccc} r(\cdot;\boldsymbol{\theta}): & \mathcal{X} & \longrightarrow & \{1,\dots,K\} \\ & \boldsymbol{x}_{n+1} & \longmapsto & r(\boldsymbol{x}_{n+1};\boldsymbol{\theta}) \end{array}$$

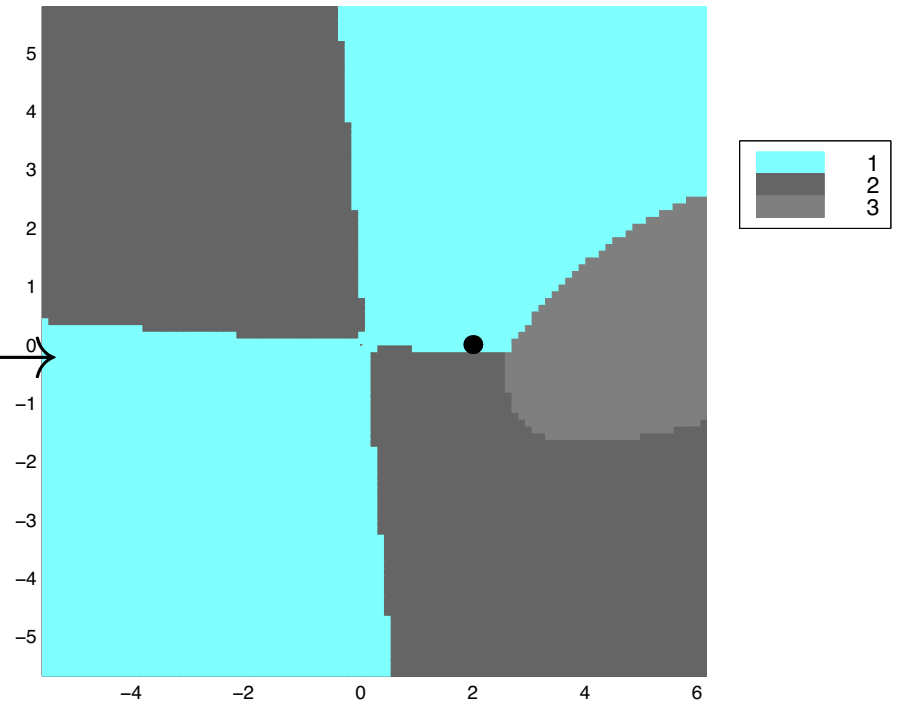
Mixed, missing, uncertain

Individuals $\boldsymbol{x}^o$				Partition $\boldsymbol{z}$			$\Leftrightarrow$	Group
?	0.5	red	5	0	1	0	$\Leftrightarrow$	G_2
0.3	0.1	green	3	1	0	0	$\Leftrightarrow$	G_1
0.3	0.6	{red,green}	3	1	0	0	$\Leftrightarrow$	G_1
0.9	[0.25 0.45]	red	?	0	0	1	$\Leftrightarrow$	G_3
↓	↓	↓	↓					
continuous	continuous	categorical	integer					

The mixture model answer for supervised classification (2/2)



$(\boldsymbol{x}^o, \boldsymbol{z})$ and $\boldsymbol{x}^o_{n+1}$



$(\hat{r}, \hat{\boldsymbol{z}}_{n+1}, \hat{\boldsymbol{x}}^m)$

The mixture model answer for semi-supervised classification (1/2)

- Aim: estimation of an allocation rule $r(\cdot;\boldsymbol{\theta})$ from $(\boldsymbol{x}^o, \boldsymbol{z}^o)$

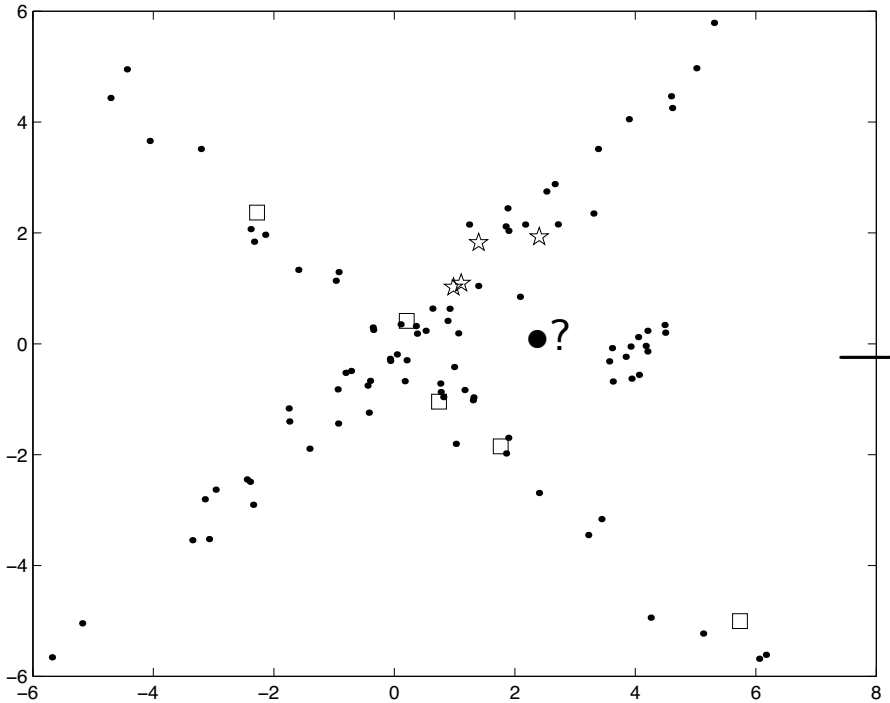
$$r(\cdot;\boldsymbol{\theta}): \quad \begin{array}{ccc} \mathcal{X} & \longrightarrow & \{1,\dots,K\} \\ \boldsymbol{x}_{n+1} & \longmapsto & r(\boldsymbol{x}_{n+1};\boldsymbol{\theta}) \end{array}$$

- Happen when $\boldsymbol{x}$ is cheaper than $\boldsymbol{z}$

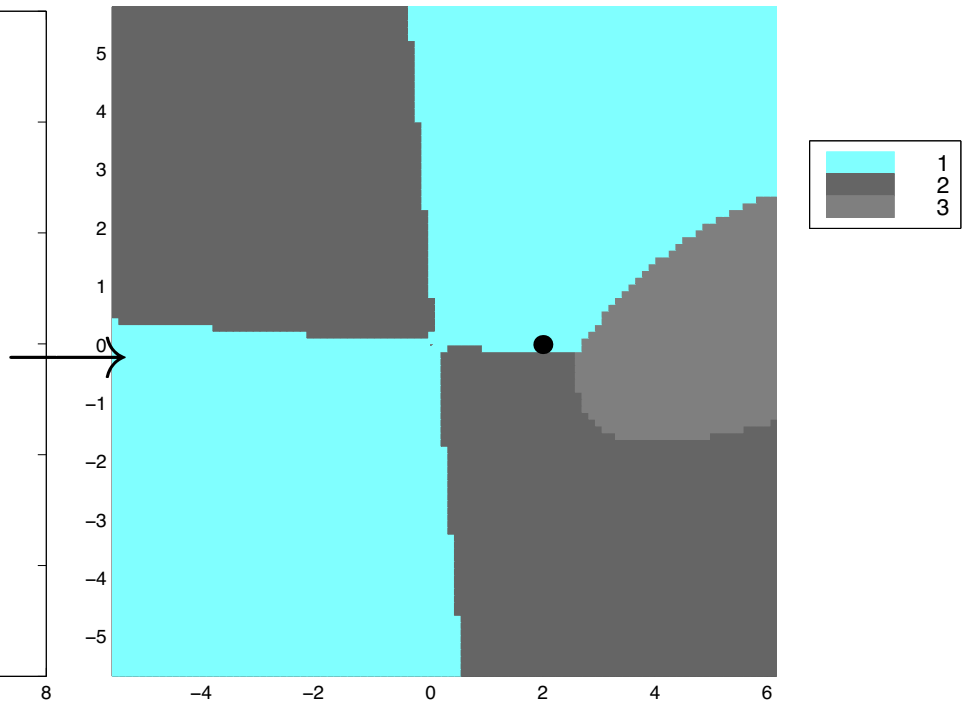
Mixed, missing, uncertain

Individuals $\boldsymbol{x}^o$				Partition $\boldsymbol{z}^o$			$\Leftrightarrow$	Group
?	0.5	red	5	0	?	?	$\Leftrightarrow$	G_2 or G_3
0.3	0.1	green	3	1	0	0	$\Leftrightarrow$	G_1
0.3	0.6	{red,green}	3	?	?	?	$\Leftrightarrow$	???
0.9	[0.25 0.45]	red	?	0	0	1	$\Leftrightarrow$	G_3
↓	↓	↓	↓					
continuous	continuous	categorical	integer					

The mixture model answer for semi-supervised classification (2/2)



$(\boldsymbol{x}^o, \boldsymbol{z}^o)$



$(\hat{r}, \hat{\boldsymbol{z}}^m, \hat{\boldsymbol{x}}^m)$

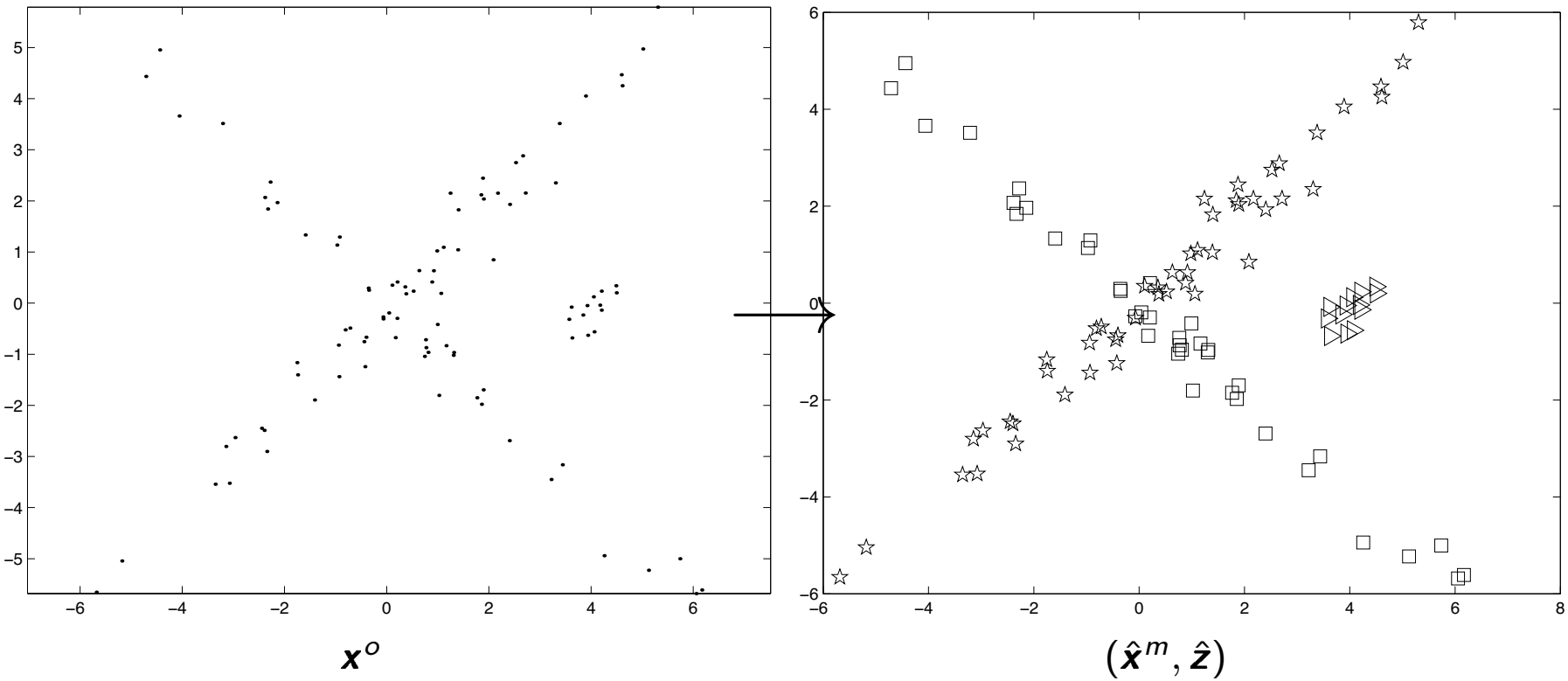
The mixture model answer for unsupervised classification (1/2)

Aim: estimation of $\boldsymbol{z}^m = \boldsymbol{z}$ from $\boldsymbol{x}^o$

Mixed, missing, uncertain

Individuals $\boldsymbol{x}^o$				Partition $\boldsymbol{z}$			$\Leftrightarrow$	Clusters
?	0.5	red	5	?	?	?	$\Leftrightarrow$	???
0.3	0.1	green	3	?	?	?	$\Leftrightarrow$	???
0.3	0.6	{red,green}	3	?	?	?	$\Leftrightarrow$	???
0.9	[0.25 0.45]	red	?	?	?	?	$\Leftrightarrow$	???
↓	↓	↓	↓					
continuous	continuous	categorical	integer					

The mixture model answer for unsupervised classification (2/2)



The mixture model answer in $\{\emptyset$,semi,un$\}$ classification

- Rigorous definition of a group:

$$\boldsymbol{x}_1 \in G_k \quad \Leftrightarrow \quad z_{1k} = 1$$

- Maximum *a posteriori* (MAP):

$$t_k(\boldsymbol{x}_1; \boldsymbol{\theta}) = \mathrm{p}(z_{1k} = 1 | \boldsymbol{x}_1) = \frac{\pi_k \mathrm{p}(\boldsymbol{x}_1; \boldsymbol{\alpha}_k)}{\mathrm{p}(\boldsymbol{x}_1; \boldsymbol{\theta})}$$

$$r(\boldsymbol{x}_1; \boldsymbol{\theta}) = \arg \max_{k=\{1,\ldots,K\}} t_k(\boldsymbol{x}_1; \boldsymbol{\theta})$$

$$\hat{z}_{1r(\boldsymbol{x}_1;\boldsymbol{\theta})} = 1$$

The central question is now to estimate $\boldsymbol{\theta}$. We use the maximum observed log-likelihood principle under the MAR principle:

$$\hat{\boldsymbol{\theta}} = \arg \max_{\boldsymbol{\theta} \in \Theta} \ell(\boldsymbol{\theta}; \boldsymbol{x}^o, \boldsymbol{z}^o)$$

EM and SEM algorithms

EM

- Initialisation: $\boldsymbol{\theta}^{(0)}$
- Iteration (q):
  - E-step: compute $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(q)}) = \mathrm{E}[\ell_c(\boldsymbol{\theta}; \boldsymbol{x}^o, \boldsymbol{x}^m, \boldsymbol{z}^o, \boldsymbol{z}^m))|\boldsymbol{x}^o, \boldsymbol{z}^o; \boldsymbol{\theta}^{(q)}]$
  - M-step: maximize $\boldsymbol{\theta}^{(q+1)} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(q)})$
- Stopping rule: iteration number Q or criterion stability

SEM

- Initialisation: $\boldsymbol{\theta}^{(0)}$
- Iteration (q):
  - SE-step: draw $(\boldsymbol{x}^{m(q)}, \boldsymbol{z}^{m(q)})$ from $\mathrm{p}(\boldsymbol{x}^m, \boldsymbol{z}^o|\boldsymbol{x}^o, z^o; \boldsymbol{\theta}^{(q)})$
  - M-step: maximize $\boldsymbol{\theta}^{(q+1)} = \arg\max_{\boldsymbol{\theta}} \ell_c(\boldsymbol{\theta}; \boldsymbol{x}^o, \boldsymbol{x}^{m(q)}, \boldsymbol{z}^o, \boldsymbol{z}^{m(q)})$
- Stopping rule: iteration number Q

See Part I for more information...

Theoretical model selection criteria

The most widespread principle

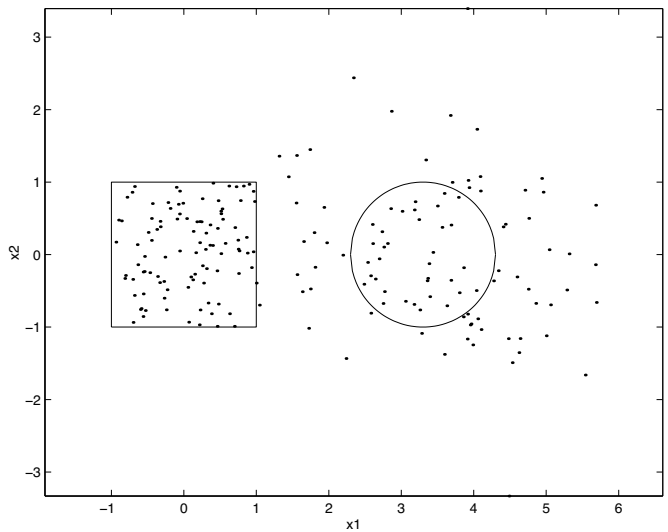
$$\underbrace{\text{Criterion}}_{\text{to be maximized}} = \underbrace{\text{maximum observed log-likelihood}}_{\text{model-data adequacy}} - \underbrace{\boxed{\text{penalty}}}_{\text{"cost" of the model}}$$

criterion	penalty	interpretation	user purpose
	general criteria in statistics		
AIC	ν	model complexity	(semi-)supervised
BIC	$0.5\nu \ln(n)$	model complexity	density estimation
	specific criterion for the clustering aim		
ICL	$0.5\nu \ln(n)$ $- \sum_{i,k} \hat{z}_{ik} \ln t_{ik}(\hat{\boldsymbol{\theta}})$	model complexity + partition entropy	clustering

N.B.: in a (semi-) supervised context, it is also possible to use the predictive error rate

The ICL criterion: robustness to model misspecification

- A bivariate mixture of a uniform and a Gaussian cluster:
  - non-Gaussian component: $\pi_1 = 0.5$, $\mathrm{p}_1(\boldsymbol{x}_1) = 0.25\ \mathbf{I}_{[-1,1]}(x^1)\ \mathbf{I}_{[-1,1]}(x^2)$
  - Gaussian component: $\pi_2 = 0.5$, $\mu_2 = (3.3, 0)'$, $\boldsymbol{\Sigma}_2 = I$
- 50 simulated data sets of size $n = 200$



K	1	2	3	4	5
BIC	.	**60**	.	32	8
ICL	.	**100**	.	.	.

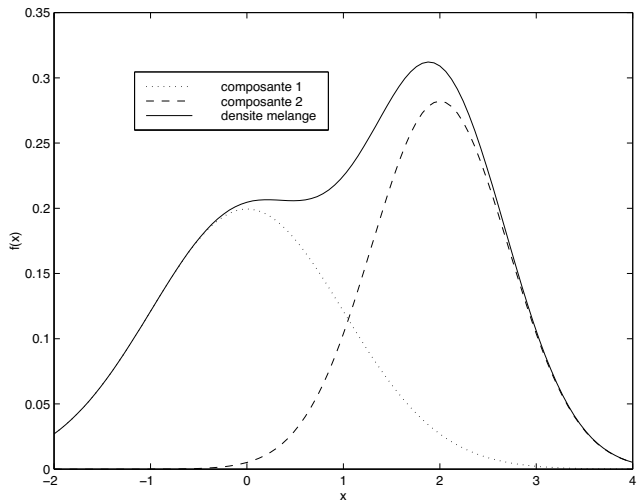
Outline

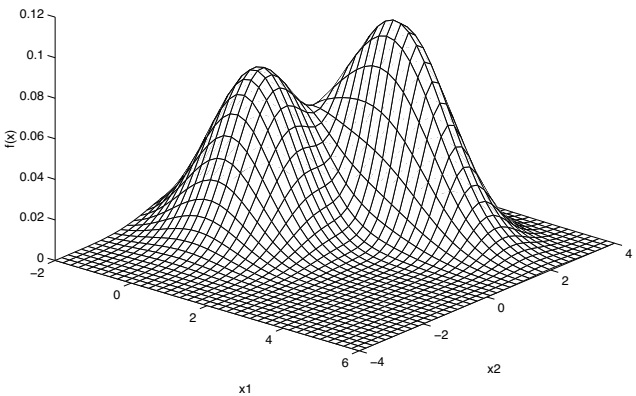
Gaussian mixtures are classical for numerical data ($\mathcal{X} = \mathbb{R}^d$)

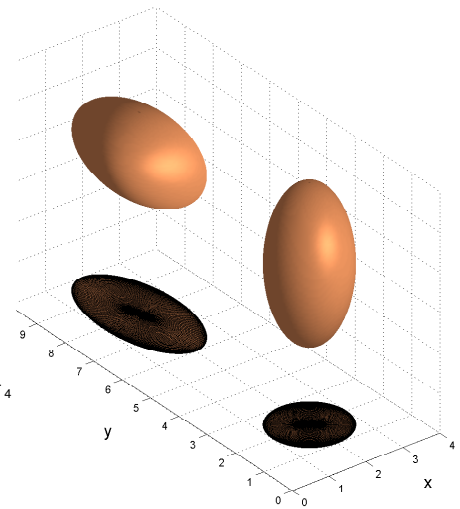
d-variate Gaussian mixture model

$$p(\boldsymbol{x}_1;\boldsymbol{\theta}) = \sum_{k=1}^{K} \pi_k \underbrace{\frac{1}{(2\pi)^{d/2}|\Sigma_k|^{1/2}} \exp\Big(-\frac{1}{2}(\boldsymbol{x}_1-\mu_k)'\Sigma_k^{-1}(\boldsymbol{x}_1-\mu_k)\Big)}_{p(\boldsymbol{x};\mu_k,\Sigma_k)}$$

$$\mathrm{p}(\cdot;\boldsymbol{\alpha}_k) = \mathrm{N}_d(\mu_k,\boldsymbol{\Sigma}_k) \quad \text{where} \quad \boldsymbol{\alpha}_k = (\underbrace{\mu_k}_{\text{center}}, \underbrace{\boldsymbol{\Sigma}_k}_{\text{dispersion}})$$







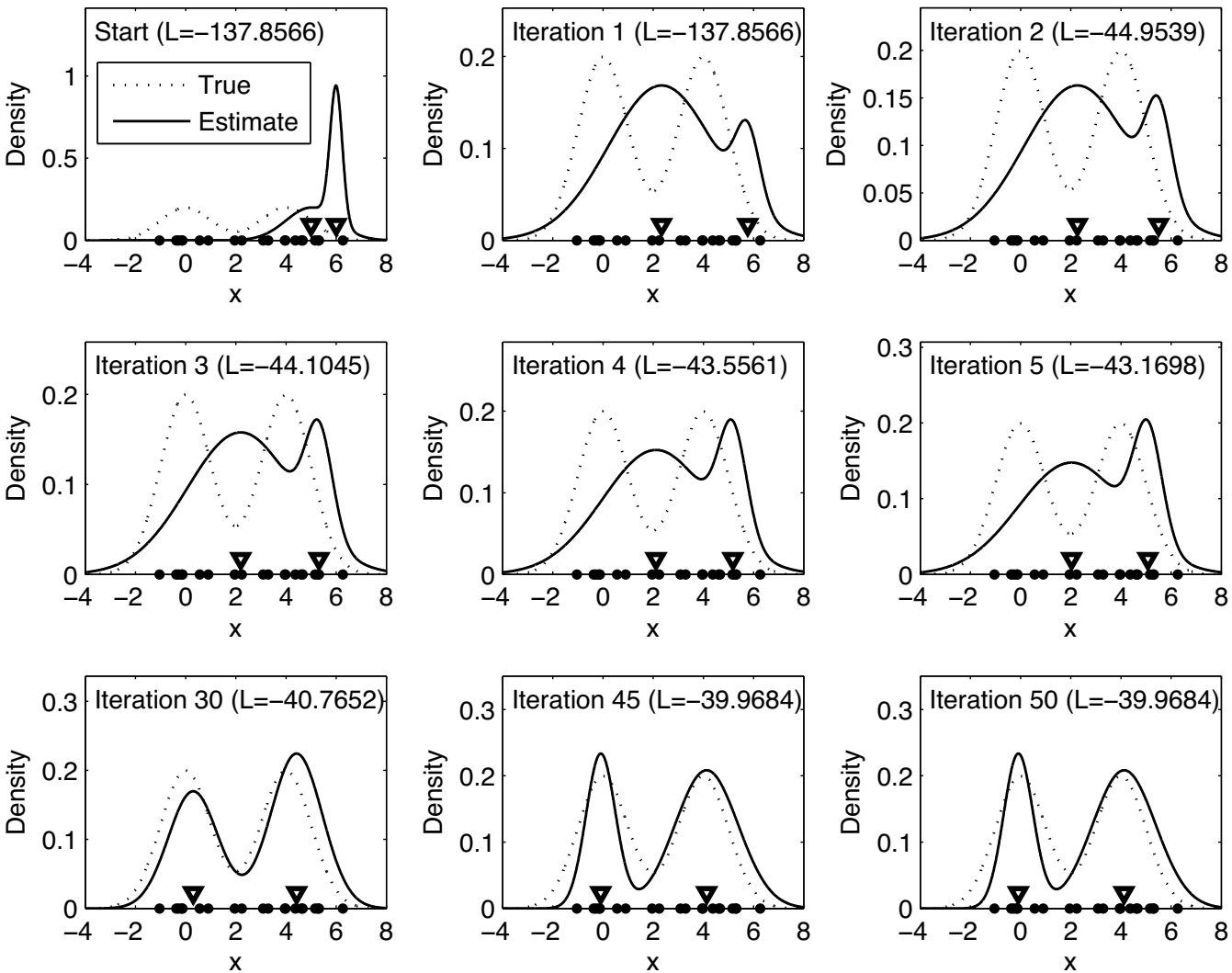
EM with complete data $\boldsymbol{x}$: E-step

Just compute all conditional probabilities $t_{ik}(\boldsymbol{\theta}^{(q)}) = t_k(\boldsymbol{x}_i; \boldsymbol{\theta}^{(q)})$.

EM with complete data $\boldsymbol{x}$: M-step

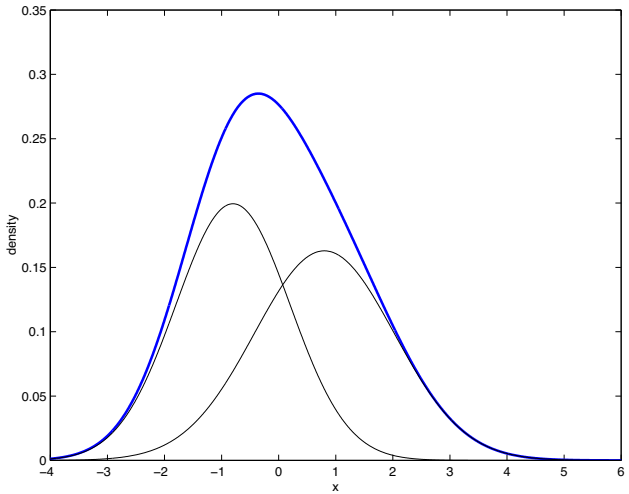
$$
\begin{aligned}
n_k^{(q)} &= \sum_{i=1}^{n} t_{ik}(\boldsymbol{\theta}^{(q)}) \\
\\
\pi_k^{(q+1)} &= \frac{n_k^{(q)}}{n} \\
\boldsymbol{\mu}_k^{(q+1)} &= \frac{1}{n_k^{(q)}} \left(\sum_{i=1}^{n} t_{ik}(\boldsymbol{\theta}^{(q)}) \mathbf{x}_i \right) \\
\boldsymbol{\Sigma}_k^{(q+1)} &= \frac{1}{n_k^{(q)}} \left(\sum_{i=1}^{n} t_{ik}(\boldsymbol{\theta}^{(q)}) (\mathbf{x}_i - \boldsymbol{\mu}_k^{(q+1)})(\mathbf{x}_i - \boldsymbol{\mu}_k^{(q+1)})' \right)
\end{aligned}
$$

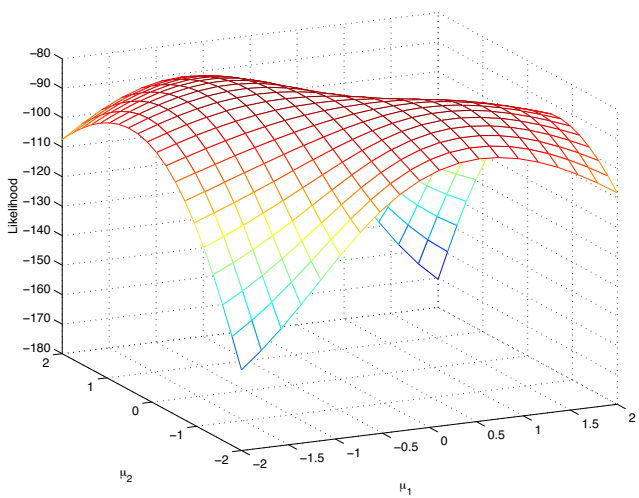
Example of EM in the univariate case with complete data $\boldsymbol{x}$

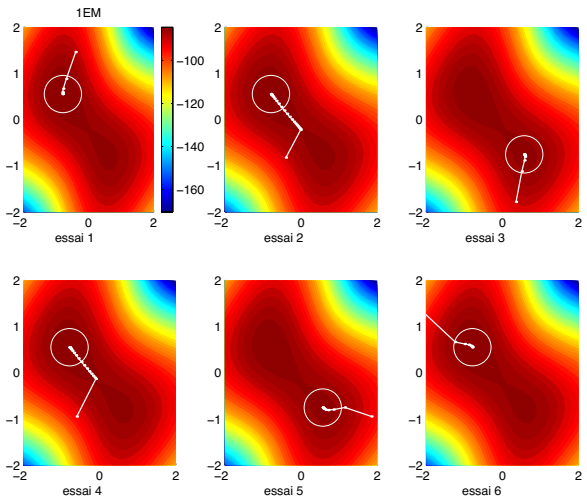


Note : low at the beginning but increase of the log-likelihood

Local maxima

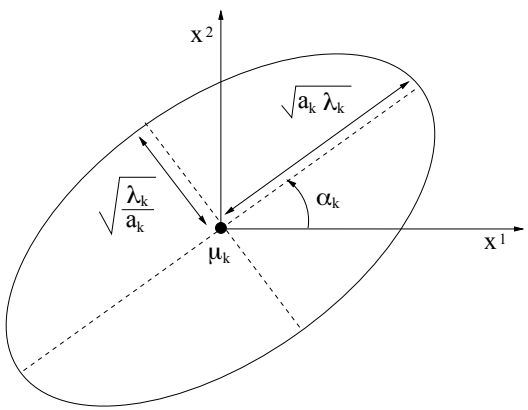


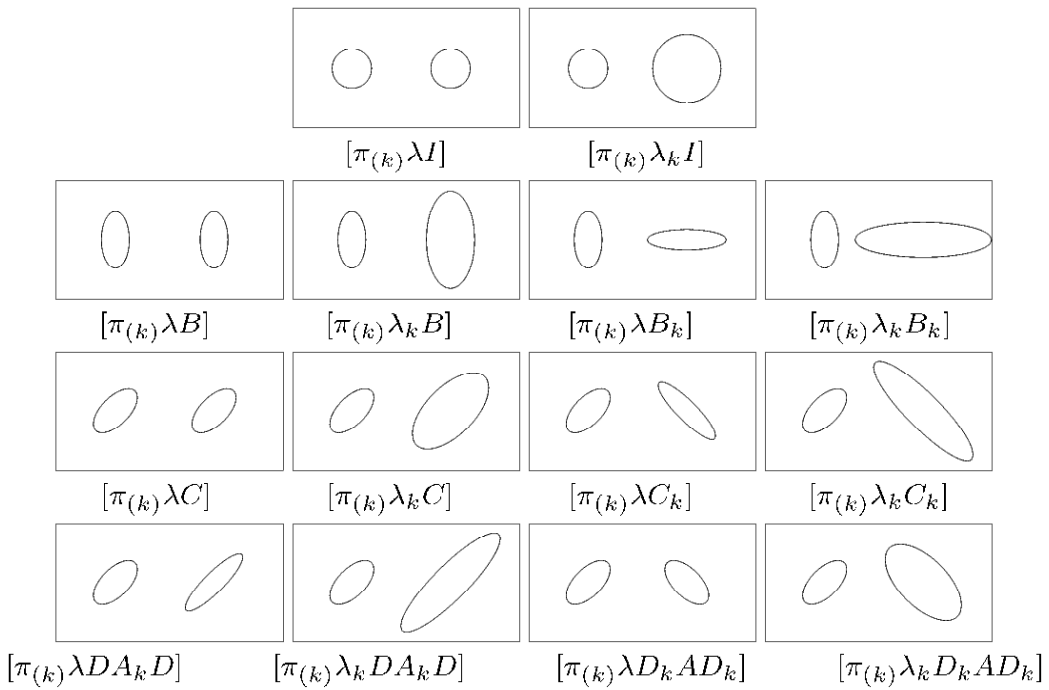




Constraints on $\mathbf{\Sigma}_k$

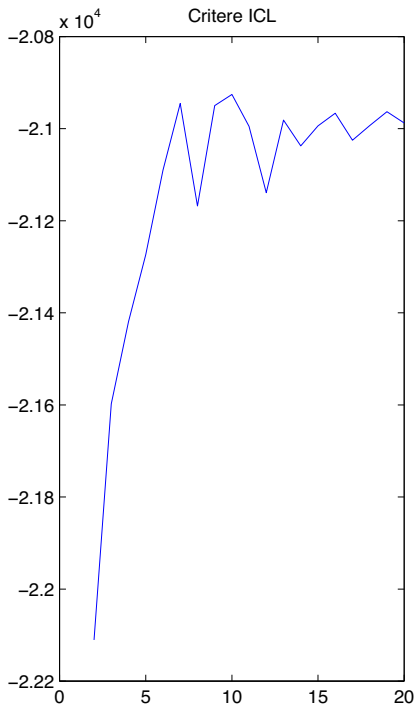
$$\mathbf{\Sigma}_k = \underbrace{\lambda_k}_{\text{volume}} \cdot \underbrace{\mathbf{D}_k}_{\text{orientation}} \cdot \underbrace{\mathbf{A}_k}_{\text{shape}} \cdot \mathbf{D}_k'$$

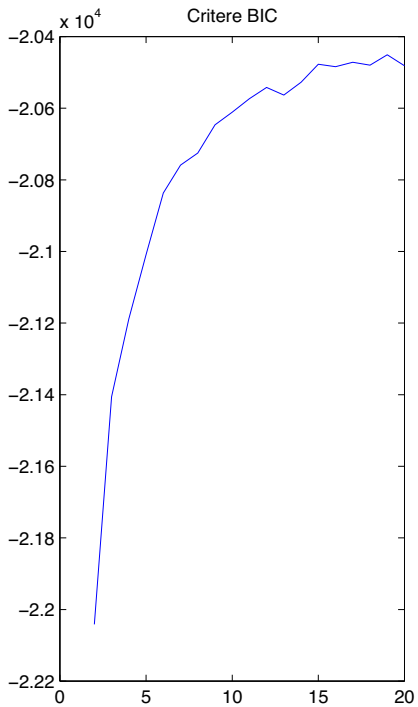


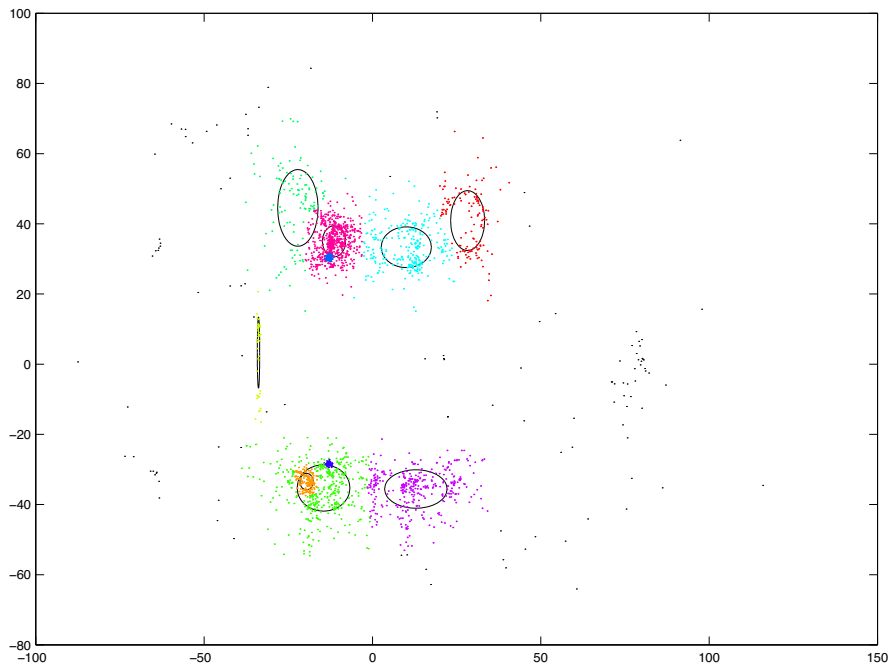


Acoustic emission control example with diagonal constraints

- Data: $n = 2\,061$ event locations in a rectangle of $\mathbb{R}^2$ representing the vessel
- Model: Diagonal Gaussian mixture + uniform (noise)
- Groups: sound locations = vessel defects







EM with incomplete data $\boldsymbol{x}^o$: E-step

$\boldsymbol{\theta}$ and $\boldsymbol{\theta}^+$ the parameters for two successive steps (*idem* for missing data)

$$
\begin{aligned}
z_{ik}^+ &= \mathrm{p}(z_{ik} = 1 | \boldsymbol{x}_i^o; \boldsymbol{\theta}) = \frac{\pi_k \mathrm{p}(\boldsymbol{x}_i^o; \mu_k, \boldsymbol{\Sigma}_k)}{\mathrm{p}(\boldsymbol{x}_i^o; \boldsymbol{\theta})} \\
\boldsymbol{x}_{ik}^{m+} &= \mathrm{E}\left[\boldsymbol{x}_i^m | \boldsymbol{x}_i^o, z_{ik} = 1; \boldsymbol{\theta}\right] = \mu_{ik}^m + \boldsymbol{\Sigma}_{ik}^{mo} \left(\boldsymbol{\Sigma}_{ik}^{oo}\right)^{-1} \left(\boldsymbol{x}_i^o - \mu_{ik}^o\right)
\end{aligned}
$$

where

- $o_i \subseteq \{1, \ldots, d\}$ the set of the observed variables for $\boldsymbol{x}_i$
- $\boldsymbol{x}_i^o$ the corresponding observed data
- m_i the set of the missing variables for $\boldsymbol{x}_i$
- μ_{ik}^o the sub-vector of μ_k associated to index o_i (the same for m_i)
- $\boldsymbol{\Sigma}_{ik}^o$ the sub-matrix of $\boldsymbol{\Sigma}_k$ associated to row o_i and columns m_i (the same for any other combination)

Interpretation

- z_{ik}^+: conditional probability membership given the available information $\boldsymbol{x}_i^o$.
- $\boldsymbol{x}_{ik}^{m+}$: conditional imputation of the missing data given the cluster.

EM with incomplete data $\boldsymbol{x}^o$: M-step

$$\begin{aligned}
\pi_k^+ &= \frac{1}{n_k^+}\sum_{i=1}^{n} z_{ik}^+, \ \boldsymbol{\mu}_k^+ = \frac{1}{n_k^+}\sum_{i=1}^{n} z_{ik}^+ \boldsymbol{x}_{ik}^+ \\
\boldsymbol{\Sigma}_k^+ &= \frac{1}{n_k^+}\sum_{i=1}^{n} z_{ik}^+ \left[(\boldsymbol{x}_{ik}^+ - \boldsymbol{\mu}_k^+)(\boldsymbol{x}_{ik}^+ - \boldsymbol{\mu}_k^+)' + \boldsymbol{\Sigma}_{ik}^+\right]
\end{aligned}$$

where $n_k^+ = \sum_{i=1}^n z_{ik}^+$, $\boldsymbol{x}_{ik}^+ = \begin{pmatrix} \boldsymbol{x}_i^o \\ \boldsymbol{x}_{ik}^{m+} \end{pmatrix}$, $\boldsymbol{\Sigma}_{ik}^+ = \begin{pmatrix} 0_i^o & 0_i^{om} \\ 0_i^{mo} & \boldsymbol{\Sigma}_{ik}^{m+} \end{pmatrix}$ with 0 the $d \times d$ null matrix, and $\boldsymbol{\Sigma}_{ik}^{m+} = \boldsymbol{\Sigma}_{ik}^{mo}\left(\boldsymbol{\Sigma}_{ik}^{o}\right)^{-1}\boldsymbol{\Sigma}_{ik}^{om}$.

Interpretation of $\boldsymbol{\Sigma}_{ik}^{m+}$

Variance correction due to the under-estimation of variability caused by the imputation of missing data.

Outline

Categorical data: latent class model

- categorical variables: d variables with l_j modalities each, $\boldsymbol{x}_{ij} \in \{0,1\}^{l_j}$ and

$$x_{jh} = 1 \quad \Leftrightarrow \quad \text{variable } j \text{ of } \boldsymbol{x}_i \text{ takes modality } h$$

- Conditional independence:

$$\mathrm{p}(\boldsymbol{x}_i; \boldsymbol{\alpha}_k) = \prod_{j=1}^{d} \prod_{h=1}^{l_j} (\alpha_{kjh})^{x_{ijh}}$$

and

$$\alpha_{kjh} = \mathrm{p}(x_{ijh} = 1 | z_{ik} = 1)$$

with $\boldsymbol{\alpha}_k = (\alpha_{kjh}; j = 1, \ldots, d; h = 1, \ldots, l_j)$

EM illustration with binary data: SPAM E-mail Database[4]

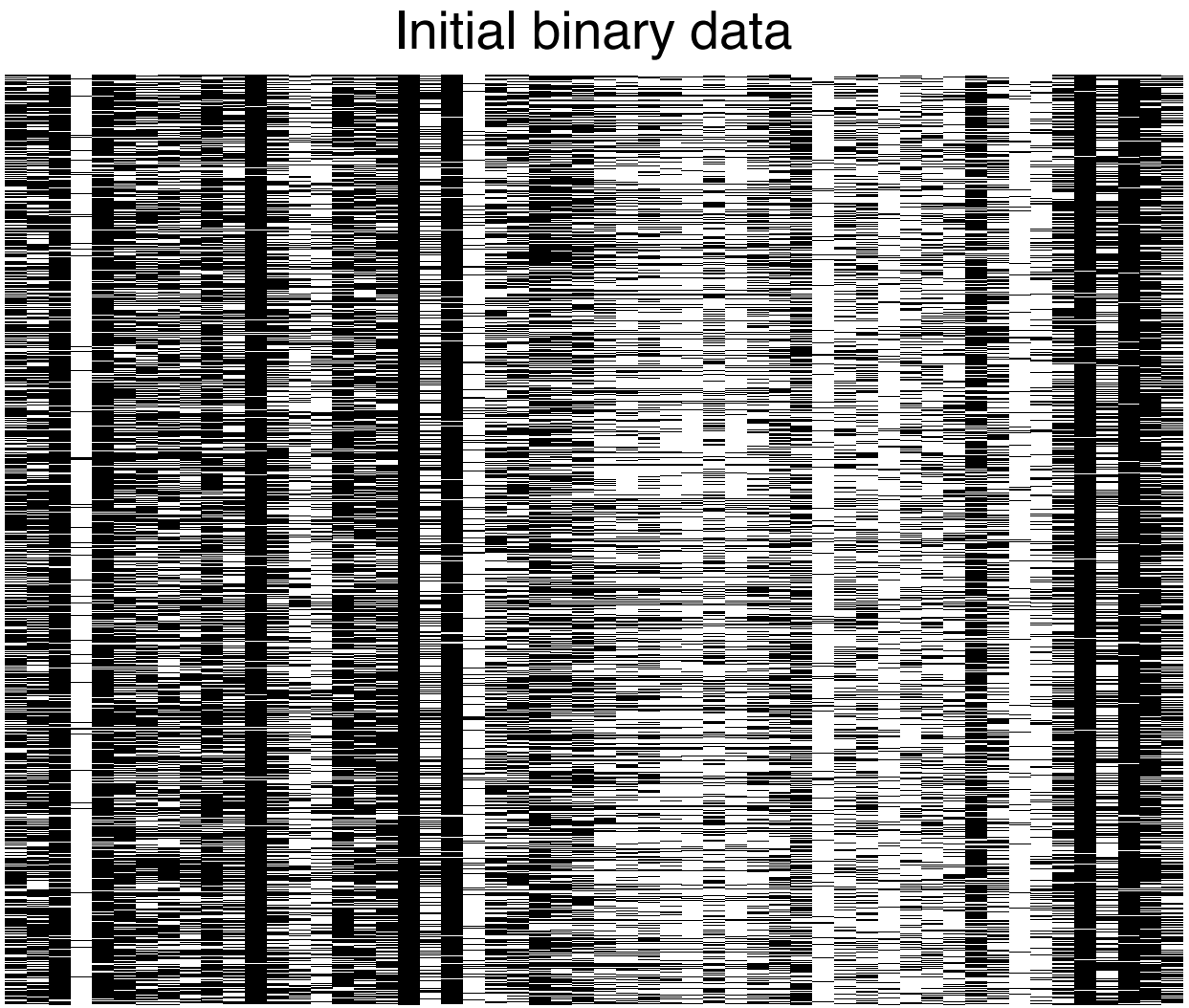
- $n = 4\,601$ e-mails composed by 1813 "spams" and 2 788 "good e-mails"
- $d = 48 + 6 = 54$ continuous descriptors[3]
  - 48 percentages that a given word appears in an e-mail ("make", "you'...)
  - 6 percentages that a given char appears in an e-mail (";", "$"...)
- Transformation of continuous descriptors into binary descriptors

$$x_{ij} = \begin{cases} 1 & \text{if word/char } j \text{ appears in e-mail } i \\ 0 & \text{otherwise} \end{cases}$$

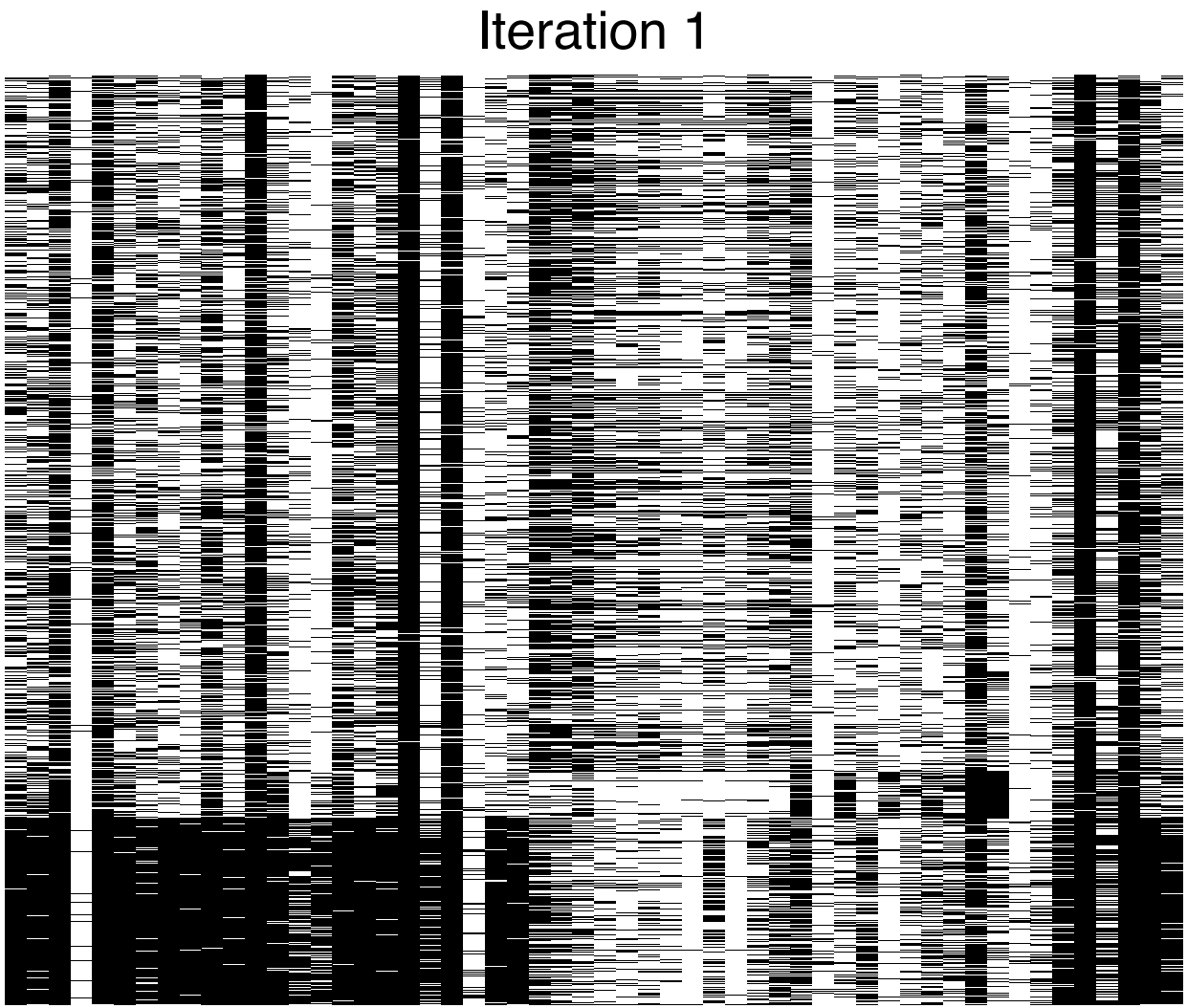
[3] There are 3 other continuous descriptors we do not use

[4] https://archive.ics.uci.edu/ml/machine-learning-databases/spambase/

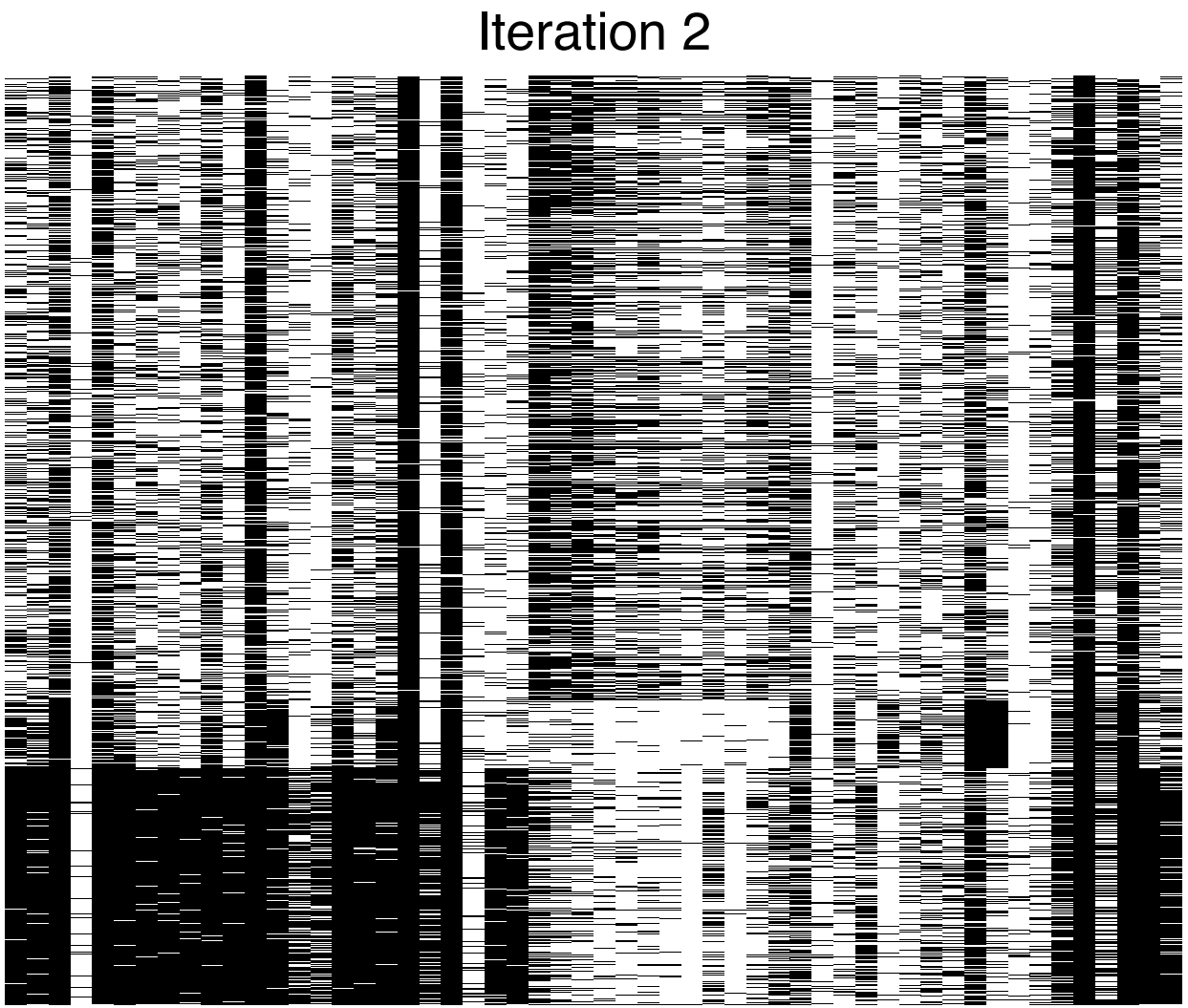
An EM run with a binary data set



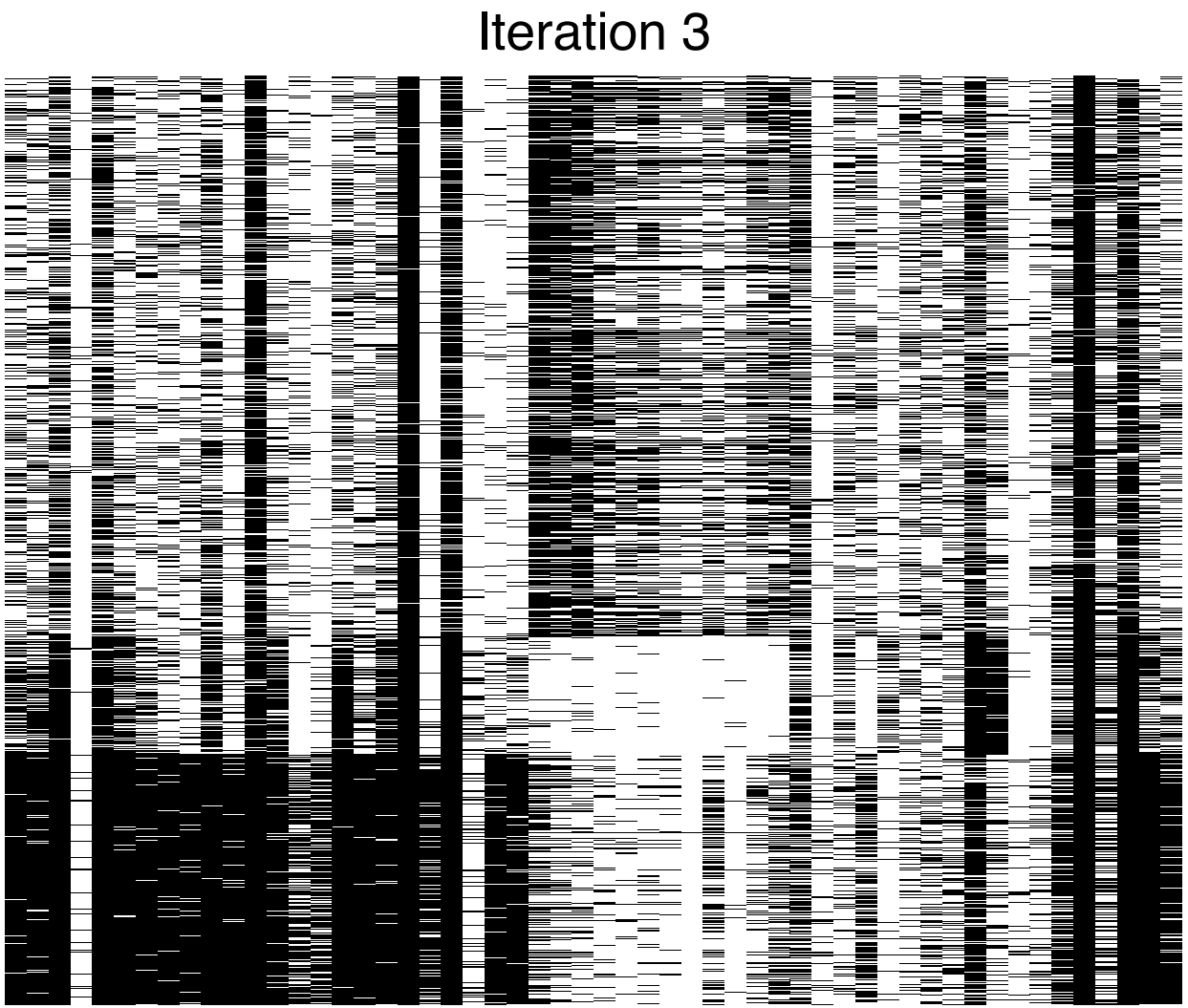
An EM run with a binary data set



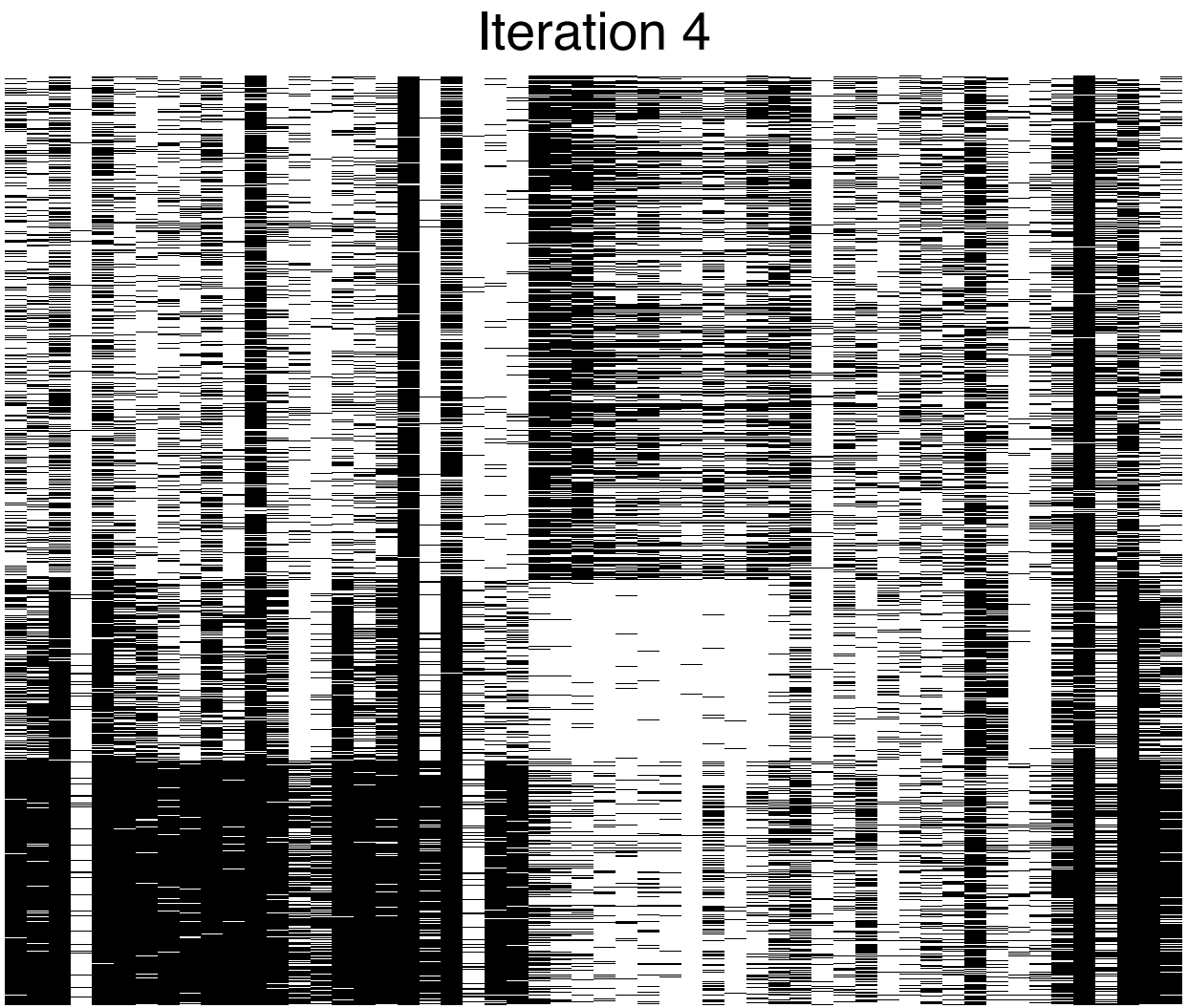
An EM run with a binary data set



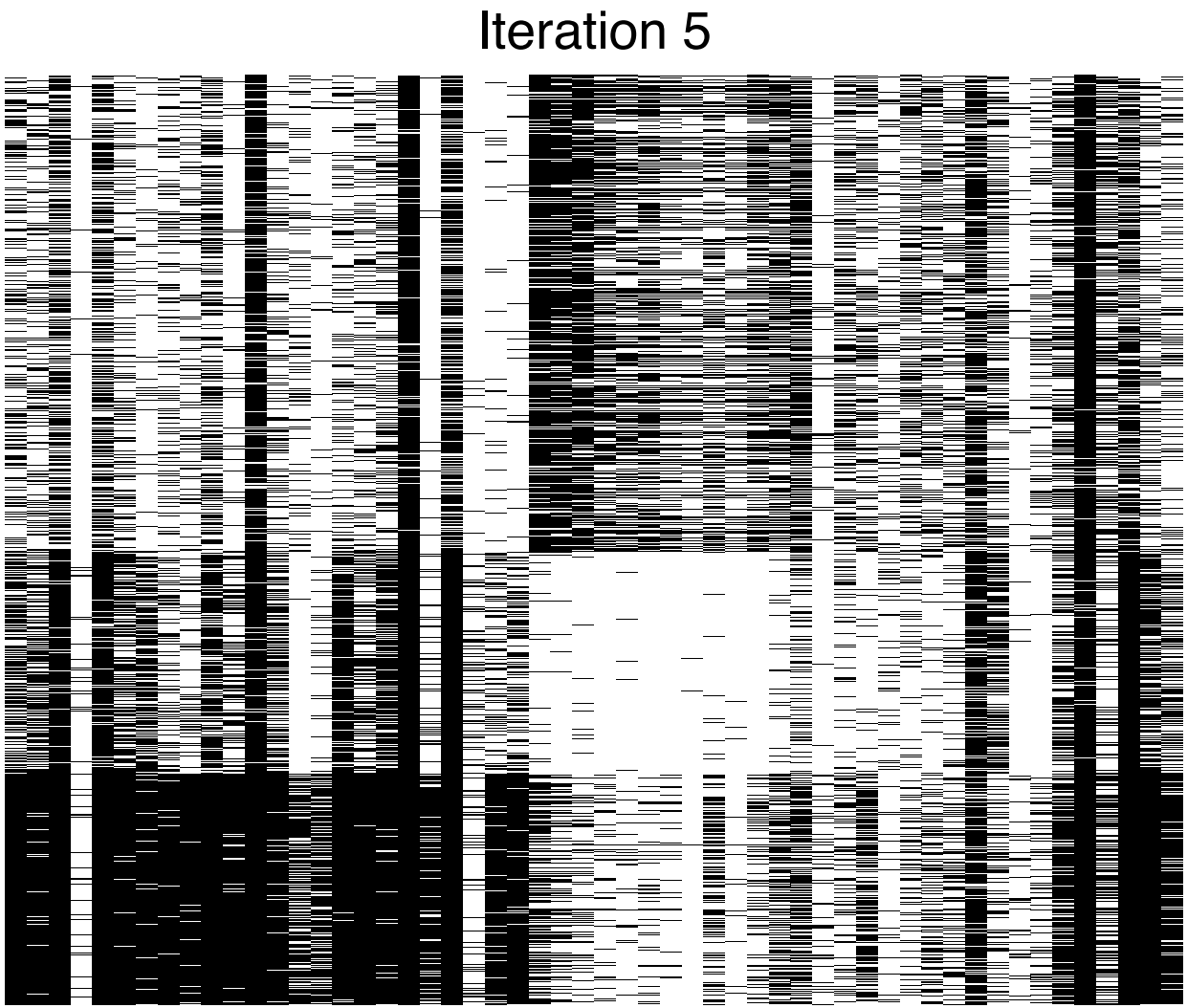
An EM run with a binary data set



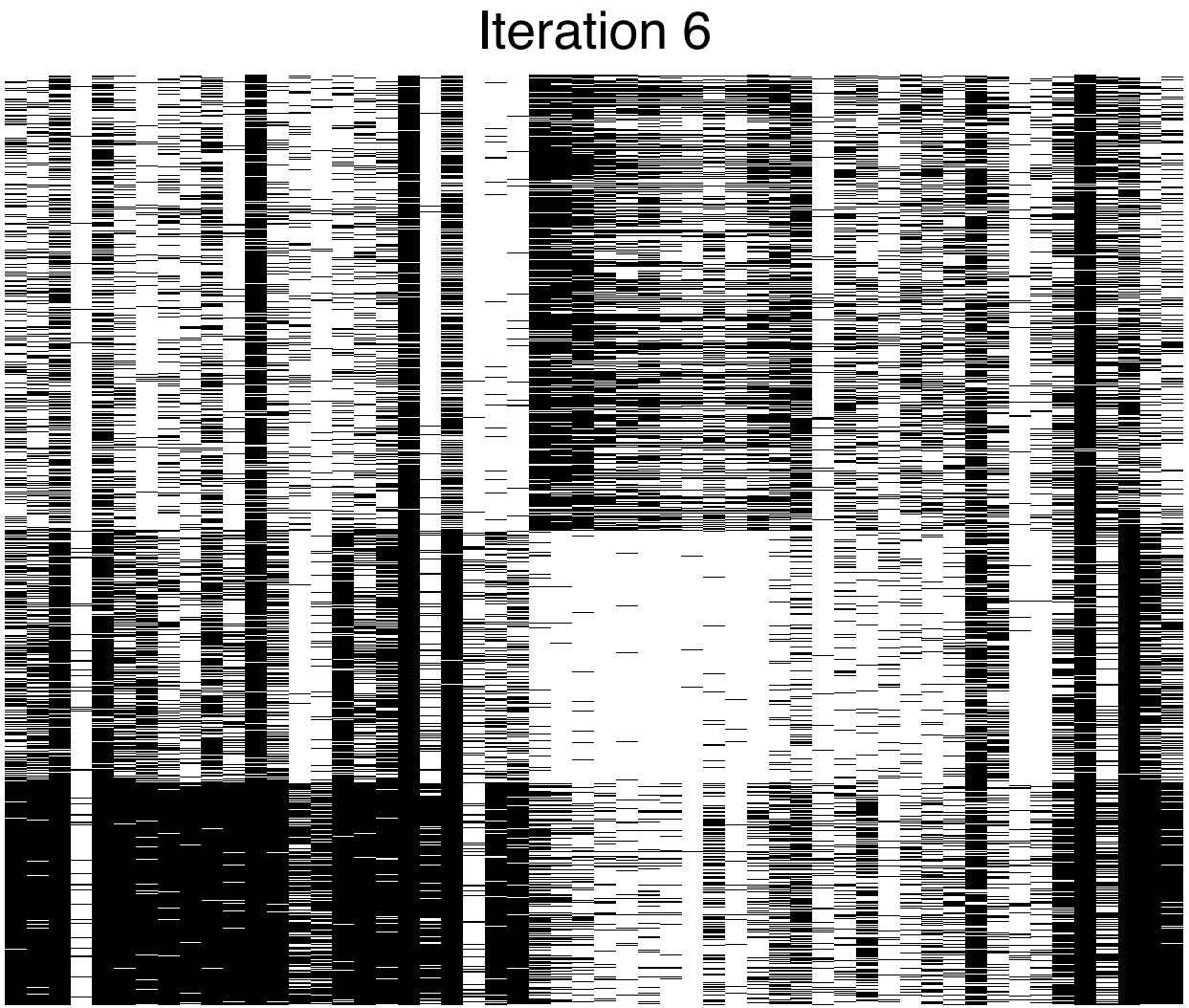
An EM run with a binary data set



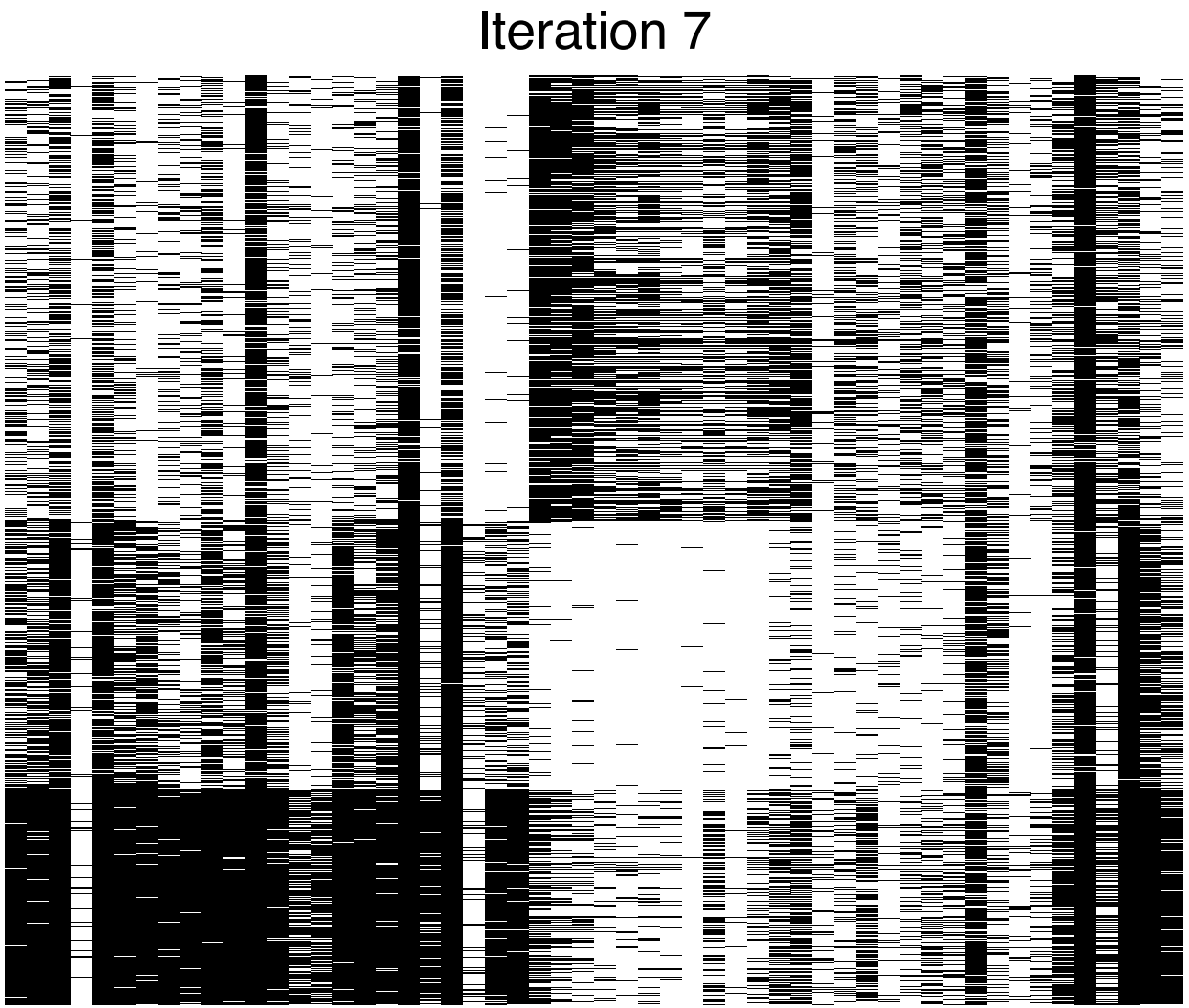
An EM run with a binary data set



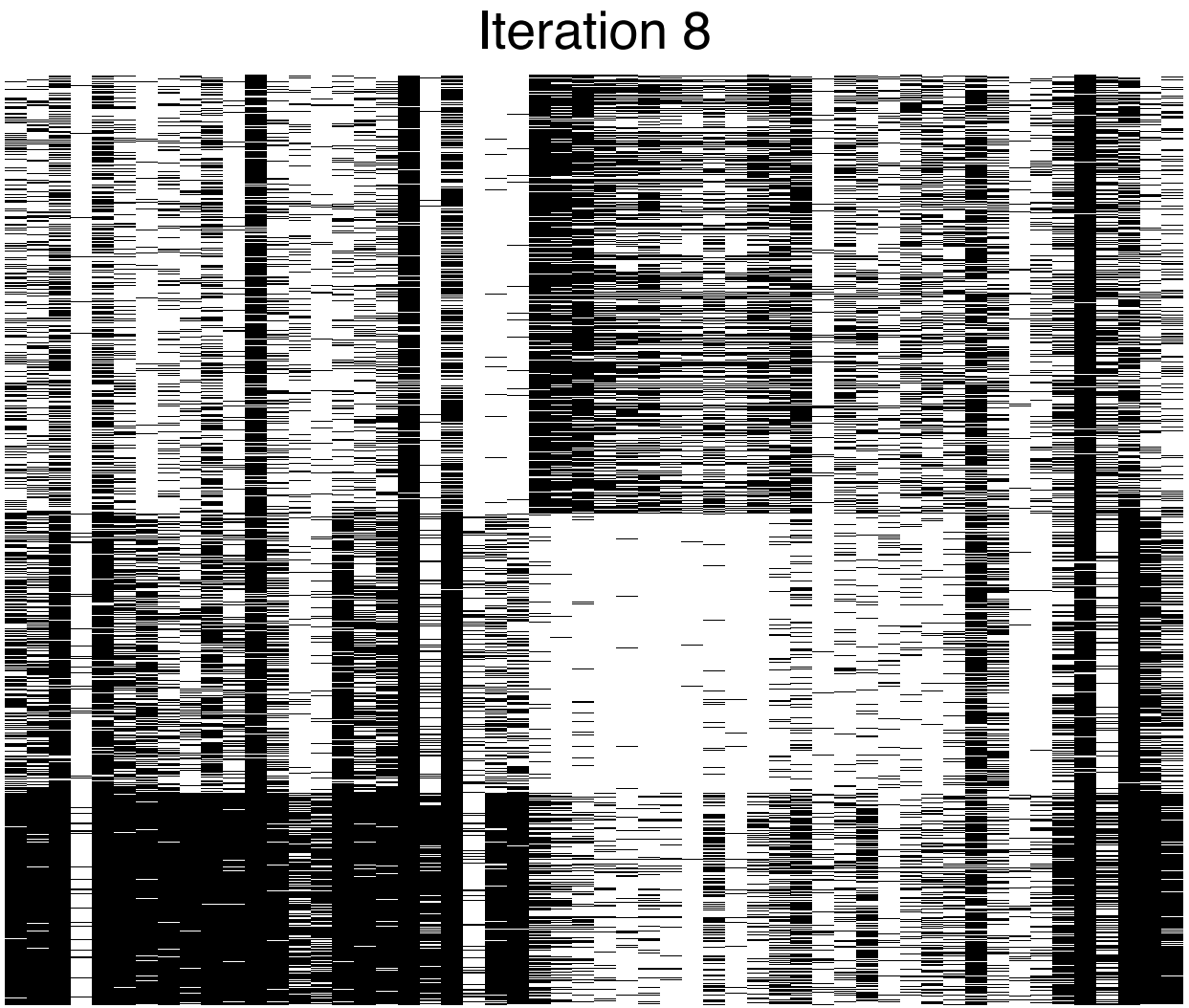
An EM run with a binary data set



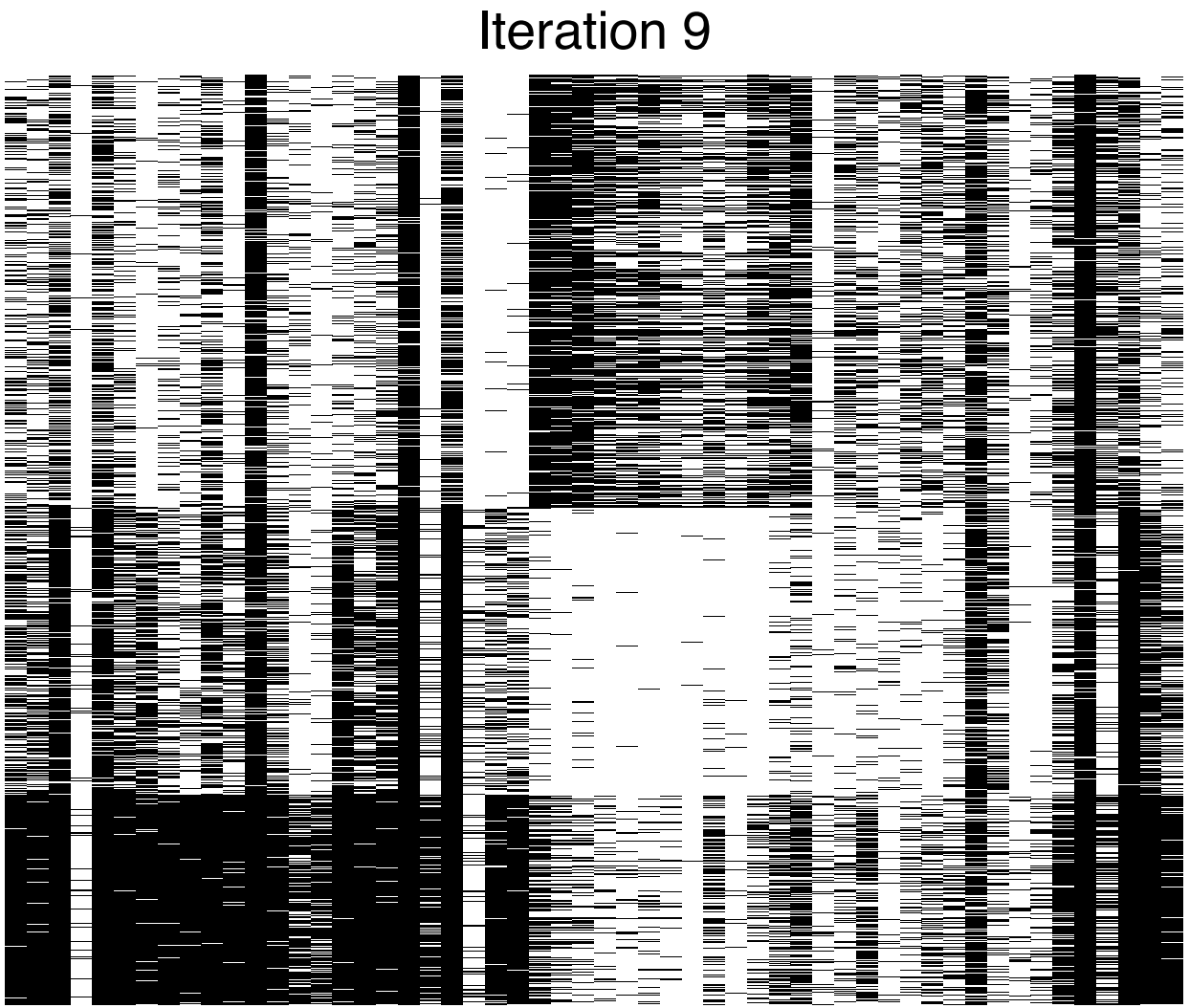
An EM run with a binary data set



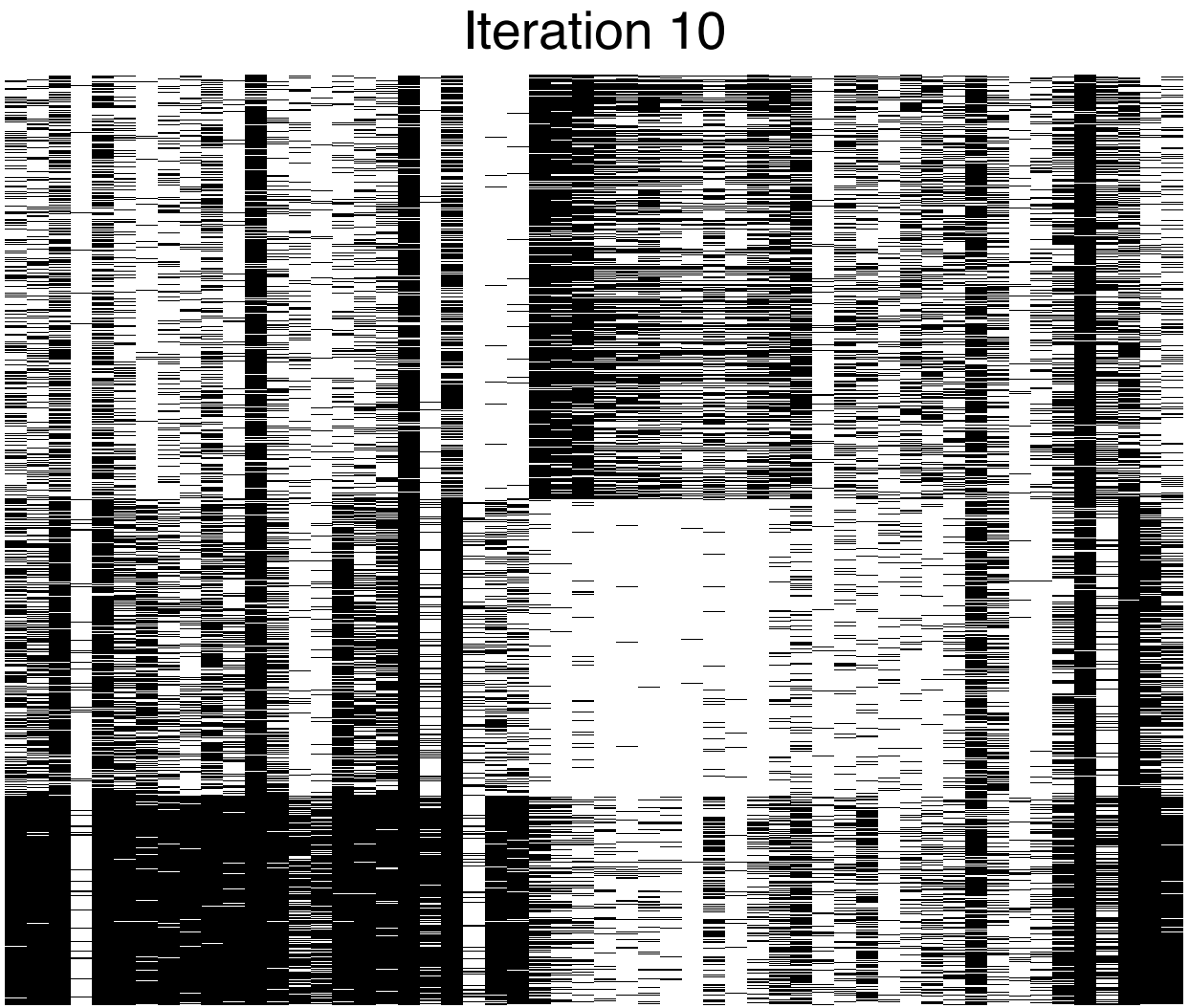
An EM run with a binary data set



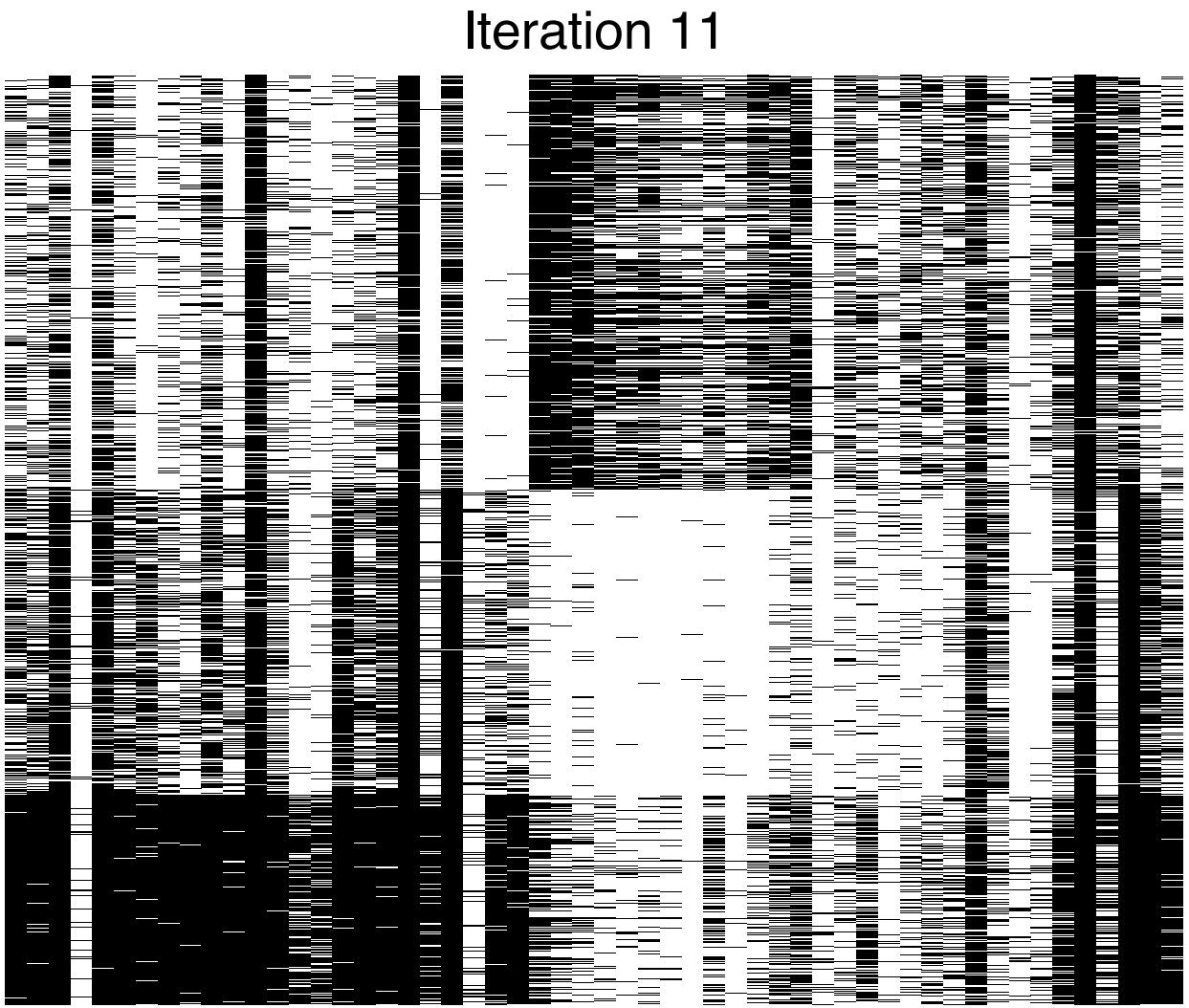
An EM run with a binary data set



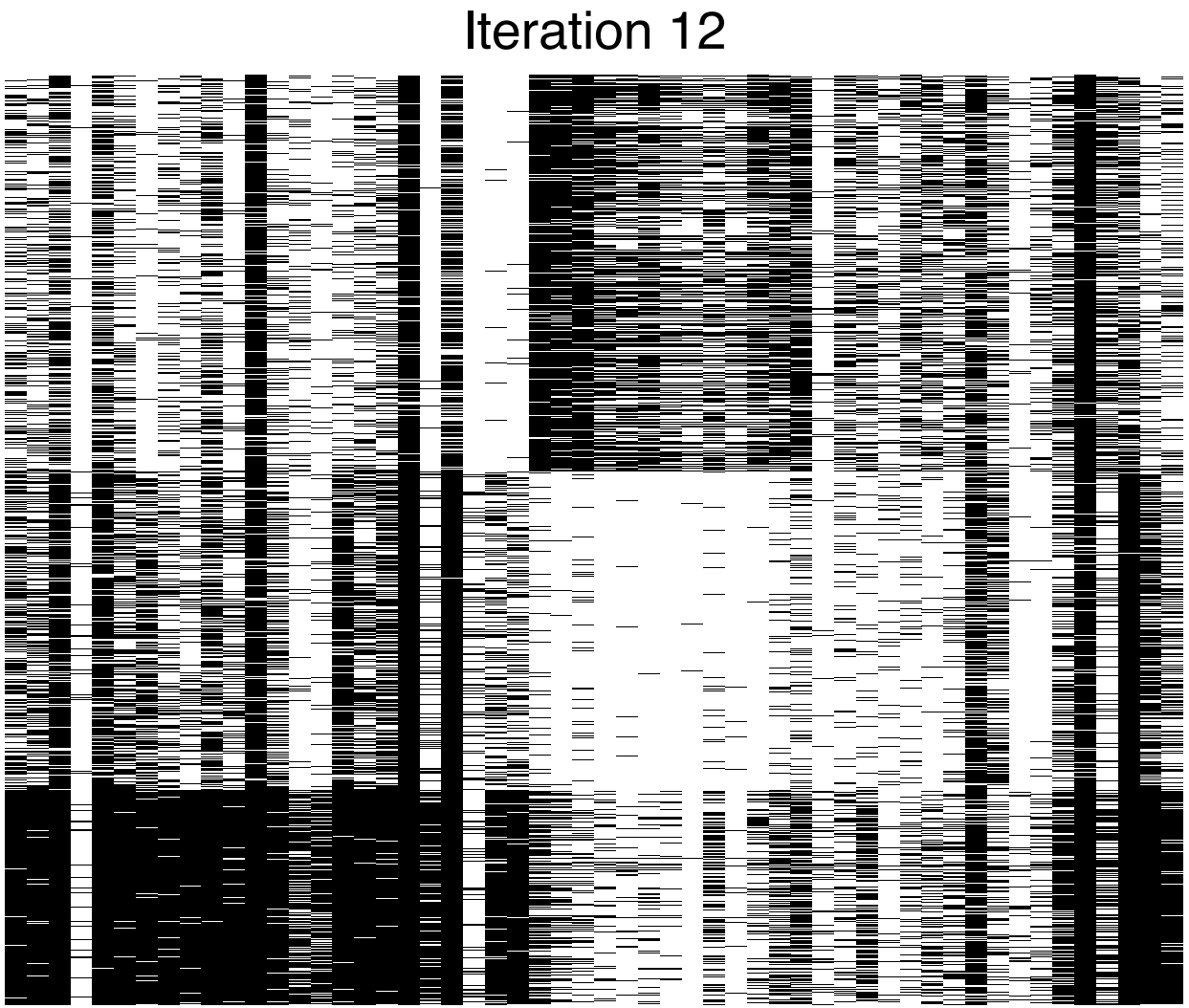
An EM run with a binary data set



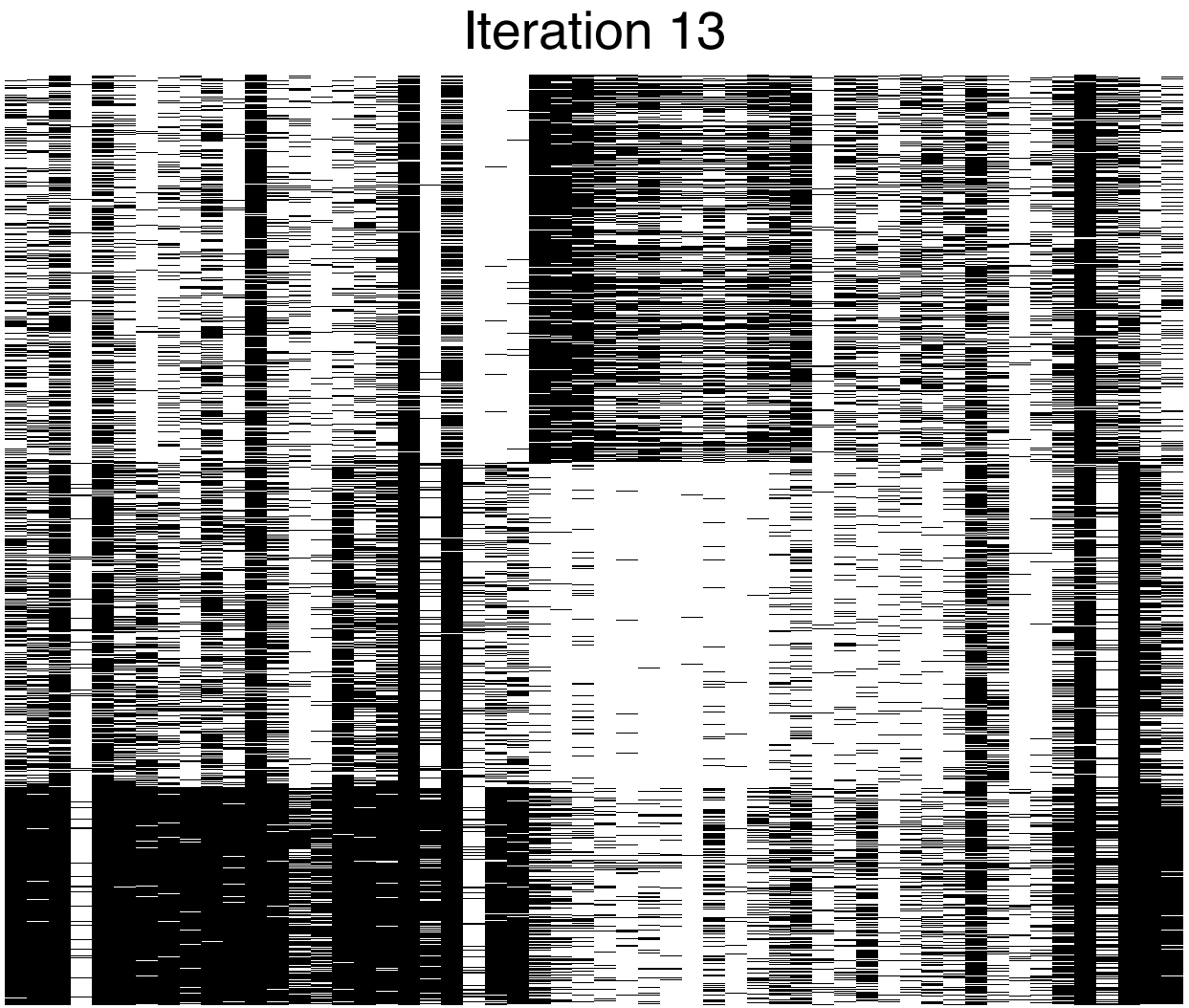
An EM run with a binary data set



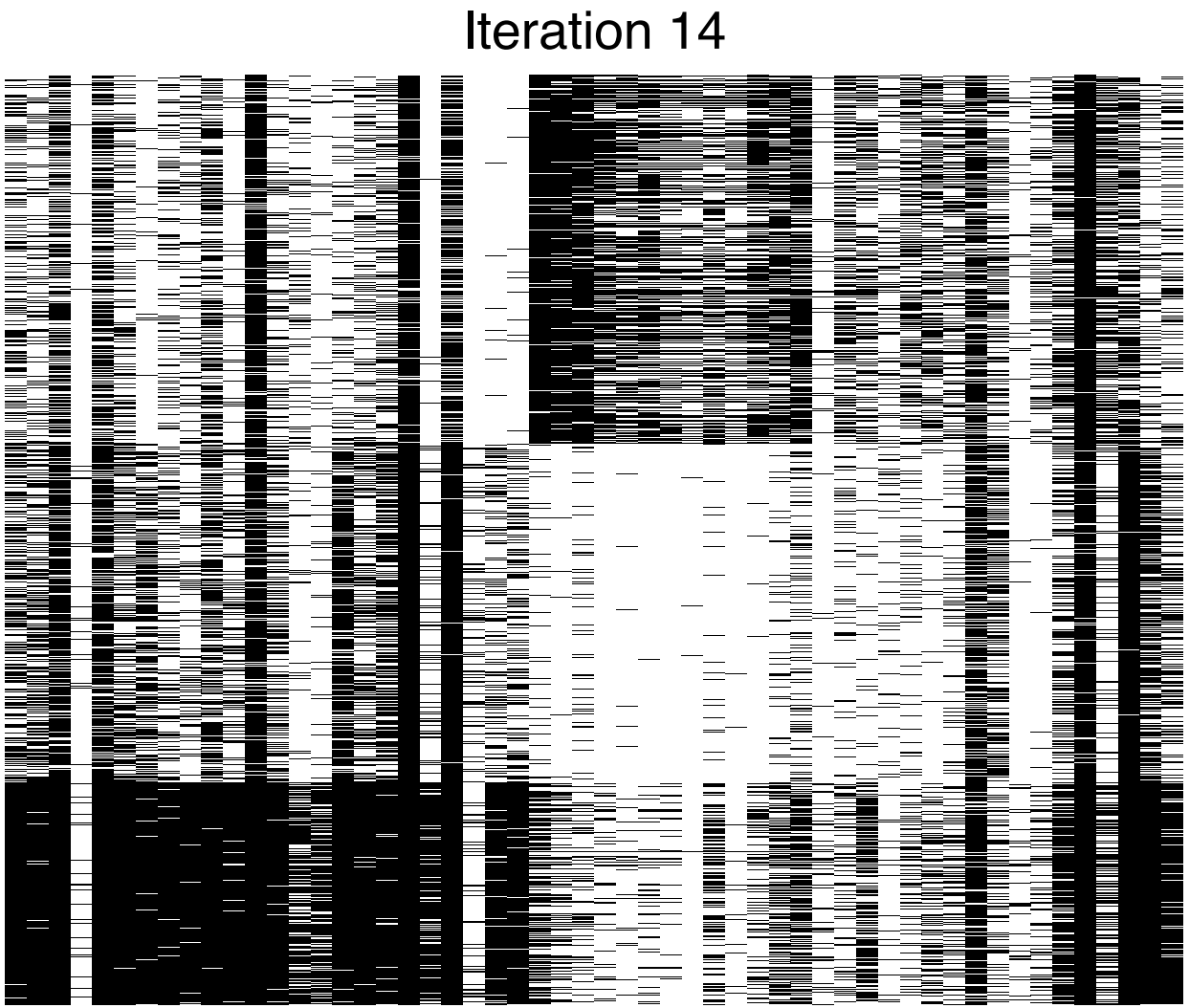
An EM run with a binary data set



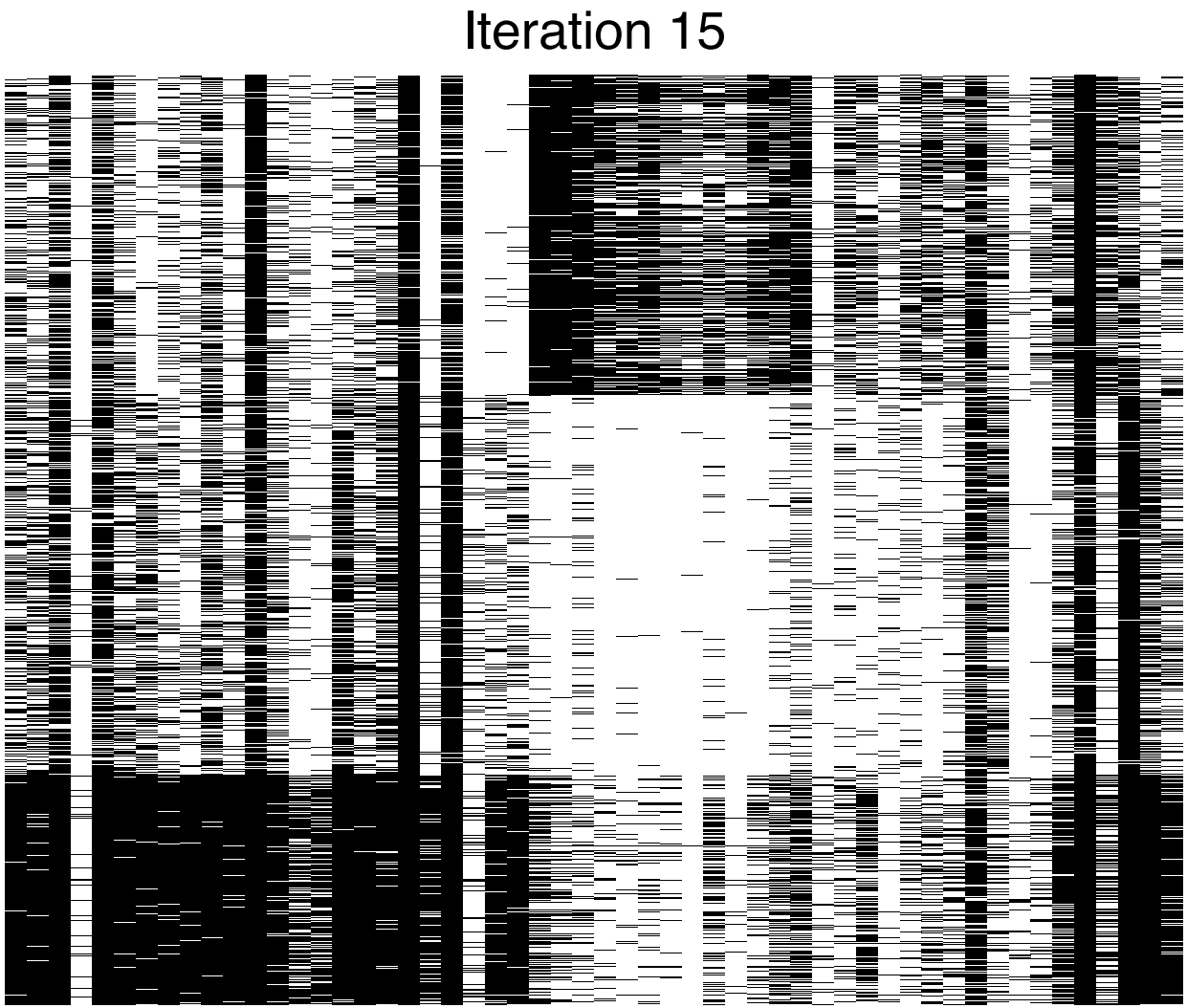
An EM run with a binary data set



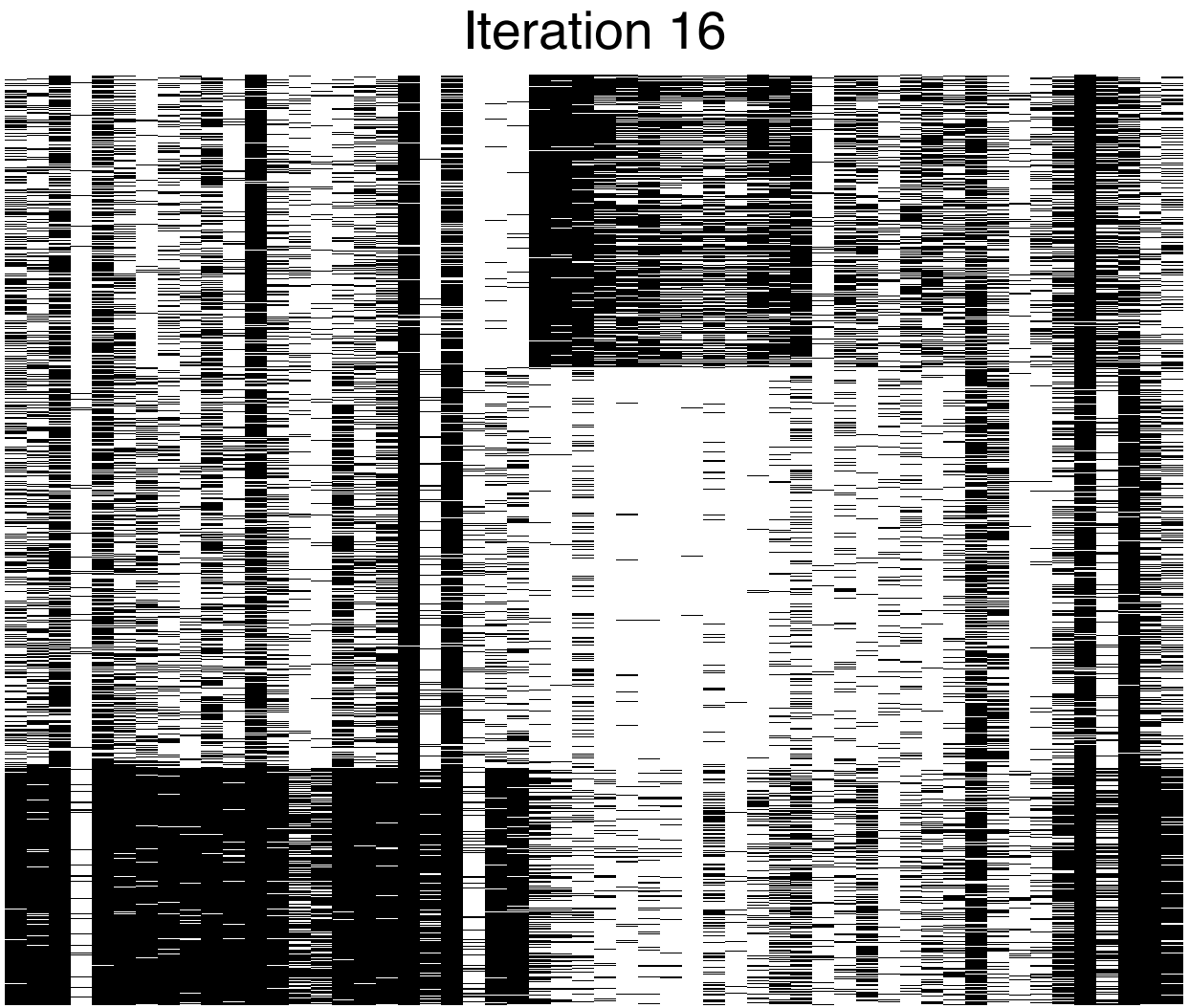
An EM run with a binary data set



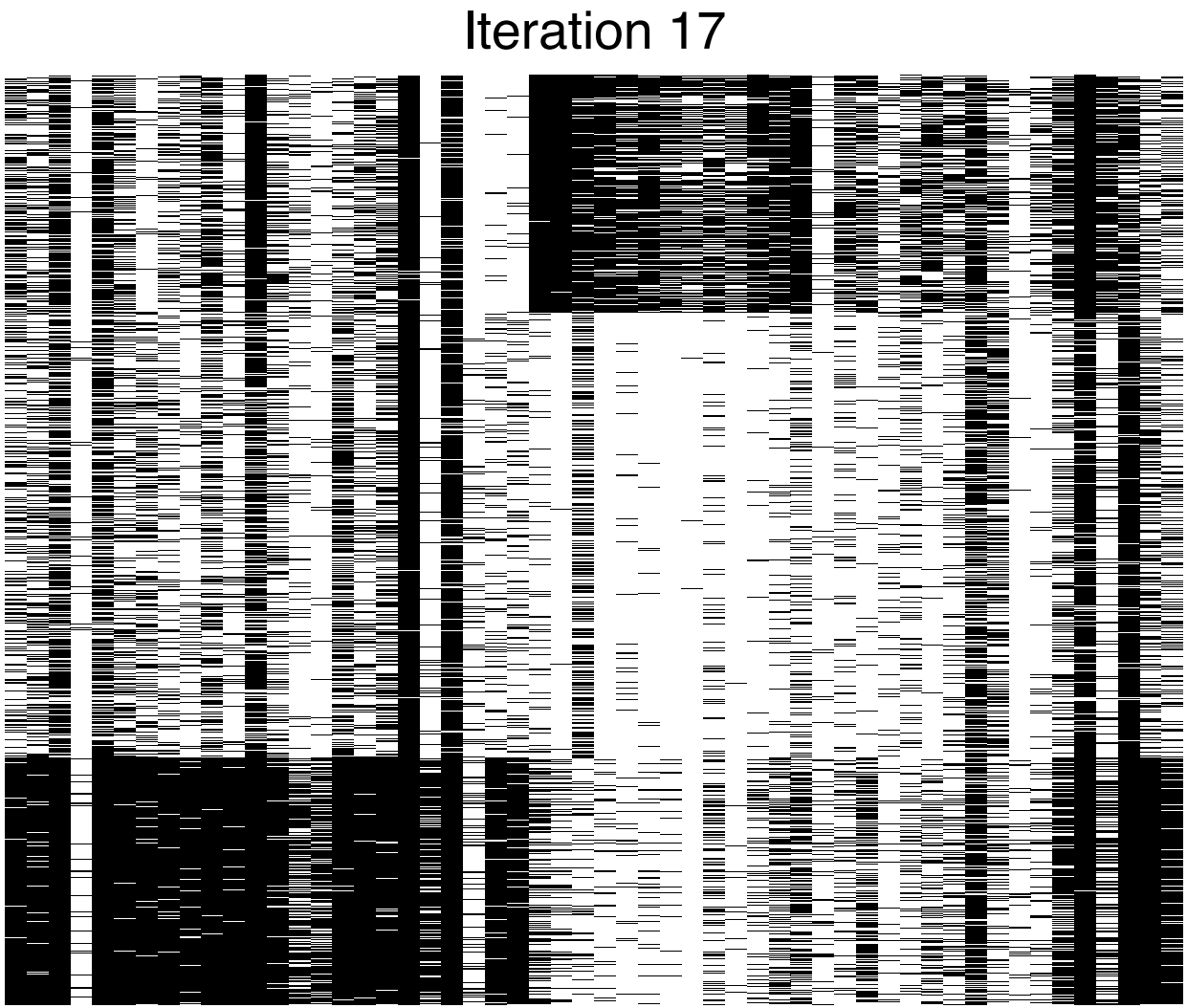
An EM run with a binary data set



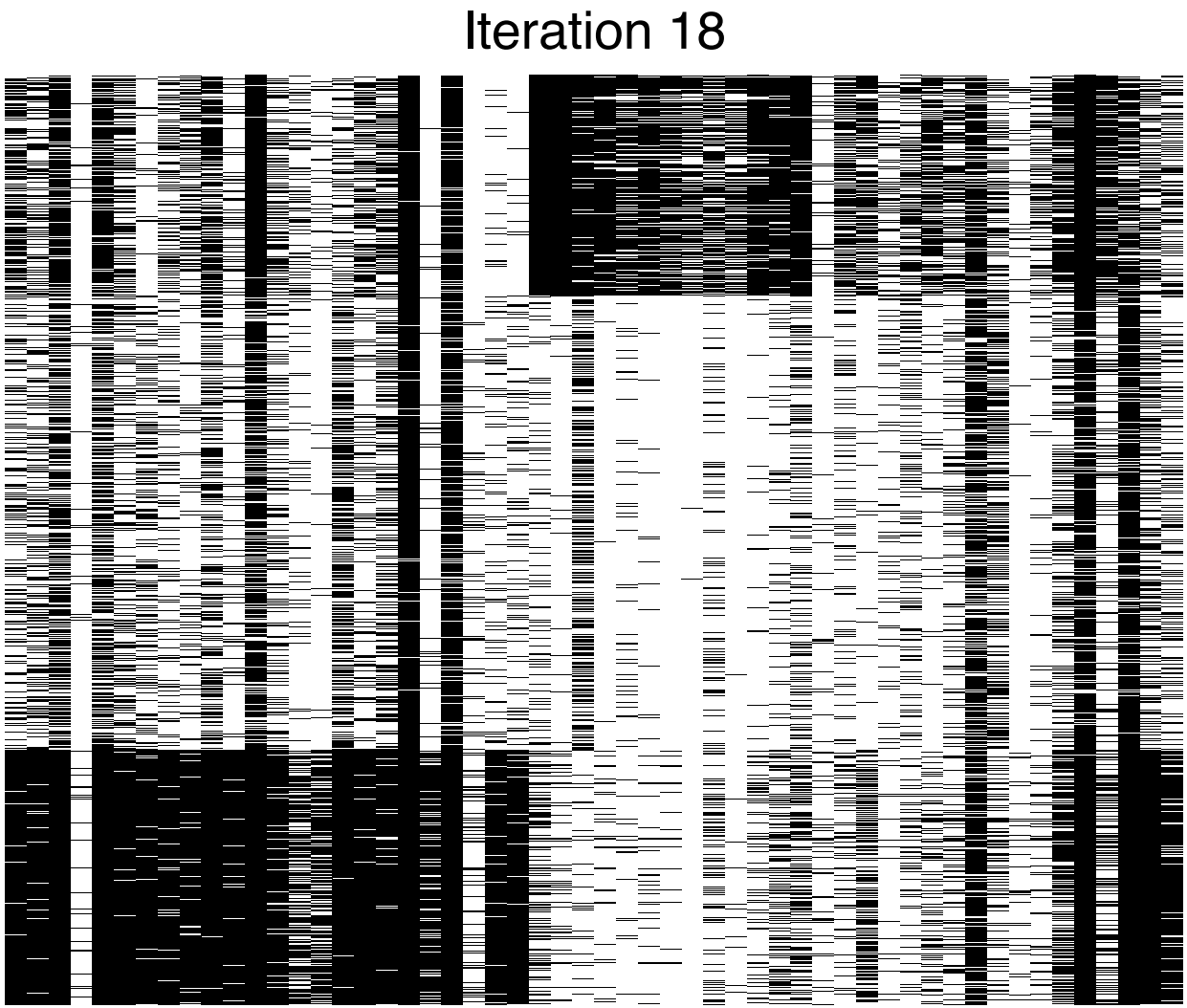
An EM run with a binary data set



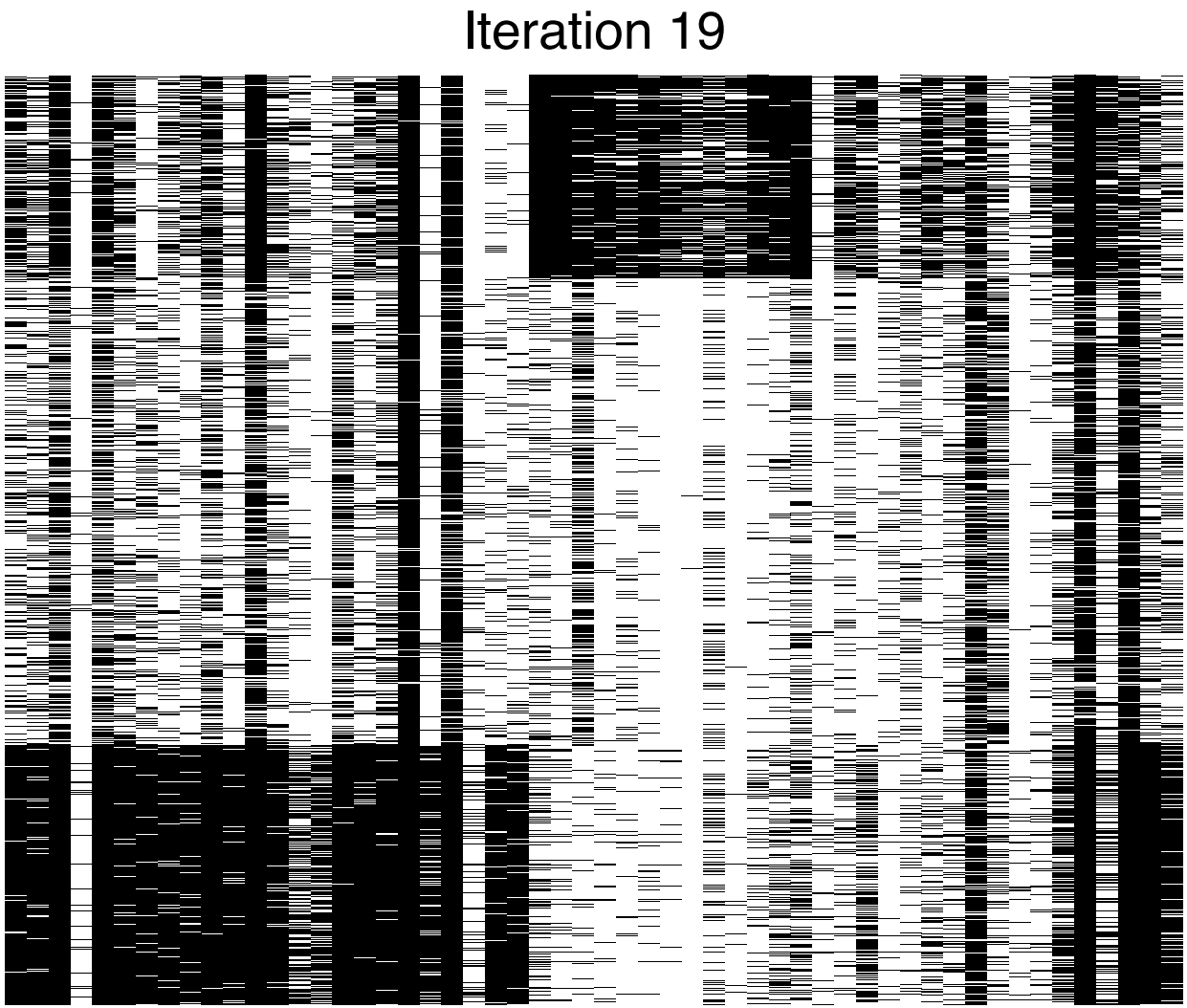
An EM run with a binary data set



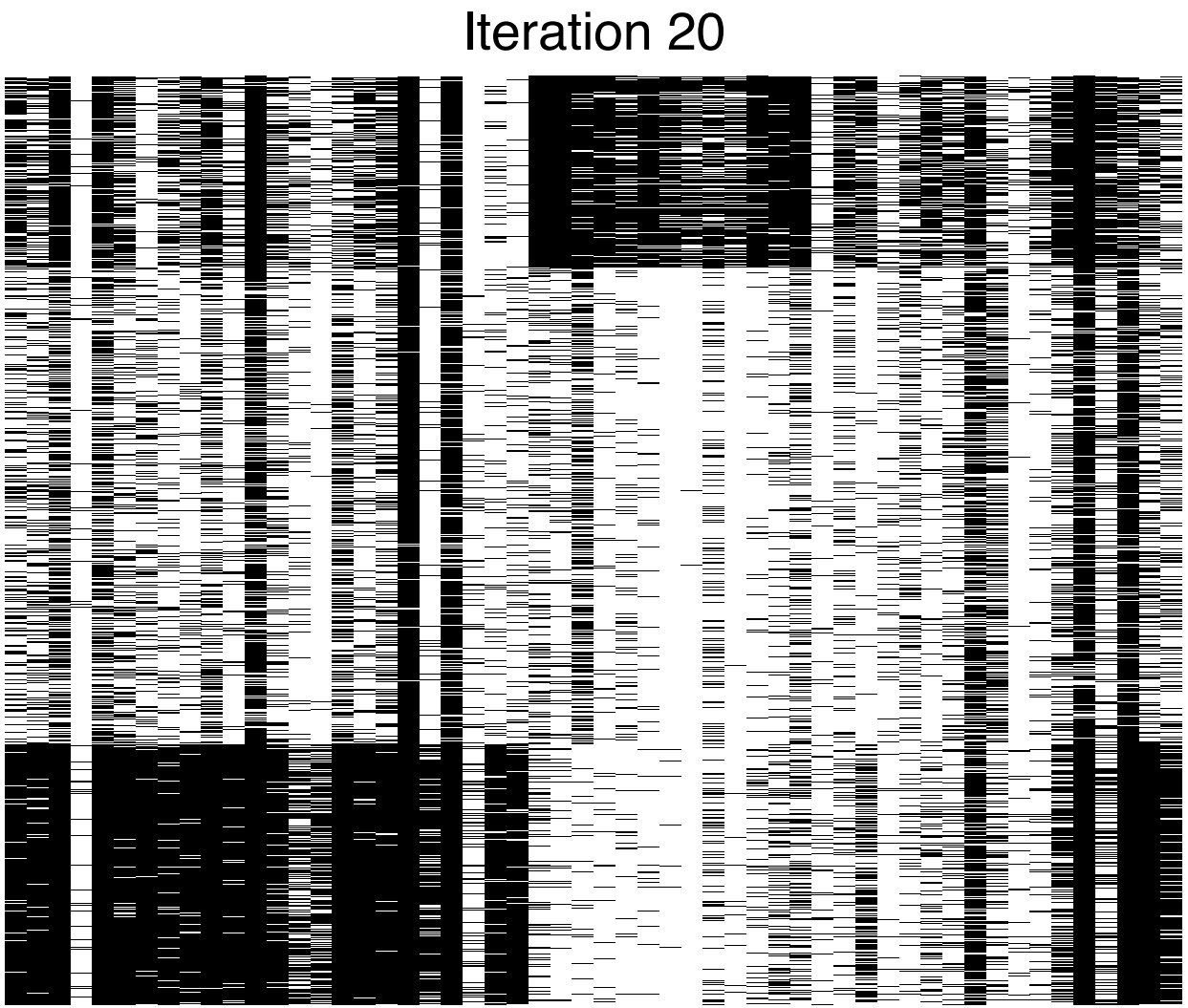
An EM run with a binary data set



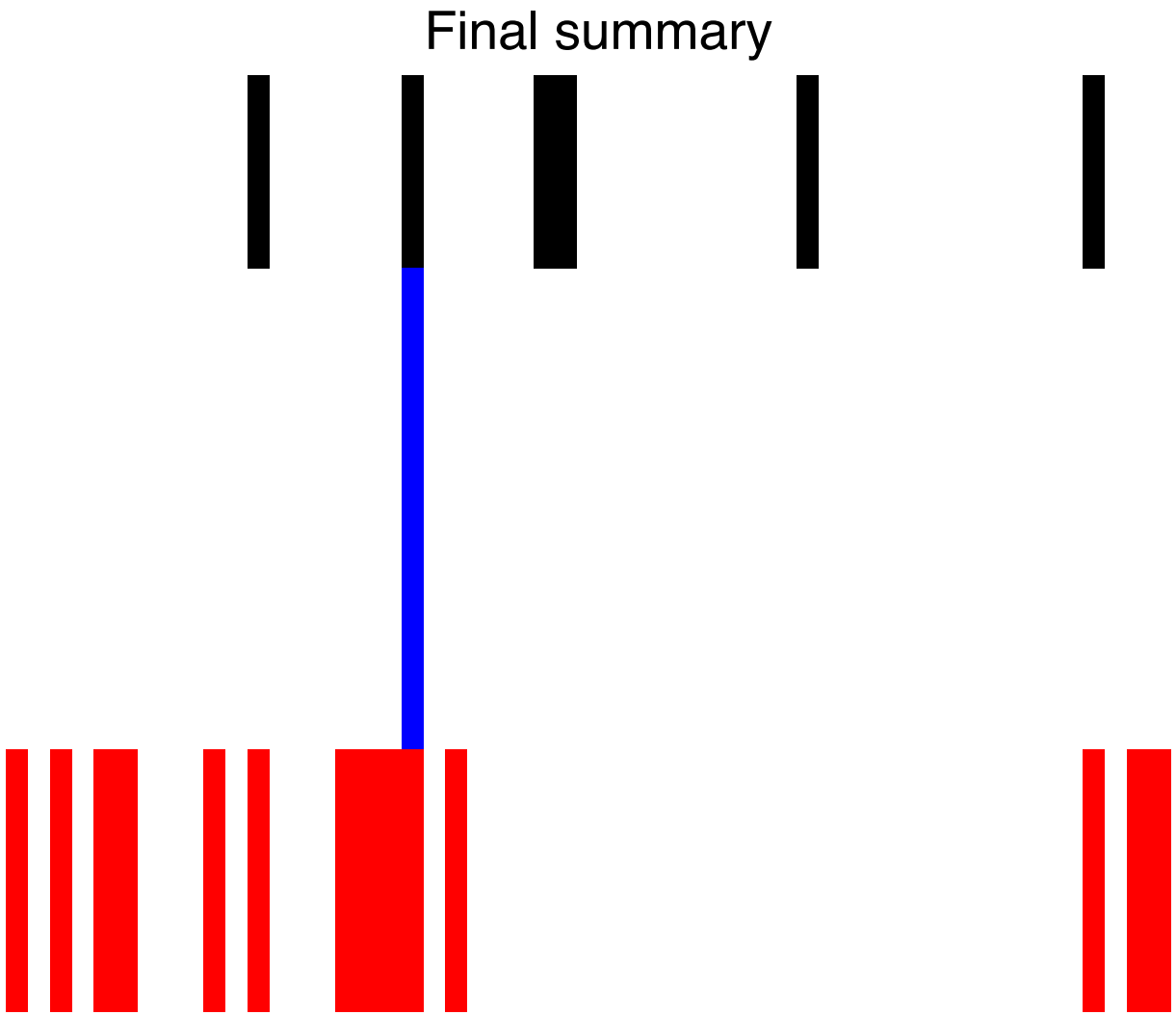
An EM run with a binary data set



An EM run with a binary data set



An EM run with a binary data set



Integer: Poisson mixture model

- integer variables: d variables $\boldsymbol{x}_{ij} \in \mathbb{N}$
- Intra conditional independence:

$$\mathrm{p}(\boldsymbol{x}_i; \boldsymbol{\alpha}_k) = \prod_{j=1}^{d} \frac{(\alpha_{kj})^{x_{ij}}}{\alpha_{kj}!} e^{-\alpha_{kj}}$$

Mixed data: classical approaches

Usually, unify data type by transformation:

- Quantify continuous variables: loose some information
- MCA of categorical variable: loose the meaning
- ...

Proposal

Model-based directly on raw data

Mixed data: conditional independence everywhere

The aim is to combine continuous, categorical and integer data

$$\boldsymbol{x}_1 = (\boldsymbol{x}_1^{cont}, \boldsymbol{x}_1^{cat}, \boldsymbol{x}_1^{int})$$

The proposed solution is to mixed all types by inter-type conditional independence

$$\mathrm{p}(\boldsymbol{x}_1; \boldsymbol{\alpha}_k) = \mathrm{p}(\boldsymbol{x}_1^{cont}; \boldsymbol{\alpha}_k^{cont}) \times \mathrm{p}(\boldsymbol{x}_1^{cat}; \boldsymbol{\alpha}_k^{cat}) \times \mathrm{p}(\boldsymbol{x}_1^{int}; \boldsymbol{\alpha}_k^{int})$$

In addition, for symmetry between types, intra-type conditional independence

Only need to define the univariate pdf for each variable type!

- Continuous: Gaussian
- Categorical: multinomial
- Integer: Poisson

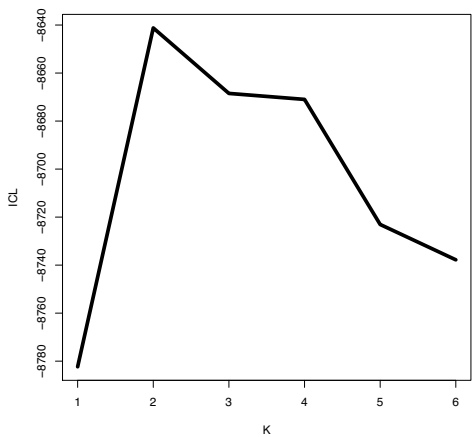
Example of mixed case: Cancer dataset with more missing data

Add artificially $\approx 30\%$ missing data with a MCAR design
Then compare two strategies of imputation:

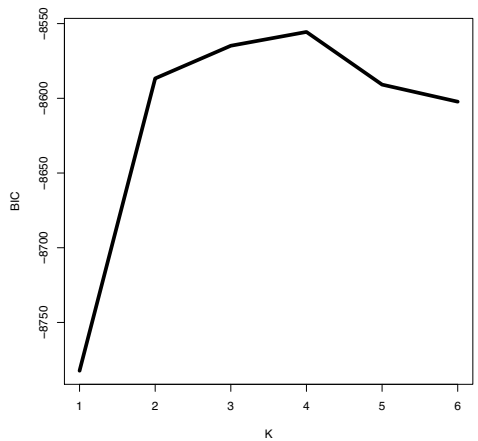
- **Strategy "mice"**: dataset completed by `mice`

```
> data.imp=mice(data)
> data.comp.mice=complete(data.imp)
```

- **Strategy "full MixtComp"**: MixtComp on the observed (no completed) dataset



ICL
$\hat{K} = 2$



BIC
$\hat{K} = 4$

Example of mixed case: imputation accuracy

- Continuous variables: mean of absolute difference between x and $\hat{x}$

var.	mice	MixtComp ($K = 2$)	MixtComp ($K = 4$)
Age	8.907143	5.546571	5.526861
Wt	13.51656	9.779485	9.731182
SBP	2.103226	1.788152	1.795820
DBP	1.317568	1.165201	1.169672
HG	21.67568	14.83514	14.51291
SZ	1.714899	1.160546	1.158105
SG	1.979866	1.386841	1.416053
AP	1.359299	1.027513	1.009126
Global mean	6.5718	4.5862	4.5400

- Categorical variable: mean of the proportion of difference between x and $\hat{x}$

var.	mice	MixtComp ($K = 2$)	MixtComp ($K = 4$)
PF	0.1904762	0.0952381	0.0952381
HX	0.4121622	0.4391892	0.4121622
EKG	0.7564103	0.6858974	0.7179487
BM	0.1081081	0.1486486	0.1216216
Global mean	0.3668	0.3422	0.3367

Outline

Notations *ranking* and *ordering*

Rank definition

A rank consists of sorting l objects following a preference order.

Example: sort three holidays places
$\mathcal{O}_1$: countryside, $\mathcal{O}_2$: mountain et $\mathcal{O}_3$: sea
$\Rightarrow$ 1st sea, 2nd countryside et 3rd mountain

Notations

- Ordering: $\boldsymbol{x} = (3,1,2) = (\overset{1^{st}}{\mathcal{O}_3}, \overset{2^{nd}}{\mathcal{O}_1}, \overset{3^{th}}{\mathcal{O}_2})$
- Ranking: $\boldsymbol{x}^{-1} = (2,3,1) = (\overset{\mathcal{O}_1}{2^{nd}}, \overset{\mathcal{O}_2}{3^{th}}, \overset{\mathcal{O}_3}{1^{st}})$

$\boldsymbol{x}, \boldsymbol{x}^{-1} \in \mathcal{P}_l$ (permutations of the first l integers).

Interest of ranking data

Human activities implying preferences or choices

Web pages sorting	Sport
Sociology	Politics
Economy	Psychology
Biology	Marketing . . .

Missing data are frequent

- When many objects to be sorted, some rankings can be incomplete
- Ex-æquo values can also seen as incomplete rankings

Univariate ISR (Insertion Sorting Rank) model

Données

- $\boldsymbol{x} = (x_1, \ldots, x_l)$: observed rankings
- $\boldsymbol{y} = (y_1, \ldots, y_l)$: latent order presentation of objects

Hypothèse

$\boldsymbol{x}$ arises from a insertion sorting algorithm with parameters

- $\mu = (\mu_1, \ldots, \mu_l)$: reference ranking (position)
- $\alpha \in [0, 1]$: proba. of good comparison by pair (dispersion)

Associated distribution, after averaging over unknown y

$$\text{pr}(\boldsymbol{x}; \mu, \pi) = \frac{1}{l} \sum_{\boldsymbol{y}} \alpha^{\text{good}(\boldsymbol{x},\boldsymbol{y},\mu)} \, (1-\alpha)^{\text{bad}(\boldsymbol{x},\boldsymbol{y},\mu)}$$

ISR properties

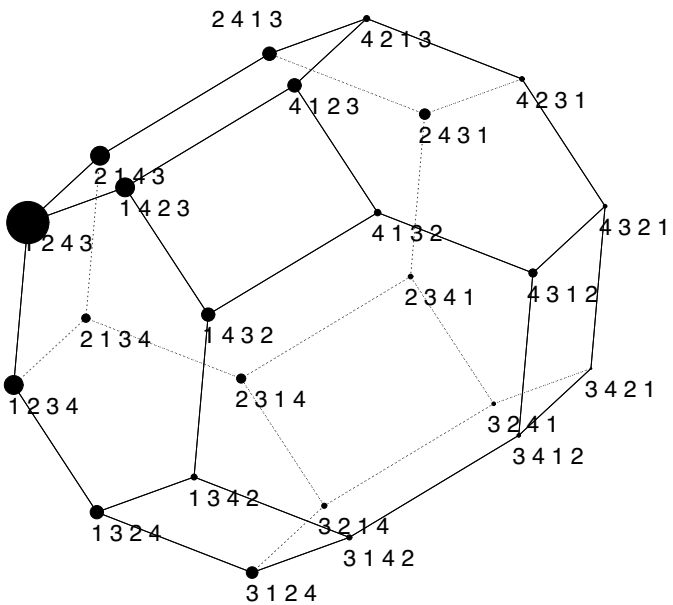
Symetry

$\text{pr}(\boldsymbol{x}; \bar{\mu}, 1-\alpha) = \text{pr}(\boldsymbol{x}; \mu, \alpha) \quad \Rightarrow \quad \alpha \in [\frac{1}{2}, 1]$

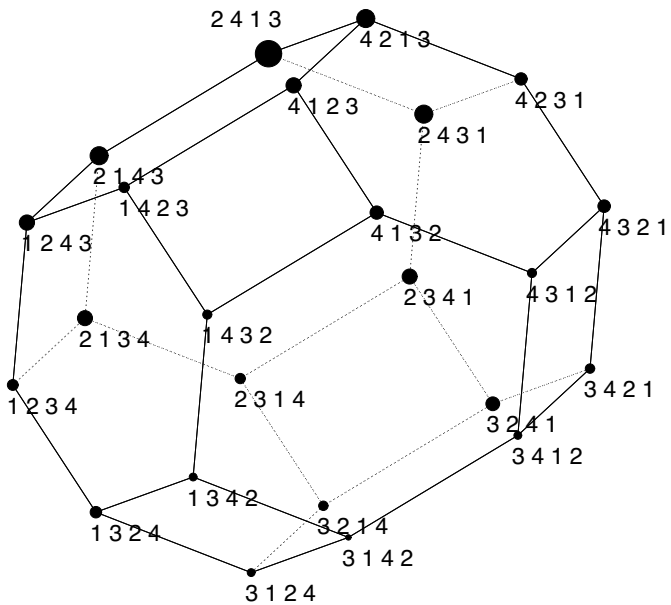
Bon comportement

- μ is the mode and $\bar{\mu}$ is the anti-mode ($\alpha > \frac{1}{2}$)
- $\text{pr}(\mu; \mu, \alpha) - \text{pr}(\boldsymbol{x}; \mu, \alpha)$ is a non-decreasing function of α:
  the largest is α, the sharper is the distribution around its mode
- the parameters (μ, α) are identifiable if $\alpha > \frac{1}{2}$
- the distribution is uniform for $\pi = \frac{1}{2}$, Dirac at μ if $\alpha = 1$

ISR distribution illustration



$\mu = (1, 2, 4, 3)$ et $\alpha = 0.83$



$\mu = (2, 4, 1, 3)$ et $\alpha = 0.68$

Extension to the ISR multivariate mixture model[5]

Multivariate ranks

- Dimension d: $\boldsymbol{x}_i = (\boldsymbol{x}_{i1}, \ldots, \boldsymbol{x}_{id})$
- l_j objects per dimension $(1 \leq j \leq d)$: $\boldsymbol{x}_{ij} = (x_{ij1}, \ldots, x_{ijl_j})$

K-mixture of multivariate ISR

Hyp. of class conditional independence

$$\mathrm{p}(\boldsymbol{x}_i; \boldsymbol{\theta}) = \underbrace{\sum_{k=1}^{K} \pi_k}_{\text{mixture}} \underbrace{\prod_{j=1}^{d} \overbrace{\mathrm{p}(\boldsymbol{x}_{ij}; \mu_{kj}, \alpha_{kj})}^{\mathrm{ISR}(\mu_{kj}, \alpha_{kj}) \text{ univariate}}}_{\text{ISR multivariate}}$$

- Proportions π_k: $\pi_k \in [0, 1]$ et $\sum_{k=1}^{K} \pi_k = 1$
- Whole parameter: $\boldsymbol{\theta} = (\alpha_{kj}, \mu_{kj}, \pi_k)_{k=1,\ldots,K\ ;j=1,\ldots,d}$

[5] J. Jacques and C. Biernacki (2014). Model-based clustering for multivariate partial ranking data. *Journal of Statistical Planning and Inference*, 149, 201–217.

Maximum likelihood estimation

- EM & SEM: impossible since combinatorics difficulty because of permutations
- SEM within Gibbs: embed a Gibbs algorithm within the SE-step

Utility of ISR mixtures: data

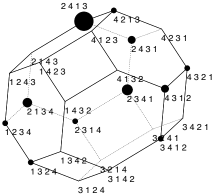
4 quizz proposed to 30 students (GIS4 - Polytech'Lille)

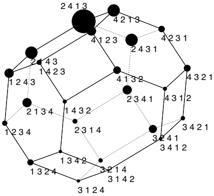
- Sort chronologically these writers
  1. Hugo 2. Molière 3. Camus 4. Rousseau
- Sort these countries by increasing order of their number of victories at the football world cup
  1. France 2. Germany 3. Brasil 4. Italy
- Sort these numbers by increasing order
  1. $\pi/3$ 2. $\ln 1$ 3. e^2 4. $(1+\sqrt{5})/2$
- Sort chronologically these movies of Tarantino
  1. Inglorious Basterds 2. Pulp Fiction 3. Reservoir Dogs 4. Jackie Brown

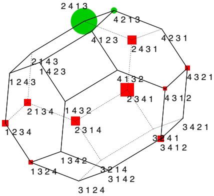
Utility of ISR mixtures: Quizz 1

ort chronologically these writers

1. Hugo 2. Molière 3. Camus 4. Rousseau





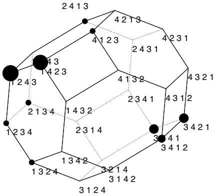


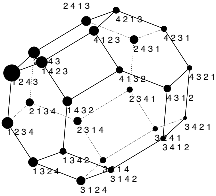
Empiric

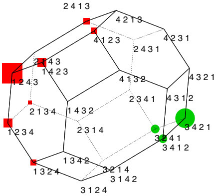
	ISR	Mixture of ISR	
μ_k	(2, 4, 1, 3)	(2, 4, 1, 3)	(2, 3, 4, 1)
α_k	0.80	0.95	0.71
π_k		0.5	0.5
BIC	152.3	**148.0**	

Utility of ISR mixtures: Quizz 2

Sort these countries by increasing order of their number of victories at the football world cup : 1. France, 2. Germany, 3. Brasil, 4. Italy







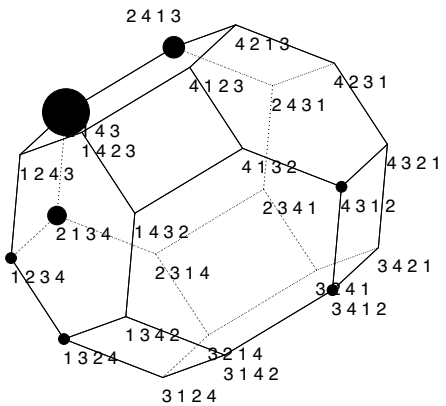
Empiric

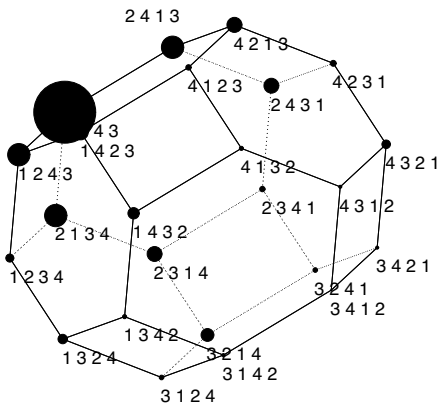
	ISR	Mixture of ISR	
μ_k	$(1,2,4,3)$	$(1,2,4,3)$	$(3,4,2,1)$
α_k	0.69	0.85	0.84
π_k		0.73	0.27
BIC	179.1	**160.6**	

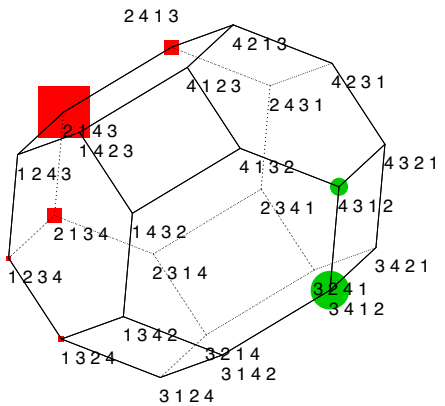
Utility of ISR mixtures: Quizz 3

Sort these numbers by increasing order

1. $\pi/3$ 2. $\ln 1$ 3. e^2 4. $(1+\sqrt{5})/2$







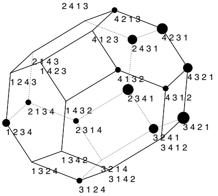
Empiric

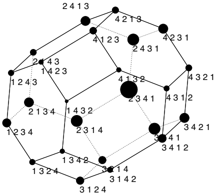
	ISR	Mixture of ISR	
μ_k	$(2,1,4,3)$	$(2,1,4,3)$	$(3,4,1,2)$
α_k	0.69	0.92	0.82
π_k		0.92	0.08
BIC	111.4	**106.3**	

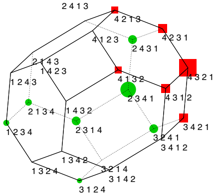
Utility of ISR mixtures: Quizz 4

Sort chronologically these movies of Tarantino

1.Inglorious Bastards 2.Pulp Fiction 3.Reservoir Dogs 4.Jackie Brown







Empiric | ISR | Mixture of ISR

	ISR	Mixture of ISR	
μ_k	$(2,3,4,1)$	$(2,3,4,1)$	$(4,3,2,1)$
α_k	0.69	0.76	0.79
π_k		0.61	0.39
BIC	177.0	**173.8**	

Social comparison theory (1/4)

Questionnaire

Which things do you prefer to compare with other children of your age? Put a 1 in front of what you prefer to compare most, a 2 in front of what you prefer next, and so on. More than 3 is not necessary, but is allowed:

1. your popularity,
2. how well you do in sports,
3. your appearance,
4. how much money you can spend,
5. how you are feeling,
6. your parents,
7. your clothes,
8. your grades at school,
9. how well you can express your opinions,
10. your hobby's,
11. how "courageous" you are,
12. how smart you are,
13. the kind of friends you have.

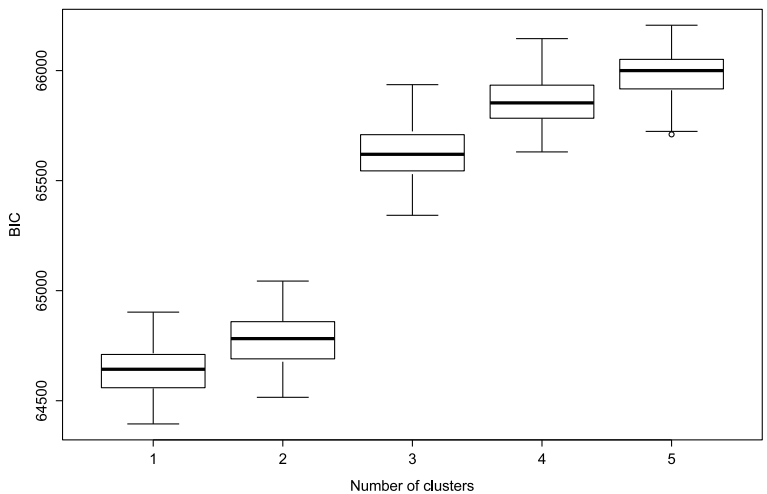
Data

- $n = 1\,567$ students
- $l = 13$ objects to be compared ($d = 1$)
- 85% of partial ranks
- among 15% of full ranks, 20% contain ties

Social comparison theory (2/4)

Package Rankcluster

```
R> res=rankclust(x,l,1:5)
```



- Log-likelihood estimated by the harmonic mean (intractable closed-form)
- BIC "hesitates" between one and two groups: many missing data...

Social comparison theory (3/4)

```
****************************************************************
Number of clusters: 1
****************************************************************
proportion: 1
mu:
cl1    8 12 1 11 6 13 3 2 4 7 9 10 5
alpha:
cl1 0.6478778
```

- π low $\Rightarrow$ μ not significant $\Rightarrow$ nearly uniform distribution
- Example 1st/2nd: $P(\mathcal{O}_3 \succ \mathcal{O}_8) = 0.54 \rightarrow$ low!
- Example 1st/last: $P(\mathcal{O}_3 \succ \mathcal{O}_6) = 0.62 \rightarrow$ low!
- Conclusion: no global preference

Social comparison theory (4/4)

```
****************************************************************
Number of clusters: 2
****************************************************************
proportion: 0.9261865 0.0738135
mu:
cl1    9 5 3 10  4 8 6 1 11  7 12 13 2
cl2    4 1 2  8 13 7 5 3  9 12 11 10 6
alpha:
cl1 0.6583201
cl2 0.7977076
```

- Group 1 large ($\pi_1 = 93\%$) :
  - Looks like the one-group case
  - No real preference ($\alpha_1 = 0.65$ low)
- Group 2 small ($\pi_1 = 7\%$) :
  - Preference more apparent ($\alpha_2 = 0.80$ greater)
  - Example 1st/2nd: $P(\mathcal{O}_2 \succ \mathcal{O}_3) = 0.64 \rightarrow$ more significant!
  - Example 1st/last: $P(\mathcal{O}_3 \succ \mathcal{O}_5) = 0.79 \rightarrow$ significant !
- Conclusion: no global preference, but locally a small group

sports $\succ$ appearance $\succ$ grades $\succ$ popularity $\succ$ . . .

Eurovision Song Contest (1/2)

Principle

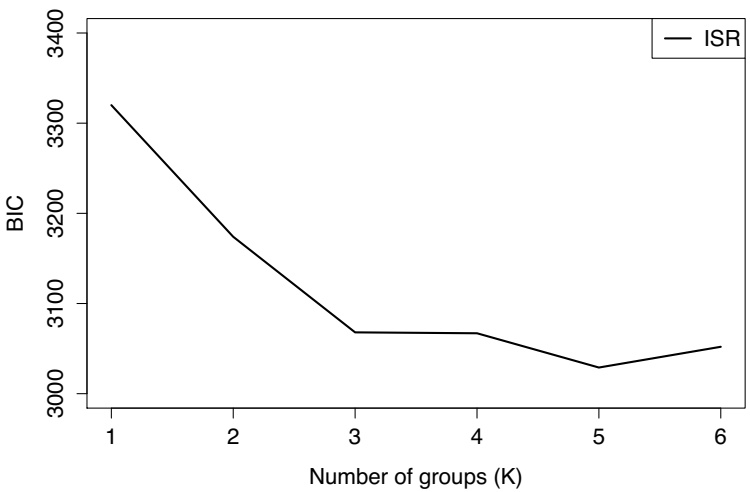
- Biggest musical competition in the world (around 40 countries)
- Each member country submits a song broadcast live...
- ...then rank these 10 favorite foreign songs

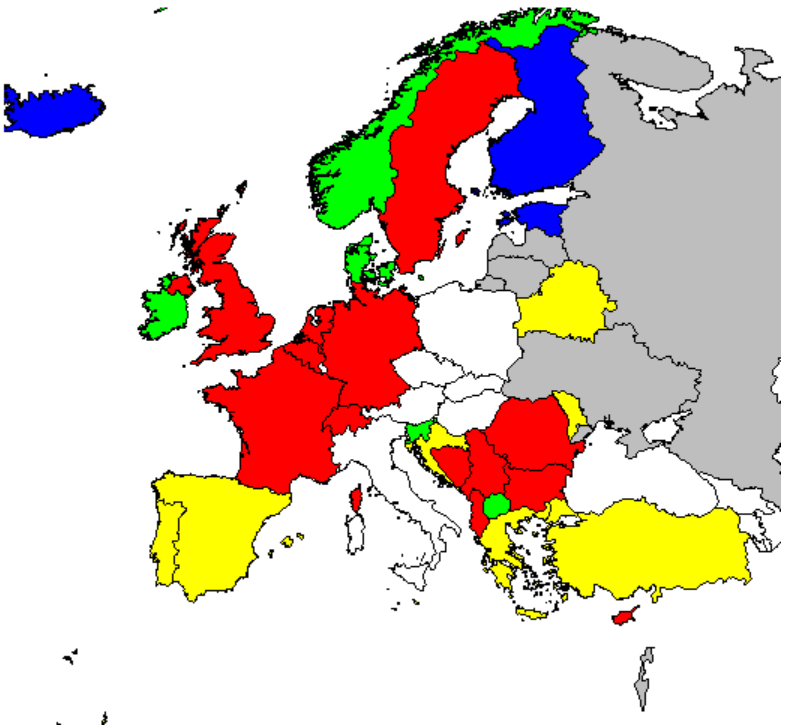
Data

- Votes of $n = 34$ participating countries between 2007 and 2012 ($d = 6$ years)
- Only 8 countries participated in the 6 finals:
  1: France, 2: Germany, 3: Greece, 4: Romania, 5: Russia, 6: Spain, 7: Ukraine, 8: UK
- Only the votes for these 8 countries are considered: $l_j = 8$
- 57.7% of ranking elements are missing!

Eurovision Song Contest (2/2)

Package Rankclust en C++ interfaced with R





Revelation of geographic alliances. . .

Outline

RMixtComp: pitch

- Mixture Models with Mixed and (Partially) Missing Data
- 8 models for real, categorical, counting, functional and ranking data
- SEM algorithm with MAR assumption
- Available on the CRAN repository:
  `https://cran.r-project.org/web/packages/RMixtComp/index.html`

RMixtComp: syntax/allowed missing data

allowed missing value types for each model

	Categorical_pjk	Gaussian_sjk	Poisson_k	LatentClass
? (completely missing)	X	X	X	X
$\{a, b, c\}$ (finite number of values authorized)	X			X
$[a : b]$ (bounded interval)		X		
$[-inf : b]$ (semi-bounded interval)		X		
$[a : +inf]$ (semi-bounded interval)		X		

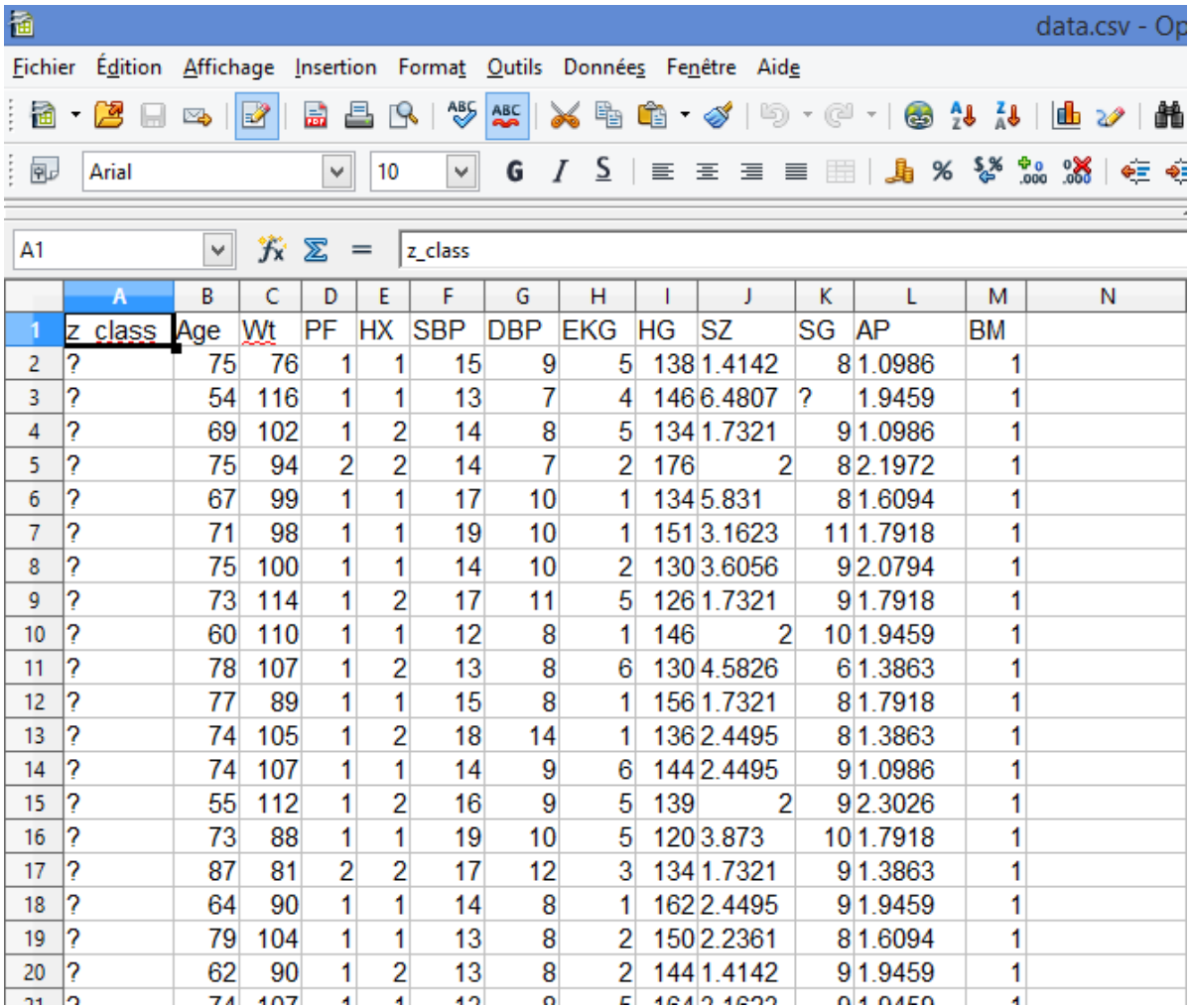
RMixtComp: overview of the output R format

```
res
    strategy
        nbTrialInInit
        nbBurnInIter
        nbIter
        nbGibbsBurnInIter
        nbGibbsIter
    mixture
        nbCluster
        nbFreeParameters
        lnObservedLikelihood
        lnSemiCompletedLikelihood
        lnCompletedLikelihood
        BIC
        ICL
        runTime
        nbSample
        warnLog
    variable
        data
            z_class
                completed !!! <- imputed classes
                stat !!! <- a posteriori distribution of class for each individual (= p(z_i / x_i))
            categorical1
                completed
                stat
            categorical2, etc ...
        param
            z_class
                stat !!! <- model proportions and quantiles
                log
            categorical1
                stat
                log
            categorical2, etc ...
```

Note that the `z_class` variable contains all the information pertaining to the latent classes:

- `res$variable$data$sample$completed` contains the imputation for the class, $\hat{z}_i$
- `res$variable$data$sample$stat` contains the estimated a posteriori probabilities, $\hat{t}_{ik}$
- `res$variable$param$z_class$stat` contains the proportions, $\hat{\pi}_k$

Prostate data set: overview of data file prostate.csv


data.csv - Op

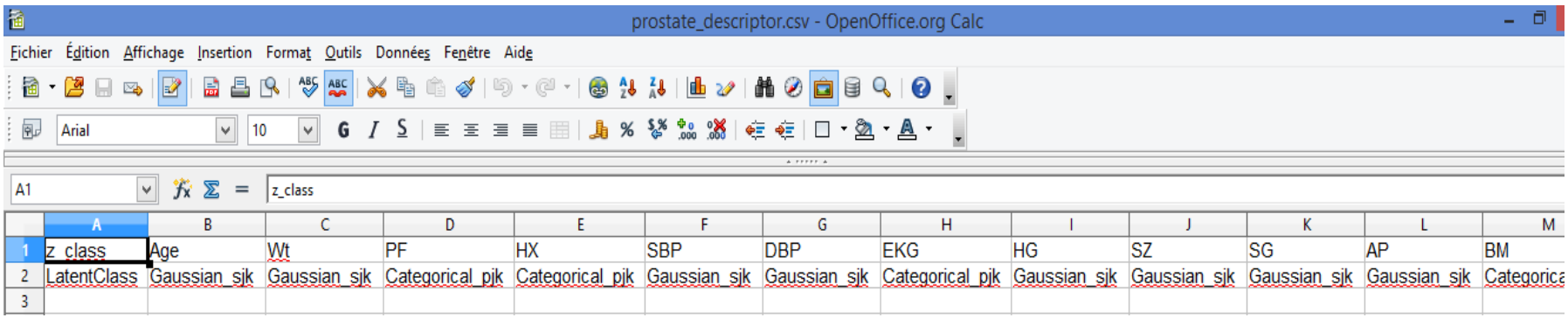
Fichier Édition Affichage Insertion Format Outils Données Fenêtre Aide

Arial 10

A1 z_class

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	z_class	Age	Wt	PF	HX	SBP	DBP	EKG	HG	SZ	SG	AP	BM	
2	?	75	76	1	1	15	9	5	138	1.4142	8	1.0986	1	
3	?	54	116	1	1	13	7	4	146	6.4807	?	1.9459	1	
4	?	69	102	1	2	14	8	5	134	1.7321	9	1.0986	1	
5	?	75	94	2	2	14	7	2	176	2	8	2.1972	1	
6	?	67	99	1	1	17	10	1	134	5.831	8	1.6094	1	
7	?	71	98	1	1	19	10	1	151	3.1623	11	1.7918	1	
8	?	75	100	1	1	14	10	2	130	3.6056	9	2.0794	1	
9	?	73	114	1	2	17	11	5	126	1.7321	9	1.7918	1	
10	?	60	110	1	1	12	8	1	146	2	10	1.9459	1	
11	?	78	107	1	2	13	8	6	130	4.5826	6	1.3863	1	
12	?	77	89	1	1	15	8	1	156	1.7321	8	1.7918	1	
13	?	74	105	1	2	18	14	1	136	2.4495	8	1.3863	1	
14	?	74	107	1	1	14	9	6	144	2.4495	9	1.0986	1	
15	?	55	112	1	2	16	9	5	139	2	9	2.3026	1	
16	?	73	88	1	1	19	10	5	120	3.873	10	1.7918	1	
17	?	87	81	2	2	17	12	3	134	1.7321	9	1.3863	1	
18	?	64	90	1	1	14	8	1	162	2.4495	9	1.9459	1	
19	?	79	104	1	1	13	8	2	150	2.2361	8	1.6094	1	
20	?	62	90	1	2	13	8	2	144	1.4142	9	1.9459	1	

Prostate data set: overview of the variable descriptors



	A	B	C	D	E	F	G	H	I	J	K	L	M
1	z_class	Age	Wt	PF	HX	SBP	DBP	EKG	HG	SZ	SG	AP	BM
2	LatentClass	Gaussian_sjk	Gaussian_sjk	Categorical_pjk	Categorical_pjk	Gaussian_sjk	Gaussian_sjk	Categorical_pjk	Gaussian_sjk	Gaussian_sjk	Gaussian_sjk	Gaussian_sjk	Categorica
3													

Prostate data set: overview of the output multiple imputation by MixtComp

```
> res$variable$data$Age$completed[8]
[1] 70.62032
> res$variable$data$Age$stat[[1]]
[[1]]
[1] 8

[[2]]
[1] 70.62032

[[3]]
[1] 58.24255

[[4]]
[1] 83.86463
```

cont.

```
> res$variable$data$EKG$completed[[5]]
[1] 1
> res$variable$data$EKG$stat[[1]]
[[1]]
[1] 5

[[2]]
[1] 1

[[3]]
[1] 0.41

[[4]]
[1] 5

[[5]]
[1] 0.29

[[6]]
[1] 6

[[7]]
[1] 0.15

[[8]]
[1] 2

[[9]]
[1] 0.07

[[10]]
[1] 3

[[11]]
[1] 0.05
```

cat.

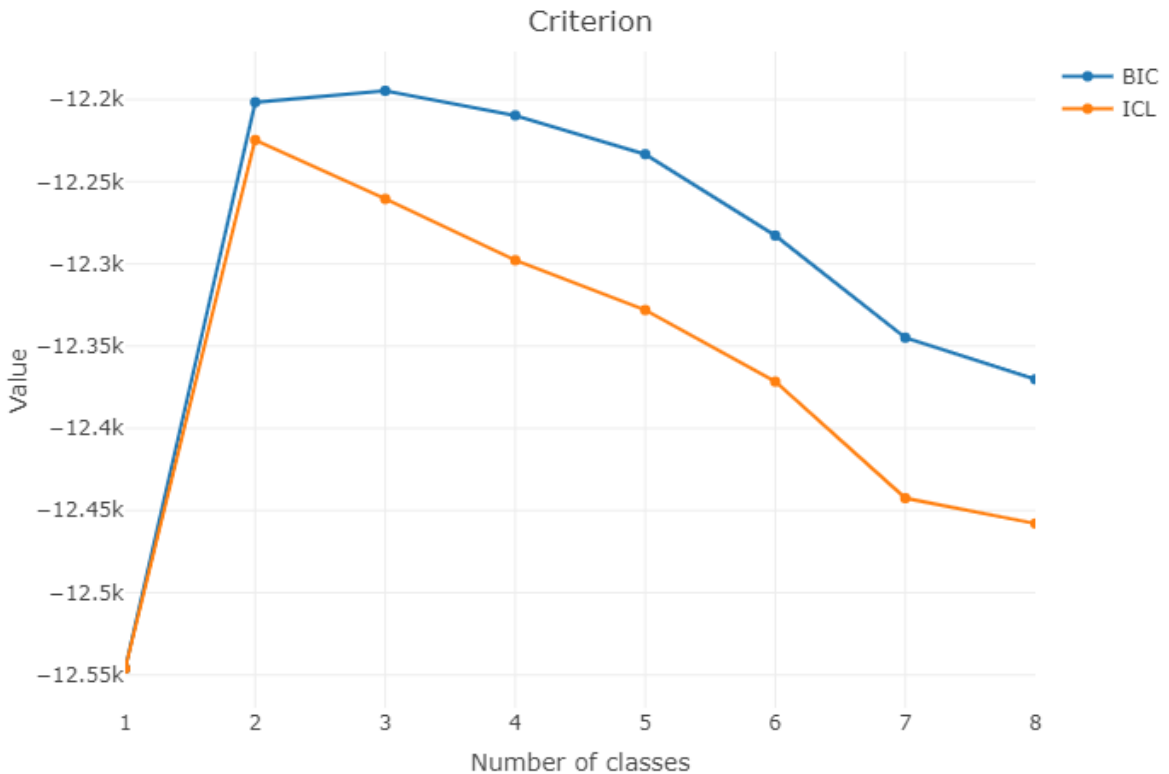
Prostate data set: ready for the practical activity with RMixtComp!

- Learning step:
  - Run RMixtComp with $K \in \{1, \ldots, 8\}$
  - Check graphically K value retained by BIC and ICL
  - Check the imputed values (and there confidence interval) of continuous/categorical missing values
- Prediction step:
  - Transform Patient 1 as follows: Age is missing, Wt is uncertain within [70,80], EKG is uncertain within $\{4, 5\}$
  - For this transformed Patient 1: retain $K = 3$, estimate his/her class, Age, Wt, EKG (and have a look at the related confidence intervals)

Prostate data set: R code for learning

```
data <- read.table("prostate.csv", sep = ";", header = TRUE)
head(data)
library(RMixtComp)

# Define the distribution used for each variable.
model <- list(Age = "Gaussian", Wt = "Gaussian", PF = "Multinomial",
              HX = "Multinomial", SBP = "Gaussian", DBP = "Gaussian",
              EKG = "Multinomial", HG = "Gaussian", SZ = "Gaussian",
              SG = "Gaussian", AP = "Gaussian", BM = "Multinomial")

# Define the SEM algorithm's parameters
algo <- list(nbBurnInIter = 50,
             nbIter = 100,
             nbGibbsBurnInIter = 50,
             nbGibbsIter = 100,
             nInitPerClass = floor(nrow(data)/2),
             nSemTry = 5,
             confidenceLevel = 0.95,
             ratioStableCriterion = 0.99,
             nStableCriterion = 10)

# Choose the desired number of classes and the number of runs for each given number of classes.
nClass <- 1:8
nRun <- 3
res <- mixtCompLearn(data, model, algo, nClass = nClass, criterion = "ICL", nRun = nRun, nCore = 1)

# Draw the criterion value (BIC and ICL) for each model that was built. The higher the value (close to 0) the better the model
plotCrit(res, pkg = "plotly")

# See estimation of all the missing Age values (idem for other variables)
res$variable$data$Age$completed # imputed
res$variable$data$Age$stat  # confidence interval
res$variable$data$BM$completed # imputed
res$variable$data$BM$stat  # confidence interval
```

Prostate data set: plot of ICL and BIC values



Prostate data set: R code for prediction

```
# Choose the number of classes to study in the following.
K <- 3
resK <- extractMixtCompObject(res, K)

# Prediction
Patient1=data[1,]
Patient1["Age"]="?"
Patient1["Wt"]="[70:80]"
Patient1["EKG"]="{4,5}"
resPredict <- mixtCompPredict(Patient1,resLearn=resK)

# See output of prediction
resPredict$variable$data$z_class$stat
resPredict$variable$data$z_class$completed
resPredict$variable$data$Age$completed
resPredict$variable$data$Age$stat
resPredict$variable$data$Wt$completed
resPredict$variable$data$Wt$stat
resPredict$variable$data$EKG$completed
resPredict$variable$data$EKG$stat
```

Prostate data set: output results from Patient 1

```
> resPredict$variable$data$z_class$stat
     k: 1 k: 2 k: 3
[1,] 0.69    0 0.31
> resPredict$variable$data$z_class$completed
[1] 1
> resPredict$variable$data$Age$completed
[1] 70.70071
> resPredict$variable$data$Age$stat
 index   median q 2.500000% q 97.500000%
     1 70.70071    55.20562     85.44931
> resPredict$variable$data$Wt$completed
[1] 76.78755
> resPredict$variable$data$Wt$stat
 index   median q 2.500000% q 97.500000%
     1 76.78755     71.3697     79.84079
> resPredict$variable$data$EKG$completed
[1] 5
> resPredict$variable$data$EKG$stat
$`1`
 modality probability
        5        0.81
        4        0.19
```

Compare to true values of Patient 1...

Outline

Rankcluster: pitch

- Implementation of the ISR model-based clustering algorithm for ranking data
- Multivariate rankings as well as partial rankings are taken into account
- This algorithm is based on an extension of the Insertion Sorting Rank (ISR) model for ranking data, which is a meaningful and effective model parametrized by a position parameter (the modal ranking) and a dispersion paramete
- The heterogeneity of the rank population is modelled by a mixture of ISR, whereas conditional independence assumption is considered for multivariate rankings
- Available on the CRAN repository: `https://cran.r-project.org/web/packages/Rankcluster/index.html`

Rankcluster: format of data and big4 data set

Multivariate Ranks

For multivariate ranks, the differents variable are combined by column and an extra parameter (`m`) indicates the size of each dimension.

```
data(big4)
head(big4$data)
#>           A.uefa B.uefa C.uefa D.uefa A.pl B.pl C.pl D.pl
#> 1992-1993      1      2      3      4    1    2    3    4
#> 1993-1994      1      3      2      4    1    3    2    4
#> 1994-1995      1      3      2      4    1    2    4    3
#> 1995-1996      1      3      2      4    1    2    3    4
#> 1996-1997      1      2      3      4    1    3    2    4
#> 1997-1998      1      3      2      4    2    3    1    4
big4$m
#> [1] 4 4
```

The `big4` dataset is composed of the rankings (in ranking notation) of the "Big Four" English football teams (A: Manchester, B: Liverpool, C: Arsenal, D: Chelsea) to the English Championship (Premier League) and according to the UEFA coefficients (statistics used in Europe for ranking and seeding teams in international competitions), from 1993 to 2013.

Each variable corresponds to the ranking of four elements, so `m = c(4, 4)`. In the `data` matrix, the first four columns correspond to the rankings in Premier League and the four next to the ranking accoding to the uefa coefficient.

Partial Missing Ranks

Rankcluster manages partial missing ranks. Missing positions are denoted by `0`.

For example `5 0 1 2 0` indicates that the position of the second and fifth objects are unknown.

Ranks with tied positions

Rankcluster manages tied positions in ranks. Tied position are replaced by the lowest position they share.

For example, assume there are five objects to rank. If the output rank in ranking notation is `4 3 4 1 1`, the `1` for both the objects number 4 and 5 indicates that either object 4 is in first position and object 5 in second or object 5 in second position and object 4 in first. Then the object number 2 is in third position, then objects 1 and 3 are in fourth and fifth or fifth and fourth.

Big4 data set: ready for the practical activity with Rankcluster!

- Check that in 2001 Arsenal and Chelsea had the same UEFA coefficient and then are tied for the first ranking dimension
- Consider that in 1992-1993, we only know the rank of the winner for UEFA
- Run a clustering for $K \in \{1, \ldots, 3\}$
- Check the retained K value with BIC and ICL
- Check the parameters of the mixture for $K = 2$ and the related partition

Big4 set: R code

```
install.packages("Rankcluster")
library(Rankcluster)

# see the dataset
data(big4)$data

# consider that in 1992-1993, we only know the rank of the winner for UEFA
big4$data[1,2:4]=0

# clustering
res=rankclust(big4$data,m=big4$m,K=1:3,Qsem=1000,Bsem=100,Ql=500,Bl=50,maxTry=20,run=5)

# see results: K=2 is retained
res
```

Big4 set: output for $K = 2$

```
ll= -104.6243
bic = 236.6493
icl = 238.3433
proportion: 0.370478 0.629522
mu:
    dim1       dim2
cl1    1 3 2 4    1 3 2 4
cl2    1 3 4 2    1 4 3 2

pi:
         dim1      dim2
cl1 1.0000000 0.7707837
cl2 0.6968134 0.7769508

partition:
 [1] 1 1 1 1 2 1 2 2 2 2 1 1 2 2 2 2 2 1 2 2 2

tik:
           [,1]        [,2]
 [1,] 0.9936257 0.006374303
 [2,] 0.9932910 0.006709006
 [3,] 0.9634028 0.036597242
 [4,] 0.9931952 0.006804807
 [5,] 0.0000000 1.000000000
 [6,] 0.9936359 0.006364105
 [7,] 0.0000000 1.000000000
 [8,] 0.0000000 1.000000000
 [9,] 0.0000000 1.000000000
[10,] 0.0000000 1.000000000
...
```

End of Part II