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**Coupling and synchronization dynamics in endogenous business
cycles models**

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1 Introduction

This communication deals with the conditions of synchronization of coupled nonlinear economic oscillators. We intend to show, building on recent advances of synchronization theory, that within nonlinear business cycle theory with the same isolated structural equations, very rich dynamical patterns can emerge. Extending the standard nonlinear flexible accelerator-multiplier framework of the business cycle to the case of coupled economies, we analyze the role the coupling technology and the existence of delays in the emergence of different scenario, ranging from complete to partial or transient synchronization. Thus, the implementation of new advances on coupling and synchronization analysis may substantially contribute to renewing the interest in nonlinear economic dynamics.

An abundant empirical literature on the synchronization issue of economic activities within and between sectors, international trade, financial markets and business cycles is today available. A general agreement has emerged on the existence of strong and robust positive relationships between trade, financial linkages and synchronization (see e.g. Kim 2015, Hanus and Vachaa 2016). But scholars disagree as to whether there is or not a tendency to decoupling in national business cycles before the crisis. Several studies also find a significant convergence of fluctuations among the group of industrial economies and the group of emerging economies with a concomitant decline in the coupling between the two (Partial synchronization hypothesis). In addition, regions with strong financial links appears to be more significantly synchronized (Kose, Otrok and Prasad 2010, Karadimitopoulou and Leo-Lesdema 2013). Finally, another interesting new stylized fact refers to the property of intermittency and wave-like fluctuations (Barnett and Dalkir, 2007, Crowley and Schultz 2012). The few theoretical exceptions consider a given network topology with diffusive coupling and limit dynamic analysis to the determination of the relevant range for the coupling coefficient achieving full synchronization (Li 2009, Barré, Raybaut and Torre 2012, Raybaut 2012).

This contribution takes a different perspective. Our aim is mainly to investigate theoretically the synchronization issue of economic activities within an endogenous business cycle framework. As regards economic analysis, it is well established that business cycles generally display asymmetric oscillations, where upturns or expansions differ from contractions in intensity and duration. Thus, the potential value of relaxation oscillations to capture these features was recognized early, and the use of the Lienard-Van der Pol or forced Van der Pol equations became standard in postwar nonlinear multiplier-accelerator business cycle models. From this standpoint, our approach follows the tradition initiated by Richard Goodwin concerning the

importance of nonlinearities and coupling within economic dynamics (Duan 2011, Abraham and Nivala 2012, Zambelli 2012).

Synchronization is an inherently collective dynamics phenomenon investigated within networks of coupled entities. The nature of the collective dynamics depends on the properties of the isolated entities and the characteristics of the coupling. The latter refer to the coupling function, its intensity and range of the interaction network (local or global coupling) and its intensity (coupling strength). The number of studies is currently growing in recent years with recent discoveries of new and often surprising forms of collective patterns of network synchronization not anticipated in previous studies (see e.g. the special issue of *Chaos* ed. by Abrams, Pecora and Motter in 2016). Kuramoto's seminal model is still studied extensively, notably with different networks of interaction, noise, delays, heterogeneity and frustration (see e.g. Strogatz's 2000, Dorfler and Bullo 2014 review articles).

But, different types of models are also analyzed in the recent theoretical literature on coupling and synchronization dynamics. In particular various classes of coupled dynamical systems like Van der Pol, forced Van der Pol-Duffing or Rössler oscillators are largely investigated. These works connect directly with the formal frameworks used in some classical nonlinear business cycle models. There is also specific connections with some issues in the modern economic literature on synchronization, like partial synchronization or intermittency. Thus, it seems interesting to re-examine these approaches in the light of these recent advances on coupling dynamics.

From this standpoint, this contribution is organized as follows. We first shortly mention some recent advances in synchronization dynamics which seem important for our issue. Second, we consider the isolated frameworks for two main traditions in nonlinear business cycle modeling, one associated to Kalecki-Kaldor and the other to Goodwin. Third, we introduce a coupling and present some main provisional numerical findings for these two approaches.

2 Some selected advances in synchronization dynamics

Understanding the different and often surprising forms that may emerge with coupled oscillators is still a major topic of current research (see e.g. Kori, Kuramoto and al. 2014, Abrams and al. 2016, Schmidt 2015). Chimera states are a good example of such puzzling patterns appearing in networks of identical oscillators with nonlocal coupling schemes. They exhibit a hybrid structure combining the existence of domains of synchronized and

de-synchronized elements (Panaggio and Abrams 2015). Different types of such chimera or chimera-like states have been identified in recent studies under nonlocal symmetric coupling or global coupling with limit cycles or chaotic orbits (Bera, Majhi, Ghosh and Perc, 2017, Rakshit, Bera, Perc and Ghosh 2017). The mechanisms behind the emergence of these patterns are not totally clarified and differ for different coupling configurations. Earlier studies indicated that the oscillators should be non locally coupled in order to generate these patterns. The introduction of non local coupling or hierarchical networks promotes the emergence of complex spatio-temporal patterns (Ulonska, et al. 2016). The presence of delays (transmission delays in interactions) has also drastic effects on the dynamics. But global coupling can also break the symmetry of population into synchronous and non synchronous parts. Accordingly chimera and chimera-like states have also been observed in globally coupled systems. Indeed, multistability and delay-induced multistability seems to play a crucial role with globally coupled systems (Chandrasekar et al. 2014, Premalatha, Chandrasekar, Senthilvelan and Lakshmanan 2015, Röhlm, Lüdge and Schneider 2017). A combination of attractive and repulsive feedbacks in coupling was also shown to break the symmetry of globally coupled oscillators (Yeldesbay and Pikrosky 2014, Mishra et al. 2015). The effects an external signal (forced system) are also studied in order to understand whether synchronization can be improved by means of this signal. Generally, it acts as a disturbance factor deterring synchronization within a group of agents (Alderisio, Bardy and Bernardo 2016).

These features are intriguing since they emerge with identical oscillators and different types of dynamical systems. In particular, they have been studied systematically in networks of identical Van der Pol oscillators (Barron 2016). In a recent study, Ulonska and al. (2016) consider Van der Pol oscillators coupled via networks of increasing hierarchical nature, showing that hierarchical coupling and large network clustering promote the existence of different types of chimera states. Chandrasekar and al. (2014), explore analytically the reason explaining the occurrence of chimera states in globally coupled oscillators. Considering two different nonlinear dynamical systems, coupled forced Van der Pol oscillators and coupled Rössler systems, they identify a mechanism of intensity induced chimera states. They find that the effect of intensity is to cause multistability by increasing the number of fixed points. This in turn increases the number of multistable attractors, and their stability is determined by the strength of coupling. This causes the coexistence of different collective states. Their findings show in particular that intensity induced chimera is generic to both periodic and chaotic systems.

The introduction of delays on the state variables of neighbors in the network is also considered(see e.g. Maia, Macau, Pereira and Yanchuk 2017). These

delays correspond to transmission delays induced by limited communication, transportation, or cognitive resources, or to lags between local decisions and their execution related to inertia in reactions of the agents or to technology and time to build. In business cycles theory, the role played by these delays on the dynamics is well-established since the pioneering works of Kalecki and Frisch and Holmes in the thirties (Szydłowski, Krawiec and Tobola 2001). These delays critically affect synchronization of the global system. Even with diffusive coupling, the synchronized dynamics is no longer the same as in the isolated case. Delays enrich the dynamics, especially in the discrete case, where a system of homogeneous oscillators unstable when isolated can be stabilized by coupling. In the case of Van der Pol oscillators with a delay, it can be shown that slight modifications of the value of this delay is critical for the existence of interesting forms of synchronization.

3 Two representative isolated frameworks in non-linear business cycle theory

3.1 The Kalecki-Kaldor tradition

The well-known representatives approaches in nonlinear business cycle theory, one associated to Kalecki-Kaldor and the other to Goodwin have been widely investigated in isolated framework since the 50's.

Let mention the Japanese pioneering mathematical advances in non-linear dynamics developed notably by Yasui, Morishima and Ichimura examined in detail in Velupillai (2008). As we may recall that these contributions were in the 1950s the first to investigate closely the question of the existence and uniqueness of a limit cycle in the business cycle models of Kaldor, Hicks and Goodwin with Lienard or van der Pol equations and Levinson and Smith (1942) criteria resorting to controversial ad hoc assumptions. These findings were synthetised by Ichimura (1955) in his classical attempt to develop a general non-linear macrodynamic theory of economic fluctuations along Keynesian lines and becomes the prototype framework for the development of endogenous business cycle theory in the Kalecki-Kaldor tradition. Then, numerous reformulations within nonlinear dynamical system theory followed this approach during the renewal of nonlinear business cycle modeling. Let only mention, Chang and Smyth (1971), Gabisch and Lorenz (1989), Galeotti and Gori (1989), Flaschel (2009), Szydłowski and Krawiec (1998), Kaddar and Alaoui (2008). These studies prove the existence of one, or at at least one, limit cycle referring to Levinson and Smith theorem, Poincaré-Bendixon theorem or Hopf bifurcation (see Arena and Raybaut 1995, Raybaut, 2014).

In this perspective, let us consider a simple synthetic framework with time to build θ in the capital accumulation equation, $\dot{K} = \mathbf{I}(Y_{t-\theta}, K) - \delta K$.

The dynamics of global income \dot{Y} is governed by the discrepancy between investment and saving, where the investment function is necessarily non linear. Assume that the investment function is additively separable and linear in capital, $\mathbf{I} = I(Y) - \beta K$ and linear saving $S = sY$.

Then, after simple manipulations, the dynamical system in $Y - K$ can be written as a second order time-delay differential equation :

$$\ddot{Y} + f(Y)\dot{Y} + g(Y) = 0 \quad (1)$$

with,

$$f(Y) = \beta + \delta + \alpha(s - I'_Y), \quad g(Y) = \alpha(\beta I_{t-\theta}) - (\beta + \delta)(I - sY)$$

Notice that without time to build ($\theta = 0$) and with $\delta = 0$, this equation can be reduced to a classic Van der Pol equation as shown by Yasui (1953).

3.2 The nonlinear accelerator tradition : Goodwin's approach

The second milestone in the development of nonlinear business cycle theory is given by Goodwin's attempts in the fifties. As well known, his nonlinear accelerator models provide one of the very few examples of forced oscillators in macrodynamics combining Keynesian and some rough Schumpeterian insights (see Sordi 2003, Raybaut 2014).

By introducing lags in the multiplier process (ε) and time to build (θ) Goodwin (1951) obtains his Raleigh type sd order differential equation

$$\varepsilon\theta\ddot{y} + (\varepsilon + s\theta)\dot{y} - \Phi(\dot{y}) + sy = 0$$

where y stands for global income, s is the propensity to save and $\Phi(\dot{y})$ is Goodwin's nonlinear investment function.

This equation can be easily transformed in a VdP equation. By differentiating with respect to time and substituting x for \dot{y} , we obtain :

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \quad (2)$$

where, $f(x) = \frac{1}{\varepsilon\theta}[\varepsilon + s\theta - \Phi'(x)]$ and $g(x) = \frac{1}{\varepsilon\theta}sx$. Assume that $g(x)$ is odd with respect to x . In addition, a unique limit cycle exists in isolation if $f(x)$ is even with $f(0) < 0$ and $f''(0) > 0$.

As suggested by Goodwin (1946), the addition of a forcing effect to relation (2) (roughly) captures Schumpeter's idea of clustering of innovations (Sordi 2003). We have

$$\ddot{x} + f(x)\dot{x} + g(x)h(t) = 0$$

Where the forcing effect $h(t)$ is a periodic function of time. Then the model in isolation behaves like a forced VdP-Duffing equation. As long as the forcing is weak (small amplitude of h), the dynamics does change qualitatively (limit cycle). With strong forcing, irregular oscillations may emerge (chaotic attractor).

4 Coupling and synchronization patterns

As in other complex phenomena, collective dynamics with coupled oscillators is based on the properties of isolated entities and their interactions. As shown in the growing literature on this issue, synchronization depends namely on, (i) the interaction network topology (local or global coupling), (ii) the coupling function which captures interaction forms, (iii) the coupling coefficient measuring the coupling strength. Consequently, we focus on the role of these elements and on the existence of time delays in the emergence of different configurations, ranging from complete to partial or transient synchronization.

4.1 The coupling technology

Consider a network of n identical coupled oscillators defined by relation (1) or (2) above and different network topologies defined respectively by their given adjacency matrices $\mathbf{A} = \{a_{ij}\}$.

We investigate global coupling where $a_{ij} = 1$ if $i \neq j$ and 0 otherwise, and other configurations with nonlocal coupling.

From this standpoint, let define the new variables, $u = Y$, $v = \dot{Y}$ and $\mathbf{x}_i = (u_i, v_i)^T$, $i = 1, \dots, n$. Then, the dynamics of each isolated oscillator is governed by :

$$F(\mathbf{x}) = \begin{pmatrix} v \\ -vf(u) - g(u) \end{pmatrix}$$

The coupling range is defined by c_i the number of neighbors of i and the coupling strength by a parameter σ .

Coupling is diffusive and captured by the 2×2 real matrix $H = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$

This coupling function involves two components (or feedbacks) referring to the variables u and v via the two parameters a and b (see Ulonska et al. 2016, Omelchenko et al. 2015).

Then, the n dynamical coupled equations for the 2-dimensional phase space variables \mathbf{x}_i are :

$$\dot{\mathbf{x}}_i = F(\mathbf{x}_i) + \frac{\sigma}{c_i} \sum_j A_{ij} \mathbf{H}(\mathbf{x}_j - \mathbf{x}_i) \quad \text{with } i = 1, \dots, n$$

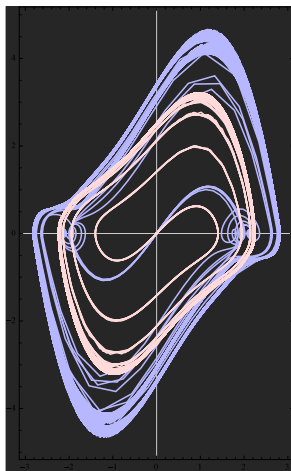
In the simplified case of the Kaldor-Kalecki model with $\delta = 0$, the isolated map admits as in a VdP equation a unique stationary solution $(0, 0)$.

In the general cases, other stationary solutions may coexist due to the non linear investment function I in g . For the Goodwin's model, interesting patterns are induced by the forcing effect.

4.2 Chimera, clusters and other patterns : some examples

We give some examples of numerical findings obtained in a network of intermediary size $n = 100$. The investment function is specified by a sigmoid. These examples focuss on chimera, clusters and the introduction of a delay in the coupling.

Example of Phase diagram Kalecki-Kaldor



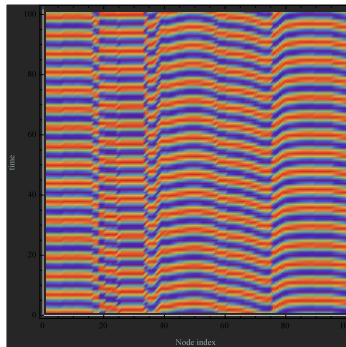
4.3 Coupling with delay in the Kalecki-Kaldor framework

Let now introduce a transmission delays in interactions τ in the coupling function.

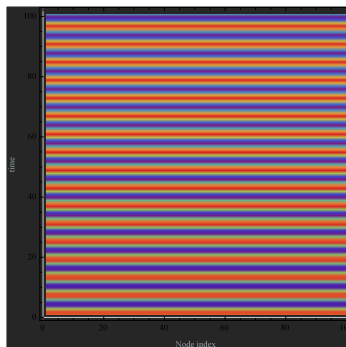
Our simulation results presented below are in line with recent findings on VdP systems with nonlocal hierarchical coupling (e.g. Ulonska et al. 2016, Omelchenko et al. 2015). They confirm that their impact in terms of synchronization are very contrasted and strongly depend on the delay.

4.3.1 Synchronization patterns for different delays

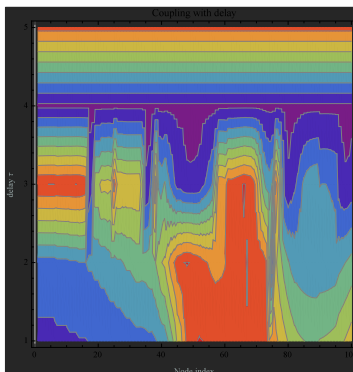
We first give below two examples of density plots for the u_i with $\tau = 3$ and $\tau = 6$. The x axis displays the nodes indexes $i = 1, \dots, 100$ and the y axis the axis snapshot. The last figure displays a synthetic density plot with $\tau \in [1, 6]$.



For $\tau = 3$, the complex patterns are stable or reinforced.



For $\tau = 6$, full synchronization is obtained.



Synthetic density plot for u_i . The x axis displays the nodes i and the y axis the delays $\tau \in [1, 6]$

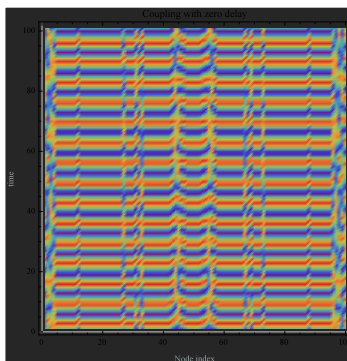
Varying the delay may amplify or discard the complex patterns. For small delays, chimera states are improved, for large delays, complete synchronization is obtained. The explanation of this phenomena proposed in Omelchenko et al. (2015) is that delays interact with the intrinsic period of the cycle, giving rise to some resonance effect when the delay is an integer of the intrinsic period of the oscillators.

4.4 Chimera and clusters in Goodwin's framework

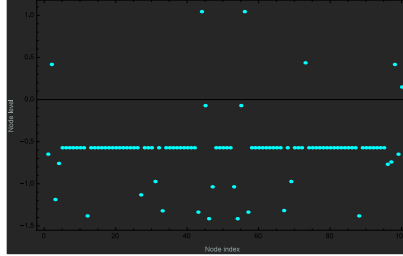
We now consider the dynamics of globally n coupled oscillators of Goodwin's type. From this standpoint, we make the same substitution in the variables $u = x$ and $v = \dot{x}$ and retain the coupling technology presented above. The main difference consists in the introduction of a forcing term $F \text{Sin}[\omega t]$ and global coupling.

4.5 Example of Chimera

The rest of the paper illustrates the emergence of Chimera and clusters in the Goodwin's framework with coupling described above.



Density plot for the u_i where the x axis displays the nodes index i and the y axis a time snapshot $\sigma = 1, b = 1, a = -0.05$

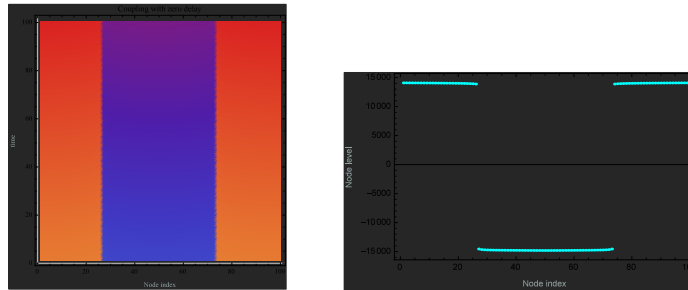


The x axis displays the nodes indexes and the y axis the levels of the u_i

We now consider the emergence of different clusters in this framework when a becomes negative enough. The left figures display a density plot for the u_i , where the x axis displays the nodes i and the y axis a time snapshot. On the right figures, the x axis display the nodes i and the y axis the levels of the u_i .

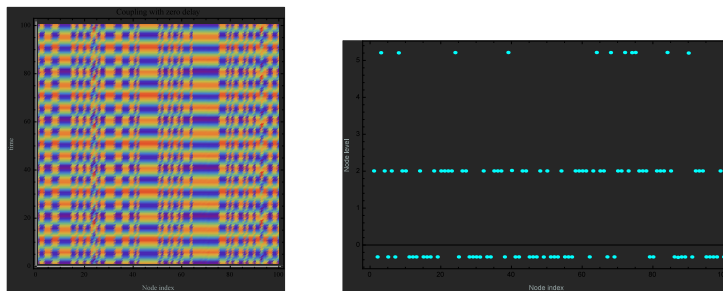
4.6 Emergence of 2 clusters $a = -0.1$

The figure below displays the emergence of 2 clusters :



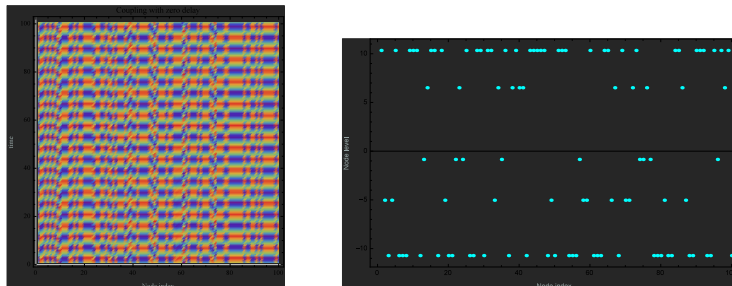
4.7 Emergence of 3 clusters $a = -0.47$

The figure below gives an example with 3 clusters :



4.8 Emergence of 5 clusters $a = -1.87$

The figure below gives an example with 5 clusters :



The figures above exemplify the emergence of different clusters. The population of oscillators splits into 2,3 or more coherent parts as the negative parameter a decreases. Within each cluster synchronization is achieved. An interplay between negative and positive feedbacks captured by parameters a and b seems necessary to obtain unsynchronized patterns, chimera or clusters in this framework with global coupling.

5 Concluding remarks

The implementation of complex tools on coupling and synchronization dynamics may substantially contribute to the renewal of the nonlinear perspective in dynamic economic theory, mainly characterized today by the use of linear stochastic equilibrium models. To clarify these provisional findings, we have to compute in each case some "strength of spatiotemporal incoherence" indicators between nodes developed recently. Some analytical investigations on stability are also possible using existing results. Extensions to other models, notably coupling dynamics in endogenous equilibrium business cycles and growth frameworks are also in progress.

A complementary approach is to consider a growing network in which connections and nodes are either lost or added to the network. Different approaches of networks under structural change have been developed. However, in these studies structural changes in the network topology is generally considered as an external event (Anzo and Barajas-Ramirez, 2014). From an economic point of view, structural change is intrinsic to the coupled dynamics of industries, prices and input-output network dynamics of a given economy. Structural dynamics necessitates to combine endogenous and exogenous mechanisms (see Arena and Porta 2012, Lorentz 2015). In this perspective, one possibility is to consider a growing network subject to an overall constraint of synchronization (Wang et al. 2015). Starting from a stable system of synchronized oscillators, the aim is to understand the effect of the growth of the

network on stability or collapse of the global system. This line of research focusses on different time scales involved in the dynamics. Two key time scales will be considered, one associated with the transient synchronization dynamics referring to short run economic fluctuations or business cycles and the other associated with the evolution of the network, referring to long run economic growth and structural change. Depending on a synchronization threshold, the introduction of a new node can increase or decrease (removing the de-synchronized nodes) the size of the network. Consequently, the size of the overall system cannot expand indefinitely and its growth rate is no longer a monotonous function of time, which offers a promising framework for modeling structural economic dynamics.

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