HAL open science

Constant Q-curvature metrics with a singularity

Tobias König, Paul Laurain

To cite this version:

Tobias König, Paul Laurain. Constant Q-curvature metrics with a singularity. 2021. hal-03505336

HAL Id: hal-03505336

https://hal.science/hal-03505336v1

Preprint submitted on 30 Dec 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

CONSTANT Q-CURVATURE METRICS WITH A SINGULARITY

TOBIAS KÖNIG AND PAUL LAURAIN

Abstract. For dimensions $n \geq 3$, we classify singular solutions to the generalized Liouville equation $(-\Delta)^{n/2}u = e^{nu}$ on $\mathbb{R}^n \setminus \{0\}$ with the finite integral condition $\int_{\mathbb{R}^n} e^{nu} < \infty$ in terms of their behavior at 0 and ∞. These solutions correspond to metrics of constant Q-curvature which are singular in the origin. Conversely, we give an optimal existence result for radial solutions. This extends some recent results on solutions with singularities of logarithmic type to allow for singularities of arbitrary order. As a key tool to the existence result, we derive a new weighted Moser–Trudinger inequality for radial functions.

1. Introduction

Let $n \geq 2$. Our goal is to understand the structure of the set of solutions to the equation

$$(-\Delta)^{n/2}u = e^{nu} \qquad \text{on } \mathbb{R}^n \setminus \{0\}, \qquad \Lambda := \int_{\mathbb{R}^n} e^{nu}\,\mathrm{d}y < \infty, \tag{1.1}$$

which may present a singularity at the origin.

For a function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the expression $(-\Delta)^{n/2}u$ is to be understood as the tempered distribution satisfying

$$\langle (-\Delta)^{n/2}u, \varphi\rangle = \int_{\mathbb{R}^n} u(x)(-\Delta)^{n/2}\varphi(x)\,\mathrm{d}x \tag{1.2}$$

for every $\varphi \in C_c^\infty(\mathbb{R}^n)$. If n is odd, the hypothesis $\int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{2n}}\,\mathrm{d}x < \infty$ needs to be added, since $(-\Delta)^{n/2}\varphi$ is not compactly supported in these cases. We refer to [26] for more details and basic regularity results. In fact, every solution to (1.1) belongs to $C^\infty(\mathbb{R}^n \setminus \{0\})$, by the proof of [26, Theorem 2.1].

For $n = 2$, equation (1.1) is the classical Liouville equation, whose solutions u correspond to metrics of constant Gauss curvature on $\mathbb{R}^2 \setminus \{0\}$. In higher dimensions, equation (1.1) plays a similar role in connection with the notion of Q-curvature. Indeed if one considers a smooth compact Riemannian surface (Σ, g) with Gauss curvature K_g,

© 2021 by the authors. This paper may be reproduced, in its entirety, for non-commercial purposes.

Date: October 20, 2021

Partial support through ANR BLADE-JC ANR-18-CE40-002 is acknowledged.

then by a conformal change of metric $g_u = e^{2u}g$, the curvature changes as follows

$$-\Delta_g u + K_g = K_{g_u} e^{2u}.$$

Here, $\Delta_g = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j)$ is the Beltrami Laplacian, which is a conformally invariant operator in dimension 2 in the sense that $\Delta_{g_u} = e^{-2u}\Delta_g$. Relying on this conformal invariance property, Paneitz [41] (and then Graham, Jenne, Mason and Sparling [21]) proved that, on a given Riemannian manifold (M^n, g), with $n \geq 2$, there exists a unique differential[1] operator P_n of order n such that

$$P_g^n = (-\Delta_g)^{\frac{n}{2}} + \text{lower order operator},$$

and

$$P_{g_u}^n = e^{-nu} P_g^n.$$

For ξ the Euclidean metric on $\mathbb{R}^n$, we can check that $P_\xi^n = (-\Delta)^{n/2}$ on $(\mathbb{R}^n, \xi)$. Hence, the notion of Gauss curvature is generalized by the one of Q-curvature, in the sense that we have the following identity

$$P_g^n(u) + Q_g = e^{nu} Q_{g_u},$$

where Q_g, the Q-curvature, depends only on the curvature and its derivatives, for instance for $n = 4$, we have

$$Q_g = -\frac{1}{6}\left(\Delta_g R_g - R_g^2 + 3|\mathrm{Ric}_g|^2\right),$$

where R_g and Ric_g are the scalar and the Ricci curvature. On the sphere $\mathbb{S}^n$ the Paneitz operator has a simple expression in terms of the Laplacian and its eigenvalues, see Section 3. For instance for $n = 4$, we have, for g_c the standard metric of $\mathbb{S}^4$,

$$P_{g_c}^4 = -\Delta_{g_c}(-\Delta_{g_c} + 2),$$

see [9] for more geometrical details.

Hence, solutions to (1.1) correspond to metrics which are conformal to the Euclidean metric on $\mathbb{R}^n \setminus \{0\}$ and have constant Q-curvature equal to one. When the behaviour of u near 0 is $u \sim \gamma \ln(|x|)$, the metric $e^{nu}\xi$ can be interpreted as a conical metric, see [14], but if the u blow faster at the origin the nature of singularity is closer to that of an essential singularity. This is to our knowlege an inexplored field from the geometric point of view.

A fundamental property of equation (1.1), which we shall use several times in this paper, is its conformal invariance. In particular, if u solves (1.1), then its inversion

$$\bar{u}(x) = u\left(\frac{x}{|x|^2}\right) - 2\ln(|x|). \tag{1.3}$$

is also a solution to (1.1), of same mass Λ as u.

[1]Pseudodifferential when the dimension is odd.

1.1. **Overview of the problem.** In the past decades, entire solutions to the equation

$$(-\Delta)^{n/2}u = e^{nu} \qquad \text{on } \mathbb{R}^n, \qquad \Lambda := \int_{\mathbb{R}^n} e^{nu}\,\mathrm{d}y < \infty, \tag{1.4}$$

have been intensely studied for all values of $n \in \mathbb{N}^*$.

A fundamental observation is that the family of 'bubble' functions

$$u(x) = \ln\left(\frac{2((n-1)!)^{1/n}\lambda}{1+\lambda^2|x-x_0|^2}\right), \qquad \lambda > 0, \quad x_0 \in \mathbb{R}^n, \tag{1.5}$$

solve (1.4), for any $n \in \mathbb{N}^*$. These solutions all have the same mass

$$\Lambda \equiv (n-1)! \int_{\mathbb{R}^n} \left(\frac{2}{1+|x|^2}\right)^n \mathrm{d}x = (n-1)!|\mathbb{S}^n| =: \Lambda_1. \tag{1.6}$$

By the classification result of Chen and Lin [11], these are all solutions to (1.4) in the classical case $n = 2$. If $n = 1$, the same is true, see [31] and references therein.

It has been put to our attention by Pierre-Damien Thizy that in fact the $n = 2$ case was already essentially known by Liouville himself, the proof relies on the interpretation of this problem in terms of holomorphic functions, see [8, Theorem 1] and [4, p. 27].

An important feature of equation (1.4), which was first noticed in the fundamental paper [30] by Lin, is that this classification results ceases to be true in higher dimensions $n \geq 3$. To explain this phenomenon in more detail, for a solution u to (1.4), we introduce its 'normalized version'

$$v(x) := \frac{1}{\gamma_n}\int_{\mathbb{R}^n} \ln\left(\frac{|y|}{|x-y|}\right) e^{nu(y)}\,\mathrm{d}y. \tag{1.7}$$

Here the constant

$$\gamma_n := \frac{(n-1)!}{2}|\mathbb{S}^n| = \frac{\Lambda_1}{2}$$

is chosen such that $(-\Delta)^{n/2}\ln\left(\frac{1}{|x|}\right) = \gamma_n\delta_0$ in the sense of distributions.

Theorem A ([30, 34, 28, 22])**.** *Let $n \geq 3$ and let u be a solution to* (1.4)*. Then there exists a polynomial p, bounded from above, of even degree at most $n-1$, such that*

$$u(x) = v(x) + p(x), \qquad x \in \mathbb{R}^n,$$

where v is as in (1.7)*.*

Moreover, as $|x| \to \infty$,

$$v(x) = -\frac{\Lambda}{\gamma_n}\ln(|x|) + o(\ln|x|) \tag{1.8}$$

If $u(x) = o(|x|^2)$ as $|x| \to \infty$, or if $n = 3, 4$ and $\Lambda = \Lambda_1$, then u is necessarily of the form (1.5)*.*

Finally, if $n = 3, 4$, then necessarily $\Lambda \leq \Lambda_1$.

Conversely, the following summary of existence results shows that the classification from Theorem 1.1 is essentially optimal.

In fact, it turns out that one can simultaneously prescribe the mass Λ and its asmyptotic polynomial p at infinity, provided $\Lambda \in (0, \Lambda_1)$ and p fulfills

$$p(x) \to -\infty \quad \text{as } |x| \to \infty, \qquad \deg p \leq n-1. \tag{1.9}$$

Theorem B ([10, 46, 27, 23, 35, 24])**.** *Let $n \geq 3$. Given any $\Lambda \in (0, \Lambda_1)$ and any polynomial p satisfying* (1.9)*, there exists a solution u to* (1.4) *such that*

$$u(x) = p(x) - \frac{\Lambda}{\gamma_n} \ln(|x|) + o(\ln|x|) \qquad \text{as } |x| \to \infty.$$

Let $n \geq 5$. Given any $\Lambda \in (0, \infty)$ and any radially symmetric polynomial p satisfying (1.9)*, there exist numbers $c_1, c_2 > 0$ and a solution u to* (1.4) *such that*

$$u(x) = p(x) + c_1|x|^2 - c_2|x|^4 - \frac{\Lambda}{\gamma_n} \ln(|x|) + o(\ln|x|) \qquad \text{as } |x| \to \infty.$$

These findings are in remarkable contrast to the equation $(-\Delta)^k u = u^{\frac{n+2k}{n-2k}}, u > 0$ on $\mathbb{R}^n$, with $n/2 > k \in \mathbb{N}^*$, which is a closely related conformally invariant of (1.1). Indeed, for this equation the classification results in [6, 30, 45] guarantee that every solution has the standard bubble form corresponding to (1.5), for every $n \geq 2$ and $k < n/2$.

Compared to the case of entire solutions to (1.4), until recently singular solutions to (1.1) have received much less attention, with the exception of $n = 2$.

If $n = 2$, it has been shown in [42] that for (1.1) a classification result analogous to the entire case holds. Indeed, every solution is of the form

$$w(z) = \ln\left(\frac{2(\alpha+1)\lambda}{1 + \lambda^2|z^{\alpha+1} - \zeta|^2}\right), \qquad \lambda > 0,\ \zeta \in \mathbb{C},\ \alpha > -1,$$

in complex notation, and where necessarily $\zeta = 0$ if $\alpha \notin \mathbb{N}_0$.

Still when $n = 2$, we mention the papers [33, 7], where existence of solutions to a singular equation corresponding to (1.1) on general Riemannian surfaces has been derived using refined variational arguments and improved Moser-Trudinger inequalities.

1.2. **Main results.** The main point of our results is to give a version of both the classification and existence results from Theorems A and B for the case of a general point singularity, i.e. for equation (1.1) instead of (1.4).

Similar to before, for any solution u to (1.1), we define

$$v_u(x) := v(x) := \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \ln\left(\frac{1+|y|}{|x-y|}\right) e^{nu(y)}\, \mathrm{d}y. \tag{1.10}$$

The following theorem classifies all singular solutions to (1.1) in terms of their asymptotic behavior at 0 and infinity.

Theorem 1.1. *Let $n \geq 3$ and let u be a solution to* (1.1)*. Then there exist $\beta \in \mathbb{R}$ and upper-bounded polynomials p and q, of degree at most $n-1$, such that*

$$u(x) = v(x) + p(x) + q\left(\frac{x}{|x|^2}\right) + \beta \ln(|x|). \tag{1.11}$$

Moreover

$$\lim_{|x|\to\infty} \frac{v(x)}{\ln(|x|)} = -\frac{\Lambda}{\gamma_n} \qquad \text{and} \qquad \lim_{x\to 0} \frac{v(x)}{\ln(|x|)} = 0.$$

For solutions to a special case of equation (1.1), namely

$$(-\Delta)^{n/2} u = e^{nu} - \beta\gamma_n \delta_0, \qquad \int_{\mathbb{R}^n} e^{nu} < \infty, \tag{1.12}$$

Theorem 1.1 has been proved in [26]. However, even without appealing to the sophisticated existence result in Theorem 1.2 below, it is easy to see that there are solutions to (1.1) which do not satisfy (1.12). For instance, for u is as in Theorem B with $p(x) = -|x|^2$, the inversion $\bar{u}$ defined in (1.3) solves (1.1) and has $q(x) = -|x|^2$ in (1.11). On the other hand, by [26] every solution to (1.12) behaves like $\beta \ln |x|$ near zero and thus has necessarily $q \equiv 0$. For $n = 4$, by ODE arguments in the spirit of [14], one can also easily construct solutions to (1.1) with both p and q non-zero, which thus cannot fulfil (1.12) even after inversion.

Let us now discuss existence results for (1.1) more systematically. It is apparent from the results stated above that a major challenge is to prescribe the mass $\Lambda > 0$ of the solution, simultaneously with its asymptotic behavior, for Λ in the largest possible range.

Our next theorem shows that in the singular setting of equation (1.1), this can be done for every possible value of Λ, at least for radial functions.

Theorem 1.2. *Let $n \geq 3$, and let p be a radial polynomial satisfying* (1.9)*. Suppose either that*

(a) *q is another radial polynomial satisfying* (1.9)*, $\beta \in \mathbb{R}$ and $\Lambda > 0$ or that*

(b) *$q \equiv 0$, $\beta > -1$ and $\Lambda \in (0, \Lambda_1(1+\beta))$.*

If p is radial-decreasing, we may also take $\Lambda = \Lambda_1(1+\beta)$ in (b).

Then there exists a solution u to (1.1) *satisfying*

$$u(x) = q\left(\frac{x}{|x|^2}\right) + \beta \ln(|x|) + o(\ln(|x|)) \qquad \text{as } |x| \to 0$$

and

$$u(x) = p(x) + \left(\beta - \frac{\Lambda}{\gamma_n}\right)\ln(|x|) + o(\ln(|x|)) \qquad \text{as } |x| \to \infty.$$

As in the case of entire solutions, this shows that the space of singular solutions to (1.1) becomes more complex as the order of derivative increases. Again, this is in stark contrast to $(-\Delta)^k u = u^{\frac{n+2k}{n-2k}}, u > 0$ on $\mathbb{R}^n \setminus \{0\}$ for $n/2 > k \in \mathbb{N}^*$. Indeed, when $k = 2$ it is proved in [17] that all solutions to this equation are given by a two-parameter family of functions $u_{a,T}(x) = |x|^{(4-n)/2} v_a(\log|x| + T)$, which presents no additional degrees of freedom compared to the basic case $k = 1$ analyzed by [6, 43].

Part (b) of Theorem 1.2, with $p(x) = -|x|^2$, is proved in [26, Theorem 1.2]. Our more general version answers, for the case of radial p, an open question mentioned in [26, Section 1.1]. Since in this case the functions one obtains grow logarithmically at the origin, these are indeed also solutions to (1.12). Except for the treatment of the limit case $\Lambda = \Lambda_1(1+\beta)$ in part (b), we use a different method of proof. Namely, the proof in [26] proceeds via a fixed point argument, while ours is based on a variational method, closer to the works [10] or [23]. Our method allows to treat the case $q \not\equiv 0$ (which we did not succeed in via the fixed point method of [26]) and thus remove the upper bound on Λ. On the other hand, variants of this fixed point argument also yield existence results for non-radial solutions to equation (1.12) [26, Theorem 1.3], respectively to its generalization $(-\Delta)^{n/2} u = e^{nu} - \sum_{l=1}^m \beta_l \gamma_n \delta_{P_l}$ with several singularities in points $P_1, ..., P_m$ [25, Theorem 1.2], which have no counterpart in our paper. A variational existence argument in the spirit of our paper covering the case of several singularities when $n = 4$ can however be found in [14]. We also mention [32], where existence results are derived by yet another different method.

The main tool in our approach to prove Theorem 1.2 is a weighted Moser-Trudinger inequality for radial functions on the sphere $\mathbb{S}^n$, which may be of independent interest and which we introduce now.

Let $Q : \mathbb{S}^n \to [0, \infty)$ be a nonnegative weight function on $\mathbb{S}^n$ with the property that

$$Q(\eta) \le C d(\eta, N)^\beta \exp(-d(\eta, S)^{-\sigma}) \tag{1.13}$$

for some $\sigma, C > 0$, $\beta \in \mathbb{R}$. Here, we denote by $d(\eta, \xi)$ the geodesic distance between two points $\eta, \xi \in \mathbb{S}^n$ on the sphere. We use $N = e_{n+1}$ and $S = -e_{n+1}$ to denote the north and south pole of $\mathbb{S}^n$.

Moreover, for $s \in (0, n)$ we define

$$K_{n,s} = \frac{\Gamma(\frac{n-s}{2})}{\Gamma(\frac{s}{2}) 2^s \pi^{n/2}}. \tag{1.14}$$

This constant is chosen such that the Green's function of $(-\Delta)^{s/2}$ on $\mathbb{R}^n$ is given by $K_{n,s}|x-y|^{-n+s}$.

We prove the following inequality.

Theorem 1.3. *Let Q be as in* (1.13)*. Then there is $C > 0$ such that*

$$\int_{\mathbb{S}^n} \exp\left(\frac{n+\beta}{|\mathbb{S}^{n-1}|} K_{n,\frac{n}{2}}^{-2} u^2\right) Q(\eta) \leq C. \tag{1.15}$$

uniformly for $u \in C^\infty(\mathbb{S}^n)$ radial with $\|P_n^{1/2} u\|_{L^2(\mathbb{S}^n)} \leq 1$ and $\int_{\mathbb{S}^n} u = 0$. Moreover, the constant $\frac{n+\beta}{|\mathbb{S}^{n-1}|} K_{n,\frac{n}{2}}^{-2}$ is sharp in the sense that if it is replaced by $\gamma > \frac{n+\beta}{|\mathbb{S}^{n-1}|} K_{n,\frac{n}{2}}^{-2}$, then the constant C is no longer uniform in u.

Theorem 1.3 is in fact a special case of a more general inequality valid for any order of derivative strictly between 0 and n. Indeed, the operator $P_n^{1/2}$ can be replaced by either P_s with $s \in (0, n)$ or by $P_{2s}^{1/2}$ with $s \in (0, n/2)$, where P_s is the Paneitz operator on $\mathbb{S}^n$ of order s; see (3.2) for its definition. To keep the introduction focused, we defer a full statement to Theorems 5.2 and 5.3 below.

We stress that if $\beta > 0$, the best constant in the above theorems is strictly larger than the constant $\frac{n}{|\mathbb{S}^{n-1}|} K_n^{-\frac{n}{n-s}}$ of the standard Moser-Trudinger inequality. In this case, the restriction to radial functions is thus truly essential. Indeed, the improved inequality must fail for a sequence of functions saturating the sharp constant in the unweighted inequality and concentrating in a point near which the weight Q is regular. Since such a point is different from the poles, such functions must of course necessarily be non-radial.

A simple variation of the arguments leading to Theorem 1.3 (respectively, Theorems 5.2 and 5.3) yields a proof of a corresponding weighted Moser-Trudinger inequality for radial functions on balls of $\mathbb{R}^n$.

Theorem 1.4. *Let $s \in (0, n)$, $\beta \in \mathbb{R}$ and $R > 0$. Then there is $C = C(R) > 0$ such that*

$$\int_{B_R} \exp\left(\frac{n+\beta}{|\mathbb{S}^{n-1}|} K_{n,s}^{-\frac{n}{n-s}} |u|^{\frac{n}{n-s}}\right) |x|^\beta \, \mathrm{d}x \leq C \tag{1.16}$$

for every $u \in C_0^\infty(B_R)$ radial with $\|(-\Delta)^{s/2} u\|_{L^{\frac{n}{s}}(B_R)} \leq 1$.

This generalizes the fractional Moser-Trudinger inequality of Martinazzi [36] to the weighted (and radial) setting. For even dimensions n and $s = n/2$, Theorem 1.4, including the case $\beta > 0$, has been proved in the recent paper [40] by a somewhat different method going back to [44]. However, since this method does not pass through a fractional integral formulation (compare the discussion below and Theorem 5.1), it does not seem to extend easily to the general case of fractional derivatives of arbitrary order. Similar weighted inequalities on all of Euclidean space appear in [13, 12].

Let us discuss in some more detail our proof strategy on the example of Theorem 1.4. The general strategy, going back to Adams [2], is to estimate u pointwise by the

convolution $v = G_s * (-\Delta)^{s/2} u$, where G_s is the Green's function of $(-\Delta)^{s/2}$ and prove a corresponding estimate on the exponential integral of that convolution; see Theorem 5.1 below.

When $\beta < 0$ in Theorem 1.4, the hypothesis of radial symmetry can be dropped by combining Martinazzi's Green's function estimate from [36] with the weighted fractional integral inequality of Lam and Lu [29, Theorem 1.1]. Indeed, in this case the weight $|x|^\beta$ is radial-decreasing. The authors of [29] can thus continue to follow Adams' proof, which consists in replacing v by its symmetric decreasing rearrangement v^* and using a variant of O'Neil's lemma [39] to get appropriate pointwise estimates on v^*.

If $\beta > 0$, the weight $|x|^\beta$ is radial-*increasing* and thus the rearrangement argument breaks down. Somewhat counterintuitively, this remains a challenge even if one restricts a priori to radial functions. Indeed, the terms which arise from transforming to radial variables turn out to have a slightly different structure than the O'Neil's-type estimate mentioned above, which leads to error terms of a different nature; see Remark 6.4 below for more details. Overcoming the non-availability of the rearrangement argument just described in the case $\beta > 0$ is in fact one of the main achievements of our proof. We are able to deal with the new error terms through a new improvement of the well-known Adams-Garsia's lemma [2, Lemma 1] stated as Lemma 6.1 below.

In the setting on $\mathbb{S}^n$ of Theorem 1.3 (respectively Theorems 5.2 and 5.3), we strongly believe that when $-n < \beta < 0$, a symmetrization argument can equally allow to drop the radial symmetry assumption. To our knowledge, this has only been carried out rigorously for the special case $n = 4$ and $s = 2$ in the recent paper [14].

2. Classification results

In this section we prove Theorem 1.1. We split the proof into two propositions to be proved in the following two subsections.

2.1. The behavior of $u - v$.

The following proposition is the main result of this subsection.

Proposition 2.1. *Let u solve* (1.1) *and let v be defined by* (1.10)*. Then*

$$u(x) - v(x) = p(x) + q\left(\frac{x}{|x|^2}\right) + \beta \ln(|x|)$$

for some polynomials p, q and some $\beta \in \mathbb{R}$. Moreover, p and q are bounded from above with $\deg p, \deg q \leq n-1$*. If n is even, then* $\deg p, \deg q \leq n-2$*.*

Lemma 2.2. *The function v defined in* (1.10) *satisfies*

$$(-\Delta)^{n/2} v = e^{nu} \qquad \text{on } \mathbb{R}^n$$

in the sense of distributions.

Proof. This proof is very standard and amounts just to reproving that $\frac{1}{\gamma_n}\ln(\frac{1}{|x-y|})$ is the Green's function for $(-\Delta)^{n/2}$ on $\mathbb{R}^n$. To emphasize its generality, we write $f := e^{nu} \in L^1(\mathbb{R}^n)$. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$. Then by Fubini

$$\int_{\mathbb{R}^n} v(x)(-\Delta)^{n/2}\varphi(x)\,\mathrm{d}x = \frac{1}{\gamma_n}\int_{\mathbb{R}^n} f(y)\left(\int_{\mathbb{R}^n} \ln\left(\frac{1+|y|}{|x-y|}\right)(-\Delta)^{n/2}\varphi(x)\,\mathrm{d}x\right)\mathrm{d}y.$$

Firstly, we observe that

$$\int_{\mathbb{R}^n} \ln(1+|y|)(-\Delta)^{n/2}\varphi(x)\,\mathrm{d}x = 0$$

Then we are left with showing that

$$\frac{1}{\gamma_n}\int_{\mathbb{R}^n} \ln\left(\frac{1}{|x-y|}\right)(-\Delta)^{n/2}\varphi(x)\,\mathrm{d}x = \varphi(y).$$

Since φ is smooth, this can be done by spliting the integral into the part over $B_\varepsilon(y)$ and over its complement, integrate by parts and use that $(-\Delta)^i\frac{1}{|x-y|}$ is integrable on $\mathbb{R}^n$ for $i = 0, 1, ..., m-1$. □

Lemma 2.3. *Let u solve* (1.1) *and let v be defined as in* (1.10). *Then $v(x) \geq 0$ if $|x| \leq 1$. Moreover, for all $|x| \geq 1$ we have*

$$v(x) \geq -\frac{\Lambda}{\gamma_n}\ln(|x|). \tag{2.1}$$

Proof. The proof of (2.1) is identical to [26, Lemma 3.1]: Let $|x| \geq 1$, then

$$|x-y| \leq |x| + |y| \leq |x|(1+|y|)$$

and therefore

$$\ln\left(\frac{1+|y|}{|x-y|}\right) \geq \ln\left(\frac{1}{|x|}\right).$$

Similarly, if $|x| \leq 1$, we have

$$|x-y| \leq |x| + |y| \leq 1 + |y|$$

and therefore

$$\ln\left(\frac{1+|y|}{|x-y|}\right) \geq \ln(1) = 0.$$

Inserting these estimates into (1.10), we obtain the conclusion. □

We can now give the

Proof of Proposition 2.1. We first prove the proposition under the assumption that $n \geq 4$ is even and then give the necessary modifications in case $n \geq 3$ is odd.

Step 1. Bôcher's Theorem. By Lemma 2.2, we know that $\eta := u - v$ satisfies $(-\Delta)^{n/2}\eta = 0$ on $\mathbb{R}^n \setminus \{0\}$. In particular, η is smooth on $\mathbb{R}^n \setminus \{0\}$.

Then the generalized Bôcher theorem in [18] (applied on a sequence of balls $B_R(0)$ with radii R tending to ∞) implies that

$$\eta = p + \sum_{\alpha \leq s+n-1} c_\alpha D^\alpha \ln\left(\frac{1}{|x|}\right) \qquad \text{on} \quad \mathbb{R}^n \setminus \{0\},$$

where $p \in C^\infty(\mathbb{R}^n)$ satisfies $(-\Delta)^{n/2}p = 0$ on $\mathbb{R}^n$ and s is some non-negative integer such that

$$\int_{B_1(0)\setminus\{0\}} \eta^+(x)|x|^s \,\mathrm{d}x < \infty.$$

Using Lemma 2.3 we can estimate, for $|x| \leq 1$, $\eta^+(x) \leq u^+(x) \leq e^{nu^+(x)}$. This is integrable, so we can take $s = 0$. Since it is easily proved by induction that $D^k \ln\left(\frac{1}{|x|}\right) = q_k\left(\frac{x}{|x|^2}\right)$ for every $|k| \geq 1$, where q_k is a polynomial of degree at most $|k|$, we conclude that

$$u(x) = v(x) + p(x) + q\left(\frac{x}{|x|^2}\right) + \beta \ln(|x|) \tag{2.2}$$

with $(-\Delta)^{n/2}p = 0$ on $\mathbb{R}^n$ and $\deg q \leq n-1$.

Step 2. Inversion.

Again, for definiteness suppose first that n is even. We shall show in this step that p must be a polynomial of degree at most $n-1$ as well. To do so, we make use of the conformal invariance of (1.1). More precisely, the Kelvin transform $\bar{u}(x) := u\left(\frac{x}{|x|^2}\right) - 2\ln(|x|)$ is also a solution to (1.1) with same mass Λ and, by the above, can be written as

$$\bar{u}(x) = v_{\bar{u}}(x) + \tilde{p}(x) + \tilde{q}\left(\frac{x}{|x|^2}\right) + \tilde{\beta}\ln(|x|),$$

for some $\tilde{\beta} \in \mathbb{R}$, with $v_{\bar{u}}$ given by (1.10). Moreover, a change of variables in the definition of v shows the relation

$$v_{\bar{u}}(x) = v_u\left(\frac{x}{|x|^2}\right) - \frac{\Lambda}{\gamma_n}\ln(|x|). \tag{2.3}$$

Putting these identities together, we find

$$\tilde{p}(x) + \tilde{q}\left(\frac{x}{|x|^2}\right) + \left(\tilde{\beta} - \frac{\Lambda}{\gamma_n}\right)\ln(|x|) = p\left(\frac{x}{|x|^2}\right) + q(x) - \beta\ln(|x|) \tag{2.4}$$

for all $x \neq 0$, where $(-\Delta)^{n/2}\tilde{p} = 0$ and $\deg \tilde{q} \leq n-1$. We claim that (2.4) implies $\tilde{q} = p$. Indeed, by (2.4) we have

$$\limsup_{|x|\to\infty} \frac{p(x)}{|x|^{n-1}} = \limsup_{|x|\to 0} |x|^{n-1} p\left(\frac{x}{|x|^2}\right) = \limsup_{|x|\to 0} |x|^{n-1}\tilde{q}\left(\frac{x}{|x|^2}\right) < \infty$$

because $\deg \tilde{q} \leq n-1$. Thus $p(x) \leq 1 + |x|^{n-1}$ on $\mathbb{R}^n$ and the generalized Liouville theorem [34, Theorem 5] implies that p is a polynomial of degree $\deg p \leq n-1$. Once we know that both $\tilde{q}$ and p must be polynomials, it is straightforward to deduce from (2.4) that in fact $\tilde{q} = p$.

(By exchanging the roles of u and $\bar{u}$, we may of course deduce as well that $\tilde{p} = q$.)

Step 3. Upper-boundedness and optimal degree. We finish the proof by showing that p and q are bounded from above. In particular, p and q must be of even degree and thus $\deg p, \deg q \leq n-2$. Since $q = \tilde{p}$ as in Step 2, it suffices to prove that p is bounded from above.

If p is unbounded from above, inspired by lemma 11 in [34], then (see Theorem 3.1 in [Gorin] and [26, proof of Lemma 3.3]) there is $s > 0$ and a sequence x_k with $|x_k| \to \infty$ such that $p(x_k) \geq |x_k|^s$. Since $\deg p \leq n-1$, we have $|\nabla p(y)| \lesssim |y|^{n-2}$ for all $|y|$ large enough. Thus, using also Lemma 2.3, we have

$$u(x) \geq p(x) + \left(\beta - \frac{\Lambda}{\gamma_n}\right)\ln(|x|) - C \gtrsim |x_k|^s$$

for all x in a ball $B_{\rho_k}(x_k)$ of radius $\rho_k \gtrsim |x_k|^{\frac{s}{n-2}}$. Therefore, for some $c, d > 0$

$$\int_{\mathbb{R}^n} e^{nu}dx \geq \int_{B_{\rho_k}(x_k)} e^{nu}dx \geq \rho_k^n e^{nc|x_k|^s} \geq d|x_k|^{\frac{ns}{n-2}} e^{nc|x_k|^s} \to \infty \qquad \text{as } k \to \infty.$$

This contradiction proves that p must be bounded from above.

Step 4. n odd. If n is odd, we use that $(-\Delta)^{\frac{n+1}{2}}\eta = 0$ in Step 1 instead. Using the Bôcher theorem as above, we find that u satisfies (2.2), but with $\deg q \leq n$. The argument from Step 2 then gives $\deg p \leq n$. Repeating Step 3, p and q are bounded from above. In particular, they are of even degree, hence $\deg p, \deg q \leq n-1$. □

2.2. **The asymptotic behavior of v at ∞.** The purpose of this subsection is to establish the leading-order behavior of $v(x)$ as $|x| \to \infty$.

Proposition 2.4. $\lim_{|x|\to\infty} \frac{v_u(x)}{\ln(|x|)} = -\frac{\Lambda}{\gamma_n}$.

We recapitulate in what follows the strategy in [26], which carries over to our case and yields a proof of Proposition 2.4. In Section 2.3 below we present an alternative approach, which in our opinion is simpler and more direct, but which does not work in the general case, except if $n = 3, 4$.

Recall that by Lemma 2.3, $v(x) \geq -\frac{\Lambda}{\gamma_n}\ln(|x|)$ for all $|x| \geq 1$. The following lemma yields a first step towards the desired reverse inequality.

Lemma 2.5. *For every $\varepsilon > 0$ there is $R > 0$ such that for all $|x| \geq R$*

$$v(x) \leq \left(-\frac{\Lambda}{\gamma_n} + \varepsilon\right)\ln(|x|) + \int_{B(x,1)} \ln\left(\frac{1}{|x-y|}\right) e^{nu(y)}\,\mathrm{d}y. \tag{2.5}$$

Proof. See [30, proof of (2.11)]. There $n = 4$, but the proof is the same for general dimension $n \geq 3$, compare [34, Lemma 9] □

The following lemma corresponds to [26, Lemma 3.5]. (For simplicity, we only consider a radius equal to one.)

Lemma 2.6. *For every $t \in [1,\infty)$ and $\varepsilon \in (0, \frac{\Lambda}{\gamma_n})$, there is $C > 0$ such that for all $x \in \mathbb{R}^n$,*

$$\int_{B(x,1)} e^{tv(y)}\,\mathrm{d}y \leq \frac{C}{|x|^{(\frac{\Lambda}{\gamma_n}-\varepsilon)t}}.$$

Proof of Lemma 2.6. The proof is identical to [26, Proof of Lemma 3.5], with the only difference that the term $|y|^{n\beta}e^{nu(y)}$ is replaced by $e^{nu(y)}$. The proof works exactly the same because $\int_{\mathbb{R}^n}|y|^{n\beta}e^{nu(y)}\,\mathrm{d}y$ is assumed to be finite in the normalization of [26], where in our normalization this assumption becomes $\Lambda = \int_{\mathbb{R}^n} e^{nu(y)}\,\mathrm{d}y < \infty$. □

Lemma 2.6 is already enough to prove Proposition 2.4 in the case when the coefficient β of $\ln(|x|)$ in Proposition 2.1 satisfies $\beta < \frac{\Lambda}{\gamma_n}$.

Proof of Proposition 2.4 in the case $\beta < \frac{\Lambda}{\gamma_n}$. In view of Lemmas 2.3 and 2.5, it remains to control the integral on the right side of (2.5). For this purpose, we claim that there is $r > 1$, $R > 0$ such that for all $|x| \geq R$,

$$\|e^{nu}\|_{L^r(B(x,1))} \leq C. \tag{2.6}$$

Indeed, by Hölder's inequality this implies

$$\int_{B(x.1)} \ln\left(\frac{1}{|x-y|}\right) e^{nu(y)}\,\mathrm{d}y \leq \left(\int_{B(x,1)} \left(\ln\left(\frac{1}{|x-y|}\right)\right)^{\frac{r}{r-1}}\mathrm{d}y\right)^{\frac{r-1}{r}} \|e^{nu}\|_{L^r(B(x,1))} \leq C. \tag{2.7}$$

Hence

$$\limsup_{|x|\to\infty} \frac{v(x)}{\ln(|x|)} \leq -\frac{\Lambda}{\gamma_n} + \varepsilon$$

and since $\varepsilon > 0$ was arbitrary, the proof is complete.

We now prove (2.6). Since p and q from Proposition 2.1 are upper-bounded, and by Lemma 2.6 applied with $q = nr$,

$$\int_{B(x,1)} e^{nru} \lesssim |x|^{n\beta r} \int_{B(x,1)} e^{nrv} \lesssim |x|^{rn(\beta - \frac{\Lambda}{\gamma_n} + \varepsilon)}.$$

Since $\beta < \frac{\Lambda}{\gamma_n}$ by assumption, we can pick $\varepsilon > 0$ so small that $\beta - \frac{\Lambda}{\gamma_n} + \varepsilon < 0$. Thus

$$\int_{B(x,1)} e^{nru} \to 0$$

as $|x| \to \infty$. In particular (2.6) holds. □

It remains to prove Proposition 2.4 in the case $\beta \geq \frac{\Lambda}{\gamma_n}$. The additional ingredient for this is the following Hölder-norm estimate.

Lemma 2.7. *Let u solve* (1.1) *and let v be defined by* (1.10)*. Suppose that $\beta \geq \frac{\Lambda}{\gamma_n}$, where β is as in Proposition 2.1. Then as $|x| \to \infty$,*

$$[v]_{C^{0,(\ln(|x|+1))^{-1}}(B(x,1))} = o(\ln(|x|+1)).$$

Proof. Again, the proof is identical to [26, proof of Lemma 3.6], up to replacing the term $|y|^{n\beta} e^{nu(y)}$ by $e^{nu(y)}$. One proceeds by first proving the Campanato-type estimate

$$\sup_{\rho \in (0,4]} \frac{1}{\rho^{n + \frac{1}{\ln(|x|)}}} \int_{B(x,\rho)} \left| v(y) - \frac{1}{|B(x,\rho)|} \int_{B(x,\rho)} v(z)\, \mathrm{d}z \right| \mathrm{d}y \to 0, \qquad \text{as } |x| \to \infty,$$

and then transforming this to the desired Hölder-type bound by a standard argument.

Let us point out that thanks to the classification result from Proposition 2.1 and upper-boundedness of the polynomial q the bound

$$e^{nu(y)} \lesssim |x|^{n\beta} e^{nv(y)} e^{np(y)},$$

which is used in [26, proof of Lemma 3.6] still holds. Also, the bound from [26, Lemma 3.3]

$$\int_{B(x,\rho)} e^{qp} \lesssim |x|^{n(\frac{\Lambda}{\gamma_n} - \beta)},$$

for every $q \geq 1$, $|x| \geq 1$ and ρ small enough (uniformly in x and ρ), still holds, by the same proof as given there. □

We can now use Lemma 2.7 to give the proof of Proposition 2.4 in the remaining case $\beta \geq \frac{\Lambda}{\gamma_n}$.

Proof of Proposition 2.4 if $\beta \geq \frac{\Lambda}{\gamma_n}$. By contradiction, in view of Lemma 2.3, assume that there is $\delta > 0$ and a sequence $|x_k| \to \infty$ such that

$$v(x_k) \geq \left(-\frac{\Lambda}{\gamma_n} + 3\delta \right) \ln(|x_k|).$$

By Lemma 2.7, for any $y \in B(x_k, 1)$, we have

$$v(y) = v(x_k) + o(\ln(|x_k|)) \geq \left(-\frac{\Lambda}{\gamma_n} + 2\delta\right) \ln(|x_k|).$$

Thus

$$\int_{B(x_k,1)} e^{nv(y)} \, \mathrm{d}y \geq |B_1||x_k|^{-n(\frac{\Lambda}{\gamma_n} - 2\delta)}.$$

On the other hand, by Lemma 2.6 with $\varepsilon = \delta$,

$$\int_{B(x_k,1)} e^{nv(y)} \, \mathrm{d}y \lesssim |x_k|^{-n(\frac{\Lambda}{\gamma_n} - \delta)},$$

a contradiction. □

Proof of Theorem 1.1. Theorem 1.1 follows from Propositions 2.1 and 2.4. □

2.3. **The asymptotic behavior of v at ∞: an alternative approach.** Given the relatively lengthy and involved proof of Proposition 2.4 in the previous subsection, we think that it is interesting to mention here an alternative strategy to find the asymptotic behavior of v at ∞. It exploits the conformal invariance of equation (1.1) and relies on the following simple-looking property of polynomials on $\mathbb{R}^n$.

Conjecture 2.8. Let q be a polynomial on $\mathbb{R}^n$. If $\sigma > -n$ is such that $\int_{\mathbb{R}^n \setminus B_1} |x|^\sigma e^{q(x)} \, \mathrm{d}x < \infty$, then there is $\varepsilon > 0$ such that $\int_{\mathbb{R}^n \setminus B_1} |x|^{\sigma+\varepsilon} e^{q(x)} \, \mathrm{d}x < \infty$.

Admitting that Conjecture 2.8 is true, a straightforward proof of Proposition 2.4 goes as follows.

Proof of Proposition 2.4 given Conjecture 2.8. By conformal invariance, $\bar{u} = u(x/|x|^2) - 2\ln(|x|)$, is again a solution to (1.1). In view of (2.3), it suffices to prove that $v_{\bar{u}}$ is bounded near 0. After exchanging $\bar{u}$ for u, this follows from the definition (1.10) of v, if we can show.

$$\int_{B_1} e^{nsu(x)} \, \mathrm{d}x < \infty \tag{2.8}$$

for some $s > 1$.

By the classification result from Proposition 2.1, we have

$$e^{nsu} = e^{nsv} |x|^{n\beta s} e^{ns(p(x) + q(x/|x|^2))}. \tag{2.9}$$

Since p and q are bounded from above, the last factor is uniformly bounded on B_1. Moreover, arguing as in [26, Proof of Theorem 2.1], for every $t > 1$ we have

$$\int_{B_1} e^{ntv(x)} \, \mathrm{d}x < \infty. \tag{2.10}$$

If $\beta > -1$,we thus have $e^{nsu} \in L^1(B_1)$ by Hölder, for $s > 1$ small enough that $\beta s > -1$.

We may thus assume in what follows that $\beta \leq -1$ and observe that $v \geq 0$ on B_1, because

$$\ln\left(\frac{1+|y|}{|x-y|}\right) \geq 0 \qquad \text{for every} \quad x \in B_1,\, y \in \mathbb{R}^n.$$

By integrability of e^{nu}, boundedness of p and Lemma 2.3 we thus have

$$\infty > \int_{\mathbb{R}^n} e^{nu} \geq \int_{B_1} |x|^{n\beta} e^{nq\left(\frac{x}{|x|^2}\right)}\,\mathrm{d}x = \int_{\mathbb{R}^n\setminus B_1} |x|^{-n(\beta+2)} e^{nq(x)}\,\mathrm{d}x.$$

By Conjecture 2.8, there is $\varepsilon > 0$ such that

$$\infty > \int_{\mathbb{R}^n\setminus B_1} |x|^{-n(\beta-\varepsilon+2)} e^{nq(x)}\,\mathrm{d}x = \int_{B_1} |x|^{n(\beta-\varepsilon)} e^{-nq(\frac{x}{|x|^2})}\,\mathrm{d}x.$$

Using this together with (2.10) and upper-boundedness of p and q, we easily infer from (2.9) and Hölder that $e^{nsu} \in L^1(B_1)$ for every $1 < s < \frac{\beta-\varepsilon}{\beta}$. This completes the proof of (2.8), and thus of Proposition 2.4. □

We are unfortunately not able to prove Conjecture 2.8 for general polynomials q. But we have the following partial result.

Proposition 2.9. *Suppose either that* $\deg q \leq 3$*, or that*

$$\begin{cases} \textit{there is } k \in [0,n] \textit{ such that } q(x) \textit{ does not depend on } z := (x_{k+1},...,x_n) \\ \textit{and } q(y) \to -\infty \textit{ as } y := (x_1,...,x_k) \to \infty. \end{cases} \tag{2.11}$$

Then Conjecture 2.8 holds.

The choices $k = 0$ and $k = n$ in (2.11) cover the cases of polynomials which are constant, respectively tend to $-\infty$.

On the other hand, an example of a polynomial q which is not covered by the assumption of Proposition 2.9 is given by $q(x) = -x_1^2x_2^2 - x_2^2x_3^2 - ... - x_{n-1}^2x_n^2$. This polynomial is bounded above and depends non-trivially on all variables, but it does not tend to $-\infty$ as $|x| \to \infty$ because it vanishes on every coordinate axis.

We point out that if $n = 3, 4$, then the asymptotic polynomial q of any solution u to (1.1) satisfies $\deg q \leq 2$ by Proposition 2.1. Therefore the alternative approach presented in this subsection yields a complete proof of Proposition 2.4 in dimensions $n = 3, 4$.

Proof of Proposition 2.9. We claim that under assumption (2.11), the integral $\int_{\mathbb{R}^n\setminus B_1} |x|^\sigma e^{q(x)}\,\mathrm{d}x$ is finite if and only if

$$\sigma < -n + k. \tag{2.12}$$

Clearly, this implies Conjecture (2.8).

Let us first prove the 'only if' part. Writing B_1^k for the unit ball of $\mathbb{R}^k$, we observe that $\mathbb{R}^k \setminus B_1^k \times \mathbb{R}^{n-k} \setminus B_1^{n-k} \subset \mathbb{R}^n \setminus B_1$. Therefore finiteness of the integral implies

$$\infty > \int_{\mathbb{R}^n \setminus B_1} |x|^\sigma e^{q(x)} \,\mathrm{d}x \geq \int_{\mathbb{R}^k \setminus B_1^k} e^{q(y)} \left(\int_{\mathbb{R}^{n-k} \setminus B_{n-k}} (|y|^2 + |z|^2)^{\sigma/2} \,\mathrm{d}z \right) \mathrm{d}y$$

Thus for a.e. $y \in \mathbb{R}^k \setminus B_1^k$, we must have

$$\int_{\mathbb{R}^{n-k} \setminus B_{n-k}} (|y|^2 + |z|^2)^{\sigma/2} \,\mathrm{d}z < \infty.$$

which yields (2.12).

Conversely, suppose that (2.12) holds. Then

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_1} |x|^\sigma e^{q(x)} \,\mathrm{d}x = &\int_{\mathbb{R}^k \setminus B_1^k} e^{q(y)} \left(\int_{\mathbb{R}^{n-k}} (|y|^2 + |z|^2)^{\sigma/2)} \,\mathrm{d}z \right) \mathrm{d}y \\ &+ \int_{B_1^k} e^{q(y)} \left(\int_{\mathbb{R}^{n-k} \setminus B^{n-k}_{\sqrt{1-|y|^2}}} (|y|^2 + |z|^2)^{\sigma/2} \,\mathrm{d}z \right) \mathrm{d}y. \end{aligned}$$

The first summand equals

$$\int_{\mathbb{R}^k \setminus B_1^k} e^{q(y)} |y|^{(n-k)+\sigma} \left(\int_{\mathbb{R}^{n-k}} (1 + |z|^2)^{\sigma/2} \,\mathrm{d}z \right) \mathrm{d}y < \infty,$$

since by (2.11), [20, Theorem 3.1] yields $q(y) \lesssim -|y|^s + C$ for some $s > 0$ as $|y| \to \infty$ on $\mathbb{R}^k$ and since by (2.12),

$$\int_{\mathbb{R}^{n-k}} (1 + |z|^2)^{\sigma/2} \,\mathrm{d}z < \infty.$$

To bound the second summand, we need to control the $\mathrm{d}z$-integral as $|y| \to 0$. We compute, for $|y|$ small enough,

$$\begin{aligned} &\int_{\mathbb{R}^{n-k} \setminus B^{n-k}_{\sqrt{1-|y|^2}}} (|y|^2 + |z|^2)^{\sigma/2} \,\mathrm{d}z \\ &\leq |y|^{n-k+\sigma} \int_{\mathbb{R}^{n-k} \setminus B_{\frac{\sqrt{1-|y|^2}}{|y|^2}}} |z|^{\sigma/2} \,\mathrm{d}z \\ &\leq |\mathbb{S}^{n-k-1}| |y|^{n-k+\sigma} \left(\frac{2}{|y|} \right)^{n-k+\sigma} \lesssim 1 \end{aligned}$$

as $|y| \to 0$. Hence the second summand is bounded as well, which completes the proof of the claimed equivalence.

Finally, assume that $\deg q \leq 3$. Arguing as in Step 3 of the proof of Proposition 2.1, we find that if $\int_{\mathbb{R}^n \setminus B_1} |x|^\sigma e^{nq(x)} \,\mathrm{d}x < \infty$, then q must be upper-bounded, and hence

have even degree. If $\deg q = 0$, assumption (2.11) holds with $k = 0$. If $\deg q = 2$, then q can be written, up to an orthogonal coordinate transformation, as

$$q(x) = -\sum_{i=1}^{n} a_i (x_i - x_0)^2 + c_0,$$

for some $a_i \geq 0$. Thus assumption (2.11) holds as well, with k being the number of non-zero coefficients a_i. □

3. The setting on the sphere $\mathbb{S}^n$

To facilitate the variational argument leading to Theorem 1.2, we shall pass to an equivalent problem on the sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$. In this brief section, we introduce the necessary preliminaries for this.

Let us denote $\mathcal{S} : \mathbb{R}^n \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ the inverse stereographic projection, i.e.

$$\mathcal{S}_j(x) = \frac{2x_j}{1+|x|^2}\,,\ j = 1, \dots, n\,, \qquad \mathcal{S}_{n+1}(x) = \frac{1-|x|^2}{1+|x|^2}\,.$$

We denote its Jacobian in a point $x \in \mathbb{R}^n$ by $J_{\mathcal{S}}(x) := (2/(1+|x|^2))^n$ and that of $\mathcal{S}^{-1}$ in a point $\eta \in \mathbb{S}^n$ by $J_{\mathcal{S}^{-1}}(\eta)$. Moreover, we call a function v on $\mathbb{S}^n$ *radial* if $v(\mathcal{S}(x))$ is a radial function on $\mathbb{R}^n$.

We now introduce the differential operators P_{2s} mentioned in the introduction. For this, recall that every $u \in L^2(\mathbb{S}^n)$ can be uniquely written as

$$u = \sum_{l=0}^{\infty} \sum_{m=1}^{N_l} u_{lm} Y_{lm} \qquad \text{for some } u_{lm} \in \mathbb{R}. \tag{3.1}$$

Here Y_{lm} are a fixed $L^2(\mathbb{S}^n)$-orthonormal basis of spherical harmonics, i.e. eigenfunctions of the Laplace-Beltrami operator $(-\Delta)_{\mathbb{S}^n}$ satisfying $(-\Delta)_{\mathbb{S}^n} Y_{lm} = \lambda_l Y_{lm}$, with $\lambda_l = l(l+n-1)$ being the l-th eigenvalue of multiplicity $N_l \in \mathbb{N}$. For $s > 0$, we introduce the $2s$-th order Paneitz operator P_{2s} by defining

$$P_{2s} u := \sum_{l,m} \frac{\Gamma(l + \frac{n}{2} + s)}{\Gamma(l + \frac{n}{2} - s)} u_{lm} Y_{lm} \tag{3.2}$$

for every $u \in L^2(\mathbb{S}^n)$ such that the right side converges in $L^2(\mathbb{S}^n)$. Here and below, the sum is the same as in (3.1) and we interpret $\Gamma(-n)^{-1} = 0$ for $n \in \mathbb{N}_0$. A conventional alternative expression for P_{2s} in terms of $(-\Delta)_{\mathbb{S}^n}$ is

$$A_{2s} = \frac{\Gamma(B + \frac{1}{2} + s)}{\Gamma(B + \frac{1}{2} - s)}, \qquad \text{for} \qquad B = \sqrt{-\Delta_{\mathbb{S}^n} + \tfrac{(n-1)^2}{4}}\,. \tag{3.3}$$

It is clear from (3.2) that for $s \in (0, n/2]$, the operator P_{2s} is nonnegative. Thus we may define its square root by

$$P_{2s}^{1/2} u := \sum_{l,m} \left(\frac{\Gamma(l + \frac{n}{2} + s)}{\Gamma(l + \frac{n}{2} - s)} \right)^{1/2} u_{lm} Y_{lm} \tag{3.4}$$

for every u such that the right side converges in $L^2(\mathbb{S}^n)$.

A crucial feature of P_{2s} is its conformal transformation property (see e.g. [19, eq. (4.11)])

$$(-\Delta)^s (J_{\mathcal{S}}^{\frac{n-2s}{2n}} u \circ \mathcal{S}) = J_{\mathcal{S}}(x)^{\frac{n+2s}{2n}} \left((P_{2s} u) \circ \mathcal{S} \right)(x) \qquad \text{for every } u \in C^\infty(\mathbb{S}^n). \tag{3.5}$$

In particular, for $s = n/2$, writing $w = u \circ \mathcal{S}$, (3.5) implies

$$\int_{\mathbb{S}^n} |P_n^{1/2} u|^2 = \int_{\mathbb{S}^n} u P_n u = \int_{\mathbb{R}^n} w \left((P_n u) \circ \mathcal{S} \right) J_{\mathcal{S}} = \int_{\mathbb{R}^n} w (-\Delta)^{n/2} w. \tag{3.6}$$

We finish by defining the Sobolev space

$$H^{n/2}(\mathbb{S}^n) := \left\{ u \in L^2(\mathbb{S}^n) \,:\, \|u\|^2_{H^{n/2}(\mathbb{S}^n)} := \|u\|^2_{L^2(\mathbb{S}^n)} + \|P_n^{1/2} u\|^2_{L^2(\mathbb{S}^n)} < \infty \right\} \tag{3.7}$$

which will play a role in the proof of Theorem 1.2. By (3.4) and Stirling's formula, the norm $\|u\|_{H^{n/2}(\mathbb{S}^n)}$ is equivalent to the standard $H^{n/2}(\mathbb{S}^n)$ norm $(\sum_{l,m} l^n u_{lm}^2)^{1/2}$. We denote by $H^{n/2}_{\mathrm{rad}}(\mathbb{S}^n)$ the subspace of $H^{n/2}(\mathbb{S}^n)$ consisting of radial functions.

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 using a variational argument, which, similarly to earlier works like [10] and [23], it is convenient to carry out on the n-dimensional sphere $\mathbb{S}^n$.

We divide the proof into several steps.

Proof of Theorem 1.2. Step 1. Reduction to a variational equation.

Let $\Lambda > 0$, $\beta \in \mathbb{R}$ and p, q (radial polynomials of degree at most $n-1$ tending to $-\infty$) be given as in the statement of Theorem 1.2.

Moreover, fix a function $u_0 \in C^\infty(\mathbb{R}^n)$ such that $u_0(x) = -\ln(|x|)$ for all $|x| \geq 1$ and denote $\varphi_0 := (-\Delta)^{-n/2} u_0$.

Let

$$K(x) := |x|^{n\beta} \exp\left(n \left(p(x) + q\left(\frac{x}{|x|^2}\right) + \frac{\Lambda}{\gamma_n} u_0 \right) \right).$$

For $w \in H^{n/2}(\mathbb{R}^n)$, let $c_w \in \mathbb{R}$ be the unique number such that

$$\int_{\mathbb{R}^n} K e^{n(w + c_w)} = \Lambda, \qquad \text{i.e.} \qquad e^{n c_w} = \frac{\Lambda}{\int_{\mathbb{R}^n} K e^{nw}}.$$

Equipped with these notations, we claim that to prove Theorem 1.2, it suffices to find $w \in H^{n/2}(\mathbb{R}^n)$ with $w(x) = o(\ln(|x|))$ as $|x| \to \infty$ such that

$$(-\Delta)^{n/2} w = K e^{n(w+c_w)} - \frac{\Lambda}{\gamma_n}\varphi_0. \tag{4.1}$$

Indeed, then

$$u(x) := w(x) + p(x) + q\left(\frac{x}{|x|^2}\right) + \beta \ln(|x|) + \frac{\Lambda}{\gamma_n} u_0(x) + c_w$$

fulfills

$$(-\Delta)^{n/2} u(x) = (-\Delta)^{n/2} w(x) + \frac{\Lambda}{\gamma_n}\varphi_0 = K e^{n(w+c_w)} = e^{nu}$$

as well as

$$\int_{\mathbb{R}^n} e^{nu} = \int_{\mathbb{R}^n} K e^{n(w+c_w)} = \Lambda.$$

Step 2. Transforming to the sphere.

Set

$$Q := J_{\mathcal{S}^{-1}} K \circ \mathcal{S}^{-1}, \qquad \psi_0 := J_{\mathcal{S}^{-1}} \varphi_0 \circ \mathcal{S}^{-1}$$

so that $\int_{\mathbb{R}^n} K e^{nw} = \int_{\mathbb{S}^n} Q e^{nu}$. By (3.5), equation (4.1) is then equivalent to

$$P_n u = -\frac{\Lambda}{\gamma_n}\psi_0 + \Lambda \frac{Q e^{nu}}{\int_{\mathbb{S}^n} Q e^{nu}}. \tag{4.2}$$

Solutions to (4.2) are given by critical points of the functional

$$I[u] := \frac{1}{2}\|P_n^{1/2} u\|^2 + \frac{\Lambda}{\gamma_n}\int_{\mathbb{S}^n} \psi_0 u - \frac{\Lambda}{n}\ln\left(\int_{\mathbb{S}^n} Q e^{nu}\right).$$

We will find such a critical point by solving the variational problem

$$\inf\left\{ I[u] \,:\, u \in H^{n/2}_{\mathrm{rad}}(\mathbb{S}^n)\right\}. \tag{4.3}$$

Notice that the infimum is taken over radial functions only.

We claim that any minimizer u_0 of (4.3) is a critical point of $I[u]$, i.e. for any test function $\phi \in H^{n/2}(\mathbb{S}^n)$, it satisfies

$$\int_{\mathbb{R}^n} P_n^{1/2} u_0 P_n^{1/2}\phi + \frac{\Lambda}{\gamma_n}\int_{\mathbb{S}^n}\psi_0\phi - \frac{\Lambda}{\int_{\mathbb{S}^n} Q e^{nu_0}}\int_{\mathbb{S}^n} Q e^{nu_0}\phi = 0. \tag{4.4}$$

If ϕ is itself radial, this is clear because u_0 is a minimizer over radial functions. The validity of (4.4) for general $\phi \in H^{n/2}(\mathbb{S}^n)$ follows since u_0, ψ_0 and q are radial and because P_n preserves radial functions.

Step 3. Minimizing $I[u]$.

As explained in the previous step, it remains to prove that a minimizer for the problem (4.3) exists. We shall give the proof using the direct method of the calculus of

variations, making crucial use of the improved Moser-Trudinger inequality in the form of Theorem 1.3.

Indeed, we claim that there is a constant $C > 0$ only depending on n, q and Λ such that

$$I[u] \geq \frac{1}{4}\|P_n^{1/2}u\|_2^2 - C \tag{4.5}$$

for all $u \in H^{n/2}_{\mathrm{rad}}(\mathbb{S}^n)$. To prove (4.5), first note that $P_n^{1/2}c = 0$ and

$$I[u+c] = I[u], \qquad \text{for any constant} \quad c \in \mathbb{R}, \tag{4.6}$$

as a consequence of the fact that $\int_{\mathbb{S}^n} \psi_0 = \int_{\mathbb{R}^n} \varphi_0 = \gamma_n$, see [23, Lemma 2.3]. Thus we may assume $\int_{\mathbb{S}^n} u = 0$. For such u, the Poincaré-type inequality

$$\|u\|_2^2 \leq C\|P_n^{1/2}u\|_2^2, \tag{4.7}$$

holds as a consequence of the definition of $P_n^{1/2}$ in (3.4). Thus we can bound the subcritical term by

$$\frac{\Lambda}{\gamma_n}\left|\int_{\mathbb{S}^n} \psi_0 u\right| \leq C\|\psi_0\|_2\|P_n^{1/2}u\|_2 \leq \frac{1}{4}\|P_n^{1/2}u\|_2^2 + C\|\psi_0\|_2^2.$$

Next, we bound the critical term $\ln\left(\int_{\mathbb{S}^n} Qe^{nu}\right)$ as follows. For any $\gamma > 0$, $nu \leq \frac{n^2}{4\gamma}\|P_n^{1/2}u\|_2^2 + \gamma\frac{u^2}{\|P_n^{1/2}u\|_2^2}$. Hence

$$\int_{\mathbb{S}^n} Qe^{nu} \leq \exp\left(\frac{n^2}{4\gamma}\|P_n^{1/2}u\|_2^2\right)\int_{\mathbb{S}^n} Qe^{\gamma\frac{u^2}{\|P_n^{1/2}u\|_2^2}}.$$

Thus, by Theorem 1.3 with $s = n/2$, since $Q(\eta) \leq C\exp -d(\eta, S)^{-\sigma}d(\eta, N)^{\beta}$ for any $\beta > 0$, we have for any $\gamma > 0$ that

$$\frac{\Lambda}{n}\ln\left(\int_{\mathbb{S}^n} Qe^{nu}\right) \leq C + \frac{n\Lambda}{4\gamma}\|P_n^{1/2}u\|_2^2.$$

Choosing $\gamma > 0$ so large that $\frac{n\Lambda}{4\gamma} < \frac{1}{4}$ and combining the above estimates, (4.5) follows.

We can now prove that (4.3) admits a minimizer. Let $(u_k)_{k\in\mathbb{N}} \subset H^{n/2}_{\mathrm{rad}}(\mathbb{S}^n)$ be a minimizing sequence for (4.3). By (4.6), we may again assume that $\int_{\mathbb{S}^n} u_k = 0$ for all $k \in \mathbb{N}$. By (4.5), $\|P_n^{1/2}u_k\|_2$ is uniformly bounded. By (4.7), so is $\|u_k\|_2$, and hence (u_k) is uniformly bounded in $H^{n/2}_{\mathrm{rad}}(\mathbb{S}^n)$. Up to extracting a subsequence, we may therefore assume $u_n \rightharpoonup u_0$ for some $u_0 \in H^{n/2}_{\mathrm{rad}}(\mathbb{S}^n)$.

Then

$$\|u_0\|_{H^{n/2}(\mathbb{S}^n)} \leq \liminf_{k\to\infty}\|u_k\|_{H^{n/2}(\mathbb{S}^n)}.$$

Moreover, since $u \mapsto u$ and $u \mapsto e^u$ defines compact embeddings from $H^{n/2}(\mathbb{S}^n)$ into $L^2(\mathbb{S}^n)$ (for the latter property see [28, Proposition 7]),

$$\|u_0\|_2 = \lim_{k\to\infty}\|u_k\|_2, \quad \int_{\mathbb{S}^n}\psi_0 u_0 = \lim_{k\to\infty}\int_{\mathbb{S}^n}\psi_0 u_k$$

and

$$\ln\left(\int_{\mathbb{S}^n} Qe^{nu_0}\right) = \lim_{k\to\infty} \ln\left(\int_{\mathbb{S}^n} Qe^{nu_k}\right).$$

In view of the definition of $\|u\|_{H^{n/2}}$ in (3.7), this implies

$$\|P_n^{1/2}u_0\|_2 \le \liminf_{k\to\infty} \|P_n^{1/2}u_k\|_2$$

and thus u_0 is a minimizer for I_0.

Step 4. Regularity

The argument that provides regularity is identical to [23]. Somewhat more precisely, by using the regularity theory for P_n that can easily be deduced from its representation on spherical harmonics [23, Lemmas 2.5 and 2.6] together with the fact that $\psi_0 \in C^{2n+1}(\mathbb{S}^n)$, one proves $u_0 \in C^{2n+1}(\mathbb{S}^n)$. Since u_0 is continuous at the south pole, the function $w_0 := u_0 \circ \mathcal{S}$ has a limit at infinity, in particular the condition $w(x) = o(\ln(|x|))$ as $|x| \to \infty$ is satisfied. Moreover, w_0 fulfills

$$(-\Delta)^{n/2} w_0 = Ke^{n(w_0+c_{w_0})} - \frac{\Lambda}{\gamma_n}\varphi_0.$$

Since $\varphi_0 \in C^\infty(\mathbb{R}^n)$, bootstrapping gives $w_0 \in C^\infty(\mathbb{R}^n)$. Thus we have found a solution $w = w_0$ as described in Step 1 and the proof is complete. □

Proof of Theorem 1.2(b). For $\Lambda < \Lambda_1(1+\beta)$, the proof of part (b) is identical to the proof of (a), except for the use of Theorem 1.3. Indeed, in this case we have $Q(\eta) \le C\mathrm{dist}(\eta, N)^{n\beta} e^{-\mathrm{dist}(\eta,S)^{-\sigma}}$ for some $\sigma > 0$ and some given $\beta > -1$. Theorem 1.3 thus yields

$$\frac{\Lambda}{n}\ln\left(\int_{\mathbb{S}^n} Qe^{nu}\right) \le C + \frac{n\Lambda}{4\gamma}\|P_n^{1/2}u\|^2,$$

with

$$\gamma = \frac{n+\beta}{|\mathbb{S}^{n-1}|} K_{n,\frac{n}{2}}^{-2}.$$

Since $\Lambda < (1+\beta)\Lambda_1$, recalling the numerical values of Λ_1 and $K_{n,n/2}$ given in (1.6) and (1.14), a direct computation shows that

$$\delta := \frac{1}{2} - \frac{n\Lambda}{4\gamma} > 0.$$

The rest of the proof for the subcritical case $\Lambda < \Lambda_1(1+\beta)$ proceeds as above.

In the critical case $\Lambda = \Lambda_1(1+\beta)$, our argument clearly breaks down. But we can take a sequence of solutions u_k having mass $\Lambda_k \nearrow \Lambda_1(1+\beta)$ and pass to the limit $k \to \infty$. Since $q \equiv 0$, we are in the setting of [26, Proof of Theorem 1.2], and we can use their argument to conclude the existence of a solution u having the limit mass $\Lambda = \Lambda_1(1+\beta)$. Indeed, the argument in [26] is written for the special case $p(x) = -|x|^2$. But it suffices to observe that the assertion of [26, Proposition A.2] continues to hold for every p which is radial-decreasing. Namely, if u solves (1.1) and

(1.11) with $q \equiv 0$ and p radial-decreasing, then the Pohozaev identity [26, Proposition A.1] implies

$$\frac{\Lambda}{\gamma_n}(\Lambda - 2\gamma_n) = 2\beta\Lambda + 2\int_{\mathbb{R}^n} x \cdot \nabla p(x) K(x) e^{nu(x)}.$$

Now if p is non-constant radial-decreasing, the last summand is strictly negative, which after rearranging terms yields the strict inequality

$$\Lambda < \Lambda_1(1+\beta).$$

As in [26], this can be used to lead [26, eq. (28)] to a contradiction and then conclude. □

5. Proof of Theorems 1.3 and 1.4

We now turn to prove the Moser-Trudinger inequalities from Theorems 1.3 and 1.4. As briefly explained in the introduction, following Adams' classical paper [2] we shall derive them from a dual inequality on convolution-type operators which we introduce now in detail.

Let $s \in (0,n)$. For $f \in L^{\frac{n}{s}}(\mathbb{S}^n)$, we denote

$$Tf(\eta) := \int_{\mathbb{S}^n} k(\eta,\xi) f(\xi)\, \mathrm{d}\sigma(\xi). \tag{5.1}$$

Here we assume that the kernel $k(\eta,\xi)$ satisfies

$$k(\eta,\xi) = d(\eta,\xi)^{-n+s}\left(1 + \mathcal{O}\left(d(\eta,\xi)\right)^{\alpha}\right). \tag{5.2}$$

for some fixed constant $\alpha > 0$. (We denote by $\mathcal{O}(d(\eta,\xi)^\alpha)$ a quantity $h(\eta,\xi)$ with the property that $|h(\eta,\xi)| d(\eta,\xi)^{-\alpha}$ is bounded uniformly in $\eta,\xi \in \mathbb{S}^n$ with $\eta \neq \xi$.)

Moreover, we assume that $k(\eta,\xi)$ only depends on the geodesic distance $d(\eta,\xi)$. As a consequence, by change of variables, Tf is radial (in the sense given in Section 3) if f is.

We are going to prove the following weighted Moser-Trudinger-Adams inequality of convolution type, valid for radial functions.

Theorem 5.1. *Let Q be as in* (1.13) *and let T be given by* (5.1)*, for some k satisfying* (5.2)*. Then there is $C > 0$ such that*

$$\int_{\mathbb{S}^n} \exp\left(\frac{n+\beta}{|\mathbb{S}^{n-1}|}|Tf|^{\frac{n}{n-s}}\right) Q(\eta) \leq C$$

uniformly for $f \in L^{\frac{n}{s}}_{rad}(\mathbb{S}^n)$ with $\|f\|_{L^{\frac{n}{s}}(\mathbb{S}^n)} \leq 1$. Moreover, the constant $\frac{n+\beta}{|\mathbb{S}^{n-1}|}$ is sharp, in the sense that if it is replaced by $\gamma > \frac{n+\beta}{|\mathbb{S}^{n-1}|}$, then the constant C is no longer uniform in f.

As in Adams' paper [2], we may deduce from this the Moser-Trudinger-Adams inequality in its differential form, more precisely Theorem 1.4 and the following two theorems. Recall that the constant $K_{n,s}$ is given in (1.14) and that the operators P_s and $P_{2s}^{1/2}$ are defined in (3.2) and (3.4).

Theorem 5.2. *Let $s \in (0, \frac{n}{2}]$ and let Q be as in (1.13). Then there is $C > 0$ such that*

$$\int_{\mathbb{S}^n} \exp\left(\frac{n+\beta}{|\mathbb{S}^{n-1}|} K_{n,s}^{-\frac{n}{n-s}} |u|^{\frac{n}{n-s}}\right) Q(\eta) \leq C. \tag{5.3}$$

uniformly for $u \in C^\infty(\mathbb{S}^n)$ radial with $\|P_{2s}^{1/2} u\|_{L^{n/s}(\mathbb{S}^n)} \leq 1$. If $s = n/2$, we assume additionally $\int_{\mathbb{S}^n} u = 0$. Moreover, the constant $\frac{n+\beta}{|\mathbb{S}^{n-1}|} K_n^{-\frac{n}{n-s}}$ is sharp in the sense that if it is replaced by $\gamma > \frac{n+\beta}{|\mathbb{S}^{n-1}|} K_{n,s}^{-\frac{n}{n-s}}$, then the constant C is no longer uniform in u.

Clearly, Theorem 5.2 contains Theorem 1.3 as its special case $s = n/2$.

Theorem 5.3. *Let $s \in (0, n)$ and let Q be as in (1.13). Then there is $C > 0$ such that*

$$\int_{\mathbb{S}^n} \exp\left(\frac{n+\beta}{|\mathbb{S}^{n-1}|} K_{n,s}^{-\frac{n}{n-s}} |u|^{\frac{n}{n-s}}\right) Q(\eta) \leq C. \tag{5.4}$$

uniformly for $u \in C^\infty(\mathbb{S}^n)$ radial with $\|P_s u\|_{L^{n/s}(\mathbb{S}^n)} \leq 1$. Moreover, the constant $\frac{n+\beta}{|\mathbb{S}^{n-1}|} K_n^{-\frac{n}{n-s}}$ is sharp in the sense that if it is replaced by $\gamma > \frac{n+\beta}{|\mathbb{S}^{n-1}|} K_{n,s}^{-\frac{n}{n-s}}$, then the constant C is no longer uniform in u.

Indeed, Theorems 5.2 and 5.3 follow from Theorem 5.1 by expressing u using the Green's function of the operators $P_{2s}^{1/2}$ and P_s respectively. It only remains to check that the resulting convolution integral satisfies (5.1) and (5.2), with $f = P_{2s}^{1/2} u$ and $f = P_s u$, respectively.

5.1. **Estimates on the Green's function and proof of Theorem 1.3.** In this subsection, we prove the following expansion of the Green's functions of $P_{2s}^{1/2}$ and P_s, which allows us to deduce Theorem 1.3, 5.2 and 5.3 from Theorem 5.1. We recall that the operators P_s and $P_{2s}^{1/2}$ have been defined in (3.2) and (3.4) respectively.

Lemma 5.4. *Let $0 < s \leq \frac{n}{2}$. There exists a function $G_s : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{R}$ with the following properties.*

(1) *For every $u \in C^\infty(\mathbb{S}^n)$ (with $\int_{\mathbb{S}^n} u = 0$ if $s = \frac{n}{2}$), one has*

$$u(\eta) = \int_{\mathbb{S}^n} G_s(\eta, \xi) P_{2s}^{1/2} u(\xi) \, \mathrm{d}\sigma(\xi),$$

(2) *For every $f \in C^\infty(\mathbb{S}^n)$, the function $u(\eta) := \int_{\mathbb{S}^n} G_s(\eta, \xi) f(\xi)$ satisfies $u \in C^\infty(\mathbb{S}^n)$ (and $\int_{\mathbb{S}^n} u = 0$ if $s = \frac{n}{2}$) and $P_{2s}^{1/2} u = f$.*

(3) *$G_s(\eta,\xi)$ only depends on $d(\eta,\xi)$. Near the diagonal, one has the expansion*

$$G_s(\eta,\xi) = K_{n,s} d(\eta,\xi)^{-n+s}\left(1+\mathcal{O}\left(d(\eta,\xi)\right)^{\alpha}\right), \tag{5.5}$$

for some $\alpha > 0$.

Moreover, the same statements hold (without the condition that $\int u = 0$) if $P_{2s}^{1/2}$ is replaced by P_s, and $s \in (0,n)$.

For P_s for $s \in (0,n)$, Lemma 5.4 follows directly from the fact that the Green's function of P_s is explicitly given by

$$G_s(\eta,\xi) = K_{n,s}|\eta-\xi|^{-n+s}, \tag{5.6}$$

see e.g. [5].

For $P_{2s}^{1/2}$ and $s \in (0,n/2]$, the argument is a generalization of the case $s = \frac{n}{2}$, see the proof of [38, Proposition 2.2].

Proof. Let us assume that $s < \frac{n}{2}$, the proof is readily the same if $s = \frac{n}{2}$ by adding the mean value. Let $w \in C^\infty(S^n)$, we have

$$w(\eta) = \int_{S^n} G(\eta,\xi)(P_{2s}w)(\xi)\, d\sigma(\xi),$$

where G is the Green function of P_{2s}. By self-adjointness of P_{2s} we have

$$w(\eta) = \int_{S^n} (P_{2s}^{\frac{1}{2}}G)(\eta,\xi)(P_{2s}^{\frac{1}{2}}w)(\xi)\, d\sigma(\xi).$$

Thus $G_s = P_{2s}^{\frac{1}{2}}G$ is the function we are looking for. Let us check the estimate along the diagonal. By classical result on pseudo-differential operators, see [1], the principal symbols satisfy $\sigma(P_{2s}^{\frac{1}{2}}) = \sigma(P_{2s})^{\frac{1}{2}}$, hence $P_{2s}^{\frac{1}{2}} = (-\Delta_{\mathbb{S}^n})^{s/2} +$ lower order operator. By using (5.6) (respectively $G(\eta,\xi) = K_{n,s}\ln\left(\frac{1}{d(\eta,\xi)}\right) + O(1)$ for $s = n/2$), we easily deduce the desired estimate. □

Proof of Theorem 5.2 and Theorem 5.3. We only prove Theorem 5.2, since the proof of Theorem 5.3 is identical after replacing $P_{2s}^{1/2}$ by P_s. Under the assumptions of Theorem 5.2, by Lemma 5.4 we may write

$$u(\eta) = \int_{\mathbb{S}^n} G_s(\eta,\xi) f(\xi)\, \mathrm{d}\xi =: Tf(\eta), \tag{5.7}$$

where $f := P_{2s}^{1/2}u$ and $G_s(\eta,\xi)$ is the Green's function associated with $P_{2s}^{1/2}$.

Still by Lemma 5.4, the kernel $k(\eta,\xi) = G_s(\eta,\xi)$ satisfies (5.2). Moreover, $\|f\|_{L^{n/s}(\mathbb{S}^n)} = \|P_{2s}^{1/2}u\|_{L^{n/s}(\mathbb{S}^n)} \le 1$ by assumption. Hence

$$\int_{\mathbb{S}^n} \exp\left(\frac{n+\beta}{|\mathbb{S}^{n-1}|}|u|^{\frac{n}{n-s}}\right) Q(\eta)\, \mathrm{d}\sigma(\eta) = \int_{\mathbb{S}^n} \exp\left(\frac{n+\beta}{|\mathbb{S}^{n-1}|}|Tf|^{\frac{n}{n-s}}\right) Q(\eta)\, \mathrm{d}\sigma(\eta) \le C$$

by Theorem 5.1.

The sharpness of the constant is again a direct consequence of the sharpness statement in Theorem 5.1. Indeed, let $\gamma > \frac{n+\beta}{|\mathbb{S}^{n-1}|}$.Then by Theorem 5.1 there exist f_k radial with $\|f_k\|_{L^{n/s}(\mathbb{S}^n)} \leq 1$ such that

$$\int_{\mathbb{S}^n} \exp\left(\gamma |u_k|^{\frac{n}{n-s}}\right) Q(\eta)\,\mathrm{d}\eta \to \infty$$

as $k \to \infty$, where we wrote $u_k := Tf_k$ with Tf_k defined as in (5.7). Since u_k are radial and $\|P_{2s}^{1/2} u_k\|_{L^{n/s}(\mathbb{S}^n)} = \|f_k\|_{L^{n/s}(\mathbb{S}^n)} \leq 1$, this completes the proof. □

6. Proof of Theorem 5.1

6.1. **An improved Adams' lemma.** The core of the proof of Theorem 5.1 is the following one-dimensional calculus lemma stated as Lemma 6.1 below. To state it, for $p > 1$ we denote by $p' = \frac{p}{p-1}$ the Hölder conjugate exponent of p.

Lemma 6.1. *Let $1 < p < \infty$, and let $a : \mathbb{R} \times [0,\infty) \to [0,\infty)$ be a function such that*

$$a(w,t) \leq \begin{cases} 1 + g(w,t) & \text{for } w \in [0,t], \\ h(w,t) & \text{for } w \in \mathbb{R} \setminus [0,t], \end{cases}$$

for some non-negative functions $g(w,t)$ and $h(w,t)$ with the property that

$$\int_0^t g(w,t) + g(w,t)^{p'}\,\mathrm{d}w + \int_{\mathbb{R}\setminus[0,t]} h(w,t)^{p'}\,\mathrm{d}w \leq b < \infty, \tag{6.1}$$

uniformly in $t \in [0,\infty)$.

For $\phi : \mathbb{R} \to (0,\infty)$ satisfying $\int_{\mathbb{R}} \phi(w)^p\,\mathrm{d}w \leq 1$, let

$$F(t) := \left(\int_{\mathbb{R}} a(w,t)\phi(w)\,\mathrm{d}w\right)^{p'} - t.$$

Then for every $\tilde{\alpha} > 0$ there is $C > 0$ not depending on ϕ such that

$$\int_0^\infty e^{-\tilde{\alpha} F(t)}\,\mathrm{d}t \leq C.$$

In its basic form, i.e. with $g(w,t) \equiv h(w,t) \equiv 0$, this lemma goes back to Moser [37]. The inclusion of the term $h(w,t)$, which is of fundamental importance, is due to Adams [2, Lemma 1]. Adams' version has since been extended in various directions, among others, in [15, 3, 16]. Our new observation is that the error term $g(w,t)$ below may depend on t as long as $g(\cdot,t)$ satisfies suitable integral bounds on the interval $[0,t]$, uniformly in $t > 0$. A typical example of an error term satisfying (6.1) below, and in fact the one we shall use in the simplest case of the proof of Theorem 5.1, would be

$$g(w,t) = Ce^{-w} + Ce^{w-t},$$

Indeed, [15, 16] are able deal with a term of type e^{-w}, which arises essentially from the error term in (5.2). The second type e^{w-t} only arises in our case, as a consequence of the lack of a rearrangement argument, as explained in the introduction. Notice that this term cannot be estimated (for $w \in [0,t]$, uniformly in $t > 0$) by any integrable function of w alone. However, as we check in the following, the proof of Adams' lemma allows to include such a term as well. We believe that this observation can be useful in a wider context to prove Moser–Trudinger-type inequalities in settings where rearrangement is not available, notably in the presence of weights.

Our proof follows Adams' paper [2, proof of Lemma 1]. The adaptations we make are close to [15, proof of Lemma 3.2]. Because of these similarities, we only give a sketch of the proof of Lemma 6.1. The reader may consult [2, 15] for more details.

Proof. For $\lambda \in \mathbb{R}$ we set $E_\lambda := \{t \geq 0 \,:\, F(t) \leq \lambda\}$ and denote by $|E_\lambda|$ its Lebesgue measure. Since

$$\int_0^\infty e^{-\tilde{\alpha}F(t)}\,\mathrm{d}t = \tilde{\alpha}\int_{\mathbb{R}} |E_\lambda| e^{-\tilde{\alpha}\lambda}\,\mathrm{d}\lambda,$$

the lemma clearly follows if we can prove the following two assertions.

(a) There is $c > 0$ such that $F(t) \geq -c$ for all $t \geq 0$. Moreover, if $t \in E_\lambda$, then there are constants d, A, B such that

$$(t+d)^{1/p}\left(\int_{\mathbb{R}\setminus[0,t]} \phi(w)^p\,\mathrm{d}w\right)^{1/p} \leq A|\lambda|^{1/p} + B. \tag{6.2}$$

(b) There are constants C, D such that for every $\lambda \geq -c$,

$$|E_\lambda| \leq C|\lambda| + D.$$

Proof of (a). Let $t \in E_\lambda$. Then by the definition of E_λ, the bound on $a(w,t)$ and Hölder's inequality,

$$\begin{aligned} t - \lambda &\leq \left(\int_{\mathbb{R}} a(w,t)\phi(w)\,\mathrm{d}w\right)^{p'} \\ &\leq \left((1-L(t)^p)^{1/p}\left(\int_0^t (1+g(w,t))^{p'}\,\mathrm{d}w\right)^{1/p'} + L(t)\left(\int_{\mathbb{R}\setminus[0,t]} h(w,t)^{p'}\,\mathrm{d}w\right)^{1/p'}\right)^{p'}, \end{aligned}$$

where we abbreviated

$$L(t) := \left(\int_{\mathbb{R}\setminus[0,t]} \phi(w)^p\,\mathrm{d}w\right)^{1/p}.$$

The assumptions on g imply that

$$\int_0^t (1+g(w,t))^{p'}\,\mathrm{d}s \leq t + d,$$

where d is a constant only depending on b. We thus obtain

$$t-\lambda \leq \Big((1-L(t)^p)^{1/p}(t+d)^{1/p'}+bL(t)\Big)^{p'}.$$

This is the same estimate as [2, eq. (15)]. Now both assertions of (a) follow by arguing as in [2].

Proof of (b). Suppose that $t_1, t_2 \in E_\lambda$ are such that $\lambda < t_1 < t_2$. We shall show that $t_2 - t_1 \leq C|\lambda|$ for all λ large enough, which clearly implies (b). Indeed, similarly to the above

$$\begin{aligned}(t_2-\lambda)^{1/p'} \leq &\left(\int_0^{t_1}(1+g(w,t_2))^{p'}\,\mathrm{d}w\right)^{1/p'}\left(\int_0^{t_1}\phi(w)^p\,\mathrm{d}w\right)^{1/p}\\ &+\left(\int_{t_1}^{t_2}(1+g(w,t_2))^{p'}\,\mathrm{d}w\right)^{1/p'}L(t_2)+\left(\int_{t_1}^{t_2}h(w,t)^{p'}\,\mathrm{d}w\right)^{1/2}L(t_2).\end{aligned}$$

By the assumptions on $g(w,t)$ we can estimate

$$\int_0^{t_1}(1+g(w,t_2))^{p'}\,\mathrm{d}w = t_1 + d\int_0^{t_2} g(w,t)\,\mathrm{d}w + d\int_0^{t_2} g(w,t)^{p'}\,\mathrm{d}w \leq t_1 + d$$

and, analogously,

$$\left(\int_{t_1}^{t_2}(1+g(w,t_2))^{p'}\,\mathrm{d}w\right)^{1/p'} \leq (t_2-t_1+d)^{1/p'} \leq (t_2-t_1)^{1/p'}+d^{1/p'}.$$

Combining everything and abbreviating $\delta := (t_2-t_1)^{1/p'}$, we obtain

$$t_2-\lambda \leq \Big((t_1+d)^{1/p'}+(\delta+b)L(t_1)\Big)^{p'}$$

Again, this is the same estimate that occurs in [2]. Using the inequality (6.2) from (a) permits to conclude as in [2]. □

6.2. **Transforming to $\mathbb{R}^n$.** As a first step in the proof of Theorem 5.1, we recast the setting on $\mathbb{R}^n$ using stereographic projection.

We first observe $d_{\mathbb{S}^n}(\eta,\xi) = |\eta-\xi|(1+\mathcal{O}(|\eta-\xi|^\alpha))$ for suitable $\alpha > 0$ (Here $|\eta-\xi|$ denotes the distance in $\mathbb{R}^{n+1}$.) Together with (5.2), up to taking a smaller $\alpha > 0$, it is easy to deduce from this that Tf defined by (5.1) satisfies

$$Tf(\eta) \leq \int_{\mathbb{S}^n} |\eta-\xi|^{-n+s}\,(1+C|\eta-\xi|^\alpha)\,f(\xi)\,\mathrm{d}\sigma(\xi). \tag{6.3}$$

Now define

$$\tilde{f}(x) := f(\mathcal{S}(x))J_{\mathcal{S}}(x)^{\frac{s}{n}},$$

so that $\|\tilde{f}\|_{L^{\frac{n}{s}}(\mathbb{R}^n)} = \|f\|_{L^{\frac{n}{s}}(\mathbb{S}^n)}$. Since the distance transforms as

$$|\mathcal{S}(x)-\mathcal{S}(y)| = J_{\mathcal{S}}(x)^{1/2n}|x-y|J_{\mathcal{S}}(y)^{1/2n},$$

by change of variables we obtain from (6.3) that

$$\tilde{T}\tilde{f}(x) := Tf(\mathcal{S}(x)) \leq \int_{\mathbb{R}^n} |\mathcal{S}(x)-\mathcal{S}(y)|^{-n+s}\left(1+C|\mathcal{S}(x)-\mathcal{S}(y)|^{\alpha}\right)\tilde{f}(y)J_{\mathcal{S}}(y)^{\frac{n-s}{n}}\,\mathrm{d}y$$
$$= \int_{\mathbb{R}^n} |x-y|^{-n+s}\left(\frac{1+|x|^2}{1+|y|^2}\right)^{\frac{n-s}{2}}\left(1+C|x-y|^{\alpha}\left((1+|x|^2)(1+|y|^2)\right)^{-\frac{\alpha}{2}}\right)\tilde{f}(y)\,\mathrm{d}y.$$

By the same change of variables in the integral appearing in Theorem 5.1, we see that it suffices to prove

$$\int_{\mathbb{R}^n} \exp\left(\frac{n+\beta}{|\mathbb{S}^{n-1}|}|\tilde{T}\tilde{f}|^{\frac{n}{n-s}}\right)K(x)\,\mathrm{d}x \leq C \tag{6.4}$$

uniformly in $\tilde{f} \in L^{\frac{n}{s}}_{\mathrm{rad}}(\mathbb{R}^n)$ with $\|\tilde{f}\|_{\frac{n}{s}} \leq 1$. Here $K(x) := Q(\mathcal{S}(x))J_{\mathcal{S}}(x)$ satisfies

$$K(x) \leq C|x|^{\beta}\exp(-|x|^{\sigma}) \tag{6.5}$$

for some $C, \sigma > 0$, $\beta \in \mathbb{R}$.

6.3. **Reducing to a local inequality.** In this and the following sections, we will work exclusively in the setting of $\mathbb{R}^n$. For ease of notation, we will drop the tilde and write henceforth, in view of the estimate in the previous section,

$$\begin{aligned} Tf(x) = &\int_{\mathbb{R}^n} |x-y|^{-n+s}\left(\frac{1+|x|^2}{1+|y|^2}\right)^{\frac{n-s}{2}} f(y)\,\mathrm{d}y \\ &+ C\int_{\mathbb{R}^n} |x-y|^{-n+s+\alpha}\frac{(1+|x|^2)^{\frac{n-s-\alpha}{2}}}{(1+|y|^2)^{\frac{n-s+\alpha}{2}}} f(y)\,\mathrm{d}y. \end{aligned} \tag{6.6}$$

We may and will always assume without explicit mention that $f \geq 0$ and that $\alpha > 0$ is so small that $n-s-\alpha > 0$.

In this step we will use the exponential decay at infinity of the weight K to control the integral in (6.4) in the region $\mathbb{R}^n \setminus B_1$. Indeed, this is a simple consequence of the following pointwise bound.

Lemma 6.2. *Let Tf be defined by (6.6) for some $f \in L^{\frac{n}{s}}_{rad}(\mathbb{R}^n)$. Then there is $C > 0$ not depending on f such that*

$$Tf(x) \leq C(1+\ln(|x|)) \qquad \textit{for all } |x| \geq 1.$$

Using this growth bound and (6.5), we find that

$$\int_{\mathbb{R}^n\setminus B_1} \exp\left(\frac{n+\beta}{|\mathbb{S}^{n-1}|}|Tf|^{\frac{n}{n-s}}\right)K(x)\,\mathrm{d}x \leq C\int_{\mathbb{R}^n\setminus B_1}\exp\left(C(1-|x|^{\sigma/2}\right)\mathrm{d}x \leq C$$

uniformly in f. Hence in what follows, we will only need to prove the local inequality

$$\int_{B_1} \exp\left(\frac{n+\beta}{|\mathbb{S}^{n-1}|}|Tf|^{\frac{n}{n-s}}\right)K(x)\,\mathrm{d}x \leq C.$$

Proof of Lemma 6.2. Let us prove that the first summand of $Tf(x)$ in (6.6) satisfied the claimed bound. In radial variables $r = |x|$ and $\rho = |y|$, it reads

$$\begin{aligned}
&\int_0^\infty \left(\int_{\mathbb{S}^{n-1}} |re_1 - \rho\omega|^{-n+s}\,\mathrm{d}\sigma(\omega)\right)\left(\frac{1+r^2}{1+\rho^2}\right)^{\frac{n-s}{2}} f(\rho)\rho^{n-1}\,\mathrm{d}\rho \\
&= |\mathbb{S}^{n-1}| r^{-n+s}\int_0^r g_{n-s}\left(\frac{\rho}{r}\right)\left(\frac{1+r^2}{1+\rho^2}\right)^{\frac{n-s}{2}} f(\rho)\rho^{n-1}\,\mathrm{d}\rho \\
&\quad + |\mathbb{S}^{n-1}|\int_r^\infty g_{n-s}\left(\frac{r}{\rho}\right)\left(\frac{1+r^2}{1+\rho^2}\right)^{\frac{n-s}{2}} f(\rho)\rho^{s-1}\,\mathrm{d}\rho,
\end{aligned}$$

where g_{n-s} is as in Lemma 6.3 below. First suppose that $s > 1$, then by Lemma 6.3, g_{n-s} is bounded on $[0,1]$. Thus, using that $\int_0^\infty f(\rho)^{\frac{n}{s}}\rho^{n-1}\,\mathrm{d}\rho \lesssim 1$, by Hölder we get that the first summand of $Tf(x)$ is bounded by a constant times

$$\left(\int_0^r \frac{\rho^{n-1}}{(1+\rho^2)^{n/2}}\,\mathrm{d}\rho\right)^{\frac{n-s}{n}} + (1+r^2)^{\frac{n-s}{2}}\left(\int_r^\infty \frac{\rho^{-1}}{(1+\rho^2)^{n/2}}\,\mathrm{d}\rho\right)^{\frac{n-s}{n}}.$$

Since

$$\int_0^r \frac{\rho^{n-1}}{(1+\rho^2)^{n/2}}\,\mathrm{d}\rho \le C(1+\ln(r))$$

and

$$\int_r^\infty \frac{\rho^{-1}}{(1+\rho^2)^{n/2}}\,\mathrm{d}\rho \le Cr^{-n}$$

for every $r \ge 1$, we obtain the conclusion.

If $s < 1$, then by Lemma 6.3, $g_{n-s}(R) \le C(1-R)^{s-1}$. Inserting this bound and arguing as before, by Hölder the first summand of $Tf(x)$ is bounded by a constant times

$$\left(\int_0^r \left(1-\frac{\rho}{r}\right)^{(s-1)\frac{n}{n-s}} \frac{\rho^{n-1}}{(1+\rho^2)^{n/2}}\,\mathrm{d}\rho\right)^{\frac{n-s}{n}} + (1+r^2)^{\frac{n-s}{2}}\left(\int_r^\infty \left(1-\frac{r}{\rho}\right)^{(s-1)\frac{n}{n-s}} \frac{\rho^{-1}}{(1+\rho^2)^{n/2}}\,\mathrm{d}\rho\right)^{\frac{n-s}{n}}.$$

$$\begin{aligned}
&\int_0^r \left(1-\frac{\rho}{r}\right)^{(s-1)\frac{n}{n-s}} \frac{\rho^{n-1}}{(1+\rho^2)^{n/2}}\,\mathrm{d}\rho \le C + \int_1^r \left(1-\frac{\rho}{r}\right)^{(s-1)\frac{n}{n-s}} \rho^{-1}\,\mathrm{d}\rho \\
&= C + \int_{1/r}^1 (1-\tau)^{(s-1)\frac{n}{n-s}}\tau^{-1}\,\mathrm{d}\tau \le C(1+\ln(r)),
\end{aligned}$$

where we changed variables $\tau = \frac{\rho}{r}$ and noticed that $(s-1)\frac{n}{n-s} > -1$ because $n \ge 2$. Similarly,

$$\begin{aligned}
&\int_r^\infty \left(1-\frac{r}{\rho}\right)^{(s-1)\frac{n}{n-s}} \frac{\rho^{-1}}{(1+\rho^2)^{n/2}}\,\mathrm{d}\rho \le C\int_r^\infty \left(1-\frac{r}{\rho}\right)^{(s-1)\frac{n}{n-s}} \rho^{-n-1}\,\mathrm{d}\rho \\
&= Cr^{-n}\int_0^1 (1-\tau)^{(s-1)\frac{n}{n-s}}\tau^{n-1}\,\mathrm{d}\tau \le Cr^{-n}
\end{aligned}$$

by changing variables $\tau = \frac{r}{\rho}$. Altogether, this shows that the first summand of $Tf(x)$ is bounded by $C(1 + \ln(|x|))$.

The second summand of $Tf(x)$ in (6.6) can be treated by the same argument, after simply estimating $(1 + |y|^2)^{-\alpha} \leq 1$ and replacing s by $s + \alpha$. □

Lemma 6.3. *For $n \geq 2$, let $0 < \alpha < n$ and define*

$$g_\alpha(R) := \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} |e_1 - R\omega|^{-\alpha} \, \mathrm{d}\sigma(\omega). \tag{6.7}$$

Then for $R \in [0, 1)$, the function $g_\alpha(R)$ satisfies

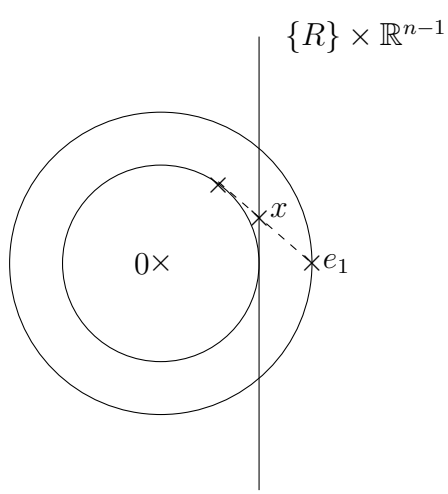
$$g_\alpha(R) \lesssim \begin{cases} 1 & \text{if } \alpha \in (0, n-1), \\ 1 + \ln\left(\frac{1}{1-R}\right) & \text{if } \alpha = n-1, \\ (1-R)^{-\alpha+n-1} & \text{if } \alpha \in (n-1, n). \end{cases}$$

Moreover, on $[0, 1)$, $g_\alpha(R)$ is

- *increasing if $\alpha \in (n-2, n)$,*
- *constant if $\alpha = n-2$ and*
- *decreasing if $\alpha \in (0, n-2)$.*

In particular, if $\alpha \in (0, n-2]$, then $\sup_{R \in [0,1]} g_\alpha(R) = g_\alpha(0) = 1$.

Proof. Let us first prove the claimed bounds on g_α. Clearly, $g_\alpha(R)$ is bounded away from $R = 1$ for every $\alpha \in (0, n)$. Considering the following picture,



Then identifying $\{R\} \times \mathbb{R}^{n-1}$ with $\mathbb{R}^{n-1} \times \{0\}$ and writing $d = 1 - R$, we clearly have

$$\begin{aligned} g(R) &\leq \int_{\{x \in \mathbb{R}^{n-1} : |x| \leq 1\}} |de_n - x|^{-\alpha} \, \mathrm{d}x + C \\ &= d^{n-1-\alpha} \int_{\{x \in \mathbb{R}^{n-1} : |x| \leq 1/d\}} |e_n - x|^{-\alpha} \, \mathrm{d}x + C \\ &\leq d^{n-1-\alpha} \int_{\{x \in \mathbb{R}^{n-1} : 1 \leq |x| \leq 1/d\}} |x|^{-\alpha} \, \mathrm{d}x + C \end{aligned}$$

As $d \to 0$, up to a constant, the last integral is bounded by 1 if $\alpha < n-1$, by $\ln(1/d)$ if $\alpha = n-1$ and by $d^{\alpha-n+1}$ if $\alpha > n-1$. This yields the claimed bound in each case.

To prove the claimed monotonicity behavior, we observe that the function $g_\alpha(R)$ is the mean value of the function $f(y) := |e_1 - y|^{-\alpha}$ over a sphere around the origin of radius R. Now for $r = |x| \neq 0$,

$$\Delta(|x|^{-\alpha}) = \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{n-1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \right) r^{-\alpha} = \alpha(\alpha + 2 - n) r^{-\alpha-2}.$$

Hence, in the open ball of radius 1, f is

- subharmonic if $\alpha \in (n-2, n)$,
- harmonic if $\alpha = n-2$ and
- superharmonic if $\alpha \in (0, n-2)$.

Thus the monotonicity of g_α follows from the mean-value theorem for harmonic functions, respectively the mean-value inequalities for sub- and superharmonic functions. □

Remark 6.4. The calculations in this section help to further elucidate the discussion following the statement of Lemma 6.1 about the new type of error term arising in our setting. We claim that the estimate obtained from of the O'Neil's-type inequality used by Adams in [2] does not extend to functions which are radial, but not necessarily decreasing. Indeed, said estimate [2, bottom of p.390] would read, in our notation and for $f \in L^{\frac{n}{s}}_{\mathrm{rad}}(B_1)$, say,

$$Tf(r) \leq |\mathbb{S}^{n-1}| \left(\frac{n}{s} r^{-n+s} \int_0^r f(\rho) \rho^{n-1} \, \mathrm{d}\rho + \int_r^1 f(\rho) \rho^{s-1} \, \mathrm{d}\rho \right). \tag{6.8}$$

On the other hand, even when we drop the terms $\frac{1+r^2}{1+\rho^2}$ due to the conformal factors $J_{\mathcal{S}}$ (i.e. when we work with the simpler convolution kernel $|x-y|^{-n+s}$ in a setting which does not come from $\mathbb{S}^n$), the calculations in the proof of Lemma 6.2 yield, for every $f \in L^{\frac{n}{s}}_{\mathrm{rad}}(B_1)$, the identity

$$Tf(r) = |\mathbb{S}^{n-1}| \left(r^{-n+s} \int_0^r g_{n-s}\left(\frac{\rho}{r}\right) f(\rho) \rho^{n-1} \, \mathrm{d}\rho + \int_r^1 g_{n-s}\left(\frac{r}{\rho}\right) f(\rho) \rho^{s-1} \, \mathrm{d}\rho \right).$$

In the case when $s < 2$, Lemma 6.3 states that $g_{n-s}(R) > g_{n-s}(0)$ for every $R \in (0,1)$. So if $f \equiv 0$ on B_{r_0}, for some $r_0 \in (0,1)$, it is plain that (6.8) is violated for every $r \leq r_0$.

6.4. **Proof of Theorem 5.1.** In this section, we give the main argument in the proof of Theorem 5.1. Recall that by what we have shown in Sections 6.2 and 6.3, it only remains to prove

$$\int_{B_1} \exp\left(\frac{n+\beta}{|\mathbb{S}^{n-1}|}|Tf|^{\frac{n}{n-s}}\right)|x|^{\beta}\,\mathrm{d}x \leq C, \tag{6.9}$$

uniformly for all $f \in L^{\frac{n}{s}}_{\mathrm{rad}}(\mathbb{R}^n)$ with $\|f\|_{L^{\frac{n}{s}}(\mathbb{R}^n)} \leq 1$. Here, Tf is given by (6.6).

Our strategy is to reduce the proof of (6.9) to the one-dimensional bound from Lemma 6.1. To achieve this, we pass to radial logarithmic coordinates. Writing $|x| = e^{-t}$, the inequality in (6.9) transforms to

$$\int_0^{\infty} \exp\left((n+\beta)\left(\left|\frac{1}{|\mathbb{S}^{n-1}|^{\frac{n-s}{n}}}(Tf)(e^{-t})\right|^{\frac{n}{n-s}} - t\right)\right)\mathrm{d}t \leq C. \tag{6.10}$$

Now, define

$$\phi(w) := |\mathbb{S}^{n-1}|^{\frac{s}{n}} f(e^{-w})e^{-sw},$$

so that

$$\int_{\mathbb{R}} |\phi(w)|^{\frac{n}{s}}\,\mathrm{d}w = \int_{\mathbb{R}^n} |f(x)|^{\frac{n}{s}}\,\mathrm{d}x \leq 1.$$

Similarly to the calculations in Lemma 6.2, writing $|x| = e^{-t}$ and $|y| = e^{-w}$, from (6.6) we obtain

$$\frac{1}{|\mathbb{S}^{n-1}|^{\frac{n-s}{n}}}(Tf)(e^{-t}) = u(t) + v(t) \tag{6.11}$$

with

$$u(t) := \int_{-\infty}^{t} g_{n-s}(e^{w-t})\left(\frac{1+e^{-2t}}{1+e^{-2w}}\right)^{\frac{n-s}{2}}\phi(w)\,\mathrm{d}w + \int_t^{\infty} g_{n-s}(e^{t-w})e^{(n-s)(t-w)}\left(\frac{1+e^{-2t}}{1+e^{-2w}}\right)^{\frac{n-s}{2}}\phi(w)\,\mathrm{d}w$$

and

$$\begin{aligned} v(t) := C\int_{-\infty}^{t} & g_{n-s-\alpha}(e^{w-t})\frac{(1+e^{-2t})^{\frac{n-s-\alpha}{2}}}{(1+e^{-2w})^{\frac{n-s+\alpha}{2}}}\phi(w)\,\mathrm{d}w \\ & + C\int_t^{\infty} g_{n-s-\alpha}(e^{t-w})e^{(n-s-\alpha)(t-w)}e^{-\alpha w}\frac{(1+e^{-2t})^{\frac{n-s-\alpha}{2}}}{(1+e^{-2w})^{\frac{n-s+\alpha}{2}}}\phi(w)\,\mathrm{d}w. \end{aligned}$$

We claim that for every $t \geq 0$

$$\frac{1}{|\mathbb{S}^{n-1}|^{\frac{n-s}{n}}}(Tf)(e^{-t}) = \int_{\mathbb{R}} a(w,t)\phi(w)\,\mathrm{d}w, \tag{6.12}$$

where $a(w,t)$ satisfies the assumptions of Lemma 6.1. Now an application of that lemma (with $\tilde{\alpha} = n+\beta$ and $p = \frac{n}{s}$) concludes the proof of (6.10), and hence of Theorem 5.1.

It thus remains to prove (6.12). Let us give the bounds for $u(t)$ in detail. For clarity we first treat the simplest case where $s > 1$, so that $g_{n-s} \leq C$ on all of $[0,1]$ by Lemma 6.3. For $w \in (-\infty, 0]$, we then have

$$g_{n-s}(e^{w-t})\left(\frac{1+e^{-2t}}{1+e^{-2w}}\right)^{\frac{n-s}{2}} \leq Ce^{(n-s)w}.$$

For $w \in (0,t]$, we simply estimate $\frac{1+e^{-2t}}{1+e^{-2w}} \leq 1$. Moreover, for $R = e^{w-t}$ we estimate $g_{n,s}(R) = g_{n,s}(0) + \int_0^R g'_{n,s}(r)\,\mathrm{d}r \leq 1 + CR$ if R is near zero and $g(R) \leq C \leq CR$ if R is away from zero. Altogether, this yields

$$g_{n-s}(e^{w-t})\left(\frac{1+e^{-2t}}{1+e^{-2w}}\right)^{\frac{n-s}{2}} \leq g_{n-s}(0) + Ce^{w-t} = 1 + Ce^{w-t}.$$

Finally, for $w \in (t,\infty)$, we estimate $g_{n-s}(e^{t-w}) \leq C$ and

$$e^{(n-s)(t-w)}\left(\frac{1+e^{-2t}}{1+e^{-2w}}\right)^{\frac{n-s}{2}} = \left(\frac{e^{2t}+1}{e^{2w}+1}\right)^{\frac{n-s}{2}} \leq \left(e^{2(t-w)} + e^{-2w}\right)^{\frac{n-s}{2}} \leq Ce^{(n-s)(t-w)}.$$

All these bounds clearly satisfy the conditions (6.1) of Lemma 6.1.

If $s \leq 1$, we need to be slightly more careful due to the unboundedness of g_{n-s} near 1. We give the argument for $s < 1$, the adaptation to $s = 1$ is straightforward. For $w \in (-\infty, -\ln(2)]$, we have $e^{w-t} \leq \frac{1}{2}$, so that $g_{n-s}(e^{w-t})$ is uniformly bounded and we can repeat the above estimate. For $w \in (-\ln(2), 0]$ however, we have the modified bound

$$g_{n-s}(e^{w-t})\left(\frac{1+e^{-2t}}{1+e^{-2w}}\right)^{\frac{n-s}{2}} \leq (1-e^{w-t})^{s-1}$$

by Lemma 6.3. But this error term still satisfies (6.1) because

$$\int_{-\ln(2)}^{0} (1-e^{w-t})^{(s-1)\frac{n}{n-s}} \leq C,$$

as a consequence of the inequality $(s-1)\frac{n}{n-s} > -1$. Similar arguments justify the validity (6.1) when $w \in (0,t]$ and when $w \in (t,\infty)$.

The term $v(t)$ can be treated by the same argument with s replaced by $s-\alpha$, after estimating $(1+e^{-2w})^{-\alpha} \leq 1$ and $e^{-\alpha w}(1+e^{-2w})^{-\alpha} \leq 1$ for $w \in \mathbb{R}$.

This proves (6.12). Hence the proof of Theorem 5.1, up to the sharpness assertion proved in Section 6.5 below, is complete. □

Remark 6.5. In the above argument, we only used the bounds on g_{n-s} given by Lemma 6.3, but not its monotonicity behavior. In fact, using the latter, our proof simplifies in

the case $s \geq 2$. Indeed, in these cases $g(R)$ is bounded and nonincreasing on $R \in [0,1]$ by Lemma 6.3. This means that we can replace the estimate $g(e^{s-t}) \leq 1 + Ce^{s-t}$ from above by the simpler $g(e^{s-t}) \leq g(0) = 1$. The point is that thanks to the absence of a t-dependent term like e^{s-t} in the error function $g(w,t)$ from Lemma 6.1, we can use directly Fontana's version of Adams' lemma instead of our improved one; see also the discussion before Lemma 6.1. It is hence only for small orders of derivative $s < 2$ that the new extension of Adams' lemma given in Lemma 6.1 is truly decisive. Compare also Remark 6.4.

6.5. **Sharpness of the constant.** We now complete the proof of Theorem 5.1 by proving the assertion on sharpness of the constant. By (5.2) and after projecting to $\mathbb{R}^n$ as in Section 6.2, let us thus suppose that $\gamma > 0$ is such that

$$\sup_{0 \leq f \in L^{\frac{n}{s}}_{\mathrm{rad}}(\mathbb{R}^n),\, \|f\|_{\frac{n}{s}} \leq 1} \int_{\mathbb{R}^n} \exp\left(\gamma T f(\eta)^{\frac{n}{n-s}}\right) |x|^\beta \exp(-|x|^\sigma)\, \mathrm{d}x \leq C < \infty, \tag{6.13}$$

where Tf satisfies

$$\begin{aligned} Tf(x) \geq & \int_{\mathbb{R}^n} |x-y|^{-n+s} \left(\frac{1+|x|^2}{1+|y|^2}\right)^{\frac{n-s}{2}} f(y)\, \mathrm{d}y \\ & - C \int_{\mathbb{R}^n} |x-y|^{-n+s+\alpha} \frac{(1+|x|^2)^{\frac{n-s-\alpha}{2}}}{(1+|y|^2)^{\frac{n-s+\alpha}{2}}} f(y)\, \mathrm{d}y \end{aligned} \tag{6.14}$$

for some $\alpha > 0$.

We shall show that necessarily

$$\gamma \leq \frac{n+\beta}{|\mathbb{S}^{n-1}|} \tag{6.15}$$

by following the argument of [2, proof of Theorem 2], where we replace the Riesz potential of f by Tf. Let $0 < r < R$ and let $f \in L^{\frac{n}{s}}(B_R)$ be any function such that $Tf \geq 1$ on B_r. Testing (6.13) with $f/\|f\|_{\frac{n}{s}}$ gives

$$C \geq \exp\left(\gamma \|f\|_{\frac{n}{s}}^{-\frac{n}{n-s}}\right) \int_{B_r} |x|^\beta = \exp\left(\gamma \|f\|_{\frac{n}{s}}^{-\frac{n}{n-s}}\right) |\mathbb{S}^{n-1}| \frac{r^{n+\beta}}{n+\beta},$$

that is,

$$\gamma \leq \left(\ln(C) + (\beta+n) \ln\left(\frac{R}{r}\right)\right) \|f\|_{\frac{n}{s}}^{\frac{n}{n-s}} \tag{6.16}$$

We now make a suitable choice of f. For given $r \in (0,R)$, define

$$f_{r,R}(x) := \begin{cases} |\mathbb{S}^{n-1}|^{-1} \left(\ln\left(\frac{R}{r}\right)\right)^{-1} |x|^{-s} & \text{for } x \in B_R \setminus B_r, \\ 0 & \text{otherwise .} \end{cases}$$

Let $\varepsilon > 0$ be given. In view of the expression of Tf in (6.14), by choosing $R = R(\varepsilon) > 0$ small enough we may assume that

$$\begin{aligned} Tf_{r,R}(x) &\geq (1-\varepsilon)\int_{B_R} |x-y|^{-n+s} f_{r,R}(y)\,\mathrm{d}y - C\int_{B_R} |x-y|^{-n+s+\alpha} f_{r,R}(y)\,\mathrm{d}y \\ &\geq (1-2\varepsilon)|\mathbb{S}^{n-1}|^{-1}\left(\ln\left(\frac{R}{r}\right)\right)^{-1}\int_{B_R\setminus B_r} |x-y|^{-n+s}|y|^{-s}\,\mathrm{d}y \end{aligned}$$

At fixed $\varepsilon > 0$ and $R = R(\varepsilon) > 0$, it follows from the argument in [2, p.392] (after rescaling by R) that there is $r_0(\varepsilon, R) > 0$ such that for every $r < r_0(\varepsilon, R)$, one has

$$\int_{B_R\setminus B_r} |x-y|^{-n+s}|y|^{-s}\,\mathrm{d}y \geq (1-\varepsilon)|\mathbb{S}^{n-1}|\ln\left(\frac{R}{r}\right).$$

Hence,

$$Tf_{r,R}(x) \geq (1-3\varepsilon) \qquad \text{for all } x \in B_r.$$

We conclude that, for $R = R(\varepsilon)$ and every $r < r_0(\varepsilon, R)$, the function $f := (1-3\varepsilon)^{-1} f_{r,R}$ is an admissible function satisfying $Tf \geq 1$ on B_r, and hence (6.16). Evaluating

$$\int_{B_R} f_r^{\frac{n}{s}} = |\mathbb{S}^{n-1}|^{-\frac{n-s}{s}}(\ln(R/r))^{-\frac{n-s}{s}},$$

and letting $r \to 0$, inequality (6.16) becomes

$$\gamma \leq (1-3\varepsilon)^{-\frac{n}{n-s}}\frac{n+\beta}{|\mathbb{S}^{n-1}|}.$$

Since $\varepsilon > 0$ was arbitrary, this completes the proof of (6.15). □

6.6. **Proof of Theorem 1.4.** The proof of Theorem 1.4 follows the same strategy as that of Theorem 5.1, Theorem 5.2 and Theorem 5.3. Since moreover it is simpler due to the presence of fewer error terms, we shall be brief. By [36, Propositions 7 and 8], we have the estimate

$$|u(x)| \leq K_{n,s}\int_{B_R} |x-y|^{-n+s} f(y)\,\mathrm{d}y =: K_{n,s}(I_{n-s} * f)(x),$$

where $f := (-\Delta)^{s/2}u|_{B_R}$. (Notice that unless s is an even integer, $(-\Delta)^{s/2}u$ is in general not supported in B_R.) Now we are in the situation of Section 6.4, only that Tf is replaced by the simpler $I_{n-s} * f$. Setting

$$\phi(s) = |\mathbb{S}^{n-1}|^{\frac{s}{n}} f(Re^{-w})(Re^{-w})^s,$$

and changing variables $|x| = Re^{-t}$, $|y| = Re^{-w}$, as above we find

$$\frac{1}{|\mathbb{S}^{n-1}|^{\frac{n-s}{n}}}(I_{n-s}*f)(Re^{-t}) = \int_0^t g_{n-s}(e^{w-t})\phi(w)\,\mathrm{d}w + \int_t^\infty g_{n-s}(e^{t-w})e^{(n-s)(t-w)}\phi(w)\,\mathrm{d}w,$$

which satisfies the assumptions of Lemma 6.1 by the same estimates as in the proof of Theorem 5.1. Hence

$$\int_{B_R} \exp\left(\frac{n+\beta}{|\mathbb{S}^{n-1}|} K_{n,s}^{-\frac{n}{n-s}} |u|^{\frac{n}{n-s}}\right) |x|^\beta \,\mathrm{d}x \leq \int_{B_R} \exp\left(\frac{n+\beta}{|\mathbb{S}^{n-1}|} |I_{n-s} * f(x)|^{\frac{n}{n-s}}\right) |x|^\beta \,\mathrm{d}x$$

$$= R^{n+\beta} |\mathbb{S}^{n-1}| \int_0^\infty \exp\left((n+\beta)\left(\left|\frac{1}{|\mathbb{S}^{n-1}|^{\frac{n-s}{n}}} (I_{n-s} * f)(Re^{-t})\right|^{\frac{n}{n-s}} - t\right)\right) \mathrm{d}t \leq C(R).$$

This completes the proof of Theorem 1.4.

Remark 6.6. It might be tempting to think that analogously to Theorem 1.3, a 'global' version of inequality (1.16) might hold on all of $\mathbb{R}^n$ in the presence of an exponentially decaying weight at infinity, i.e. that

$$\int_{\mathbb{R}^n} \exp\left(\gamma |u|^{\frac{n}{n-s}}\right) |x|^\beta \exp(-|x|^\sigma) \,\mathrm{d}x \leq C \tag{6.17}$$

uniformly in u radial with $\|(-\Delta)^{s/2} u\|_{L^{n/s}(\mathbb{R}^n)} \leq 1$. However, this is wrong for any choice of $\gamma, \sigma > 0$ and $\beta \in \mathbb{R}$. A counterexample is given by the family $u_R(x) := (-\Delta)^{-s/2} f_R(x) = K_{n,s} \int_{\mathbb{R}^n} |x-y|^{-n+s} f_R(y) \,\mathrm{d}y$, for the $L^{\frac{n}{s}}(\mathbb{R}^n)$-normalized functions $f_R(y) = (|\mathbb{S}^{n-1}| \ln(R))^{-\frac{s}{n}} 1_{B_R \setminus B_1}(y) |y|^{-s}$. Indeed, since $|x-y| \leq 2|y|$ whenever $|x| \leq 1 \leq |y|$, we have $u_R(x) \gtrsim (\ln(R))^{\frac{n-s}{n}}$ for every $x \in B_1$. Thus the left side of (6.17) with $u = u_R$ is unbounded as $R \to \infty$.

References

[1] S. Alinhac, P. Gérard, *Pseudo-differential operators and the Nash-Moser theorem.* Graduate Studies in Mathematics **82** (2007), viii+168.

[2] D. R. Adams, *A sharp inequality of J. Moser for higher order derivatives.* Ann. of Math. **128** (1988), 385–398

[3] A. Alvino, V. Ferone, G. Trombetti, *Moser-type inequalities in Lorentz spaces.* Potential Anal. **5** (1996), no. 3, 273–299.

[4] C. Bandle, *Isoperimetric inequalities and applications*, Monographs and Studies in Mathematics, **7**, Pitman (Advanced Publishing Program), Boston, Mass.-London, 1980.

[5] W. Beckner, *Sharp Sobolev Inequalities on the Sphere and the Moser–Trudinger Inequality* Ann. of Math. **138** (1993), no. 1, 213–242.

[6] L. Caffarelli, B. Gidas, J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth.* Comm. Pure Appl. Math. **42** (1989), no. 3, 271–297.

[7] A. Carlotto, A. Malchiodi, *Weighted barycentric sets and singular Liouville equations on compact surfaces.* J. Funct. Anal. **262** (2012), no. 2, 409–450.

[8] Chou, K. S. and Wan, Tom Yau-Heng, *Asymptotic radial symmetry for solutions of* $\Delta u + e^u = 0$ *in a punctured disc*,Pacific J. Math., **163**, (1994), no.2,269–276.

[9] S-Y. A. Chang, *Non-linear elliptic equations in conformal geometry.* Zurich Lectures in Advanced Mathematics (2004), viii+92.

[10] S.-Y. A. Chang, W. Chen, *A note on a class of higher order conformally covariant equations.* Discrete Contin. Dyn. Syst. **63** (2001) 275–281.

[11] W. Chen, C. Li, *Classification of solutions of some nonlinear elliptic equations.* Duke Math. J. **63** (1991), no. 3, 615–622.

[12] M. del Pino, J. Dolbeault, *The Euclidean Onofri inequality in higher dimensions.* Int. Math. Res. Not. IMRN 2013, no. 15, 3600–3611.

[13] J. Dolbeault, M. J. Esteban, G. Tarantello, *The role of Onofri type inequalities in the symmetry properties of extremals for Caffarelli-Kohn-Nirenberg inequalities, in two space dimensions.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **7** (2008), no. 2, 313–341.

[14] H. Fang, B. Ma, *Constant Q-curvature metrics on conic 4-manifolds.* Advances in Calculus of Variations, vol. , no. , 2020

[15] L. Fontana, *Sharp borderline Sobolev inequalities on compact Riemannian manifolds.* Comment. Math. Helv. **68** (1993), no. 3, 415–454.

[16] L. Fontana, C. Morpurgo, *Adams inequalities on measure spaces.* Adv. Math. **226** (2011), no. 6, 5066–5119.

[17] R. L. Frank, T. König, *Classification of positive singular solutions to a nonlinear biharmonic equation with critical exponent.* Anal. PDE **12** (2019), no. 4, 1101–1113.

[18] T. Futamura, K. Kishi, and Y. Mizuta, *A generalization of Bôcher's theorem for polyharmonic functions*, Hiroshima Math. J. **31** (2001), no. 1, 59–70.

[19] M. del Mar González, *Recent progress on the fractional Laplacian in conformal geometry.* Recent developments in nonlocal theory, 236–273, De Gruyter, Berlin, 2018.

[20] E. A. Gorin, *Asymptotic properties of polynomials and algrebraic functions of several variables.* Russ. Math. Surv. **16** (1961), 93–119.

[21] C.R. Graham, R. Jenne, L. Mason, and G. Sparling, *Conformally invariant powers of the Laplacian, I:existence.* J. Lond. Math. Soc. **46** (1992), 557–565.

[22] A. Hyder, *Structure of conformal metrics on Rn with constant Q-curvature*, Differential Integral Equations **32** (2019), no. 7-8, 423–454.

[23] A. Hyder, *Existence of entire solutions to a fractional Liouville equation in R^n*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **27** (2016), no. 1, 1–14.

[24] A. Hyder, *Conformally Euclidean metrics on R^n with arbitrary total Q-curvature.* Anal. PDE **10** (2017), no. 3, 635–652.

[25] A. Hyder, C.-S. Lin, J. Wei, *The $SU(3)$ Toda system with multiple singular sources.* Pacific Journal of Mathematics **305** (2020), 645–666

[26] A. Hyder, G. Mancini, L. Martinazzi, *Local and nonlocal singular Liouville equations in Euclidean spaces*, IMRN (2021), no. 15, 11393–11425

[27] A. Hyder, L. Martinazzi, *Conformal metrics on R^{2m} with constant Q-curvature, prescribed volume and asymptotic behavior*, Discrete Contin. Dyn. Syst. **35** (2015), no. 1, 283–299.

[28] T. Jin, A. Maalaoui, L. Martinazzi, J. Xiong, *Existence and asymptotics for solutions of a nonlocal Q-curvature equation in dimension three*, Calc. Var. Partial Differential Equations **52** (2015), no. 3-4, 469–488.

[29] N. Lam, G. Lu, *Sharp singular Adams inequalities in high order Sobolev spaces.* Methods Appl. Anal. **19** (2012), no. 3, 243–266.

[30] C.-S. Lin, *A classification of solutions of a conformally invariant fourth order equation in R^n.* Comment. Math. Helv. **73** (1998), no. 2, 206–231.

[31] F. Da Lio, L. Martinazzi, *The nonlocal Liouville-type equation in $\mathbb{R}$ and conformal immersions of the disk with boundary singularities.* Calc. Var. Partial Differ. Equ. **56** (2017), no. 5, paper no. 152.

[32] A. Maalaoui, *Prescribing the Q-curvature on the sphere with conical singularities.* Discrete Contin. Dyn. Syst. **36** (2016), no. 11, 6307–6330.

[33] A. Malchiodi,D. Ruiz, *New improved Moser-Trudinger inequalities and singular Liouville equations on compact surfaces.* Geom. Funct. Anal. **21** (2011), no. 5, 1196–1217.

[34] L. Martinazzi, *Classification of solutions to the higher order Liouville's equation on* $\mathbb{R}^{2m}$, Math. Z. **263** (2009), no. 2, 307–329.

[35] L. Martinazzi, *Conformal metrics on* $\mathbb{R}^{2m}$ *with constant* Q*-curvature and large volume.* Ann. Inst. H. Poincaré Anal. Non Linéaire **30** (2013), no. 6, 969–982.

[36] L. Martinazzi, *Fractional Adams-Moser-Trudinger type inequalities.* Nonlinear Anal. **127** (2015), 263–278.

[37] J. Moser, *A sharp form of an inequality by N. Trudinger.* Indiana Univ. Math. J. **20** (1970/71), 1077–1092.

[38] C. B. Ndiaye, Cheikh Birahim, *Constant* Q*-curvature metrics in arbitrary dimension.* J. Funct. Anal. **251** (2007), no. 1,1–58.

[39] R. O'Neil, *Convolution operators and* $L(p,q)$ *spaces.* Duke Math. J. **30** (1963), 129–142.

[40] J. M. do Ó, A. C. Macedo, J. F. de Oliveira, *A sharp Adams-type inequality for weighted Sobolev spaces.* Q. J. Math. **71** (2020), no. 2, 517–538.

[41] S. Paneitz, *A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds. Symmetry Integrability.* Geom. Methods Appl. 4, Paper 36 (2008), preprint (1983).

[42] J. Prajapat, G. Tarantello, *On a class of elliptic problems in* $\mathbb{R}^2$*: symmetry and uniqueness results.* Proc. Roy. Soc. Edinburgh Sect. A **131** (2001), no. 4, 967–985.

[43] R. M. Schoen, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics.* Topics in calculus of variations (Montecatini Terme, 1987), edited by M. Giaquinta, Lecture Notes in Math. 1365, Springer, 1989, 120–154.

[44] C. Tarsi, *Adams' inequality and limiting Sobolev embeddings into Zygmund spaces.* Potential Anal. **37** (2012), no. 4, 353–385.

[45] J. Wei, X. Xu, *Classification of solutions of higher order conformally invariant equations.* Math. Ann. **313** (1999), no. 2, 207–228.

[46] J. Wei, D. Ye, *Nonradial solutions for a conformally invariant fourth order equation in* $\mathbb{R}^4$. Calc. Var. **32** (2008), 373–386.

(Tobias König) Institut de Mathématiques de Jussieu, Université de Paris, Bâtiment Sophie Germain, Boite Courrier 7012, 8 Place Aurélie Nemours, 75205 Paris Cedex 13, France

Email address: koenig@imj-prg.fr

(Paul Laurain) Institut de Mathématiques de Jussieu, Université de Paris, Bâtiment Sophie Germain, Case 7052, 75205 Paris Cedex 13, France & DMA, Ecole normale supérieure, CNRS, PSL Research University, 75005 Paris.

Email address: paul.laurain@imj-prg.fr