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REVERSE CONFORMALLY INVARIANT SOBOLEV INEQUALITIES ON THE SPHERE

RUPERT L. FRANK, TOBIAS KÖNIG, AND HANLI TANG

Abstract. We consider the optimization problem corresponding to the sharp constant in a conformally invariant Sobolev inequality on the n-sphere involving an operator of order $2s > n$. In this case the Sobolev exponent is negative. Our results extend existing ones to noninteger values of s and settle the question of validity of a corresponding inequality in all dimensions $n \geq 2$.

1. Introduction and main results

We are interested in sharp constants in conformally invariant Sobolev inequalities. The classical version of this inequality concerns powers $(-\Delta)^s$ of the Laplacian in $\mathbb{R}^n$ with a real parameter $0 < s < \frac{n}{2}$ and it reads

$$\left\|(-\Delta)^{s/2}U\right\|_2^2 \geq \mathcal{S}_{s,n}\|U\|^2_{\frac{2n}{n-2s}} \qquad \text{for all } U \in \dot{H}^s(\mathbb{R}^n) \tag{1}$$

with

$$\mathcal{S}_{s,n} = (4\pi)^s \, \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(\frac{n-2s}{2})} \left(\frac{\Gamma(\frac{n}{2})}{\Gamma(n)}\right)^{2s/n} = \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(\frac{n-2s}{2})} \, |\mathbb{S}^n|^{2s/n} \,. \tag{2}$$

This inequality was proved in an equivalent, dual form by Lieb in [38], where also the cases of equality were characterized. Moreover, in that work a fundamental property of (1), namely its conformal invariance, was discovered and exploited. This result extends the earlier result in the local case $s = 1$ going back to [44, 45, 2, 47].

Since $\mathbb{R}^n$ (or rather $\mathbb{R}^n \cup \{\infty\}$) and $\mathbb{S}^n$ are conformally equivalent, there is an equivalent version of (1) on $\mathbb{S}^n$. This form was found explicitly by Beckner in [5, Eq. (19)], namely,

$$\left\|A_{2s}^{1/2}u\right\|_2^2 \geq \mathcal{S}_{s,n}\|u\|^2_{\frac{2n}{n-2s}} \qquad \text{for all } u \in H^s(\mathbb{S}^n) \tag{3}$$

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with the same constant $\mathcal{S}_{s,n}$ as in (2) and with

$$A_{2s} = \frac{\Gamma(B+\frac{1}{2}+s)}{\Gamma(B+\frac{1}{2}-s)} \qquad \text{and} \qquad B = \sqrt{-\Delta_{\mathbb{S}^n} + \tfrac{(n-1)^2}{4}}\,. \tag{4}$$

Note that the operators B and A_{2s} act diagonally in any basis of spherical harmonics, and on spherical harmonics of degree $\ell \in \mathbb{N}_0 = \{0,1,2,\dots\}$, the operator B acts by multiplication with $\ell + \frac{n-1}{2}$ and, consequently, A_{2s} acts by multiplication with

$$\alpha_{2s,n}(\ell) = \frac{\Gamma(\ell+\frac{n}{2}+s)}{\Gamma(\ell+\frac{n}{2}-s)}\,. \tag{5}$$

The operators A_{2s} can be thought of as $(-\Delta_{\mathbb{S}^n})^s$ perturbed by lower order terms. For integer s, they are related to the GJMS operators in conformal geometry [30].

Note that as $s \nearrow \frac{n}{2}$, the integrability exponent $\frac{2n}{n-2s}$ in (1) and (3) tends to $+\infty$. In [5] Beckner derived a conformally invariant endpoint inequality for $s = \frac{n}{2}$, which extends [36, 42, 43]; see also [10] for an equivalent, dual inequality. In passing, we mention that in [4] Beckner also proved a conformally invariant endpoint inequality for $s = 0$.

Our goal in this paper is to investigate the range

$$s > \frac{n}{2}\,.$$

Note that in this case the integrability exponent $\frac{2n}{n-2s}$ in (3) is negative, and therefore we will restrict ourselves to functions that are positive almost everywhere. It is because of this sign change that we call the inequalities in this paper 'reverse' Sobolev inequalities.

The operators A_{2s} are well-defined in the whole range $s > 0$, provided one sets $\alpha_{2s,n}(\ell) = 0$ when the denominator in (5) has a pole. Note, however, that the operators A_{2s} are no longer positive definite and therefore we define

$$a_{2s}[u] := \sum_{\ell\in\mathbb{N}_0} \alpha_{2s,n}(\ell)\|P_\ell u\|_2^2 \qquad \text{for all } u \in H^s(\mathbb{S}^n)\,,$$

where P_ℓ is the projection onto spherical harmonics of degree ℓ. Note that when $s \le \frac{n}{2}$, then $a_{2s}[u] = \|A_{2s}^{1/2}u\|_2^2$ for all $u \in H^s(\mathbb{S}^n)$.

In the following we will study inequalities of the type

$$a_{2s}[u] \ge \mathcal{S}_{s,n}\left(\int_{\mathbb{S}^n} u^{-\frac{2n}{2s-n}}\,d\omega\right)^{-\frac{2s-n}{n}} \qquad \text{for all } 0 < u \in H^s(\mathbb{S}^n)\,. \tag{6}$$

We are interested in whether such an inequality holds with some finite constant $\mathcal{S}_{s,n}$ (not necessarily positive) and, if so, what the optimal value of this constant is.

A first inequality of this type, corresponding to $s = 1$ in $n = 1$, is shown in [23] and reads

$$\int_{-\pi}^{\pi}\left((u')^2 - \frac{1}{4}u^2\right)d\theta \ge -\pi^2\left(\int_{-\pi}^{\pi} u^{-2}\,d\theta\right)^{-1} \qquad \text{for all } u \in H^1(\mathbb{R}/2\pi\mathbb{Z})\,.$$

An independent proof of this inequality and a characterization of the cases of equality appears in [1]. The case $s = 2$ in $n = 3$ is analyzed in [48]; see also [33]. The paper [32] by Hang treats all cases $s \in \mathbb{N} \cap (\frac{n}{2}, \infty)$ in general dimensions $n \geq 1$ (here, $\mathbb{N} = \{1, 2, 3, \ldots\}$); for the cases $s = 1, 2$ in $n = 1$, see also [41]. All these cases treated so far correspond to integer s, when A_{2s} is a differential operator.

In the above mentioned works it was established that inequality (6) is valid, with the constant given by (2), when restricted to positive functions, provided that $s = \frac{n+1}{2}$, $\frac{n+3}{2}$ when n is odd and $s = \frac{n}{2} + \mathbb{N}_0$ when n is even. For odd n equality is achieved exactly for the constant function, modulo conformal transformations, and for even n exactly for positive linear combinations of spherical harmonics of degree $\leq s - \frac{n}{2}$. Moreover, a rather surprising result in [32] is that for odd n and $s \in \frac{n+5}{2} + \mathbb{N}_0$, the infimum

$$I_{2s,n} := \inf_{0<u\in H^s(\mathbb{S}^n)} \left(\int_{\mathbb{S}^n} u^{-\frac{2n}{2s-n}} d\omega \right)^{\frac{2s-n}{n}} a_{2s}[u] \tag{7}$$

is *not* achieved and, in fact, there is not even a local minimum. As far as we know, this is one of the very few instances of conformally invariant functional inequalities on $\mathbb{S}^n$ without minimizers.

While the fundamental works [23, 1, 48, 33, 32] answer many questions concerning the family of inequalities (6), two natural ones remain open. (a) Do these results extend to all real values of the parameter $s > \frac{n}{2}$ and, if so, where does the transition between existence and nonexistence of a minimizer occur? (b) If there is no minimizer for (7), what is the value of the infimum?

In this paper we completely answer question (a) and, in dimension $n \geq 2$, also question (b).

The following two theorems are our main results.

Theorem 1. *Let $n \geq 1$ and $s \in (\frac{n}{2}, \frac{n+4}{2}) \cup (\frac{n}{2} + \mathbb{N})$. Then for all $0 \leq u \in H^s(\mathbb{S}^n)$ with $u^{-\frac{2n}{2s-n}} \in L^1(\mathbb{S}^n)$,*

$$a_{2s}[u] \left(\int_{\mathbb{S}^n} u^{-\frac{2n}{2s-n}} d\omega \right)^{\frac{2s-n}{n}} \geq \frac{\Gamma(\frac{n}{2} + s)}{\Gamma(\frac{n}{2} - s)} \, |\mathbb{S}^n|^{\frac{2s}{n}} . \tag{8}$$

For $s \in (\frac{n}{2}, \frac{n+4}{2}) \setminus \{\frac{n+2}{2}\}$, equality is attained if and only if

$$u(\omega) = c\,(1 - \zeta \cdot \omega)^{\frac{2s-n}{2}}$$

for some $c > 0$ and $\zeta \in \mathbb{R}^{n+1}$ with $|\zeta| < 1$. For $s \in \frac{n}{2} + \mathbb{N}$, equality is attained if and only if u is in the linear span of spherical harmonics of degree $\leq s - \frac{n}{2}$.

Note that the constant on the right side of (8) coincides with $\mathcal{S}_{s,n}$ in (2). It is negative for $s \in (\frac{n}{2}, \frac{n+2}{2})$, positive for $s \in (\frac{n+2}{2}, \frac{n+4}{2})$ and zero for $s \in \frac{n}{2} + \mathbb{N}$.

Theorem 2. *Let $n \geq 1$ and $s \in (\frac{n+4}{2}, \infty) \setminus (\frac{n}{2} + \mathbb{N})$. Then the infimum (7) is not attained. If, in addition, $n \geq 2$, then $I_{2s,n} = -\infty$.*

Theorem 1 for $s \in \frac{n}{2} + \mathbb{N}$ is almost immediate from the definition of a_{2s}. In order to prove the theorem for $s \in (\frac{n}{2}, \frac{n+4}{2}) \setminus \{\frac{n+2}{2}\}$ we follow closely the strategy of Hang [32]. Namely, first we prove existence of a minimizer and then we apply a result of Li [37] characterizing all solutions to the corresponding Euler–Lagrange equation. In the proof of existence of minimizers one has to deal with the noncompact symmetry group of conformal transformations. To rule out loss of compactness modulo symmetries, an important role is played by the fact that $a_{2s}[u] \geq 0$ if u vanishes at a point (together with its gradient if $s \in (\frac{n+2}{2}, \frac{n+4}{2})$). Similar results already appeared in [23, 48, 33, 32], where the authors dealt with local operators and could exploit integration by parts. In Proposition 5 we prove the corresponding fact for general $s \in (\frac{n}{2}, \frac{n+4}{2})$. We also proceed by going to $\mathbb{R}^n$, but the proof for noninteger s is quite a bit more involved.

The first part of Theorem 2 follows again closely the strategy of Hang [32] and also uses the result of Li [37]. The second part answers a question that was left open in [32] even in the integer case. The idea is to find a function $u \in H^s(\mathbb{S}^n)$ such that $u^{-\frac{2n}{2s-n}} \notin L^1(\mathbb{S}^n)$ and $a_{2s}[u] < 0$. Then using $u + \varepsilon$ as trial functions for the infimum (7) yields the assertion in Theorem 2. The function u that we choose vanishes to sufficiently high order on the equator $\{\omega_{n+1} = 1\}$. Showing that the quadratic form is negative on this function, requires some rather explicit analysis involving spherical harmonics. The seemingly simpler question of whether $I_{2s,1}$ is finite or not for $n = 1$, remains open.

Background and open problems. We end this introduction by putting our results into perspective and by mentioning some open problems.

The work of Dou and Zhu [22] spiked a lot of interest in reversed Hardy–Littlewood–Sobolev (HLS) inequalities. The conformally invariant case of these inequalities states that for $\mu > 0$ and nonnegative functions F, G on $\mathbb{R}^n$,

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} F(x)|x-y|^\mu G(y)\, dx\, dy \geq \mathcal{H}_{\mu,n} \|F\|_{\frac{2n}{2n+\mu}} \|G\|_{\frac{2n}{2n+\mu}} \,. \tag{9}$$

The optimal constant $\mathcal{H}_{\mu,n} > 0$ and all optimizing functions F, G were obtained in [22]; see also [40, 16]. By conformal invariance, (9) has an equivalent version on $\mathbb{S}^n$, namely,

$$\iint_{\mathbb{S}^n \times \mathbb{S}^n} f(\omega)|\omega-\omega'|^\mu g(\omega')\, d\omega\, d\omega' \geq \mathcal{H}_{\mu,n} \|f\|_{\frac{2n}{2n+\mu}} \|g\|_{\frac{2n}{2n+\mu}} \,. \tag{10}$$

For open questions in a non-conformally invariant case motivated by aggregation-diffusion equations, see [11, 12].

While the (usual) HLS inequality studied in [38] is equivalent to the Sobolev inequality (1), there seems to be no relation between (10) and the inequality (6). This is despite the fact that the integral kernel $|\omega - \omega'|^\mu$ appearing in (10) is a multiple of the Green's function of the operator A_{2s} with $\mu = 2s - n$; see the proof of Lemma 3. The fundamental difference between the (usual) HLS inequality and the reverse one is that the kernel is positive definite in the former case, but not in the latter. For inequalities

(9) and (10) optimizers exist for *all* $\mu > 0$ and one does not have an analogue of the nonexistence phenomenon in our Theorem 2.

As we mentioned before, in our proof of Theorem 1 for $s \in (\frac{n}{2}, \frac{n+4}{2}) \setminus \{\frac{n+2}{2}\}$ we apply a result of Li [37] and to do so, we use a relation between the Euler–Lagrange equations corresponding to (10) and (6). Interestingly, the analogue of this relation on Euclidean space may fail in the excluded case $s = \frac{n+2}{2}$; for an example with $n = 2$ and $s = 2$, see [49].

Besides finding optimal constants and characterizing optimizers, a natural problem is to characterize all positive solutions of the corresponding Euler–Lagrange equation. For the Sobolev inequality (1) or, equivalently, for the corresponding HLS inequality, this was accomplished in [18]; see also [37]. The latter paper also contains a characterization of solutions to the Euler–Lagrange equation corresponding to (9) and, in fact, just as in [32] this will be a major ingredient in our proof of Theorem 1. For related classification results, see [19, 35, 21, 26] and references therein. In connection with this we emphasize that our Theorem 2 does not exclude that for $s \in (\frac{5}{2}, \infty) \setminus (\frac{1}{2} + \mathbb{N})$ in $n = 1$, the infimum in (7) is attained for $u \in H^s(\mathbb{S})$ with $u^{-\frac{2}{2s-1}} \in L^1(\mathbb{S})$ and $\min u = 0$. We find it unlikely that such u exist, but we cannot exclude their existence via [37] since the Euler–Lagrange equation then only holds in $\mathbb{S} \setminus \{u = 0\}$.

After the works of Brezis and Lieb [8] and Bianchi and Egnell [6] and, in particular, in the last decade there has been an immense body of work concerning the quantitative stability of Sobolev and isoperimetric inequalities; see, e.g., [29, 17, 9, 28, 24, 7, 25] and references therein. It is natural to ask whether there are such stable versions of Theorem 1. The computations with the linearization in the proof of Theorem 2 suggest that the answer is affirmative for $s \in (\frac{n}{2}, \frac{n+4}{2}) \setminus \{\frac{n+2}{2}\}$, but the precise form of such a purported inequality is unclear since the form $a_{2s}[u]$ is not positive semidefinite.

Finally, we would like to mention the relation between the problem studied in this paper and conformal geometry. The sharp constant in the Sobolev inequality (1) with $s = 1$ appears as a compactness threshold in the Yamabe problem on general manifolds [2]. The latter concerns the scalar curvature. Similarly, the case $s = 2$ is related to the Q-curvature [34] and generalized Q curvatures were introduced in [31] for $0 < s < \frac{n}{2}$; see also [14, 13, 15]. While (generalized) Q-curvature problems were originally considered for $s \leq \frac{n}{2}$, they are also meaningful for $s > \frac{n}{2}$ and, in fact, this was the original motivation for [48, 33]. Our Theorem 1 says that for $s \in (\frac{n}{2}, \frac{n+4}{2}) \cup (\frac{n}{2} + \mathbb{N})$, within the conformal class of the standard metric $g_{\mathbb{S}^n}$ on $\mathbb{S}^n$ and under the volume constraint $\mathrm{vol}_g(\mathbb{S}^n) = |\mathbb{S}^n|$, the standard metric maximizes the total generalized Q-curvature, defined by

$$Q_{2s,g} = -\frac{2}{2s-n}\, u^{\frac{2s+n}{2s-n}} A_{2s} u \qquad \text{if } g = u^{-\frac{4}{2s-n}} g_{\mathbb{S}^n}\,.$$

Our Theorem 1 plays the same role for the fractional order problems in [14, 13, 15] as the results in [48, 33] do in the Q-curvature problem on three-dimensional manifolds.

2. Preliminaries

2.1. **Conformal invariance.** In this subsection $n \geq 1$ and $s > \frac{n}{2}$ are fixed. Let Φ be a conformal transformation of $\mathbb{S}^n$ and, for a function u on $\mathbb{S}^n$, set

$$u_\Phi(\omega) = J_\Phi(\omega)^{-\frac{2s-n}{2n}} u(\Phi(\omega)) \,.$$

Clearly, if u is nonnegative and measurable, then

$$\int_{\mathbb{S}^n} u_\Phi^{-\frac{2n}{2s-n}} \, d\omega = \int_{\mathbb{S}^n} u^{-\frac{2n}{2s-n}} \, d\omega \,.$$

Lemma 3. *If $u \in H^s(\mathbb{S}^n)$, then $u_\Phi \in H^s(\mathbb{S}^n)$ and*

$$a_{2s}[u_\Phi] = a_{2s}[u] \,.$$

Proof. We prove the lemma under the assumption $s \notin \frac{n}{2} + \mathbb{N}$, which implies the general result by a limiting argument. This assumption implies that $\alpha_{2s,n}(\ell) \neq 0$ for all $\ell \in \mathbb{N}_0$. Moreover, by Stirling's formula, $\alpha_{2s,n}(\ell)$ grows like ℓ^{2s}. Thus, $A_{2s} = \Gamma(B + \frac{1}{2} + s)/\Gamma(B + \frac{1}{2} - s)$ is invertible as an operator from $H^{-s}(\mathbb{S}^n)$ to $H^s(\mathbb{S}^n)$. The Funk–Hecke formula implies that if Y is a spherical harmonic of degree $\ell \in \mathbb{N}_0$, then

$$\int_{\mathbb{S}^n} |\omega - \omega'|^{2s-n} Y(\omega') \, d\omega' = \frac{2^{2s} \pi^{\frac{n}{2}} \, \Gamma(s)}{\Gamma(\frac{n}{2} - s)} \frac{1}{\alpha_{2s,n}(\ell)} Y(\omega) \,;$$

see [5, Eq. 17] and also [27, Cor. 4.3]. Consequently, A_{2s}^{-1} is an integral operator with integral kernel

$$\frac{\Gamma(\frac{n}{2} - s)}{2^{2s} \pi^{\frac{n}{2}} \, \Gamma(s)} \, |\omega - \omega'|^{2s-n} \,.$$

Using this formula, together with the fact that

$$J_\Phi(\omega)^{\frac{1}{n}} |\omega - \omega'|^2 J_\Phi(\omega')^{\frac{1}{n}} = |\Phi(\omega) - \Phi(\omega')|^2 \,,$$

we easily see that for any $v \in H^{-s}(\mathbb{S}^n)$

$$A_{2s}^{-1} v^\Phi = (A_{2s}^{-1} v)_\Phi \,,$$

where we set

$$v^\Phi(\omega) := J_\Phi(\omega)^{\frac{2s+n}{2n}} v(\Phi(\omega)) \,.$$

This is equivalent to

$$A_{2s} u_\Phi = (A_{2s} u)^\Phi \,.$$

Multiplying this formula by u_Φ and integrating we obtain the claim. $\square$

2.2. **Stereographic projection.** In the previous subsection we considered the behavior of A_{2s} under a conformal transformation of $\mathbb{S}^n$. In this subsection we consider its behavior under stereographic projection. Throughout this subsection we fix $n \geq 1$ and $s \in (0,\infty) \setminus (\frac{n}{2} + \mathbb{N}_0)$.

We introduce the (inverse) stereographic projection $\mathcal{S} : \mathbb{R}^n \to \mathbb{S}^n$ by

$$\mathcal{S}_j(x) = \frac{2x_j}{1+|x|^2}\,,\ j = 1,\dots,n\,, \qquad \mathcal{S}_{n+1}(x) = \frac{1-|x|^2}{1+|x|^2}\,.$$

Given a function u on $\mathbb{S}^n$, we define two functions $u_{\mathcal{S}}$ and $u^{\mathcal{S}}$ on $\mathbb{R}^n$ by

$$u_{\mathcal{S}}(x) = \left(\frac{1+|x|^2}{2}\right)^{\frac{2s-n}{2}} u(\mathcal{S}(x))\,, \qquad u^{\mathcal{S}}(x) = \left(\frac{1+|x|^2}{2}\right)^{-\frac{2s+n}{2}} u(\mathcal{S}(x))\,. \tag{11}$$

Note that, since $(2/(1+|x|^2))^n$ is the Jacobian of $\mathcal{S}$, these formulas are similar to those appearing in Lemma 3 and its proof.

Lemma 4. *Let $s = N + \sigma$ with $N \in \mathbb{N}_0$ and $\sigma \in [0,1)$. If $n \geq 2$, then*

$$(-\Delta)^{-\sigma}\,(A_{2s}u)^{\mathcal{S}} = (-\Delta)^N u_{\mathcal{S}} \qquad \text{for all } u \in C^\infty(\mathbb{S}^n)\,, \tag{12}$$

The same identity holds if $n = 1$ and $\sigma \in [0,\frac{1}{2})$. If $n = 1$ and $\sigma \in (\frac{1}{2},1)$, then

$$(-\tfrac{d^2}{dx^2})^{-\sigma+\frac{1}{2}} H\,(A_{2s}u)^{\mathcal{S}} = \tfrac{d^{2N+1}}{dx^{2N+1}} u_{\mathcal{S}} \qquad \text{for all } u \in C^\infty(\mathbb{S})\,, \tag{13}$$

where H is multiplication in Fourier space by $i\xi/|\xi|$.

The proof will show that both sides of (12) and (13) are continuous functions and that the identities hold pointwise.

We note that (12) and (13) are precise versions of the 'heuristic formula'

$$(A_{2s}u)^{\mathcal{S}} = (-\Delta)^s u_{\mathcal{S}}\,, \tag{14}$$

which is analogous to the formula in the proof of Lemma 3. When trying to directly prove (14), we ran into technical problems concerning the convolution of two tempered distributions. This can be circumvented by proving the less elegant formulas (12) and (13), which are just as good for our purposes.

Proof. Step 1. As a preparation we prove the following assertion, still assuming $s \in (0,\infty) \setminus (\frac{n}{2} + \mathbb{N}_0)$. If $n \geq 2$, then, for any measurable f on $\mathbb{R}^n$ such that $|f(x)| \lesssim \langle x\rangle^{-2s-n}$,

$$(-\Delta)^N \int_{\mathbb{R}^n} |x-x'|^{2s-n} f(x')\,dx' = \frac{2^{2s}\pi^{\frac{n}{2}}\Gamma(s)}{\Gamma(\frac{n}{2}-s)}\,((-\Delta)^{-\sigma}f)(x)\,.$$

For $n = 1$ and $\sigma \in [0,\frac{1}{2})$, the same assertion is true, while for $\sigma \in (\frac{1}{2},1)$ one has

$$\frac{d^{2N+1}}{dx^{2N+1}} \int_{\mathbb{R}} |x-x'|^{2s-1} f(x')\,dx' = \frac{2^{2s}\pi^{\frac{1}{2}}\Gamma(s)}{\Gamma(\frac{1}{2}-s)}\,\left((-\tfrac{d^2}{dx^2})^{-\sigma+\frac{1}{2}} Hf\right)(x)\,.$$

We prove this by induction on N. For $N=0$ and, if $n=1$, $s<\frac12$, this is a standard result; see, e.g., [39, Theorem 5.9 and Corollary 5.10]. For $n=1$ and $\frac12<s<1$, using dominated convergence one easily sees that $x\mapsto\int_{\mathbb{R}}|x-x'|^{2s-1}f(x')\,dx'$ is C^1 and

$$\frac{d}{dx}\int_{\mathbb{R}}|x-x'|^{2s-1}f(x')\,dx'=(2s-1)\int_{\mathbb{R}}|x-x'|^{2s-3}(x-x')f(x')\,dx'\,.$$

Note that the integral kernel on the right side is locally integrable. The claimed identity then follows from the identity

$$(2s-1)|x|^{2s-3}x=\frac{2^{2s}\pi^{\frac12}\Gamma(s)}{\Gamma(\frac12-s)}\int_{\mathbb{R}}|\xi|^{-2s}i\xi e^{i\xi x}\,d\xi\,,$$

where the right side exists as an improper Riemann integral. This identity can either be proved directly by moving the integration contour to the positive imaginary axis and using identities for the Gamma function, or by analytic continuation from the identity implicit in the above proof for $0<s<\frac12$.

Now let us assume $N\geq 1$. Using dominated convergence, one easily verifies that $x\mapsto\int_{\mathbb{R}^n}|x-x'|^{2s-n}f(x')\,dx'$ is C^2 and that

$$\Delta\int_{\mathbb{R}^n}|x-x'|^{2s-n}f(x')\,dx'=4(s-\tfrac{n}{2})(s-1)\int_{\mathbb{R}^n}|x-x'|^{2s-n-2}f(x')\,dx'\,.$$

By induction, one concludes that, if either $n\geq 2$ or if $n=1$ and $\sigma<\frac12$,

$$\begin{aligned}(-\Delta)^N\int_{\mathbb{R}^n}|x-x'|^{2s-n}f(x')\,dx'&=-4(s-\tfrac{n}{2})(s-1)(-\Delta)^{N-1}\int_{\mathbb{R}^n}|x-x'|^{2s-n-2}f(x')\,dx'\\&=-4(s-\tfrac{n}{2})(s-1)\frac{2^{2(s-1)}\pi^{\frac{n}{2}}\Gamma(s-1)}{\Gamma(\frac{n}{2}-s+1)}\left((-\Delta)^{-\sigma}f\right)(x)\\&=\frac{2^{2s}\pi^{\frac{n}{2}}\Gamma(s)}{\Gamma(\frac{n}{2}-s)}\left((-\Delta)^{-\sigma}f\right)(x)\,.\end{aligned}$$

The proof for $n=1$ and $\frac12<\sigma<1$ is similar. This proves the claimed formula.

Step 2. It remains to prove (12) and (13). Let $u\in C^\infty(\mathbb{S}^n)$. Then $A_{2s}u\in C^\infty(\mathbb{S}^n)$ (indeed, for any $\sigma>0$, $u\in H^\sigma(\mathbb{S}^n)$, so $A_{2s}u\in H^{\sigma-2s}(\mathbb{S}^n)$) and, using the explicit integral kernel of A_{2s}^{-1} from the proof of Lemma 3,

$$u(\omega)=\frac{\Gamma(\frac{n}{2}-s)}{2^{2s}\pi^{\frac{n}{2}}\Gamma(s)}\int_{\mathbb{S}^n}|\omega-\omega'|^{2s-n}(A_{2s}u)(\omega')\,d\omega'\,.$$

Thus, using

$$\frac{1+|x|^2}{2}\,|\mathcal{S}(x)-\mathcal{S}(x')|^2\,\frac{1+|x'|^2}{2}=|x-x'|^2$$

and changing variables, we obtain

$$u_{\mathcal{S}}(x)=\frac{\Gamma(\frac{n}{2}-s)}{2^{2s}\pi^{\frac{n}{2}}\Gamma(s)}\int_{\mathbb{R}^n}|x-x'|^{2s-n}(A_{2s}u)^{\mathcal{S}}(x')\,dx'\,.$$

Note that $|(A_{2s}u)^{\mathcal{S}}(x)| \le \|A_{2s}u\|_\infty (2/(1+|x|^2))^{(2s+n)/2}$, so, in particular, the integral on the right side converges absolutely. Now the result from Step 1 is applicable and we obtain (12) and (13). This concludes the proof of the proposition. □

2.3. **Positivity under a vanishing condition.** A crucial role in our proof of existence of a minimizer is played by the following

Proposition 5. *Let $n \ge 1$ and $s \in (\frac{n}{2}, \frac{n+4}{2})$. For all $u \in H^s(\mathbb{S}^n)$ with $u(S) = 0$ and, if $s > \frac{n+2}{2}$, $\nabla u(S) = 0$, one has*

$$a_{2s}[u] \ge 0\,.$$

Moreover, if $s \in (\frac{n}{2}, \frac{n+2}{2})$, then equality holds if and only if for some $c \in \mathbb{R}$,

$$u(\omega) = c\,(1+\omega_{n+1})^{\frac{2s-n}{2}} \qquad \text{for all } \omega \in \mathbb{S}^n\,.$$

and if $s \in [\frac{n+2}{2}, \frac{n+4}{2})$, then equality holds if and only if for some $c \in \mathbb{R}$, $b \in \mathbb{R}^n$,

$$u(\omega) = c\,(1+\omega_{n+1})^{\frac{2s-n}{2}} + (1+\omega_{n+1})^{\frac{2s-n-2}{2}} b\cdot\omega' \qquad \text{for all } \omega = (\omega', \omega_{n+1}) \in \mathbb{S}^n\,.$$

Here $S = (0,\dots,0,-1)$ denotes the south pole.

Proof. If $s = \frac{n+2}{2}$, then $\alpha_{2s,n}(\ell) \ge 0$ for all $\ell \in \mathbb{N}_0$ with equality if and only if $\ell \le 1$. This proves immediately the claimed inequality as well as the characterization of the cases of equality. Thus, in the following we assume that $s \ne \frac{n+2}{2}$.

First, let $u \in C_c^\infty(\mathbb{S}^n \setminus \{S\})$, so that $u_{\mathcal{S}} \in C_c^\infty(\mathbb{R}^n)$. If $n \ge 2$, or if $n = 1$ and $s \in [1, \frac{3}{2}) \cup [2, \frac{5}{2})$, we multiply (12) by $(-\Delta)^\sigma u_{\mathcal{S}}$ and integrate to obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{u_{\mathcal{S}}}|^2\, d\xi &= \int_{\mathbb{R}^n} ((-\Delta)^\sigma u_{\mathcal{S}})((-\Delta)^N u_{\mathcal{S}})\, dx = \int_{\mathbb{R}^n} u_{\mathcal{S}} (A_{2s}u)^{\mathcal{S}}\, dx = \int_{\mathbb{S}^n} u A_{2s} u\, d\omega \\ &= a_{2s}[u]\,, \end{aligned} \tag{15}$$

where the Fourier transform is defined by

$$\widehat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\, dx\,.$$

For $n = 1$ and $s \in (\frac{1}{2}, 1) \cup (\frac{3}{2}, 2)$ we multiply (13) by $(-\frac{d^2}{dx^2})^{\sigma - \frac{1}{2}} \overline{H u_{\mathcal{S}}}$ and obtain the same identity (15). Since the left side in (15) is nonnegative, we obtain the inequality in the first part of the proposition for $u \in C_c^\infty(\mathbb{S}^n \setminus \{S\})$.

We abbreviate

$$\mathcal{Q} := \begin{cases} \{u \in H^s(\mathbb{S}^n) :\ u(S) = 0\} & \text{if } s \in (\frac{n}{2}, \frac{n+2}{2})\,, \\ \{u \in H^s(\mathbb{S}^n) :\ u(S) = 0,\ \nabla u(S) = 0\} & \text{if } s \in (\frac{n+2}{2}, \frac{n+4}{2})\,. \end{cases}$$

Our goal is to extend the identity (15) to $u \in \mathcal{Q}$ and use this extension to characterize the cases of equality. It is well-known that the set $C_c^\infty(\mathbb{S}^n \setminus \{S\})$ is dense in $\mathcal{Q}$ with respect to the norm in $H^s(\mathbb{S}^n)$. Moreover, a_{2s} is continuous with respect to the norm in $H^s(\mathbb{S}^n)$. This immediately implies that $a_{2s}[u] \ge 0$ for all $u \in \mathcal{Q}$.

Let $u \in \mathcal{Q}$ and let $(u_j) \subset C_c^\infty(\mathbb{S}^n \setminus \{0\})$ be a sequence that converges to $u \in \mathcal{Q}$ in $H^s(\mathbb{S}^n)$. In particular, (u_j) converges to u in $L^2(\mathbb{S}^n)$ and, by a change of variables,

$((u_j)_{\mathcal{S}})$ converges to $u_{\mathcal{S}}$ in $L^2(\mathbb{R}^n, (2/(1+|x|^2))^{2s}\,dx)$. In particular, $((u_j)_{\mathcal{S}})$ converges to $u_{\mathcal{S}}$ in the sense of tempered distributions, and therefore $(\widehat{(u_j)_{\mathcal{S}}})$ converges to $\widehat{u_{\mathcal{S}}}$ in the sense of tempered distributions. On the other hand, the fact that (u_j) is a Cauchy sequence in $H^s(\mathbb{S}^n)$, the identity (15) and the H^s-continuity of a_{2s} imply that $(\widehat{(u_j)_{\mathcal{S}}})$ is a Cauchy sequence in $L^2(\mathbb{R}^n, |\xi|^{2s}\,d\xi)$ and therefore convergent. A standard argument (namely, interlacing two Cauchy sequences) shows that the limit is independent of the approximating sequence. We deduce from this that the restriction of the distribution $\widehat{u_{\mathcal{S}}}$ to $\mathbb{R}^n \setminus \{0\}$ coincides with a function and that this function belongs to $L^2(\mathbb{R}^n, |\xi|^{2s}\,d\xi)$. Moreover, identity (15) remains valid for $u \in \mathcal{Q}$, provided the integral on the left side is restricted to $\mathbb{R}^n \setminus \{0\}$ and $\widehat{u_{\mathcal{S}}}$ on the left side is interpreted as the restriction of the corresponding distribution to this set.

In particular, if $a_{2s}[u] = 0$ for some $u \in \mathcal{Q}$, then the distribution $\widehat{u_{\mathcal{S}}}$ vanishes on $\mathbb{R}^n \setminus \{0\}$ and therefore, by a well-known theorem about distributions, $\widehat{u_{\mathcal{S}}}$ coincides with a finite sum of derivatives of a Dirac delta distribution at the origin. Thus, $u_{\mathcal{S}}$ is a polynomial.

By Morrey's inequality we have $u \in C^{s-\frac{n}{2}}(\mathbb{S}^n)$ if $s \in (\frac{n}{2}, \frac{n+2}{2})$ and $u \in C^{1,s-\frac{n+2}{2}}(\mathbb{S}^n)$ if $s \in (\frac{n+2}{2}, \frac{n+4}{2})$ and therefore in either case, the vanishing conditions imply that

$$|u(\omega)| \lesssim |\omega - S|^{s-\frac{n}{2}} \qquad \text{for all } \omega \in \mathbb{S}^n\,,$$

that is,

$$|u_{\mathcal{S}}(x)| \lesssim \left(\frac{1+|x|^2}{2}\right)^{s-\frac{n}{2}} |\mathcal{S}(x) - S|^{s-\frac{n}{2}} = \left(1+|x|^2\right)^{\frac{1}{2}(s-\frac{n}{2})} \qquad \text{for all } x \in \mathbb{R}^n\,. \tag{16}$$

If $s < \frac{n+2}{2}$, then the right side grows sublinearly and therefore $u_{\mathcal{S}}$, being a polynomial, is equal to a constant c. Now $u_{\mathcal{S}}(x) = c$ is equivalent to $u(\omega) = c(1+\omega_{n+1})^{(2s-n)/2}$, as claimed. If $s \in (\frac{n+2}{2}, \frac{n+4}{2})$, then the right side in (16) grows subquadratically and therefore $u_{\mathcal{S}}$ is affine linear, that is, $u_{\mathcal{S}}(x) = c + b \cdot x$. This is equivalent to the form given in the proposition. □

3. Proof of Theorem 1

In this section, we prove our first main theorem, whose nontrivial part says that for $s \in (\frac{n}{2}, \frac{n+4}{2}) \setminus \{\frac{n+2}{2}\}$ the infimum $I_{2s,n}$ in (7) is achieved precisely by the constant function and its images under the group of conformal transformations. Similarly as in [32] we proceed in two steps, namely first showing that the infimum is achieved and then characterizing the functions where the infimum is achieved.

3.1. **Existence of a minimizer.**

Proposition 6. *Let $n \geq 1$ and $s \in (\frac{n}{2}, \frac{n+4}{2}) \setminus \{\frac{n+2}{2}\}$. Let $(u_j) \subset H^s(\mathbb{S}^n)$ be a sequence of nonnegative functions with $u_j^{-\frac{2n}{2s-n}} \in L^1(\mathbb{S}^n)$ and*

$$\lim_{j\to 0} a_{2s}[u_j] \left(\int_{\mathbb{S}^n} u_j^{-\frac{2n}{2s-n}}\,d\omega\right)^{\frac{2s-n}{n}} = I_{2s,n}\,.$$

Then there is a sequence (Φ_j) of conformal transformations of $\mathbb{S}^n$ and a sequence $(c_j) \subset \mathbb{R}_+$ such that, after passing to a subsequence, the functions $c_j(u_j)_{\Phi_j}$ converge in $H^s(\mathbb{S}^n)$ to an everywhere positive function that minimizes $I_{2s,n}$.

Proof. Step 1. After multiplying u_j by a positive constant and after a rotation (which can be implemented as a conformal transformation), we may assume that for all j,

$$\max_{\mathbb{S}^n} u_j = 1 \qquad \text{and} \qquad u_j(S) = \min_{\mathbb{S}^n} u_j \,.$$

Here $S = (0, \dots, 0, -1)$ denotes the south pole and later $N = (0, \dots, 0, 1)$ will denote the north pole

Let us show that (u_j) is bounded in $H^s(\mathbb{S}^n)$. By the minimizing property, there is a $C \geq 0$ such that for all j,

$$a_{2s}[u_j] \left(\int_{\mathbb{S}^n} u_j^{-\frac{2n}{2s-n}} \, d\omega \right)^{\frac{2s-n}{n}} \leq C \,. \tag{17}$$

Thus, by our normalization,

$$a_{2s}[u_j] \leq C \left(\int_{\mathbb{S}^n} u_j^{-\frac{2n}{2s-n}} \, d\omega \right)^{-\frac{2s-n}{n}} \leq C|\mathbb{S}^n|^{-\frac{2s-n}{n}} \,.$$

On the other hand, by Stirling's formula, $\alpha_{2s,n}(\ell)$ grows like ℓ^{2s}. Since $\alpha_{2s,n}(\ell) \geq 0$ for all $\ell \geq 1$ if $s \in (\frac{n}{2}, \frac{n+2}{2})$ and for all $\ell \geq 2$ if $s \in (\frac{n+2}{2}, \frac{n+4}{2})$ and since the remaining finite rank terms are bounded in $L^2(\mathbb{S}^n)$, we see that for all $v \in H^s(\mathbb{S}^n)$,

$$a_{2s}[v] \geq c\|v\|^2_{H^s(\mathbb{S}^n)} - C'\|v\|^2_{L^2(\mathbb{S}^n)} \tag{18}$$

with $c > 0$ and $C' < \infty$. Combining these inequalities we obtain

$$c\|u_j\|^2_{H^s(\mathbb{S}^n)} \leq C'\|u_j\|^2_{L^2(\mathbb{S}^n)} + C|\mathbb{S}^n|^{-\frac{2s-n}{n}} \leq C'|\mathbb{S}^n| + C|\mathbb{S}^n|^{-\frac{2s-n}{n}} \,,$$

which proves the claimed boundedness.

Thus, after passing to a subsequence, we may assume that (u_j) converges weakly in $H^s(\mathbb{S}^n)$ to some u. By Morrey's inequality and the Arzelà–Ascoli lemma, (u_j) converges strongly to u in $C(\mathbb{S}^n)$. As a consequence, $u \geq 0$ and

$$\max_{\mathbb{S}^n} u = 1 \qquad \text{and} \qquad u(S) = \min_{\mathbb{S}^n} u \,.$$

We note that a_{2s} is lower semicontinuous with respect to weak convergence in $H^s(\mathbb{S}^n)$. Indeed, this is clear for the positive part of the functional a_{2s} and its negative part is finite rank and therefore continuous. As a consequence of lower semicontinuity,

$$\liminf_{j\to\infty} a_{2s}[u_j] \geq a_{2s}[u] \,.$$

Step 2. If we have $u > 0$ on $\mathbb{S}^n$, then $u_j^{-1} \to u^{-1}$ uniformly on $\mathbb{S}^n$ and consequently

$$\int_{\mathbb{S}^n} u_j^{-\frac{2n}{2s-n}} \, d\omega \to \int_{\mathbb{S}^n} u^{-\frac{2n}{2s-n}} \, d\omega \,.$$

This, together with the lower semicontinuity of a_{2s} implies that u is a minimizer. Moreover, setting $r_j := u_j - u$ and using weak convergence in $H^s(\mathbb{S}^n)$, we find

$$a_{2s}[u_j] = a_{2s}[u] + a_{2s}[r_j] + o(1)$$

and therefore

$$a_{2s}[u_j] \left(\int_{\mathbb{S}^n} u_j^{-\frac{2n}{2s-n}} d\omega \right)^{\frac{2s-n}{n}} = a_{2s}[u] \left(\int_{\mathbb{S}^n} u^{-\frac{2n}{2s-n}} d\omega \right)^{\frac{2s-n}{n}} + a_{2s}[r_j] \left(\int_{\mathbb{S}^n} u^{-\frac{2n}{2s-n}} d\omega \right)^{\frac{2s-n}{n}} + o(1)\,.$$

Since the left side converges to $I_{2s,n}$ and the first term on the right side is equal to $I_{2s,n}$, we find that $a_{2s}[r_j] \to 0$. By (18) and the strong convergence of r_j in $L^2(\mathbb{S}^n)$, we infer that $r_j \to 0$ strongly in $H^s(\mathbb{S}^n)$, that is, $u_j \to u$ in $H^s(\mathbb{S}^n)$, as claimed.

Step 3. Thus, in what follows we assume that $\min u = 0$. Our goal will be to show that after a conformal transformation and multiplication by a constant we can make the u_j converge to a positive limit, which will be a minimizer.

We observe that

$$a_{2s}[u] \leq 0\,. \tag{19}$$

In fact, for $s \in (\frac{n}{2}, \frac{n+2}{2})$ the infimum $I_{2s,n}$ is negative, as can be seen by taking a constant trial function, and therefore $a_{2s}[u_j]$ is negative for all sufficiently large j. Thus, (19) follows by lower semicontinuity. On the other hand, for $s \in (\frac{n+2}{2}, \frac{n+4}{2})$ we have by Morrey's inequality $u \in C^{1,s-\frac{n+2}{2}}(\mathbb{S}^n)$. Since $u(S) = \min u = 0$ and $u \geq 0$ we have $\nabla u(S) = 0$ and consequently,

$$u(\omega) \leq C_u |\omega - S|^{s-\frac{n}{2}} \qquad \text{for all } \omega \in \mathbb{S}^n\,.$$

Thus, by Fatou's lemma,

$$\liminf_{j\to\infty} \int_{\mathbb{S}^n} u_j^{-\frac{2n}{2s-n}}\, d\omega \geq \int_{\mathbb{S}^n} u^{-\frac{2n}{2s-n}}\, d\omega \geq C_u^{-\frac{2n}{2s-n}} \int_{\mathbb{S}^n} |\omega - S|^{-n}\, d\omega = \infty\,,$$

that is, $\int_{\mathbb{S}^n} u_j^{-\frac{2n}{2s-n}}\, d\omega \to \infty$. Inserting this into (17) we obtain $\limsup_{j\to\infty} a_{2s}[u_j] \leq 0$ and then (19) follows again by lower semicontinuity.

On the other hand, by the first part of Proposition 5, the fact that $u(S) = 0$ (and $\nabla u(S) = 0$ if $s > \frac{n+2}{2}$) implies that $a_{2s}[u] \geq 0$ and therefore, in view of (19), the second part of Proposition 5 implies that

$$u(\omega) = 2^{-\frac{2s-n}{2}} (1 + \omega_{n+1})^{\frac{2s-n}{2}} \qquad \text{for all } \omega \in \mathbb{S}^n\,.$$

(Here we used the normalization $\max u = 1$ to determine the constant and, in case $s > \frac{n+2}{2}$ we use positivity of u to deduce that $b = 0$.)

With a sequence $(\lambda_j) \subset (0, \infty)$ to be determined later, we now consider the conformal transformation Φ_j of $\mathbb{S}^n$ given by $\Phi_j := \mathcal{S}\mathcal{D}_{\lambda_j}\mathcal{S}^{-1}$, where $\mathcal{D}_{\lambda_j}$ is dilation on $\mathbb{R}^n$ by λ_j, that is $(\mathcal{D}_{\lambda_j})(x) = \lambda_j x$, and $\mathcal{S}$ is the (inverse) stereographic projection. We set

$$v_j(\omega) := J_{\Phi_j}(\omega)^{-\frac{2s-n}{2n}} u_j(\Phi_j(\omega))$$

and

$$\tilde{u}_j := \frac{v_j}{\max v_j}\,.$$

By conformal invariance (Lemma 3) and homogeneity, $(\tilde{u}_j)$ is a minimizing sequence for $I_{2s,n}$ and it is normalized by $\max \tilde{u}_j = 1$. We argue as before and, after passing to a subsequence, we may assume that $(\tilde{u}_j)$ converges weakly in $H^s(\mathbb{S}^n)$ and strongly in $C(\mathbb{S}^n)$ to some $\tilde{u}$.

Note that $\tilde{u}$ depends on the choice of the sequence (λ_j). We claim that, for an appropriate choice of (λ_j) (where for each j, λ_j only depends on u_j), we have $\tilde{u} > 0$. Once this is shown, we obtain in the same way as before that $\tilde{u}$ is a minimizer and that the convergence is strong in $H^s(\mathbb{S}^n)$, so we are done.

We argue by contradiction and assume that there is a $\xi \in \mathbb{S}^n$ such that $\tilde{u}(\xi) = 0$. Then, still arguing as before, but with a rotated version of Proposition 5,

$$\tilde{u}(\omega) = 2^{-\frac{2s-n}{2}}\,(1-\xi\cdot\omega)^{\frac{2s-n}{2}} \qquad \text{for all } \omega \in \mathbb{S}^n\,. \tag{20}$$

We now compute explicitly

$$J_{\Phi_j}(\omega) = \left(\frac{2\lambda_j}{1+\omega_{n+1}+\lambda_j^2(1-\omega_{n+1})}\right)^n$$

and, using $\Phi_j(N) = N$ and $\Phi_j(S) = S$, we obtain

$$v_j(N) = \lambda_j^{-\frac{2s-n}{2}} u_j(N) \qquad \text{and} \qquad v_j(S) = \lambda_j^{\frac{2s-n}{2}} u_j(S)\,.$$

Thus, if we choose

$$\lambda_j := \left(\frac{u_j(N)}{u_j(S)}\right)^{\frac{1}{2s-n}}\,,$$

then $\tilde{u}_j(S) = \tilde{u}_j(N)$ and, in the limit, $\tilde{u}(S) = \tilde{u}(N)$. By (20), this implies that $\xi_{n+1} = 0$ and $\tilde{u}(N) = 2^{-\frac{2s-n}{2}} > 0$.

Since $\min u_j = u_j(S)$, we have for any $\omega \in \mathbb{S}^n$,

$$\begin{aligned}\tilde{u}_j(\omega) &\geq \frac{u_j(S)}{\max v_j}\left(\frac{1+\omega_{n+1}+\lambda_j^2(1-\omega_{n+1})}{2\lambda_j}\right)^{\frac{2s-n}{2}}\\ &= \tilde{u}_j(N)\left(\frac{\lambda_j^{-2}(1+\omega_{n+1})+1-\omega_{n+1}}{2}\right)^{\frac{2s-n}{2}}\,.\end{aligned}$$

Since $\tilde{u}_j(N) \to \tilde{u}(N) = 2^{-\frac{2s-n}{2}}$ and since $\lambda_j \to +\infty$ (because $u_j(S) \to u(S) = 0$ and $u_j(N) \to u(N) = 1$), we infer that in the limit

$$\tilde{u}(\omega) \geq 2^{2s-n}\,(1-\omega_{n+1})^{\frac{2s-n}{2}}\,.$$

In particular, $\tilde{u}(\xi) \geq 2^{2s-n} > 0$, a contradiction. This completes the proof. □

A variation of part of the above argument allows us to show the following.

Lemma 7. *Let $n \geq 1$ and $s \in (\frac{n}{2}, \frac{n+4}{2}) \setminus \{\frac{n+2}{2}\}$. Assume that $u \in H^s(\mathbb{S}^n)$ is nonnegative with $u^{-\frac{2n}{2s-n}} \in L^1(\mathbb{S}^n)$ and*

$$a_{2s}[u] \left(\int_{\mathbb{S}^n} u^{-\frac{2n}{2s-n}} \, d\omega \right)^{\frac{2s-n}{n}} = I_{2s,n} \, .$$

Then u is everywhere positive.

Proof. We argue by contradiction and assume that $\min u = 0$. Then, by a rotated version of Proposition 5, $a_{2s}[u] \geq 0$. Since $a_{2s}[1] < 0$ for $s \in (\frac{n}{2}, \frac{n+2}{2})$, we immediately obtain a contradiction in that case. On the other hand, if $s \in (\frac{n+2}{2}, \frac{n+4}{2})$, then similarly as in the previous proof $u(\omega) \lesssim |\omega - \xi|^{s-\frac{n}{2}}$ for all $\omega \in \mathbb{S}^n$ and some $\xi \in \mathbb{S}^n$ and consequently $u^{-\frac{2n}{2s-n}} \notin L^1(\mathbb{S}^n)$, which is again a contradiction. □

3.2. **Proof of Theorem 1.** First, let $s \in \frac{n}{2} + \mathbb{N}$. Then $\alpha_{2s,n}(\ell) \geq 0$ for all $\ell \in \mathbb{N}_0$ with equality if and only if $\ell \leq s - \frac{n}{2}$. This immediately proves the inequality and the characterization of cases of equality.

In the remainder of this proof, let $s \in (\frac{n}{2}, \frac{n+4}{2}) \setminus \{\frac{n+2}{2}\}$. Then, according to Proposition 6, there is a minimizer u for the infimum $I_{2s,n}$. Conversely, assume that $0 \leq u \in H^s(\mathbb{S}^n)$ with $u^{-\frac{2n}{2s-n}} \in L^1(\mathbb{S}^n)$ realized equality in (8). Then by Lemma 7 $u > 0$ and using this, it is easy to derive the Euler–Lagrange equation

$$A_{2s} u = \lambda \, u^{-\frac{2s+n}{2s-n}} \, , \qquad \lambda = \frac{a_{2s}[u]}{\int_{\mathbb{S}^n} u^{-\frac{2n}{2s-n}} \, d\omega} \, .$$

(The form of the Euler–Lagrange multiplier follows by integrating the equation against u.) The equation holds a-priori in $H^{-s}(\mathbb{S}^n)$, but since the right side is square-integrable (in fact, Hölder continuous), a standard bootstrap argument yields that $u \in C^\infty(\mathbb{S}^n)$. Applying the inverse A_{2s}^{-1} to both sides as in the proof of Lemma 3, we find

$$u = \lambda \, A_{2s}^{-1} u^{-\frac{2s+n}{2s-n}} = \lambda \, \frac{\Gamma(\frac{n}{2} - s)}{2^{2s} \pi^{\frac{n}{2}} \Gamma(s)} \int_{\mathbb{S}^n} |\omega - \omega'|^{2s-n} u(\omega')^{-\frac{2s+n}{2s-n}} \, d\omega' \, .$$

Taking into account the sign of $\Gamma(\frac{n}{2} - s)$, we see that $\lambda < 0$ if $s \in (\frac{n}{2}, \frac{n+2}{2})$ and $\lambda > 0$ if $s \in (\frac{n+2}{2}, \frac{n+4}{2})$. Defining $u_{\mathcal{S}}$ and $u^{\mathcal{S}}$ as in (11) and arguing as in Step 2 of the proof of Lemma 4, we obtain

$$\begin{aligned} u_{\mathcal{S}}(x) &= \lambda \, \frac{\Gamma(\frac{n}{2} - s)}{2^{2s} \pi^{\frac{n}{2}} \Gamma(s)} \int_{\mathbb{R}^n} |x - x'|^{2s-n} (u^{-\frac{2s+n}{2s-n}})^{\mathcal{S}}(x') \, dx' \\ &= \lambda \, \frac{\Gamma(\frac{n}{2} - s)}{2^{2s} \pi^{\frac{n}{2}} \Gamma(s)} \int_{\mathbb{R}^n} |x - x'|^{2s-n} (u_{\mathcal{S}}(x'))^{-\frac{2s+n}{2s-n}} \, dx' \, . \end{aligned}$$

Applying [37, Theorem 1.5] to a suitable multiple of $u_{\mathcal{S}}$ (at this point we use the sign of λ), we find that, for some $a \in \mathbb{R}^n$, $b, c > 0$,

$$u_{\mathcal{S}}(x) = c \left(\frac{b^2 + |x - a|^2}{2b} \right)^{\frac{2s-n}{2}} \, .$$

This means that, with $\zeta := (2\eta - b^2(1+\eta_{n+1})e_{n+1})/(2+b^2(1+\eta_{n+1}))$ and $\eta := S(a)$,

$$u(\omega) = c\left(\frac{1-\zeta\cdot\omega}{\sqrt{1-|\zeta|^2}}\right)^{\frac{2s-n}{2}},$$

which is of the form claimed in the theorem.

Conversely, if u is of the form in the theorem, then $u = cJ_\Phi^{-(2s-n)/(2n)} = c1_\Phi$ for some conformal transformation Φ of $\mathbb{S}^n$ and some $c > 0$. (This follows, for instance, by reversing the above computation, namely by showing that $u_{\mathcal{S}}$ is a multiple of a translation and dilation of $((1+|x|^2)/2)^{(2s-n)/2}$.) Then $\int u^{-2n/(2s-n)}\,d\omega = c^{-2n/(2s-n)}\int 1^{-2n/(2s-n)}\,d\omega$ and, by Lemma 3, $a_{2s}[u] = c^2 a_{2s}[1]$. In particular, the value of the left side of (8) is independent of ζ and c and since, according to what we showed before, the left side is minimal for some ζ and c, it is in fact minimal for every ζ and c. This concludes the proof of Theorem 1.

4. Proof of Theorem 2

In this section we prove our second main result, which says that for $s \in (\frac{n+4}{2}, +\infty)\setminus(\frac{n}{2}+\mathbb{N})$ there is no minimizer for the infimum $I_{2s,n}$ in (7) and that, at least for $n \geq 2$, one has $I_{2s,n} = -\infty$. These two assertions are proved in the following two subsections.

4.1. **Local instability.** We prove more than what is stated in Theorem 2, namely that the quantity in (7) does not even have a local minimizer $0 < u \in H^s(\mathbb{S}^n)$. Indeed, if such a local minimizer would exist, we could repeat the argument in the proof of Theorem 1 and would infer that this minimizer is necessarily of the form $c1_\Phi$ for a conformal transformation Φ of $\mathbb{S}^n$ and a constant $c > 0$. Since the minimization problem is homogeneous and conformally invariant (Lemma 3), it therefore suffices to show that the constant function 1 is not a local minimizer. We do this by showing that the second variation is not positive semidefinite.

A simple computation shows that for every $\varphi \in H^s(\mathbb{S}^n)$, as $t \to 0$,

$$a[1+t\varphi]\left(\int_{\mathbb{S}^n}(1+t\varphi)^{-\frac{2n}{2s-n}}d\omega\right)^{\frac{2s-n}{n}} = a[1]|\mathbb{S}^n|^{\frac{2s-n}{n}} + t^2|\mathbb{S}^n|^{\frac{2s-n}{n}}H[\varphi] + o(t^2)$$

with

$$H[\varphi] := a_{2s}[\varphi] + \frac{2s+n}{2s-n}\frac{a_{2s}[1]}{|\mathbb{S}^n|}\int_{\mathbb{S}^n}\varphi^2\,d\omega + \frac{4(s-n)}{2s-n}\frac{a_{2s}[1]}{|\mathbb{S}^n|^2}\left(\int_{\mathbb{S}^n}\varphi\,d\omega\right)^2 - 4\,\frac{a_{2s}[1,\varphi]}{|\mathbb{S}^n|}\int_{\mathbb{S}^n}\varphi\,d\omega\,.$$

Here $a_{2s}[\cdot,\cdot]$ is the natural bilinear form associated to $a_{2s}[\cdot]$. This can be rewritten as

$$H[\varphi] = a_{2s}[\varphi] + \frac{2s+n}{2s-n}\,\alpha_{2s,n}(0)\int_{\mathbb{S}^n}\varphi^2\,d\omega - \frac{4s}{2s-n}\,\alpha_{2s,n}(0)\left(\int_{\mathbb{S}^n}\varphi\,d\omega\right)^2.$$

If $2k < s - \frac{n}{2} < 2k+1$ for some $k \in \mathbb{N}$, we choose φ to be an L^2-normalized spherical harmonic of degree two and obtain

$$H[\varphi] = \alpha_{2s,n}(2) + \frac{2s+n}{2s-n}\alpha_{2s,n}(0) = \frac{\Gamma(2+\frac{n}{2}+s)}{\Gamma(2+\frac{n}{2}-s)} + \frac{2s+n}{2s-n}\frac{\Gamma(\frac{n}{2}+s)}{\Gamma(\frac{n}{2}-s)} = 2s\frac{\Gamma(1+\frac{n}{2}+s)}{\Gamma(2+\frac{n}{2}-s)}.$$

Since $\Gamma(2+\frac{n}{2}-s) < 0$, we have $H[\varphi] < 0$, which shows the instability of 1.

If $2k+1 < s - \frac{n}{2} < 2(k+1)$ for some $k \in \mathbb{N}$, we choose φ to be an L^2-normalized spherical harmonic of degree three and obtain

$$\begin{aligned} H[\varphi] &= \alpha_{2s,n}(3) + \frac{2s+n}{2s-n}\alpha_{2s,n}(0) = \frac{\Gamma(3+\frac{n}{2}+s)}{\Gamma(3+\frac{n}{2}-s)} + \frac{2s+n}{2s-n}\frac{\Gamma(\frac{n}{2}+s)}{\Gamma(\frac{n}{2}-s)} \\ &= 2s(n+3)\frac{\Gamma(1+\frac{n}{2}+s)}{\Gamma(3+\frac{n}{2}-s)}. \end{aligned}$$

Since $\Gamma(3+\frac{n}{2}-s) < 0$, we have $H[\varphi] < 0$, which shows again the instability of 1.

4.2. **Global instability.** We now complete the proof of Theorem 2 by showing that

$$I_{2s,n} = -\infty \qquad \text{if } n \geq 2 \quad \text{and} \quad s \in (\tfrac{n+4}{2}, \infty) \setminus (\tfrac{n}{2} + \mathbb{N}). \tag{21}$$

We will give a separate proof for the following two subcases

$$2K < s - \frac{n}{2} < 2K+1 \quad \text{for some } K \in \mathbb{N}\,, \tag{22}$$

$$2K+1 < s - \frac{n}{2} < 2K+2 \quad \text{for some } K \in \mathbb{N}\,. \tag{23}$$

Note that in the first case, we have $\alpha_{2s,n}(2k) < 0$ for all $k = 0, \dots, K$ and $\alpha_{2s,n}(\ell) > 0$ for all other ℓ. In the second case, we have $\alpha_{2s,n}(2k+1) < 0$ for all $k = 0, \dots, K$ and $\alpha_{2s,n}(\ell) > 0$ for all other ℓ.

For both cases, we will give a test function $u \geq 0$ such that

$$-\infty < a_{2s}[u] < 0 \qquad \text{and} \qquad \int_{\mathbb{S}^n} u^{-\frac{2n}{2s-n}}\, d\omega = +\infty\,. \tag{24}$$

This essentially proves (21), except that the function may not satisfy the strict inequality $u > 0$. But we can simply take $u + \varepsilon$ with a constant $\varepsilon > 0$ as a trial function for the infimum and let $\varepsilon \to 0_+$.

The case (22). Let $K \in \mathbb{N}$ be as in (22). We shall show that (24) holds for the function

$$u(\omega) = \omega_{n+1}^{2K}.$$

The integral condition is simple. Using spherical coordinates with $\omega_{n+1} = \cos\theta$, $\theta \in [0, \pi]$, and changing variables $t = \cos\theta \in [-1, 1]$ we obtain

$$\int_{\mathbb{S}^n} u^{-\frac{2n}{2s-n}}\, d\omega = |\mathbb{S}^{n-1}| \int_0^\pi |\cos\theta|^{-\frac{4nK}{2s-n}} \sin^{n-1}\theta\, d\theta = |\mathbb{S}^{n-1}| \int_{-1}^1 |t|^{-\frac{4nK}{2s-n}} (1-t^2)^{\frac{n-2}{2}}\, dt\,.$$

This integral is divergent if and only if $\frac{4nK}{2s-n} \geq 1$. In view of (22), this is the case if and only if $n \geq 2$.

Next, we show that $a_{2s}[u] < 0$. We claim that

$$u(\omega) = \sum_{k=0}^{K} c_k C_{2k}^{(\frac{n-1}{2})}(\omega_{n+1}) \,, \tag{25}$$

where $C_\ell^{(\alpha)}$ are the Gegenbauer (or ultraspherical) polynomials and where $c_k \in \mathbb{R}$. It is well known (see, for instance, [46, Thm. IV.2.14]) that the function $C_\ell^{(\frac{n-1}{2})}(\omega_{n+1})$ is a spherical harmonic of degree ℓ, namely a so-called zonal harmonic.

Note that we claim that in the spherical harmonic expansion (25) of u there are only terms of even degree at most K. As noted above, the condition (22) then guarantees that $\alpha_{2s,n}(2k) < 0$ for all $k = 0, ..., K$ and thus

$$a_{2s}[u] = \sum_{k=0}^{K} \alpha_{2s,n}(2k)\, c_k^2 \|Y_{2k}\|^2_{L^2(\mathbb{S}^n)} < 0 \,,$$

as desired.

We recall two standard facts about the Gegenbauer polynomials. First, $C_\ell^{(\alpha)}$ is a polynomial of exact degree ℓ and second, $C_\ell^{(\alpha)}$ has the same parity as ℓ, see [20, Eq. 18.5.10]. That is, for every k,

$$C_{2k}^{(\frac{n-1}{2})}(t) = a_{k,k} t^{2k} + a_{k,k-1} t^{2k-2} + \ldots + a_{k,0}$$

with $a_{k,k} \neq 0$. Thus, the desired expansion

$$t^{2K} = \sum_{k=0}^{K} c_k\, C_{2k}^{(\frac{n-1}{2})}(t) \qquad \text{for all } t \in [-1,1] \,. \tag{26}$$

can be equivalently rewritten, with respect to the basis $\{t^{2K}, t^{2K-2}, \ldots, t^2, 1\}$ of even polynomials on $[-1,1]$ of order at most $2K$, as

$$\begin{pmatrix} a_{K,K} & 0 & ... & 0 \\ a_{K,K-1} & a_{K-1,K-1} & ... & 0 \\ & & ... & \\ a_{K,0} & a_{K-1,0} & ... & a_{0,0} \end{pmatrix} \begin{pmatrix} c_K \\ c_{K-1} \\ ... \\ c_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ ... \\ 0 \end{pmatrix} .$$

Since the matrix on the left side is of lower-triangular form with non-zero diagonal entries, its determinant is non-zero. Hence there are (unique) numbers $c_0, ..., c_K \in \mathbb{R}$ such that (26), and hence (25), holds. This completes the proof in the case (22).

The case (23). In this case, the above argument becomes a bit more involved because we need to work with the more complicated test function

$$u(\omega) := \omega_{n+1}^{2K} - \omega_{n+1}^{2K+1},$$

where K is as in (23). Indeed, since $\alpha_{2s,n}(\ell) < 0$ if and only if $\ell = 1, 3, ..., 2K+1$, the test function needs to contain an 'odd' component like ω_{n+1}^{2K+1} to achieve $a_{2s}[u] < 0$. On the other hand, the 'even' term ω_{n+1}^{2K} is necessary to ensure $u \geq 0$.

Let us verify divergence of the integral. With the same change of variables as before we obtain

$$\int_{\mathbb{S}^n} u^{-\frac{2n}{2s-n}}\,d\omega = |\mathbb{S}^{n-1}| \int_0^\pi |\cos\theta|^{-\frac{4nK}{2s-n}} (1-\cos\theta)^{-\frac{2n}{2s-n}} \sin^{n-1}\theta\,d\theta$$
$$= |\mathbb{S}^{n-1}| \int_{-1}^1 |t|^{-\frac{4nK}{2s-n}} (1-t)^{-\frac{2n}{2s-n}} (1-t^2)^{\frac{n-2}{2}}\,dt\,.$$

The integral is divergent (at $t=0$) if and only if $\frac{4nK}{2s-n} \geq 1$. In view of (23), this is the case if and only if $n \geq 2$.

Next, we show that $a_{2s}[u] < 0$. By using the properties of Gegenbauer polynomials as in the case (22), we find

$$t^{2K} = \sum_{k=0}^K c_k C_{2k}^{(\frac{n-1}{2})}(t) \qquad \text{and} \qquad t^{2K+1} = \sum_{k=0}^K d_k C_{2k+1}^{(\frac{n-1}{2})}(t) \tag{27}$$

for suitable coefficients $c_k, d_k \in \mathbb{R}$. (In fact, in this case we shall need to find c_k and d_k explicitly, see (29) and Lemma 8 below.) Therefore

$$u(\omega) = \sum_{k=0}^K \Big(c_k C_{2k}^{(\frac{n-1}{2})}(\omega_{n+1}) - d_k C_{2k+1}^{(\frac{n-1}{2})}(\omega_{n+1}) \Big)\,.$$

Since a_{2s} is diagonal with respect to spherical harmonics, we thus obtain

$$a_{2s,n}[u] = \sum_{k=0}^K \Big(\alpha_{2s,n}(2k) c_k^2 \|C_{2k}^{(\frac{n-1}{2})}(\omega_{n+1})\|^2_{L^2(\mathbb{S}^n)} + \alpha_{2s,n}(2k+1) d_k^2 \|C_{2k+1}^{(\frac{n-1}{2})}(\omega_{n+1})\|^2_{L^2(\mathbb{S}^n)} \Big).$$

By (5), we have the relation $\alpha_{2s,n}(2k+1) = -\frac{2s+n+4k}{2s-n-4k}\alpha_{2s,n}(2k)$, where $\frac{2s+n+4k}{2s-n-4k} > 0$ for all $k = 0, ..., K$, thanks to (23). Hence

$$a_{2s,n}[u] = \sum_{k=0}^K \alpha_{2s,n}(2k) \Big(c_k^2 \|C_{2k}^{(\frac{n-1}{2})}(\omega_{n+1})\|^2_{L^2(\mathbb{S}^n)} - \frac{2s+n+4k}{2s-n-4k} d_k^2 \|C_{2k+1}^{(\frac{n-1}{2})}(\omega_{n+1})\|^2_{L^2(\mathbb{S}^n)} \Big). \tag{28}$$

In view of $\alpha_{2s,n}(2k) > 0$ for all $k = 0, ..., K$ by (23), the desired inequality $a_{2s}[u] < 0$ follows if we can show that the difference in brackets is strictly negative for every $k = 0, ..., K$. To do so, we claim that the coefficients c_k and d_k are related by

$$d_k = c_k \frac{(2K+1)(4k+n+1)}{(2K+2k+n+1)(4k+n-1)} \tag{29}$$

and the L^2-norms of the spherical harmonics by

$$\|C_{2k+1}^{(\frac{n-1}{2})}(\omega_{n+1})\|^2_{L^2(\mathbb{S}^n)} = \|C_{2k}^{(\frac{n-1}{2})}(\omega_{n+1})\|^2_{L^2(\mathbb{S}^n)} \frac{(2k+n-1)(4k+n-1)}{(2k+1)(4k+n+1)}. \tag{30}$$

We defer the details of these computations to Lemma 8 below.

Inserting (29) and (30) into (28), the inequality we need to verify reduces to

$$1 - \frac{2s+n+4k}{2s-n-4k} \cdot \frac{(2K+1)^2(2k+n-1)(4k+n+1)}{(2K+2k+n+1)^2(2k+1)(4k+n-1)} < 0. \tag{31}$$

Since $2s - n < 4K + 4$ by (23), and since $t \mapsto \frac{t+2n+4k}{t-4k}$ is strictly decreasing, we can estimate

$$\frac{2s+n+4k}{2s-n-4k} > \frac{4K+4+2n+4k}{4K+4-4k} = \frac{2K+2+n+2k}{2K+2-2k}.$$

Hence to show (31), it suffices to prove

$$\frac{2K+2+n+2k}{2K+2-2k}\frac{(2K+1)^2(2k+n-1)(4k+n+1)}{(2K+2k+n+1)^2(2k+1)(4k+n-1)} \geq 1 \tag{32}$$

for all integers $n \geq 2$, $K \geq 1$ and $0 \leq k \leq K$.

The rest of the proof will be devoted to establishing (32) by considering several cases separately.

Let us first assume that $k \leq K - 1$. We write the left side of (32) as

$$\frac{(4k+n+1)(2K+2+n+2k)}{(2K+2k+n+1)^2} \cdot \frac{2k+n-1}{4k+n-1} \cdot \frac{(2K+1)^2}{(2K+2-2k)(2k+1)},$$

and notice that the first factor is a decreasing function of $n \geq 2$ (see Lemma 9 below; here is where the assumption $k \leq K-1$ enters) and the second factor is a decreasing function of $n \geq 2$. Thus, if $k \leq K-1$, it suffices to prove (32) for $n = 2$, which we write as

$$\frac{F(K,k)}{G(K,k)} := \frac{4k+3}{4k+1} \cdot \frac{2K+1}{2K+2k+3} \cdot \frac{2K+1}{2K-2k+2} \cdot \frac{2K+2k+4}{2K+2k+3} \geq 1. \tag{33}$$

If $k = 0$, then for all $K \geq 1$

$$\frac{F(K,0)}{G(K,0)} = 3 \cdot \frac{(2K+1)^2}{(2K+3)^2} \cdot \frac{K+2}{K+1} \geq 3 \cdot \frac{3^2}{5^2} \cdot 1 > 1\,,$$

so we may assume $k \geq 1$ in the following. To solve this case, we resort to explicit calculation. We compute

$$\frac{1}{2}F(K,k) = (16k+12)K^3 + (16k^2+60k+36)K^2 + (16k^2+48k+27)K + (4k+3)(k+2)$$

and

$$\begin{aligned}\frac{1}{2}G(K,k) = {}& (16k+4)K^3 + (16k^2+68k+16)K^2 + (-16k^3+28k^2+92k+21)K \\ & - (k-1)(2k+3)^2(4k+1).\end{aligned}$$

Hence for every $k \geq 1$, dropping the positive constant term and using $k \leq K-1$, we get

$$\begin{aligned}\frac{1}{4K}(F(K,k) - G(K,k)) &\geq 4K^2 + (10-4k)K + (8k^3 - 6k^2 - 22k + 3) \\ &\geq 8k^3 - 6k^2 - 12k + 13 =: P(k)\,.\end{aligned}$$

We have $P(k) \geq 0$ for all $1 \leq k \leq K-1$ because $P'(k) = 24k^2 - 12k - 12 \geq 0$ for $k \geq 1$ and $P(1) = 3 > 0$. This finishes the proof of (33), and hence of (32), in the case $k \leq K-1$.

Let us finally give the proof of (32) when $k = K$. If $K \geq 2$, we estimate the left side of (32) by

$$\frac{2K+1}{2} \cdot \frac{2K+n-1}{4K+n-1} \cdot \frac{4K+n+2}{4K+n+1} \geq \frac{2K+1}{2} \cdot \frac{1}{2} \geq \frac{5}{4} > 1. \tag{34}$$

If $K = 1$ and $n \geq 3$, since $\frac{2K+n-1}{4K+n-1}$ is increasing in n, the left side of (34) can be estimated by

$$\frac{3}{2} \cdot \frac{2+n-1}{4+n-1} \cdot \frac{4+n+2}{4+n+1} > \frac{3}{2} \cdot \frac{4}{6} = 1.$$

Finally, if $K = 1$ and $n = 2$, by a direct calculation the left side of (34) equals $\frac{36}{35} > 1$.

The proof of Theorem 2 is now complete. □

We finally prove the two lemmas that we referred to in the proof.

Lemma 8. *The coefficients c_k and d_k in* (27) *are given by*

$$c_k = \frac{2^{-2K}(2k+\frac{n-1}{2})\Gamma(2K+1)\Gamma(\frac{n-1}{2})}{\Gamma(K+k+\frac{n+1}{2})\Gamma(K-k+1)},$$

$$d_k = \frac{2^{-2K-1}(2k+\frac{n+1}{2})\Gamma(2K+2)\Gamma(\frac{n-1}{2})}{\Gamma(K+k+\frac{n+3}{2})\Gamma(K-k+1)}.$$

Moreover,

$$\int_{-1}^{1} |C_\ell^{(\frac{n-1}{2})}(t)|^2 (1-t^2)^{\frac{n-2}{2}}\, dt = \frac{\pi\, 2^{2-n}\, \Gamma(n-1+\ell)}{\ell!\, (\ell+\frac{n-1}{2})\, \Gamma(\frac{n-1}{2})^2}. \tag{35}$$

In particular, identities (29) *and* (30) *hold.*

Proof. Formula (35) is the special case $\alpha = \frac{n-1}{2}$ of the following general formula [20, Table 18.3.1]

$$\int_{-1}^{1} |C_\ell^{(\alpha)}(t)|^2 (1-t^2)^{\alpha-\frac{1}{2}}\, dt = \frac{\pi 2^{1-2\alpha}\Gamma(2\alpha+\ell)}{\ell!(\ell+\alpha)\Gamma(\alpha)^2}.$$

Observing that by change of variables $t = \omega_{n+1}$,

$$\|C_\ell^{(\frac{n-1}{2})}(\omega_n)\|^2_{L^2(\mathbb{S}^n)} = |\mathbb{S}^{n-1}| \int_{-1}^{1} |C_\ell^{(\frac{n-1}{2})}(t)|^2 (1-t^2)^{\frac{n-2}{2}}\, dt,$$

identity (30) follows readily from (35) by a direct computation.

To obtain the expression for c_k, recall that, at fixed α, the Gegenbauer polynomials $C_\ell^{(\alpha)}$ are pairwise orthogonal on the space $L^2\left((-1,1),(1-t^2)^{\alpha-\frac{1}{2}}\, dt\right)$, see e.g. [20, Table 18.3.1]. Integrating $t^{2K} = \sum_{l=0}^{K} c_l C_{2l}^{(\frac{n-1}{2})}(t)$ against $C_{2k}^{(\frac{n-1}{2})}(t)(1-t^2)^{\frac{n-2}{2}}$, we thus find

$$\int_{-1}^{1} t^{2K} C_{2k}^{(\frac{n-1}{2})}(t)(1-t^2)^{\frac{n-2}{2}}\, dt = c_k \int_{-1}^{1} |C_\ell^{(\frac{n-1}{2})}(t)|^2 (1-t^2)^{\frac{n-2}{2}}\, dt.$$

The integral on the right side is given in (35) and the integral on the left side is [20, Eq. 18.17.37]

$$\int_{-1}^{1} t^{2K} C_{2k}^{(\frac{n-1}{2})}(t)(1-t^2)^{\frac{n-2}{2}}\, dt = \frac{\pi 2^{2-n-2K}\Gamma(2k+n-1)\Gamma(2K+1)}{(2k)!\Gamma(\frac{n-1}{2})\Gamma(K+k+\frac{n+1}{2})\Gamma(K-k+1)}.$$

The expression for d_k follows analogously, using again (35) and [20, Eq. 18.17.37]. Finally, a direct computation gives identity (29). □

Lemma 9. *Suppose that* $0 \le k \le K-1$. *Then* $n \mapsto \frac{(4k+n+1)(2K+2+n+2k)}{(2K+2k+n+1)^2}$ *is increasing in* $n \ge 2$.

Proof. More generally, let $f(n) := \frac{(a+n)(b+n)}{(c+n)^2}$ with $a, b, c \ge -1$, $a \le c \le b$. We claim that if

$$c - a \ge b - c, \tag{36}$$

then $f(n)$ is increasing in $n \ge 2$, unless when $a = b = c$. Applying this claim with $a = 4k+1$, $b = 2K+2k+2$ and $c = 2K+2k+1$, condition (36) becomes $k \le K - \frac{1}{2}$ and the lemma follows.

To prove the claim, write

$$f(n) = \left(1 - \frac{c-a}{c+n}\right)\left(1 + \frac{b-c}{c+n}\right) = 1 + \frac{a+b-2c}{c+n} - \frac{(c-a)(b-c)}{(c+n)^2}.$$

The last summand is nonpositive by assumption, hence nondecreasing in n. The second summand is nondecreasing in n if $a+b-2c \le 0$, which is just (36). If $a = b = c$ does not hold, then one of the summands is increasing. □

References

[1] J. Ai, K.-S. Chou, J. Wei, *Self-similar solutions for the anisotropic affine curve shortening problem.* Calc. Var. Partial Differential Equations **13** (2001), no. 3, 311–337.

[2] T. Aubin, *Problèmes isoperimétriques et espaces de Sobolev.* J. Differ. Geometry **11** (1976), 573–598.

[3] T. Aubin, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire.* J. Math. Pures Appl. (9) **55** (1976), no. 3, 269–296.

[4] W. Beckner, *Sobolev inequalities, the Poisson semigroup, and analysis on the sphere* $\mathbb{S}$. Proc. Nat. Acad. Sci. U.S.A. **89** (1992), no. 11, 4816–4819.

[5] W. Beckner, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality.* Ann. of Math. (2) **138** (1993), no. 1, 213–242.

[6] G. Bianchi, H. Egnell, *A note on the Sobolev inequality.* J. Funct. Anal. **100** (1991), no. 1, 18–24.

[7] M. Bonforte, J. Dolbeault, B. Nazaret, N. Simonov, *Stability in Gagliardo–Nirenberg–Sobolev inequalities: flows, regularity and the entropy method.* Preprint (2021), arXiv:2007.03674.

[8] H. Brezis, E. H. Lieb, *Sobolev inequalities with remainder terms.* J. Funct. Anal. **62** (1985), no. 1, 73–86.

[9] E. A. Carlen, R. L. Frank, E. H. Lieb, *Stability estimates for the lowest eigenvalue of a Schrödinger operator.* Geom. Funct. Anal. **24** (2014), no. 1, 63–84.

[10] E. Carlen, M. Loss, *Competing symmetries, the logarithmic HLS inequality and Onofri's inequality on* $\mathbb{S}^n$. Geom. Funct. Anal. **2** (1992), no. 1, 90–104.

[11] J. A. Carrillo, M. Delgadino, J. Dolbeault, R. L. Frank, F. Hoffmann, *Reverse Hardy–Littlewood–Sobolev inequalities.* J. Math. Pures Appl. (9) **132** (2019), 133–165.

[12] J. A. Carrillo, M. Delgadino, R. L. Frank, M. Lewin, *Fast diffusion leads to partial mass concentration in Keller–Segel type stationary solutions.* Preprint (2020), arXiv:2012.08586.

[13] J. S. Case, S.–Y. A. Chang, *On fractional GJMS operators.* Comm. Pure Appl. Math. **69** (2016), no. 6, 1017–1061.

[14] S.–Y. A. Chang, M. d. M. González, *Fractional Laplacian in conformal geometry.* Adv. Math. **226** (2011), no. 2, 1410–1432.

[15] S.–Y. A. Chang, R. A. Yang, *On a class of non-local operators in conformal geometry.* Chin. Ann. Math. Ser. B **38** (2017), no. 1, 215–234.

[16] L. Chen, Z. Liu, G. Lu, C. Tao, *Reverse Stein–Weiss inequalities and existence of their extremal functions.* Trans. Amer. Math. Soc. **370** (2018), no. 12, 8429–8450.

[17] S. Chen, R. L. Frank, T. Weth, *Remainder terms in the fractional Sobolev inequality.* Indiana Univ. Math. J. **62** (2013), no. 4, 1381–1397.

[18] W. Chen, C. Li, B. Ou, *Classification of solutions for an integral equation.* Comm. Pure Appl. Math. **59** (2006), no. 3, 330–343.

[19] Y. S. Choi, X. Xu, *Nonlinear biharmonic equations with negative exponents.* J. Differential Equations **246** (2009), no. 1, 216–234.

[20] *NIST Digital Library of Mathematical Functions.* http://dlmf.nist.gov/, Release 1.1.1 of 2021-03-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[21] J. Dou, Q. Guo, M. Zhu, *Subcritical approach to sharp Hardy–Littlewood–Sobolev type inequalities on the upper half space.* Adv. Math. **312** (2017), 1–45.

[22] J. Dou, M. Zhu, *Reversed Hardy–Littewood–Sobolev inequality.* Int. Math. Res. Not. IMRN 2015, no. 19, 9696–9726.

[23] P. Exner, E. M. Harrell, M. Loss, *Optimal eigenvalues for some Laplacians and Schrödinger operators depending on curvature.* Mathematical results in quantum mechanics (Prague, 1998), 47–58, Oper. Theory Adv. Appl., **108**, Birkhäuser, Basel, 1999.

[24] A. Figalli, Y. R.–Y. Zhang, *Sharp gradient stability for the Sobolev inequality.* Preprint (2020), arXiv:2003.04037.

[25] R. L. Frank, *Degenerate stability of some Sobolev inequalities.* Ann. Inst. H. Poincaré Anal. Non Linéaire, to appear. Preprint (2021), arXiv:2107.11608.

[26] R. L. Frank, T. König, H. Tang, *Classification of solutions of an equation related to a conformal log Sobolev inequality.* Adv. Math. **375** (2020), 107395, 27 pp.

[27] R. L. Frank, E. H. Lieb, *A new, rearrangement-free proof of the sharp Hardy–Littlewood–Sobolev inequality.* Spectral theory, function spaces and inequalities, 55–67, Oper. Theory Adv. Appl. **219**, Birkhäuser/Springer Basel AG, Basel, 2012.

[28] N. Fusco, *The quantitative isoperimetric inequality and related topics.* Bull. Math. Sci. **5** (2015), no. 3, 517–607.

[29] N. Fusco, F. Maggi, A. Pratelli, *The sharp quantitative isoperimetric inequality.* Ann. of Math. (2) **168** (2008), no. 3, 941–980.

[30] C. R. Graham, R. Jenne, L. J. Mason, G. A. Sparling, *Conformally invariant powers of the Laplacian. I. Existence.* J. London Math. Soc. (2) **46** (1992), no. 3, 557–565.

[31] C. R. Graham, M. Zworski, *Scattering matrix in conformal geometry.* Invent. Math. **152** (2003), no. 1, 89–118.

[32] F. Hang, *On the higher order conformal covariant operators on the sphere.* Commun. Contemp. Math. **9** (2007), no. 3, 279–299.

[33] F. Hang, P. C. Yang, *The Sobolev inequality for Paneitz operator on three manifolds.* Calc. Var. Partial Differential Equations **21** (2004), no. 1, 57–83.

[34] F. Hang, P. C. Yang, *Lectures on the fourth-order Q curvature equation.* Geometric analysis around scalar curvatures, 1–33, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., 31, World Sci. Publ., Hackensack, NJ, 2016.

[35] A. Hyder, J. Wei, *Non-radial solutions to a bi-harmonic equation with negative exponent.* Calc. Var. Partial Differential Equations **58** (2019), no. 6, Paper No. 198, 11 pp.

[36] N. A. Lebedev, I. M. Milin, *An inequality.* Vestnik Leningrad. Univ. **20** (1965), no. 19, 157–158.
[37] Y. Y. Li, *Remark on some conformally invariant integral equations: the method of moving spheres.* J. Eur. Math. Soc. (JEMS) **6** (2004), no. 2, 153–180.
[38] E. H. Lieb, *Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities.* Ann. of Math. (2) **118** (1983), no. 2, 349–374.
[39] E. H. Lieb, M. Loss, *Analysis. Second edition.* Graduate Studies in Mathematics **14**, American Mathematical Society, Providence, RI, 2001.
[40] Q. A. Ngô, V. H. Nguyen, *Sharp reversed Hardy–Littlewood–Sobolev inequality on* $\mathbb{R}^n$. Israel J. Math. **220** (2017), no. 1, 189–223.
[41] Y. Ni, M. Zhu, *Steady states for one-dimensional curvature flows.* Commun. Contemp. Math. **10** (2008), no. 2, 155–179.
[42] E. Onofri, *On the positivity of the effective action in a theory of random surfaces.* Comm. Math. Phys. **86** (1982), no. 3, 321–326.
[43] B. Osgood, R. Phillips, P. Sarnak, *Extremals of determinants of Laplacians.* J. Funct. Anal. **80** (1988), no. 1, 148–211.
[44] E. Rodemich, *The Sobolev inequality with best possible constant.* Analysis Seminar Caltech, Spring 1966.
[45] G. Rosen, *Minimum value for c in the Sobolev inequality* $\|\varphi^3\| \leq c\|\nabla\varphi\|^3$. SIAM J. Appl. Math. **21** (1971), 30–32.
[46] E. M. Stein, G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces.* PMS **32**, Princeton University Press, 1971.
[47] G. Talenti, *Best constants in Sobolev inequality.* Ann. Mat. Pura Appl. **110** (1976), 353–372.
[48] P. Yang, M. Zhu, *On the Paneitz energy on standard three sphere.* ESAIM Control Optim. Calc. Var. **10** (2004), no. 2, 211–223.
[49] M. Zhu, *Prescribing integral curvature equation.* Differential Integral Equations **29** (2016), no. 9-10, 889–904.

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