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CLASSIFICATION OF SOLUTIONS OF AN EQUATION RELATED TO A CONFORMAL LOG SOBOLEV INEQUALITY

RUPERT L. FRANK, TOBIAS KÖNIG, AND HANLI TANG

Abstract. We classify all finite energy solutions of an equation which arises as the Euler–Lagrange equation of a conformally invariant logarithmic Sobolev inequality on the sphere due to Beckner. Our proof uses an extension of the method of moving spheres from $\mathbb{R}^n$ to $\mathbb{S}^n$ and a classification result of Li and Zhu. Along the way we prove a small volume maximum principle and a strong maximum principle for the underlying operator which is closely related to the logarithmic Laplacian.

1. Introduction

1.1. **Main result.** The motivation of this paper is Beckner's logarithmic Sobolev inequality on $\mathbb{S}^n$ with sharp constant [1, 3]. It states that

$$\iint_{\mathbb{S}^n\times\mathbb{S}^n} \frac{|v(\omega)-v(\eta)|^2}{|\omega-\eta|^n}\,\mathrm{d}\omega\,\mathrm{d}\eta \geqslant C_n \int_{\mathbb{S}^n} |v(\omega)|^2 \ln\frac{|v(\omega)|^2|\mathbb{S}^n|}{\|v\|_2^2}\,\mathrm{d}\omega \tag{1}$$

with

$$C_n = \frac{4}{n}\frac{\pi^{n/2}}{\Gamma(n/2)}\,. \tag{2}$$

Here and in the following, $\mathrm{d}\omega$ denotes the surface measure induced by the embedding of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, i.e., $\int_{\mathbb{S}^n}\mathrm{d}\omega = |\mathbb{S}^n| = 2\pi^{\frac{n+1}{2}}/\Gamma(\frac{n+1}{2})$.

Note that, by Jensen's inequality and convexity of $x\mapsto x\ln x$, the right side of (1) is nonnegative and vanishes if and only if $|v|$ is constant. Inequality (1) is a limiting form of the Sobolev inequalities and, in the spirit of these inequalities, it states that functions with some regularity (quantified by the finiteness of the left side) have some improved integrabiliy properties (quantified by the finiteness of the right side). Beckner used inequality (1) to prove an optimal hypercontractivity bound for the Poisson semigroup on the sphere. A remarkable feature of inequality (1) is its conformal invariance, which we will discuss below in detail.

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In [3] Beckner showed that equality holds in (1) if and only if

$$v(\omega) = c \left(\frac{\sqrt{1-|\zeta|^2}}{1-\zeta\cdot\omega} \right)^{n/2} \tag{3}$$

for some $\zeta \in \mathbb{R}^{n+1}$ with $|\zeta| < 1$ and some $c \in \mathbb{R}$.

Our goal in this paper is to classify all nonnegative solutions u of the equation

$$P.V. \int_{\mathbb{S}^n} \frac{u(\omega)-u(\eta)}{|\omega-\eta|^n}\, \mathrm{d}\eta = C_n u(\omega) \ln u(\omega) \qquad \text{in } \mathbb{S}^n. \tag{4}$$

This equation arises, after a suitable normalization, as the Euler–Lagrange equation of the optimization problem corresponding to (1).

Because of the principal value in (4) we interpret this equation in the weak sense. The maximal class of functions for which (1) holds is

$$\mathcal{D} := \left\{ v \in L^2(\mathbb{S}^n) : \iint_{\mathbb{S}^n\times\mathbb{S}^n} \frac{|v(\omega)-v(\eta)|^2}{|\omega-\eta|^n}\, \mathrm{d}\omega\, \mathrm{d}\eta < \infty \right\}.$$

We say that a nonnegative function $u \in \mathcal{D}$ on $\mathbb{S}^n$ is a *weak solution* of (4) if

$$\frac{1}{2} \iint_{\mathbb{S}^n\times\mathbb{S}^n} \frac{(\varphi(\omega)-\varphi(\eta))\,(u(\omega)-u(\eta))}{|\omega-\eta|^n}\, \mathrm{d}\omega\, \mathrm{d}\eta = C_n \int_{\mathbb{S}^n} \varphi(\omega)\, u(\omega) \ln u(\omega)\, \mathrm{d}\omega$$

for every $\varphi \in \mathcal{D}$.

Clearly, the constant function $u \equiv 1$ is a weak solution of (4). Because of the conformal invariance, which equation (4) inherits from inequality (1), see Lemma 2, all elements in the orbit of the constant function $u \equiv 1$ under the conformal group are also weak solutions. One can show that these are precisely the functions of the form (3) with $c = 1$. Our main result is that these are *all* the finite energy solutions of (4).

Theorem 1. *Let $0 \not\equiv u \in \mathcal{D}$ be a nonnegative weak solution of equation* (4)*. Then*

$$u(\omega) = \left(\frac{\sqrt{1-|\zeta|^2}}{1-\zeta\cdot\omega} \right)^{n/2}$$

for some $\zeta \in \mathbb{R}^{n+1}$ with $|\zeta| < 1$.

As we will explain in the next subsection, this result and its proof are in the spirit of similar classification results for conformally invariant equations. Groundbreaking results in the local case were obtained by Gidas, Ni and Nirenberg [13] and Caffarelli, Gidas and Spruck [4]. In the nonlocal case similar results were first obtained by Chen, Li and Ou [7] and Li [17] and we refer to these works for further references.

We follow a general strategy that was pioneered by Li and Zhu [19]; see also [18]. The basic observation there is that a symmetry result together with the conformal invariance of the equation forces the solutions to be of the claimed form. More precisely, the proof proceeds in two steps. In a first step one uses the method of moving planes or its variant, the method of moving spheres, in order to show symmetry of positive solutions. The symmetry in question respects the conformal invariance of the equation. The second step employs a powerful lemma by Li and Zhu [19] which classifies the sufficently regular functions which have the conformal symmetry property established in the first step.

The adaptation of these methods to the present setting, however, encounters several difficulties. One of these comes from the fact that functions in $\mathcal{D}$ have only a very limited regularity. In fact, the left side of (1) is comparable to

$$\Big(v, \big(\ln(-\Delta_{\mathbb{S}^n}+1)\big)\, v\Big)$$

with the inner product in $L^2(\mathbb{S}^n)$, see (13) below. Thus, the linear operator on the left side of (4) is reminiscent of the logarithmic Laplacian, studied recently in [5] on a domain in Euclidean space; see also [9] for a related functional. The work [5] contains some regularity results, but we have not been able to use these to deduce that solutions u of (4) are continuous. Therefore, we need to perform the method of moving spheres in the energy space. While this can be carried out in an elegant and concise way in the case of $(-\Delta)^{\pm s}$ [7], the proof of the corresponding small volume maximum principle in our setting is rather involved and constitutes one of the main achievements in this paper; see Section 3. The missing regularity also prevents us from directly applying the classification lemma by Li and Zhu [19]. Instead, we use its extension in [11] to measures; see Section 5.

We believe that the techniques that we develop in this paper can be useful in similar problems and that they illustrate, in particular, how to prove classification theorems in problems with conformal invariance without first establishing regularity results.

1.2. **Background.** In order to put this problem into context, let us recall Lieb's sharp form [20] of the Hardy–Littlewood–Sobolev inequality, which states that, if $0<\lambda<n$, then for any $f\in L^{2n/(2n-\lambda)}(\mathbb{R}^n)$,

$$\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{f(x)\,f(y)}{|x-y|^\lambda}\,\mathrm{d}x\,\mathrm{d}y \leqslant \mathcal{C}_{\lambda,n}\left(\int_{\mathbb{R}^n}|f|^{2n/(2n-\lambda)}\,\mathrm{d}x\right)^{(2n-\lambda)/n} \tag{5}$$

with

$$\mathcal{C}_{\lambda,n}=\pi^{\lambda/2}\frac{\Gamma(\frac{n-\lambda}{2})}{\Gamma(n+\frac{\lambda}{2})}\left(\frac{\Gamma(n)}{\Gamma(\frac{n}{2})}\right)^{1-\lambda/n}.$$

Moreover, equality in (5) holds if and only if

$$f(x) = c\left(\frac{2b}{b^2+|x-a|^2}\right)^{(2n-\lambda)/2} \tag{6}$$

for some $a \in \mathbb{R}^n$, $b > 0$ and $c \in \mathbb{R}$. The Euler–Lagrange equation of the optimization problem related to (5) reads, in a suitable normalization,

$$\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^\lambda}\,\mathrm{d}y = |f(x)|^{-2(n-\lambda)/(2n-\lambda)} f(x) \qquad \text{in } \mathbb{R}^n\,. \tag{7}$$

Lieb posed the classification of positive solutions of (7) as an open problem, which was finally solved by Chen, Li and Ou [7] and Li [17]. They showed that the only positive solutions in $L^{2n/(2n-\lambda)}_{\mathrm{loc}}(\mathbb{R}^n)$ of (7) are given by (6) with $a \in \mathbb{R}^n$, $b > 0$ and with a constant c depending only on λ and n.

Writing $|x-y|^{-\lambda}$ in (5) as a constant times $\int_{\mathbb{R}^n} |x-z|^{-(n+\lambda)/2}|z-y|^{-(n+\lambda)/2}\,\mathrm{d}z$ and recognizing $|\cdot|^{-(n+\lambda)/2}$ as a constant times the Green's function of $(-\Delta)^{(n-\lambda)/4}$, we see by duality, putting $\lambda = n-2s$, that (5) is equivalent to the sharp Sobolev inequality, namely, if $0 < s < n/2$, then for all $u \in \dot{H}^s(\mathbb{R}^n)$,

$$\|(-\Delta)^{s/2}u\|_2^2 \geqslant \mathcal{S}_{s,n}\left(\int_{\mathbb{R}^n} |u|^{2n/(n-2s)}\,\mathrm{d}x\right)^{(n-2s)/n} \tag{8}$$

with

$$\mathcal{S}_{s,n} = (4\pi)^s\,\frac{\Gamma(\frac{n+2s}{2})}{\Gamma(\frac{n-2s}{2})}\left(\frac{\Gamma(\frac{n}{2})}{\Gamma(n)}\right)^{2s/n} = \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(\frac{n-2s}{2})}\,|\mathbb{S}^n|^{2s/n}\,.$$

Moreover, equality holds if and only if

$$u(x) = c\left(\frac{2b}{b^2+|x-a|^2}\right)^{(n-2s)/2}\,.$$

By integrating the Euler–Lagrange equation corresponding to (8) against $|x-y|^{-(n-2s)}$, we obtain (7) with f replaced by a multiple of $|u|^{4s/(n-2s)}u$. This leads to a classification of all positive solutions in $\dot{H}^s(\mathbb{R}^n)$ of the corresponding Euler–Lagrange equation [7].

A crucial step in Lieb's proof of the sharp inequality (5) and the classification of its optimizers was the observation that it is equivalent to the following sharp inequality on $\mathbb{S}^n$,

$$\iint_{\mathbb{S}^n\times\mathbb{S}^n} \frac{g(\omega)\,g(\eta)}{|\omega-\eta|^\lambda}\,\mathrm{d}\omega\,\mathrm{d}\eta \leqslant \mathcal{C}_{\lambda,n}\left(\int_{\mathbb{S}^n} |g|^{2n/(2n-\lambda)}\,\mathrm{d}\omega\right)^{(2n-\lambda)/n}\,. \tag{9}$$

In fact, each side of (9) equals the corresponding side in (5) if

$$f(x) = \left(\frac{2}{1+|x|^2}\right)^{(2n-\lambda)/2} g(\mathcal{S}(x))\,,$$

where $\mathcal{S} : \mathbb{R}^n \to \mathbb{S}^n$ is the inverse stereographic projection; see (22) below. This transformation yields also a characterization of optimizers and of positive solutions of the Euler–Lagrange equation corresponding to (9). The functions f in (6) become

$$g(x) = c \left(\frac{\sqrt{1-|\zeta|^2}}{1-\zeta\cdot\omega} \right)^{(2n-\lambda)/2} \tag{10}$$

with $\zeta \in \mathbb{R}^{n+1}$ such that $|\zeta| < 1$. More explicitly, there is a bijection between such ζ and parameters $a \in \mathbb{R}^n$, $b > 0$ in (6) given by $\zeta = (2\eta - b^2(1+\eta_{n+1})e_{n+1})/(2+b^2(1+\eta_{n+1}))$ with $\eta = \mathcal{S}(a)$.

Beckner [2, Eq. (19)] observed that, in the same sense as (8) is the dual of (5), the dual of (9) is

$$\left\| A_{2s}^{1/2} v \right\|_2^2 \geqslant \mathcal{S}_{s,n} \|v\|_q^2 \tag{11}$$

with

$$A_{2s} = \frac{\Gamma(B+\frac{1}{2}+s)}{\Gamma(B+\frac{1}{2}-s)} \qquad \text{and} \qquad B = \sqrt{-\Delta_{\mathbb{S}^n} + \tfrac{(n-1)^2}{4}}\,. \tag{12}$$

The operators A_{2s} are special cases of the GJMS operators in conformal geometry [14]. The duality between (9) and (11) and the known results about the former yield a characterization of optimizers and of positive solutions of the Euler–Lagrange equation corresponding to (11).

The relation between these inequalities and classification results and the problem studied in this paper is as follows. Inequality (11) becomes an equality as $s \to 0$. Differentiating at $s = 0$, Beckner [1] obtained the inequality

$$\Big(v, \big(\psi(B+\tfrac{1}{2}) - \psi(\tfrac{n}{2})\big)\, v \Big) \geqslant \frac{1}{n} \int_{\mathbb{S}^n} |v(\omega)|^2 \ln \frac{|v(\omega)|^2 |\mathbb{S}^n|}{\|v\|_2^2}\, \mathrm{d}\omega\,,$$

where $\psi = \Gamma'/\Gamma$ is the digamma function. Using the Funk–Hecke formula one can show that

$$\Big(v, \big(\psi(B+\tfrac{1}{2}) - \psi(\tfrac{n}{2})\big)\, v \Big) = \frac{1}{n\, C_n} \iint_{\mathbb{S}^n\times\mathbb{S}^n} \frac{|v(\omega)-v(\eta)|^2}{|\omega-\eta|^n}\, \mathrm{d}\omega\, \mathrm{d}\eta\,, \tag{13}$$

which yields (1). Alternatively, one can subtract

$$\int_{\mathbb{S}^n} \frac{d\omega}{|\omega - e|^\lambda} \|g\|_2^2$$

(with $e \in \mathbb{S}^n$ arbitrary) from the left side of (9) and pass to the limit $\lambda \to n$. From the characterization of optimizers in (9) or (11) (or by a simple computation), one finds that the functions in (3) are optimizers in (1). Because of the limiting argument, however, uniqueness of these optimizers requires a separate argument [3].

Similarly, characterization of the solutions of the Euler–Lagrange equations corresponding to (9) or (11) does not yield the characterization of solutions of the limiting equation (4). This is what we achieve in the present paper.

1.3. **Notation.** For $u, v \in \mathcal{D}$, we put

$$\mathcal{E}[u,v] := \frac{1}{2}\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{(u(\xi)-u(\eta))(v(\xi)-v(\eta))}{|\xi-\eta|^n}\,\mathrm{d}\xi\,\mathrm{d}\eta\,.$$

Moreover, if u is sufficiently regular (for instance, Dini continuous), then we introduce

$$Hu(\xi) := P.V.\int_{\mathbb{S}^n} \frac{u(\xi)-u(\eta)}{|\xi-\eta|^n}\,\mathrm{d}\eta\,. \tag{14}$$

Note that in this case, for any $v \in \mathcal{D}$,

$$\int_{\mathbb{S}^n} v(\xi)(Hu)(\xi)\,\mathrm{d}\xi = \mathcal{E}[v,u]\,.$$

2. Preliminaries

In this section we prove conformal invariance of equation (4). Moreover, we introduce the necessary notation for the conformal maps which our argument relies on, namely inversion and reflection on $\mathbb{R}^n$ and stereographic projection from $\mathbb{S}^n$ to $\mathbb{R}^n$.

2.1. **Conformal invariance.** For a general conformal map $\Phi : X \to Y$ with determinant $J_\Phi(x) := |\det D\Phi(x)|$ and a function $u \in L^2(Y)$, we define the pullback of u under Φ by

$$u_\Phi(x) := J_\Phi(x)^{1/2} u(\Phi(x)), \qquad x \in X. \tag{15}$$

This definition is chosen so that $\|u_\Phi\|_{L^2(X)} = \|u\|_{L^2(Y)}$.

The following lemma shows that equation (4) is conformally invariant. This is crucial for our approach.

Lemma 2. *Let $u, v \in \mathcal{D}$ and let Φ be a conformal map on $\mathbb{S}^n$. Then $u_\Phi, v_\Phi \in \mathcal{D}$ and we have*

$$\mathcal{E}[u_\Phi, v_\Phi] = \mathcal{E}[u,v] + C_n \int_{\mathbb{S}^n} uv \ln J_{\Phi^{-1}}^{-1/2}\,\mathrm{d}\xi \tag{16}$$

and, in particular, in the weak sense,

$$H(u_\Phi) = (Hu)_\Phi + C_n u_\Phi \ln J_\Phi^{\frac{1}{2}}\,. \tag{17}$$

Moreover, if u is a weak solution to (4), then so is u_Φ.

To avoid confusion, we emphasize that in the second term on the right side of (16), Φ^{-1} denotes the inverse of the map Φ, while $J_{\Phi^{-1}}^{-1/2}$ denotes $1/\sqrt{J_{\Phi^{-1}}}$.

Proof. Step 1. For $s > 0$, denote by $\mathcal{P}_{2s}$ the operator given by

$$\mathcal{P}_{2s}u(\xi) := \frac{\Gamma(\frac{n-2s}{2})}{2^{2s}\pi^{\frac{n}{2}}\Gamma(s)} \int_{\mathbb{S}^n} \frac{u(\eta)}{|\xi-\eta|^{n-2s}}\,\mathrm{d}\eta. \tag{18}$$

This operator fulfills the conformal invariance property

$$\mathcal{P}_{2s}\left(J_\Phi^{\frac{n+2s}{2n}} u\circ\Phi\right) = J_\Phi^{\frac{n-2s}{2n}}(\mathcal{P}_{2s}u)\circ\Phi. \tag{19}$$

Indeed, this follows from a straightforward change of variables together with the transformation rule

$$J_\Phi(\xi)^{\frac{1}{n}}|\xi-\eta|^2 J_\Phi(\eta)^{\frac{1}{n}} = |\Phi(\xi)-\Phi(\eta)|^2, \tag{20}$$

which holds because Φ is conformal.

We next give the action of $\mathcal{P}_{2s}$ on spherical harmonics. We denote by $(Y_{l,m})$ an orthonormal basis of $L^2(\mathbb{S}^n)$ composed of real spherical harmonics. The index l runs through $\mathbb{N}_0$ and denotes the degree of the spherical harmonic. The index m runs through a certain index set of cardinality depending on l and labels the degeneracy of spherical harmonics of degree l.

By the Funk-Hecke formula (see [2, Eq. (17)] and also [12, Corollary 4.3]) we have

$$\mathcal{P}_{2s}Y_{l,m} = \frac{\Gamma(l+n/2-s)}{\Gamma(l+n/2+s)}Y_{l,m}.$$

Expanding $u \in L^2(\mathbb{S}^n)$ in terms of the spherical harmonics,

$$u = \sum_{l,m} u_{l,m}Y_{l,m} \qquad \text{with} \qquad u_{l,m} = \int_{\mathbb{S}^n} uY_{l,m}\,\mathrm{d}\eta, \tag{21}$$

by the Funk-Hecke formula (see [12, Corollary 4.3]) we have the representation

$$\mathcal{P}_{2s}u = \sum_{l,m} u_{l,m}\mathcal{P}_{2s}Y_{l,m} = \sum_{l,m} u_{l,m}\frac{\Gamma(l+n/2-s)}{\Gamma(l+n/2+s)}Y_{l,m}\,.$$

In passing, we note that the right side is equal to $A_{2s}^{-1}u$ with the operator A_{2s} from (12).

We denote by $\mathcal{F}$ the space of functions u on $\mathbb{S}^n$ such that only finitely many coefficients $u_{l,m}$ in (21) are nonzero. All the above computations are justified, in particular, for such functions. Moreover, acting on such functions one has $\lim_{s\to 0}\mathcal{P}_{2s} = 1 =: \mathcal{P}_0$.

Step 2. We now prove (17) for $u \in \mathcal{F}$. For such u, we may differentiate the identity (19) with respect to s at $s = 0$. We note that

$$\mathcal{P}_0 = 1\,, \qquad \dot{\mathcal{P}}_0 = -\frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}}H - \frac{\dot{\Gamma}(\frac{n}{2})}{\Gamma(\frac{n}{2})}\,.$$

(Here and in the following, we use a dot to denote the derivative with respect to s.) The latter follows from the identity

$$Hu(\xi) = \lim_{s\to 0} \int_{\mathbb{S}^n} \frac{u(\xi)-u(\eta)}{|\xi-\eta|^{n-2s}}\,\mathrm{d}\eta = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \lim_{s\to 0} \frac{1}{2s}\left(\frac{\Gamma(\frac{n}{2}-s)}{\Gamma(\frac{n}{2}+s)} - \mathcal{P}_{2s}\right) u$$

by expanding the quotient of gamma functions.

For the left side of (19), we obtain

$$\frac{d}{ds}\mathcal{P}_{2s}\left(J_\Phi^{\frac{n+2s}{2n}}\, u\circ\Phi\right) = 2\,\dot{\mathcal{P}}_{2s}\left(J_\Phi^{\frac{n+2s}{2n}}\, u\circ\Phi\right) + \frac{1}{n}\mathcal{P}_{2s}\left(J_\Phi^{\frac{n+2s}{2n}}\, \ln J_\Phi\; u\circ\Phi\right)$$

and therefore, at $s=0$,

$$\frac{d}{ds}|_{s=0}\mathcal{P}_{2s}\left(J_\Phi^{\frac{n+2s}{2n}}\, u\circ\Phi\right) = -2\left(\frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}}H + \frac{\dot\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2})}\right)\left(J_\Phi^{\frac{1}{2}} u\circ\Phi\right) + \frac{1}{n}\left(J_\Phi^{\frac{1}{2}}\ln J_\Phi\; u\circ\Phi\right).$$

For the right side of (19), we obtain

$$\frac{d}{ds} J_\Phi^{\frac{n-2s}{2n}}\,(\mathcal{P}_{2s}u)\circ\Phi = -\frac{1}{n} J_\Phi^{\frac{n-2s}{2n}}\,\ln J_\Phi\,(\mathcal{P}_{2s}u)\circ\Phi + 2\,J_\Phi^{\frac{n-2s}{2n}}\,\left(\dot{\mathcal{P}}_{2s}u\right)\circ\Phi$$

and therefore, at $s=0$,

$$\frac{d}{ds}|_{s=0} J_\Phi^{\frac{n-2s}{2n}}\,(\mathcal{P}_{2s}u)\circ\Phi = -\frac{1}{n} J_\Phi^{\frac{1}{2}}\ln J_\Phi\; u\circ\Phi - 2\,J_\Phi^{\frac{1}{2}}\left(\left(\frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}}H + \frac{\dot\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2})}\right)u\right)\circ\Phi\,.$$

Combining these two identities and recalling $u_\Phi = J_\Phi^{1/2} u\circ\Phi$, we arrive at equation (17), understood in a pointwise sense.

Now let $v\in\mathcal{F}$, multiply (17) by v_Φ and integrate over $\mathbb{S}^n$. After a change of variables $\xi\mapsto\Phi(\xi)$ on the right hand side, we obtain the desired identity (16) for all $u,v\in\mathcal{F}$.

Step 3. We now remove the apriori assumption $u,v\in\mathcal{F}$.

From the representation (13) we see that $\mathcal{F}\subset\mathcal{D}$ is dense in $\mathcal{D}$ with respect to the norm $\sqrt{\mathcal{E}[u,u]+\|u\|_2^2}$.

We need to show that the second term on the right side of (16) is harmless. By the classification of conformal maps of $\mathbb{S}^n$, we know that $J_{\Phi^{-1}}(\xi) = (\sqrt{1-|\zeta|^2}/(1-\zeta\cdot\xi))^n$ for some $\zeta\in\mathbb{R}^{n+1}$ with $|\zeta|<1$. Thus,

$$J_{\Phi^{-1}}(\xi) \geqslant \left(\frac{\sqrt{1-|\zeta|^2}}{1+|\zeta|}\right)^n = \left(\frac{1-|\zeta|}{1+|\zeta|}\right)^{n/2}.$$

This implies

$$C_n\int_{\mathbb{S}^n} u^2\ln J_{\Phi^{-1}}^{-1/2}\,\mathrm{d}\xi \leqslant \frac{n\,C_n}{4}\left(\ln\frac{1+|\zeta|}{1-|\zeta|}\right)\|u\|_2^2 = C_\Phi\,\|u\|_2^2$$

and

$$\mathcal{E}[u_\Phi,u_\Phi] \leqslant \mathcal{E}[u,u] + C_\Phi\,\|u\|_2^2\,.$$

This bound, together with a standard approximation argument, shows that $u_\Phi \in \mathcal{D}$ whenever $u \in \mathcal{D}$ and that (16) holds for every $u, v \in \mathcal{D}$.

Step 4. To obtain the statement on conformal invariance of equation (4), let $\varphi \in \mathcal{D}$ and set $v := \varphi_{\Phi^{-1}}$. Then we compute, using (16),

$$\begin{aligned}\mathcal{E}[\varphi, u_\Phi] = \mathcal{E}[v_\Phi, u_\Phi] &= \mathcal{E}[v, u] + C_n \int_{\mathbb{S}^n} vu \ln J_{\Phi^{-1}}^{-1/2} \, \mathrm{d}\xi \\ &= C_n \int_{\mathbb{S}^n} vu \ln u \, \mathrm{d}\xi + C_n \int_{\mathbb{S}^n} vu \ln J_{\Phi^{-1}}^{-1/2} \, \mathrm{d}\xi = C_n \int_{\mathbb{S}^n} \varphi u_\Phi \ln u_\Phi \, \mathrm{d}\xi.\end{aligned}$$

This finishes the proof. □

2.2. **Some conformal maps.** Our argument will make use of certain one-parameter families of conformal transformations of $\mathbb{S}^n$. It is natural to define these maps on $\mathbb{R}^n$ and then to lift them to the sphere via a stereographic projection. The families in questions are inversions in spheres (with fixed center and varying radii) and reflections in hyperplanes (with fixed normal and varying positions).

Let us set up our notation. On $\mathbb{R}^n$, the inversion about the sphere $\partial B_\lambda(x_0)$ with center $x_0 \in \mathbb{R}^n$ and radius $\lambda > 0$ is given by

$$I_{\lambda,x_0} : \mathbb{R}^n \backslash \{x_0\} \to \mathbb{R}^n \backslash \{x_0\}\,, \qquad I_{\lambda,x_0}(x) = \frac{\lambda^2(x - x_0)}{|x - x_0|^2} + x_0\,.$$

Similarly, if $H_{\alpha,e} := \{x \in \mathbb{R}^n \,:\, x \cdot e > \alpha\}$ denotes the halfspace with normal $e \in \mathbb{S}^{n-1}$ and position $\alpha \in \mathbb{R}$, the reflection about the hyperplane $\partial H_{\alpha,e}$ is given by

$$R_{\alpha,e} : \mathbb{R}^n \to \mathbb{R}^n\,, \qquad R_{\alpha,e}(x) := x + 2(\alpha - x \cdot e)e\,.$$

The inverse stereographic projection $\mathcal{S} : \mathbb{R}^n \to \mathbb{S}^n \backslash \{S\}$, where $S = -e_{n+1}$ denotes the southpole, is given by

$$(\mathcal{S}(x))_i = \frac{2x_i}{1 + |x|^2}, \quad i = 1, ..., n, \quad (\mathcal{S}(x))_{n+1} = \frac{1 - |x|^2}{1 + |x|^2}. \tag{22}$$

Correspondingly, the stereographic projection is given by $\mathcal{S}^{-1} : \mathbb{S}^n \backslash \{S\} \to \mathbb{R}^n$,

$$(\mathcal{S}^{-1}(\xi))_i = \frac{\xi_i}{1 + \xi_{n+1}}, \quad i = 1, ..., n.$$

Using stereographic projection, we now lift the inversions and reflections to $\mathbb{S}^n$. For any $\lambda > 0$ and $\xi_0 \in \mathbb{S}^n \backslash \{S\}$ we set

$$\Phi_{\lambda,\xi_0} := \mathcal{S} \circ I_{\lambda,x_0} \circ \mathcal{S}^{-1} : \mathbb{S}^n \backslash \{\xi_0, S\} \to \mathbb{S}^n \backslash \{\xi_0, S\}. \tag{23}$$

Here and in the following, the relation $\mathcal{S}(x_0) = \xi_0$ is understood. The map Φ_{λ,ξ_0} is conformal, being a composition of conformal maps. We abbreviate

$$J_{\lambda,\xi_0}(\eta) := |\det D\Phi_{\lambda,\xi_0}(\eta)| \qquad \text{and} \qquad \Sigma_{\lambda,\xi_0} := \mathcal{S}(B_\lambda(x_0))\,.$$

Similarly, for any $\alpha \in \mathbb{R}$ and $e \in \mathbb{S}^{n-1}$ we set

$$\Psi_{\alpha,e} := \mathcal{S} \circ R_{\alpha,e} \circ \mathcal{S}^{-1} : \mathbb{S}^n \backslash \{S\} \to \mathbb{S}^n \backslash \{S\}. \tag{24}$$

We abbreviate

$$J_{\alpha,e}(\eta) := |\det D\Psi_{\alpha,e}(\eta)|\,.$$

To motivate the following lemma, we recall that when applying the method of moving planes in $\mathbb{R}^n$ in integral form, the following inequality is fundamental,

$$|x-y| < |x - R_{0,e}(y)| \qquad \text{for all } x, y \in H_{0,e}\,. \tag{25}$$

The following lemma yields the corresponding inequalities for the lifted maps Φ_{λ,ξ_0} and $\Psi_{\alpha,e}$ on $\mathbb{S}^n$.

Lemma 3. (i) *Let $\lambda > 0$ and let $\xi_0 \in \mathbb{S}^n \backslash \{S\}$. Then*

$$\frac{|J_{\lambda,\xi_0}(\eta)|^{1/2}}{|\xi - \Phi_{\lambda,\xi_0}(\eta)|^n} - \frac{1}{|\xi-\eta|^n} < 0 \qquad \textit{for all } \xi, \eta \in \Sigma_{\lambda,\xi_0} \textit{ with } \xi \neq \eta. \tag{26}$$

(ii) *Let $\alpha \in \mathbb{R}$ and let $e \in \mathbb{S}^{n-1}$. Then*

$$\frac{|J_{\alpha,e}(\eta)|^{1/2}}{|\xi - \Psi_{\alpha,e}(\eta)|^n} - \frac{1}{|\xi-\eta|^n} < 0 \qquad \textit{for all } \xi, \eta \in \mathcal{S}(H_{\alpha,e}) \textit{ with } \xi \neq \eta.$$

Proof. Inequality (26) is equivalent to the inequality

$$|\xi - \eta| < |\xi - \Phi_{\lambda,\xi_0}(\eta)| |J_{\lambda,\xi_0}(\eta)|^{-1/2n} \qquad \text{for all } \xi, \eta \in \Sigma_{\lambda,\xi_0}. \tag{27}$$

Our strategy is to deduce this inequality from (25) (with $e = e_n$, say). We observe that there is a conformal map $\mathcal{B}_{\lambda,x_0} : \mathbb{R}^n \backslash \{x_0, x_0 - \lambda e_n\} \to \mathbb{R}^n \backslash \{x_0, x_0 - \lambda e_n\}$, which maps the ball $B_\lambda(x_0)$ to the half-space H_{0,e_n} and which is such that $I_{\lambda,x_0} = \mathcal{B}_{\lambda,x_0}^{-1} \circ R_{0,e_n} \circ \mathcal{B}_{\lambda,x_0}$. (See [11, Section 2.2] for details; the map $\mathcal{B}_{\lambda,x_0}$ is the map $\mathcal{B}$ given there, composed with a dilation and a translation which map $B_\lambda(x_0)$ to $B_1(0)$.)

Therefore, letting $\mathcal{T} = \mathcal{T}_{\lambda,x_0} = \mathcal{B}_{\lambda,x_0} \circ \mathcal{S}^{-1}$, we can write $\Phi_{\lambda,\xi_0} = \mathcal{T}^{-1} \circ R_{0,e_n} \circ \mathcal{T}$. If we use the formula $|\mathcal{T}(\xi) - \mathcal{T}(\eta)| = |\det D\mathcal{T}(\xi)|^{1/2n} |\det D\mathcal{T}(\eta)|^{1/2n} |\xi - \eta|$ (valid for any conformal map, see (20)) and the chain rule for the determinantal factors, we get that (27) is equivalent to

$$|\mathcal{T}(\xi) - \mathcal{T}(\eta)| < |\mathcal{T}(\xi) - \mathcal{T}(\Phi_{\lambda,x_0}(\eta))| |\det D(\mathcal{T} \circ \Phi_{\lambda,x_0} \circ \mathcal{T}^{-1})(\mathcal{T}(\eta))|^{\frac{1}{2n}} \text{ for all } \xi, \eta \in \Sigma_{\lambda,\xi_0}.$$

Setting $x = \mathcal{T}(\xi)$ and $y = \mathcal{T}(\eta)$ and observing that $\mathcal{T}(\Sigma_{\lambda,\xi_0}) = H_{0,e_n}$, this simplifies to

$$|x-y| < |x - R_{0,e_n}(y)| |\det DR_{0,e_n}(y)|^{1/2n} \qquad \text{for all } x, y \in H_{0,e_n},$$

which is just (25) because $|\det DR_{0,e_n}(y)| = 1$ for all $y \in \mathbb{R}^n$.

The proof of (ii) is similar, but simpler (instead of $\mathcal{T}$ one can take simply $\mathcal{S}^{-1}$) and we omit it. □

3. Maximum principles for antisymmetric functions

This section serves as a preparation for the moving spheres argument carried out in Section 4 below. Here we will derive two maximum principles which will be the technical heart of the moving spheres method. An important point, which makes them well suited for the moving spheres application, is that both Lemmas 4 and 5 only hold for antisymmetric functions. Here, with the notation introduced in Section 2, we call a function w on $\mathbb{S}^n$ *antisymmetric* (with respect to the conformal maps Φ_{λ,ξ_0} resp. $\Psi_{\alpha,e}$) if

$$w(\eta) = -w_{\Phi_{\lambda,\xi_0}}(\eta) \qquad \text{for a.e.} \quad \eta \in \Sigma_{\lambda,\xi_0},$$

respectively,

$$w(\eta) = -w_{\Psi_{\alpha,e}}(\eta) \qquad \text{for a.e.} \quad \eta \in \mathcal{S}(H_{\alpha,e}).$$

The use of maximum principles is fundamental in the method of moving planes and the role of antisymmetry in these maximum principles becomes particularly important when applied to nonlocal equations. Antisymmetric maximum principles are implicit, among others, in [7, 17, 21, 10, 16] and were made explicit in [15, 6, 8]. A particular feature of our result is that we deal with weak solutions without further regularity assumptions and with very weak conditions on the potential. This makes, for instance, the proof of the strong maximum principle from Lemma 5 below considerably more involved than its counterpart in [6].

For clarity of exposition, we will state and prove the lemmas in this section only for functions antisymmetric with respect to Φ_{λ,ξ_0}. We leave it to the reader to check that their statements and proofs remain valid when Φ_{λ,ξ_0} and Σ_{λ,ξ_0} are replaced by $\Psi_{\alpha,e}$ and $\mathcal{S}(H_{\alpha,e})$.

The first lemma states a maximum principle which is valid for sets of sufficiently small volume. We recall that the constant C_n was defined in (2).

Lemma 4 (Small volume maximum principle)**.** *Let $\lambda > 0$ and $\xi_0 \in \mathbb{S}^n \backslash \{S\}$, let $\Omega \subset \Sigma_{\lambda,\xi_0}$ be measurable and let $V : \Omega \to \mathbb{R}$ be a measurable function with*

$$\int_\Omega e^{2V_-/C_n} < |\mathbb{S}^n| \,.$$

If $w \in \mathcal{D}$ is antisymmetric with respect to Σ_{λ,ξ_0} and satisfies

$$\mathcal{E}[\varphi, w] + \int_\Omega \varphi V w \geqslant 0 \quad \textit{for any } 0 \leqslant \varphi \in \mathcal{D} \textit{ with } \varphi = 0 \textit{ on } \Omega^c \tag{28}$$

and

$$w \geqslant 0 \quad \textit{a.e. on } \Sigma_{\lambda,\xi_0} \backslash \Omega, \tag{29}$$

then $w \geqslant 0$ a.e. on Ω.

We prove Lemma 4 in Section 3.1 below.

The second lemma gives a strong maximum principle.

Lemma 5 (Strong maximum principle)**.** *Let $\lambda > 0$ and $\xi_0 \in \mathbb{S}^n \backslash \{S\}$ and let $V : \Sigma_{\lambda,\xi_0} \to \mathbb{R}$ be a measurable function. If $w \in \mathcal{D}$ is antisymmetric with respect to Σ_{λ,ξ_0}, satisfies $V_+ \min\{w, 1\} \in L^1_{\mathrm{loc}}(\Sigma_{\lambda,\xi_0})$, as well as*

$$\mathcal{E}[\varphi, w] + \int_{\Sigma_{\lambda,\xi_0}} \varphi V w \geqslant 0 \qquad \text{for all } 0 \leqslant \varphi \in \mathcal{D} \text{ with } \varphi = 0 \text{ on } (\Sigma_{\lambda,\xi_0})^c \tag{30}$$

and

$$w \geqslant 0 \quad \text{a.e. on } \Sigma_{\lambda,\xi_0}, \tag{31}$$

then either $w \equiv 0$ on $\mathbb{S}^n$ or $w > 0$ a.e. on Σ_{λ,ξ_0}.

We prove Lemma 5 in Section 3.3 below.

3.1. **Proof of Lemma 4.** To prove Lemma 4, we use the following inequality. It is well known, but we include a proof for the sake of completeness.

Lemma 6. *Let f and g be measurable functions on a measured space (X, μ) such that $f \geqslant 0$ and $\int_X f \,\mathrm{d}\mu = 1$. Assume that $\int_X f g_- \,\mathrm{d}\mu < \infty$. Then*

$$\int_X f g \,\mathrm{d}\mu \leqslant \int_X f \ln f \,\mathrm{d}\mu + \ln \int_X e^g \,\mathrm{d}\mu \,.$$

Proof. From the elementary inequality $e^\alpha \geqslant 1 + \alpha$ for all $\alpha \in \mathbb{R}$, we obtain

$$e^{g - \ln f - \int_X f(g - \ln f)\,\mathrm{d}\mu} \geqslant 1 + g - \ln f - \int_X f(g - \ln f)\,\mathrm{d}\mu \,.$$

Multiplying this inequality by f and integrating we obtain

$$\int_X e^{g - \int_X f(g - \ln f)\,\mathrm{d}\mu} \,\mathrm{d}\mu \geqslant 1 \,,$$

which is the claimed inequality. □

Lemma 7. *Let $\lambda > 0$ and $\xi_0 \in \mathbb{S}^n \backslash \{S\}$ and let $w \in \mathcal{D}$ be antisymmetric with respect to Σ_{λ,ξ_0}. Then $v := \mathbb{1}_{\Sigma_{\lambda,\xi_0}} w_- \in \mathcal{D}$ and*

$$\mathcal{E}[v, v] \leqslant -\mathcal{E}[v, w] \,.$$

Proof. For $\epsilon > 0$ set

$$\begin{aligned} E_\epsilon := {} & \{(\xi,\eta) \in \Sigma_{\lambda,\xi_0} \times \Sigma_{\lambda,\xi_0} : |\xi - \eta| < \epsilon\} \cup \{(\xi,\eta) \in \Sigma_{\lambda,\xi_0} \times \Sigma^c_{\lambda,\xi_0} : |\xi - \Phi_{\lambda,\xi_0}(\eta)| < \epsilon\} \\ & \cup \{(\xi,\eta) \in \Sigma^c_{\lambda,\xi_0} \times \Sigma_{\lambda,\xi_0} : |\Phi_{\lambda,\xi_0}(\xi) - \eta| < \epsilon\} \\ & \cup \{(\xi,\eta) \in \Sigma^c_{\lambda,\xi_0} \times \Sigma^c_{\lambda,\xi_0} : |\Phi_{\lambda,\xi_0}(\xi) - \Phi_{\lambda,\xi_0}(\eta)| < \epsilon\} \,, \end{aligned}$$

$$k_\epsilon(\xi,\eta) := \mathbb{1}_{E^c_\epsilon}(\xi,\eta)\, |\xi - \eta|^{-n}$$

and

$$\mathcal{E}_\epsilon[f, g] := \frac{1}{2} \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} (u(\xi) - u(\eta))(v(\xi) - v(\eta)) k_\epsilon(\xi,\eta) \,\mathrm{d}\xi \,\mathrm{d}\eta \,.$$

Since k_ϵ is bounded, both $\mathcal{E}_\epsilon[v,v]$ and $\mathcal{E}_\epsilon[w,v]$ are finite and we have

$$\begin{aligned}
&\mathcal{E}_\epsilon[v,v]+\mathcal{E}_\epsilon[v,w]\\
&=\frac{1}{2}\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}\big(|v(\xi)-v(\eta)|^2+[v(\xi)-v(\eta)][w(\xi)-w(\eta)]\big)\,k_\epsilon(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&=\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}\big(v(\xi)(v(\xi)+w(\xi))-v(\xi)(v(\eta)+w(\eta))\big)\,k_\epsilon(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&=-\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}v(\xi)(v(\eta)+w(\eta))k_\epsilon(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,,
\end{aligned}$$

where we used the fact that $v(\xi)(v(\xi)+w(\xi))=0$ on $\mathbb{S}^n$.

By a change of variables and the antisymmetry of w, we have

$$\begin{aligned}
&-\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}v(\xi)(v(\eta)+w(\eta))k_\epsilon(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&=-\int_{\Sigma_{\lambda,\xi_0}}\int_{\Sigma_{\lambda,\xi_0}}w(\xi)_-w(\eta)_+k_\epsilon(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta+\int_{\Sigma^c_{\lambda,\xi_0}}\int_{\Sigma_{\lambda,\xi_0}}w(\xi)_-w(\eta)_-k_\epsilon(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&\quad-\int_{\Sigma^c_{\lambda,\xi_0}}\int_{\Sigma_{\lambda,\xi_0}}w(\xi)_-w(\eta)_+k_\epsilon(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&=-\int_{\Sigma_{\lambda,\xi_0}}\int_{\Sigma_{\lambda,\xi_0}}w(\xi)_-w(\eta)_+\Big(k_\epsilon(\xi,\eta)-J_{\lambda,\xi_0}(\eta)^{1/2}k_\epsilon(\xi,\Phi_{\lambda,\xi_0}(\eta))\Big)\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&\quad-\int_{\Sigma^c_{\lambda,\xi_0}}\int_{\Sigma_{\lambda,\xi_0}}w(\xi)_-w(\eta)_+k_\epsilon(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,.
\end{aligned}$$

The second double integral on the right side is clearly nonnegative. Moreover, it follows from Lemma 3 and the choice of E_ϵ (more precisely, the fact that $(\xi,\Phi(\eta))\in E_\epsilon$ implies $(\xi,\eta)\in E_\epsilon$) that

$$k_\epsilon(\xi,\eta)-J_{\lambda,\xi_0}(\eta)^{1/2}k_\epsilon(\xi,\Phi_{\lambda,\xi_0}(\eta))\geqslant 0\qquad\text{for all }\xi,\eta\in\Sigma_{\lambda,\xi_0}\,.$$

Therefore also the first double integral on the right side is nonnegative and we conclude that

$$\mathcal{E}_\epsilon[v,v]+\mathcal{E}_\epsilon[v,w]\leqslant 0\,. \tag{32}$$

By the Schwarz inequality, we have $-\mathcal{E}_\epsilon[v,w]\leqslant\sqrt{\mathcal{E}_\epsilon[v,v]}\sqrt{\mathcal{E}_\epsilon[w,w]}$ and therefore the previous inequality implies that

$$\mathcal{E}_\epsilon[v,v]\leqslant\mathcal{E}_\epsilon[w,w]\,.$$

Since $\mathcal{E}_\epsilon[w,w]\leqslant\mathcal{E}[w,w]<\infty$, we can let $\epsilon\to 0$ and use monotone convergence to deduce that $\mathcal{E}[v,v]<\infty$, that is, $v\in\mathcal{D}$. With this information we can return to (32) and let $\epsilon\to 0$ to obtain the inequality in the lemma. □

Now we can give the proof of the small volume maximum principle.

Proof of Lemma 4. Let λ, ξ_0, w and Ω be as in the assumptions and denote $v = \mathbb{1}_\Omega w_-$. Assumption (29) implies that $v = \mathbb{1}_{\Sigma_{\lambda,\xi_0}} w_-$ and therefore, by Lemma 7, $v \in \mathcal{D}$ and $\mathcal{E}[v,v] \leqslant \mathcal{E}[v,w]$. Combining this with assumption (28) (with $\varphi = v$), we obtain

$$\mathcal{E}[v,v] \leqslant -\mathcal{E}[v,w] \leqslant \int_\Omega vVw = -\int_\Omega Vw_-^2 \leqslant \int_\Omega V_- w_-^2 \,. \tag{33}$$

On the other hand, by Beckner's inequality (1),

$$\mathcal{E}[v,v] = \frac{1}{2}\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{|v(\xi)-v(\eta)|^2}{|\xi-\eta|^n}\,\mathrm{d}\xi\,\mathrm{d}\eta \geqslant \frac{C_n}{2}\int_\Omega w_-^2 \ln \frac{w_-^2|\mathbb{S}^n|}{\|w_-1_\Omega\|_2^2}\,\mathrm{d}\eta. \tag{34}$$

We now argue by contradiction and assume that $v \not\equiv 0$. Then we may define $\rho = \|w_-\mathbb{1}_\Omega\|_2^{-2}\mathbb{1}_\Omega w_-^2$ and rewrite inequalities (33) and (34) as

$$\int_\Omega \rho\ln\rho\,\mathrm{d}\eta + \ln|\mathbb{S}^n| \leqslant \frac{2}{C_n}\int_\Omega V_-\rho\,\mathrm{d}\eta. \tag{35}$$

Notice that $\rho \geqslant 0$ and $\int_{\mathbb{S}^n}\rho = 1$. Therefore we may apply Lemma 6 with $f = \rho$, $g = \frac{2}{C_n}V_-$ and $X = \Omega$ to (35) and deduce that

$$|\mathbb{S}^n| \leqslant \int_\Omega e^{\frac{2V_-}{C_n}}\,\mathrm{d}\eta\,.$$

This contradicts the assumption of the lemma and therefore we conclude that $v \equiv 0$, that is, $w \geqslant 0$ a. e. on Ω. □

3.2. **A general form of the strong maximum principle.** We deduce Lemma 5 from the following strong maximum principle, which holds for a general interaction kernel k and arbitrary, not necessarily antisymmetric, functions w.

More precisely, let (X,d,μ) be a metric measure space and suppose that the kernel $k : X\times X \to (0,\infty)\cup\{\infty\}$ is such that

$$\iint_{K\times X} k(x,y)d(x,y)^2\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) < \infty \qquad \text{for any compact } K\subset X\,. \tag{36}$$

For simplicity of notation, we will also assume that $k(x,y) = k(y,x)$ for all $x,y\in X$, although this is not really necessary. For a measurable function u on X, let

$$\mathcal{I}[w] := \frac{1}{2}\iint_{X\times X} k(x,y)(w(x)-w(y))^2\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)$$

and set

$$D(\mathcal{I}) := \{w : X\to\mathbb{R} :\ w \text{ measurable}\,,\ \mathcal{I}[w]<\infty\}.$$

Moreover, for $v,w\in D(\mathcal{I})$, let

$$\mathcal{I}[v,w] := \frac{1}{2}\iint_{X\times X} k(x,y)(v(x)-v(y))(w(x)-w(y))\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\,.$$

With this notation, we can state the following general strong maximum principle.

Proposition 8. *Let k satisfy* (36) *and let U be a measurable function on X. Assume that $0 \leqslant v \in D(\mathcal{I})$ satisfies $U_+ \min\{v,1\} \in L^1_{\mathrm{loc}}(X)$ and*

$$\mathcal{I}[\varphi, v] + \int_X \varphi U v \,\mathrm{d}\mu \geqslant 0 \qquad \textit{for all } 0 \leqslant \varphi \in D(\mathcal{I}) \textit{ with compact support}. \tag{37}$$

Then either $v \equiv 0$ or $v > 0$ a.e. on X.

In the proof of Lemma 5, we will use this proposition in a setting where in fact $U_+ v \in L^1_{\mathrm{loc}}(X)$. While this may at first look less natural than the assumption $U_+ \in L^1_{\mathrm{loc}}(X)$, the difference is crucial in our application, where U depends in a nonlinear fashion on v.

We begin with a technical lemma about the form domain $D(\mathcal{I})$.

Lemma 9. *Let k satisfy* (36) *and let $w \in D(\mathcal{I})$.*

(a) *If $w \geqslant 0$, then $(w+\epsilon)^{-1} \in D(\mathcal{I})$ for all $\epsilon > 0$.*

(b) *If w is bounded and ζ is Lipschitz function on X with compact support, then $\zeta w \in D(\mathcal{I})$.*

Proof of Lemma 9. To prove (a), we write

$$\left(\frac{1}{w(x)+\epsilon} - \frac{1}{w(y)+\epsilon}\right)^2 = \frac{(w(x)-w(y))^2}{(w(x)+\epsilon)^2(w(y)+\epsilon)^2} \leqslant \frac{1}{\epsilon^4}(w(x)-w(y))^2 .$$

Thus,

$$\mathcal{I}[(w+\epsilon)^{-1}] \leqslant \epsilon^{-4}\, \mathcal{I}[w] .$$

To prove (b), we first note that $\mathcal{I}[\zeta] < \infty$. Indeed, if $K := \operatorname{supp}\zeta$ and L is the Lipschitz constant of ζ, then

$$\begin{aligned}\mathcal{I}[\zeta] &\leqslant \int_K \int_X \big(\zeta(x)-\zeta(y)\big)^2 k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x) \\ &\leqslant L^2 \int_K \int_X d(x,y)^2 k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x) < \infty\end{aligned}$$

by (36). Now we bound

$$\big|\zeta(x)w(x) - \zeta(y)w(y)\big| \leqslant \|\zeta\|_\infty \big|w(x)-w(y)\big| + \|w\|_\infty \big|\zeta(x)-\zeta(y)\big|$$

and obtain

$$\mathcal{I}[\zeta w] \leqslant \|\zeta\|_\infty^2\, \mathcal{I}[w] + \|w\|_\infty^2\, \mathcal{I}[\zeta] + 2\|\zeta\|_\infty \|w\|_\infty \sqrt{I[w]\,\mathcal{I}[\zeta]} .$$

This proves the lemma. □

Proof of Proposition 8. Let ζ be a Lipschitz function on X with compact support. By the first part of Lemma 9, the function $(v+\epsilon)^{-1}$ belongs to $D(\mathcal{I})$ and is bounded, so by the second part, with ζ replaced by ζ^2, the function $\varphi = \zeta^2/(v+\epsilon)$ belongs to $D(\mathcal{I})$.

We write

$$\begin{aligned}&\big(\varphi(x)-\varphi(y)\big)\big(v(x)-v(y)\big)\\ &= \frac{-(\zeta(x)v(y)-\zeta(y)v(x))^2 + v(x)v(y)(\zeta(x)-\zeta(y))^2 + \epsilon(v(x)-v(y))(\zeta(x)^2-\zeta(y)^2)}{(v(x)+\epsilon)(v(y)+\epsilon)}.\end{aligned}$$

Using also

$$\int_X \varphi U v \leqslant \int_X U_+ \zeta^2 \frac{v}{v+\epsilon}\,,$$

we obtain from (37) that

$$\frac{1}{2}\iint_{X\times X} \frac{(\zeta(x)v(y)-\zeta(y)v(x))^2}{(v(x)+\epsilon)(v(y)+\epsilon)} k(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \leqslant \sum_{k=1}^{3} I_k(\epsilon) \tag{38}$$

with

$$\begin{aligned}I_1(\epsilon) &= \int_X U_+\zeta^2 \frac{v}{v+\epsilon}\,,\\ I_2(\epsilon) &= \frac{1}{2}\iint_{X\times X} \frac{v(x)v(y)(\zeta(x)-\zeta(y))^2}{(v(x)+\epsilon)(v(y)+\epsilon)} k(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\,,\\ I_3(\epsilon) &= \frac{1}{2}\iint_{X\times X} \frac{\epsilon(v(x)-v(y))(\zeta(x)^2-\zeta(y)^2)}{(v(x)+\epsilon)(v(y)+\epsilon)} k(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\,.\end{aligned}$$

We bound the left side of (38) from below. Setting $Z := \{v=0\}$, we have

$$\begin{aligned}&\frac{1}{2}\iint_{X\times X} \frac{(\zeta(x)v(y)-\zeta(y)v(x))^2}{(v(x)+\epsilon)(v(y)+\epsilon)} k(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\\ &\geqslant \iint_{Z^c\times Z} \frac{(\zeta(x)v(y)-\zeta(y)v(x))^2}{(v(x)+\epsilon)(v(y)+\epsilon)} k(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\\ &= \epsilon^{-1}\int_{Z^c} \frac{v(x)^2}{v(x)+\epsilon}\kappa(x)\,\mathrm{d}\mu(x)\end{aligned}$$

with

$$\kappa(x) := \int_Z \zeta(y)^2 k(x,y)\,\mathrm{d}\mu(y)\,.$$

By dominated convergence, we have

$$\lim_{\epsilon\to 0}\int_{Z^c} \frac{v(x)^2}{v(x)+\epsilon}\kappa(x)\,\mathrm{d}\mu(x) = \int_{Z^c} v(x)\kappa(x)\,\mathrm{d}\mu(x)\,,$$

and therefore, by (38),

$$\int_{Z^c} v(x)\kappa(x)\,\mathrm{d}\mu(x) \leqslant \liminf_{\epsilon\to 0} \sum_{k=1}^{3} \epsilon I_k(\epsilon)\,. \tag{39}$$

Let us show that $\lim_{\epsilon\to 0} \epsilon I_k(\epsilon) = 0$ for $k = 1, 2, 3$. We write the integrand of $\epsilon I_1(\epsilon)$ as

$$U_+\zeta^2 \min\{v, 1\} \left(\frac{\epsilon}{v+\epsilon} \mathbb{1}_{\{0<v<1\}} + \epsilon \frac{v}{v+\epsilon} \mathbb{1}_{\{v\geqslant 1\}} \right).$$

By assumption, the product in front of the parentheses is integrable. The factor in parentheses is $\leqslant 1$ if $\epsilon \leqslant 1$ and tends to zero pointwise. Therefore, by dominated convergence, $\epsilon I_1(\epsilon) \to 0$. Moreover, we can simply bound $I_2(\epsilon) \leqslant \mathcal{I}[\zeta]$, which is finite as shown in the proof of Lemma 9. Thus, $\epsilon I_2(\epsilon) \to 0$. Finally, the integrand of $\epsilon I_3(\epsilon)$ is bounded, in absolute value, by

$$2\|\zeta\|_\infty |v(x) - v(y)||\zeta(x) - \zeta(y)|k(x, y)\,,$$

which is integrable. Moreover, this integrand tends pointwise to zero. Thus, by dominated convergence, $\epsilon I_3(\epsilon) \to 0$.

Returning to (39), we infer that

$$\int_{Z^c} v(x)\kappa(x)\,\mathrm{d}\mu(x) = 0\,. \tag{40}$$

Assume now that Z has positive measure. Then we can choose the function ζ in such a way that $\zeta^2 \mathbb{1}_Z$ is not identically zero. Then, since $k > 0$ on $X \times X$, we have $\kappa > 0$ on X. Thus, by (40), $|Z^c| = 0$, that is, $v \equiv 0$. This completes the proof. □

Remark 10. *There is also a global version of Proposition 8. Namely, the same conclusion holds without an underlying metric and without assumption* (36)*, provided one has the global integrability* $U_+ \min\{v, 1\} \in L^1(X)$ *and the compact support condition in* (37) *is dropped. This follows by the same proof with* $\zeta \equiv 1$.

3.3. **Proof of Lemma 5.** It remains to reduce Lemma 5 to the general maximum principle from Proposition 8 from the previous subsection. To do so, we use antisymmetry of w to express the quadratic form $\mathcal{E}[\varphi, w]$ as a double integral over the region Σ_{λ,ξ_0}, plus a multiplicative term. We drop in the following the subscript from Σ_{λ,ξ_0} and Φ_{λ,ξ_0} to ease notation. Moreover, we set

$$l(\xi, \eta) := \frac{1}{|\xi - \eta|^n} - \frac{J_\Phi^{1/2}(\eta)}{|\xi - \Phi(\eta)|^n}.$$

Notice that $l(\xi, \eta) > 0$ for every $\xi, \eta \in \Sigma$ by Lemma 3. Moreover, by (20), we have $l(\xi, \eta) = l(\eta, \xi)$ for all $\xi, \eta \in \Sigma$. For functions u, v on Σ, we define the quadratic

form

$$\tilde{\mathcal{E}}[u,v] := \frac{1}{2} \iint\limits_{\Sigma\times\Sigma} l(\xi,\eta)(u(\xi)-u(\eta))(v(\xi)-v(\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{41}$$

on the domain

$$\tilde{\mathcal{D}} := \{u \in L^2(\Sigma) \,:\, \tilde{\mathcal{E}}[u,u] < \infty\}.$$

Lemma 11. *Let $w \in \mathcal{D}$ be antisymmetric with respect to Φ and let $\varphi \in \mathcal{D}$ with $\varphi = 0$ on Σ^c. Then*

$$\mathcal{E}[\varphi, w] = \tilde{\mathcal{E}}[\varphi, w|_\Sigma] + \int_\Sigma \varphi(\xi)\tilde{V}(\xi)w(\xi)\,\mathrm{d}\xi,$$

with

$$\tilde{V}(\xi) = \int_\Sigma \frac{J_\Phi(\eta)^{1/2}}{|\xi-\Phi(\eta)|^n}(1+J_\Phi(\eta)^{1/2})\,\mathrm{d}\eta.$$

Proof. We write

$$\mathcal{E}[w,\varphi] = \frac{1}{2}\iint\limits_{\Sigma\times\Sigma} \frac{(\varphi(\xi)-\varphi(\eta))(w(\xi)-w(\eta))}{|\xi-\eta|^n}\,\mathrm{d}\xi\,\mathrm{d}\eta + \iint\limits_{\Sigma\times\Sigma^c} \frac{\varphi(\xi)(w(\xi)-w(\eta))}{|\xi-\eta|^n}\,\mathrm{d}\xi\,\mathrm{d}\eta\,. \tag{42}$$

The second integral on the right side is a sum of two terms, corresponding to $w(\xi)$ and $w(\eta)$, respectively. Since

$$\int_{\Sigma^c} \frac{\mathrm{d}\eta}{|\xi-\eta|^n} = \int_\Sigma \frac{J_\Phi(\eta)\,\mathrm{d}\eta}{|\xi-\Phi(\eta)|^n}\,,$$

the first term becomes

$$\iint\limits_{\Sigma\times\Sigma^c} \frac{\varphi(\xi)w(\xi)}{|\xi-\eta|^n}\,\mathrm{d}\xi\,\mathrm{d}\eta = \int_\Sigma \varphi(\xi)w(\xi)\left(\int_\Sigma \frac{J_\Phi(\eta)\,\mathrm{d}\eta}{|\xi-\Phi(\eta)|^n}\right)\mathrm{d}\xi\,.$$

This is one contribution of the $\tilde{V}$ term.

Let us discuss the second contribution coming from the second integral on the right side of (42). By antisymmetry and a change of variables, we have

$$\int_{\Sigma^c} \frac{w(\eta)}{|\xi-\eta|^n}\,\mathrm{d}\eta = -\int_\Sigma \frac{J_\Phi(\eta)^{1/2}}{|\xi-\Phi(\eta)|^n} w(\eta)\,\mathrm{d}\eta\,,$$

and therefore, by symmetry,

$$\begin{aligned}\iint\limits_{\Sigma\times\Sigma^c} \frac{\varphi(\xi)w(\eta)}{|\xi-\eta|^n}\,\mathrm{d}\xi\,\mathrm{d}\eta &= -\frac{1}{2}\iint\limits_{\Sigma\times\Sigma} \frac{J_\Phi(\eta)^{1/2}}{|\xi-\Phi(\eta)|^n}\big(\varphi(\xi)w(\eta)+\varphi(\eta)w(\xi)\big)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &= \frac{1}{2}\iint\limits_{\Sigma\times\Sigma} \frac{J_\Phi(\eta)^{1/2}}{|\xi-\Phi(\eta)|^n}(\varphi(\xi)-\varphi(\eta))(w(\xi)-w(\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\quad -\frac{1}{2}\iint\limits_{\Sigma\times\Sigma} \frac{J_\Phi(\eta)^{1/2}}{|\xi-\Phi(\eta)|^n}\big(\varphi(\xi)w(\xi)+\varphi(\eta)w(\eta)\big)\,\mathrm{d}\xi\,\mathrm{d}\eta\,.\end{aligned}$$

In this expression, the first double integral combines with the first double integral on the right side of (42) to give the term $\tilde{\mathcal{E}}[\varphi, w|_\Sigma]$ in the lemma. Moreover, by symmetry the second double integral equals

$$\frac{1}{2}\iint\limits_{\Sigma\times\Sigma} \frac{J_\Phi(\eta)^{1/2}}{|\xi-\Phi(\eta)|^n}\big(\varphi(\xi)w(\xi)+\varphi(\eta)w(\eta)\big)\,\mathrm{d}\xi\,\mathrm{d}\eta = \int_\Sigma \varphi(\xi)w(\xi)\left(\int_\Sigma \frac{J_\Phi(\eta)^{1/2}\,\mathrm{d}\eta}{|\xi-\Phi(\eta)|^n}\right)\mathrm{d}\xi\,,$$

which is the remaining contribution to the $\tilde{V}$ term. Collecting all terms we arrive at the formula in the lemma. □

Proof of Lemma 5. We are going to apply Lemma 8 with $\mathcal{I} = \tilde{\mathcal{E}}$, $k = l$, $X = \Sigma_{\lambda,\xi_0}$ and $U = V + \tilde{V}$. Let us check that the assumptions of Lemma 8 are satisfied.

By Lemma 11, we have, for any $0 \leqslant \varphi \in \mathcal{D}$ with $\varphi = 0$ on $(\Sigma_{\lambda,\xi_0})^c$,

$$\tilde{\mathcal{E}}[\varphi, w] + \int_{\Sigma_{\lambda,\xi_0}} \varphi(\xi)U(\xi)w(\xi)\,\mathrm{d}\xi \geqslant 0\,. \tag{43}$$

Next, we observe that for any compact subset $C \subset \Sigma_{\lambda,\xi_0}$, there is $M > 0$ such that we have the uniform bound

$$|\xi-\Phi(\eta)|^{-n} \leqslant M \quad \text{for} \quad \xi \in C, \eta \in \Sigma_{\lambda,\xi_0}, \tag{44}$$

This has two consequences. First, if φ is compactly supported on Σ_{λ,ξ_0}, it is easy to deduce from (44) that $\varphi \in \mathcal{D}$ if and only if $\varphi \in \tilde{\mathcal{D}}$. Therefore, (43) holds for all compactly supported $\varphi \in \tilde{\mathcal{D}}$.

Second, it follows from (44) that $\tilde{V}$ is bounded on C and hence $\tilde{V}\min\{v,1\} \in L^1(C)$. Since, moreover, $V_+\min\{w,1\} \in L^1(C)$ by assumption, we have $U_+\min\{w,1\} \in L^1(C)$ for every compact $C \subset \Sigma_{\lambda,\xi_0}$.

Thus, all the assumptions of Lemma 8 are satisfied and we conclude by that lemma. □

4. Symmetry by the method of moving spheres

In this section, we prove a symmetry result for solutions of (4). We will deduce this by the method of moving spheres using the preliminaries introduced so far, in particular the maximum principles from Section 3.

The method of moving spheres is well-established on $\mathbb{R}^n$ and consists in comparing the values of a solution to some equation with its (suitably defined) inversion about a certain sphere $\partial B_\lambda(x_0) \subset \mathbb{R}^n$. Using stereographic projection, we lift this procedure to $\mathbb{S}^n$. Namely, for any solution to (4) and $\lambda > 0$, $\xi_0 \in \mathbb{S}^n \backslash \{S\}$, we will compare u on the set $\Sigma_{\lambda,\xi_0} = \mathcal{S}(B_\lambda(\mathcal{S}^{-1}(\xi_0)))$ with its reflected version $u_{\Phi_{\lambda,\xi_0}}$. Recall that the map Φ_{λ,ξ_0} has been introduced in (23) and the definition of u_Φ has been given in (15).

At the same time we need to consider the reflection of u about (stereographically projected) planes, i.e., $u_{\Psi_{\alpha,e}}$ for $e \in \mathbb{S}^{n-1}$, $\alpha \in \mathbb{R}$, with $\Psi_{\alpha,e}$ defined in (24).

The following is the main result of this section.

Theorem 12. *Let $u \geqslant 0$ be a weak solution to (4). Then the following holds.*

(i) *For every $\xi_0 \in \mathbb{S}^n \backslash \{S\}$, there is a $\lambda_0 = \lambda_0(\xi_0) > 0$ such that $u_{\Phi_{\lambda_0,\xi_0}} \equiv u$.*

(ii) *For every $e \in \mathbb{S}^{n-1}$, there is $a = a(e) \in \mathbb{R}$ such that $u_{\Psi_{\alpha,e}} \equiv u$.*

As in Section 3, since the arguments to prove parts (i) and (ii) are very similar and of comparable difficulty, for clarity of exposition we focus in the following on proving part (i) of Theorem 12. The reader is invited to check that all arguments given in the rest of the present section remain valid when Φ_{λ,ξ_0} is replaced by $\Psi_{\alpha,e}$ and therefore yield a proof of part (ii) as well.

4.1. **The moving spheres argument.** In this subsection, we fix $\xi_0 \in \mathbb{S}^n \backslash \{S\}$ and let $\lambda > 0$ vary. We abbreviate

$$u_{\lambda,\xi_0} := u_{\Phi_{\lambda,\xi_0}}.$$

We will prove Theorem 12 by analyzing the positivity of the difference

$$w_{\lambda,\xi_0} := u_{\lambda,\xi_0} - u$$

on Σ_{λ,ξ_0}. Since $\Phi^2_{\lambda,\xi_0} = \mathrm{id}_{\mathbb{S}^n \backslash \{\xi_0, S\}}$, the function w_{λ,ξ_0} is antisymmetric with respect to Φ_{λ,ξ_0}. By the conformal invariance proved in Lemma 2, both u and u_{λ,ξ_0} are weak solutions of (4) and therefore the function w_{λ,ξ_0} satisfies

$$\mathcal{E}[\varphi, w_{\lambda,\xi_0}] = \int_{\mathbb{S}^n} \varphi(\xi) h(\xi) w_{\lambda,\xi_0}(\xi)\, \mathrm{d}\xi \qquad \text{for all } \varphi \in \mathcal{D} \tag{45}$$

with

$$h(\xi) := \begin{cases} \frac{g(u_{\lambda,\xi_0}(\xi)) - g(u(\xi))}{u_{\lambda,\xi_0}(\xi) - u(\xi)} & \text{if } u_{\lambda,\xi_0}(\xi) \neq u(\xi)\,, \\ g'(u(\xi)) & \text{if } u_{\lambda,\xi_0}(\xi) = u(\xi)\,, \end{cases} \qquad \text{and} \qquad g(u) := C_n u \ln u\,.$$

Convexity of g implies that

$$h(\xi)w_{\lambda,\xi_0}(\xi) \geqslant g'(u(\xi))w_{\lambda,\xi_0}(\xi) \qquad \text{if } u_{\lambda,\xi_0}(\xi) \leqslant u(\xi)$$

and a simple computation shows that

$$h(\xi)w_{\lambda,\xi_0}(\xi) \geqslant -e^{-1} \qquad \text{if } u_{\lambda,\xi_0}(\xi) \geqslant u(\xi)\,.$$

Thus, setting

$$\Sigma^-_{\lambda,\xi_0} := \{\eta \in \Sigma_{\lambda,\xi_0} \,:\, w_{\lambda,\xi_0}(\eta) < 0\},$$

and

$$V(\xi) := -g'(u(\xi))\ \mathbb{1}_{\Sigma^-_{\lambda,\xi_0}}(\xi) + (ew_{\lambda,\xi_0}(\xi))^{-1}\mathbb{1}_{\Sigma_{\lambda,\xi_0}\setminus\Sigma^-_{\lambda,\xi_0}}(\xi)\,,$$

we have $hw_{\lambda,\xi_0} \geqslant -Vw_{\lambda,\xi_0}$ on Σ_{λ,ξ_0} and, consequently,

$$\mathcal{E}[\varphi, w_{\lambda,\xi_0}] + \langle \varphi, Vw_{\lambda,\xi_0}\rangle \geqslant 0 \qquad \text{for all } 0 \leqslant \varphi \in \mathcal{D} \text{ with } \varphi = 0 \text{ on } (\Sigma_{\lambda,\xi_0})^c\,. \tag{46}$$

The first step in the method of moving spheres is the following application of the small volume maximum principle from Lemma 4.

Lemma 13 (Starting the sphere)**.** *Let $\xi_0 \in \mathbb{S}^n\setminus\{S\}$ be fixed. Then for every $\lambda > 0$ small enough, we have $w_{\lambda,\xi_0} \geqslant 0$ a.e. on Σ_{λ,ξ_0}.*

Proof. We will apply Lemma 4 with $\Omega = \Sigma^-_{\lambda,\xi_0}$. As remarked before, w_{λ,ξ_0} is antisymmetric. Assumption (28) follows from (46) and Assumption (29) follows by definition of $\Omega = \Sigma^-_{\lambda,\xi_0}$. Finally,

$$\int_\Omega e^{2V_-/C_n} = e^2 \int_{\{u>e^{-1}\}\cap\Sigma^-_{\lambda,\xi_0}} u^2 \leqslant e^2 \int_{\Sigma^-_{\lambda,\xi_0}} u^2\,.$$

Since $\mathbb{1}_{\Sigma^-_{\lambda,\xi_0}} \to 0$ a.e. as $\lambda \to 0$ and $u \in L^2(\mathbb{S}^n)$, we deduce from dominated convergence that

$$\int_\Omega e^{2V_-/C_n} < |\mathbb{S}^n| \qquad \text{for all sufficiently small } \lambda > 0\,.$$

Thus, Lemma 4 implies that $w_{\lambda,\xi_0} \geqslant 0$ a.e. on Σ^-_{λ,ξ_0}, so $|\Sigma^-_{\lambda,\xi_0}| = 0$, which is the assertion of the lemma. □

Due to Lemma 13, the 'critical scale' associated to ξ_0,

$$\lambda_0(\xi_0) := \sup\left\{\lambda > 0 \,:\, w_{\mu,\xi_0}(\eta) \geqslant 0 \text{ for all } 0 < \mu < \lambda \text{ and almost every } \eta \in \Sigma_{\mu,\xi_0}\right\}, \tag{47}$$

is well-defined with $\lambda_0(\xi_0) \in (0, \infty]$.

Proof of Theorem 12. We recall that $\xi_0 \in \mathbb{S}^n\setminus\{S\}$ is fixed.

First, let us prove $\lambda_0(\xi_0) < \infty$ by contradiction. Assuming that $\lambda_0(\xi_0) = +\infty$, we can choose $\lambda_i > 0$ with $\lambda_i \to +\infty$ and $u_{\lambda_i,\xi_0} - u = w_{\lambda_i,\xi_0} \geqslant 0$ a.e. on Σ_{λ_i,ξ_0}. Integrating over Σ_{λ_i,ξ_0} and changing variables, we obtain

$$\int_{\Sigma_{\lambda_i,\xi_0}} u(\eta)^2 \,\mathrm{d}\eta \leqslant \int_{\Sigma_{\lambda_i,\xi_0}} J_{\lambda_i,\xi_0}(\eta) u(\Phi_{\lambda_i,\xi_0}\eta)^2 \,\mathrm{d}\eta = \int_{\mathbb{S}^n\setminus\Sigma_{\lambda_i,\xi_0}} u(\eta)^2 \,\mathrm{d}\eta\,,$$

that is,

$$\int_{\mathbb{S}^n\setminus\Sigma_{\lambda_i,\xi_0}} u(\eta)^2 \,\mathrm{d}\eta \geqslant \frac{1}{2}\int_{\mathbb{S}^n} u(\eta)^2 \,\mathrm{d}\eta\,.$$

Since $\mathbb{1}_{\mathbb{S}^n\setminus\Sigma_{\lambda,\xi_0}} \to 0$ a.e. as $\lambda \to 0$ and $u \in L^2(\mathbb{S}^n)$, dominated convergence implies that the left side tends to zero as $i \to \infty$. This contradicts the assumption $u \not\equiv 0$. Thus, we have shown that $\lambda_0 := \lambda_0(\xi_0) < \infty$.

Next, we prove that $w_{\lambda_0,\xi_0} \geqslant 0$ a.e. on Σ_{λ_0,ξ_0}. By continuity of the map $\lambda \mapsto w_{\lambda,\xi_0}$ into $L^2(\mathbb{S}^n)$, we have, up to a subsequence, that $w_{\lambda,\xi_0} \to w_{\lambda_0,\xi_0}$ a.e. on Σ_{λ_0,ξ_0} as $\lambda \nearrow \lambda_0$ from below. Consequently, by the definition of λ_0 we have $w_{\lambda_0,\xi_0} \geqslant 0$ a.e. on Σ_{λ_0,ξ_0}.

Next, we claim that either $w_{\lambda_0,\xi_0} \equiv 0$ or $w_{\lambda_0,\xi_0} > 0$ a.e. on Σ_{λ_0,ξ_0}. We will deduce this from Lemma 5. Assumption (30) follows from (46) and we have already verified assumption (31). Finally, $V_+ w_{\lambda_0,\xi_0} \leqslant e^{-1}$ is bounded. Therefore Lemma 5 is applicable and yields the claimed dichotomy.

Finally, in order to show that $w_{\lambda_0,\xi_0} \equiv 0$, we argue by contradiction and assume that $w_{\lambda_0,\xi_0} > 0$ a.e. on Σ_{λ_0,ξ_0}. Similarly as in the proof of Lemma 13 we choose $\Omega = \Sigma^-_{\lambda,\xi_0}$ and bound, for $\lambda > \lambda_0$,

$$\int_\Omega e^{2V_-/C_n} \leqslant e^2 \int_{\Sigma_{\lambda_0,\xi_0}\cap\{w_{\lambda,\xi_0}<0\}} u^2 + e^2 \int_{\Sigma_{\lambda,\xi_0}\setminus\Sigma_{\lambda_0,\xi_0}} u^2\,.$$

Since $w_{\lambda,\xi_0} \to w_{\lambda_0,\xi_0}$ a.e. on Σ_{λ_0,ξ_0} as $\lambda \to \lambda_0$ and $w_{\lambda_0,\xi_0} > 0$ a.e. on Σ_{λ_0,ξ_0}, we have $\mathbb{1}_{\{w_{\lambda,\xi_0}<0\}} \to 0$ a.e. on Σ_{λ_0,ξ_0} as $\lambda \to \lambda_0$. Moreover, clearly, $\mathbb{1}_{\Sigma_{\lambda,\xi_0}\setminus\Sigma_{\lambda_0,\xi_0}} \to 0$ a.e. as $\lambda \searrow \lambda_0$. By dominated convergence these facts, together with $u \in L^2(\mathbb{S}^n)$, imply that

$$\int_\Omega e^{2V_-/C_n} < |\mathbb{S}^n| \qquad \text{for all sufficiently small } \lambda - \lambda_0 > 0\,.$$

The small volume maximum principle from Lemma 4 therefore implies that $w_{\lambda,\xi_0} \geqslant 0$ a.e. on Σ_{λ,ξ_0} for all sufficiently small $\lambda - \lambda_0 > 0$. This contradicts the definition of $\lambda_0(\xi_0)$ from (47) and therefore proves that $w_{\lambda_0,\xi_0} \equiv 0$, as claimed. □

5. Proof of the main result

In this section we use the symmetry of u derived in Theorem 12 via the method of moving spheres in order to deduce that u must be of the form claimed in Theorem 1. This will be a consequence of the symmetry result of Li and Zhu [19] in the generalized form stated in [11]. Actually, the theorem in [11] is for arbitrary finite measures, but

we shall only quote a version for the case of measures which are absolutely continuous with respect to Lebesgue measure; see the remark after [11, Theorem 1.4].

Theorem 14 ([11, Theorem 1.4]). *Let $v \in L^2(\mathbb{R}^n)$ be nonnegative. Assume that for any $x_0 \in \mathbb{R}^n$ there is a $\lambda > 0$ such that*

$$v(x) = v_{I_{\lambda,x_0}}(x) \qquad \text{for almost every} \quad x \in \mathbb{R}^n \tag{48}$$

and for any $e \in \mathbb{S}^{n-1}$ there is an $\alpha \in \mathbb{R}$ such that

$$v(x) = v_{R_{\alpha,e}}(x) \qquad \text{for almost every} \quad x \in \mathbb{R}^n. \tag{49}$$

Then there are $a \in \mathbb{R}^n$, $b > 0$ and $c \geqslant 0$ such that

$$v(x) = c\left(\frac{2b}{b^2 + |x-a|^2}\right)^{n/2}. \tag{50}$$

We can now give the proof of our main result.

Proof of Theorem 1. From Theorem 12 we deduce immediately that the function $v = u_{\mathcal{S}}$ (in the notation of (15)) satisfies the assumptions of Theorem 14. Therefore, v is of the form (50) for some $a \in \mathbb{R}^n$, $b > 0$ and $c \geqslant 0$. A computation shows that

$$u(\omega) = c\left(\frac{\sqrt{1-|\zeta|^2}}{1-\zeta\cdot\omega}\right)^{n/2}$$

with a certain $\zeta \in \mathbb{R}^{n+1}$ with $|\zeta| < 1$ which is given explicitly in terms of a and b; see the discussion after (10). Thus, there is a conformal mapping Φ on $\mathbb{S}^n$ (corresponding via stereographic projection to translation by a and dilation by b on $\mathbb{R}^n$) such that

$$u = cJ_\Phi^{1/2} = c\mathbb{1}_\Phi\,.$$

Here $\mathbb{1}$ is the function on $\mathbb{S}^n$ which is constant one and $\mathbb{1}_\Phi$ refers to notation (15). By equation (4) and its conformal invariance given in (17), we have

$$C_n u \ln u = Hu = cH\mathbb{1}_\Phi = c\left((H\mathbb{1})_\Phi + C_n\mathbb{1}_\Phi \ln J_\Phi^{1/2}\right) = C_n u \ln\frac{u}{c}\,.$$

This implies $c = 1$ and concludes the proof of the theorem. □

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