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# New Spaces in Mathematics Formal and Conceptual Reflections\*

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## Introduction

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## 1 A Brief History of Space

Space is a central notion in both mathematics and physics and has always been at the heart of their interactions. From Greek geometry to Galileo experiences, mathematics and physics have been rooted in constructions performed in the same ambient physical space. But both mathematics and physics have eventually left the safe experience of this common ground for more abstract notions of space.

The 17th century witnessed the development of projective geometry and the strange, yet effective, idea of points at infinity. This century witnessed also the advent of analytic geometry (with its use of coordinates) with Descartes and of differential calculus with Newton and Leibniz. Both have led to an approach of geometry fundamentally based on the manipulation of algebraic formulas. The capacity to manipulate spaces without relying on a spatial intuition has laid the foundations for one of the most important revolutions in geometry: the conception of spaces of arbitrary dimension. During the same time, the successful geometrization of astronomy, optics, and mechanics anchored physics to the paradigm of a differential Euclidean space.

The 18th century essentially develops differential calculus for spaces of high dimensions. Analytic geometry and physics converge into analytical mechanics. This is a revolution that introduces abstract spaces in physics (like the six-dimensional spaces of trajectories) and reduces the actual physical space to be a mere starting point out of which other relevant spaces can be constructed. In

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mathematics, the invention of complex numbers laid the groundwork for the future algebraic geometry. In turn, the contradictions of logarithm theory and the study of polyhedra and graphs planted the seeds of algebraic topology. Moreover, analytical methods forecast a new notion of space: the infinite dimensional spaces of functions.

In the 19th century, geometry exploded of diversity. The use of local coordinates in analytical mechanics gave rise to the intrinsic theory of manifolds and the fundamental local–global dialectic. The points at infinity, the points with complex coordinates, and the multiple points of intersection theory are all unified in the framework of algebraic geometry. The geometric study of linear equations led to the notion of vector space. The development of Lie group theory created a completely new branch of geometry centered on the characterization of the symmetries of spaces. The construction of models for non-Euclidean geometry revived the old synthetic/axiomatic geometry, and the development of analysis prepares the notions of metric spaces. In physics, thermodynamics and electromagnetism are successfully developed within the framework of the differential calculus of  $\mathbb{R}^n$ . The latter entangles space with time in an unusual way, but the geometric paradigm of classical mechanics remains well secured.

The mathematics of the 20th century started with the successful definition of topological spaces. At the heart of the notion of space are now the set of its points, the open subsets, and the continuous-discontinuous dialectic. From function spaces to manifolds and unseparated spaces, topological spaces are powerful enough to unify many kinds of spaces. Lie groups and the paradigm of symmetry are also everywhere, from differential equations to manifolds and linear algebra. Another major revolution was the discovery that the spaces of high dimensions have specific shapes and can be different from each other. This qualitative study of spaces gave birth to algebraic topology and its two branches of homotopy and homology theories. With higher-dimensional spaces and algebraic topology, figures have essentially disappeared from geometry books and geometry has become the study of spaces that cannot be “seen” anymore.

In physics, the contradictions raised by the constancy of the speed of light and the spectrum of the black body were the source of a schism on the role of space. Relativity grounded the geometry of physics in the new dynamical object that is spacetime and successfully formalized gravity in purely geometrico-differential terms. Yang-Mills theories extended this program to the other (electromagnetic and nuclear) fundamental interactions. On the other side, the formalism of quantum mechanics required abandoning geometric intuition and, rather, focusing on algebras of operators. Despite the fundamental role played by symmetries and Lie group theory in both theories, the geometric unity of physics was to a certain extent broken.

By the middle of the 20th century, mathematics and physics are much better structured than they were at the beginning. The notions of sets, topological spaces, manifolds (Riemannian or not), algebraic varieties and vector spaces organize the geography of mathematical spaces. General relativity, classical mechanics, and quantum mechanics divide the physical space in three scales,

each with its own geometric formalism. The situation is summarized in [Tables 1](#) and [2](#). All things seems to fall into place and, for our purposes in this book, we shall refer to this situation as the *classical paradigm of space*. The conceptual categories that organize this paradigm on the mathematical side are points, open and closed subsets, coordinates and functions, local/global, measure of distances, continuity/discontinuity, infinitesimal variations, and approximation. On the physical side, the classical paradigm relies on a differentiable spacetime, trajectories and fields, infinitesimal equations, and symmetries and covariance. The intuition of space as been pushed far away from the original intuition of the ambient physical space, but in a clear continuity.

The evolution of the notion of space in mathematics and physics has continued until now. However, the results of these developments are less universally known in the mathematical and physical communities where the common background stays, even nowadays, the classical paradigm. It is the purpose of this book and its companion [\[3\]](#) to illustrate and explain some of these “post-classical” developments.

Table 1: The “classical” kinds of mathematical spaces

<i>Basic structures</i>	<i>Topology</i>	<i>Differential geometry</i>	<i>Linear spaces</i>	<i>Algebraic geometry</i>
Sets	Topological spaces	Differential manifolds, Lie groups	Vector spaces, function spaces, modules over rings	Algebraic varieties, algebraic groups
Pre-orders, equivalence relations	Metric spaces	Riemannian manifolds	Banach, Fréchet, Hilbert spaces	

## 2 Contemporary Mathematical Spaces

### 2.1 Algebraic Topology

One of the most important geometric achievements of the postclassical period is the revisitation of *algebraic topology* (homotopy and homology theory) in terms of higher category theory. Homotopy theory evolved from the definition of the fundamental group of a space (and its applications to classify covering spaces and to explain the multiple values of analytic continuations) to a general study

Table 2: The “classical” kinds of physical spaces

	<i>General relativity</i> <i>(large scale)</i>	<i>Mechanics and thermodynamics</i> <i>(medium scale)</i>	<i>Quantum mechanics</i> <i>(small scale)</i>
<i>Ambient space and time</i>	Lorentzian 4-manifold	Galilean spacetime $\mathbb{R} \times \mathbb{R}^3$	Galilean or Poincaré Lie group
<i>Phase spaces</i>	Spaces with action of the local Poincaré group	Manifolds with action of the Galilean group	Representations of various Lie groups in Hilbert spaces

of continuous maps and spaces up to continuous deformations (homotopies and homotopy equivalences) [28, 46]. The central object ended up to be that of the *homotopy type* of a space, that is, the equivalent classes of this space up to homotopy equivalence. Homological algebra evolved from a computation of numbers and groups to a calculus of resolutions of modules over a ring (or sheaves of such) [18, 66]. The notion of abelian category put some order in this calculus [14, 39], but it is only with triangulated categories that a central object emerged: *chain complexes* up to quasi-isomorphisms [26, 97]. In a separated approach, the axiomatisation of homology theories in terms of functors had also led to a new kind of object: *spectra*, of which chain complexes are a particular instance [1, 60, 81]. Any space defines both a homotopy type and a spectrum (its *stable homotopy type*) but until the 1970s, the nature of these two objects was somehow elusive.

The development in the 1970s of *homotopical algebra* (i.e. , model category theory) provided for the first time a unified framework for both homotopy and homology theories [34, 76]. But even with this unification, the theory was still highly technical and, many times, ad hoc. The concepts that revealed the meaning of these constructions were only found in the 1980s, when higher category theory emerged [12, 31, 41, 75]. The main progress was to understand that homotopy types of spaces were the same thing as  $\infty$ -groupoids, that is, a particular kind higher category in which all morphisms are invertible (see Chapter 5). By viewing homotopy types as  $\infty$ -groupoids, it was possible to revisit homotopical algebra from the standpoint provided by the whole conceptual apparatus of higher category theory. This has provided conceptual simplification of many of the homotopical constructions, but this story lies beyond the scope of this book (see [20, 64]). We have limited our study to the the utilization of  $\infty$ -groupoids in geometry, namely, in topos theory (Chapter 4), in stack theory (Chapter 8), and in the theory of derived schemes (Chapter 9). We have also

included a chapter explaining how  $\infty$ -groupoids have permitted us to revisit the foundations of mathematics (Chapter 6). Moreover, Chapters 4 and 5 of the companion volume, *New Spaces in Physics* [3], show how  $\infty$ -groupoids are useful in symplectic geometry and physics.

## 2.2 Algebraic Geometry

The field of geometry that has undergone the deepest postclassical transformation is *algebraic geometry*. From the 1950s to the 1980s, Grothendieck's school brought many definitions and improvements for the objects of algebraic geometry. The definition of Zariski spectra and schemes as ringed spaces permitted for the first time the unification of all the notions of algebraic varieties. Moreover, the notion of affine scheme provided a perfect duality between some geometric objects and arbitrary commutative rings of coordinates [42]. An important difference that schemes have with manifolds is the fact that they can accommodate singular points. This singular structure is encoded algebraically by the existence of nilpotent elements in the ring of local coordinates, a feature that is possible only if arbitrary rings are considered. Nilpotent elements provide an efficient infinitesimal calculus, which is one of the nicest achievements of algebraic geometry (see [24] and volume IV of [42]).<sup>1</sup>

The theory of ringed spaces was efficient to define general schemes by pasting of affine schemes. However, motivated by the study of algebraic groups and the construction of moduli spaces, schemes were almost immediately redefined as functors, making the previous construction somehow superfluous (see Chapter 7 on the functor of points and [27, 40]). Later on, the definition of étale spectra of rings (which was needed to define cohomology theories with étale descent) came back to a definition in terms of ringed spaces with the difference that the base space was now a *topos* (see Chapter 4 and [6, 26, 67]). The functorial point of view continued to be used simultaneously.<sup>2</sup>

The definitive approach to construct *moduli spaces* (e.g. , the space of curves or bundles on a given space) was eventually found with *stacks*, which are a variation on the notion of sheaf (see Chapter 8 and [4, 25, 36, 41, 58, 85]). Essentially, stacks provide a notion of space where the set of points is enhanced into a groupoid of points. This feature makes them perfectly suited to classifying objects (such as curves or bundles) together with their symmetries. From a geometric point of view, stack theory is a formalism intended to deal properly with the possible singularities created by taking a quotient (see Chapter 9).

The most recent development has been *derived algebraic geometry*. This formalism enhances the theory of stacks in order also to tame the singularities created by nontransverse intersections (see Chapter 9 and [65, 90, 91]).

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<sup>1</sup>This calculus is also at the core of *synthetic differential geometry*; see Chapter 2.

<sup>2</sup>For example, the notion of a connection on a singular scheme  $X$  was successfully defined by means of the *de Rham shape* of  $X$ , which is the quotient of  $X$  by the equivalence relation identifying two infinitesimally closed points. The result of such a quotient is not a scheme, but it can be described nicely as a sheaf on schemes (see Chapters 4 and 5 of *New Spaces in Physics* [15, 79] and [24, 84]).

At the end of the story, derived algebraic geometry provides by far the most sophisticated notion of space ever invented.<sup>3</sup> Derived stacks have become a powerful archetype for a new paradigm of geometric spaces (see chapter 9 and [51, 61, 73, 88, 92]). However, so many turns in only 60 years have been hard to follow, and the community of algebraic geometers is largely spread out between different technologies and viewpoints on its objects.

An important field related to algebraic geometry is *complex geometry*. In comparison with their differential analogs, complex manifolds have the problem that they admit too few globally defined holomorphic functions. This has deeply grounded the field in sheaf theory and cohomological methods and kept it close to algebraic geometry, where the same methods were used for similar reasons [37, 47, 83, 98]. Nonetheless, complex manifolds have not really evolve into more sophisticated types of spaces (incorporating singularities and points with symmetries). The recent rise of *derived analytic geometry* might change this [62, 73, 74].

Algebraic geometry depends on the existence of a well-defined dictionary between the geometric features of affine schemes and the algebraic features of commutative rings. This successful translation has led to several attempts to generalize it for other kinds of algebraic structures. The most famous attempt is given by the geometry of noncommutative rings. The attempts to build an actual topological space (a spectrum) from a noncommutative rings have not been entirely satisfactory [77, 95]<sup>4</sup>, but the dual attempt to characterize geometric features in noncommutative terms has had more successes (see Chapters 10 and Chapter 1 of the companion volume [21], and references therein). However, some important geometric notions are absent from both these approaches (e.g., open subsets, étale maps, the local/global dialectic), preventing a geometric intuition of noncommutative features in classical terms. Other offsprings of algebraic geometry have been *relative geometry*, which develops a geometry for various contexts of commutative monoids (see Chapter 7 and [92]), the geometry of *Berkovich spaces* dual to non-Archimedean fields [10, 13], the *tropical geometry* dual to tropical semirings [38, 69], and the conjectural *geometry over the field with one element* [22, 30, 86].

## 2.3 Topology

The notion of topological space has been robust enough to successfully deal with some of the new spaces invented in the second half of the 20th century such as fractals, strange attractors, and nonseparated spaces (such as the Zariski spectrum of a commutative ring). Even the study of topological spaces by means of rings of continuous functions (motivated by Stone and Gelfand dualities) has not introduced new objects [35].

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<sup>3</sup>Algebraic geometry was able to deal successfully with “multiple points with complex coordinates at infinity”; derived algebraic geometry added to these features the possibility to work with quotients by nonfree group actions and self-intersection of such points.

<sup>4</sup>Mostly by lack of functoriality of the spectra.

Nonetheless, new spaces have been invented for the needs of topology. For example, the close relationship between topology and intuitionist logic à la Heyting has led to *locale theory*, a variation on topological spaces well suited to define interpretations of logical theories (see Chapter 4 and [48, 96]). Also, in algebraic geometry, the remarkable analogy between the Galois theory of fields and the theory of covering spaces [29] has motivated the search for a functor associating a topological space to a commutative ring (a spectrum) which could transport, so to speak, the former theory into the latter. The Zariski spectrum fails to satisfy this, and the proper answer was found with the *étale spectrum*. However, étale spectra could no longer be defined as topological spaces anymore but rather were defined as topoi [6, 26]. Essentially, a *topos* is a new kind of space defined by its category of sheaves instead of its poset of open subspaces. This broader definition led to many new topological objects that are not topological spaces (see Chapter 4 and [49, 67]).

Another motivation for enhancing the notion of topological space was the study of badly separated spaces [2], for example, spaces that have many points but a trivial topology, such as the irrational torus  $\mathbb{T}_\alpha := \mathbb{R}/(\mathbb{Z} \oplus \alpha\mathbb{Z})$  ( $\alpha \notin \mathbb{Q}$ ), the leaf spaces of foliations with dense leaves, or even bizarre quotients such as  $\mathbb{R}/\mathbb{R}_{dis}$  (the continuous  $\mathbb{R}$  quotiented by the discrete  $\mathbb{R}$ ). The theory of topoi turned out to be well suited to studying these spaces.<sup>5</sup> But other methods have been developed, like topological sheaves and stacks (inspired by algebraic geometry) [11, 19, 23] or noncommutative geometry à la Connes (see Chapter 1 of the companion volume [21], and references therein), diffeologies (see Chapter 1, and references therein), or orbifolds and Lie groupoids [59, 70, 78, 94].

## 2.4 Differential geometry

*Differential geometry* has not escaped the development of new types of spaces, but the size of the field has perhaps kept most of it within the classical paradigm. From Riemannian geometry to knot theory, the basic notion is still that of manifold. Overall, the field does not seem to be in a hurry to incorporate the developments of algebraic geometry (e.g., duality algebra/geometry, singular spaces, functorial approach to moduli spaces and infinite dimensions, relativization with respect to a base space, tangent complexes). Many attempts have been made to improve manifolds, but none of them seems to have become central. An example is *diffeology* theory, which provides a nice framework to deal with infinite-dimensional spaces as well as quotients (see Chapter 1, and references therein). Another one is *synthetic differential geometry*, which enhances the notion of manifold by authorizing singular points and nilpotent coordinates (see Chapter 2, and references therein).<sup>6</sup> Related approaches have tried to ground differential geometry in the algebraic notion of  $C^\infty$ -ring [51, 71, 72]. The most successful new notion of differentiable space is perhaps that of *orbifolds*, motivated among other things by Thurston’s geometrization program [78, 89]. Orbifolds have

<sup>5</sup>They are called *étendues* in topos theory, see [6, 49].

<sup>6</sup>Synthetic differential geometry, as its name suggests, also promulgates an axiomatic approach to geometry.

brought to the field some tools from higher category theory like stacks [59, 70] and equivariant homotopy theory [80].

Another domain using such methods is microlocal analysis, where sheaves and their derived categories are of great help for dealing with the problem of extending local solutions of differential equations (see Chapter 3, and references therein).

The most impressive display of postclassical methods in differential geometry can be found in *symplectic geometry* (together with the related fields of *Poisson* and *contact geometries*). Symplectic geometry is a contemporary descendent of analytical mechanics. The notions of symplectic manifold and their Lagrangian submanifolds have given a new geometrical meaning to many constructions of mechanics (e.g., extremal principles and generating functions, covariant phase spaces, Noether symmetries and reduction [43, 53, 87]) A central operation in the theory is *symplectic reduction*, which combines the restriction to a subspace of a symplectic manifold with a group quotient [68]. Since these two operations might create singularities, symplectic geometry has been forced to deal with both nontransverse intersections and quotients of nonfree group actions. These issues have led to the use of new formalisms, such as cohomological methods [44, 57], Lie groupoids and stacks [99, 100], and, eventually, derived geometry (see Chapter 4 of the companion volume and [16, 93]). Also, the application of symplectic geometry to physics has imported many methods from higher category theory: cohomological methods in deformation quantization [17, 56], Fukaya categories in mirror symmetry [55, 82], and, more recently a whole new interpretation of gauge theory in terms of stacks (see Chapter 5 of the companion volume, and references therein). In fact, more than a simpler user of higher categories and derived algebraic geometry, symplectic geometry has been an important catalyzer in the development of these theories.

Another important innovation with respect to the notion of manifold has been the interpretation of manifolds with boundaries and cobordisms in terms of *higher categories with duals*, a viewpoint that was inspired by topological field theories in physics [7, 8, 63]. In the same way that homotopy theory has transformed topological spaces into tools that can be used to work with  $\infty$ -groupoids, this view on cobordisms does not address manifolds as an object on its own but rather as a tool to encode the combinatorial structure of some higher categories.

## 2.5 Conclusion

We have referred to the understanding of the notion of space in the middle of the 20th century as the “classical paradigm”. This raises the question of whether the many evolutions undergone by the notion of space since then qualify as a new paradigm. As the previous presentation and Table 3 illustrate, the classical geography of mathematical spaces is still pertinent today. The conceptual categories organizing the intuition of space have not fundamentally changed (points, functions, local/global dialectic, etc.) If something radical has changed, it will

not be found there.

In our opinion, the most important postclassical change has in fact not concerned spaces directly – although it had a tremendous impact of them – but *sets*. If there has been a paradigm shift in mathematics, it has been the enhancement of *set theory* in *category theory* (in which we include higher categories). Category theory is responsible for most of the new spatial features:

1. The most important change has been that sets of points have been enhanced in categories of points (in particular, points can have symmetries).
2. The definition of a space by means of a poset of open subsets has been enhanced in a definition by means of categories of sheaves (topoi, dg-categories, stable categories, etc.).
3. Functions with values in set-based objects (numbers, manifolds, etc.) have been enhanced by functions with values in category-based objects (stacks, moduli spaces, etc.).
4. Many spaces are defined as functors (schemes, moduli spaces, stacks, diffeologies, etc.).
5. Homotopy types are now seen as  $\infty$ -groupoids.
6. Also, the relation with logic and axiomatization is made by means of categorical semantics for logical theories.

In the classical paradigm, sets can be thought as the most primitive notion of space – collecting things together in a minimalist way – from which other notions of space are formally derived. In the new paradigm, categories, and particularly higher categories, are the new primitive spatial notion from which the others are derived. Nowadays, categories are everywhere in topology and geometry, from the definition of the basic objects to the problems and methods of study. The reader will realize that category theory is central in *all* the chapters of this volume.

Table 3: Classical and “new” kinds of mathematical spaces

	<i>Basic structures</i>	<i>Topology</i>	<i>Differential geometry</i>	<i>Linear spaces</i>	<i>Algebraic geometry</i>	<i>New geometries</i>
<i>Classical notions (1-categorical)</i>	Sets  Pre-orders, equivalence relations	Topological spaces  Metric spaces	Differential manifolds, Lie groups  Riemannian manifolds	Vector spaces, function spaces, modules over rings  Banach, Fréchet, Hilbert spaces	Algebraic varieties, algebraic groups	—
<i>New 1-categorical notions</i>	—	Locales	Diffeologies, SDG, $C^\infty$ -manifolds	—	Zariski spectra, schemes, algebraic spaces	Noncommutative spaces (Combes), super-manifolds, tropical geometry
<i>New <math>\infty</math>-categorical notions</i>	Groupoids, cellular complexes, $\infty$ -groupoids (homotopy types)	Topoi, $\infty$ -topoi	Lie groupoids, orbifolds, stacks, moduli spaces, derived manifolds	Chain complexes, spectra (homology theories)	Étale spectra, algebraic stacks, derived schemes	Noncommutative spaces (Kontsevich)

## 3 Summaries of the Chapters

### 3.1 Differential Geometry

#### 3.1.1 An Introduction to Diffeology (Patrick Iglesias-Zemmour)

The theory of diffeologies – started by the French mathematician J.-M. Souriau in the early 1980s – provides a formal setting in which the main tools of differential calculus can be extended to infinite-dimensional spaces (such as spaces of functions between manifolds or symmetry group of manifolds). Recall that a manifold  $M$  of dimension  $n$  can be described by the data given by all differentiable maps  $\mathbb{R}^n \rightarrow M$  which are open immersions. A diffeology  $X$  will be similarly described by the data of all differentiable maps  $\mathbb{R}^n \rightarrow M$  (called *plots*), but without the assumption that the maps  $\mathbb{R}^n \rightarrow X$  have to be open immersions, and without the restriction that  $n$  has to be fixed. For example, if  $E$  is an infinite-dimensional topological vector space, the corresponding diffeology is defined by the data given by all (nonlinear) differentiable maps  $\mathbb{R}^n \rightarrow E$  for all  $\mathbb{R}^n$ . In analogy with the fact that an infinite set is always the union of its finite subsets, a diffeology can be understood as the “union” of all its finite-dimensional plots.<sup>7</sup>

This definition permits the definition of fiber bundles, differential forms, de Rham cohomology, and other classical notions of manifold theory. The methods to do so are very close to the sheaf theoretic methods of Chapter 7 but with a more classical flavor. Diffeologies provide an efficient setting extending the classical notion of manifolds at a rather low technical cost. Despite their original application to infinite-dimensional spaces, they have proved also to be well suited to defining the differentiable structure of some “bad quotients” (such as the irrational torus  $\mathbb{T}_\alpha = \mathbb{R}/(\mathbb{Z} \oplus \alpha\mathbb{Z})$  for  $\alpha \notin \mathbb{Q}$  or other leaf spaces of dense foliations) and of manifolds with boundaries.

#### 3.1.2 New Methods for Old Spaces: Synthetic Differential Geometry (Anders Kock)

*Synthetic differential geometry* (SDG in what follows) started with Lawvere’s work on continuum mechanics in the 1960s and relies on two main ideas. First, SDG provides a *synthetic* – or *axiomatic* – framework for differential geometry. In the same way that points and lines are just assumed and not constructed in Euclidean geometry, manifolds are just assumed collectively as primitive objects in SDG. This idea is opposed to the analytic description of manifolds individually in terms of coordinates. The central object of SDG is a ring object  $R$  playing the role of the field  $\mathbb{R}$  of real numbers. The axioms are chosen so that the theory recovers all classical constructions (tangent vectors, differential forms, connections) and more.

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<sup>7</sup>This point of view becomes clearer when diffeologies are defined as sheaves over the category of manifolds [9]. In this framework, the notion of “union” is given by the categorical notion of *colimit*. By Yoneda lemma, any sheaf on a category  $C$  is always a colimit of objects of  $C$ . A diffeology is then a colimit of  $\mathbb{R}^n$ s.

The second idea is to provide a setting encompassing manifolds with singularities (like the cusp  $\{(x, y) | x^2 = y^3\}$ ). The definition of singular objects is inspired from algebraic geometry, where singular points can be defined by the property to have local coordinates that are *nilpotent*. Having nilpotent elements is the main difference between  $R$  and  $\mathbb{R}$  in SDG. The entire differential calculus can be deployed from these elements. For example, the subspace  $D_1 = \{x \in R | x^2 = 0\}$  of elements of square zero plays the role of the first-order infinitesimal neighborhood of 0 in  $R$ . Then a tangent vector of a manifold  $M$  is simply a map  $D_1 \rightarrow M$ . The space  $D_1$  has a canonical point, which is 0, and the base point of the vector is simply the image of 0.<sup>8</sup>

The classical construction of  $\mathbb{R}$  does not allow nilpotent elements, and the requirement of SDG may seem strange.<sup>9</sup> But the definition of tangent vectors shows that their introduction simplifies classical constructions. Other examples are given by the pleasant definition of differential forms and affine connections (see the chapter).

### 3.1.3 Microlocal Analysis and Beyond (Pierre Schapira)

An important problem with differential equations is to know if a solution over a domain  $U$  can be extended over a bigger domain  $V$ , and this problem can naturally be formulated in terms of sheaves. Recall that the notion of a sheaf on a space  $X$  encodes the data of local functions defined on  $X$ . Let  $U$  be an open domain,  $\sigma$  a section of a sheaf  $F$  on  $U$ , and  $x$  a point of the boundary of  $U$ . The problem at stake can be locally formulated as follows: *is it possible to find a neighborhood of  $x$  and an extension of  $\sigma$  in this neighborhood?*

By developing the work of Sato and Hormander from the 1970s, Kashiwara and Schapira’s microlocal analysis tackles this question on manifolds. The differentiable structure of a manifold  $M$  provides tools to answer the problem. If  $U$  has a smooth boundary, the tangent hyperplane at  $x$  is always the kernel of a differential form  $p$  that is negative on  $U$  and positive outside (sometimes called the codirection of the hyperplane). The question about the propagation of sections can then be formulated in terms of  $p$  and no longer  $U$ . The *microsupport* of a sheaf  $F$  is the set of points  $(x, p)$  of  $T^*M$  through which the sections of  $F$  cannot be extended uniquely. In other terms, microlocal analysis introduces a notion of locality that refers not only to the points of the manifold (classical locality) but also to the codirections around that point (microlocality).

A remarkable result is that the microsupport of a sheaf  $F$  is within the zero section of  $T^*M$  if and only if  $F$  is a locally constant sheaf. This is analogous to the fact that the graph of the differential  $df$  of a function  $f : M \rightarrow \mathbb{R}$  is in the zero section of  $T^*M$  if and only if  $f$  is a locally constant function. This fact suggests that the microsupport can be understood as a sort of “derivative” of

<sup>8</sup>Recall that in classical differential calculus, a vector is defined as an equivalence classes of paths having the same 1-jet. But this definition cannot work at singular points, since singular points are precisely points with 1-jets not integrable into actual paths.

<sup>9</sup>It might help to look at the explicit model, closer the classical analytical approach, given in terms of sheaves in [71].

the sheaf  $F$ . From this perspective, microlocal analysis may be the beginning of a differential calculus for sheaves on manifolds.

## 3.2 Topology and Algebraic Topology

### 3.2.1 Topologie (Mathieu Anel and André Joyal)

This chapter is about the two evolutions of the notion of topological space, which are *locales* and *topoi*. The *theory of locales* – also known as *point-free topology* – is rooted in the close relationship between topology and intuitionist logic. The main idea in defining a locale is to forget the underlying set of points of a topological space (hence the name of the field) and to define locales directly by their *frames* of open subsets, which are a commutative ring-like structures. The category of locales is then formally defined as the opposite of that of frames. The theory is then based on a dictionary between geometric features of locales and their translation in algebraic features of frames, very much as in algebraic geometry. Locales have some new features compared to topological spaces. For example, any intersection of dense subspaces is always dense, and there exist nontrivial locales with an empty set of points.<sup>10</sup> Overall, locales provide a nicer topological setting than topological spaces, but the two notions are too close, and the latter is too well established for the former to pretend to replace it.

The notion of *topos* is similarly defined as dual to an algebraic structure that the authors call a *logos*. A logos is intuitively a category of sheaves on a space, and it is equipped with operations that make it look like a commutative ring. As for locales, the theory of topoi provides a dictionary between geometric and algebraic features. Every topological space or locale defines a topos, but there are much more topoi than topological spaces. A big difference between topoi and classical spaces is that the former can have a category of points instead of a mere set. In particular, there exists a topos  $\mathbb{A}$  whose category of points is the category of sets. This topos plays a central role in the theory since one can describe the logos  $Sh(X)$  of sheaves on a topos  $X$  as the category of morphisms of topoi  $X \rightarrow \mathbb{A}$ . In other words, a topos is the object dual to an algebra of functions with values in the “space of sets”.

The theory of topoi has mostly been popular in logic, where it turned out to be well suited to providing interpretations of higher-order theories. The topological aspects of topos theory are less known. Nevertheless, the theory of topoi is also quite useful in topology, where it can encode badly separated spaces, such as foliation spaces<sup>11</sup> or some moduli spaces. It also provides a nice setting where the homotopy and homology theories of spaces can be defined.

### 3.2.2 Spaces as Infinity-Groupoids (Timothy Porter)

The homotopy theory of topological spaces grew from the study of the fundamental group  $\pi_1(X)$  of a connected space  $X$  to the definition of a whole

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<sup>10</sup>For this reason, it is a pun to refer to point-free topology as “pointless” topology [50].

<sup>11</sup>See the notion of *étendue* in [6, 49].

collection of homotopy groups  $\pi_n(X)$  indexed by natural numbers. For this reason, homotopy types of spaces (i.e., topological spaces up to weak homotopy equivalence) were first understood as an algebraic structure akin to groups. As Porter explains in his chapter, the quest for this structure has led to many definitions, but they were never able to encompass all homotopy types. Significant progress was made when it was understood that the notion of algebraic structure based on functional relations (such as the composition law) and conditions written as equations (such as associativity) was too strict. Another kind of algebra was needed to capture the features of homotopy theory: functions had to be multivalued (correspondences), and equations had to be replaced by the existence of paths.

The problem starts when one tries to define the composition of paths in a space: not only are there many ways to define such a composition but none of them are associative. The classical solution is to look at paths up to homotopy for which the composition becomes uniquely defined and associative. However, this strategy truncates the higher homotopical structure and only captures  $\pi_1(X)$ . The insight was to recognize that the existence of multiple compositions for paths was not a problem but a feature of the theory. The composition does not exist uniquely, but between two choices of compositions, there always exists a homotopy; moreover, between any two such homotopies, there always exists a higher homotopy, and so on. In other words, the regular structure was found when all the possible compositions were considered together and not individually: the composition becomes “unique” because the space of compositions is *contractible*. The same idea can be used to deal with associativity.<sup>12</sup> The existence of homotopies is formally encoded by *lifting conditions*, for example, those defining *Kan complexes*, which are one of the best definitions of  $\infty$ -groupoids.

Another important step forward was made when it was understood that homotopy types of spaces were a particular kind of higher category called  $\infty$ -groupoids. Even if this equivalence does not entail a simplification of the definition of homotopy types/ $\infty$ -groupoids, it has permitted us to understand many constructions of homotopy theory in the light of concepts coming from higher category theory.

### 3.2.3 Homotopy Type Theory: The Logic of Space (Michael Shulman)

The most unexpected consequence of the idea that homotopy types are  $\infty$ -groupoids has been in foundations of mathematics. The development of  $\infty$ -groupoids in algebraic topology and algebraic geometry has produced algebraic objects such as groups or rings with an underlying  $\infty$ -groupoid instead of an underlying set. This has led topologists and geometers to the idea that  $\infty$ -groupoids are objects as fundamental as sets and could be used as a primitive notion (or “background structure” in Shulman’s terms) to build other mathematical objects.

<sup>12</sup>This idea has been promoted to an important working philosophy: whenever some choice has to be made, the good choice is to consider all choices together.

A similar idea was found in logic when Martin-Löf’s theory of *dependent types with identity types* – which has been designed as a language for set theory – was given a successful interpretation in terms of  $\infty$ -groupoids by Awodey–Warren [5] and Voevodsky [52, 45]. More precisely, the homotopical idea that the paths between two elements  $a$  and  $b$  of a space  $X$  are the elements of the path space  $\Omega_{a,b}X$  turned out to be perfectly suited to encoding the logical idea that proofs of equality between two terms  $a$  and  $b$  of a type  $X$  should be the terms of an identity type  $a =_X b$ . *Homotopy type theory* is the offspring of homotopy theory and type theory based on this idea.

From a logical perspective, this homotopical interpretation has provided a completely new understanding of types and their axioms. In particular, thinking the proofs of an equality  $a =_X b$  as paths has explained why such a proof need not be unique (not all paths are homotopic).<sup>13</sup> For the mathematician, the homotopic semantic of type theory has offered the luxury of a formal language to work with homotopy types, which is an alternative to higher category theory, independent of set theory and implementable on proof assistants.

In addition to semantics in terms of sets, logical theories and type theories also have interpretations in terms of topological spaces. When these semantics are crossed with the homotopical semantics, these give rise to interpretations in higher topological stacks, that is,  $\infty$ -groupoids enriched over topological spaces. Moreover, variations can be defined where stacks are defined in the context of differential geometry or algebraic geometry. In this way, type theory provides a common language for a variety of geometric contexts. It is in this sense that type theory is indeed a “logic of spaces”.

### 3.3 Algebraic Geometry

#### 3.3.1 Sheaves and Functors of Points (Michel Vaquié)

Locally, a differential manifold looks like  $\mathbb{R}^n$ , and any manifold can be obtained by pasting the elementary pieces  $\mathbb{R}^n$ . Moreover, every manifold can be embedded in some sufficiently large  $\mathbb{R}^n$ . The situation is not as simple in algebraic geometry. Only the affine schemes dual to commutative rings can be embedded in affine spaces  $\mathbb{A}^n$ . Also, because schemes can have singularities, it is false that every scheme is locally like  $\mathbb{A}^n$ .<sup>14</sup> A general scheme is defined as a pasting of affine schemes, and, unfortunately, there is no simpler and smaller class of objects (like the  $\mathbb{R}^n$ s of the handlebodies of Morse theory) that could also generate all schemes.

This is where the general methods of category theory are helpful. Given the category  $\text{Aff}$  of affine schemes, there is a simple description of all possible past-

<sup>13</sup>When types are interpreted as sets, the type  $a =_X b$  is either empty or a one-point set; however, this fact cannot be deduced from the axioms of type theory. This posited a long-standing puzzle. The homotopical interpretation solved it since it provided a model where this is not true.

<sup>14</sup>This is only true for smooth schemes and for the étale topology.

ings of objects of  $\text{Aff}$ : they are the presheaves on  $\text{Aff}$ .<sup>15</sup> Presheaves are functors and this is where the *functor of points* approach has its roots. Any presheaf is the pasting of a diagram of affine schemes, but not every presheaf is nice enough to be considered as a geometric object (for instance, they do not all have tangent spaces). This is why presheaves satisfying extra *geometric conditions* are considered. Among these conditions, there is always a *sheaf condition*, which guarantees that the embedding of affine schemes into the geometric presheaves preserves the pastings of affine schemes that are affine. The other conditions are conditions on the type of pasting producing geometric objects (for example, the only pastings allowed for schemes are along open subsets).

The framework given by the “functors of points” is efficient enough to provide a definition of schemes different than the classical presentation by ringed spaces. This functorial framework turned out to be perfectly suited to study *moduli spaces* classifying some structure  $S$  (e.g., the Hilbert schemes that classify all the closed subschemes of a given scheme). By definition, moduli spaces define almost tautologically a presheaf on  $\text{Aff}$ : the value of the functor on an affine scheme  $X$  is the set of families of objects with the structure  $S$  parameterized by  $X$  (see the next chapter on stacks for an example). Once this presheaf is defined, the *moduli problem* is to know whether it is the functor of points of a scheme (or another kind of geometric object, such as an *algebraic space* [54]). This setting has been efficient to construct many moduli spaces: projective spaces, Grassmannians, flag manifolds, Hilbert schemes, Picard schemes, and so on.

### 3.3.2 Stacks (Nicole Mestrano and Carlos Simpson)

When a moduli space is intended to classify objects that have symmetries, the sheaves with values in *sets* have to be replaced by sheaves with values in groupoids (*stacks*), where the morphisms encode the corresponding symmetries. The paradigmatic example in Mestrano and Simpson’s chapter is given by the *moduli space of curves*.<sup>16</sup> A family of curves parametrized by an affine scheme  $X$ <sup>17</sup> is a bundle  $E \rightarrow X$  whose fibers are curves. Such a family can intuitively be thought of as a function on  $X$  with values in some “space of curves”  $M$ . The *moduli problem* of curves is to construct this space  $M$ . The corresponding functor of points sends an affine scheme  $X$  to the set  $\mathcal{M}(X)$  of bundles of curves over  $X$ , and the question is whether there exists a scheme  $M$  such that  $\mathcal{M}(X)$  is in bijection with the set of morphisms  $X \rightarrow M$ . Unfortunately, the answer is negative. A necessary condition for the functor  $\mathcal{M}$  to be represented by a scheme  $M$  is that it be a sheaf, but this is not the case. The problem comes

<sup>15</sup>Presheaves are functors  $\text{Aff}^{\text{op}} \rightarrow \text{Set}$ . The category of presheaves on a category  $C$  has the universal property to be the free completion of  $C$  for colimits, i.e. , for pasting.

<sup>16</sup>The problem at stake is really to study Riemann surfaces. They are called *curves* by algebraic geometers because such surfaces are in fact of dimension 1 relative to the field of complex numbers.

<sup>17</sup>The reader unfamiliar with scheme theory can assume that they are manifolds since the peculiarities of schemes will not play a role here. Although the chapter presents the moduli problem of curves in the framework of algebraic geometry, the problem can also be formulated in differential or complex geometries.

from the fact the data of bundles on an open cover  $U_i$  of  $X$  is patched together into a bundle on the whole of  $X$  by using *isomorphisms* on  $U_i \cap U_j$ . But the sheaf condition only patches them up to *equality* on  $U_i \cap U_j$ , which is too strict. This problem can be bypassed by incorporating the data associated to the isomorphisms between bundles into the values of the functor  $\mathcal{M}$ . This means that  $\mathcal{M}(X)$  – rather than being a mere set – is now the *groupoid* of curve bundles together with their isomorphisms. This new functor does satisfy a sheaf-like condition: the *stack condition* (see the chapter). When the identifications between the objects are unique, the functor is valued in groupoids that are just sets, and the stack is simply a sheaf.

An important class of stacks are the *geometric stacks* (e.g., Deligne-Mumford and Artin stacks). These are the stacks that admit *local coordinates* (*atlases*) and for which it is possible to define tangent spaces, local dimensions, and so on. The moduli stack of curves is an example of a geometric stack. Stacks have been defined first in algebraic geometry, but they are a general notion that also plays a role in differential geometry (orbifolds) and in topology.

### 3.3.3 The geometry of Ambiguity: An Introduction to Derived Geometry (Mathieu Anel)

Two fundamental geometric operations are intersecting subspaces and taking quotients. Both these operations can create singular points. Intersection singularities include the *multiple points* appearing in nontransverse intersections (like intersecting a circle with one of its tangent line) and the singular points of a function (like the cusp of  $x^3 = y^2$ ). Quotient singularities are typically created by fixed points of group actions (like the origin in the action of  $\mathbb{Z}/2\mathbb{Z}$  on the affine line  $\mathbb{A}^1$  or that of  $SO(2)$  on the affine plane  $\mathbb{A}^2$ ). The notion of scheme is able to deal efficiently with multiple points and singular points of functions but finds a limit when self-intersections are involved.<sup>18</sup> Also, the quotients of group actions do not in general have local coordinates when computed in schemes, but a solution was found with stacks. Because of the use of groupoids, stacks are perfectly suited to keeping track of the possible multiple identifications between the points of a quotient (isotropy groups). This has the consequence that quotients do have local coordinates (atlas) when computed in stacks. This property has been another incentive for introducing stacks (the quotients  $\mathbb{A}^1/(\mathbb{Z}/2\mathbb{Z})$  or  $\mathbb{A}^2/SO(2)$  are geometric stacks).

Both types of singular points have in common that the tangent space at a singular point has a dimension which is strictly bigger than the local dimension of the space around it.<sup>19</sup> In practice, these tangent spaces are always computed as the zeroth homology group of some chain complex whose Euler characteristic is always the good local dimension. A point is regular if and only if the homology of this “tangent complex” is concentrated in degree 0. When the

<sup>18</sup>The “nonschematic” intersections are the ones for which Serre intersection formula, involving the derived tensor product, has been invented.

<sup>19</sup>This is actually how singular points are defined in algebraic geometry. Intuitively, this is a way to say that not all 1-jets of paths can be integrated into actual paths.

point is singular, the positive homology of the tangent complex measures the “quotient complexity” of the singularity, and the negative homology measure the “intersection complexity”.

Derived geometry can then be understood as an enhancement of stack theory (which already deals satisfactorily with quotient singularities) that incorporates tools permitting us to deal with intersection singularities and justify the computation of tangent complexes. As with stacks, the methods to do so are deeply rooted in homotopy theory (see the chapter for why and how).

### 3.3.4 Geometry in dg-Categories (Maxim Kontsevich)

The 20th century has witnessed the rise of many algebraic methods to study spaces. Beyond the classical correspondence between spaces and rings of functions/coordinates, other algebraic devices have been invented, notably *categories of sheaves*. Sheaves of sets are the core of topos theory, sheaves of  $\infty$ -groupoids are the core of higher topos theory and homotopy theory, sheaves of modules over a ring are widely used in algebraic and complex geometry, and sheaves of chain complexes are central in homology theory. Beyond sheaves, categories have also become central in the study of representations of groups and algebras.

Overall, these constructions of categories provide a unity of structure to topology, geometry, and algebra which has brought M. Kontsevich to an original geometric program of unified notions and methods for these three fields. The common structure is that of a category enriched over chain complexes (differential graded category or dg-category) that is *derived* from abelian categories (in the sense of derived functors and derived categories). The categories produced from topology or geometry are usually equipped with a monoidal structure (often commutative), but those from algebra may not. This leads Kontsevich to make the leap to the noncommutative realm not only by forgetting the commutativity of the monoidal structure but by removing entirely the monoidal structure. The basic object of the resulting *derived noncommutative geometry* is, then, simply a bare dg-category.<sup>20</sup>

The first part of the chapter presents the main examples in algebraic topology, algebraic geometry, and algebra from which Kontsevich draws his inspiration. The second part continues with a list of geometric notions that survive the removal of the monoidal structure (e.g., smoothness, properness, finiteness, deformations). The third part continues with new features and methods specifically provided by this new context. The resulting definitions permit the extension of the notions to nongeometric contexts, thereby enriching the comprehension of these contexts with a geometric intuition. Dually, the study of actual geometrical objects without the constraint to preserve the monoidal structure brings a freedom of operations that is similar to what complex numbers bring to the study of real numbers.

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<sup>20</sup>Perhaps the name of “nonmonoidal geometry” would be more accurate than “noncommutative geometry”.

Table 4: Motivations for the new mathematical spaces

	Infinite dimension (function spaces)	Better local structure (singular points)	Better quotients	Better intersections	Better basic objects than sets	Moduli spaces	Axiomatic approach	Geometry/Algebra duality	Application to physics
Diff. Geom.	Diffeology [Ch. 1]		a few						some
	SDG [Ch. 2]	some		some			good	partial	
	Micro-local analysis [Ch. 3]	some							
Top. & Alg. Top.	Locales [Ch. 4]			some		some		yes	
	Topos [Ch. 4]		some	some		some		yes	
	$\infty$ -Groupoids [Ch. 5]		good		good				
	HoTT [Ch. 6]		some		good		good		
Alg. Geom.	Schemes[Ch. 7]	good	some	some		a few		partial	
	Sheaves[Ch. 7]	good	some			some			
	Stacks[Ch. 8]		some	good		good			good
	Derived geometry [Ch. 9]	good	good	good	good		good	partial	good
	NCG [Ch. 10]		some					partial	some

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The history of the evolution of space in mathematics and physics attempted here does not pretend to be comprehensive. For various reasons, important chapters are missing (orbifolds, tropical geometry; Berkovich spaces;  $\mathbb{F}_1$ -geometry; motives; quantales; constructive analysis and its real numbers; formal topology; or other approaches to quantum gravity like causal sets, group field theory, dynamical triangulations) and we apologize to the reader who might have expected to find these in this work.

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