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New Spaces in Physics Formal and Conceptual Reflections*

Introduction

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1 New Spaces in Physics

Two fundamental scientific revolutions took place in physics during the first decades of the 20th century: Einstein’s geometric description of the gravitational interaction by means of the *general theory of relativity* and the development of *quantum mechanics*. These two revolutions radically modified our understanding of the laws that rule the physical phenomena taking place at opposite (astrophysical and microscopic) spatiotemporal scales.

On the one hand, general relativity introduced into physics essential geometric ideas and tools mainly developed during the 19th century in pure mathematics, notably differential geometry, Riemannian geometry, and tensor calculus. Moreover, general relativity provided the motivating example of the general program – launched by H. Weyl around 1918 – intended to provide similar geometric descriptions of the other fundamental (electromagnetic and nuclear) interactions. This “geometrization program” was finally achieved in the 1950s in the framework of the *Yang-Mills theories* and acquired a solid mathematical foundation and geometric interpretation with the theory of Cartan (for general relativity) and Ehresmann (for Yang-Mills theories) connections on principal fiber bundles.¹ Both general relativity and Yang-Mills theories define the so-called *gauge theories of fundamental interactions*, where the term *gauge* refers to the fact that these theories are endowed with a *local symmetry* associated to the possibility to choose different coordinate systems (“gauges”) at each spatiotemporal location. Moreover, this geometrization of the fundamental interactions

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¹For a history of the path that led from general relativity to Yang-Mills theories (and a collection of some of the corresponding seminal papers), see [31].

provided the cornerstone of the so-called *standard model* of elementary particles and the associated attempts to unify the four fundamental interactions (where the most celebrated success of this program up to now was the Glashow-Weinberg-Salam unification of the electromagnetic and weak interactions).

On the other hand, quantum mechanics – with its utilization of noncommutative operator algebras on Hilbert spaces – has a strong algebraic flavor that has obstructed to a certain extent the construction of a conceptual interpretation based on a geometric intuition. The main obstacle to the comprehension of quantum mechanics in geometric terms is given by the *non-commutative* character of the algebras of quantum observables. Indeed, this central feature of the quantum formalism has as a consequence that – differently from the *commutative* algebras of classical observables – the quantum observables cannot be understood as functions on an “ordinary” space. This essential feature of quantum mechanics introduces a sort of discontinuity between this theory on the one hand and both classical mechanics (which relies on a solid geometric intuition) and the gauge theories of the fundamental interactions on the other.

Roughly speaking, the main lines of research leading to new notions of space in physics after the quantum and the relativistic revolutions can be understood as attempts to understand quantum mechanics in more geometric terms on the one hand and to quantize general relativity on the other. Let us consider first the “geometrization” of quantum mechanics. Is it possible to construct non-commutative quantum algebras out of geometric structures? What would it be gained by doing so? First, it is worth stressing that quantum mechanics is a formalism that – up to now – could not be endowed with an unanimously accepted conceptual interpretation, being the landscape of competing interpretations populated with radically different conceptual schemes. Now, casting quantum mechanics in more geometric terms redounds in a gain of a conceptual and more intuitive understanding that might pave the road for solving this interpretative conundrum. For instance, the *geometric quantization* formalism developed by Kirillov, Kostant, and Souriau presents quantum mechanics in the same geometric formalism – the theory of connections on fiber bundles – used in gauge theories (see [7, 21, 22, 36]). Now, since gauge theories are better understood than quantum mechanics from a conceptual standpoint, geometric quantization provides a useful bridge to transport this conceptual clarity to quantum mechanics. Second, since classical mechanics relies on a clear geometric basis, the geometrization of quantum mechanics might improve the comprehension of the relationship between quantum mechanics and classical mechanics.

Among the different ways according to which mathematicians can construct non-commutative algebras from geometry, three constructions became relevant in physics, namely

- the deformation of a ring of functions (giving rise in particular to the *deformation quantization* of a Poisson manifold; see for instance [37] and references therein),

- the endomorphisms of a fiber bundle (giving rise in particular to the *geometric quantization* of a symplectic manifold [7, 21, 22, 36]).
- the convolution algebra of a groupoid (giving rise to *non-commutative methods* [8]),

Now, both deformation quantization and geometric quantization strongly rely on the *symplectic formulation of classical mechanics*. Here a main player for the development of physical geometry during the 20th century enters the scene, *symplectic geometry*. Thanks to the work of mathematicians like Arnold, Maslov, Souriau, and Weinstein, among others, the explosion of research in symplectic geometry during the 20th century led to a deep transformation of our comprehension of classical mechanics (see for instance [1, 4, 12, 24, 25, 36]). In the framework of this symplectic geometrization of classical mechanics, fundamental new notions and theories were introduced, such as for instance Souriau’s *moment map* [28, 36], the Marsden-Weinstein’s *symplectic reduction* [26], and Weinstein’s *symplectic “category”* and *Lagrangian correspondences* [38]. In the wake of this symplectic refoundation of classical mechanics, it is also worth mentioning the development of the theory of variational calculus on jet bundles and the development of multisymplectic geometry launched by De Donder, Weyl and continued – more recently – by Kijowski among others. In this extended context, important new notions were introduced, such as the *covariant phase space*, the *Peierls bracket*, and the *variational bicomplex* (see for instance [9, 10, 20, 29, 36, 40]).

From a conceptual standpoint, the great importance of symplectic (and Poisson) geometry is that it encodes what we could call the *classical seeds* of quantum mechanics. By doing so, the development of symplectic geometry allowed to significantly reduce the gap between classical and quantum mechanics. It could even be argued that symplectic geometry opened the path to the comprehension of quantum mechanics as a continuous extension of classical mechanics and no longer as a sort of “new paradigm” discontinuously separated from the classical one (see Schreiber’s contribution in Chapter 5). For instance, both in *deformation quantization* and in *geometric quantization*, classical structures (namely, the Poisson structure and the symplectic structure respectively) encode fundamental quantum features. While in deformation quantization the Poisson structure provides the first term of the “quantum” deformation (in the formal parameter \hbar) of the commutative algebra of functions on a phase space, in geometric quantization the symplectic structure defines the curvature (on the prequantization fiber bundle) that explains the noncommutativity of quantum operators.² Moreover, one of the central facts of symplectic geometry is the existence of a correspondence defined by the symplectic structure between observables (functions on a phase space) and what could be called *classical operators* (Hamiltonian vector fields). In this way, the fundamental role played

²It is worth noting that this is in complete analogy to the fact that in general relativity and Yang-Mills theories, the noncommutativity of parallel transports results from the presence of a nontrivial curvature.

by operators in mechanics – far from being a quantum innovation – is already a central feature of classical mechanics.³ It is also worth mentioning that the category-theoretic “points” of a symplectic manifold are given by its Lagrangian submanifolds.⁴ According to Guillemin and Sternberg, the notion of Lagrangian submanifold encodes the classical seeds of the quantum indeterminacies: “*The Heisenberg uncertainty principle says that it is impossible to determine simultaneously the position and momentum of a quantum-mechanical particle. This can be rephrased as follows: the smallest subsets of classical phase space in which the presence of a quantum-mechanical particle can be detected are its Lagrangian submanifolds. For this reason it makes sense to regard the Lagrangian submanifolds of phase space [rather than its set-theoretic points] as being its true ‘points’.*” [13] In this way, it could be argued that if the notion of localization in phase space (in the sense of “being at a certain point” of phase space) is not defined with respect to its set-theoretic points but rather with respect to the Lagrangian “points”, then Heisenberg indeterminacy principle does not forbid a localization of a quantum particle in phase space. All in all, these different insights brought forward by the development of symplectic geometry are permitting us to progressively sublimate the simplistic opposition between the supposedly stable and well-understood realm of classical mechanics and the still-unsolved conceptual problems posited by quantum mechanics. By pushing this line of thought to its limit, it could even be argued that the missing insights permitting us to construct a satisfactory conceptual interpretation of quantum mechanics might stem from a better comprehension of classical mechanics and its symplectic foundations. In this sense, the explosion of research in symplectic geometry is pulling back the problem of interpreting quantum mechanics to an unexpected problem: the problem of reinterpreting classical mechanics.

Another direct repercussion on geometry elicited by the development of quantum mechanics is given by the study of hypothetical “spaces” supporting (or dual to) *noncommutative* “algebras of functions”. The new branch of geometry known as *noncommutative geometry* might have been inspired by the capacity to generate new notions of space associated to the *geometry-algebra dualities*, that is, to the dualities between spaces and the algebras of “functions” on them (like for instance the duality between affine schemes and commutative rings or the Gelfand-Naimark duality between compact Hausdorff topological spaces and commutative unital C^* -algebras). Indeed, the geometry-algebra dualities naturally lead to the introduction of new spaces by means of the following pattern: given a particular instantiation of a geometry-algebra duality, one can generalize the corresponding algebra of functions – by passing for instance to

³In the framework of geometric quantization, quantum operators are in fact defined by means of a vertical extension (where *vertical* means in the direction of the fibers of the corresponding prequantization fiber bundle) of these classical operators (see for instance [7]).

⁴Considered from the standpoint of category theory, the Lagrangian submanifolds of a symplectic manifold (M, ω) are the $(*, 0)$ -points of M in Weinstein’s symplectic “category” (where $(*, 0)$ is the trivial symplectic manifold), that is the morphisms (Lagrangian correspondences) $(*, 0) \rightarrow (M, \omega)$.

noncommutative algebras – and try to interpret the new algebra as an “algebra of functions” on a generalized space. However, it is not clear to what extent the noncommutative approaches to geometry do really produce “non-commutative spaces” dual to the corresponding algebras. An alternative way to understand noncommutative geometry could be the following. Given “ordinary” (commutative) spaces one can define non-commutative invariants. These invariants do not always allow us to reconstruct the space, but they encode nonetheless certain important geometric aspects like *properness* or *smoothness* (see, for instance, Kontsevich’s chapter in the companion volume *New Spaces in Mathematics* [17]). The important fact is that these noncommutative invariants endowed with a geometric meaning permit us to introduce certain geometric concepts and intuitions into the realm of noncommutative algebra.

The formulation of quantum mechanics and general relativity naturally leads to the *quantum gravity* program, that is, to the different research programs intended to quantize general relativity (like for instance superstring theory, loop quantum gravity, semiclassical quantum gravity, causal sets, dynamical triangulations, lattice quantum gravity, and the asymptotic safety program among others⁵). The general expression *quantize general relativity* denotes here both the application of standard quantization methods (e.g. canonical quantization, path integral, etc.) to general relativity in its Lagrangian or Hamiltonian formulation and the direct construction of a theory out of which general relativity and the continuum description of spacetime is supposed to emerge in some “classical” approximation.

The supposed necessity to quantize general relativity can be justified on different grounds, like for instance

- the idea that quantum gravity is required to deal with spacetime singularities taking place at very high energies and very small scales (such as the Big Bang and black hole singularities);
- the fact that while general relativity describes (by means of the Einstein field equations) the coupling between *classical* matter and the geometry of spacetime, all matter is currently described in the framework of *quantum* field theory;
- the idea that the unification between gravity and the other *quantum* gauge fields carrying the electromagnetic and nuclear interactions requires us also to describe gravity in quantum terms – by taking into account that the nongravitational interactions are mediated by the so-called *gauge bosons* (like the photon for the electromagnetic interaction), this argumentative line led (mainly in the framework of perturbative string theory) to the postulation of a hypothetical massless spin-2 particle that mediates the

⁵For an overview of different approaches to quantum gravity see for instance [27] and references therein.

gravitational interaction, the *graviton*;⁶

- the arguments based on the finite character of black hole entropy (see for instance [34]).

Besides these particular motivations, a more straightforward argument is the following. Since

- a) general relativity is already a *classical* theory in the sense that it can be cast in the terms of classical (Hamiltonian or Lagrangian) mechanics (e.g. ADM formalism, Einstein-Hilbert action); and
- b) classical mechanics has been superseded by (or extended to) quantum mechanics,

then general relativity has to be recast in quantum-mechanical terms.

Despite the still highly speculative nature of the field, research in quantum gravity has already had a significant impact on mathematical geometry. First, string theory already had important repercussions on research in pure geometry (e.g. mirror symmetry, Gromov-Witten invariants and enumerative geometry, etc.; see for instance [5, 18, 19]). Second, research in quantum gravity opened the field of *quantum geometry*, that is, the study of different geometric structures, out of which the classical and continuum spacetime geometry described by general relativity can be reobtained in some form of “classical” limit. In very general terms the field of quantum geometry explores ideas such as

- a fundamental discretization of spacetime (an idea that goes back to Riemann [32] and reappears in almost every approach to quantum gravity);
- spaces described by noncommutative coordinates (e.g. noncommutative geometry);
- quantum indeterminacies and fluctuations of geometric quantities;
- linear superpositions of geometries.

For instance (as Mariño explains in Chapter 9), string theory addresses different forms of deformation (stringy, quantum) of classical Riemannian geometry resulting from the quantum description of dynamical extended objects (strings and eventually p -branes). In turn, loop quantum gravity studies certain geometric structures – the canonical spin networks and the covariant spin foams – arising from a more or less direct quantization of general relativity (see Han’s contribution in Chapter 8). Other approaches explore the possibility of understanding the classical and continuum description of spacetime geometry – as well as geometric notions like *dimension* and *locality* – as an emergent description arising from *nongeometric* or *pregeometric* (a term introduced by Wheeler

⁶It is worth noting that a straightforward application of the perturbative methods of quantum field theory to the gravitational interaction leads to a perturbative nonrenormalizability. This obstacle has been the main motivation for the development of *nonperturbative approaches* to quantum gravity.

[39]) degrees of freedom. Examples of these supposed pre-geometric structures are the *causal sets* – i.e. sets representing spacetime events endowed with an order relation encoding the causal structure – [11, 35] or combinatorial structures like simplicial objects and graphs (e.g. *quantum graphity* [16]). However, the characterization of these structures as non- or pregeometric is problematic (do they really “*break loose at the start from all mention of geometry and distance?*” [39]) and it might seem more appropriate to state that the different “pregeometric” scenarios proposed thus far remove certain geometric features of the classical and continuum description of spacetime conveyed by general relativity (e.g. continuity, differential structure, distance, dimensionality or locality).

Let us consider now in some detail the different chapters of this volume.

2 Summaries of the Chapters

1 Noncommutative and Supercommutative Geometries

1.1 *Noncommutative Geometry, the Spectral Standpoint (Alain Connes)* The construction of quotients of spaces has been an important source of definitions of new notions of space. The space of leaves of a dense foliation does not have enough open subsets to be described as a manifold or even as a topological space. The spaces of orbits of group actions that are not free have singularities that a topology or a differential structure cannot encode. Several methods have been invented to work with these objects, some using category theory (e.g. sheaves and stacks, topoi, diffeologies), others algebra. The non-commutative geometry of A. Connes belongs to this later class. The basic idea is to replace the commutative ring of observable functions on the quotient by the noncommutative convolution algebra of the foliation or the group action. This construction is justified by the fact that, when the quotient exists, the categories of modules over the function ring or over the convolution algebra coincide.⁷ However the latter construction is better behaved than the former.

From a more conceptual standpoint, the basic principle of Connes’ non-commutative geometry is to substitute the equivalence relation associated to a quotient operation by the corresponding *action groupoid* of identifications. The main difference between an equivalence relation and a groupoid is that the latter keeps track of the fact that different points might be identified in many different ways (which includes *a fortiori* the particular case of possible non-trivial stabilizers). In this sense, an equivalence relation can be understood as a truncated groupoid where the possibly multiple concrete identifications between two elements are collapsed to the abstract fact that they are equivalent. This transition from equivalence relations to groupoids leads to the consideration of a particular noncommutative algebraic structure, namely the *convolution algebra on the action groupoid* (where the noncommutativity is a direct consequence of the noncommutativity of compositions in the groupoid). As it was stressed by Connes in Ref.[8, §1.1, pp.40-45] this kind of noncommutative algebras was

⁷Technically, they are Morita-equivalent algebras.

implicitly discovered by Heisenberg in the seminal 1925 article in which he proposed the matrix formulation of quantum mechanics [14].⁸

Noncommutative geometry consists in defining a certain number of geometric notions (infinitesimal calculus, integration and measure theory, metric, etc.) in terms of algebras which are not necessarily commutative. The central notion is that of *spectral triplets* (A, \mathcal{H}, D) encoding a “space” with a metric and a measure theory.⁹ The commutative algebra of functions on a Riemannian (spin^c) manifold is reinterpreted by Connes as an algebra of operators acting on a Hilbert space of spinors, and the inverse line element of the Riemannian structure is encoded (in Connes’ *distance formula*) by the corresponding Dirac operator. Now, the central insight is that this setting remains valid when we substitute the commutative algebra functions by a noncommutative algebra of operators acting on a Hilbert space.

It is also worth noting that Connes’ version of noncommutative geometry is also motivated by the problem of quantizing gravity and unifying the four fundamental interactions. The inverse line element defined by the Dirac operator D encodes not only the gravitational interaction (associated as usual to the metric), but also the electromagnetic, and nuclear – weak and strong – interactions (which are associated to the inner fluctuations of the metric). This results in a successful derivation of the Lagrangian of the standard model from a Lorentzian spacetime crossed with a specific finite noncommutative space. Interestingly enough, the different physical forces are unified by means of the metric structure of the noncommutative space, thereby giving rise to a sort of generalized gravity theory.

1.2 *The Logic of Quantum Mechanics (Revisited)* (Klaas Landsman) Landsman’s contribution can be inscribed among the attempts to generalize the classical notions of space by using the framework provided by the geometry-algebra duality. Starting with

- the (constructive versions of the) Gelfand-Naimark duality between *commutative* unital C^* -algebras and compact Hausdorff topological spaces; and
- the Stone duality between the category of boolean lattices (with homomorphisms of orthocomplemented lattices as arrows) and totally disconnected compact Hausdorff spaces (Stone spaces),

⁸In Heisenberg’s matrix formulation, the relations between physical quantities is governed by the noncommutative algebra of matrices that represent these quantities. Connes argued that the Ritz-Rydberg combination principle that models the experimental results provided by atomic spectroscopy (which were incompatible with the classical predictions) can be encoded in a groupoid of frequencies whose convolution algebra is nothing but the algebra of matrices discovered by Heisenberg.

⁹More precisely, a general spectral triplet (A, \mathcal{H}, D) is given by a $*$ -algebra A endowed with a representation by bounded operators on a Hilbert space \mathcal{H} and an unbounded self-adjoint Dirac operator D acting on \mathcal{H} and encoding a generalized notion of distance that extends the Riemannian notion of distance to the noncommutative realm.

Landsman moves forward to the intuitionistic/noncommutative realm by addressing

- the Priestley duality between bounded distributive lattices and Priestley spaces; and
- the Esakia duality between Heyting algebras and Esakia spaces.

The ultimate goal of this progression is a conjectured duality between arbitrary unital C^* -algebras and some Heyting algebras. The result of this work in progress would be the construction of a model of an *intuitionistic quantum logic* that has the opposite features than the Birkhoff and von Neumann’s quantum logic [6]. This means that such an intuitionistic quantum logic is distributive (which paves the way to an interpretation of the logical operations \wedge and \vee as a disjunction and a conjunction, respectively), but does not keep the law of the excluded middle (which, according to Landsman, matches quantum features such as Schrödinger cat situations).

Interestingly enough this construction of an intuitionistic quantum logic can be related to topos theory. Briefly, we can associate to any unital C^* -algebra A the topos of covariant functors $\mathcal{C}(A) \rightarrow \mathit{Set}$ on the posetal category $\mathcal{C}(A)$ of all unital commutative subalgebras of A .

1.3 Supergeometry in Mathematics and Physics (Mikhail Kapranov) Kapranov’s contribution addresses the quandaries of supergeometry in mathematics and supersymmetry in physics from an original homotopical perspective. According to Kapranov, the challenge posited by supergeometry and supersymmetry is to understand the formal and conceptual structures underlying the \pm sign rules that govern the supercommutation structures in both mathematics and physics. These structures involve vector spaces with a $\mathbb{Z}/2\mathbb{Z}$ -grading together with a monoidal structure involving Koszul’s sign rule. Now, an important caveat is here necessary: the similarities between formalisms discovered by physicists and mathematicians might sometimes be misleading. According to Kapranov, an instance of this danger is provided by these supercommutative structures. Indeed, a careful comparative study of supercommutative structures in mathematics and physics leads Kapranov to conclude that the formal similarity should not lead to an identification: the $\mathbb{Z}/2\mathbb{Z}$ of mathematicians is not the same as the $\mathbb{Z}/2\mathbb{Z}$ of physicists.

From a mathematical standpoint, supergeometry is the study of geometric objects whose rings of functions are commutative superalgebras $A = A^{\bar{0}} \oplus A^{\bar{1}}$ composed of even and odd elements subjected to the corresponding supercommutation rules. In this way, supergeometry can be added to the list of new geometries (along noncommutative spaces, topoi, and so forth) associated to the attempts to generalize the standard commutative algebras of functions of the geometry-algebra dualities. What Kapranov calls the *principle of naturality of supers* states that supercommutative algebras – rather than being a non-commutative generalization – can be understood as a natural “super” extension of commutative algebra itself (which is implicit in the term *super*-commutative

rather than *non-commutative*).¹⁰ From a physical standpoint, Kapranov understands supersymmetry as a particular case of the heuristic *principle of square roots* according to which certain quantities of “immediate physical interest” (e.g., real quantities like the probability density in quantum mechanics) are bilinear combinations of more fundamental quantities (e.g., complex quantities like the wave function in quantum mechanics).

The original idea addressed by Kapranov in his contribution is that supercommutative structures are nothing but the tip of an homotopical iceberg associated to the first level of the *sphere spectrum* \mathbb{S} (where the group $\{\pm\}$ associated to the sign rules is the first homotopy group of \mathbb{S}). To explain this, M. Kapranov proposes to push further the comprehension of the group of integers. Classically, \mathbb{Z} is the free commutative group on one generator. However, this is no longer the case in a homotopical or higher categorical context. If sets are replaced by ∞ -groupoids (see Chapters 5, 6 and 9 of the companion volume *New Spaces in Mathematics* [2, 30, 33]), the free commutative group on one generator is no longer \mathbb{Z} but the *sphere spectrum* \mathbb{S} .¹¹ As Kapranov writes in the wake of Grothendieck, the sphere spectrum – being the homotopic version of the ring \mathbb{Z} of entire numbers – “is the most fundamental commutative object”, that is (we could say) the “supercommutative” object. By doing so, Kapranov establishes an unexpected link between supercommutative structures in mathematics and physics on the one hand and the homotopical reconceptualization of the abstract notion of identity in terms of concrete (and possibly multiple) identifications on the other hand.

The sphere spectrum is a nontrivial homotopy type, whose homotopy invariants are the so-called *stable homotopy groups of spheres*. The π_0 of this homotopy type recovers \mathbb{Z} , but its π_1 and π_2 are both $\mathbb{Z}/2\mathbb{Z}$. This last feature of the sphere spectrum allows Kapranov to explain the similarity between the supercommutative structures of mathematicians and physicists. Both are working with a sign rule, but these are controlled by different levels of the homotopy of the sphere spectrum (the 1-truncation for mathematicians and 2-truncation for physicists). Kapranov conclusion is that the practice of mathematicians and physicists can be unified by considering the new notion of vector spaces graded by \mathbb{S} (rather than \mathbb{Z}), a notion which does not yet formally exist and would “open a fantastic possibility of higher super-mathematics” (sic). The chapter finishes by a sketch of what this theory could be.

¹⁰For instance, given a supercommutative algebra $A = A^{\bar{0}} \oplus A^{\bar{1}}$, the even part $A^{\bar{0}}$ defines an ordinary affine scheme $\text{Spec}(A^{\bar{0}})$ and the odd part (being nilpotent) enriches this scheme by adding an “infinitesimal neighborhood” to it.

¹¹Intuitively, the difference is that the products ab and ba are equal in \mathbb{Z} , whereas they are only homotopic in \mathbb{S} . It follows that a square a^2 inherits a non-trivial loop in the space \mathbb{S} . In this regard, \mathbb{Z} is constructed from the space \mathbb{S} by contracting these loops into trivial loops. But by doing so, \mathbb{Z} is no longer described as a free object, but rather as an object with relations.

2 Symplectic Geometry

2.1 *Derived Stacks in Symplectic Geometry (Damien Calaque)* An important recent development in the history of the relations between symplectic geometry and mechanics is provided by the “derived” enhancement of symplectic geometry discussed in Chapter 4.

The reformulation of mechanics in terms of symplectic geometry led to a number of problems related to the limitations of the notion of manifold. For example, a fundamental operation in symplectic geometry is the *symplectic reduction* of a Hamiltonian group action on a symplectic manifold. This process combines a *restriction* to a subspace of phase space and a *projection* to a quotient space (called, in the physics jargon, *constraint surface* and *reduced phase space*, respectively) [15, 26]. Now, these two operations might produce singularities. The result of this symplectic reduction is again a symplectic manifold when there are no singularities, but the definition of a symplectic structure around a singular point becomes problematic when singularities are present.

The recent development of derived geometry (see Chapter 9 of *New Spaces in Mathematics*) has succeeded in defining a general notion of singular symplectic space in the context of algebraic geometry.¹² The introduction of these new symplectic spaces allowed us to regularize important features of the theory: Lagrangian correspondences (i.e., the morphisms in Weinstein’s symplectic category) can always be composed, symplectic reductions are always symplectic, and the symplectic structure built from the transgression construction always exists. The “derivation” of symplectic geometry has also led to some new features. First, the extension of the notion of symplectic structure to the notion of *shifted symplectic structure* allowed us to realize that certain important spaces (like the intersection of any two Lagrangian correspondences, the quotient stack \mathfrak{g}^*/G of a coadjoint action, and the classifying space $BG = */G$) are in fact endowed with shifted symplectic structures. Moreover, Lagrangian correspondences are no longer given by subspaces, but rather by general maps which are not necessarily injective.¹³

2.2 *Higher Prequantum Geometry (Urs Schreiber)* As we have said before, the standard (and too simplistic) demarcation line between a supposedly well-understood realm of classical mechanics and the quandaries of quantum physics has been blurred by both the explosion in the second half of the 20th century

¹²It is worth noting that methods developed for dealing with constrained Hamiltonian systems (like the BRST cohomological reformulation of symplectic reduction or the BV formalism [15, 23]) already implicitly encode ideas coming from the domain of “derived mathematics”. In the framework of derived geometry, the ad hoc (co)homological methods used in physics are interpreted as a way to deal with singular points: degenerated systems of constraint equations produce nontransverse intersections and degenerated symmetries produce nonfree group actions and singular quotients. The (co)homological structure generated by the so-called *ghosts* and *anti-ghosts* of the BRST formalism serves to regularize these possible pathological situations.

¹³Interestingly enough, moment maps are particular instances of these generalized Lagrangian correspondences.

of research on *symplectic* and *Poisson geometries* and the development of quantization formalisms that strongly rely on this geometric description of classical mechanics, notably, *geometric quantization* and *deformation quantization*.

Schreiber’s Chapter 5 moves forward in the direction of sublating the opposition between classical and quantum mechanics

- by showing that the prequantization construction in geometric quantization (i.e., the definition of a linear fiber bundle on the phase space with a connection whose curvature is defined by the symplectic form) can be understood as a *global lifting of local data*; and
- by developing this understanding of prequantum geometry for *covariant* field theories, that is, for theories that are local in spacetime M rather than in space.

Briefly, the main idea is to generalize the definition of the Lagrangian (of a theory that describes physical fields given by sections of a bundle $\varphi : E \rightarrow M$ over spacetime) from *globally defined Lagrangian* to *families of local Lagrangians endowed with gluing data*. Technically, this amounts to define a Lagrangian as a function on E with values in the *moduli space of Čech-Deligne cocycles*, which classify *gerbes with connections*. Roughly speaking, a *gerbe* is a generalization of a G -principal fiber bundle where the fibers are not isomorphic to G but rather to the classifying spaces $B^n G$ for some n . Geometrically, this means that the topological twists are not introduced at the level of the identifications between the fibers (given by G -valued transition functions on the twofold intersections $U_{i_1} \cap U_{i_2}$, where U_{i_n} are open sets of a covering) but rather at the level of the higher cocycle consistency conditions defined on the higher n -fold intersections $U_{i_1} \cap \dots \cap U_{i_n}$.

The resulting *higher prequantum geometry* allows us to prequantize local field theory in an explicitly *local* (or *covariant*) and *gauge-invariant* manner. It is a major conceptual step for the comprehension of mechanics to understand the prequantization construction (which yields the noncommutative algebra of quantum operators) in terms of what we could characterize as *global classical mechanics*. A related important feature of Schreiber’s formalism is that the global degrees of freedom given by the topology of the bundle (e.g., the instanton sector) – rather than being fixed in an ad hoc manner – are incorporated in the definition of the generalized Lagrangians as functions with values in a classifying space. In this sense, Schreiber’s formalism gives a further step toward fulfilling the heuristic principle of “background independence”, that is, the principle according to which physical theories have to be as free as possible from the presupposition of ad hoc geometric background structures.

3 Spacetime

3.1 *Struggles with the Continuum (John C. Baez)* An important motivation for constructing a mathematical description of physical nature is given by the possibility of making predictions of future states by means of computations.

In general, such computations requires to introduce idealizations or approximations (for instance, it is easier to represent planets as points rather than 3-dimensional objects). In particular, the representation of spacetime by means of the continuum of real numbers seems to be one of these idealizations. Even if spacetime might well be discrete at small scales, it is helpful to assume that it is continuous in almost all branches of physics. In particular, this assumption permits us to use differential calculus and approximations by series expansions. Now, while making idealizations is an essential component of scientific activity, idealizations might lead to inconsistencies when pushed beyond its limits (e.g., forces or speeds can become infinite, series expansion can diverge) or become an obstruction to the construction of a rigorous formulation of the theories at stake.

In Chapter 6, Baez analyzes a number of problems posited in different branches of physics by the assumption of spacetime continuity, for instance, the problems related to the collision and noncollision singularities in Newtonian point particles interacting gravitationally, the problem of self-forces in electromagnetic theory (exerted by the field created by a particle on itself), the renormalization of infinities in quantum field theory, and the questions and problems posited by (black hole and cosmological) singularities in general relativity. All of these problems are related to the possibility of having arbitrarily small or large real numbers (typically, small distances creating infinite forces).

Overall, Baez's chapter shows that the interactions between mathematical theories, physical theories, and reality are more complex than what is commonly assumed. Theories can be incomplete, be inconsistent, or fail to make predictions in certain regimes, even if they are in agreement (at least to a certain extent) with experiments. Some of the problems posited by the assumption of continuity have led to important solutions (e.g., the discretization of energy in quantum mechanics), but most of them are still open. While in pure mathematics certain research programs address the problems posited by the mathematical continuum (e.g., nonstandard analysis, synthetic differential geometry, constructivism, finitism, ultrafinitism), Baez and many physicists believe that a successful theory of quantum gravity might shed new light on the problems associated to the assumption of spacetime continuity.

3.2 Twistor Theory: a Geometric Perspective for Describing the Physical World (Roger Penrose) Penrose's twistor theory provides a new approach to fundamental physics in which quantum mechanics and relativity theory are combined in an original way that departs from the more established quantum gravity approaches. Rather than combining them by applying (for instance) quantization techniques to general relativity, the communicating vessel between the two theories is surprisingly provided by complex analysis and holomorphic geometry.

Twistor theory provides a framework in which spacetime is a derived notion with respect to an underlying arena provided by complex *twistor geometry*. One of the central intuitions of twistor theory is that ray-lights (endowed with an angular momentum twist) should be considered more fundamental than spacetime

points. In a sense, this amounts to take seriously the fact that spacetime events connected by a ray-light are “separated” by a null spatiotemporal interval. In turn, this projective stance amounts to ascribe a more fundamental importance to the physics of massless particles and the corresponding conformal invariance.

A central feature of twistor theory is that it is specifically adapted to the fact that the physical spacetime “*that macroscopically presents to us*” has $3 + 1$ dimensions. This feature of the theory contrasts both with approaches like string theory that require extra space dimensions and with the mathematical drive toward full generality. In particular, the central role played by complex numbers in quantum mechanics is directly related by Penrose to the fact that physical space is of dimension 3. In turn, the *Riemann sphere* that provides this relation also shows up in a relativistic context, now as the *celestial sphere* surrounding an observer. Twistor theory is based on the intuition according to which this dual role played by the Riemann sphere – far from being a mere coincidence – provides an essential hint toward a unification of both theories based on 2-spinor calculus.

Besides unfolding the different trends of thought that led to twistor theory, Penrose also introduces in Chapter 7 a recent attempt (called *palatial twistor theory*) intended to overcome the *googly problem* related to the left-handed (or anti-self dual) character of the gravitational fields in Penrose’s nonlinear graviton construction.

3.3 Quantum Geometry of Space (Muxin Han) One possible strategy to construct a *quantum theory of gravity* is to apply a quantization formalism to the canonical (constrained) formulation of general relativity (for instance, the ADM Hamiltonian formalism). Differently from Yang-Mills theories (where the dynamical variable is a connection), the fundamental dynamical variable in the standard Einstein-Hilbert formulation of general relativity is the 3-metric of the corresponding spatial hypersurfaces of a four-dimensional spacetime manifold M_4 (with respect to a chosen foliation of M_4 in three-dimensional hypersurfaces M_3^t). An important step forward intended to overcome the impasses of the original quantum geometrodynamics program (impasses which are mainly associated to the nonlinear nature of the corresponding constraints) was given by Ashtekar’s reformulation of general relativity as a dynamical theory of $SU(2)$ connections. The main step of this program is the definition of the *quantum configuration space* $\overline{\mathcal{A}}$ of gravity as a space of $SU(2)$ -holonomies around spatial “loops” defined by “generalized” or “distributional connections”.¹⁴ This means that the wave functions of the quantum theory will be given by functions of these $SU(2)$ -holonomies

$$\psi_\gamma(A) = \psi(h_{e_1}(A), \dots, h_{e_n}(A)),$$

¹⁴The distributional nature of the connections means that the usual smoothness assumption on the connections was dropped, that is, the map $e \mapsto h_e(A)$ can be discontinuous, in order to cope with the problem of defining a diffeomorphism invariant measure on the configuration space.

where γ is a graph (or “network”) immersed in M_3 , (e_1, \dots, e_n) are the n edges of γ , and $h_{e_i}(A)$ is the holonomy of A along the edge e_i . The main advantage of this connection-theoretical reformulation of general relativity is that it brings Einstein’s theory closer to the Yang-Mills theories that describe the other fundamental interactions and whose quantization is well understood.

Along these lines, *loop quantum gravity* was developed – thanks to the efforts of physicists like Ashtekar, Rovelli, Smolin, Thiemann, and Lewandowski, among others – as a *background independent* and *nonperturbative* approach to quantum gravity out of which a well-defined picture of quantum space and quantum spacetime emerged in the form of the *spin networks* and *spinfoams* states, respectively. Since the spin network states are common eigenvectors of the area and the volume operators with discrete spectra, loop quantum gravity provides a concrete description of a discrete quantum geometry obtained by quantizing a quantum configuration space of distributional $SU(2)$ -connections.

3.4 Stringy Geometry and Emergent Space (Marcos Mariño) The problem of quantizing gravity has also triggered research in theoretical physics aiming to extend the physics of fundamental point-particles to higher dimensional objects such as strings, membranes, and p -branes. In particular, string theory provides a unified theory of the fundamental interactions with propagating gravitons (and black hole solutions), that is to say, a unified theory of the four fundamental interactions including a perturbative theory of quantum gravity. As Mariño explains in his contribution, this perturbative quantum theory of gravity can be geometrically understood in terms of a 2-parameter *stringy* and *quantum* deformation of classical Riemannian geometry (where the deformation parameters are the string length l_s and the string coupling constant g_{st} , respectively). In concrete examples, classical Riemannian geometry “emerges” as an effective description in the limit of a point-particle approximation ($l_s \rightarrow 0$) of noninteracting ($g_{st} \rightarrow 0$) strings.

Regarding the stringy deformation, a manifold X – rather than being “probed” by points (limit case in which $l_s \rightarrow 0$) – is endowed with all possible maps from Riemann surfaces to X , where these maps describe the possible worldsheets of the string. In the simpler case given by *topological string theory*, the quantum theory of strings embedded in a Calabi-Yau manifold X gave rise to the celebrated results in enumerative geometry associated to the Gromov-Witten invariants.

In the case of a nonzero string coupling constant g_{st} , quantum corrections associated to Riemann surfaces with a nonzero genus have to be considered. Since the corresponding genus expansion is divergent, a nonperturbative formulation of string theory is needed (the formulation of which remains problematic). Now, it was understood that a nonperturbative formulation of string theory requires us to consider higher dimensional objects (p -branes) that are “invisible” from the standpoint of the perturbative theory. One of the most fruitful avenues of research toward a nonperturbative formulation of string theory is given by the so-called *holographic dualities*, notably Maldacena’s AdS/CFT correspondence

(where AdS/CFT stands for *anti-de Sitter/conformal field theory*). These are dualities between string theories (and the corresponding spacetime gravitational physics) on the one hand and quantum gauge theories in lower dimension on the other.

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The history of the evolution of space in mathematics and physics attempted here does not pretend to be comprehensive. For various reasons, important chapters are missing (orbifolds, tropical geometry; Berkovich spaces \mathbb{F}_1 -geometry; motives; quantales; constructive analysis and its real numbers; formal topology; or other approaches to quantum gravity like causal sets, group field theory, or dynamical triangulations) and we apologize to the reader that might have expected to find these in this work.

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