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DISTRIBUTION OF ELEMENTS OF A FLOOR FUNCTION SET IN ARITHMETICAL PROGRESSION

YAHUI YU AND JIE WU

ABSTRACT. Let $[t]$ be the integral part of the real number t . The aim of this short note is to study the distribution of elements of the set $\mathcal{S}(x) := \{\left[\frac{x}{n}\right] : 1 \leq n \leq x\}$ in the arithmetical progression $\{a + dq\}_{d \geq 0}$. Our result is as follows: the asymptotic formula

$$(0.1) \quad S(x; q, a) := \sum_{\substack{m \in \mathcal{S}(x) \\ m \equiv a \pmod{q}}} 1 = \frac{2\sqrt{x}}{q} + O((x/q)^{1/3} \log x)$$

holds uniformly for $x \geq 3$, $1 \leq q \leq x^{1/4}/(\log x)^{3/2}$ and $1 \leq a \leq q$, where the implied constant is absolute. The special case of (0.1) with fixed q and $a = q$ confirms a recent numeric test of Heyman.

1. INTRODUCTION

As usual, denote by $[t]$ the integral part of the real number t . Recently Heyman [2] quantifies the cardinal of the set

$$(1.1) \quad \mathcal{S}(x) := \left\{ \left[\frac{x}{n} \right] : 1 \leq n \leq x \right\}.$$

Noticing that

$$\mathcal{S}(x) = \left\{ m \in \mathbb{N} : m = \left[\frac{x}{n} \right] \text{ for some } n \leq x \right\},$$

by elementary argument he proved that (see [2, Theorems 1 and 2]):

$$(1.2) \quad S(x) := |\mathcal{S}(x)| = [\sqrt{4x+1}] - 1 = 2\sqrt{x} + O(1)$$

for $x \rightarrow \infty$. Subsequently in his another article [3, Theorem 1], he also investigated the number of primes in the set $\mathcal{S}(x)$:

$$(1.3) \quad \begin{aligned} \pi_{\mathcal{S}}(x) &:= \left| \left\{ \left[\frac{x}{n} \right] : 1 \leq n \leq x \text{ and } \left[\frac{x}{n} \right] \text{ is prime} \right\} \right| \\ &= \frac{2\sqrt{x}}{\log \sqrt{x}} + O\left(\frac{\sqrt{x}}{(\log x)^2} \right), \quad \text{as } x \rightarrow \infty. \end{aligned}$$

This can be considered as analogue of the prime number theorem for the set $\mathcal{S}(x)$. Very recently Ma and Wu [6] sharpened this result by proving the strong form of the prime number theorem for $\mathcal{S}(x)$:

$$(1.4) \quad \pi_{\mathcal{S}}(x) = \text{Li}_{\mathcal{S}}(x) + O\left(\sqrt{x} e^{-c(\log x)^{3/5}(\log \log x)^{-1/5}}\right),$$

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where $c > 0$ is a positive constant and

$$\text{Lis}(x) := \int_2^{\sqrt{x}} \frac{dt}{\log t} + \int_2^{\sqrt{x}} \frac{dt}{\log(x/t)}.$$

Some related results have been obtained in [4, 5].

In [2], Heyman also proposed to consider a more general problem than (1.2), i.e. to study the asymptotic behaviour of the cardinal $S(x; q)$ of the set

$$\mathcal{S}(x; q) := \left\{ \left[\frac{x}{n} \right] : 1 \leq n \leq x \text{ and } q \mid \left[\frac{x}{n} \right] \right\}$$

for each fixed integer $q \geq 1$. Let $\psi(t) := t - [t] - \frac{1}{2}$ and $b := (\sqrt{4x+1} - 1)/2$. Heyman first showed that (see [2, Lemma 3])

$$S(x; q) = \frac{4\sqrt{x}}{3q} + \sum_{r=1}^{\lfloor x/b \rfloor} \sum_{d=(x-r)/(qr)}^{x/(qr)} \left(\psi\left(\frac{x}{dq+1}\right) - \psi\left(\frac{x}{dq}\right) \right) + O(1)$$

and then wrote: “Calculating various sums using Maple suggests that the double sum cannot successfully be bound. In fact Maple suggests that the double sum is asymptotically equivalent to $2\sqrt{x}/(3q)$. If this argument is correct then

$$(1.5) \quad S(x; q) \sim \frac{2\sqrt{x}}{q} \quad (x \rightarrow \infty),$$

as one would expect heuristically.”

The aim of this short note is to prove (1.5). In fact we can consider a more general problem: for $1 \leq a \leq q$, study the distribution of elements of the set $\mathcal{S}(x)$ in the arithmetical progression $\{a + dq\}_{d \geq 0}$. Define

$$\mathcal{S}(x; q, a) := \left\{ \left[\frac{x}{n} \right] : 1 \leq n \leq x \text{ and } \left[\frac{x}{n} \right] \equiv a \pmod{q} \right\}$$

and

$$S(x; q, a) := |\mathcal{S}(x; q, a)|.$$

Our result is as follows.

Theorem 1. *Under the previous notation, we have*

$$(1.6) \quad S(x; q, a) = \frac{2\sqrt{x}}{q} + O((x/q)^{1/3} \log x)$$

uniformly for $x \geq 3$, $1 \leq q \leq x^{1/4}/(\log x)^{3/2}$ and $1 \leq a \leq q$, where the implied constant is absolute.

Since $S(x; q, q) = S(x; q)$, Heyman’s expected result (1.5) is a special case of our (1.6) with fixed q . It is worth to notice that the result of Theorem 1 implies

$$S(x; q, a) \sim \frac{2\sqrt{x}}{q}$$

uniformly for $1 \leq q = o(x^{1/4}/(\log x)^{3/2})$ and $1 \leq a \leq q$. We didn’t make an effort to get the best possible exponent, and further improvements of the constant $\frac{1}{4}$ are possible.

It seems interesting to establish analogues of the Dirichlet theorem or more general the Siegel-Walfisz theorem, the Brun-Titchmarsh theorem, the Bombieri-Vinogradov theorem for the set $\mathcal{S}(x)$. We shall leave these problems to another occasion.

2. PRELIMINARY LEMMAS

In this section, we shall cite two lemmas, which will be needed in the next section. The first one is due to Vaaler (see [1, Theorem A.6]).

Lemma 2.1. *Let $\psi(t) := t - [t] - \frac{1}{2}$. For $x \geq 1$ and $H \geq 1$, we have*

$$\psi(x) = - \sum_{1 \leq |h| \leq H} \Phi\left(\frac{h}{H+1}\right) \frac{e(hx)}{2\pi i h} + R_H(x),$$

where $e(t) := e^{2\pi i t}$, $\Phi(t) := \pi t(1 - |t|) \cot(\pi t) + |t|$ and the error term $R_H(x)$ satisfies

$$(2.1) \quad |R_H(x)| \leq \frac{1}{2H+2} \sum_{0 \leq |h| \leq H} \left(1 - \frac{|h|}{H+1}\right) e(hx).$$

Lemma 2.2. *For $1 \leq a \leq q$ and $\delta \in \{0, 1\}$, define*

$$(2.2) \quad \mathfrak{S}_\delta(D, D') := \sum_{D < d \leq D'} \psi\left(\frac{x}{dq + a + \delta}\right).$$

If (κ, λ) is an exponent pair, then we have

$$(2.3) \quad \mathfrak{S}_\delta(D, D') \ll (x^\kappa D^{-\kappa+\lambda} q^{-\kappa})^{1/(1+\kappa)} + x^\kappa D^{-2\kappa+\lambda} q^{-\kappa} + x^{-1} D^2 q$$

uniformly for $1 \leq a \leq q$, $(\sqrt{x} - a)/q < D \leq (x/q)^{2/3}$ and $D < D' \leq 2D$, where the implied constant depends on (κ, λ) at most.

Proof. Using Lemma 2.1, we can write

$$(2.4) \quad \mathfrak{S}_\delta(D, D') = -\frac{1}{2\pi i} \left(\mathfrak{S}_\delta^b(D, D') + \overline{\mathfrak{S}_\delta^b(D, D')} \right) + \mathfrak{S}_\delta^\dagger(D, D'),$$

where $H \leq D$ and

$$\begin{aligned} \mathfrak{S}_\delta^b(D, D') &:= \sum_{h \leq H} \frac{1}{h} \Phi\left(\frac{h}{H+1}\right) \sum_{D < d \leq D'} e\left(\frac{hx}{dq + a + \delta}\right), \\ \mathfrak{S}_\delta^\dagger(D, D') &:= \sum_{D < d \leq D'} R_H\left(\frac{x}{dq + a + \delta}\right). \end{aligned}$$

Inverting the order of summations and applying the exponent pair (κ, λ) to the sum over d , it follows that

$$(2.5) \quad \begin{aligned} \mathfrak{S}_\delta^b(D, D') &\ll \sum_{h \leq H} \frac{1}{h} \left(\left(\frac{xh}{D^2 q} \right)^\kappa D^\lambda + \left(\frac{xh}{D^2 q} \right)^{-1} \right) \\ &\ll x^\kappa D^{-2\kappa+\lambda} q^{-\kappa} H^\kappa + x^{-1} D^2 q. \end{aligned}$$

On the other hand, (2.1) of Lemma 2.1 allows us to derive that

$$\begin{aligned} |\mathfrak{S}_\delta^\dagger(D, D')| &\leq \sum_{D < d \leq D'} \left| R_H \left(\frac{x}{dq + a + \delta} \right) \right| \\ &\leq \frac{1}{2H+2} \sum_{0 \leq |h| \leq H} \left(1 - \frac{|h|}{H+1} \right) \sum_{D < d \leq D'} e \left(\frac{xh}{dq + a + \delta} \right). \end{aligned}$$

When $h \neq 0$, as before we apply the exponent pair (κ, λ) to the sum over d and obtain

$$(2.6) \quad \mathfrak{S}_\delta^\dagger(D, D') \ll DH^{-1} + x^\kappa D^{-2\kappa+\lambda} q^{-\kappa} H^\kappa + x^{-1} D^2 q.$$

Inserting (2.5) and (2.6) into (2.4), it follows that

$$\mathfrak{S}_\delta(D, D') \ll DH^{-1} + x^\kappa D^{-2\kappa+\lambda} q^{-\kappa} H^\kappa + x^{-1} D^2 q$$

for $H \leq D$. Optimising H on $[1, D]$, we obtain the required inequality (2.3). \square

3. PROOF OF THEOREM 1

If $\left[\frac{x}{n}\right] = m = dq + a$ with $0 \leq d \leq (x-a)/q$, then $x/(dq + a + 1) < n \leq x/(dq + a)$. Thus we can write

$$\begin{aligned} (3.1) \quad S(x; q, a) &= \sum_{d \leq (x-a)/q} \mathbb{1} \left(\left[\frac{x}{dq + a} \right] - \left[\frac{x}{dq + a + 1} \right] > 0 \right) + O(1) \\ &= S_1(x; q, a) + S_2(x; q, a) + O(1), \end{aligned}$$

where $\mathbb{1} = 1$ if the statement is true and 0 otherwise, and

$$\begin{aligned} S_1(x; q, a) &:= \sum_{d \leq (\sqrt{x}-a)/q} \mathbb{1} \left(\left[\frac{x}{dq + a} \right] - \left[\frac{x}{dq + a + 1} \right] > 0 \right), \\ S_2(x; q, a) &:= \sum_{(\sqrt{x}-a)/q < d \leq (x-a)/q} \mathbb{1} \left(\left[\frac{x}{dq + a} \right] - \left[\frac{x}{dq + a + 1} \right] > 0 \right). \end{aligned}$$

For $d \leq (\sqrt{x}-a-1)/q$, we have

$$\left[\frac{x}{dq + a} \right] - \left[\frac{x}{dq + a + 1} \right] > \frac{x}{(dq + a)(dq + a + 1)} - 1 > 0.$$

Thus we have

$$(3.2) \quad S_1(x; q, a) = \frac{\sqrt{x}}{q} + O(1)$$

for $x \geq 3$, where the implied constant is absolute.

Next we treat $S_2(x; q, a)$. Noticing that for $d > (\sqrt{x}-a)/q$ we have

$$0 < \frac{x}{dq + a} - \frac{x}{dq + a + 1} = \frac{x}{(dq + a)(dq + a + 1)} < 1,$$

the quantity $\left[\frac{x}{dq+a}\right] - \left[\frac{x}{dq+a+1}\right]$ can only equal to 0 or 1. On the other hand, for $d \geq (x/q)^{2/3}$, then $dq + a = \left[\frac{x}{n}\right]$ for some $n \leq (x/q)^{1/3}$. Thus we can write

$$S_2(x; q, a) = \sum_{(\sqrt{x}-a)/q < d \leq (x/q)^{2/3}} \left(\left[\frac{x}{dq+a}\right] - \left[\frac{x}{dq+a+1}\right] \right) + O((x/q)^{1/3}).$$

Noticing that

$$\left[\frac{x}{dq+a}\right] - \left[\frac{x}{dq+a+1}\right] = \frac{x}{dq+a} - \frac{x}{dq+a+1} - \psi\left(\frac{x}{dq+a}\right) + \psi\left(\frac{x}{dq+a+1}\right),$$

it follows that

$$(3.3) \quad S_2(x; q, a) = S_{2,1}(x; q, a) - S_{2,2}^{(0)}(x; q, a) + S_{2,2}^{(1)}(x; q, a) + O((x/q)^{1/3}),$$

where

$$S_{2,1}(x; q, a) := \sum_{(\sqrt{x}-a)/q < p \leq (x/q)^{2/3}} \left(\frac{x}{dq+a} - \frac{x}{dq+a+1} \right),$$

$$S_{2,2}^{(\delta)}(x; q, a) := \sum_{(\sqrt{x}-a)/q < d \leq (x/q)^{2/3}} \psi\left(\frac{x}{dq+a+\delta}\right).$$

Firstly an elementary computation shows that

$$(3.4) \quad S_{2,1}(x; q, a) = \sum_{(\sqrt{x}-a)/q < d \leq (x/q)^{2/3}} \frac{x}{d^2 q^2} + O(1) = \frac{\sqrt{x}}{q} + O((x/q^4)^{1/3})$$

for $x \rightarrow \infty$.

It remains to bound $S_{2,2}^{(\delta)}(x; q, a)$. According to [1, Theorem 3.10]), $(\frac{1}{2}, \frac{1}{2})$ is an exponent pair. Thus we can take $(\kappa, \lambda) = (\frac{1}{2}, \frac{1}{2})$ in (2.3) of Lemma 2.2 to get

$$\mathfrak{S}_\delta(D, D') \ll (x/q)^{1/3} + (xD^{-1}q^{-1})^{1/2} + x^{-1}D^2q$$

uniformly for $1 \leq a \leq q$, $(\sqrt{x}-a)/q < D \leq (x/q)^{2/3}$ and $D < D' \leq 2D$. Using $(\sqrt{x}-a)/q < D \leq (x/q)^{2/3}$, we easily see that the preceding inequality implies

$$\mathfrak{S}_\delta(D, D') \ll (x/q)^{1/3}$$

uniformly for $1 \leq a \leq q$, $(\sqrt{x}-a)/q < D \leq (x/q)^{2/3}$ and $D < D' \leq 2D$. From this, we can derive that

$$(3.5) \quad S_{2,2}^{(\delta)}(x; q) \ll (\log x) \max_{(\sqrt{x}-a)/q < D \leq (x/q)^{2/3}} |\mathfrak{S}_\delta(D, 2D)|$$

$$\ll (x/q)^{1/3} \log x.$$

Inserting (3.4) and (3.5) into (3.3), we obtain

$$(3.6) \quad S_2(x; q, a) = \frac{\sqrt{x}}{q} + O((x/q)^{1/3} \log x)$$

for $x \rightarrow \infty$.

Now the required result follows from (3.1), (3.2) and (3.6). \square

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