STABILITY AND LARGE-TIME BEHAVIOR FOR THE 2D BOUSSINESQ SYSTEM WITH VERTICAL DISSIPATION AND HORIZONTAL THERMAL DIFFUSION

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STABILITY AND LARGE-TIME BEHAVIOR FOR THE 2D BOUSSINESQ SYSTEM WITH VERTICAL DISSIPATION AND HORIZONTAL THERMAL DIFFUSION

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Abstract. This paper addresses the stability and large-time behavior problem on the perturbations near the hydrostatic balance of the two dimensional Boussinesq system, taking into account vertical dissipation and horizontal thermal diffusion. The spatial framework Ω is defined as $\mathbb{T}\times\mathbb{R}$, where $\mathbb{T}$ spans $[0,1]$, representing the 1D periodic box, while $\mathbb{R}$ denotes the whole line. The results outlined in this article confirm the fact that the temperature can actually have a stabilizing effect on the buoyancy-driven fluids. The stability and long-time behavior issues discussed here are difficult due to the lack of the horizontal dissipation and vertical thermal diffusion. By formulating in the appropriate energy functional and implementing the orthogonal decomposition of the velocity and the temperature into their horizontal averages and oscillation parts, we are able to make up for the missing regularization and establish the nonlinear stability in the Sobolev space $H^2(\Omega)$ and acheive the algebraic decay rates for the oscillation parts in the H^1-norm.

1. Introduction

This paper focuses on the following 2D anisotropic Boussinesq system

$$\begin{cases} \partial_t U + U\cdot\nabla U = -\nabla P + \nu\,\partial_{22}U + g_0\Theta\mathbf{e}_2, \quad x\in\Omega,\ t>0,\\ \partial_t\Theta + U\cdot\nabla\Theta = \eta\,\partial_{11}\Theta,\\ \nabla\cdot U = 0, \end{cases} \tag{1.1}$$

where U denotes the fluid velocity, P the pressure, Θ the temperature, $\nu>0$ and $\eta>0$ are parameters representing the kinematic viscosity and the thermal diffusivity, respectively. Here $\mathbf{e}_2=(0,1)$ is the unit vector in the vertical direction, g_0 is a non zero constant and the spatial domain Ω is taken to be

$$\Omega=\mathbb{T}\times\mathbb{R},$$

with $\mathbb{T}=[0,1]$ being a 1D periodic box and $\mathbb{R}$ being the whole line. This partially dissipated system models anisotropic buoyancy-driven fluids in the circumstance when the horizontal dissipation and the vertical thermal diffusion are negligible [24].

The Boussinesq systems stand out as the most commonly employed models for studying atmospheric and oceanographic flows (see, e.g., [4], [12], [22]). Recent research has been focused on addressing two fundamental challenges related to these

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equations, namely, global existence and regularity problem and the stability problem on perturbations near various physically relevant equilibrium states (see, e.g., [1], [2], [5], [9], [10], [13], [14], [15], [16], [17], [18], [19], [20], [21], [28]).

This work intends to show the $H^2(\Omega)-$stability and examine the the large-time behavior of perturbations near the hydrostatic equilibrium (U_{he}, Θ_{he}) with

$$U_{he} = 0, \quad \Theta_{he} = g_0 x_2.$$

For the velocity U_{he}, the momentum equation is fulfilled when the pressure gradient is balanced by the buoyancy force, namely

$$-\nabla P_{he} + g_0 \Theta_{he}\, \mathbf{e}_2 = 0 \quad \text{or} \quad P_{he} = \frac{1}{2} g_0^2 x_2^2.$$

To examine the stability problem, we need first to write down the equations for the perturbation (u, p, θ), where

$$u = U - U_{he}, \quad p = P - P_{he} \quad \text{and} \quad \theta = \Theta - \Theta_{he}.$$

It is evident from equations (1.1) that (u, p, θ) satisfies the following anisotropic Boussinesq equations with vertical dissipation and horizontal thermal diffusion

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu\, \partial_{22} u + g_0 \theta \mathbf{e}_2, \quad x \in \Omega,\ t > 0, \\ \partial_t \theta + u \cdot \nabla \theta + g_0 u_2 = \eta\, \partial_{11} \theta, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x). \end{cases} \tag{1.2}$$

The difference between the original system (1.1) and the system governing the perturbations (1.2) is that the temperature equation in (1.2) contains $g_0 u_2$. Without this extra term, the L^2-norm of the velocity u to (1.1) can grow in time due to the buoyancy forcing term $g_0 \theta \mathbf{e}_2$. With even full dissipation and thermal diffusion, as taken in [3], solutions of the 3D Boussinesq equations can actually grow in time. This term in (1.2) contributes to balancing $g_0 \theta \mathbf{e}_2$ in the energy estimates. Consequently, the buoyancy forcing ceases to have a destabilizing impact in (1.2). In cases where dissipation is degenerate and is only one-directional as in (1.1), it is not clear how the solution would behave.

When the spacial domain is the whole space $\mathbb{R}^2$, the lack of the horizontal dissipation complicates the control of the growth of the vorticity $\omega = \nabla \times u$, satisfying

$$\partial_t \omega + u \cdot \nabla \omega = \nu\, \partial_{22} \omega + g_0 \partial_1 \theta, \quad x \in \mathbb{R}^2,\ t > 0. \tag{1.3}$$

More precisely, it is feasible to derive a uniform bound on the L^2-norm of the vorticity ω itself. Nonetheless, controlling the L^2-norm of the gradient of the vorticity, $\nabla \omega$, does not seem achievable. In particular, if the temperature θ is zero, (1.3) reduces to the 2D Navier-Stokes equation with degenerate dissipation,

$$\partial_t \omega + u \cdot \nabla \omega = \nu\, \partial_{22} \omega, \quad x \in \mathbb{R}^2,\ t > 0. \tag{1.4}$$

While (1.4) always has a unique global solution ω for any initial data $\omega_0 \in H^1(\mathbb{R}^2)$, the question of whether $\|\nabla \omega(t)\|_{L^2}$ for the solution ω of (1.4) grows with respect to t remains an open problem.

Furthermore, when there is no dissipation at all, namely when $\nu = 0$, (1.4) takes the form of the 2D Euler vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega = 0, \quad x \in \mathbb{R}^2, \ t > 0. \tag{1.5}$$

As pointed out in many works (see, e.g., [8], [13], [31]), $\nabla\omega(t)$ of (1.5) can grow even double exponentially in time. Particularly, the velocity of the 2D Euler equations in the Sobolev space H^2 is not stable. Conversely, solutions to the 2D Navier-Stokes equations with full dissipation

$$\partial_t \omega + u \cdot \nabla \omega = \nu \, \Delta \omega, \quad x \in \mathbb{R}^2, \ t > 0$$

decays algebraically in time, as shown by Schonbek (see, e.g., [25], [26]). The abscence of the horizontal dissipation in (1.4) hinders our ability to follow the approach used for the fully dissipative Navier-Stokes equations. Specifically, when applying the energy method to bound $\|\nabla\omega(t)\|_{L^2}$, namely

$$\frac{1}{2}\frac{d}{dt}\|\nabla\omega(t)\|_{L^2}^2 + \nu\|\partial_2\nabla\omega(t)\|_{L^2}^2 = -\int \nabla\omega \cdot \nabla u \cdot \nabla\omega \, dx, \tag{1.6}$$

the one-directional dissipation is not enough to control the nonlinearity. The challenge lies in acquiring a suitable upper bound for the term on the right-hand side of (1.6). To effectively leverage the anisotropic dissipation, we naturally decompose this term further into four component terms.

$$\begin{aligned}\int \nabla\omega \cdot \nabla u \cdot \nabla\omega \, dx = &\int \partial_1 u_1 \, (\partial_1\omega)^2 \, dx + \int \partial_1 u_2 \, \partial_1\omega \, \partial_2\omega \, dx \\ &+ \int \partial_2 u_1 \, \partial_1\omega \, \partial_2\omega \, dx + \int \partial_2 u_2 \, (\partial_2\omega)^2 \, dx.\end{aligned} \tag{1.7}$$

Without horizontal dissipation, establishing a time-integrable upper bound for the first two terms in (1.7) is not possible.

When dealing with the stability problem on (1.2), we come across the same nonlinear term presented in (1.7). Fortunately, the smoothing and stabilization effect of the temperature through the coupling and interaction allows us to solve the stability problem in (1.2). To reveal these effects, we start by eliminating the pressure term in (1.2). Applying the Helmholtz-Leray projection $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ to the velocity equation in (1.2), we get

$$\partial_t u = \nu\partial_{22} u + \mathbb{P}(g_0\theta\mathbf{e}_2) - \mathbb{P}(u \cdot \nabla u). \tag{1.8}$$

Using the definition of the Leray projection $\mathbb{P}$,

$$\mathbb{P}(g_0\theta\mathbf{e}_2) = g_0\theta\mathbf{e}_2 - \nabla\Delta^{-1}\nabla \cdot (g_0\theta\mathbf{e}_2) = g_0 \begin{bmatrix} -\partial_1\partial_2\Delta^{-1}\theta \\ \theta - \partial_2^2\Delta^{-1}\theta \end{bmatrix}. \tag{1.9}$$

Substituting (1.9) into (1.8) and expressing (1.8) in terms of its component equations, yields

$$\begin{cases} \partial_t u_1 = \nu \, \partial_{22} u_1 - g_0\partial_1\partial_2\Delta^{-1}\theta + N_1, \\ \partial_t u_2 = \nu \, \partial_{22} u_2 + g_0\partial_1\partial_1\Delta^{-1}\theta + N_2, \end{cases} \tag{1.10}$$

where N_1 and N_2 represent the nonlinear terms,

$$N_1 = -(u \cdot \nabla u_1 - \partial_1\Delta^{-1}\nabla \cdot (u \cdot \nabla u)), \quad N_2 = -(u \cdot \nabla u_2 - \partial_2\Delta^{-1}\nabla \cdot (u \cdot \nabla u)).$$

Differentiating the first equation of (1.10) with respect to t, we get

$$\partial_{tt}u_1 = \nu\partial_{22}\partial_t u_1 - g_0\partial_1\partial_2\Delta^{-1}\partial_t\theta + \partial_t N_1.$$

Using the equation of θ in (1.2), we substitute $\partial_t\theta$ in the above equation with $\eta\,\partial_{11}\theta - g_0u_2 - u\cdot\nabla\theta$ to write

$$\partial_{tt}u_1 = \nu\partial_{22}\partial_t u_1 + g_0^2\partial_1\partial_2\Delta^{-1}u_2 - g_0\eta\,\partial_{11}\partial_1\partial_2\Delta^{-1}\theta + g_0\partial_1\partial_2\Delta^{-1}(u\cdot\nabla\theta) + \partial_t N_1.$$

Additionally, replacing $g_0\partial_1\partial_2\Delta^{-1}\theta$ by the first component equation of (1.10), namely

$$-g_0\partial_1\partial_2\Delta^{-1}\theta = \partial_t u_1 - \nu\,\partial_{22}u_1 - N_1,$$

we find

$$\begin{aligned}\partial_{tt}u_1 = \nu\partial_{22}\partial_t u_1 + g_0^2\partial_1\partial_2\Delta^{-1}u_2 + \eta\,\partial_{11}(\partial_t u_1 - \nu\,\partial_{22}u_1 - N_1)\\ +g_0\,\partial_1\partial_2\Delta^{-1}(u\cdot\nabla\theta) + \partial_t N_1,\end{aligned}$$

which in turn gives, due to the divergence-free condition $\partial_2 u_2 = -\partial_1 u_1$,

$$\partial_{tt}u_1 - (\eta\partial_{11} + \nu\partial_{22})\partial_t u_1 + \nu\eta\partial_{11}\partial_{22}u_1 + g_0^2\partial_{11}\Delta^{-1}u_1 = N_3, \tag{1.11}$$

where N_3 is the nonlinear term,

$$N_3 = (\partial_t - \eta\partial_{11})N_1 + g_0\partial_1\partial_2\Delta^{-1}(u\cdot\nabla\theta).$$

Following the same procedure, we can easily show that u_2 and θ satisfy

$$\begin{aligned}&\partial_{tt}u_2 - (\eta\partial_{11} + \nu\partial_{22})\partial_t u_2 + \nu\eta\partial_{11}\partial_{22}u_2 + g_0^2\partial_{11}\Delta^{-1}u_2 = N_4,\\ &\partial_{tt}\theta - (\eta\partial_{11} + \nu\partial_{22})\partial_t\theta + \nu\eta\partial_{11}\partial_{22}\theta + g_0^2\partial_{11}\Delta^{-1}\theta = N_5\end{aligned} \tag{1.12}$$

with

$$\begin{aligned}N_4 &= (\partial_t - \eta\partial_{11})N_2 - g_0\partial_1\partial_1\Delta^{-1}(u\cdot\nabla\theta),\\ N_5 &= -(\partial_t - \nu\partial_{22})(u\cdot\nabla\theta) - g_0N_2.\end{aligned}$$

Then, merging (1.11) and (1.12) and expressing them into the velocity vector form, we have reformulated (1.2) into the following new system

$$\begin{cases}\partial_{tt}u - (\eta\partial_{11} + \nu\partial_{22})\partial_t u + \nu\eta\partial_{11}\partial_{22}u + g_0^2\partial_{11}\Delta^{-1}u = N_6,\\ \partial_{tt}\theta - (\eta\partial_{11} + \nu\partial_{22})\partial_t\theta + \nu\eta\partial_{11}\partial_{22}\theta + g_0^2\partial_{11}\Delta^{-1}\theta = N_5,\end{cases} \tag{1.13}$$

where

$$N_6 = (N_3, N_4) = -(\partial_t - \eta\partial_{11})\mathbb{P}(u\cdot\nabla u) + g_0\nabla^\perp\partial_1\Delta^{-1}(u\cdot\nabla\theta)$$

with $\nabla^\perp = (\partial_2, -\partial_1)$. By applying the curl of the velocity equation, we can likewise transform (3.44) into a system of ω and θ,

$$\begin{cases}\partial_{tt}\omega - (\eta\partial_{11} + \nu\partial_{22})\partial_t\omega + \nu\eta\partial_{11}\partial_{22}\omega + g_0^2\partial_{11}\Delta^{-1}\omega = N_7,\\ \partial_{tt}\theta - (\eta\partial_{11} + \nu\partial_{22})\partial_t\theta + \nu\eta\partial_{11}\partial_{22}\theta + g_0^2\partial_{11}\Delta^{-1}\theta = N_5,\end{cases}$$

with

$$N_7 = -(\partial_t - \eta\partial_{11})(u\cdot\nabla\omega) - g_0\partial_1(u\cdot\nabla\theta).$$

Remarkably, we observe that all physical quantities u, θ and ω obey the same damped degenerate wave equation, differing only in their respective nonlinear terms. Compared to the original system of (u, θ) in (1.2), the wave equations (1.13) reveal the underlying smoothing and stabilization hidden in (1.2). In (1.2), where horizontal

dissipation is absent in the velocity field, the wave structure implies that the temperature can stabilize the fluids by creating the horizontal regularization through the coupling and interaction. By taking advantage of these effects, the stability problem on (1.2) was recently established by Ben Said and al in [1] when the spacial domain is the whole plane $\mathbb{R}^2$. However, the large time behaviour of the solution in $\mathbb{R}^2$ remains a mystery. When the spatial domain is $\Omega = \mathbb{T} \times \mathbb{R}$, this paper also proves the stability of (1.2). Additionally, we analyze the large-time behavior of the solutions. The core idea involves breaking down a physical quantity into its horizontal average and the associated oscillation. Specifically, for a function $f = f(x_1, x_2)$ defined on $\mathbb{T} \times \mathbb{R}$ and integrable in x_1 over the 1D periodic box $\mathbb{T} = [0, 1]$, we define its horizontal average $\overline{f}$ by

$$\overline{f}(x_2) = \int_{\mathbb{T}} f(x_1, x_2) dx_1, \tag{1.14}$$

and we write,

$$f = \overline{f} + \widetilde{f}. \tag{1.15}$$

Note here that, the horizontal average $\overline{f}$ corresponds to the zeroth Fourier mode of f while $\widetilde{f}$ contains all non-zero Fourier modes.

The decomposition (1.15) possesses distinct properties. To begin with, this decomposition is orthogonal in the Sobolev space $H^k(\Omega)$ for any non-negative integer. This implies that the H^k-norms of $\overline{f}$ and $\widetilde{f}$ are bounded by the H^k-norm of f. Furthermore, this decomposition commutes with derivatives, and $\overline{f}$ and $\widetilde{f}$ of a divergence-free vector field f are also divergence-free. An essential property to be frequently used in our estimates is that $\widetilde{f}$ admits strong versions of the Poincaré type inequality

$$\|\widetilde{f}\|_{L^2(\Omega)} \leq C \|\partial_1 \widetilde{f}\|_{L^2(\Omega)}, \quad \|\widetilde{f}\|_{L^\infty(\Omega)} \leq C \|\partial_1 \widetilde{f}\|_{H^1(\Omega)}.$$

Applying this decomposition to the velocity field u and the temperature θ, namely writing

$$u = \overline{u} + \widetilde{u}, \quad \theta = \overline{\theta} + \widetilde{\theta}$$

and exploiting the aforementioned properties we can effectively handle the nonlinear terms in (1.7) appropriately, even when there is only vertical dissipation. More precisely, the following theorems hold. Theorem 1.1 establishes the H^2-stability while Theorem 1.2 specifies the decay rates of the oscillation part $(\widetilde{u}, \widetilde{\theta})$.

Theorem 1.1. *Let $\mathbb{T} = [0, 1]$ be a 1D periodic box and let $\Omega = \mathbb{T} \times \mathbb{R}$. Assume $u_0, \theta_0 \in H^2(\Omega)$ and $\nabla \cdot u_0 = 0$. Then there exists $\varepsilon = \varepsilon(\nu, \eta) > 0$ such that, if*

$$\|u_0\|_{H^2} + \|\theta_0\|_{H^2} \leq \varepsilon,$$

then (1.2) has a unique global solution (u, θ) that remains uniformly bounded for all time, for any $t \geq 0$,

$$\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 + 2\nu \int_0^t \|\partial_2 u(\tau)\|_{H^2}^2 d\tau$$

$$+ 2\eta \int_0^t \|\partial_1\theta(\tau)\|_{H^2}^2 d\tau + C(\nu,\eta)\int_0^t \|g_0\partial_1 u_2\|_{L^2}^2 d\tau \leq C\varepsilon^2$$

where $C(\nu,\eta)$ and $C>0$ are constants.

Theorem 1.1 states that any small initial perturbation, in the H^2-sense, leads to a unique global, in time, solution of (1.2) that remains small in H^2 for all time t. Furthermore, it implies that the time-integral of $\|\partial_1 u_2(\tau)\|_{L^2}^2$ is finite.

The following Theorem asserts that the oscillation portion $(\widetilde{u},\widetilde{\theta})$ decays to zero algebraically in time in the H^1-norm. This result aligns with the stratification phenomenon of buoyancy driven fluids. Additionally, it affirms the observation derived from the numerical simulations presented in [9], indicating that the temperature becomes horizontally homogeneous and stratify in the vertical direction over time.

Theorem 1.2. *Let $u_0,\theta_0 \in H^2(\Omega)$ with $\nabla\cdot u_0 = 0$. Assume that (u_0,θ_0) satisfies*

$$\|u_0\|_{H^2} + \|\theta_0\|_{H^2} \leq \varepsilon,$$

for sufficiently small $\varepsilon>0$. Let (u,θ) be the corresponding solution of (1.2) with g_0 negative constant. Then the oscillation part $(\widetilde{u},\widetilde{\theta})$ satisfies the following algebraic decay in time,

$$\|\widetilde{u}\|_{H^1} + \|\widetilde{\theta}\|_{H^1} \leq c(1+t)^{-\frac{1}{2}},$$

for some constant $c>0$ and for all $t\geq 0$. In addition, $(\widetilde{u},\widetilde{\theta})$ has the asymptotic behavior, as $t\to\infty$,

$$t\left(\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2\right) \to 0.$$

According to Theorem 1.2, the solution (u,θ) of (1.2) approaches its horizontal average $(\overline{u},\overline{\theta})$ asymptotically, and eventually, the Boussinesq equations (1.2) evolves to the following 1D system

$$\begin{cases} \partial_t \overline{u} + \overline{u\cdot\nabla\widetilde{u}} + \begin{pmatrix} 0 \\ \partial_2\overline{p}\end{pmatrix} = g_0\begin{pmatrix}0\\ \overline{\theta}\end{pmatrix} + \nu\partial_2^2\overline{u}, \\ \partial_t\overline{\theta} + \overline{u\cdot\nabla\widetilde{\theta}} = 0. \end{cases}$$

We briefly outline the proofs for Theorem 1.1 and Theorem 1.2. As the local, in time, well-posedness on (1.2) in the Sobolev setting $H^2(\Omega)$ can be established using standard approaches such as Friedrichs' Fourier cutoff (see, e.g., [23]), the proof of Theorem 1.1 is essentially reduced to demonstrating the global, in time, *a priori* bound on the solution in $H^2(\Omega)$. To do so, we make use of the bootstrapping argument (see [29], p 20). To set it up, we introduce the following energy functional for the H^2-solution,

$$\begin{aligned} E(t) = \max_{0\leq\tau\leq t}(\|u(\tau)\|_{H^2}^2 + \|\theta(\tau)\|_{H^2}^2) + 2\nu\int_0^t \|\partial_2 u\|_{H^2}^2 d\tau \\ +2\eta\int_0^t \|\partial_1\theta\|_{H^2}^2 d\tau + \delta\int_0^t \|g_0\partial_1 u_2\|_{L^2}^2\, d\tau, \end{aligned} \tag{1.16}$$

where $\delta > 0$ is a suitably selected small parameter. Our central objective, is to show that, for a constant C uniform and for all $t > 0$,

$$E(t) \le C\, E(0) + C\, E(t)^{\frac{3}{2}}. \tag{1.17}$$

To prove (1.17), we should make full use of the extra regularization resulting from the wave structure in (1.13). Furthermore, the control on the time integral of the horizontal derivative of the velocity field, namely

$$\int_0^t \|g_0 \partial_1 u_2(\tau)\|_{L^2}^2 \, d\tau \tag{1.18}$$

plays an improtant role our proof. Note here, that the uniform boundedness of (1.18) is not a consequence of the vertical dissipation in the velocity equation but due to the interaction with the temperature equation. In fact, using the equation of θ in (3.61), we represent $g_0 \partial_1 u_2$ as,

$$g_0 \partial_1 u_2 = -\partial_t \partial_1 \theta - \partial_1 (u \cdot \nabla \theta) + \eta \partial_{111} \theta,$$

then

$$\|g_0 \partial_1 u_2\|_{L^2}^2 = -g_0 \int \partial_t \partial_1 \theta \, \partial_1 u_2 \, dx - g_0 \int \partial_1 u_2 \, \partial_1 (u \cdot \nabla \theta) \, dx + \eta g_0 \int \partial_1 u_2 \, \partial_{111} \theta \, dx.$$

Hence, the time integrability of $\|g_0 \partial_1 u_2\|_{L^2}^2$ is converted to the time integrability of other terms. This phenomenon of extra regularization and time integrability, resulting from the coupling, is also observed in other models of partial differential equations, such as the Oldroyd-B system (see [7], [11]).

Once (1.17) is proven, the bootstrapping argument then gives that, if

$$E(0) = \|(u_0, \theta_0)\|_{H^2}^2 \le \varepsilon^2$$

for some sufficiently small $\varepsilon > 0$, then $E(t)$ remains uniformly small for all time, namely

$$E(t) \le C\, \varepsilon^2 \tag{1.19}$$

for a constant $C > 0$ and for all $t \ge 0$. In particular, (1.19) yields the desired global H^2-bound on the solution (u, θ). We leave details on the application of the bootstrapping argument in the proof of Theorem 1.1 in Section 3.

To demonstrate the algebraic decay rates on the H^1-norm of the oscillation component, as stated in Theorem 1.2, we initially take the difference of (1.2) and its horizontal average, to write down the system governing the oscillation part $(\widetilde{u}, \widetilde{\theta})$

$$\begin{cases} \partial_t \widetilde{u} + \widetilde{u \cdot \nabla \widetilde{u}} + \widetilde{u}_2 \partial_2 \overline{u} - \nu \partial_2^2 \widetilde{u} + \nabla \widetilde{p} = g_0 \widetilde{\theta} e_2, \\ \partial_t \widetilde{\theta} + \widetilde{u \cdot \nabla \widetilde{\theta}} + \widetilde{u}_2 \partial_2 \overline{\theta} - \eta \partial_1^2 \widetilde{\theta} + g_0 \widetilde{u}_2 = 0. \end{cases} \tag{1.20}$$

Controling the H^1-norm of $(\widetilde{u}, \widetilde{\theta})$ naturally involves estimating the L^2-norms $\|(\widetilde{u}, \widetilde{\theta})\|_{L^2}$ and $\|(\nabla \widetilde{u}, \nabla \widetilde{\theta})\|_{L^2}$. Here, one major difficulty is that the equation of $\widetilde{u}$ has only vertical dissipation, however, the aforementioned Poincaré inequality can only bound a function in terms of its horizontal derivatives. Consequently, some of the nonlinear parts associated with $\widetilde{u}$ can not be bounded suitably and require the upper bounds

involving $\|\widetilde{u}_2\|_{L^2}$. To handle these terms, we seek extra smoothing and stabilizing effect on $\widetilde{u}_2$ by exploiting the coupling in (1.20). Specifically, we introduce the following extra term along with the H^1-norm to form a Lyapunov functional,

$$-\delta(\widetilde{u}_2, \widetilde{\theta}),$$

where $\delta > 0$ is a small constant and $(\widetilde{u}_2, \widetilde{\theta})$ denotes the L^2-inner product. The time derivative of this terme produces $\delta\|\widetilde{u}_2\|_{L^2}^2$, which help balance $\|\widetilde{u}_2\|_{L^2}^2$ from the nonlinearity. Then, applying anisotropic inequalities presented in section 2, we demonstrate the following energy inequality.

$$\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2 - \delta(\widetilde{u}_2, \widetilde{\theta})\Big) + \nu\|\partial_2\widetilde{u}\|_{H^1}^2 + \eta\|\partial_1\widetilde{\theta}\|_{H^1}^2 + \frac{\delta}{4}\|\widetilde{u}_2\|_{L^2}^2 \leq 0,$$

resulting the desired algebraic decay stated in Theorem 1.2. More details are given in Section 4.

The subsequent sections are organized as follows: Section 2 presents various anisotropic inequalities and some crucial properties related to the orthogonal decomposition, including the Poincaré type inequality for the oscillation portion $\widetilde{f}$. Section 3 is dedicated to the proof of Theorem 1.1 and Section 4 proves Theorem 1.2.

2. Preliminaries

This Section serves as preparation for the proof of Theorems 1.1 and 1.2. Lemma 2.1 through Lemma 2.5 provide several frequently used facts on the orthogonal decomposition. While Lemma 2.6 presents a precise decay rate for a nonnegative integrable function, which is also monotonic in a generalized sense.

We start first, by presenting some basic properties of $\overline{f}$ and $\widetilde{f}$.

Lemma 2.1. *Let $\Omega = \mathbb{T} \times \mathbb{R}$. Assume that f defined on Ω is sufficiently regular, say $f \in H^2(\Omega)$. Let $\overline{f}$ and $\widetilde{f}$ be defined as in (1.14) and (1.15). Then*
(a) The average operator $\overline{f}$ and the oscillation operator $\widetilde{f}$ commute with partial derivatives,

$$\overline{\partial_1 f} = \partial_1\overline{f} = 0, \quad \overline{\partial_2 f} = \partial_2\overline{f}, \quad \widetilde{\partial_1 f} = \partial_1\widetilde{f}, \quad \widetilde{\partial_2 f} = \partial_2\widetilde{f}, \quad \overline{\widetilde{f}} = 0.$$

(b) If f is a divergence-free vector field, namely $\nabla \cdot f = 0$, then $\overline{f}$ and $\widetilde{f}$ are also divergence-free,

$$\nabla \cdot \overline{f} = 0 \quad \textit{and} \quad \nabla \cdot \widetilde{f} = 0.$$

(c) $\overline{f}$ and $\widetilde{f}$ are orthogonal in $\dot{H}^k$ for any integer $k \geq 0$, namely

$$(\overline{f}, \widetilde{f})_{\dot{H}^k(\Omega)} := \int_\Omega \overline{D^k f} \cdot \widetilde{D^k f} dx = 0, \quad \|f\|_{\dot{H}^k(\Omega)}^2 = \|\overline{f}\|_{\dot{H}^k(\Omega)}^2 + \|\widetilde{f}\|_{\dot{H}^k(\Omega)}^2.$$

In particular,

$$\|\overline{f}\|_{\dot{H}^k(\Omega)} \leq \|f\|_{\dot{H}^k(\Omega)} \quad \textit{and} \quad \|\widetilde{f}\|_{\dot{H}^k(\Omega)} \leq \|f\|_{\dot{H}^k(\Omega)}.$$

The orthogonality is actually more general and holds for any integrable functions,

$$\int_\Omega \overline{f} \cdot \widetilde{g}\, dx = 0.$$

Lemma 2.1 can be proven easily using the definition of $\overline{f}$ and $\widetilde{f}$.

The next Lemma compares the 1D Sobolev inequalities on the whole line $\mathbb{R}$ and on bounded domains.

Lemma 2.2. *For any 1D function $f \in H^1(\mathbb{R})$,*

$$\|f\|_{L^\infty(\mathbb{R})} \le \sqrt{2}\, \|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\, \|f'\|_{L^2(\mathbb{R})}^{\frac{1}{2}}.$$

For any bounded domain such as $\mathbb{T} = [0,1]$ and $f \in H^1(\mathbb{T})$,

$$\|f\|_{L^\infty(\mathbb{T})} \le \sqrt{2}\, \|f\|_{L^2(\mathbb{T})}^{\frac{1}{2}}\, \|f'\|_{L^2(\mathbb{T})}^{\frac{1}{2}} + \|f\|_{L^2(\mathbb{T})},$$

in particular, if the function f has mean zero such as the oscillation part $\tilde{f}$,

$$\|f\|_{L^\infty(\mathbb{T})} \le C\, \|f\|_{L^2(\mathbb{T})}^{\frac{1}{2}}\, \|f'\|_{L^2(\mathbb{T})}^{\frac{1}{2}}.$$

The following lemma presents anisotropic upper bounds for triple products as well as for the L^∞-norm on the domain Ω. Anisotropic Sobolev inequalities are powerful tools for dealing with anisotropic models. The whole space version of these type of inequalities has previously been used in [6] in the 2D cases and in [30] in the 3D case.

Lemma 2.3. *Let $\Omega = \mathbb{T} \times \mathbb{R}$. For any $f, g, h \in L^2(\Omega)$ with $\partial_1 f \in L^2(\Omega)$ and $\partial_2 g \in L^2(\Omega)$, then*

$$\Big| \int_\Omega fgh\, dx \Big| \le C \|f\|_{L^2}^{\frac{1}{2}} (\|f\|_{L^2} + \|\partial_1 f\|_{L^2})^{\frac{1}{2}} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_2 g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}\,. \tag{2.1}$$

For any $f \in H^2(\Omega)$, we have

$$\begin{aligned}\|f\|_{L^\infty(\Omega)} \le & C \|f\|_{L^2}^{\frac{1}{4}} (\|f\|_{L^2} + \|\partial_1 f\|_{L^2})^{\frac{1}{4}} \|\partial_2 f\|_{L^2}^{\frac{1}{4}} \\ & \times (\|\partial_2 f\|_{L^2} + \|\partial_1 \partial_2 f\|_{L^2})^{\frac{1}{4}}.\end{aligned}$$

Replacing f in Lemma 2.3 by its oscillation portion $\widetilde{f}$, the lower-order part in (2.1) can be dropped, as presented in the next Lemma.

Lemma 2.4. *Let $\Omega = \mathbb{T} \times \mathbb{R}$. For any $f, g, h \in L^2(\Omega)$ with $\partial_1 f \in L^2(\Omega)$ and $\partial_2 g \in L^2(\Omega)$, then*

$$\Big| \int_\Omega \widetilde{f} gh\, dx \Big| \le C \|\widetilde{f}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{f}\|_{L^2}^{\frac{1}{2}} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_2 g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}. \tag{2.2}$$

For any $f \in H^2(\Omega)$, we have

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \le C \|\widetilde{f}\|_{L^2}^{\frac{1}{4}} \|\partial_1 \widetilde{f}\|_{L^2}^{\frac{1}{4}} \|\partial_2 \widetilde{f}\|_{L^2}^{\frac{1}{4}} \|\partial_1 \partial_2 \widetilde{f}\|_{L^2}^{\frac{1}{4}}\,.$$

The subsequent Lemma states that the oscillation component $\widetilde{f}$ verifies a strong Poincaré type inequality with the upper bound expressed in terms of $\partial_1 \widetilde{f}$ rather than $\nabla \widetilde{f}$.

Lemma 2.5. *Let $\overline{f}$ and $\widetilde{f}$ be defined as in (1.14) and (1.15). If $\|\partial_1 \widetilde{f}\|_{L^2(\Omega)} < \infty$, then*

$$\|\widetilde{f}\|_{L^2(\Omega)} \leq C \|\partial_1 \widetilde{f}\|_{L^2(\Omega)},$$

where C is a pure constant. In addition, if $\|\partial_1 \widetilde{f}\|_{H^1(\Omega)} < \infty$, then

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \leq C \|\partial_1 \widetilde{f}\|_{H^1(\Omega)}.$$

As a direct consequence of Lemma 2.5 and the inequality (2.2), one has

$$\Big| \int_\Omega \widetilde{f} g h \, dx \Big| \leq C \|\partial_1 \widetilde{f}\|_{L^2} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_2 g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}. \tag{2.3}$$

We refer the readers to [10] for detailed proofs of Lemmas 2.1, 2.3, 2.4 and 2.5.

The last lemma precises an explicit decay rate in (2.5) for functions that are integrable and are decreasing in a general sense, namely (2.4).

Lemma 2.6. *Let $f = f(t)$ be a nonnegative function satisfying , for two constants $C_0 > 0$ and $C_1 > 0$,*

$$\int_0^\infty f(\tau) d\tau < C_0 \quad \text{and} \quad f(t) \leq C_1 f(s) \quad \text{for any} \quad 0 \leq s < t. \tag{2.4}$$

Then, for $C_2 = \max\{2C_1 f(0), 4C_0 C_1\}$ and for any $t > 0$,

$$f(t) \leq C_2 (1+t)^{-1}. \tag{2.5}$$

Furthermore, $f(t)$ has the following large-time asymptotic behavior,

$$\lim_{t \to +\infty} t\, f(t) = 0.$$

A detailed proof of Lemma 2.6 can be found in [21].

3. The H^2 Nonlinear Stability

This section proves Theorem. 1.1.

Proof. The proof is naturally divided into two major parts. The first part is for the existence, while the second part is for the uniqueness of solutions to (1.2).

To prove the global existence of solutions, it suffices to establish the energy inequality in (1.17) with $E(t)$ being defined in (1.16). This process consists of two main parts. The first is to estimate the H^2-norm of (u, θ) while the second is to estimate $\|\partial_1 u_2\|_{L^2}^2$ and its time integral.

Note that, for a divergence-free vector field u, namely $\nabla \cdot u = 0$, we have

$$\|\nabla u\|_{L^2} = \|\omega\|_{L^2}, \quad \|\Delta u\|_{L^2} = \|\nabla \omega\|_{L^2},$$

where $\omega = \nabla \times u$ is the vorticity. Then, the H^2-norm of u is equivalent to the sum of the L^2-norms of u, ω and $\nabla\omega$.
Taking the L^2-inner product of (u, θ) with the first two equations in (1.2), we find that the L^2-norm of (u, θ) obeys

$$\begin{aligned}&\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + 2\nu \int_0^t \|\partial_2 u(\tau)\|_{L^2}^2 \, d\tau + 2\eta \int_0^t \|\partial_1 \theta(\tau)\|_{L^2}^2 d\tau \\ &= \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2. \end{aligned} \tag{3.1}$$

Next, we estimate the L^2-norm of $(\omega, \nabla\theta)$. We make use of the vorticity equation and the temperature equation,

$$\begin{aligned}&\partial_t \omega + u \cdot \nabla \omega = \nu \partial_{22} \omega + g_0 \partial_1 \theta, \\ &\partial_t \theta + u \cdot \nabla \theta + g_0 u_2 = \eta \partial_{11} \theta. \end{aligned} \tag{3.2}$$

Dotting the equations of ω and $\nabla\theta$ by $(\omega, \nabla\theta)$, yields

$$\frac{1}{2}\frac{d}{dt}(\|\omega\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \nu\|\partial_2\omega\|_{L^2}^2 + \eta\|\partial_1\nabla\theta\|_{L^2}^2 = I_1 + I_2, \tag{3.3}$$

where

$$I_1 = g_0 \int (\partial_1\theta \, \omega - \nabla u_2 \cdot \nabla\theta) \, dx, \quad I_2 = -\int \nabla\theta \cdot \nabla u \cdot \nabla\theta \, dx.$$

Then, expressing ω and u in terms of the stream function ψ, namely $\omega = \Delta\psi$ and $u = \nabla^\perp\psi := (-\partial_2\psi, \partial_1\psi)$, we get

$$\begin{aligned} I_1 &= g_0 \int (\partial_1\theta \, \omega - \nabla u_2 \cdot \nabla\theta) \, dx = g_0 \int (\partial_1\theta\Delta\psi - \nabla\partial_1\psi \cdot \nabla\theta) \, dx \\ &= g_0 \int (-\theta \, \Delta\partial_1\psi + \Delta\partial_1\psi \, \theta) \, dx = 0. \end{aligned}$$

We further write I_2 into four terms,

$$\begin{aligned} I_2 \, &= \, -\int (\partial_1 u_1 (\partial_1\theta)^2 + \partial_1 u_2 \partial_1\theta\partial_2\theta + \partial_2 u_1 \partial_1\theta\partial_2\theta + \partial_2 u_2 (\partial_2\theta)^2) \, dx \\ &:= I_{21} + I_{22} + I_{23} + I_{24}. \end{aligned} \tag{3.4}$$

The key point here is to obtain upper bounds for the terms on the right-hand side of (3.4) that are time integrable. By Lemmas 2.1, 2.4 and Young's inequality, I_{21}, I_{22} and I_{23} can be bounded as follows

$$\begin{aligned} I_{21} : &= -\int \partial_1 u_1 (\partial_1\theta)^2 dx = -\int \partial_1 \widetilde{u_1} (\partial_1\theta)^2 dx \\ &\le c\|\partial_1\theta\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1\theta\|_{L^2}^{\frac{1}{2}} \|\partial_1\theta\|_{L^2}^{\frac{1}{2}} \|\partial_1\partial_1\theta\|_{L^2}^{\frac{1}{2}} \|\partial_1\widetilde{u_1}\|_{L^2} \\ &\le c \, \|u\|_{H^2} \, \|\partial_1\theta\|_{H^2}^2, \end{aligned} \tag{3.5}$$

$$\begin{aligned} I_{22} : &= -\int \partial_1 u_2 \partial_1\theta\partial_2\theta dx = -\int \partial_1 u_2 \partial_1\widetilde{\theta}\partial_2\theta dx \\ &\le c \, \|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_2\theta\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_2\theta\|_{L^2}^{\frac{1}{2}} \|\partial_1 u_2\|_{L^2} \\ &\le c \, \|\partial_1\theta\|_{H^2} \|\theta\|_{H^2} \|\partial_1 u_2\|_{L^2} \end{aligned}$$

$$\leq c\,\|\theta\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2\Big), \tag{3.6}$$

$$\begin{aligned} I_{23} :&= -\int \partial_2 u_1\partial_1\theta\partial_2\theta dx = -\int \partial_2 u_1\partial_1\widetilde{\theta}\partial_2\theta dx\\ &\leq c\,\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2 u_1\|_{L^2}^{\frac12}\|\partial_2\partial_2 u_1\|_{L^2}^{\frac12}\|\partial_2\theta\|_{L^2}\\ &\leq c\,\|\partial_1\theta\|_{H^2}\|\partial_2 u\|_{H^2}\|\theta\|_{H^2}\\ &\leq c\,\|\theta\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2+\|\partial_2 u\|_{H^2}\Big). \end{aligned} \tag{3.7}$$

Using the divergence-free condition $\nabla\cdot u=0$, integration by parts and Lemmas 2.1 and 2.5, we obtain

$$\begin{aligned} I_{24} :&= -\int \partial_2 u_2(\partial_2\theta)^2 dx = \int \partial_1 u_1(\partial_2\theta)^2 dx\\ &= -2\int \widetilde{u_1}\,\partial_2\theta\,\partial_1\partial_2\theta\,dx\\ &\leq c\,\|\partial_2\theta\|_{L^2}\|\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_1\partial_2\theta\|_{L^2}^{\frac12}\|\partial_2\partial_1\partial_2\theta\|_{L^2}^{\frac12}\\ &\leq c\,\|\theta\|_{H^2}\underbrace{\|\partial_1\widetilde{u_1}\|_{L^2}}_{=\|\partial_2\widetilde{u_2}\|_{L^2}}\|\partial_1\theta\|_{H^2}\\ &\leq c\,\|\theta\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2\Big). \end{aligned} \tag{3.8}$$

Hence, collecting the upper bounds on I_2 and inserting them in (3.3), we find

$$\begin{aligned} &\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)+2\nu\|\partial_2\nabla u\|_{L^2}^2+2\eta\|\partial_1\nabla\theta\|_{L^2}^2\\ &\qquad\leq c\,\|(u,\theta)\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2\Big). \end{aligned} \tag{3.9}$$

Thus, integrating (3.9) over $[0,t]$ and combining with (3.1), we get

$$\begin{aligned} &\|(u,\theta)\|_{H^1}^2+2\nu\int_0^t\|\partial_2 u(s)\|_{H^1}^2 ds+2\eta\int_0^t\|\partial_1\theta(s)\|_{H^1}^2 ds\\ &\leq \|(u_0,\theta_0)\|_{H^1}^2+\,c\int_0^t\|(u,\theta)\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2\Big)\,d\tau\\ &\leq E(0)+\,c\,E(t)^{\frac32}. \end{aligned} \tag{3.10}$$

To bound the H^2-norm of (u,θ), it then remains to control the L^2-norm of $(\nabla\omega,\Delta\theta)$. Applying ∇ to the first equation of (3.2) then dotting with $\nabla\omega$, and applying Δ to the second equation of (3.2) then dotting with $\Delta\theta$, we find

$$\frac12\frac{d}{dt}(\|\nabla\omega\|_{L^2}^2+\|\Delta\theta(t)\|_{L^2}^2)+\nu\|\partial_2\nabla\omega\|_{L^2}^2+\eta\|\partial_1\Delta\theta\|_{L^2}^2=J_1+J_2+J_3, \tag{3.11}$$

with

$$J_1=g_0\int(\nabla\partial_1\theta\cdot\nabla\omega-\Delta u_2\Delta\theta)\,dx,$$

$$J_2 = -\int \nabla\omega \cdot \nabla u \cdot \nabla\omega \, dx,$$
$$J_3 = -\int \Delta\theta \cdot \Delta(u \cdot \nabla\theta) dx.$$

Similarly, we need to obtaining an upper bound for that is time integrable for each term in (3.11). Writing ω and u in terms of the stream function ψ, namely $\omega = \Delta\psi$ and $u = \nabla^\perp\psi := (-\partial_2\psi, \partial_1\psi)$, we have

$$\begin{aligned} J_1 &= g_0 \int (\nabla\partial_1\theta \cdot \nabla\omega - \Delta u_2 \Delta\theta)\, dx = g_0 \int (\nabla\partial_1\theta \cdot \nabla\omega - \Delta\partial_1\psi\Delta\theta)\, dx \\ &= g_0 \int (\nabla\partial_1\theta \cdot \nabla\omega - \partial_1\omega\, \Delta\theta)\, dx = g_0 \int (\nabla\partial_1\theta \cdot \nabla\omega + \partial_1\nabla\omega\, \cdot \nabla\theta)\, dx \\ &= g_0 \int \partial_1(\nabla\theta \cdot \nabla\omega)\, dx = 0. \end{aligned}$$

After integration by parts, we decompose J_3 it into four pieces,

$$\begin{aligned} J_3 = &-\int \Delta\theta\, \Delta u_1\, \partial_1\theta\, dx - \int \Delta\theta\, \Delta u_2\, \partial_2\theta\, dx \\ &- 2\int \Delta\theta\, \nabla u_1 \cdot \partial_1\nabla\theta\, dx - 2\int \Delta\theta\, \nabla u_2 \cdot \partial_2\nabla\theta\, dx \\ &:= J_{31} + J_{32} + J_{33} + J_{34}. \end{aligned} \tag{3.12}$$

To deal with J_{31}, we make use of the orthogonal decompositions $u = \overline{u} + \widetilde{u}$ and $\theta = \overline{\theta} + \widetilde{\theta}$ to write

$$\begin{aligned} J_{31} :&= -\int \Delta\theta\Delta u_1\partial_1\theta dx = -\int \Delta\theta\Delta u_1\partial_1\widetilde{\theta} dx \\ &= -\int \Delta\theta\partial_{11}u_1\partial_1\widetilde{\theta} dx - \int \Delta\theta\partial_{22}u_1\partial_1\widetilde{\theta} dx \\ &= \int \Delta\theta\partial_{12}u_2\partial_1\widetilde{\theta} dx - \int \Delta\theta\partial_{22}u_1\partial_1\widetilde{\theta} dx \\ &:= J_{311} + J_{312}. \end{aligned} \tag{3.13}$$

Applying Lemma 2.4 we obtain,

$$\begin{aligned} J_{311} :&= \int \Delta\theta\partial_{12}u_2\partial_1\widetilde{\theta} dx \\ &\le c\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_{12}u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_{12}u_2\|_{L^2}^{\frac{1}{2}}\|\Delta\theta\|_{L^2} \\ &\le c\|\theta\|_{H^2}\|\partial_1\theta\|_{H^2}\|\partial_2 u\|_{H^2} \\ &\le c\|\theta\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\Big), \end{aligned} \tag{3.14}$$

$$\begin{aligned} J_{312} :&= -\int \Delta\theta\partial_{22}u_1\partial_1\widetilde{\theta} dx \\ &\le c\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_{22}u_1\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_{22}u_1\|_{L^2}^{\frac{1}{2}}\|\Delta\theta\|_{L^2} \\ &\le c\|\theta\|_{H^2}\|\partial_1\theta\|_{H^2}\|\partial_2 u\|_{H^2} \end{aligned}$$

$$\leq c\|\theta\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\Big). \tag{3.15}$$

Inserting the upper bounds for J_{311} and J_{312} in (3.13) yields

$$J_{31}\leq c\|\theta\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\Big). \tag{3.16}$$

To deal with J_{32}, we divide it first into two terms,

$$\begin{aligned}
J_{32} &= -\int \Delta\theta\Delta u_2\partial_2\theta dx\\
&= -\int \partial_1\partial_1\theta\,\Delta u_2\,\partial_2\theta dx-\int \partial_2\partial_2\theta\,\Delta u_2\,\partial_2\theta\,dx\\
&= -\int \partial_1\partial_1\theta\,\Delta u_2\partial_2\,\theta\,dx+\frac{1}{2}\int \Delta\partial_2 u_2\,(\partial_2\theta)^2\,dx\\
&= -\int \partial_1\partial_1\theta\,\Delta u_2\,\partial_2\theta\,dx-\frac{1}{2}\int \Delta\partial_1 u_1\,(\partial_2\theta)^2 dx\\
&= -\int \partial_1\partial_1\theta\,\Delta u_2\,\partial_2\theta\,dx+\int \Delta u_1\,\partial_2\theta\,\partial_1\partial_2\theta dx\\
&= J_{321}+J_{322}.
\end{aligned} \tag{3.17}$$

Invoking the decompositions of u and θ, we can rewrite J_{321} as,

$$\begin{aligned}
J_{321} &:= -\int \partial_1\partial_1\theta\,\Delta u_2\,\partial_2\theta\,dx\\
&= -\int \partial_1\partial_1\widetilde{\theta}\,\partial_{11}\widetilde{u_2}\,\partial_2\overline{\theta}\,dx-\int \partial_1\partial_1\widetilde{\theta}\,\partial_{11}\widetilde{u_2}\,\partial_2\widetilde{\theta}\,dx-\int \partial_1\partial_1\widetilde{\theta}\,\partial_{22}u_2\,\partial_2\theta\,dx\\
&:= J_{3211}+J_{3212}+J_{3213}.
\end{aligned} \tag{3.18}$$

The three terms in J_{321} can be bounded as follows. By interation by parts, Lemma 2.1, Hölder's inequality, Lemma 2.2 and Young's inequality,

$$\begin{aligned}
J_{3211} &:= -\int \partial_1\partial_1\widetilde{\theta}\,\partial_{11}\widetilde{u_2}\,\partial_2\overline{\theta}\,dx\\
&= \int \partial_1\partial_{11}\widetilde{\theta}\,\partial_1\widetilde{u_2}\,\partial_2\overline{\theta}\,dx\\
&= \int_{\mathbb{R}}\partial_2\overline{\theta}\Big(\int_{\mathbb{T}}\partial_1\partial_{11}\widetilde{\theta}\,\partial_1\widetilde{u_2}dx_1\Big)dx_2\\
&\leq \int_{\mathbb{R}}|\partial_2\overline{\theta}|\|\partial_1\widetilde{u_2}\|_{L^2_{x_1}}\|\partial_1\partial_{11}\widetilde{\theta}\|_{L^2_{x_1}}dx_2\\
&\leq \|\partial_2\overline{\theta}\|_{L^\infty_{x_2}}\|\partial_1\widetilde{u_2}\|_{L^2_{x_2}L^2_{x_1}}\|\partial_1\partial_{11}\widetilde{\theta}\|_{L^2_{x_2}L^2_{x_1}}\\
&\leq c\|\partial_2\overline{\theta}\|_{H^1}\|\partial_1\widetilde{u_2}\|_{L^2}\|\partial_1\partial_{11}\widetilde{\theta}\|_{L^2}\\
&\leq c\|\theta\|_{H^2}\Big(\|\partial_1 u_2\|_{L^2}^2+\|\partial_1\theta\|_{H^2}^2\Big).
\end{aligned} \tag{3.19}$$

By lemma 2.4 and then lemma 2.5

$$J_{3212} := -\int \partial_1\partial_1\widetilde{\theta}\,\partial_{11}\widetilde{u_2}\,\partial_2\widetilde{\theta}\,dx$$

$$\leq c \underbrace{\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}}_{\leq \|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}} \|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_{11}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_{11}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_{11}u_2\|_{L^2}$$
$$\leq c\|u\|_{H^2}\|\partial_1\theta\|_{H^2}^2. \tag{3.20}$$

Making use of the divergence-free condition of u, Lemmas 2.1 and 2.4, we have

$$\begin{aligned} J_{3213} &:= -\int \partial_1\partial_1\widetilde{\theta}\,\partial_{22}u_2\,\partial_2\theta\,dx \\ &= -\int \partial_1\partial_1\widetilde{\theta}\,\partial_{21}\widetilde{u_1}\,\partial_2\theta\,dx \\ &\leq c\|\partial_{21}\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{21}\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_{11}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_{11}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{L^2} \\ &\leq c\|\theta\|_{H^2}\|\partial_1\theta\|_{H^2}\|\partial_2 u\|_{H^2} \\ &\leq c\|\theta\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\Big). \end{aligned} \tag{3.21}$$

Inserting (3.19), (3.20) and (3.21) in (3.18) we obtain

$$J_{321} \leq c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2\Big). \tag{3.22}$$

We now turn to J_{322}. We further decompose it into two terms,

$$\begin{aligned} J_{322} &:= \int \Delta u_1\,\partial_2\theta\,\partial_1\partial_2\theta dx \\ &= \int \partial_{11}\widetilde{u_1}\,\partial_2\theta\,\partial_1\partial_2\widetilde{\theta}dx + \int \partial_{22}u_1\,\partial_2\theta\,\partial_1\partial_2\widetilde{\theta}dx \\ &= J_{3221} + J_{3222}. \end{aligned} \tag{3.23}$$

Due to the divergence-free condition of u and Lemma 2.4,

$$\begin{aligned} J_{3221} &:= \int \partial_{11}\widetilde{u_1}\,\partial_2\theta\,\partial_{12}\widetilde{\theta}dx \\ &= -\int \partial_{12}\widetilde{u_2}\,\partial_2\theta\,\partial_{12}\widetilde{\theta}dx \\ &\leq c\|\partial_{12}\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{12}\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_{12}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_{12}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{L^2} \\ &\leq c\|\theta\|_{H^2}\|\partial_2 u\|_{H^2}\|\partial_1\theta\|_{H^2} \\ &\leq c\|\theta\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2\Big). \end{aligned} \tag{3.24}$$

By Lemma 2.4,

$$\begin{aligned} J_{3222} &:= \int \partial_{22}u_1\,\partial_2\theta\,\partial_{12}\widetilde{\theta}dx \\ &\leq c\|\partial_{12}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{12}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_{22}u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_{22}u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{L^2} \\ &\leq c\|\theta\|_{H^2}\|\partial_2 u\|_{H^2}\|\partial_1\theta\|_{H^2} \\ &\leq c\|\theta\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2\Big). \end{aligned} \tag{3.25}$$

Combining the estimates (3.24) and (3.25) and inserting them in (3.23) we find

$$J_{322} \leq c\|\theta\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2\Big). \tag{3.26}$$

Putting (3.22) and (3.26) in (3.17) we obtain

$$J_{32} \leq c\|(u,\theta)\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2\Big). \tag{3.27}$$

The next term J_{33} is naturally split into two parts,

$$\begin{aligned} J_{33} &:= -2\int \Delta\theta\nabla u_1\cdot\partial_1\nabla\theta dx \\ &= -2\int \Delta\theta\partial_1 u_1\partial_1\partial_1\theta dx - 2\int \Delta\theta\partial_2 u_1\partial_1\partial_2\theta dx \\ &:= J_{331} + J_{332}. \end{aligned} \tag{3.28}$$

All terms can be bounded suitably. In fact, due to the divergence-free condition of u and Lemma 2.4,

$$\begin{aligned} J_{331} &:= -2\int \Delta\theta\partial_1 u_1\partial_{11}\theta dx \\ &= 2\int \Delta\theta\partial_2 u_2\partial_1\partial_1\widetilde{\theta} dx \\ &\leq c\|\partial_{11}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{11}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\Delta\theta\|_{L^2} \\ &\leq c\|\theta\|_{H^2}\|\partial_2 u\|_{H^2}\|\partial_1\theta\|_{H^2} \\ &\leq c\|\theta\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2\Big). \end{aligned} \tag{3.29}$$

J_{332} can be bounded similarly, by $\partial_1\theta = \partial_1\widetilde{\theta}$ and Lemma 2.4,

$$\begin{aligned} J_{332} &:= -2\int \Delta\theta\partial_2 u_1\partial_{12}\theta dx \\ &= -2\int \Delta\theta\partial_2 u_1\partial_{12}\widetilde{\theta} dx \\ &\leq c\|\partial_{12}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{12}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2 u_1\|_{L^2}^{\frac{1}{2}}\|\Delta\theta\|_{L^2} \\ &\leq c\|\theta\|_{H^2}\|\partial_2 u\|_{H^2}\|\partial_1\theta\|_{H^2} \\ &\leq c\|\theta\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2\Big). \end{aligned} \tag{3.30}$$

Inserting these upper bounds in (3.28) we get

$$J_{33} \leq c\|\theta\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\Big). \tag{3.31}$$

To estimate J_{34}, we first invoke the decompositions $u = \overline{u} + \widetilde{u}$, $\theta = \overline{\theta} + \widetilde{\theta}$ and Lemma 2.1, to write J_{34} as

$$J_{34} := -2\int \Delta\theta\nabla u_2\cdot\partial_2\nabla\theta dx$$

$$
\begin{aligned}
&= -2\int (\partial_1 u_2 \partial_1\partial_2\theta\Delta\theta + \partial_2 u_2\partial_2\partial_2\theta\Delta\theta)dx\\
&= -2\int \partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\Delta\theta - 2\int \partial_2 u_2\partial_2\partial_2\theta\Delta\theta dx\\
&= -2\int \partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\partial_{11}\theta dx - 2\int \partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\partial_{22}\theta dx - 2\int \partial_2 u_2\partial_2\partial_2\theta\Delta\theta dx\\
&:= J_{341} + J_{342} + J_{343}. \qquad (3.32)
\end{aligned}
$$

We start with J_{341}. By integration by parts, Lemmas 2.1 and 2.4 we have

$$
\begin{aligned}
J_{341} &:= -2\int \partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\partial_{11}\theta dx\\
&= 2\int \widetilde{u_2}\partial_1\partial_1\partial_2\widetilde{\theta}\partial_{11}\widetilde{\theta}dx\\
&\le c\|\partial_{11}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{11}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_{12}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_{11}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u_2}\|_{L^2}\\
&\le c\|u\|_{H^2}\|\partial_1\theta\|_{H^2}^2. \qquad (3.33)
\end{aligned}
$$

Using the decomposition $\theta = \overline{\theta} + \widetilde{\theta}$ we write J_{342} as,

$$
\begin{aligned}
J_{342} &:= -2\int \partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\partial_{22}\theta dx\\
&= -2\int \partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\partial_{22}\overline{\theta}dx - 2\int \partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\partial_{22}\widetilde{\theta}dx\\
&:= J_{3421} + J_{3422}. \qquad (3.34)
\end{aligned}
$$

We start with J_{3421}. Due to integration by parts, Lemma 2.1, Hölder's inequality, Lemma 2.2 and Young's inequality,

$$
\begin{aligned}
J_{3421} &:= -2\int \partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\partial_{22}\overline{\theta}dx\\
&= 2\int (\partial_2\partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta} + \partial_1\widetilde{u_2}\partial_2\partial_1\partial_2\widetilde{\theta})\partial_2\overline{\theta}\,dx\\
&= 2\int_{\mathbb{R}} \partial_2\overline{\theta}\Big(\int_{\mathbb{T}} (\partial_2\partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta} + \partial_1\widetilde{u_2}\partial_2\partial_1\partial_2\widetilde{\theta})dx_1\Big)dx_2\\
&\le c\int_{\mathbb{R}} |\partial_2\overline{\theta}|\Big(\|\partial_2\partial_1\widetilde{u_2}\|_{L^2_{x_1}}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2_{x_1}} + \|\partial_1\widetilde{u_2}\|_{L^2_{x_1}}\|\partial_2\partial_1\partial_2\widetilde{\theta}\|_{L^2_{x_1}}\Big)dx_2\\
&\le c\|\partial_2\overline{\theta}\|_{L^\infty_{x_2}}\Big(\|\partial_2\partial_1\widetilde{u_2}\|_{L^2_{x_2}L^2_{x_1}}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2_{x_2}L^2_{x_1}} + \|\partial_1\widetilde{u_2}\|_{L^2_{x_2}L^2_{x_1}}\|\partial_2\partial_1\partial_2\widetilde{\theta}\|_{L^2_{x_2}L^2_{x_1}}\Big)\\
&\le c\|\partial_2\overline{\theta}\|_{H^1}\Big(\|\partial_2\partial_1\widetilde{u_2}\|_{L^2}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2} + \|\partial_1\widetilde{u_2}\|_{L^2}\|\partial_2\partial_1\partial_2\widetilde{\theta}\|_{L^2}\Big)\\
&\le c\|\theta\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2\Big). \qquad (3.35)
\end{aligned}
$$

For J_{3422}, we apply Lemma 2.4 then Young's inequality,

$$
J_{3422} := -2\int \partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\partial_{22}\widetilde{\theta}dx
$$

$$\leq c\|\partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_{12}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_{12}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u_2}\|_{L^2}$$
$$\leq c\|\theta\|_{H^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|\partial_1\theta\|_{H^2}^{\frac{3}{2}}\|\partial_1 u_2\|_{L^2}^{\frac{1}{2}}$$
$$\leq c\|\theta\|_{H^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1\theta\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2\Big). \tag{3.36}$$

In view of (3.34), (3.35) and (3.36) we have

$$J_{342}\leq c\|(u,\theta)\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2\Big). \tag{3.37}$$

Writing J_{343} more explicitly and using $\partial_1\theta=\partial_1\widetilde{\theta}$, we have

$$J_{343}:=-2\int\partial_2 u_2\partial_2\partial_2\theta\Delta\theta dx$$
$$=-2\int\partial_2 u_2\partial_{22}\theta\partial_{11}\widetilde{\theta}dx-2\int\partial_2 u_2\partial_{22}\theta\partial_{22}\theta dx$$
$$:=J_{3431}+J_{3432}. \tag{3.38}$$

From Lemma 2.4, J_{3431} can be bounded as,

$$J_{3431}:=-2\int\partial_2 u_2\partial_{22}\theta\partial_{11}\widetilde{\theta}dx$$
$$\leq c\|\partial_{11}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{11}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_{22}\theta\|_{L^2}$$
$$\leq c\|\theta\|_{H^2}\|\partial_1\theta\|_{H^2}\|\partial_2 u\|_{H^2}$$
$$\leq c\|\theta\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\Big). \tag{3.39}$$

The estimate of J_{3432} is slightly more delicate. Due to the decomposition $\theta=\widetilde{\theta}+\overline{\theta}$, we write J_{3432} as,

$$J_{3432}:=-2\int\partial_2 u_2\partial_{22}\theta\partial_{22}\theta dx$$
$$=-2\int\partial_2 u_2\partial_{22}\overline{\theta}\partial_{22}\overline{\theta}dx-4\int\partial_2 u_2\partial_{22}\overline{\theta}\partial_{22}\widetilde{\theta}dx-2\int\partial_2 u_2\partial_{22}\widetilde{\theta}\partial_{22}\widetilde{\theta}dx$$
$$:=J_{34321}+J_{34322}+J_{34323}. \tag{3.40}$$

By $\nabla\cdot u=0$ and Lemma 2.1, the first term J_{34321} is clearly zero,

$$J_{34321}=-2\int\partial_2 u_2\partial_{22}\overline{\theta}\partial_{22}\overline{\theta}dx=2\int\partial_1\widetilde{u_1}\partial_{22}\overline{\theta}\partial_{22}\overline{\theta}dx=0. \tag{3.41}$$

Applying Lemmas 2.4 and 2.5 and Young's inequality,

$$J_{34322}:=-4\int\partial_2 u_2\partial_{22}\overline{\theta}\partial_{22}\widetilde{\theta}dx$$
$$\leq c\|\partial_{22}\overline{\theta}\|_{L^2}\|\partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2 u_2\|_{L^2}^{\frac{1}{2}}$$
$$\leq c\|\partial_{22}\overline{\theta}\|_{L^2}\|\partial_1\partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2 u_2\|_{L^2}^{\frac{1}{2}}$$
$$\leq c\|\theta\|_{H^2}\|\partial_1\theta\|_{H^2}\|\partial_2 u\|_{H^2}$$
$$\leq c\|\theta\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\Big), \tag{3.42}$$

$$
\begin{aligned}
J_{34323} &:= -4\int \partial_2 u_2 \partial_{22}\widetilde{\theta}\partial_{22}\widetilde{\theta}dx \\
&\le c\|\partial_{22}\overline{\theta}\|_{L^2}\|\partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2 u_2\|_{L^2}^{\frac{1}{2}} \\
&\le c\|\partial_{22}\overline{\theta}\|_{L^2}\|\partial_1\partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2 u_2\|_{L^2}^{\frac{1}{2}} \\
&\le c\|\theta\|_{H^2}\|\partial_1\theta\|_{H^2}\|\partial_2 u\|_{H^2} \\
&\le c\|\theta\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\Big). \qquad (3.43)
\end{aligned}
$$

The bounds for J_{3432} in (3.41), (3.42) and (3.43) lead to,

$$
J_{3432} \le c\|\theta\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\Big). \qquad (3.44)
$$

Combining (3.39) and (3.44) and inserting them in (3.38) we obtain

$$
J_{343} \le c\|\theta\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\Big). \qquad (3.45)
$$

Inserting (3.33), (3.37) and (3.45) in (3.32) we get

$$
J_{34} \le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2\Big). \qquad (3.46)
$$

Thus, by (3.16), (3.27), (3.31), (3.46), and (3.12),

$$
J_3 \le c\|(u,\theta)\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2 + \|\partial_1\theta\|_{H^2}^2\Big). \qquad (3.47)
$$

As outlined in the introduction, we need the help of an extra regularization term to bound J_2, namely,

$$
\int_0^t \|\partial_1 u_2\|_{L^2}^2\, d\tau. \qquad (3.48)
$$

In order to make efficient use of the anisotropic dissipation, we express J_2 as follows

$$
\begin{aligned}
J_2 =& -\int \partial_1 u_1\,(\partial_1\omega)^2\,dx - \int \partial_1 u_2\,\partial_1\omega\,\partial_2\omega\,dx \\
&- \int \partial_2 u_1\,\partial_1\omega\,\partial_2\omega dx - \int \partial_2 u_2\,(\partial_2\omega)^2 dx \\
=& \int \partial_2 u_2\,(\partial_1\omega)^2 dx - \int \partial_1 u_2\,\partial_1\omega\,\partial_2\omega\,dx \\
&- \int \partial_2 u_1\,\partial_1\omega\,\partial_2\omega\,dx - \int \partial_2 u_2\,(\partial_2\omega)^2\,dx \\
:=& J_{21} + J_{22} + J_{23} + J_{24}. \qquad (3.49)
\end{aligned}
$$

The terms J_{21} through J_{24} can be bounded in the following manner. Due to $\nabla\cdot u = 0$, integration by parts and Lemmas 2.1 and 2.4,

$$
\begin{aligned}
J_{21} &:= -\int \partial_1 u_1 (\partial_1\omega)^2\,dx \\
&= \int \partial_2 u_2 (\partial_1\omega)^2\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= -2\int \widetilde{u_2}\partial_1\widetilde{\omega}\partial_2\partial_1\widetilde{\omega}\,dx\\
&\leq c\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}\\
&\leq c\|u\|_{H^2}\|\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2 u\|_{H^2}^{\frac{3}{2}}\\
&\leq c\|u\|_{H^2}\Big(\|\partial_1 u_2\|_{L^2}^2+\|\partial_2 u\|_{H^2}^2\Big). \qquad (3.50)
\end{aligned}
$$

According to Lemmas 2.1 and 2.4,

$$
\begin{aligned}
J_{22} &:= -\int \partial_1 u_2\partial_1\omega\partial_2\omega\,dx\\
&= -\int \partial_1\widetilde{u_2}\partial_1\widetilde{\omega}\partial_2\omega\,dx\\
&\leq c\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_2\omega\|_{L^2}\\
&\leq c\|u\|_{H^2}\|\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2 u\|_{H^2}^{\frac{3}{2}}\\
&\leq c\|u\|_{H^2}\Big(\|\partial_1 u_2\|_{L^2}^2+\|\partial_2 u\|_{H^2}^2\Big). \qquad (3.51)
\end{aligned}
$$

To bound J_{23}, we first use the orthogonal decomposition of u_1 and ω and Lemma 2.1, to write J_{23} as

$$
\begin{aligned}
J_{23} &:= -\int \partial_2 u_1\partial_1\omega\partial_2\omega\,dx = -\int \partial_2 u_1\partial_1\widetilde{\omega}\partial_2\omega\,dx\\
&= -\int \partial_2\overline{u_1}\partial_1\widetilde{\omega}\partial_2\overline{\omega}\,dx - \int \partial_2\overline{u_1}\partial_1\widetilde{\omega}\partial_2\widetilde{\omega}\,dx - \int \partial_2\widetilde{u_1}\partial_1\widetilde{\omega}\partial_2\omega\,dx\\
&= J_{231}+J_{232}+J_{233}. \qquad (3.52)
\end{aligned}
$$

According to Lemma 2.1, the first term J_{231} is clearly zero,

$$
J_{231} := -\int \partial_2\overline{u_1}\partial_1\widetilde{\omega}\partial_2\overline{\omega}\,dx = -\int_{\mathbb{R}} \partial_2\overline{u_1}\partial_2\overline{\omega}\int_{\mathbb{T}}\partial_1\widetilde{\omega}\,dx_1dx_2 = 0. \qquad (3.53)
$$

The terms J_{232} and J_{233} can be bounded directly. By Lemma 2.4,

$$
\begin{aligned}
J_{232} &:= -\int \partial_2\overline{u_1}\partial_1\widetilde{\omega}\partial_2\widetilde{\omega}\,dx\\
&\leq c\|\partial_2\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_2\overline{u_1}\|_{L^2}\\
&\leq c\|u\|_{H^2}\|\partial_2 u\|_{H^2}^2, \qquad (3.54)
\end{aligned}
$$

$$
\begin{aligned}
J_{233} &:= -\int \partial_2\widetilde{u_1}\partial_1\widetilde{\omega}\partial_2\omega\,dx\\
&\leq c\|\partial_2\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_2\omega\|_{L^2}\\
&\leq c\|u\|_{H^2}\|\partial_2 u\|_{H^2}^2. \qquad (3.55)
\end{aligned}
$$

Inserting these upper bounds in (3.52) yields

$$
J_{23} \leq c\|u\|_{H^2}\|\partial_2 u\|_{H^2}^2. \qquad (3.56)
$$

To deal with J_{24} we use the divergence-free condition of u, Lemma 2.1, and the inequality (2.3) in Lemma 2.5

$$\begin{aligned} J_{24} &:= -\int \partial_2 u_2 (\partial_2 \omega)^2 \, dx \\ &= -\int \partial_1 \widetilde{u_1} (\partial_2 \overline{\omega} + \partial_2 \widetilde{\omega})^2 \, dx \\ &= -2\int \partial_1 \widetilde{u_1} \partial_2 \overline{\omega} \partial_2 \widetilde{\omega} \, dx - 2\int \partial_1 \widetilde{u_1} (\partial_2 \widetilde{\omega})^2 \, dx \\ &\le c\Big(\|\partial_2 \overline{\omega}\|_{L^2} + \|\partial_2 \widetilde{\omega}\|_{L^2} \Big) \|\partial_1 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \widetilde{\omega}\|_{L^2} \\ &\le c\|u\|_{H^2} \|\partial_2 u\|_{H^2}^2. \end{aligned} \tag{3.57}$$

Collecting the bounds for J_{21} through J_{24} obtained in (3.50), (3.51), (3.56) and (3.57), we obtain

$$J_2 \le c\|u\|_{H^2} \|\partial_2 u\|_{H^2}^2 \, . \tag{3.58}$$

Inserting $J_1 = 0$, (3.47) and (3.58) in (3.11), yields

$$\begin{aligned} &\frac{d}{dt}(\|\Delta u\|_{L^2}^2 + \|\Delta \theta\|_{L^2}^2) + 2\nu \|\partial_2 \Delta u\|_{L^2}^2 + 2\eta \|\partial_1 \Delta \theta\|_{L^2}^2 \\ &\le c \, \|(u,\theta)\|_{H^2} \Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1 \theta\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2 \Big). \end{aligned} \tag{3.59}$$

Integrating (3.59) over $[0,t]$, we get

$$\begin{aligned} &\|\Delta u(t)\|_{L^2}^2 + \|\Delta \theta(t)\|_{L^2}^2 + 2\nu \int_0^t \|\partial_2 \Delta u\|_{L^2}^2 d\tau + 2\eta \int_0^t \|\Delta \partial_1 \theta\|_{L^2}^2 d\tau \\ &\le \|\Delta u_0\|_{L^2}^2 + \|\Delta \theta_0\|_{L^2}^2 + c \int_0^t \|(u,\theta)\|_{H^2} \Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1 \theta\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2 \Big) \\ &\le E(0) + \, c\, E(t)^{\frac{3}{2}}. \end{aligned} \tag{3.60}$$

The subsequent step is to control the last piece in $E(t)$ defined by (1.16), namely

$$\int_0^t \|g_0 \partial_1 u_2\|_{L^2}^2 \, d\tau. \tag{3.61}$$

Our strategy is to make use of the special structure of the equation for θ in (1.2) and replace $g_0 \partial_1 u_2$ in (3.61) via the equation of θ,

$$g_0 \partial_1 u_2 = -\partial_t \partial_1 \theta - \partial_1 (u \cdot \nabla \theta) + \eta \partial_{111} \theta. \tag{3.62}$$

Multiplying (3.62) by $g_0 \partial_1 u_2$ and then integrating over Ω, we obtain

$$\begin{aligned} \|g_0 \partial_1 u_2\|_{L^2}^2 &= -g_0 \int \partial_t \partial_1 \theta \, \partial_1 u_2 \, dx - g_0 \int \partial_1 u_2 \, \partial_1 (u \cdot \nabla \theta) \, dx + g_0 \eta \int \partial_1 u_2 \, \partial_{111} \theta \, dx \\ &:= K_1 + K_2 + K_3. \end{aligned} \tag{3.63}$$

We bound K_3 as follows,

$$|K_3| \le \, \eta \|g_0 \partial_1 u_2\|_{L^2} \, \|\partial_{111} \theta\|_{L^2} \le \frac{1}{2} \|g_0 \partial_1 u_2\|_{L^2}^2 + c \, \|\partial_1 \theta\|_{H^2}^2, \tag{3.64}$$

the term with unfavorable derivative $\partial_1 u_2$ will be then absorbed by the left-hand side of (3.64).
For K_1, we first shift the time derivative

$$K_1 = -g_0 \frac{d}{dt} \int \partial_1\theta \, \partial_1 u_2 \, dx + g_0 \int \partial_1\theta \, \partial_1\partial_t u_2 \, dx := K_{11} + K_{12}. \tag{3.65}$$

Using the equation for the second component of the velocity, we write

$$\begin{aligned} K_{12} &= -g_0 \int \partial_1\partial_1\theta \, \partial_t u_2 \, dx \qquad (3.66)\\ &= -g_0 \int \partial_{11}\theta(-(u\cdot\nabla u_2) - \partial_2 p + \nu\partial_{22}u_2 + g_0\theta) \, dx \\ &= g_0 \int \partial_{11}\theta \, (u\cdot\nabla u_2) \, dx \; + g_0 \int \partial_{11}\theta \; \partial_2 p \, dx \\ &\quad -g_0\nu \int \partial_{11}\theta \; \partial_{22}u_2 \; dx \; - g_0^2 \int \partial_{11}\theta \; \theta \; dx. \qquad (3.67) \end{aligned}$$

Then, we apply the divergence operator to the velocity equation to express the pressure term as

$$p = -\Delta^{-1}\nabla\cdot(u\cdot\nabla u) + g_0\Delta^{-1}\partial_2\theta. \tag{3.68}$$

Inserting (3.68) in (3.67), we obtain

$$\begin{aligned} K_{12} &= g_0 \int \partial_{11}\theta \, (u\cdot\nabla u_2) \, dx \; + g_0 \int \partial_{11}\theta \, (-\partial_2\Delta^{-1}\nabla\cdot(u\cdot\nabla u)) \, dx \\ &\quad - g_0\nu \int \partial_{11}\theta \; \partial_{22}u_2 \; dx \; - g_0^2 \int \partial_{11}\theta \; \partial_{11}\Delta^{-1}\theta \; dx \\ &:= K_{121} + K_{122} + K_{123} + K_{124}. \end{aligned} \tag{3.69}$$

Due to Hölder's inequality and the fact that the double Riesz transform $\partial_{11}\Delta^{-1}$ is bounded on L^q for any $1 < q < \infty$ (see, e.g., [27]), we have

$$K_{124} := -g_0^2 \int \partial_1\theta \, \partial_{11}\Delta^{-1}\partial_1\theta \; dx \le c \, \|\partial_1\theta\|_{L^2} \|\partial_{11}\Delta^{-1}\partial_1\theta\|_{L^2} \le c \, \|\partial_1\theta\|_{L^2}^2. \tag{3.70}$$

Thanks to Hölder's inequality,

$$K_{123} := -g_0\nu \int \partial_{11}\theta \; \partial_{22}u_2 \; dx \le c \, \|\partial_{11}\theta\|_{L^2} \, \|\partial_{22}u_2\|_{L^2}. \tag{3.71}$$

By integration by parts, Hölder's inequality and the boundedness of the double Riesz transform,

$$\begin{aligned} K_{122} &:= g_0 \int \partial_{11}\theta \, (-\partial_2\Delta^{-1}\nabla\cdot(u\cdot\nabla u)) \, dx \\ &= g_0 \int \partial_1\theta \; \partial_{12}\Delta^{-1}\nabla\cdot(u\cdot\nabla u) \, dx \\ &\le c\|\partial_1\theta\|_{L^2} \; \|\Delta^{-1}\partial_{12}\nabla\cdot(u\cdot\nabla u)\|_{L^2} \\ &\le \; c \, \|\partial_1\theta\|_{L^2} \, \|\partial_2(u\cdot\nabla u)\|_{L^2} \\ &\le \; c \, \|\partial_1\theta\|_{L^2} \, \|\partial_2 u\cdot\nabla u + u\cdot\nabla\partial_2 u\|_{L^2} \end{aligned}$$

$$\begin{aligned}
&\leq c\,\|\partial_1\theta\|_{L^2}\,(\,\|\partial_2 u\|_{L^4}\,\|\nabla u\|_{L^4}+\|u\|_\infty\|\nabla\partial_2 u\|_{L^2})\\
&\leq c\,\|\partial_1\theta\|_{L^2}\,\|\partial_2 u\|_{H^1}\,\|\nabla u\|_{H^1}+c\,\|\partial_1\theta\|_{L^2}\,\|u\|_{H^2}\|\nabla\partial_2 u\|_{L^2}. \qquad (3.72)
\end{aligned}$$

To deal with K_{121}, we rewrite it as

$$\begin{aligned}
K_{121}&=g_0\int\partial_{11}\theta(u_1\partial_1u_2+u_2\partial_2u_2)dx\\
&=g_0\int\partial_{11}\widetilde{\theta}\;u_1\;\partial_1u_2\;dx\;+g_0\int\partial_{11}\widetilde{\theta}\;u_2\;\partial_2u_2\;dx\\
&=g_0\int\partial_{11}\widetilde{\theta}\;\widetilde{u_1}\;\partial_1\widetilde{u_2}\;dx\;+g_0\int\partial_{11}\widetilde{\theta}\;\overline{u_1}\;\partial_1\widetilde{u_2}\;dx+g_0\int\partial_{11}\widetilde{\theta}\;u_2\;\partial_2u_2\;dx\\
&:=K_{1211}+K_{1212}+K_{1213}. \qquad (3.73)
\end{aligned}$$

By Lemma 2.4, the divergence-free condition of u and Lemma 2.5,

$$\begin{aligned}
K_{1211}&:=g_0\int\partial_{11}\widetilde{\theta}\;\widetilde{u_1}\;\partial_1\widetilde{u_2}\;dx\\
&\leq c\,\|\partial_{11}\theta\|_{L^2}^{\frac12}\|\partial_2\partial_{11}\theta\|_{L^2}\;\underbrace{\|\widetilde{u_1}\|_{L^2}^{\frac12}}_{\leq\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}}\;\underbrace{\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}}_{=\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac12}}\|\partial_1\widetilde{u_2}\|_{L^2}\\
&\leq c\,\|u\|_{H^2}\,\|\partial_2u\|_{H^2}\,\|\partial_1\theta\|_{H^2}\\
&\leq c\,\|u\|_{H^2}\Big(\|\partial_2u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2\Big). \qquad (3.74)
\end{aligned}$$

Due to Lemma 2.1, Hölder's inequality, Lemma 2.2 and then Young's inequality,

$$\begin{aligned}
K_{1212}&:=g_0\int\partial_{11}\widetilde{\theta}\;\overline{u_1}\;\partial_1\widetilde{u_2}\;dx\\
&=g_0\int_{\mathbb{R}}\overline{u_1}\Big(\int_{\mathbb{T}}\partial_{11}\widetilde{\theta}\partial_1\widetilde{u_2}dx_1\Big)dx_2\\
&\leq c\int_{\mathbb{R}}|\overline{u_1}|\|\partial_{11}\widetilde{\theta}\|_{L^2_{x_1}}\|\partial_1\widetilde{u_2}\|_{L^2_{x_1}}dx_2\\
&\leq c\|\overline{u_1}\|_{L^\infty_{x_2}}\|\partial_{11}\widetilde{\theta}\|_{L^2_{x_2}L^2_{x_1}}\|\partial_1\widetilde{u_2}\|_{L^2_{x_2}L^2_{x_1}}\\
&\leq c\|\overline{u_1}\|_{H^1}\|\partial_{11}\widetilde{\theta}\|_{L^2}\|\partial_1\widetilde{u_2}\|_{L^2}\\
&\leq c\|u\|_{H^2}\Big(\|\partial_1u_2\|_{L^2}^2+\|\partial_1\theta\|_{H^2}^2\Big). \qquad (3.75)
\end{aligned}$$

According to Lemma 2.4,

$$\begin{aligned}
K_{1213}&:=g_0\int\partial_{11}\widetilde{\theta}\;u_2\;\partial_2u_2\;dx\\
&\leq c\|\partial_{11}\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\partial_{11}\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2u_2\|_{L^2}^{\frac12}\|\partial_2\;\partial_2u_2\|_{L^2}^{\frac12}\|u_2\|_{L^2}\\
&\leq c\|u\|_{H^2}\|\partial_2u\|_{H^2}\|\partial_1\theta\|_{H^2}\\
&\leq c\|u\|_{H^2}\Big(\|\partial_2u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2\Big). \qquad (3.76)
\end{aligned}$$

Inserting (3.74), (3.75) and (3.76) in (3.73) we get

$$K_{121} \leq c\|u\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2 + \|\partial_1\theta\|_{H^2}^2\Big). \tag{3.77}$$

It then follows from (3.69), (3.70), (3.71), (3.72) and (3.77) that

$$|K_{12}| \leq\ c\|u\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2 + \|\partial_1\theta\|_{H^2}^2\Big). \tag{3.78}$$

We now need to bound K_2. We first split it into four terms,

$$\begin{aligned} K_2 :&= -g_0 \int \partial_1 u_2\, \partial_1(u\cdot\nabla\theta)\, dx \\ &= -g_0 \int \partial_1 u_2\, \partial_1 u_1\, \partial_1\theta\, dx - g_0 \int \partial_1 u_2 u_1 \partial_1\partial_1\theta\, dx \\ &\quad - g_0 \int \partial_1 u_2 \partial_1 u_2 \partial_2\theta\, dx - g_0 \int \partial_1 u_2 u_2 \partial_1\partial_2\theta\, dx \\ &:= K_{21} + K_{22} + K_{23} + K_{24}. \end{aligned} \tag{3.79}$$

Due to $\partial_1\theta = \partial_1\widetilde{\theta}$, Lemma 2.4 and Young's inequality

$$\begin{aligned} K_{21} &:= -g_0 \int \partial_1 u_2\, \partial_1 u_1\, \partial_1\theta\, dx \\ &= -g_0 \int \partial_1 u_2\, \partial_1 u_1\, \partial_1\widetilde{\theta}\, dx \\ &\leq c\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2 u_2\|_{L^2}^{\frac12}\|\partial_2\partial_2 u_2\|_{L^2}^{\frac12}\|\partial_1 u_2\|_{L^2} \\ &\leq c\|u\|_{H^2}\|\partial_2 u\|_{H^2}\|\partial_1\theta\|_{H^2} \\ &\leq c\|u\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2\Big). \end{aligned} \tag{3.80}$$

Using Lemma 2.1 and invoking the decompositions $u = \overline{u} + \widetilde{u}$ we write K_{22} as

$$\begin{aligned} K_{22} &:= -g_0 \int \partial_1 u_2 u_1 \partial_1\partial_1\theta\, dx \\ &= g_0 \int \partial_{11}\widetilde{\theta}\ \widetilde{u_1}\ \partial_1\widetilde{u_2}\ dx\ + g_0 \int \partial_{11}\widetilde{\theta}\ \overline{u_1}\ \partial_1\widetilde{u_2}\ dx \\ &:= K_{221} + K_{222}. \end{aligned} \tag{3.81}$$

By Lemmas 2.4, 2.5 and the divergence-free condition of u,

$$\begin{aligned} K_{221} &:= g_0 \int \partial_{11}\widetilde{\theta}\ \widetilde{u_1}\ \partial_1\widetilde{u_2}\ dx \\ &\leq\ c\,\|\partial_{11}\theta\|_{L^2}^{\frac12}\|\partial_2\partial_{11}\theta\|_{L^2}\ \underbrace{\|\widetilde{u_1}\|_{L^2}^{\frac12}}_{\leq\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}}\ \underbrace{\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}}_{=\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac12}}\|\partial_1\widetilde{u_2}\|_{L^2} \\ &\leq\ c\,\|u\|_{H^2}\,\|\partial_2 u\|_{H^2}\,\|\partial_1\theta\|_{H^2} \\ &\leq\ c\,\|u\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}\Big). \end{aligned} \tag{3.82}$$

To bound K_{222}, we first use Lemma 2.1, Hölder's inequality and then Lemma 2.2 to obtain

$$\begin{aligned}
K_{222} &:= g_0 \int \partial_{11}\widetilde{\theta}\,\overline{u_1}\,\partial_1\widetilde{u_2}\,dx \\
&= g_0 \int_{\mathbb{R}} \overline{u_1}\Big(\int_{\mathbb{T}} \partial_{11}\widetilde{\theta}\partial_1\widetilde{u_2}dx_1\Big)dx_2 \\
&\le c\int_{\mathbb{R}} |\overline{u_1}|\|\partial_{11}\widetilde{\theta}\|_{L^2_{x_1}}\|\partial_1\widetilde{u_2}\|_{L^2_{x_1}}dx_2 \\
&\le c\|\overline{u_1}\|_{L^\infty_{x_2}}\|\partial_{11}\widetilde{\theta}\|_{L^2_{x_2}L^2_{x_1}}\|\partial_1\widetilde{u_2}\|_{L^2_{x_2}L^2_{x_1}} \\
&\le c\|\overline{u_1}\|_{H^1}\|\partial_{11}\widetilde{\theta}\|_{L^2}\|\partial_1\widetilde{u_2}\|_{L^2} \\
&\le c\|u\|_{H^2}\Big(\|\partial_1 u_2\|^2_{L^2} + \|\partial_1\theta\|^2_{H^2}\Big). \qquad (3.83)
\end{aligned}$$

Then (3.82), (3.83) and (3.81) together leads to

$$K_{22} \le c\|u\|_{H^2}\Big(\|\partial_2 u\|^2_{H^2} + \|\partial_1 u_2\|^2_{L^2} + \|\partial_1\theta\|^2_{H^2}\Big). \qquad (3.84)$$

By $\partial_1 u_2 = \partial_1\widetilde{u_2}$ and $\theta = \widetilde{\theta} + \overline{\theta}$, we rewrite K_{23} as

$$\begin{aligned}
K_{23} &:= -g_0\int \partial_1 u_2\partial_1 u_2\partial_2\theta\,dx \\
&= -g_0\int \partial_1\widetilde{u_2}\partial_1\widetilde{u_2}\partial_2\overline{\theta}\,dx - g_0\int \partial_1\widetilde{u_2}\partial_1\widetilde{u_2}\partial_2\widetilde{\theta}\,dx \\
&:= K_{231} + K_{232}. \qquad (3.85)
\end{aligned}$$

To estimate K_{231}, we make use of Lemma 2.1, Hölder's inequality and then Lemma 2.2 to get

$$\begin{aligned}
K_{231} &:= g_0\int \partial_1\widetilde{u_2}\partial_1\widetilde{u_2}\partial_2\overline{\theta}\,dx \\
&= g_0\int_{\mathbb{R}} \partial_2\overline{\theta}\Big(\int_{\mathbb{T}} \partial_1\widetilde{u_2}\partial_1\widetilde{u_2}dx_1\Big)dx_2 \\
&\le c\int_{\mathbb{R}} |\partial_2\overline{\theta}|\|\partial_1\widetilde{u_2}\|_{L^2_{x_1}}\|\partial_1\widetilde{u_2}\|_{L^2_{x_1}}dx_2 \\
&\le c\|\partial_2\overline{\theta}\|_{L^\infty_{x_2}}\|\partial_1\widetilde{u_2}\|_{L^2_{x_2}L^2_{x_1}}\|\partial_1\widetilde{u_2}\|_{L^2_{x_2}L^2_{x_1}} \\
&\le c\|\partial_2\overline{\theta}\|_{H^1}\|\partial_1\widetilde{u_2}\|^2_{L^2} \\
&\le c\|\theta\|_{H^2}\|\partial_1 u_2\|^2_{L^2}. \qquad (3.86)
\end{aligned}$$

Via Lemma 2.4,

$$\begin{aligned}
K_{232} &:= g_0\int \partial_1\widetilde{u_2}\partial_1\widetilde{u_2}\partial_2\widetilde{\theta}\,dx \\
&\le c\|\partial_2\widetilde{\theta}\|^{\frac12}_{L^2}\|\partial_1\partial_2\widetilde{\theta}\|^{\frac12}_{L^2}\|\partial_1 u_2\|^{\frac12}_{L^2}\|\partial_2\partial_1 u_2\|^{\frac12}_{L^2}\|\partial_1 u_2\|_{L^2} \\
&\le c\|\theta\|^{\frac12}_{H^2}\|u\|^{\frac12}_{H^2}\|\partial_1\theta\|^{\frac12}_{H^2}\|\partial_1 u_2\|^{\frac32}_{L^2}
\end{aligned}$$

$$\leq c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2\Big). \tag{3.87}$$

Inserting the bounds for K_{231} and K_{232} in (3.85), we find

$$K_{23}\leq c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2\Big). \tag{3.88}$$

The last term K_{24} can also be bounded due to the fact that $\overline{u_2}=0$, Lemmas 2.4 and 2.5

$$\begin{aligned}
K_{24} &:= -g_0\int \partial_1 u_2 u_2 \partial_1\partial_2\theta\,dx\\
&= -g_0\int \partial_1 u_2 \widetilde{u_2}\partial_1\partial_2\theta\,dx\\
&\leq c\ \underbrace{\|\widetilde{u_2}\|_{L^2}^{\frac12}}_{\leq\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac12}}\ \|\partial_1\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_1 u_2\|_{L^2}^{\frac12}\|\partial_2\partial_1 u_2\|_{L^2}^{\frac12}\|\partial_1\partial_2\theta\|_{L^2}\\
&\leq c\|\theta\|_{H^2}^{\frac12}\|u\|_{H^2}^{\frac12}\|\partial_1\theta\|_{H^2}^{\frac12}\|\partial_1 u_2\|_{L^2}^{\frac32}\\
&\leq c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2\Big).
\end{aligned} \tag{3.89}$$

Inserting (3.80), (3.84), (3.88), (3.89), in (3.79) we obtain

$$K_2\leq c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2+\|\partial_2 u\|_{H^2}^2\Big). \tag{3.90}$$

Collecting the bounds obtained above for K_1 through K_3 in (3.64), (3.65), (3.78) and (3.90) and inserting them in (3.63), we get

$$\begin{aligned}
\frac12\,\|g_0\partial_1 u_2\|_{L^2}^2 \leq\ & c\,\|\partial_1\theta\|_{H^2}^2 - g_0\frac{d}{dt}\int \partial_1\theta\,\partial_1 u_2\,dx\\
&+ c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2\Big).
\end{aligned} \tag{3.91}$$

Integrating (3.91) over the time interval $[0,t]$, we find

$$\begin{aligned}
\int_0^t \|g_0\partial_1 u_2\|_{L^2}^2\,d\tau \leq\ & c\int_0^t \|\partial_1\theta\|_{H^2}^2\,d\tau - 2g_0\int \partial_1\theta\,\partial_1 u_2\,dx + 2g_0\int \partial_1\theta_0\,\partial_1 u_{02}\,dx\\
&+ c\int_0^t \|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2\Big)\,d\tau\\
\leq\ & c\int_0^t \|\partial_1\theta\|_{H^2}^2\,d\tau + c\int_0^t \|\partial_2 u\|_{H^2}^2\,d\tau + c\,(\|u\|_{H^1}^2+\|\theta\|_{H^1}^2)\\
&+ c\,(\|u_0\|_{H^1}^2+\|\theta_0\|_{H^1}^2) + c\,E(t)^{\frac32}.
\end{aligned} \tag{3.92}$$

To conclude, we combine the H^1-bound in (3.10), the homogeneous H^2-bound in (3.60) and the bound for the extra regularization term in (3.92). When doing so, we need eliminate the quadratic terms on the right-hand side of (3.92) by the corresponding terms on the left-hand side, then it suffices to multiply both sides of

(3.92) by a suitable small coefficient $\delta > 0$. Taking (3.10) + (3.60) + δ (3.92), leads to

$$\begin{aligned}
&\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 + 2\nu \int_0^t \|\partial_2 u\|_{H^2}^2 d\tau + 2\eta \int_0^t \|\partial_1 \theta\|_{H^2}^2 d\tau + \delta \int_0^t \|g_0 \partial_1 u_2\|_{L^2}^2 \\
&\leq E(0) + c\, E(t)^{\frac{3}{2}} + c\,\delta\,(\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2) + c\,\delta\,(\|u_0\|_{H^2}^2 + \|\theta_0\|_{H^2}^2) \\
&\quad + c\,\delta \int_0^t \|\partial_2 u\|_{H^2}^2 d\tau + c\,\delta \int_0^t \|\partial_1 \theta\|_{H^2}^2 d\tau + c\,\delta\, E(t)^{\frac{3}{2}}. \qquad (3.93)
\end{aligned}$$

If $\delta > 0$ is chosen to be sufficiently small, say

$$c\,\delta \leq \frac{1}{2}, \quad c\,\delta \leq \nu, \quad c\,\delta \leq \eta,$$

then (3.93) gives

$$E(t) \leq C_1\, E(0) + C_2\, E(t)^{\frac{3}{2}}, \qquad (3.94)$$

where C_1 and C_2 are positive constants. The proof of the desired stability result, is then completed by applying the bootstrapping argument on (3.94). Indeed, if the initial data (u_0, θ_0), is sufficiently small, say,

$$E(0) = \|(u_0, \theta_0)\|_{H^2}^2 \leq \varepsilon^2 := \frac{1}{16 C_1 C_2^2}, \qquad (3.95)$$

then (3.94) implies

$$\|(u(t), \theta(t))\|_{H^2}^2 \leq 2 C_1\, \varepsilon^2.$$

To initiate the bootstrapping argument, we make the ansatz that, for $t \leq T$

$$E(t) \leq \frac{1}{4 C_2^2}, \qquad (3.96)$$

and we then show that $E(t)$ actually admits an even smaller bound by taking the initial H^2-norm $E(0)$ sufficiently small. In fact, Inserting (3.96) in (3.94) yields

$$\begin{aligned}
E(t) &\leq C_1\, E(0) + C_2\, E(t)^{\frac{3}{2}} \\
&\leq C_1\, \varepsilon^2 + C_2\, \frac{1}{2 C_2}\, E(t).
\end{aligned}$$

That is,

$$\frac{1}{2} E(t) \leq C_1\, \varepsilon^2 \quad \text{or} \quad E(t) \leq 2\, C_1\, \frac{1}{16 C_1 C_2^2} = \frac{1}{8 C_2^2} = 2 C_1\, \epsilon^2, \quad \text{for all } t \leq T.$$

The bootstrapping argument then assesses that (3.96) holds for all time when $E(0)$ satisfies (3.95). This establishes the global stability.

Finally, we establish the uniqueness of H^2-solutions to (1.2). Assume that $(u^{(1)}, p^{(1)}, \theta^{(1)})$ and $(u^{(2)}, p^{(2)}, \theta^{(2)})$ are two solutions of (1.2) with one of them in the H^2-regularity class say $(u^{(1)}, \theta^{(1)}) \in L^\infty(0, T; H^2)$. The difference between the two solutions (u^*, p^*, θ^*) with

$$u^* = u^{(2)} - u^{(1)}, \quad p^* = p^{(2)} - p^{(1)} \quad \text{and} \quad \theta^* = \theta^{(2)} - \theta^{(1)}$$

verifies

$$\begin{cases} \partial_t u^* + u^{(2)} \cdot \nabla u^* + u^* \cdot \nabla u^{(1)} + \nabla p^* = \nu \partial_{22} u^* + g_0 \theta^* \mathbf{e}_2, \\ \partial_t \theta^* + u^{(2)} \cdot \nabla \theta^* + u^* \cdot \nabla \theta^{(1)} + g_0 {u_2}^* = \eta \partial_{11} \theta^*, \\ \nabla \cdot u^* = 0, \\ u^*(x,0) = 0, \quad \theta^*(x,0) = 0. \end{cases} \tag{3.97}$$

We estimate the difference (u^*, p^*, θ^*) in $L^2(\Omega)$. Taking the L^2-inner product of (3.97) with (u^*, θ^*) and applying the divergence-free condition, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|(u^*,\theta^*)\|_{L^2}^2 + \nu\|\partial_2 u^*\|_{L^2}^2 + \eta\|\partial_1\theta^*\|_{L^2}^2 &= -\int u^* \cdot \nabla u^{(1)} \cdot u^* \, dx - \int u^* \cdot \nabla \theta^{(1)} \cdot \theta^* \, dx \\ &= I_1 + I_2. \end{aligned} \tag{3.98}$$

Due to Lemma 2.3 and the uniformly global bound for $\|u^{(1)}\|_{H^2}$,

$$\begin{aligned} I_1 &:= -\int u^* \cdot \nabla u^{(1)} \cdot u^* \, dx \\ &\le c \underbrace{\|\nabla u^{(1)}\|_{L^2}^{\frac{1}{2}} \Big(\|\nabla u^{(1)}\|_{L^2} + \|\partial_1 \nabla u^{(1)}\|_{L^2} \Big)^{\frac{1}{2}}}_{\le c} \|u^*\|_{L^2}^{\frac{1}{2}} \|\partial_2 u^*\|_{L^2}^{\frac{1}{2}} \|u^*\|_{L^2} \\ &\le c\|u^*\|_{L^2}^{\frac{3}{2}} \|\partial_2 u^*\|_{L^2}^{\frac{1}{2}} \\ &\le c\|u^*\|_{L^2}^2 + \frac{\nu}{4}\|\partial_2 u^*\|_{L^2}^2. \end{aligned} \tag{3.99}$$

Similarly, by Lemma 2.3 and the uniformly global bound for $\|\theta^{(1)}\|_{H^2}$,

$$\begin{aligned} I_2 &:= -\int u^* \cdot \nabla \theta^{(1)} \cdot \theta^* \, dx \\ &\le c \underbrace{\|\nabla \theta^{(1)}\|_{L^2}^{\frac{1}{2}} \Big(\|\nabla \theta^{(1)}\|_{L^2} + \|\partial_1 \nabla \theta^{(1)}\|_{L^2} \Big)^{\frac{1}{2}}}_{\le c} \|u^*\|_{L^2}^{\frac{1}{2}} \|\partial_2 u^*\|_{L^2}^{\frac{1}{2}} \|\theta^*\|_{L^2} \\ &\le c\|u^*\|_{L^2}^{\frac{1}{2}} \|\partial_2 u^*\|_{L^2}^{\frac{1}{2}} \|\theta^*\|_{L^2} \\ &\le c\|\theta^*\|_{L^2} \Big(\|u^*\|_{L^2} + \|\partial_2 u^*\|_{L^2} \Big) \\ &\le c\|\theta^*\|_{L^2}^2 + c\|u^*\|_{L^2}^2 + \frac{\nu}{4}\|\partial_2 u^*\|_{L^2}^2. \end{aligned} \tag{3.100}$$

Putting the estimates (3.99) and (3.100) in (3.98) leads to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|(u^*,\theta^*)\|_{L^2}^2 + \nu\|\partial_2 u^*\|_{L^2}^2 + \eta\|\partial_1\theta^*\|_{L^2}^2 \\ &\quad \le c\Big(\|u^*\|_{L^2}^2 + \|\theta^*\|_{L^2}^2 \Big) + \frac{\nu}{2}\|\partial_2 u^*\|_{L^2}^2 \end{aligned}$$

or

$$\frac{d}{dt}\|(u^*,\theta^*)\|_{L^2}^2 + \nu\|\partial_2 u^*\|_{L^2}^2 + \eta\|\partial_1\theta^*\|_{L^2}^2 \le c\|(u^*,\theta^*)\|_{L^2}^2. \tag{3.101}$$

Grönwall's inequality then implies,

$$\|u^*(t)\|_{L^2} = \|\theta^*(t)\|_{L^2} = 0.$$

In other words, these two solutions coincide. This finishes the proof of Theorem 1.1. □

4. Decay Rates Result

This section is devoted to the proof the decay rates presented in Theorem 1.2.

Proof of Theorem 1.2. Taking the average of the system (1.2) and using the fact that $\overline{u\cdot\nabla\overline{u}}=0$, we write the equations of $(\overline{u},\overline{\theta})$,

$$\begin{cases}\partial_t\overline{u}+\overline{u\cdot\nabla\widetilde{u}}+\begin{pmatrix}0\\ \partial_2\overline{p}\end{pmatrix}=g_0\begin{pmatrix}0\\ \overline{\theta}\end{pmatrix}+\nu\partial_2^2\overline{u}\,,\\ \partial_t\overline{\theta}+\overline{u\cdot\nabla\widetilde{\theta}}=0,\end{cases}\tag{4.1}$$

where g_0 is a negative constant. By subtracting (4.1) from (1.2), we get

$$\begin{cases}\partial_t\widetilde{u}+\widetilde{u\cdot\nabla\widetilde{u}}+\widetilde{u}_2\partial_2\overline{u}-\nu\partial_2^2\widetilde{u}+\nabla\widetilde{p}=g_0\widetilde{\theta}e_2\,,\\ \partial_t\widetilde{\theta}+\widetilde{u\cdot\nabla\widetilde{\theta}}+\widetilde{u}_2\partial_2\overline{\theta}-\eta\partial_1^2\widetilde{\theta}+g_0\widetilde{u}_2=0.\end{cases}\tag{4.2}$$

Taking the L^2-inner product of $(\widetilde{u},\widetilde{\theta})$ with (4.2) yields,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\Big)+\nu\|\partial_2\widetilde{u}\|_{L^2}^2+\eta\|\partial_1\widetilde{\theta}\|_{L^2}^2\\&=-\int\widetilde{u\cdot\nabla\widetilde{u}}\cdot\widetilde{u}dx-\int\widetilde{u}_2\partial_2\overline{u}\cdot\widetilde{u}dx-\int\widetilde{u\cdot\nabla\widetilde{\theta}}\cdot\widetilde{\theta}dx-\int\widetilde{u}_2\partial_2\overline{\theta}\cdot\widetilde{\theta}dx\\&:=A_1+A_2+A_3+A_4.\end{aligned}\tag{4.3}$$

Now, we estimate A_1 through A_4. The first term A_1 is clearly zero due to $\nabla\cdot u=0$ and Lemma 2.1,

$$A_1:=-\int\widetilde{u\cdot\nabla\widetilde{u}}\cdot\widetilde{u}dx=\underbrace{-\int u\cdot\nabla\widetilde{u}\cdot\widetilde{u}dx}_{=0}+\underbrace{\int\overline{u\cdot\nabla\widetilde{u}}\cdot\widetilde{u}dx}_{=0}=0.\tag{4.4}$$

Likewise,

$$A_3:=\int\widetilde{u\cdot\nabla\widetilde{\theta}}\cdot\widetilde{\theta}dx=0.\tag{4.5}$$

To bound A_2 we first write it as,

$$\begin{aligned}A_2&:=-\int\widetilde{u}_2\partial_2\overline{u}\cdot\widetilde{u}dx\\&:=-\int\widetilde{u}_2\partial_2\overline{u_1}\widetilde{u}_1dx-\int\widetilde{u}_2\partial_2\overline{u_2}\widetilde{u}_2dx\\&:=A_{21}+A_{22}.\end{aligned}\tag{4.6}$$

Due to the fact that $\overline{u_2}=0$ we have,

$$A_{22}=-\int\widetilde{u}_2\partial_2\overline{u_2}\widetilde{u}_2dx=0.\tag{4.7}$$

Applying Lemmas 2.4 and 2.5, the divergence-free condition of u and then Young's inequality leads to

$$\begin{aligned}
A_{21} &:= -\int \widetilde{u_2}\partial_2\overline{u_1}\widetilde{u_1}dx \\
&\leq c\|\partial_2\overline{u}\|_{L^2}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \underbrace{\|\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}}_{\leq\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}=\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}} \underbrace{\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}}_{=\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}} \\
&\leq c\|u\|_{H^2}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u}\|_{L^2}^{\frac{3}{2}} \\
&\leq c\|u\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2+\|\partial_2\widetilde{u}\|_{L^2}^2\Big). \qquad (4.8)
\end{aligned}$$

Inserting (4.7) and (4.8) in (4.6) we get

$$A_2 \leq c\|u\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2+\|\partial_2\widetilde{u}\|_{L^2}^2\Big). \qquad (4.9)$$

The last term A_4 can be bounded via Lemma 2.1, Hölder's inequality, and Lemmas 2.2 and 2.5,

$$\begin{aligned}
A_4 &:= -\int \widetilde{u_2}\partial_2\overline{\theta}\cdot\widetilde{\theta}dx \\
&= -\int_{\mathbb{R}}\partial_2\overline{\theta}\Big(\int_{\mathbb{T}}\widetilde{\theta}\widetilde{u_2}dx_1\Big)dx_2 \\
&\leq c\|\partial_2\overline{\theta}\|_{L^\infty_{x_2}}\|\widetilde{u_2}\|_{L^2}\|\widetilde{\theta}\|_{L^2} \\
&\leq c\|\partial_2\overline{\theta}\|_{H^1}\|\widetilde{u_2}\|_{L^2}\|\partial_1\widetilde{\theta}\|_{L^2} \\
&\leq c\|\theta\|_{H^2}\|\widetilde{u_2}\|_{L^2}\|\partial_1\widetilde{\theta}\|_{L^2} \\
&\leq c\|\theta\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2+\|\partial_1\widetilde{\theta}\|_{L^2}^2\Big). \qquad (4.10)
\end{aligned}$$

Combining the estimates of A_1 through A_4, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\Big)+\nu\|\partial_2\widetilde{u}\|_{L^2}^2+\eta\|\partial_1\widetilde{\theta}\|_{L^2}^2 \\
&\qquad\leq c\|(u,\theta)\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2+\|\partial_2\widetilde{u}\|_{L^2}^2+\|\partial_1\widetilde{\theta}\|_{L^2}^2\Big). \qquad (4.11)
\end{aligned}$$

Applying ∇ to (4.2), we write

$$\begin{cases}
\partial_t\nabla\widetilde{u}+\nabla\widetilde{(u\cdot\nabla u)}+\nabla(\widetilde{u_2}\partial_2\overline{u})-\nu\partial_2^2\nabla\widetilde{u}+\nabla\nabla\widetilde{p}=g_0\nabla(\widetilde{\theta}e_2)\,, \\
\partial_t\nabla\widetilde{\theta}+\nabla\widetilde{(u\cdot\nabla\theta)}+\nabla(\widetilde{u_2}\partial_2\overline{\theta})-\eta\partial_1^2\nabla\widetilde{\theta}+g_0\nabla\widetilde{u_2}=0.
\end{cases} \qquad (4.12)$$

Dotting (4.12) by $(\nabla\widetilde{u},\nabla\widetilde{\theta})$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla\widetilde{u}(t)\|_{L^2}^2+\|\nabla\widetilde{\theta}(t)\|_{L^2}^2\Big)+\nu\|\partial_2\nabla\widetilde{u}\|_{L^2}^2+\eta\|\partial_1\nabla\widetilde{\theta}\|_{L^2}^2 \\
&=-\int\nabla\widetilde{(u\cdot\nabla u)}\cdot\nabla\widetilde{u}dx-\int\nabla(\widetilde{u_2}\partial_2\overline{u})\cdot\nabla\widetilde{u}dx
\end{aligned}$$

$$
\begin{aligned}
&-\int \nabla(\widetilde{u\cdot\nabla\theta})\cdot\nabla\widetilde{\theta}dx-\int\nabla(\widetilde{u}_2\partial_2\overline{\theta})\cdot\nabla\widetilde{\theta}dx\\
&:=B_1+B_2+B_3+B_4. \qquad (4.13)
\end{aligned}
$$

The terms B_1 through B_4 can be bounded as follows. We start with B_1. Using Lemma 2.1, we write B_1 as,

$$
\begin{aligned}
B_1&:=-\int\nabla(\widetilde{u\cdot\nabla u})\cdot\nabla\widetilde{u}dx\\
&=-\int\nabla(u\cdot\nabla\widetilde{u})\cdot\nabla\widetilde{u}\,dx+\underbrace{\int\nabla(\overline{u\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{u}dx}_{=0}\\
&=-\int\partial_1u_1\partial_1\widetilde{u}\cdot\partial_1\widetilde{u}dx-\int\partial_1u_2\partial_2\widetilde{u}\cdot\partial_1\widetilde{u}dx\\
&\quad-\int\partial_2u_1\partial_1\widetilde{u}\cdot\partial_2\widetilde{u}dx-\int\partial_2u_2\partial_2\widetilde{u}\cdot\partial_2\widetilde{u}dx\\
&:=B_{11}+B_{12}+B_{13}+B_{14}. \qquad (4.14)
\end{aligned}
$$

Further, we divide the first term B_{11} into the following two integrals,

$$
\begin{aligned}
B_{11}&:=-\int\partial_1u_1\partial_1\widetilde{u}\cdot\partial_1\widetilde{u}dx\\
&=-\int\partial_1\widetilde{u}_1\partial_1\widetilde{u}_1\partial_1\widetilde{u}_1dx-\int\partial_1\widetilde{u}_1\partial_1\widetilde{u}_2\partial_1\widetilde{u}_2dx\\
&:=B_{111}+B_{112}. \qquad (4.15)
\end{aligned}
$$

By the divergence-free condition of u and Lemma 2.4

$$
\begin{aligned}
B_{111}&:=-\int\partial_1\widetilde{u}_1\partial_1\widetilde{u}_1\partial_1\widetilde{u}_1dx\\
&=\int\partial_2\widetilde{u}_2\partial_2\widetilde{u}_2\partial_2\widetilde{u}_2dx\\
&\le c\|\partial_2\widetilde{u}_2\|_{L^2}\|\partial_2\widetilde{u}_2\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{u}_2\|_{L^2}^{\frac12}\|\partial_2\widetilde{u}_2\|_{L^2}^{\frac12}\|\partial_2\partial_2\widetilde{u}_2\|_{L^2}^{\frac12}\\
&\le c\|u\|_{H^2}\|\partial_2\widetilde{u}\|_{H^1}^2. \qquad (4.16)
\end{aligned}
$$

Due to $\nabla\cdot u=0$, integration by parts and Lemma, 2.4

$$
\begin{aligned}
B_{112}&:=-\int\partial_1\widetilde{u}_1\partial_1\widetilde{u}_2\partial_1\widetilde{u}_2dx\\
&=\int\partial_2\widetilde{u}_2\partial_1\widetilde{u}_2\partial_1\widetilde{u}_2dx\\
&=2\int\widetilde{u}_2\partial_2\partial_1\widetilde{u}_2\partial_1\widetilde{u}_2dx\\
&\le c\|\partial_2\partial_1\widetilde{u}_2\|_{L^2}\|\widetilde{u}_2\|_{L^2}^{\frac12}\|\partial_2\widetilde{u}_2\|_{L^2}^{\frac12}\|\partial_1\widetilde{u}_2\|_{L^2}^{\frac12}\|\partial_1\partial_1\widetilde{u}_2\|_{L^2}^{\frac12}\\
&\le c\|u\|_{H^2}\|\widetilde{u}_2\|_{L^2}^{\frac12}\|\partial_2\widetilde{u}\|_{H^1}^{\frac32}
\end{aligned}
$$

$$\leq c\|u\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2+\|\partial_2\widetilde{u}\|_{H^1}^2\Big). \tag{4.17}$$

Inserting the upper bound for B_{111} and B_{112} in (4.15) we get

$$B_{11}\leq c\|u\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2+\|\partial_2\widetilde{u}\|_{H^1}^2\Big). \tag{4.18}$$

To deal with B_{12}, we write it first as

$$\begin{aligned} B_{12}&:=-\int\partial_1u_2\partial_2\widetilde{u}\cdot\partial_1\widetilde{u}dx\\ &=-\int\partial_1\widetilde{u_2}\partial_2\widetilde{u_1}\partial_1\widetilde{u_1}dx-\int\partial_1\widetilde{u_2}\partial_2\widetilde{u_2}\partial_1\widetilde{u_2}dx\\ &:=B_{121}+B_{122}. \end{aligned} \tag{4.19}$$

For B_{121}, we use the divergence-free condition of u and Lemma 2.4

$$\begin{aligned} B_{121}&:=-\int\partial_1\widetilde{u_2}\partial_2\widetilde{u_1}\partial_1\widetilde{u_1}dx=\int\partial_1\widetilde{u_2}\partial_2\widetilde{u_1}\partial_2\widetilde{u_2}dx\\ &\leq c\|\partial_1\widetilde{u_2}\|_{L^2}\|\partial_2\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\\ &\leq c\|u\|_{H^2}\|\partial_2\widetilde{u}\|_{H^1}^2. \end{aligned} \tag{4.20}$$

The second piece B_{122} can be bounded using integrating by parts, Lemma 2.4 and then Young's inequality

$$\begin{aligned} B_{122}&:=-\int\partial_1\widetilde{u_2}\partial_2\widetilde{u_2}\partial_1\widetilde{u_2}dx\\ &=2\int\partial_2\partial_1\widetilde{u_2}\widetilde{u_2}\partial_1\widetilde{u_2}dx\\ &\leq c\|\partial_2\partial_1\widetilde{u_2}\|_{L^2}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\\ &\leq c\|u\|_{H^2}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u}\|_{H^1}^{\frac{3}{2}}\\ &\leq c\|u\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2+\|\partial_2\widetilde{u}\|_{H^1}^2\Big) \end{aligned} \tag{4.21}$$

Combining (4.20) and (4.21) and inserting them in (4.19) we obtain

$$B_{12}\leq c\|u\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2+\|\partial_2\widetilde{u}\|_{H^1}^2\Big). \tag{4.22}$$

The term B_{13} is naturally divided into two integrals,

$$\begin{aligned} B_{13}&:=-\int\partial_2u_1\partial_1\widetilde{u}\cdot\partial_2\widetilde{u}dx\\ &=-\int\partial_2u_1\partial_1\widetilde{u_1}\partial_2\widetilde{u_1}dx-\int\partial_2u_1\partial_1\widetilde{u_2}\partial_2\widetilde{u_2}dx\\ &:=B_{131}+B_{132}. \end{aligned} \tag{4.23}$$

Due to $\nabla\cdot u=0$ and Lemma 2.4,

$$B_{131}:=-\int\partial_2u_1\partial_1\widetilde{u_1}\partial_2\widetilde{u_1}dx$$

$$
\begin{aligned}
&= \int \partial_2 u_1 \partial_2 \widetilde{u_2} \partial_2 \widetilde{u_1} dx \\
&\leq c\|\partial_2 u_1\|_{L^2} \|\partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \\
&\leq c\|u\|_{H^2} \|\partial_2 \widetilde{u}\|_{H^1}^2. \qquad (4.24)
\end{aligned}
$$

Integrating by parts, making use of Lemma 2.4 and then Young's inequality

$$
\begin{aligned}
B_{132} &:= -\int \partial_2 u_1 \partial_1 \widetilde{u_2} \partial_2 \widetilde{u_2} dx \\
&= \int \partial_1 \partial_2 \widetilde{u_1} \widetilde{u_2} \partial_2 \widetilde{u_2} dx + \int \partial_2 u_1 \widetilde{u_2} \partial_1 \partial_2 \widetilde{u_2} dx \\
&\leq c\|\partial_1 \partial_2 \widetilde{u_1}\|_{L^2} \|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \\
&\quad + c\|\partial_2 u_1\|_{L^2} \|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_1 \partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \\
&\leq c\|u\|_{H^2} \|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u}\|_{H^1}^{\frac{3}{2}} \\
&\leq c\|u\|_{H^2} \Big(\|\widetilde{u_2}\|_{L^2}^2 + \|\partial_2 \widetilde{u}\|_{H^1}^2 \Big). \qquad (4.25)
\end{aligned}
$$

Inserting the estimates (4.24) and (4.25) in (4.23) we get

$$
B_{13} \leq c\|u\|_{H^2} \Big(\|\widetilde{u_2}\|_{L^2}^2 + \|\partial_2 \widetilde{u}\|_{H^1}^2 \Big). \qquad (4.26)
$$

The last term B_{14} can be bounded directly via Lemma 2.4,

$$
\begin{aligned}
B_{14} &:= -\int \partial_2 u_2 \partial_2 \widetilde{u} \cdot \partial_2 \widetilde{u} dx \\
&\leq c\|\partial_2 u_2\|_{L^2} \|\partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\leq c\|u\|_{H^2} \|\partial_2 \widetilde{u}\|_{H^1}^2. \qquad (4.27)
\end{aligned}
$$

Collecting the upper bounds obtained in (4.18), (4.22), (4.26) and (4.27) and inserting them in (4.14), yields

$$
B_1 \leq c\|u\|_{H^2} \Big(\|\widetilde{u_2}\|_{L^2}^2 + \|\partial_2 \widetilde{u}\|_{H^1}^2 \Big). \qquad (4.28)
$$

The next term B_2 is naturally split into four parts,

$$
\begin{aligned}
B_2 &:= -\int \nabla(\widetilde{u_2} \partial_2 \overline{u}) \cdot \nabla \widetilde{u} dx \\
&= -\int \partial_1 \widetilde{u_2} \partial_2 \overline{u} \cdot \partial_1 \widetilde{u} dx - \int \partial_2 \widetilde{u_2} \partial_2 \overline{u} \cdot \partial_2 \widetilde{u} dx \\
&\quad - \int \widetilde{u_2} \partial_1 \partial_2 \overline{u} \cdot \partial_1 \widetilde{u} dx - \int \widetilde{u_2} \partial_2 \partial_2 \overline{u} \cdot \partial_2 \widetilde{u} dx \\
&:= B_{21} + B_{22} + B_{23} + B_{24}. \qquad (4.29)
\end{aligned}
$$

We rewrite B_{21} as,

$$
B_{21} := -\int \partial_1 \widetilde{u_2} \partial_2 \overline{u} \cdot \partial_1 \widetilde{u} dx
$$

$$= -\int \partial_1\widetilde{u_2}\partial_2\overline{u_1}\partial_1\widetilde{u_1}dx - \int \partial_1\widetilde{u_2}\partial_2\overline{u_2}\partial_1\widetilde{u_2}dx$$
$$:= B_{211} + B_{212}. \tag{4.30}$$

Clearly, due to $\overline{u_2} = 0$,

$$B_{212} := -\int \partial_1\widetilde{u_2}\partial_2\overline{u_2}\partial_1\widetilde{u_2}dx = 0. \tag{4.31}$$

By the divergence-free condition of u, integration by parts, Lemma 2.1, Hölder's inequality and then Lemma 2.2

$$\begin{aligned}
B_{211} &:= -\int \partial_1\widetilde{u_2}\partial_2\overline{u_1}\partial_1\widetilde{u_1}dx \\
&= \int \partial_1\widetilde{u_2}\partial_2\overline{u_1}\partial_2\widetilde{u_2}dx \\
&= \int \widetilde{u_2}\partial_2\overline{u_1}\partial_1\partial_2\widetilde{u_2}dx \\
&= \int_{\mathbb{R}} \partial_2\overline{u_1}\Big(\int_{\mathbb{T}} \widetilde{u_2}\partial_1\partial_2\widetilde{u_2}dx_1\Big)dx_2 \\
&\le c\|\partial_2\overline{u_1}\|_{L^\infty_{x_2}}\|\widetilde{u_2}\|_{L^2}\|\partial_1\partial_2\widetilde{u_2}\|_{L^2} \\
&\le c\|\partial_2\overline{u_1}\|_{H^1}\|\widetilde{u_2}\|_{L^2}\|\partial_1\partial_2\widetilde{u_2}\|_{L^2} \\
&\le c\|u\|_{H^2}\|\widetilde{u_2}\|_{L^2}\|\partial_2\widetilde{u}\|_{H^1} \\
&\le c\|\theta\|_{H^2}\Big(\|\widetilde{u_2}\|^2_{L^2} + \|\partial_2\widetilde{u}\|^2_{H^1}\Big).
\end{aligned} \tag{4.32}$$

It then follows from (4.31), (4.32) and (4.30) that

$$B_{21} \le c\|u\|_{H^2}\Big(\|\partial_2\widetilde{u}\|^2_{H^1} + \|\widetilde{u_2}\|^2_{L^2}\Big). \tag{4.33}$$

According to Lemma 2.4,

$$\begin{aligned}
B_{22} &:= -\int \partial_2\widetilde{u_2}\partial_2\overline{u}\cdot\partial_2\widetilde{u}dx \\
&\le c\|\partial_2\overline{u}\|_{L^2}\|\partial_2\widetilde{u_2}\|^{\frac12}_{L^2}\|\partial_1\partial_2\widetilde{u_2}\|^{\frac12}_{L^2}\|\partial_2\widetilde{u}\|^{\frac12}_{L^2}\|\partial_2\partial_2\widetilde{u}\|^{\frac12}_{L^2} \\
&\le c\|u\|_{H^2}\|\partial_2\widetilde{u}\|^2_{H^1}.
\end{aligned} \tag{4.34}$$

By definition of $\overline{u}$,

$$B_{23} := -\int \widetilde{u_2}\partial_1\partial_2\overline{u}\cdot\partial_1\widetilde{u}dx = 0. \tag{4.35}$$

Due Lemma 2.4 and Young's inequality,

$$\begin{aligned}
B_{24} &:= -\int \widetilde{u_2}\partial_2\partial_2\overline{u}\cdot\partial_2\widetilde{u}dx \\
&\le c\|\partial_2\partial_2\overline{u}\|_{L^2}\|\widetilde{u_2}\|^{\frac12}_{L^2}\|\partial_2\widetilde{u_2}\|^{\frac12}_{L^2}\|\partial_2\widetilde{u}\|^{\frac12}_{L^2}\|\partial_1\partial_2\widetilde{u}\|^{\frac12}_{L^2} \\
&\le c\|u\|_{H^2}\|\widetilde{u_2}\|^{\frac12}_{L^2}\|\partial_2\widetilde{u}\|^{\frac32}_{H^1}
\end{aligned}$$

$$\leq c\|u\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2+\|\partial_2\widetilde{u}\|_{H^1}^2\Big). \tag{4.36}$$

Collecting the estimates (4.33), (4.34), (4.35), (4.36) and (4.29), we get

$$B_2\leq c\|u\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2+\|\partial_2\widetilde{u}\|_{H^1}^2\Big). \tag{4.37}$$

To bound B_3, we first write $u=\overline{u}+\widetilde{u}$ and use Lemma 2.1

$$\begin{aligned}
B_3 &:= -\int \widetilde{\nabla(u\cdot\nabla\widetilde{\theta})}\cdot\nabla\widetilde{\theta}dx\\
&= -\int \nabla(u\cdot\nabla\widetilde{\theta})\cdot\nabla\widetilde{\theta}dx+\underbrace{\int\overline{\nabla(u\cdot\nabla\widetilde{\theta})}\cdot\nabla\widetilde{\theta}dx}_{=0}\\
&= -\int \partial_1\widetilde{\theta}\partial_1\widetilde{u_1}\partial_1\widetilde{\theta}dx-\int\partial_2\widetilde{\theta}\partial_1\widetilde{u_2}\partial_1\widetilde{\theta}dx\\
&\quad -\int\partial_1\widetilde{\theta}\partial_2u_1\partial_2\widetilde{\theta}dx-\int\partial_2\widetilde{\theta}\partial_2\widetilde{u_2}\partial_2\widetilde{\theta}dx\\
&:= B_{31}+B_{32}+B_{33}+B_{34}.
\end{aligned} \tag{4.38}$$

All terms in (4.38) can be bounded suitably. In fact, by Lemma 2.4,

$$\begin{aligned}
B_{31} &:= -\int\partial_1\widetilde{\theta}\partial_1\widetilde{u_1}\partial_1\widetilde{\theta}dx\\
&\leq c\|\partial_1\widetilde{u_1}\|_{L^2}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\\
&\leq c\|u\|_{H^2}\|\partial_1\widetilde{\theta}\|_{H^1}^2.
\end{aligned} \tag{4.39}$$

For B_{32}, B_{33} and B_{34} we use Lemmas 2.4 and 2.5,

$$\begin{aligned}
B_{32} &:= -\int\partial_2\widetilde{\theta}\partial_1\widetilde{u_2}\partial_1\widetilde{\theta}dx\\
&\leq c\|\partial_1\widetilde{u_2}\|_{L^2}\underbrace{\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}}_{\leq\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\\
&\leq c\|u\|_{H^2}\|\partial_1\widetilde{\theta}\|_{H^1}^2,
\end{aligned} \tag{4.40}$$

$$\begin{aligned}
B_{33} &:= -\int\partial_1\widetilde{\theta}\partial_2u_1\partial_2\widetilde{\theta}dx\\
&\leq c\|\partial_2u_1\|_{L^2}\underbrace{\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}}_{\leq\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\\
&\leq c\|u\|_{H^2}\|\partial_1\widetilde{\theta}\|_{H^1}^2,
\end{aligned} \tag{4.41}$$

and

$$B_{34} := -\int\partial_2\widetilde{\theta}\partial_2\widetilde{u_2}\partial_2\widetilde{\theta}dx$$

$$\leq c\|\partial_2\widetilde{\theta}\|_{L^2}\underbrace{\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}}_{\leq\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}$$
$$\leq c\|\theta\|_{H^2}\|\partial_2\widetilde{u}\|_{H^1}\|\partial_1\widetilde{\theta}\|_{H^1}$$
$$\leq c\|\theta\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big). \tag{4.42}$$

Combining the estimates (4.39), (4.40), (4.41), (4.42) and (4.38), we obtain

$$B_3\leq c\|(u,\theta)\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big). \tag{4.43}$$

To deal with B_4, we split it into four pieces,

$$\begin{aligned}B_4&:=-\int\nabla(\widetilde{u_2}\partial_2\overline{\theta})\cdot\nabla\widetilde{\theta}dx\\&=-\int\partial_1(\widetilde{u_2}\partial_2\overline{\theta})\cdot\partial_1\widetilde{\theta}dx-\int\partial_2(\widetilde{u_2}\partial_2\overline{\theta})\cdot\partial_2\widetilde{\theta}dx\\&=-\int\partial_1\widetilde{u_2}\partial_2\overline{\theta}\partial_1\widetilde{\theta}dx-\int\widetilde{u_2}\partial_1\partial_2\overline{\theta}\partial_1\widetilde{\theta}dx\\&\quad-\int\partial_2\widetilde{u_2}\partial_2\overline{\theta}\partial_2\widetilde{\theta}dx-\int\widetilde{u_2}\partial_2\partial_2\overline{\theta}\partial_2\widetilde{\theta}dx\\&:=B_{41}+B_{42}+B_{43}+B_{44}.\end{aligned} \tag{4.44}$$

The terms above can be bounded as follows. Due to the definition of the horizontal average $\overline{\theta}$,

$$B_{42}:=-\int\widetilde{u_2}\partial_1\partial_2\overline{\theta}\partial_1\widetilde{\theta}dx=0. \tag{4.45}$$

For B_{41}, we use integration by parts, Lemma 2.1, Hölder's inequality and then Lemma 2.2

$$\begin{aligned}B_{41}&:=-\int\partial_1\widetilde{u_2}\partial_2\overline{\theta}\partial_1\widetilde{\theta}dx\\&=-\int\widetilde{u_2}\partial_2\overline{\theta}\partial_1\partial_1\widetilde{\theta}dx\\&=\int_{\mathbb{R}}\partial_2\overline{\theta}\Big(\int_{\mathbb{T}}\widetilde{u_2}\partial_1\partial_1\widetilde{\theta}dx_1\Big)dx_2\\&\leq c\|\partial_2\overline{\theta}\|_{L^\infty_{x_2}}\|\widetilde{u_2}\|_{L^2}\|\partial_1\partial_1\widetilde{\theta}\|_{L^2}\\&\leq c\|\partial_2\overline{\theta}\|_{H^1}\|\widetilde{u_2}\|_{L^2}\|\partial_1\partial_1\widetilde{\theta}\|_{L^2}\\&\leq c\|\theta\|_{H^2}\|\widetilde{u_2}\|_{L^2}\|\partial_1\widetilde{\theta}\|_{H^1}\\&\leq c\|\theta\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big).\end{aligned} \tag{4.46}$$

The other two terms B_{43}, B_{44} can be bounded via Lemmas 2.4 and 2.5,

$$B_{43}:=-\int\partial_2\widetilde{u_2}\partial_2\overline{\theta}\partial_2\widetilde{\theta}dx$$

$$\le c\|\partial_2\overline{\theta}\|_{L^2}\underbrace{\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}}_{\le\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\widetilde{u_2}\|_{L^2}\|\partial_2\partial_2\widetilde{u_2}\|_{L^2}^{\frac12}$$
$$\le c\|\theta\|_{H^2}\|\partial_1\widetilde{\theta}\|_{H^1}\|\partial_2\widetilde{u}\|_{H^1}$$
$$\le c\|\theta\|_{H^2}\Big(\|\partial_1\widetilde{\theta}\|_{H^1}^2+\|\partial_2\widetilde{u}\|_{H^1}^2\Big), \tag{4.47}$$

$$B_{44}:=-\int\widetilde{u_2}\partial_2\partial_2\overline{\theta}\partial_2\widetilde{\theta}dx$$
$$\le c\|\partial_2\partial_2\overline{\theta}\|_{L^2}\underbrace{\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}}_{\le\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}\|\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac12}$$
$$\le c\|\theta\|_{H^2}\|\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_2\widetilde{u}\|_{H^1}^{\frac12}\|\partial_1\widetilde{\theta}\|_{H^1}$$
$$\le c\|\theta\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big). \tag{4.48}$$

Inserting all the bounds obtained above for B_{41} through B_{44} in (4.44) leads to

$$B_4\le c\|(u,\theta)\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2\Big). \tag{4.49}$$

Collecting (4.28), (4.37), (4.43) and (4.49) gives

$$\frac12\frac{d}{dt}\Big(\|\nabla\widetilde{u}(t)\|_{L^2}^2+\|\nabla\widetilde{\theta}(t)\|_{L^2}^2\Big)+\nu\|\partial_2\nabla\widetilde{u}\|_{L^2}^2+\eta\|\partial_1\nabla\widetilde{\theta}\|_{L^2}^2$$
$$\le c\|(u,\theta)\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2\Big). \tag{4.50}$$

Now, to control the norm $\|\widetilde{u_2}\|_{L^2}$ present in (4.11) and (4.50), we need to add the following term,

$$-\frac{d}{dt}\Big(\delta(\widetilde{u_2},\widetilde{\theta})\Big)=-\delta(\partial_t\widetilde{u_2},\widetilde{\theta})-\delta(\widetilde{u_2},\partial_t\widetilde{\theta}),$$

with $\delta>0$ is a small constant to be fixed at the end of the proof. Doing so, we generate an extra regularization term that helps bound $\|\widetilde{u_2}\|_{L^2}$. Note that, this stabilizing term comes from the interaction between $\widetilde{u}$ and $\widetilde{\theta}$. Due to Hölder's inequality, we have, for sufficiently small $\delta>0$,

$$\|(\widetilde{u},\widetilde{\theta})\|_{H^1}^2-\delta(\widetilde{u_2},\widetilde{\theta})\ge0.$$

Using the first equation of (4.2) and $\overline{u_2}=0$, we write

$$\partial_t\widetilde{u_2}+\widetilde{u\cdot\nabla u_2}+\underbrace{\widetilde{u_2}\partial_2\overline{u_2}}_{=0}-\nu\partial_2^2\widetilde{u_2}+\partial_2\widetilde{p}=g_0\widetilde{\theta}. \tag{4.51}$$

Applying $\nabla\cdot$ to the first equation of (4.2), we obtain

$$\nabla\cdot(\widetilde{u\cdot\nabla u})+\nabla\cdot(\widetilde{u_2}\partial_2\overline{u})+\Delta\widetilde{p}=g_0\partial_2\widetilde{\theta}. \tag{4.52}$$

Making use of (4.52), we have

$$\widetilde{p}=-\Delta^{-1}\nabla\cdot(\widetilde{u\cdot\nabla u})-\Delta^{-1}\nabla\cdot(\widetilde{u_2}\partial_2\overline{u})+g_0\Delta^{-1}\partial_2\widetilde{\theta}.$$

Then,

$$\partial_2\widetilde{p} = -\partial_2\Delta^{-1}\nabla\cdot(\widetilde{u\cdot\nabla u}) - \partial_2\Delta^{-1}\nabla\cdot(\widetilde{u}_2\partial_2\overline{u}) + g_0\partial_2\partial_2\Delta^{-1}\widetilde{\theta}. \tag{4.53}$$

By (4.51) and the second equation of (4.2), we write

$$\begin{aligned}
-\delta\frac{d}{dt}(\widetilde{u}_2,\widetilde{\theta}) &= -\delta(\partial_t\widetilde{u}_2,\widetilde{\theta}) - \delta(\widetilde{u}_2,\partial_t\widetilde{\theta}) \\
&= -\delta(g_0\widetilde{\theta} - \partial_2\widetilde{p} + \nu\partial_2^2\widetilde{u}_2 - \widetilde{u\cdot\nabla u}_2,\widetilde{\theta}) \\
&\quad - \delta(\widetilde{u}_2, -g_0\widetilde{u}_2 + \eta\partial_1^2\widetilde{\theta} - \widetilde{u}_2\partial_2\overline{\theta} - \widetilde{u\cdot\nabla\theta}) \\
&= -g_0\delta\|\widetilde{\theta}\|_{L^2}^2 + \int\partial_2\widetilde{p}\widetilde{\theta}dx - \delta\nu\int\partial_2^2\widetilde{u}_2\widetilde{\theta}dx + \delta\int\widetilde{u\cdot\nabla u}_2\widetilde{\theta}dx \\
&\quad + g_0\delta\|\widetilde{u}_2\|_{L^2}^2 - \delta\eta\int\partial_1^2\widetilde{\theta}\widetilde{u}_2dx + \delta\int\widetilde{u}_2\widetilde{u}_2\partial_2\overline{\theta}dx + \delta\int\widetilde{u\cdot\nabla\theta}\widetilde{u}_2dx \\
&:= N_1 + \cdots + N_8.
\end{aligned} \tag{4.54}$$

The terms N_1 through N_8 obey the following bounds. For N_2, we use (4.53) to rewrite it as,

$$\begin{aligned}
N_2 &:= \delta\int\partial_2\widetilde{p}\widetilde{\theta}dx \\
&= -\delta\int\partial_2\Delta^{-1}\nabla\cdot(\widetilde{u\cdot\nabla u})\cdot\widetilde{\theta}dx - \delta\int\partial_2\Delta^{-1}\nabla\cdot(\widetilde{u}_2\partial_2\overline{u})\cdot\widetilde{\theta}dx \\
&\quad + g_0\delta\int\partial_2\partial_2\Delta^{-1}\widetilde{\theta}\cdot\widetilde{\theta}dx \\
&:= N_{21} + N_{22} + N_{23}.
\end{aligned} \tag{4.55}$$

By Lemma 2.1 and integration by parts we split N_{21} into three pieces

$$\begin{aligned}
N_{21} &:= -\delta\int\partial_2\Delta^{-1}\nabla\cdot(\widetilde{u\cdot\nabla u})\cdot\widetilde{\theta}dx \\
&= -\delta\int\partial_2\Delta^{-1}\nabla\cdot(u\cdot\nabla\widetilde{u})\cdot\widetilde{\theta}dx + \underbrace{\delta\int\partial_2\Delta^{-1}\nabla\cdot(\overline{u\cdot\nabla\widetilde{u}})\cdot\widetilde{\theta}dx}_{=0} \\
&= -\delta\int\partial_2\Delta^{-1}\partial_1(u_1\partial_1\widetilde{u})\cdot\widetilde{\theta}dx - \delta\int\partial_2\Delta^{-1}\partial_2(u_2\partial_2\widetilde{u})\cdot\widetilde{\theta}dx \\
&= -\delta\int(u_1\partial_1\widetilde{u})\cdot\partial_2\Delta^{-1}\partial_1\widetilde{\theta}dx - \delta\int(u_2\partial_2\widetilde{u})\cdot\partial_2\Delta^{-1}\partial_2\widetilde{\theta}dx \qquad (4.56) \\
&= -\delta\int\partial_1\widetilde{u}_1\widetilde{u}\cdot\partial_2\Delta^{-1}\partial_1\widetilde{\theta}dx - \delta\int u_1\widetilde{u}\cdot\partial_1\partial_2\Delta^{-1}\partial_1\widetilde{\theta}dx \\
&\quad - \delta\int(u_2\partial_2\widetilde{u})\cdot\partial_2\Delta^{-1}\partial_2\widetilde{\theta}dx \\
&= N_{211} + N_{212} + N_{213}.
\end{aligned} \tag{4.57}$$

Due to $\nabla \cdot u = 0$, Lemma 2.4 and the boundedness of the Riesz transform,

$$\begin{aligned}
N_{211} &= -\delta \int \partial_1 \widetilde{u_1} \widetilde{u} \cdot \partial_2 \Delta^{-1} \partial_1 \widetilde{\theta} dx \\
&= \delta \int \partial_2 \widetilde{u_2} \widetilde{u} \cdot \partial_2 \Delta^{-1} \partial_1 \widetilde{\theta} dx \\
&\le c \|\widetilde{u}\|_{L^2} \|\partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \Delta^{-1} \partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \Delta^{-1} \partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \\
&\le c \|u\|_{H^2} \|\partial_2 \widetilde{u}\|_{H^1} \|\widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \\
&\le c \|u\|_{H^2} \|\partial_2 \widetilde{u}\|_{H^1} \|\partial_1 \widetilde{\theta}\|_{H^1} \\
&\le c \|u\|_{H^2} \Big(\|\partial_2 \widetilde{u}\|_{H^1}^2 + \|\partial_1 \widetilde{\theta}\|_{H^1}^2 \Big). \qquad (4.58)
\end{aligned}$$

According to Lemma 2.4, the boundedness of the Riesz transform, Lemma 2.5 and $\nabla \cdot u = 0$,

$$\begin{aligned}
N_{212} &= -\delta \int u_1 \widetilde{u} \cdot \partial_1 \partial_2 \Delta^{-1} \partial_1 \widetilde{\theta} dx \\
&\le c \|u_1\|_{L^2} \|\widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \Delta^{-1} \partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_1 \partial_2 \Delta^{-1} \partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \\
&\le c \|u\|_{H^2} \|\widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \\
&\le c \|u\|_{H^2} (\|\widetilde{u_1}\|_{L^2} + \|\widetilde{u_2}\|_{L^2})^{\frac{1}{2}} \|\partial_2 \widetilde{u}\|_{H^1}^{\frac{1}{2}} \|\partial_1 \widetilde{\theta}\|_{H^1} \\
&\le c \|u\|_{H^2} \Big(\underbrace{\|\widetilde{u_1}\|_{L^2}^2}_{\le \|\partial_1 \widetilde{u_1}\|_{L^2}^2 = \|\partial_2 \widetilde{u_2}\|_{L^2}^2} + \|\widetilde{u_2}\|_{L^2}^2 + \|\partial_2 \widetilde{u}\|_{H^1}^2 + \|\partial_1 \widetilde{\theta}\|_{H^1}^2 \Big) \\
&\le c \|u\|_{H^2} \Big(\|\widetilde{u_2}\|_{L^2}^2 + \|\partial_2 \widetilde{u}\|_{H^1}^2 + \|\partial_1 \widetilde{\theta}\|_{H^1}^2 \Big). \qquad (4.59)
\end{aligned}$$

Applying Lemma 2.4, the boundedness of the Riesz transform and then Lemma 2.5,

$$\begin{aligned}
N_{213} &= -\delta \int (u_2 \partial_2 \widetilde{u}) \cdot \partial_2 \Delta^{-1} \partial_2 \widetilde{\theta} dx \\
&\le c \|u_2\|_{L^2} \|\partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \Delta^{-1} \partial_2 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \Delta^{-1} \partial_2 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \\
&\le c \|u\|_{H^2} \|\partial_2 \widetilde{u}\|_{H^1} \|\widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \\
&\le c \|u\|_{H^2} \|\partial_2 \widetilde{u}\|_{H^1} \|\partial_1 \widetilde{\theta}\|_{H^1} \\
&\le c \|u\|_{H^2} \Big(\|\partial_2 \widetilde{u}\|_{H^1}^2 + \|\partial_1 \widetilde{\theta}\|_{H^1}^2 \Big). \qquad (4.60)
\end{aligned}$$

The bounds in (4.58), (4.59) and (4.60) lead to

$$N_{21} \le c \|u\|_{H^2} \Big(\|\widetilde{u_2}\|_{L^2}^2 + \|\partial_2 \widetilde{u}\|_{H^1}^2 + \|\partial_1 \widetilde{\theta}\|_{H^1}^2 \Big). \qquad (4.61)$$

Now we turn to the next term N_{22}. Using Hölder's inequality, the boundedness of the Riesz transform and Lemmas 2.1, 2.2 and 2.5

$$N_{22} := -\delta \int \partial_2 \Delta^{-1} \nabla \cdot (\widetilde{u_2} \partial_2 \overline{u}) \cdot \widetilde{\theta} dx$$

$$\begin{aligned}
&\leq c\delta\|\partial_2\Delta^{-1}\nabla\cdot(\widetilde{u_2}\partial_2\overline{u})\|_{L^2}\|\widetilde{\theta}\|_{L^2}\\
&\leq c\delta\|\widetilde{u_2}\partial_2\overline{u}\|_{L^2}\|\widetilde{\theta}\|_{L^2}\\
&\leq c\delta\|\partial_2\overline{u}\|_{L^\infty_{x_2}}\|\widetilde{u_2}\|_{L^2}\|\widetilde{\theta}\|_{L^2}\\
&\leq c\delta\|\partial_2\overline{u}\|_{H^1}\|\widetilde{u_2}\|_{L^2}\|\widetilde{\theta}\|_{L^2}\\
&\leq c\delta\|u\|_{H^2}\|\widetilde{u_2}\|_{L^2}\|\partial_1\widetilde{\theta}\|_{L^2}\\
&\leq c\delta\|u\|_{H^2}\Big(\|\widetilde{u_2}\|^2_{L^2}+\|\partial_1\widetilde{\theta}\|^2_{H^1}\Big). \qquad (4.62)
\end{aligned}$$

To deal with N_{23}, we integrate by parts, use Plancherel's theorem and then Lemma 2.5,

$$\begin{aligned}
N_{23}&:=g_0\delta\int\partial_2\partial_2\Delta^{-1}\widetilde{\theta}\cdot\widetilde{\theta}dx\\
&=g_0\delta\int\partial_2\Delta^{-\frac{1}{2}}\widetilde{\theta}\cdot\partial_2\Delta^{-\frac{1}{2}}\widetilde{\theta}dx\\
&=g_0\delta\|\partial_2\Lambda^{-1}\widetilde{\theta}\|^2_{L^2}\\
&=g_0\delta\sum_{\substack{k\in\mathbb{Z}\\k\neq0}}\int_{\mathbb{R}}\frac{\xi_2^2}{k^2+\xi_2^2}|\widehat{\widetilde{\theta}}(k,\xi_2)|^2d\xi_2\\
&\leq c\delta\sum_{\substack{k\in\mathbb{Z}\\k\neq0}}\int_{\mathbb{R}}\xi_2^2|\widehat{\widetilde{\theta}}(k,\xi_2)|^2d\xi_2=c\delta\|\partial_2\widetilde{\theta}\|^2_{L^2}\leq c\delta\|\partial_1\partial_2\widetilde{\theta}\|^2_{L^2}\leq c\delta\|\partial_1\widetilde{\theta}\|^2_{H^1}, \qquad (4.63)
\end{aligned}$$

where we denote $\Lambda=(-\Delta)^{\frac{1}{2}}$ and we have used the fact that the oscillation part $\widehat{\theta}(0,\xi_2)$ has the horizontal mode equal to 0, namely $\widehat{\widetilde{\theta}}(0,\xi_2)=0$.
Collecting (4.57), (4.62), (4.63) and (4.55), we find

$$N_2\leq c\delta\|(u,\theta)\|_{H^2}\Big(\|\partial_2\widetilde{u}\|^2_{H^1}+\|\widetilde{u_2}\|^2_{L^2}+\|\partial_1\widetilde{\theta}\|^2_{H^1}\Big)+c\delta\|\partial_1\widetilde{\theta}\|^2_{H^1}. \qquad (4.64)$$

To deal with N_3 we use $\nabla\cdot u=0$, integration by parts, Hölder's inequality and Lemma 2.5,

$$\begin{aligned}
N_3&:=-\delta\nu\int\partial_2^2\widetilde{u_2}\widetilde{\theta}dx=\delta\nu\int\partial_2\partial_1\widetilde{u_1}\widetilde{\theta}dx\\
&=-\delta\nu\int\widetilde{u_1}\partial_2\partial_1\widetilde{\theta}dx\\
&\leq\delta\nu\|\widetilde{u_1}\|_{L^2}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}\\
&\leq c\delta\Big(\underbrace{\|\widetilde{u_1}\|^2_{L^2}}_{\leq\|\partial_1\widetilde{u_1}\|^2_{L^2}=\|\partial_2\widetilde{u_2}\|^2_{L^2}}+\|\partial_2\partial_1\widetilde{\theta}\|^2_{L^2}\Big)\\
&\leq c\delta\Big(\|\partial_2\widetilde{u}\|^2_{H^1}+\|\partial_1\widetilde{\theta}\|^2_{H^1}\Big). \qquad (4.65)
\end{aligned}$$

To estimate N_4, we make use of Lemma 2.1 and integration by parts, to write it as

$$\begin{aligned} N_4 &:= \delta \int \widetilde{u \cdot \nabla \widetilde{u}_2} \widetilde{\theta} dx \\ &= \delta \int u \cdot \nabla \widetilde{u}_2 \widetilde{\theta} dx - \underbrace{\delta \int \overline{u \cdot \nabla \widetilde{u}_2} \widetilde{\theta} dx}_{=0} \\ &= \delta \int u \partial_1 \widetilde{u}_2 \widetilde{\theta} dx + \delta \int u \partial_2 \widetilde{u}_2 \widetilde{\theta} dx \\ &= -\delta \int \partial_1 \widetilde{u} \widetilde{u}_2 \widetilde{\theta} dx + \delta \int u \partial_2 \widetilde{u}_2 \widetilde{\theta} dx \\ &= N_{41} + N_{42}. \end{aligned} \tag{4.66}$$

By Lemmas 2.4 and 2.5

$$\begin{aligned} N_{41} &:= -\delta \int \partial_1 \widetilde{u} \widetilde{u}_2 \widetilde{\theta} dx \\ &\le c \|\partial_1 \widetilde{u}\|_{L^2} \|\widetilde{u}_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u}_2\|_{L^2}^{\frac{1}{2}} \|\widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \\ &\le c \|u\|_{H^2} \|\widetilde{u}_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{\theta}\|_{H^1} \\ &\le c \|u\|_{H^2} \Big(\|\partial_2 \widetilde{u}\|_{H^1}^2 + \|\partial_1 \widetilde{\theta}\|_{H^1}^2 + \|\widetilde{u}_2\|_{L^2}^2 \Big). \end{aligned} \tag{4.67}$$

Similarly,

$$\begin{aligned} N_{42} &= \delta \int u \partial_2 \widetilde{u}_2 \widetilde{\theta} dx \\ &\le c \|u\|_{L^2} \|\partial_2 \widetilde{u}_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 \widetilde{u}_2\|_{L^2}^{\frac{1}{2}} \|\widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \\ &\le c \|u\|_{H^2} \|\partial_2 \widetilde{u}\|_{H^1} \|\partial_1 \widetilde{\theta}\|_{H^1} \\ &\le c \|u\|_{H^2} \Big(\|\partial_2 \widetilde{u}\|_{H^1}^2 + \|\partial_1 \widetilde{\theta}\|_{H^1}^2 \Big). \end{aligned} \tag{4.68}$$

Inserting (4.67) and (4.68) in (4.66) we find

$$N_4 \le c\delta \|u\|_{H^2} \Big(\|\partial_2 \widetilde{u}\|_{H^1}^2 + \|\partial_1 \widetilde{\theta}\|_{H^1}^2 + \|\widetilde{u}_2\|_{L^2}^2 \Big). \tag{4.69}$$

Clearly, the term N_5 can be bounded via Lemma 2.5,

$$N_5 := -g_0 \delta \|\widetilde{\theta}\|_{L^2}^2 \le c\delta \|\partial_1 \widetilde{\theta}\|_{L^2}^2 \le c\delta \|\partial_1 \widetilde{\theta}\|_{H^1}^2. \tag{4.70}$$

Applying Hölder's inequality and Young's inequality,

$$\begin{aligned} N_6 &:= -\delta\eta \int \partial_1^2 \widetilde{\theta} \widetilde{u}_2 dx \\ &\le c\delta \|\partial_1^2 \widetilde{\theta}\|_{L^2} \|\widetilde{u}_2\|_{L^2} \\ &\le c\delta \|\partial_1 \widetilde{\theta}\|_{H^1} \|\widetilde{u}_2\|_{L^2} \\ &\le c\delta \|\partial_1 \widetilde{\theta}\|_{H^1}^2 - g_0 \frac{\delta}{4} \|\widetilde{u}_2\|_{L^2}^2. \end{aligned} \tag{4.71}$$

Using integration by parts and Lemma 2.4, we obtain

$$\begin{aligned} N_7 &:= \delta \int \widetilde{u_2}\widetilde{u_2}\partial_2\overline{\theta}dx = 2\delta \int \partial_2\widetilde{u_2}\widetilde{u_2}\overline{\theta}dx \\ &\leq c\delta\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\overline{\theta}\|_{L^2} \\ &\leq c\delta\|\partial_2\widetilde{u}\|_{L^2}^{\frac{3}{2}}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\theta\|_{H^2} \\ &\leq c\delta\|\theta\|_{H^2}(\|\partial_2\widetilde{u}\|_{H^1}^2 + \|\widetilde{u_2}\|_{L^2}^2). \end{aligned} \tag{4.72}$$

It remain to bound the last term N_8. Making use of Lemma 2.1, we divide it into three parts

$$\begin{aligned} N_8 &:= \delta \int \widetilde{u \cdot \nabla\theta}\widetilde{u_2}dx \\ &= \delta \int u \cdot \nabla\widetilde{\theta}\widetilde{u_2}dx - \underbrace{\delta \int \overline{u \cdot \nabla\widetilde{\theta}}\widetilde{u_2}dx}_{=0} \\ &= \delta \int \widetilde{u_1}\partial_1\widetilde{\theta}\widetilde{u_2}dx + \delta \int \overline{u_1}\partial_1\widetilde{\theta}\widetilde{u_2}dx + \delta \int u_2\partial_2\widetilde{\theta}\widetilde{u_2}dx \\ &:= N_{81} + N_{82} + N_{83}. \end{aligned} \tag{4.73}$$

Due to Lemmas 2.4, 2.5 and divergence-free condition of u, we have

$$\begin{aligned} N_{81} &:= \delta \int \widetilde{u_1}\partial_1\widetilde{\theta}\widetilde{u_2}dx \\ &\leq c\delta\|\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\theta}\|_{L^2} \\ &\leq c\delta\|\partial_2\widetilde{u}\|_{L^2}^{\frac{3}{2}}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\theta}\|_{L^2} \\ &\leq c\delta\|\theta\|_{H^2}(\|\partial_2\widetilde{u}\|_{H^1}^2 + \|\widetilde{u_2}\|_{L^2}^2). \end{aligned} \tag{4.74}$$

By Lemma 2.1, Hölder's inequality and Lemma 2.2,

$$\begin{aligned} N_{82} &:= \delta \int \overline{u_1}\partial_1\widetilde{\theta}\widetilde{u_2}dx \\ &\leq \delta\|\overline{u_1}\|_{L^\infty_{x_2}}\|\partial_1\widetilde{\theta}\widetilde{u_2}\|_{L^1} \\ &\leq c\delta\|\overline{u_1}\|_{L^\infty_{x_2}}\|\partial_1\widetilde{\theta}\|_{L^2}\|\widetilde{u_2}\|_{L^2} \\ &\leq c\delta\|u\|_{H^1}\|\partial_1\widetilde{\theta}\|_{L^2}\|\widetilde{u_2}\|_{L^2} \\ &\leq c\delta\|u\|_{H^2}\Big(\|\partial_1\widetilde{\theta}\|_{H^1}^2 + \|\widetilde{u_2}\|_{L^2}^2\Big). \end{aligned} \tag{4.75}$$

Due to $\overline{u_2} = 0$, integration by parts, Lemma 2.4 and Young's inequality

$$\begin{aligned} N_{83} &:= \delta \int u_2\partial_2\widetilde{\theta}\widetilde{u_2}dx = \delta \int \widetilde{u_2}\partial_2\widetilde{\theta}\widetilde{u_2}dx \\ &= 2\delta \int \partial_2\widetilde{u_2}\widetilde{\theta}\widetilde{u_2}dx \\ &\leq c\delta\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{\theta}\|_{L^2} \end{aligned}$$

$$\leq c\delta\|\partial_2\widetilde{u}\|_{H^1}^{\frac{3}{2}}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\theta\|_{H^2}$$
$$\leq c\delta\|\theta\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2\Big). \tag{4.76}$$

Inserting (4.74), (4.75) and (4.76) in (4.73) leads to

$$N_8\leq c\delta\|(u,\theta)\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big). \tag{4.77}$$

Considering (4.54) and collecting (4.64), (4.65), (4.66), (4.70), (4.71), (4.72) and (4.77), we obtain

$$\begin{aligned}-\delta\frac{d}{dt}(\widetilde{u_2},\widetilde{\theta})\leq{}& g_0\delta\|\widetilde{u_2}\|_{L^2}^2+c\delta\|(u,\theta)\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2\Big)\\ &-g_0\frac{\delta}{4}\|\widetilde{u_2}\|_{L^2}^2+c\delta\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big).\end{aligned} \tag{4.78}$$

It then follows from (4.11), (4.50) and (4.78) that

$$\begin{aligned}&\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2+\|\widetilde{\theta}\|_{H^1}^2-\delta(\widetilde{u_2},\widetilde{\theta})\Big)+2\nu\|\partial_2\widetilde{u}\|_{H^1}^2+2\eta\|\partial_1\widetilde{\theta}\|_{H^1}^2\\ &\quad\leq c\|(u,\theta)\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2\Big)\\ &\qquad+g_0\frac{3\delta}{4}\|\widetilde{u_2}\|_{L^2}^2+c\delta\|(u,\theta)\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2\Big)\\ &\qquad\quad+c\delta\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big).\end{aligned}$$

Using Theorem 1.1, if $\varepsilon>0$ is sufficiently small and $\|u_0\|_{L^2}+\|\theta_0\|_{L^2}\leq\varepsilon$, then $\|(u(t),\theta(t))\|_{H^2}\leq c\varepsilon$ and so,

$$\begin{aligned}&\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2+\|\widetilde{\theta}\|_{H^1}^2-\delta(\widetilde{u_2},\widetilde{\theta})\Big)+2\nu\|\partial_2\widetilde{u}\|_{H^1}^2+2\eta\|\partial_1\widetilde{\theta}\|_{H^1}^2\\ &\quad\leq c\epsilon\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2\Big)\\ &\qquad+g_0\frac{3\delta}{4}\|\widetilde{u_2}\|_{L^2}^2+c\delta\epsilon\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2\Big)\\ &\qquad+c\delta\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big).\end{aligned}$$

Choosing $\epsilon>0$ such that $c\epsilon\leq-g_0\min(\frac{1}{4},\frac{\delta}{4})$, we obtain

$$\begin{aligned}&\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2+\|\widetilde{\theta}\|_{H^1}^2-\delta(\widetilde{u_2},\widetilde{\theta})\Big)+2\nu\|\partial_2\widetilde{u}\|_{H^1}^2+2\eta\|\partial_1\widetilde{\theta}\|_{H^1}^2\\ &\quad\leq\frac{\delta}{4}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big)-g_0\frac{\delta}{4}\|\widetilde{u_2}\|_{L^2}^2\\ &\qquad+g_0\frac{3\delta}{4}\|\widetilde{u_2}\|_{L^2}^2-g_0\frac{\delta}{4}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2\Big)\\ &\qquad\quad+c\delta\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big)\\ &\quad\leq g_0\frac{\delta}{4}\|\widetilde{u_2}\|_{L^2}^2+c\delta\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big).\end{aligned}$$

Choosing $\delta > 0$ such that $c\delta \leq \min(\nu, \eta, \frac{c}{2})$, we get

$$\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2 - \delta(\widetilde{u_2}, \widetilde{\theta})\Big) + \nu\|\partial_2\widetilde{u}\|_{H^1}^2 + \eta\|\partial_1\widetilde{\theta}\|_{H^1}^2 - g_0\frac{\delta}{4}\|\widetilde{u_2}\|_{L^2}^2 \leq 0. \quad (4.79)$$

Due to the above choice of δ, we obtain

$$\frac{1}{2}\Big(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2\Big) - \delta(\widetilde{u_2}, \widetilde{\theta}) \geq 0.$$

or

$$\frac{1}{2}(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2) \leq \|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2 - \delta(\widetilde{u_2}, \widetilde{\theta}) \leq \frac{3}{2}(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2).$$

For any $0 \leq s \leq t$, integrating (4.79) in time leads to

$$\begin{aligned}&\frac{1}{2}(\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2) + \int_s^t (\nu\|\partial_2\widetilde{u}\|_{H^1}^2 + \eta\|\partial_1\widetilde{\theta}\|_{H^1}^2 - g_0\frac{\delta}{4}\|\widetilde{u_2}\|_{L^2}^2)\, d\tau \\ &\leq \frac{3}{2}(\|\widetilde{u}(s)\|_{H^1}^2 + \|\widetilde{\theta}(s)\|_{H^1}^2).\end{aligned}$$

Then, for any $0 \leq s \leq t$, we have

$$\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2 \leq 3(\|\widetilde{u}(s)\|_{H^1}^2 + \|\widetilde{\theta}(s)\|_{H^1}^2) \quad (4.80)$$

and

$$\int_0^\infty (\nu\|\partial_2\widetilde{u}\|_{H^1}^2 + \eta\|\partial_1\widetilde{\theta}\|_{H^1}^2 - g_0\frac{\delta}{4}\|\widetilde{u_2}\|_{L^2}^2)\, d\tau \leq C < \infty.$$

Combining with the time integral bounds from Theorem 1.1,

$$\int_0^\infty \|\partial_2 u\|_{H^2}^2\, dt < \infty, \quad \int_0^\infty \|\partial_1 u_2\|_{L^2}^2\, dt < \infty \quad \text{and} \quad \int_0^\infty \|\partial_1\theta\|_{H^2}^2\, dt < \infty,$$

we get

$$\int_0^\infty (\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2)\, dt < \infty. \quad (4.81)$$

Finally, applying Lemma 2.6 to (4.80) and (4.81) leads to

$$\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2 \leq c(1+t)^{-1},$$

and the asymptotic behavior, as $t \to \infty$,

$$t\,(\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2) \to 0.$$

This completes the proof of Theorem 1.2. □

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