STABILITY AND LARGE-TIME BEHAVIOR FOR THE 2D BOUSSINEQ SYSTEM WITH VERTICAL DISSIPATION AND HORIZONTAL THERMAL DIFFUSION

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STABILITY AND LARGE-TIME BEHAVIOR FOR THE 2D BOUSSINEQ SYSTEM WITH VERTICAL DISSIPATION AND HORIZONTAL THERMAL DIFFUSION

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Abstract. The hydrostatic balance or hydrostatic equilibrium is one of the most prominent topics in fluid dynamics, atmospherics and astrophysics. In fact, our atmosphere is mainly in hydrostatic equilibrium, between the upward-directed pressure gradient force and the downward-directed force of gravity. Understanding the stability of perturbations near the hydrostatic balance of the Boussinesq system may help gain insight into some weather phenomena. This paper is concerned with the 2D anisotropic Boussinesq equations involving only vertical dissipation and horizontal thermal diffusion. When the spatial domain is $\Omega = \mathbb{T} \times \mathbb{R}$ with $\mathbb{T} = [0, 1]$ being a 1D periodic box, this work establishes the stability and specifies the precise large-time behavior of perturbations near the hydrostatic equilibrium. The approach here is to distinguish the vertical averages of the velocity and temperature from their corresponding oscillation parts.

1. Introduction

This paper focuses on the following 2D anisotropic Boussinesq system

$$\begin{cases} \partial_t U + U \cdot \nabla U = -\nabla P + \nu\, \partial_{22} U + g_0 \Theta \mathbf{e}_2, \quad x \in \Omega,\ t > 0, \\ \partial_t \Theta + U \cdot \nabla \Theta = \eta\, \partial_{11} \Theta, \\ \nabla \cdot U = 0, \end{cases} \tag{1.1}$$

where U denotes the fluid velocity, P the pressure, Θ the temperature, $\nu > 0$ and $\eta > 0$ are parameters representing the kinematic viscosity and the thermal diffusivity, respectively. Here $\mathbf{e}_2 = (0, 1)$ is the unit vector in the vertical direction, g_0 is a non zero constant and the spatial domain Ω is taken to be

$$\Omega = \mathbb{T} \times \mathbb{R},$$

with $\mathbb{T} = [0, 1]$ being a 1D periodic box and $\mathbb{R}$ being the whole line. This partially dissipated system models anisotropic buoyancy-driven fluids in the circumstance when the horizontal dissipation and the vertical thermal diffusion are negligible [24].

The Boussinesq systems are the most widely used models for atmospheric and oceanographic flows (see, e.g., [4], [12], [22]). Over the past few years, many efforts

2010 *Mathematics Subject Classification.* 35Q35, 35Q86, 76D03, 76D50.

Key words and phrases. Boussinesq equations; Hydrostatic balance; Partial dissipation; Stability; Decay rates.

have been devoted to understanding two fundamental issues concerning the Boussinesq systems. The first is the global existence and regularity problem and the second is the stability problem on perturbations near several physically relevant steady states (see, e.g., [1], [2], [5], [9], [10], [13], [14], [15], [16], [17], [18], [19], [20], [21], [28]).

This work intends to assess the stability and the precise large-time behavior of perturbations near the hydrostatic equilibrium (U_{he}, Θ_{he}) with

$$U_{he} = 0, \quad \Theta_{he} = g_0 x_2.$$

For the static velocity U_{he}, the momentum equation is satisfied when the pressure gradient is balanced by the buoyancy force, namely

$$-\nabla P_{he} + g_0 \Theta_{he}\, \mathbf{e}_2 = 0 \quad \text{or} \quad P_{he} = \frac{1}{2} g_0^2 x_2^2.$$

To understand the stability problem, we write the equations for the perturbation (u, p, θ), where

$$u = U - U_{he}, \quad p = P - P_{he} \quad \text{and} \quad \theta = \Theta - \Theta_{he}.$$

It is easy to check that (u, p, θ) satisfies the following anisotropic Boussinesq equations with vertical dissipation and horizontal thermal diffusion

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu\, \partial_{22} u + g_0 \theta \mathbf{e}_2, \quad x \in \Omega,\ t > 0, \\ \partial_t \theta + u \cdot \nabla \theta + g_0 u_2 = \eta\, \partial_{11} \theta, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x). \end{cases} \tag{1.2}$$

The difference between the original system (1.1) and the system governing the perturbations (1.2) is that the temperature equation in (1.2) contains $g_0 u_2$. Without this term, the L^2-norm of the velocity u to (1.1) can grow in time due to the buoyancy forcing term $g_0 \theta \mathbf{e}_2$. As specified in [3], solutions of the 3D Boussinesq equations with even full dissipation and thermal diffusion can actually grow in time. This term in (1.2) helps balance out $g_0 \theta \mathbf{e}_2$ in the energy estimates. Therefore, the buoyancy forcing no longer plays a destabilizing role in (1.2). When the dissipation is degenerate and is only one-directional as in (1.1), it is not clear how the solution would behave.

When the spacial domain is the whole space $\mathbb{R}^2$, the lack of the horizontal dissipation makes it hard to control the growth of the vorticity $\omega = \nabla \times u$, which satisfies

$$\partial_t \omega + u \cdot \nabla \omega = \nu\, \partial_{22} \omega + g_0 \partial_1 \theta, \quad x \in \mathbb{R}^2,\ t > 0. \tag{1.3}$$

More precisely, we can obtain a uniform bound on the L^2-norm of the vorticity ω itself, but it does not appear possible to control the L^2-norm of the gradient of the vorticity, $\nabla \omega$. In particular, if the temperature θ were identically zero, (1.3) reduces to the 2D Navier-Stokes equation with degenerate dissipation,

$$\partial_t \omega + u \cdot \nabla \omega = \nu\, \partial_{22} \omega, \quad x \in \mathbb{R}^2,\ t > 0. \tag{1.4}$$

(1.4) always has a unique global solution ω for any initial data $\omega_0 \in H^1(\mathbb{R}^2)$, however the issue of whether $\|\nabla \omega(t)\|_{L^2}$ for the solution ω of (1.4) grows as a function of t

remains an open problem. Furthermore, when there is no dissipation at all, namely when $\nu = 0$, (1.4) becomes the 2D Euler vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega = 0, \quad x \in \mathbb{R}^2, \ t > 0. \tag{1.5}$$

As pointed out in many works (see, e.g., [8], [13], [31]), $\nabla\omega(t)$ of (1.5) can grow even double exponentially in time. In particular, the velocity of the 2D Euler equations in the Sobolev space H^2 is not stable. In contrast, solutions to the 2D Navier-Stokes equations with full dissipation

$$\partial_t \omega + u \cdot \nabla \omega = \nu \, \Delta \omega, \quad x \in \mathbb{R}^2, \ t > 0$$

decays algebraically in time, as shown by Schonbek (see, e.g., [25], [26]). The lack of the horizontal dissipation in (1.4) prevents us from mimicking the approach designed for the fully dissipative Navier-Stokes equations. In fact, when we resort to the energy method to bound $\|\nabla\omega(t)\|_{L^2}$, namely

$$\frac{1}{2}\frac{d}{dt}\|\nabla\omega(t)\|_{L^2}^2 + \nu\|\partial_2\nabla\omega(t)\|_{L^2}^2 = -\int \nabla\omega \cdot \nabla u \cdot \nabla\omega \, dx, \tag{1.6}$$

the one-directional dissipation fails to control the nonlinearity. The difficulty is how to obtain a suitable upper bound on the term on the right-hand side of (1.6). To make full use of the anisotropic dissipation, we naturally divide this term further into four component terms

$$\begin{aligned}\int \nabla\omega \cdot \nabla u \cdot \nabla\omega \, dx = &\int \partial_1 u_1 \, (\partial_1\omega)^2 \, dx + \int \partial_1 u_2 \, \partial_1\omega \, \partial_2\omega \, dx \\ &+ \int \partial_2 u_1 \, \partial_1\omega \, \partial_2\omega \, dx + \int \partial_2 u_2 \, (\partial_2\omega)^2 \, dx.\end{aligned} \tag{1.7}$$

Due to the lack of the horizontal dissipation, the first two terms in (1.7) do not admit a time-integrable upper bound. This explains the difficulty of seeking a solution ω of (1.4) in H^2 as well as lowering the exponential upper bound.

When we deal with the stability problem on (1.2), we encounter exactly the same term in (1.7). Fortunately, the smoothing and stabilization effect of the temperature through the coupling and interaction make the stability problem on (1.2) possible. To reveal these effects, we first eliminate the pressure term in (1.2). Applying the Helmholtz-Leray projection $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ to the velocity equation yields

$$\partial_t u = \nu\partial_{22} u + \mathbb{P}(g_0\theta\mathbf{e}_2) - \mathbb{P}(u \cdot \nabla u). \tag{1.8}$$

By the definition of $\mathbb{P}$,

$$\mathbb{P}(g_0\theta\mathbf{e}_2) = g_0\theta\mathbf{e}_2 - \nabla\Delta^{-1}\nabla \cdot (g_0\theta\mathbf{e}_2) = g_0 \begin{bmatrix} -\partial_1\partial_2\Delta^{-1}\theta \\ \theta - \partial_2^2\Delta^{-1}\theta \end{bmatrix}. \tag{1.9}$$

Inserting (1.9) in (1.8) and writing (1.8) in terms of its component equations, we get

$$\begin{cases} \partial_t u_1 = \nu \, \partial_{22} u_1 - g_0\partial_1\partial_2\Delta^{-1}\theta + N_1, \\ \partial_t u_2 = \nu \, \partial_{22} u_2 + g_0\partial_1\partial_1\Delta^{-1}\theta + N_2, \end{cases} \tag{1.10}$$

where N_1 and N_2 are the nonlinear terms,

$$N_1 = -(u \cdot \nabla u_1 - \partial_1 \Delta^{-1} \nabla \cdot (u \cdot \nabla u)), \quad N_2 = -(u \cdot \nabla u_2 - \partial_2 \Delta^{-1} \nabla \cdot (u \cdot \nabla u)).$$

By differentiating the first equation of (1.10) in t, we obtain

$$\partial_{tt} u_1 = \nu \partial_{22} \partial_t u_1 - g_0 \partial_1 \partial_2 \Delta^{-1} \partial_t \theta + \partial_t N_1.$$

Using the equation of θ, we replace $\partial_t \theta$ in the above equation by $\eta\, \partial_{11}\theta - g_0 u_2 - u \cdot \nabla \theta$ to write

$$\partial_{tt} u_1 = \nu \partial_{22} \partial_t u_1 + g_0^2 \partial_1 \partial_2 \Delta^{-1} u_2 - g_0 \eta\, \partial_{11} \partial_1 \partial_2 \Delta^{-1} \theta + g_0 \partial_1 \partial_2 \Delta^{-1} (u \cdot \nabla \theta) + \partial_t N_1.$$

By further making the substitution of $g_0 \partial_1 \partial_2 \Delta^{-1} \theta$ by the first equation of (1.10), namely

$$-g_0 \partial_1 \partial_2 \Delta^{-1} \theta = \partial_t u_1 - \nu\, \partial_{22} u_1 - N_1,$$

we find

$$\begin{aligned} \partial_{tt} u_1 = {} & \nu \partial_{22} \partial_t u_1 + g_0^2 \partial_1 \partial_2 \Delta^{-1} u_2 + \eta\, \partial_{11} (\partial_t u_1 - \nu\, \partial_{22} u_1 - N_1) \\ & + g_0\, \partial_1 \partial_2 \Delta^{-1} (u \cdot \nabla \theta) + \partial_t N_1, \end{aligned}$$

which yields, due to the divergence-free condition $\partial_2 u_2 = -\partial_1 u_1$,

$$\partial_{tt} u_1 - (\eta \partial_{11} + \nu \partial_{22}) \partial_t u_1 + \nu \eta \partial_{11} \partial_{22} u_1 + g_0^2 \partial_{11} \Delta^{-1} u_1 = N_3, \tag{1.11}$$

where N_3 is the nonlinear term,

$$N_3 = (\partial_t - \eta \partial_{11}) N_1 + g_0 \partial_1 \partial_2 \Delta^{-1} (u \cdot \nabla \theta).$$

Through a similar process, u_2 and θ can be shown to satisfy

$$\begin{aligned} & \partial_{tt} u_2 - (\eta \partial_{11} + \nu \partial_{22}) \partial_t u_2 + \nu \eta \partial_{11} \partial_{22} u_2 + g_0^2 \partial_{11} \Delta^{-1} u_2 = N_4, \\ & \partial_{tt} \theta - (\eta \partial_{11} + \nu \partial_{22}) \partial_t \theta + \nu \eta \partial_{11} \partial_{22} \theta + g_0^2 \partial_{11} \Delta^{-1} \theta = N_5 \end{aligned} \tag{1.12}$$

with

$$\begin{aligned} N_4 &= (\partial_t - \eta \partial_{11}) N_2 - g_0 \partial_1 \partial_1 \Delta^{-1} (u \cdot \nabla \theta), \\ N_5 &= -(\partial_t - \nu \partial_{22}) (u \cdot \nabla \theta) - g_0 N_2. \end{aligned}$$

Then, combining (1.11) and (1.12) and rewriting them into the velocity vector form, we have converted (1.2) into the following new system

$$\begin{cases} \partial_{tt} u - (\eta \partial_{11} + \nu \partial_{22}) \partial_t u + \nu \eta \partial_{11} \partial_{22} u + g_0^2 \partial_{11} \Delta^{-1} u = N_6, \\ \partial_{tt} \theta - (\eta \partial_{11} + \nu \partial_{22}) \partial_t \theta + \nu \eta \partial_{11} \partial_{22} \theta + g_0^2 \partial_{11} \Delta^{-1} \theta = N_5, \end{cases} \tag{1.13}$$

where

$$N_6 = (N_3, N_4) = -(\partial_t - \eta \partial_{11}) \mathbb{P} (u \cdot \nabla u) + g_0 \nabla^\perp \partial_1 \Delta^{-1} (u \cdot \nabla \theta)$$

with $\nabla^\perp = (\partial_2, -\partial_1)$. Taking the curl of the velocity equation, we can also convert (3.43) into a system of ω and θ,

$$\begin{cases} \partial_{tt} \omega - (\eta \partial_{11} + \nu \partial_{22}) \partial_t \omega + \nu \eta \partial_{11} \partial_{22} \omega + g_0^2 \partial_{11} \Delta^{-1} \omega = N_7, \\ \partial_{tt} \theta - (\eta \partial_{11} + \nu \partial_{22}) \partial_t \theta + \nu \eta \partial_{11} \partial_{22} \theta + g_0^2 \partial_{11} \Delta^{-1} \theta = N_5, \end{cases}$$

where

$$N_7 = -(\partial_t - \eta \partial_{11}) (u \cdot \nabla \omega) - g_0 \partial_1 (u \cdot \nabla \theta).$$

Amazingly we have found that all physical quantities u, θ and ω satisfy the same damped degenerate wave equation only with different nonlinear terms. In comparison with the original system of (u, θ) in (1.2), the wave equations (1.13) obeyed by (u, θ) helps unearth all the smoothing and stabilization hidden in the original system. The velocity in (1.2) involves only vertical dissipation, but the wave structure actually implies that the temperature can stabilize the fluids by creating the horizontal regularization via the coupling and interaction. By taking advantage of these effects, the stability problem on (1.2) was recently established by Ben Said and al in [1] when the spacial domain is the whole plane $\mathbb{R}^2$. However, the large time behaviour of the solution in $\mathbb{R}^2$ remains a mystery. When the spatial domain is $\Omega = \mathbb{T} \times \mathbb{R}$, this paper also proves the stability of (1.2). More importantly, we establish the precise large-time behavior of the solutions. The main idea here is to separate a physical quantity into its horizontal average and the corresponding oscillation. More precisely, for a function $f = f(x_1, x_2)$ defined on $\mathbb{T} \times \mathbb{R}$ that is integrable in x_1 over the 1D periodic box $\mathbb{T} = [0, 1]$, we define its horizontal average $\overline{f}$ by

$$\overline{f}(x_2) = \int_{\mathbb{T}} f(x_1, x_2) dx_1, \tag{1.14}$$

and we write,

$$f = \overline{f} + \widetilde{f}. \tag{1.15}$$

Clearly, the horizontal average $\overline{f}$ represents the zeroth Fourier mode of f while $\widetilde{f}$ contains all non-zero Fourier modes.

The decomposition (1.15) is very special. First of all, this decomposition is orthogonal in the Sobolev space $H^k(\Omega)$ for any non-negative integer. As a special consequence, the H^k-norms of $\overline{f}$ and $\widetilde{f}$ are bounded by the H^k-norm of f. Furthermore, this decomposition commutes with derivatives, and $\overline{f}$ and $\widetilde{f}$ of a divergence-free vector field f are also divergence-free. A crucial property to be frequently used in our estimates is that $\widetilde{f}$ admits strong versions of the Poincaré type inequality

$$\|\widetilde{f}\|_{L^2(\Omega)} \leq C \|\partial_1 \widetilde{f}\|_{L^2(\Omega)}, \quad \|\widetilde{f}\|_{L^\infty(\Omega)} \leq C \|\partial_1 \widetilde{f}\|_{H^1(\Omega)}.$$

By applying this decomposition to the velocity field u and the temperature θ, namely writing

$$u = \overline{u} + \widetilde{u}, \quad \theta = \overline{\theta} + \widetilde{\theta}$$

and taking advantage of the aforementioned properties we are able to deal with the nonlinear terms in (1.7) suitably, even when there is only vertical dissipation. More precisely, the following theorem holds.

The first result establishes the H^2-stability while the second result provides the decay rates of the oscillation portion $(\widetilde{u}, \widetilde{\theta})$.

Theorem 1.1. *Let $\mathbb{T} = [0, 1]$ be a 1D periodic box and let $\Omega = \mathbb{T} \times \mathbb{R}$. Assume $u_0, \theta_0 \in H^2(\Omega)$ and $\nabla \cdot u_0 = 0$. Then there exists $\varepsilon = \varepsilon(\nu, \eta) > 0$ such that, if*

$$\|u_0\|_{H^2} + \|\theta_0\|_{H^2} \leq \varepsilon,$$

then (1.2) has a unique global solution (u, θ) that remains uniformly bounded for all time, for any $t \geq 0$,

$$\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 + 2\nu \int_0^t \|\partial_2 u(\tau)\|_{H^2}^2 d\tau + 2\eta \int_0^t \|\partial_1 \theta(\tau)\|_{H^2}^2 d\tau + C(\nu, \eta) \int_0^t \|g_0 \partial_1 u_2\|_{L^2}^2 d\tau \leq C\varepsilon^2$$

where $C(\nu, \eta)$ and $C > 0$ are constants.

Theorem 1.1 rigorously assesses that any small initial perturbation (in the H^2-sense) leads to a unique global (in time) solution of (1.2) that remains small in H^2 for all time. Furthermore, it implies that $\|\partial_1 u_2(\tau)\|_{L^2}^2$ is also time integrable.

Our second Theorem states that the oscillation part $(\widetilde{u}, \widetilde{\theta})$ decays to zero algebraically in time in the H^1-norm. This result reflects the stratification phenomenon of buoyancy driven fluids. It also confirms the observation of the numerical simulations in [9]), the temperature becomes horizontally homogeneous and stratify in the vertical direction as time evolves.

Theorem 1.2. *Let $u_0, \theta_0 \in H^2(\Omega)$ with $\nabla \cdot u_0 = 0$. Assume that (u_0, θ_0) satisfies*

$$\|u_0\|_{H^2} + \|\theta_0\|_{H^2} \leq \varepsilon,$$

for sufficiently small $\varepsilon > 0$. Let (u, θ) be the corresponding solution of (1.2) with g_0 negative constant. Then the oscillation part $(\widetilde{u}, \widetilde{\theta})$ satisfies the following algebraic decay in time,

$$\|\widetilde{u}\|_{H^1} + \|\widetilde{\theta}\|_{H^1} \leq c(1+t)^{-\frac{1}{2}},$$

for some constant $c > 0$ and for all $t \geq 0$. In addition, $(\widetilde{u}, \widetilde{\theta})$ has the asymptotic behavior, as $t \to \infty$,

$$t\,(\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2) \to 0.$$

As a special consequence of this decay result, the solution (u, θ) of (1.2) approaches the horizontal average $(\overline{u}, \overline{\theta})$ asymptotically, and the Boussinesq system (1.2) evolves to the following 1D system eventually

$$\begin{cases} \partial_t \overline{u} + \overline{u \cdot \nabla \widetilde{u}} + \begin{pmatrix} 0 \\ \partial_2 \overline{p} \end{pmatrix} = g_0 \begin{pmatrix} 0 \\ \overline{\theta} \end{pmatrix} + \nu \partial_2^2 \overline{u}, \\ \partial_t \overline{\theta} + \overline{u \cdot \nabla \widetilde{\theta}} = 0. \end{cases}$$

We briefly outline the proofs for Theorem 1.1 and Theorem 1.2. Since the local (in time) well-posedness on (1.2) in the Sobolev setting $H^2(\Omega)$ can be shown via standard approaches such as Friedrichs' Fourier cutoff (see, e.g., [23]), the proof of theorem 1.1 is reduced to establishing the global (in time) *a priori* bound on the solution in $H^2(\Omega)$. The framework for proving the global H^2-bound is the bootstrapping argument. An abstract bootstrapping argument can be found in

Tao's book (see [29], p 20). To set it up, we define the following energy functional for the H^2-solution,

$$E(t) = \max_{0\le\tau\le t}(\|u(\tau)\|_{H^2}^2 + \|\theta(\tau)\|_{H^2}^2) + 2\nu\int_0^t \|\partial_2 u\|_{H^2}^2 d\tau$$
$$+2\eta\int_0^t \|\partial_1\theta\|_{H^2}^2 d\tau + \delta\int_0^t \|g_0\partial_1 u_2\|_{L^2}^2\, d\tau, \tag{1.16}$$

where $\delta > 0$ is a suitably selected small parameter. Our main efforts are devoted to proving that, for all $t > 0$,

$$E(t) \le C_1\, E(0) + C_2\, E(t)^{\frac{3}{2}}, \tag{1.17}$$

where C_1 and C_2 are two uniform constants.

To do so, we need to fully exploit the extra regularization due to the wave structure in (1.13). We should mention here that, the control on the time integral of the horizontal derivative of the velocity field, namely

$$\int_0^t \|g_0\partial_1 u_2(\tau)\|_{L^2}^2\, d\tau \tag{1.18}$$

plays an improtant role in the proof. Clearly the uniform boundedness of (1.18) is not a consequence of the vertical dissipation in the velocity equation but due to the interaction with the temperature equation. More precisely, as in (3.61), we represent $g_0\partial_1 u_2$ in terms of the rest in the equation of θ,

$$g_0\partial_1 u_2 = -\partial_t\partial_1\theta - \partial_1(u\cdot\nabla\theta) + \eta\partial_{111}\theta,$$

then

$$\|g_0\partial_1 u_2\|_{L^2}^2 = -g_0\int \partial_t\partial_1\theta\,\partial_1 u_2\, dx - g_0\int \partial_1 u_2\,\partial_1(u\cdot\nabla\theta)\, dx + \eta g_0\int \partial_1 u_2\,\partial_{111}\theta\, dx.$$

The time integrability of $\|g_0\partial_1 u_2\|_{L^2}^2$ is then converted to the time integrability of other terms. This phenomenon of extra regularization and time integrability due to the coupling also shows up in some other models of partial differential equations such as the Oldroyd-B system (see [7], [11]).

Once (1.17) is established, the bootstrapping argument then asserts that, if

$$E(0) = \|(u_0,\theta_0)\|_{H^2}^2 \le \varepsilon^2$$

for some sufficiently small $\varepsilon > 0$, then $E(t)$ remains uniformly small for all time, namely

$$E(t) \le C\,\varepsilon^2 \tag{1.19}$$

for a constant $C > 0$ and for all $t \ge 0$. In particular, (1.19) yields the desired global H^2-bound on the solution (u,θ). We leave details on the application of the bootstrapping argument in the proof of Theorem 1.1 in Section 3.

To prove the algebraic decay rates on the H^1-norm of the oscillation part stated in Theorem 1.2, we first take the difference of (1.2) and its horizontal average, to

write the system governing the oscillation $(\widetilde{u}, \widetilde{\theta})$

$$\begin{cases} \partial_t \widetilde{u} + \widetilde{u \cdot \nabla u} + \widetilde{u}_2 \partial_2 \overline{u} - \nu \partial_2^2 \widetilde{u} + \nabla \widetilde{p} = g_0 \widetilde{\theta} e_2, \\ \partial_t \widetilde{\theta} + \widetilde{u \cdot \nabla \theta} + \widetilde{u}_2 \partial_2 \overline{\theta} - \eta \partial_1^2 \widetilde{\theta} + g_0 \widetilde{u}_2 = 0. \end{cases} \tag{1.20}$$

The estimate of the H^1-norm of $(\widetilde{u}, \widetilde{\theta})$ is naturally separated into controlling the L^2-norms $\|(\widetilde{u}, \widetilde{\theta})\|_{L^2}$ and $\|(\nabla \widetilde{u}, \nabla \widetilde{\theta})\|_{L^2}$. One major difficulty is that the equation of $\widetilde{u}$ has only vertical dissipation, but the aforementioned Poincaré inequality can only bound a function in terms of its horizontal derivatives. As a consequence of this disparity, some of the nonlinear parts related to $\widetilde{u}$ can not be bounded suitably and require the upper bounds involving $\|\widetilde{u}_2\|_{L^2}$. To deal with these terms, we seek extra smoothing and stabilizing effect on $\widetilde{u}_2$ by exploiting the coupling in (1.20). More precisely, we include the following extra term along with the H^1-norm to form a Lyapunov functional,

$$-\delta(\widetilde{u}_2, \widetilde{\theta}),$$

where $\delta > 0$ is a small constant and $(\widetilde{u}_2, \widetilde{\theta})$ denotes the L^2-inner product. The time derivative of this inner product generates $\delta \|\widetilde{u}_2\|_{L^2}^2$, which help balance $\|\widetilde{u}_2\|_{L^2}^2$ from the nonlinearity. By invoking various anisotropic inequalities stated in section 2, we are able to show the following energy inequality

$$\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2 - \delta(\widetilde{u}_2, \widetilde{\theta}) \Big) + \nu \|\partial_2 \widetilde{u}\|_{H^1}^2 + \eta \|\partial_1 \widetilde{\theta}\|_{H^1}^2 + \frac{\delta}{4} \|\widetilde{u}_2\|_{L^2}^2 \leq 0,$$

which leads to the desired exponential decay in theorem 1.2. More details are presented in Section 4.

The rest of the paper provides the details outlined above. Section 2 serves as a preparation. It presents several anisotropic inequalities and some crucial properties on the orthogonal decomposition such as the Poincaré type inequality for the oscillation part $\widetilde{f}$. Section 3 proves Theorem 1.1 while Section 4 is devoted to verifying Theorem 1.2.

2. Preliminaries

This section serves as preparation for the proof of theorems 1.1 and 1.2. The first few following lemmas contain several frequently used facts on the aforementioned decomposition (1.15). While the last Lemma provides a precise decay rate for a nonnegative integrable function, which is also monotonic in a generalized sense.

First, we present basic properties of $\overline{f}$ and $\widetilde{f}$.

Lemma 2.1. *Let $\Omega = \mathbb{T} \times \mathbb{R}$. Assume that f defined on Ω is sufficiently regular, say $f \in H^2(\Omega)$. Let $\overline{f}$ and $\widetilde{f}$ be defined as in (1.14) and (1.15). Then*
(a) The average operator $\overline{f}$ and the oscillation operator $\widetilde{f}$ commute with partial derivatives,

$$\overline{\partial_1 f} = \partial_1 \overline{f} = 0, \quad \overline{\partial_2 f} = \partial_2 \overline{f}, \quad \widetilde{\partial_1 f} = \partial_1 \widetilde{f}, \quad \widetilde{\partial_2 f} = \partial_2 \widetilde{f}, \quad \overline{\widetilde{f}} = 0.$$

(b) If f is a divergence-free vector field, namely $\nabla \cdot f = 0$, then $\overline{f}$ and $\widetilde{f}$ are also divergence-free,

$$\nabla \cdot \overline{f} = 0 \quad \text{and} \quad \nabla \cdot \widetilde{f} = 0.$$

(c) $\overline{f}$ and $\widetilde{f}$ are orthogonal in $\dot{H}^k$ for any integer $k \geq 0$, namely

$$(\overline{f}, \widetilde{f})_{\dot{H}^k(\Omega)} := \int_\Omega \overline{D^k f} \cdot \widetilde{D^k f} dx = 0, \quad \|f\|^2_{\dot{H}^k(\Omega)} = \|\overline{f}\|^2_{\dot{H}^k(\Omega)} + \|\widetilde{f}\|^2_{\dot{H}^k(\Omega)}.$$

In particular,

$$\|\overline{f}\|_{\dot{H}^k(\Omega)} \leq \|f\|_{\dot{H}^k(\Omega)} \quad \text{and} \quad \|\widetilde{f}\|_{\dot{H}^k(\Omega)} \leq \|f\|_{\dot{H}^k(\Omega)}.$$

The orthogonality is actually more general and holds for any integrable functions,

$$\int_\Omega \overline{f} \cdot \widetilde{g}\, dx = 0.$$

All the items in Lemma 2.1 can be directly verified by the definition of $\overline{f}$ and $\widetilde{f}$.

The second lemma compares the 1D Sobolev inequalities on the whole line $\mathbb{R}$ and on bounded domains.

Lemma 2.2. *For any 1D function $f \in H^1(\mathbb{R})$,*

$$\|f\|_{L^\infty(\mathbb{R})} \leq \sqrt{2}\, \|f\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \|f'\|^{\frac{1}{2}}_{L^2(\mathbb{R})}.$$

For any bounded domain such as $\mathbb{T} = [0, 1]$ and $f \in H^1(\mathbb{T})$,

$$\|f\|_{L^\infty(\mathbb{T})} \leq \sqrt{2}\, \|f\|^{\frac{1}{2}}_{L^2(\mathbb{T})} \|f'\|^{\frac{1}{2}}_{L^2(\mathbb{T})} + \|f\|_{L^2(\mathbb{T})},$$

in particular, if the function f has mean zero such as the oscillation part $\tilde{f}$,

$$\|f\|_{L^\infty(\mathbb{T})} \leq C\, \|f\|^{\frac{1}{2}}_{L^2(\mathbb{T})} \|f'\|^{\frac{1}{2}}_{L^2(\mathbb{T})}.$$

The following lemma presents anisotropic upper bounds for triple products as well as for the L^∞-norm on the domain Ω. Anisotropic Sobolev inequalities are powerful tools for dealing with anisotropic models. The whole space version of these type of inequalities has previously been used in [6] in the 2D cases and in [30] in the 3D case.

Lemma 2.3. *Let $\Omega = \mathbb{T} \times \mathbb{R}$. For any $f, g, h \in L^2(\Omega)$ with $\partial_1 f \in L^2(\Omega)$ and $\partial_2 g \in L^2(\Omega)$, then*

$$\Big| \int_\Omega fgh\, dx \Big| \leq C \|f\|^{\frac{1}{2}}_{L^2} (\|f\|_{L^2} + \|\partial_1 f\|_{L^2})^{\frac{1}{2}} \|g\|^{\frac{1}{2}}_{L^2} \|\partial_2 g\|^{\frac{1}{2}}_{L^2} \|h\|_{L^2}. \tag{2.1}$$

For any $f \in H^2(\Omega)$, we have

$$\begin{aligned} \|f\|_{L^\infty(\Omega)} \leq & C \|f\|^{\frac{1}{4}}_{L^2} (\|f\|_{L^2} + \|\partial_1 f\|_{L^2})^{\frac{1}{4}} \|\partial_2 f\|^{\frac{1}{4}}_{L^2} \\ & \times (\|\partial_2 f\|_{L^2} + \|\partial_1 \partial_2 f\|_{L^2})^{\frac{1}{4}}. \end{aligned}$$

When f in Lemma 2.3 is replaced by its oscillation part $\widetilde{f}$, then the lower-order part in (2.1) can be dropped.

Lemma 2.4. *Let $\Omega = \mathbb{T} \times \mathbb{R}$. For any $f, g, h \in L^2(\Omega)$ with $\partial_1 f \in L^2(\Omega)$ and $\partial_2 g \in L^2(\Omega)$, then*

$$\Big| \int_\Omega \widetilde{f} g h \, dx \Big| \leq C \|\widetilde{f}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{f}\|_{L^2}^{\frac{1}{2}} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_2 g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}. \tag{2.2}$$

For any $f \in H^2(\Omega)$, we have

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \leq C \|\widetilde{f}\|_{L^2}^{\frac{1}{4}} \|\partial_1 \widetilde{f}\|_{L^2}^{\frac{1}{4}} \|\partial_2 \widetilde{f}\|_{L^2}^{\frac{1}{4}} \|\partial_1 \partial_2 \widetilde{f}\|_{L^2}^{\frac{1}{4}}.$$

The next lemma assesses that the oscillation part $\widetilde{f}$ obeys a strong Poincaré type inequality with the upper bound in terms of $\partial_1 \widetilde{f}$ instead of $\nabla \widetilde{f}$.

Lemma 2.5. *Let $\overline{f}$ and $\widetilde{f}$ be defined as in (1.14) and (1.15). If $\|\partial_1 \widetilde{f}\|_{L^2(\Omega)} < \infty$, then*

$$\|\widetilde{f}\|_{L^2(\Omega)} \leq C \|\partial_1 \widetilde{f}\|_{L^2(\Omega)},$$

where C is a pure constant. In addition, if $\|\partial_1 \widetilde{f}\|_{H^1(\Omega)} < \infty$, then

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \leq C \|\partial_1 \widetilde{f}\|_{H^1(\Omega)}.$$

As a direct consequence of Lemma 2.5 and the inequality (2.2), one has

$$\Big| \int_\Omega \widetilde{f} g h \, dx \Big| \leq C \|\partial_1 \widetilde{f}\|_{L^2} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_2 g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}. \tag{2.3}$$

We refer the readers to [10] for detailed proofs of Lemmas 2.1, 2.3, 2.4 and 2.5.

The last lemma precises an explicit decay rate in (2.5) for functions that are integrable and are decreasing in a general sense, namely (2.4).

Lemma 2.6. *Let $f = f(t)$ be a nonnegative function satisfying , for two constants $C_0 > 0$ and $C_1 > 0$,*

$$\int_0^\infty f(\tau) d\tau < C_0 \quad \text{and} \quad f(t) \leq C_1 f(s) \quad \text{for any} \quad 0 \leq s < t. \tag{2.4}$$

Then, for $C_2 = \max\{2C_1 f(0), 4C_0 C_1\}$ and for any $t > 0$,

$$f(t) \leq C_2 (1+t)^{-1}. \tag{2.5}$$

Furthermore, $f(t)$ has the following large-time asymptotic behavior,

$$\lim_{t \to +\infty} t\, f(t) = 0.$$

A detailed proof of Lemma 2.6 can be found in [21].

3. The H^2 Nonlinear Stability

This section proves Theorem. 1.1.

Proof. The proof is naturally divided into two major parts. The first part is for the existence, while the second part is for the uniqueness of solutions to (1.2).

To prove the global existence of solutions, it suffices to establish the energy inequality in (1.17) with $E(t)$ being defined in (1.16). This process consists of two main parts. The first is to estimate the H^2-norm of (u, θ) while the second is to estimate $\|\partial_1 u_2\|_{L^2}^2$ and its time integral.

Note that, for a divergence-free vector field u, namely $\nabla \cdot u = 0$, we have

$$\|\nabla u\|_{L^2} = \|\omega\|_{L^2}, \quad \|\Delta u\|_{L^2} = \|\nabla \omega\|_{L^2},$$

where $\omega = \nabla \times u$ is the vorticity. Then, the H^2-norm of u is equivalent to the sum of the L^2-norms of u, ω and $\nabla\omega$.
Taking the L^2-inner product of (u, θ) with the first two equations in (1.2), we find that the L^2-norm of (u, θ) obeys

$$\begin{aligned} &\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + 2\nu \int_0^t \|\partial_2 u(\tau)\|_{L^2}^2 \, d\tau + 2\eta \int_0^t \|\partial_1 \theta(\tau)\|_{L^2}^2 d\tau \\ &= \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2. \end{aligned} \tag{3.1}$$

To estimate the L^2-norm of $(\omega, \nabla\theta)$, we resort to the vorticity equation combined with the temperature equation,

$$\begin{aligned} &\partial_t \omega + u \cdot \nabla \omega = \nu \partial_{22} \omega + g_0 \partial_1 \theta, \\ &\partial_t \theta + u \cdot \nabla \theta + g_0 u_2 = \eta \partial_{11} \theta. \end{aligned} \tag{3.2}$$

Dotting the equations of ω and $\nabla\theta$ by $(\omega, \nabla\theta)$, yields

$$\frac{1}{2}\frac{d}{dt}(\|\omega\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \nu\|\partial_2\omega\|_{L^2}^2 + \eta\|\partial_1\nabla\theta\|_{L^2}^2 = I_1 + I_2, \tag{3.3}$$

where

$$I_1 = g_0 \int (\partial_1 \theta \, \omega - \nabla u_2 \cdot \nabla\theta) \, dx, \quad I_2 = -\int \nabla\theta \cdot \nabla u \cdot \nabla\theta \, dx.$$

Writing ω and u in terms of the stream function ψ, namely $\omega = \Delta\psi$ and $u = \nabla^\perp \psi := (-\partial_2\psi, \partial_1\psi)$, we have

$$\begin{aligned} I_1 &= g_0 \int (\partial_1\theta \, \omega - \nabla u_2 \cdot \nabla\theta) \, dx = g_0 \int (\partial_1\theta \Delta\psi - \nabla\partial_1\psi \cdot \nabla\theta) \, dx \\ &= g_0 \int (-\theta \, \Delta\partial_1\psi + \Delta\partial_1\psi \, \theta) \, dx = 0. \end{aligned}$$

We further write I_2 into four terms,

$$\begin{aligned} I_2 &= -\int (\partial_1 u_1 (\partial_1\theta)^2 + \partial_1 u_2 \partial_1\theta\partial_2\theta + \partial_2 u_1 \partial_1\theta\partial_2\theta + \partial_2 u_2 (\partial_2\theta)^2) \, dx \\ &:= I_{21} + I_{22} + I_{23} + I_{24}. \end{aligned}$$

The terms on the right-hand side can be bounded as follows. The key point here is to obtain upper bounds that are time integrable. I_{21}, I_{22} and I_{23} can be bounded directly. By Lemmas 2.1, 2.4 and Young's inequality,

$$\begin{aligned} I_{21} :&= -\int \partial_1 u_1 (\partial_1\theta)^2 dx = -\int \partial_1 \widetilde{u_1} (\partial_1\theta)^2 dx \\ &\le c\|\partial_1\theta\|_{L^2}^{\frac12}\|\partial_2\partial_1\theta\|_{L^2}^{\frac12}\|\partial_1\theta\|_{L^2}^{\frac12}\|\partial_1\partial_1\theta\|_{L^2}^{\frac12}\|\partial_1\widetilde{u_1}\|_{L^2} \\ &\le c\,\|u\|_{H^2}\,\|\partial_1\theta\|_{H^2}^2, \end{aligned} \tag{3.4}$$

$$\begin{aligned} I_{22} :&= -\int \partial_1 u_2 \partial_1\theta\partial_2\theta dx = -\int \partial_1 u_2 \partial_1\widetilde{\theta}\partial_2\theta dx \\ &\le c\,\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\theta\|_{L^2}^{\frac12}\|\partial_2\partial_2\theta\|_{L^2}^{\frac12}\|\partial_1 u_2\|_{L^2} \\ &\le c\,\|\partial_1\theta\|_{H^2}\|\theta\|_{H^2}\|\partial_1 u_2\|_{L^2} \\ &\le c\,\|\theta\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2\Big), \end{aligned} \tag{3.5}$$

$$\begin{aligned} I_{23} :&= -\int \partial_2 u_1 \partial_1\theta\partial_2\theta dx = -\int \partial_2 u_1 \partial_1\widetilde{\theta}\partial_2\theta dx \\ &\le c\,\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2 u_1\|_{L^2}^{\frac12}\|\partial_2\partial_2 u_1\|_{L^2}^{\frac12}\|\partial_2\theta\|_{L^2} \\ &\le c\,\|\partial_1\theta\|_{H^2}\|\partial_2 u\|_{H^2}\|\theta\|_{H^2} \\ &\le c\,\|\theta\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}\Big). \end{aligned} \tag{3.6}$$

Using the divergence-free condition $\nabla\cdot u = 0$, integration by parts and Lemmas 2.1 and 2.5, we obtain

$$\begin{aligned} I_{24} :&= -\int \partial_2 u_2 (\partial_2\theta)^2 dx = \int \partial_1 u_1 (\partial_2\theta)^2 dx \\ &= -2\int \widetilde{u_1}\,\partial_2\theta\,\partial_1\partial_2\theta\,dx \\ &\le c\,\|\partial_2\theta\|_{L^2}\|\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_1\partial_2\theta\|_{L^2}^{\frac12}\|\partial_2\partial_1\partial_2\theta\|_{L^2}^{\frac12} \\ &\le c\,\|\theta\|_{H^2}\underbrace{\|\partial_1\widetilde{u_1}\|_{L^2}}_{=\|\partial_2\widetilde{u_2}\|_{L^2}}\|\partial_1\theta\|_{H^2} \\ &\le c\,\|\theta\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2\Big). \end{aligned} \tag{3.7}$$

Collecting the upper bounds on I_2 and inserting them in (3.3), we find

$$\begin{aligned} &\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + 2\nu\|\partial_2\nabla u\|_{L^2}^2 + 2\eta\|\partial_1\nabla\theta\|_{L^2}^2 \\ &\qquad\le c\,\|(u,\theta)\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2\Big). \end{aligned} \tag{3.8}$$

Integrating (3.8) over $[0,t]$ and combining with (3.1), we obtain

$$\|(u,\theta)\|_{H^1}^2 + 2\nu\int_0^t \|\partial_2 u(s)\|_{H^1}^2 ds + 2\eta\int_0^t \|\partial_1\theta(s)\|_{H^1}^2 ds$$

$$\leq \|(u_0,\theta_0)\|_{H^1}^2 + c\int_0^t \|(u,\theta)\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2\Big)\,d\tau$$
$$\leq E(0) + c\,E(t)^{\frac{3}{2}}. \tag{3.9}$$

To control the H^2-norm of (u,θ), it then remains to bound the L^2-norm of $(\nabla\omega, \Delta\theta)$. Applying ∇ to the first equation of (3.2) and dotting with $\nabla\omega$, and apply Δ to the second equation of (3.2) and dotting with $\Delta\theta$, we find

$$\frac{1}{2}\frac{d}{dt}(\|\nabla\omega\|_{L^2}^2 + \|\Delta\theta(t)\|_{L^2}^2) + \nu\|\partial_2\nabla\omega\|_{L^2}^2 + \eta\|\partial_1\Delta\theta\|_{L^2}^2 = J_1 + J_2 + J_3, \tag{3.10}$$

where

$$J_1 = g_0\int(\nabla\partial_1\theta\cdot\nabla\omega - \Delta u_2\Delta\theta)\,dx,$$
$$J_2 = -\int\nabla\omega\cdot\nabla u\cdot\nabla\omega\,dx,$$
$$J_3 = -\int\Delta\theta\cdot\Delta(u\cdot\nabla\theta)dx.$$

The effort is still devoted to obtaining an upper bound that is time integrable for each term.

Writing ω and u in terms of the stream function ψ, namely $\omega = \Delta\psi$ and $u = \nabla^\perp\psi := (-\partial_2\psi, \partial_1\psi)$, we have

$$J_1 = g_0\int(\nabla\partial_1\theta\cdot\nabla\omega - \Delta u_2\Delta\theta)\,dx = g_0\int(\nabla\partial_1\theta\cdot\nabla\omega - \Delta\partial_1\psi\Delta\theta)\,dx$$
$$= g_0\int(\nabla\partial_1\theta\cdot\nabla\omega - \partial_1\omega\,\Delta\theta)\,dx = g_0\int(\nabla\partial_1\theta\cdot\nabla\omega + \partial_1\nabla\omega\,\cdot\nabla\theta)\,dx$$
$$= g_0\int\partial_1(\nabla\theta\cdot\nabla\omega)\,dx = 0.$$

After integration by parts, we decompose J_3 it into four pieces,

$$J_3 = -\int\Delta\theta\,\Delta u_1\,\partial_1\theta\,dx - \int\Delta\theta\,\Delta u_2\,\partial_2\theta\,dx$$
$$-2\int\Delta\theta\,\nabla u_1\cdot\partial_1\nabla\theta\,dx - 2\int\Delta\theta\,\nabla u_2\cdot\partial_2\nabla\theta\,dx$$
$$:= J_{31} + J_{32} + J_{33} + J_{34}. \tag{3.11}$$

To deal with J_{31}, we make use of the orthogonal decompositions $u = \overline{u} + \widetilde{u}$ and $\theta = \overline{\theta} + \widetilde{\theta}$ to write

$$J_{31} := -\int\Delta\theta\Delta u_1\partial_1\theta dx = -\int\Delta\theta\Delta u_1\partial_1\widetilde{\theta}dx$$
$$= -\int\Delta\theta\partial_{11}u_1\partial_1\widetilde{\theta}dx - \int\Delta\theta\partial_{22}u_1\partial_1\widetilde{\theta}dx$$
$$= \int\Delta\theta\partial_{12}u_2\partial_1\widetilde{\theta}dx - \int\Delta\theta\partial_{22}u_1\partial_1\widetilde{\theta}dx$$
$$:= J_{311} + J_{312}. \tag{3.12}$$

Applying Lemma 2.4 we obtain,

$$\begin{aligned} J_{311} &:= \int \Delta\theta \partial_{12} u_2 \partial_1 \widetilde{\theta} dx \\ &\leq c \|\partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_{12} u_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_{12} u_2\|_{L^2}^{\frac{1}{2}} \|\Delta\theta\|_{L^2} \\ &\leq c \|\theta\|_{H^2} \|\partial_1 \theta\|_{H^2} \|\partial_2 u\|_{H^2} \\ &\leq c \|\theta\|_{H^2} \Big(\|\partial_1 \theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2 \Big), \end{aligned} \tag{3.13}$$

$$\begin{aligned} J_{312} &:= -\int \Delta\theta \partial_{22} u_1 \partial_1 \widetilde{\theta} dx \\ &\leq c \|\partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_{22} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_{22} u_1\|_{L^2}^{\frac{1}{2}} \|\Delta\theta\|_{L^2} \\ &\leq c \|\theta\|_{H^2} \|\partial_1 \theta\|_{H^2} \|\partial_2 u\|_{H^2} \\ &\leq c \|\theta\|_{H^2} \Big(\|\partial_1 \theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2 \Big). \end{aligned} \tag{3.14}$$

Inserting the upper bounds for J_{311} and J_{312} in (3.12) yields

$$J_{31} \leq c \|\theta\|_{H^2} \Big(\|\partial_1 \theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2 \Big). \tag{3.15}$$

To deal with J_{32}, we divide it first into two terms,

$$\begin{aligned} J_{32} &= -\int \Delta\theta \Delta u_2 \partial_2 \theta dx \\ &= -\int \partial_1 \partial_1 \theta \, \Delta u_2 \, \partial_2 \theta dx - \int \partial_2 \partial_2 \theta \, \Delta u_2 \, \partial_2 \theta \, dx \\ &= -\int \partial_1 \partial_1 \theta \, \Delta u_2 \partial_2 \, \theta \, dx + \frac{1}{2} \int \Delta \partial_2 u_2 \, (\partial_2 \theta)^2 \, dx \\ &= -\int \partial_1 \partial_1 \theta \, \Delta u_2 \, \partial_2 \theta \, dx - \frac{1}{2} \int \Delta \partial_1 u_1 \, (\partial_2 \theta)^2 dx \\ &= -\int \partial_1 \partial_1 \theta \, \Delta u_2 \, \partial_2 \theta \, dx + \int \Delta u_1 \, \partial_2 \theta \, \partial_1 \partial_2 \theta dx \\ &= J_{321} + J_{322}. \end{aligned} \tag{3.16}$$

Invoking the decompositions of u and θ, we can rewrite J_{321} as,

$$\begin{aligned} J_{321} &:= -\int \partial_1 \partial_1 \theta \, \Delta u_2 \, \partial_2 \theta \, dx \\ &= -\int \partial_1 \partial_1 \widetilde{\theta} \, \partial_{11} \widetilde{u_2} \, \partial_2 \overline{\theta} \, dx - \int \partial_1 \partial_1 \widetilde{\theta} \, \partial_{11} \widetilde{u_2} \, \partial_2 \widetilde{\theta} \, dx - \int \partial_1 \partial_1 \widetilde{\theta} \, \partial_{22} u_2 \, \partial_2 \theta \, dx \\ &:= J_{3211} + J_{3212} + J_{3213}. \end{aligned} \tag{3.17}$$

The three terms in J_{321} can be bounded as follows. By interation by parts, Lemma 2.1, Hölder's inequality, Lemma 2.2 and Young's inequality,

$$J_{3211} := -\int \partial_1 \partial_1 \widetilde{\theta} \, \partial_{11} \widetilde{u_2} \, \partial_2 \overline{\theta} \, dx$$

$$
\begin{aligned}
&= \int \partial_1\partial_{11}\widetilde{\theta}\,\partial_1\widetilde{u_2}\,\partial_2\overline{\theta}\,dx\\
&= \int_{\mathbb{R}} \partial_2\overline{\theta}\Big(\int_{\mathbb{T}} \partial_1\partial_{11}\widetilde{\theta}\,\partial_1\widetilde{u_2}dx_1\Big)dx_2\\
&\le \int_{\mathbb{R}} |\partial_2\overline{\theta}|\|\partial_1\widetilde{u_2}\|_{L^2_{x_1}}\|\partial_1\partial_{11}\widetilde{\theta}\|_{L^2_{x_1}}dx_2\\
&\le \|\partial_2\overline{\theta}\|_{L^\infty_{x_2}}\|\partial_1\widetilde{u_2}\|_{L^2_{x_2}L^2_{x_1}}\|\partial_1\partial_{11}\widetilde{\theta}\|_{L^2_{x_2}L^2_{x_1}}\\
&\le c\|\partial_2\overline{\theta}\|_{H^1}\|\partial_1\widetilde{u_2}\|_{L^2}\|\partial_1\partial_{11}\widetilde{\theta}\|_{L^2}\\
&\le c\|\theta\|_{H^2}\Big(\|\partial_1 u_2\|^2_{L^2}+\|\partial_1\theta\|^2_{H^2}\Big). \qquad (3.18)
\end{aligned}
$$

By lemma 2.4 and then lemma 2.5

$$
\begin{aligned}
J_{3212} &:= -\int \partial_1\partial_1\widetilde{\theta}\,\partial_{11}\widetilde{u_2}\,\partial_2\widetilde{\theta}\,dx\\
&\le c\,\underbrace{\|\partial_2\widetilde{\theta}\|^{\frac12}_{L^2}}_{\le\|\partial_1\partial_2\widetilde{\theta}\|^{\frac12}_{L^2}}\|\partial_1\partial_2\widetilde{\theta}\|^{\frac12}_{L^2}\|\partial_{11}\widetilde{\theta}\|^{\frac12}_{L^2}\|\partial_2\partial_{11}\widetilde{\theta}\|^{\frac12}_{L^2}\|\partial_{11}u_2\|_{L^2}\\
&\le c\|u\|_{H^2}\|\partial_1\theta\|^2_{H^2}. \qquad (3.19)
\end{aligned}
$$

Making use of the divergence-free condition of u, Lemmas 2.1 and 2.4, we have

$$
\begin{aligned}
J_{3213} &:= -\int \partial_1\partial_1\widetilde{\theta}\,\partial_{22}u_2\,\partial_2\theta\,dx\\
&= -\int \partial_1\partial_1\widetilde{\theta}\,\partial_{21}\widetilde{u_1}\,\partial_2\theta\,dx\\
&\le c\|\partial_{21}\widetilde{u_1}\|^{\frac12}_{L^2}\|\partial_1\partial_{21}\widetilde{u_1}\|^{\frac12}_{L^2}\|\partial_{11}\widetilde{\theta}\|^{\frac12}_{L^2}\|\partial_2\partial_{11}\widetilde{\theta}\|^{\frac12}_{L^2}\|\partial_2\theta\|_{L^2}\\
&\le c\|\theta\|_{H^2}\|\partial_1\theta\|_{H^2}\|\partial_2 u\|_{H^2}\\
&\le c\|\theta\|_{H^2}\Big(\|\partial_1\theta\|^2_{H^2}+\|\partial_2 u\|^2_{H^2}\Big). \qquad (3.20)
\end{aligned}
$$

Inserting (3.18), (3.19) and (3.20) in (3.17) we obtain

$$
J_{321} \le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|^2_{H^2}+\|\partial_2 u\|^2_{H^2}+\|\partial_1 u_2\|^2_{L^2}\Big). \qquad (3.21)
$$

We now turn to J_{322}. We further decompose it into two terms,

$$
\begin{aligned}
J_{322} &:= \int \Delta u_1\,\partial_2\theta\,\partial_1\partial_2\theta dx\\
&= \int \partial_{11}\widetilde{u_1}\,\partial_2\theta\,\partial_1\partial_2\widetilde{\theta}dx + \int \partial_{22}u_1\,\partial_2\theta\,\partial_1\partial_2\widetilde{\theta}dx\\
&= J_{3221}+J_{3222}. \qquad (3.22)
\end{aligned}
$$

Due to the divergence-free condition of u and Lemma 2.4,

$$
J_{3221} := \int \partial_{11}\widetilde{u_1}\,\partial_2\theta\,\partial_{12}\widetilde{\theta}dx
$$

$$
\begin{aligned}
&= -\int \partial_{12}\widetilde{u_2}\,\partial_2\theta\,\partial_{12}\widetilde{\theta}dx\\
&\le c\|\partial_{12}\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{12}\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_{12}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_{12}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{L^2}\\
&\le c\|\theta\|_{H^2}\|\partial_2 u\|_{H^2}\|\partial_1\theta\|_{H^2}\\
&\le c\|\theta\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2\Big). \qquad (3.23)
\end{aligned}
$$

By Lemma 2.4,

$$
\begin{aligned}
J_{3222} &:= \int \partial_{22}u_1\,\partial_2\theta\,\partial_{12}\widetilde{\theta}dx\\
&\le c\|\partial_{12}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{12}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_{22}u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_{22}u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{L^2}\\
&\le c\|\theta\|_{H^2}\|\partial_2 u\|_{H^2}\|\partial_1\theta\|_{H^2}\\
&\le c\|\theta\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2\Big). \qquad (3.24)
\end{aligned}
$$

Combining the estimates (3.23) and (3.24) and inserting them in (3.22) we find

$$
J_{322} \le c\|\theta\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2\Big). \qquad (3.25)
$$

Putting (3.21) and (3.25) in (3.16) we obtain

$$
J_{32} \le c\|(u,\theta)\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2\Big). \qquad (3.26)
$$

The next term J_{33} is naturally split into two parts,

$$
\begin{aligned}
J_{33} &:= -2\int \Delta\theta\nabla u_1\cdot\partial_1\nabla\theta dx\\
&= -2\int \Delta\theta\partial_1 u_1\partial_1\partial_1\theta dx - 2\int \Delta\theta\partial_2 u_1\partial_1\partial_2\theta dx\\
&:= J_{331}+J_{332}. \qquad (3.27)
\end{aligned}
$$

All terms can be bounded suitably. In fact, due to the divergence-free condition of u and Lemma 2.4,

$$
\begin{aligned}
J_{331} &:= -2\int \Delta\theta\partial_1 u_1\partial_{11}\theta dx\\
&= 2\int \Delta\theta\partial_2 u_2\partial_1\partial_1\widetilde{\theta}dx\\
&\le c\|\partial_{11}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{11}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\Delta\theta\|_{L^2}\\
&\le c\|\theta\|_{H^2}\|\partial_2 u\|_{H^2}\|\partial_1\theta\|_{H^2}\\
&\le c\|\theta\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2\Big). \qquad (3.28)
\end{aligned}
$$

J_{332} can be bounded similarly, by $\partial_1\theta=\partial_1\widetilde{\theta}$ and Lemma 2.4,

$$
J_{332} := -2\int \Delta\theta\partial_2 u_1\partial_{12}\theta dx
$$

$$
\begin{aligned}
&= -2\int \Delta\theta\partial_2 u_1\partial_{12}\widetilde{\theta}dx\\
&\leq c\|\partial_{12}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{12}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2 u_1\|_{L^2}^{\frac{1}{2}}\|\Delta\theta\|_{L^2}\\
&\leq c\|\theta\|_{H^2}\|\partial_2 u\|_{H^2}\|\partial_1\theta\|_{H^2}\\
&\leq c\|\theta\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2\Big). \qquad (3.29)
\end{aligned}
$$

Inserting these upper bounds in (3.27) we get

$$
J_{33}\leq c\|\theta\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\Big). \qquad (3.30)
$$

To estimate J_{34}, we first invoke the decompositions $u=\overline{u}+\widetilde{u}$, $\theta=\overline{\theta}+\widetilde{\theta}$ and Lemma 2.1, to write J_{34} as

$$
\begin{aligned}
J_{34} &:= -2\int \Delta\theta\nabla u_2\cdot\partial_2\nabla\theta dx\\
&= -2\int(\partial_1 u_2\partial_1\partial_2\theta\Delta\theta+\partial_2 u_2\partial_2\partial_2\theta\Delta\theta)dx\\
&= -2\int\partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\Delta\theta-2\int\partial_2 u_2\partial_2\partial_2\theta\Delta\theta dx\\
&= -2\int\partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\partial_{11}\theta dx-2\int\partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\partial_{22}\theta dx-2\int\partial_2 u_2\partial_2\partial_2\theta\Delta\theta dx\\
&:= J_{341}+J_{342}+J_{343}. \qquad (3.31)
\end{aligned}
$$

We start with J_{341}. By integration by parts, Lemmas 2.1 and 2.4 we have

$$
\begin{aligned}
J_{341} &:= -2\int\partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\partial_{11}\theta dx\\
&= 2\int\widetilde{u_2}\partial_1\partial_1\partial_2\widetilde{\theta}\partial_{11}\widetilde{\theta}dx\\
&\leq c\|\partial_{11}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{11}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_{12}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_{11}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u_2}\|_{L^2}\\
&\leq c\|u\|_{H^2}\|\partial_1\theta\|_{H^2}^2. \qquad (3.32)
\end{aligned}
$$

Using the decomposition $\theta=\overline{\theta}+\widetilde{\theta}$ we write J_{342} as,

$$
\begin{aligned}
J_{342} &:= -2\int\partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\partial_{22}\theta dx\\
&= -2\int\partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\partial_{22}\overline{\theta}dx-2\int\partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\partial_{22}\widetilde{\theta}dx\\
&:= J_{3421}+J_{3422}. \qquad (3.33)
\end{aligned}
$$

We start with J_{3421}. Due to integration by parts, Lemma 2.1, Hölder's inequality, Lemma 2.2 and Young's inequality,

$$
\begin{aligned}
J_{3421} &:= -2\int\partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\partial_{22}\overline{\theta}dx\\
&= 2\int(\partial_2\partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}+\partial_1\widetilde{u_2}\partial_2\partial_1\partial_2\widetilde{\theta})\partial_2\overline{\theta}\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= 2\int_{\mathbb{R}} \partial_2\overline{\theta}\Big(\int_{\mathbb{T}} (\partial_2\partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta} + \partial_1\widetilde{u_2}\partial_2\partial_1\partial_2\widetilde{\theta})dx_1\Big)dx_2\\
&\leq c\int_{\mathbb{R}} |\partial_2\overline{\theta}|\Big(\|\partial_2\partial_1\widetilde{u_2}\|_{L^2_{x_1}}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2_{x_1}} + \|\partial_1\widetilde{u_2}\|_{L^2_{x_1}}\|\partial_2\partial_1\partial_2\widetilde{\theta}\|_{L^2_{x_1}}\Big)dx_2\\
&\leq c\|\partial_2\overline{\theta}\|_{L^\infty_{x_2}}\Big(\|\partial_2\partial_1\widetilde{u_2}\|_{L^2_{x_2}L^2_{x_1}}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2_{x_2}L^2_{x_1}} + \|\partial_1\widetilde{u_2}\|_{L^2_{x_2}L^2_{x_1}}\|\partial_2\partial_1\partial_2\widetilde{\theta}\|_{L^2_{x_2}L^2_{x_1}}\Big)\\
&\leq c\|\partial_2\overline{\theta}\|_{H^1}\Big(\|\partial_2\partial_1\widetilde{u_2}\|_{L^2}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2} + \|\partial_1\widetilde{u_2}\|_{L^2}\|\partial_2\partial_1\partial_2\widetilde{\theta}\|_{L^2}\Big)\\
&\leq c\|\theta\|_{H^2}\Big(\|\partial_2 u\|^2_{H^2} + \|\partial_1\theta\|^2_{H^2} + \|\partial_1 u_2\|^2_{L^2}\Big). \qquad (3.34)
\end{aligned}
$$

For J_{3422}, we apply Lemma 2.4 then Young's inequality,

$$
\begin{aligned}
J_{3422} &:= -2\int \partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\partial_{22}\widetilde{\theta}dx\\
&\leq c\|\partial_{22}\widetilde{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_1\partial_{22}\widetilde{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_{12}\widetilde{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_2\partial_{12}\widetilde{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_1\widetilde{u_2}\|_{L^2}\\
&\leq c\|\theta\|^{\frac{1}{2}}_{H^2}\|u\|^{\frac{1}{2}}_{H^2}\|\partial_1\theta\|^{\frac{3}{2}}_{H^2}\|\partial_1 u_2\|^{\frac{1}{2}}_{L^2}\\
&\leq c\|\theta\|^{\frac{1}{2}}_{H^2}\|u\|^{\frac{1}{2}}_{H^2}\Big(\|\partial_1\theta\|^2_{H^2} + \|\partial_1 u_2\|^2_{L^2}\Big). \qquad (3.35)
\end{aligned}
$$

In view of (3.33), (3.34) and (3.35) we have

$$
J_{342} \leq c\|(u,\theta)\|_{H^2}\Big(\|\partial_2 u\|^2_{H^2} + \|\partial_1\theta\|^2_{H^2} + \|\partial_1 u_2\|^2_{L^2}\Big). \qquad (3.36)
$$

Writing J_{343} more explicitly and using $\partial_1\theta = \partial_1\widetilde{\theta}$, we have

$$
\begin{aligned}
J_{343} &:= -2\int \partial_2 u_2\partial_2\partial_2\theta\Delta\theta dx\\
&= -2\int \partial_2 u_2\partial_{22}\theta\partial_{11}\widetilde{\theta}dx - 2\int \partial_2 u_2\partial_{22}\theta\partial_{22}\theta dx\\
&:= J_{3431} + J_{3432}. \qquad (3.37)
\end{aligned}
$$

From Lemma 2.4, J_{3431} can be bounded as,

$$
\begin{aligned}
J_{3431} &:= -2\int \partial_2 u_2\partial_{22}\theta\partial_{11}\widetilde{\theta}dx\\
&\leq c\|\partial_{11}\widetilde{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_1\partial_{11}\widetilde{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_2 u_2\|^{\frac{1}{2}}_{L^2}\|\partial_2\partial_2 u_2\|^{\frac{1}{2}}_{L^2}\|\partial_{22}\theta\|_{L^2}\\
&\leq c\|\theta\|_{H^2}\|\partial_1\theta\|_{H^2}\|\partial_2 u\|_{H^2}\\
&\leq c\|\theta\|_{H^2}\Big(\|\partial_1\theta\|^2_{H^2} + \|\partial_2 u\|^2_{H^2}\Big). \qquad (3.38)
\end{aligned}
$$

The estimate of J_{3432} is slightly more delicate. Due to the decomposition $\theta = \widetilde{\theta} + \overline{\theta}$, we write J_{3432} as,

$$
\begin{aligned}
J_{3432} &:= -2\int \partial_2 u_2\partial_{22}\theta\partial_{22}\theta dx\\
&= -2\int \partial_2 u_2\partial_{22}\overline{\theta}\partial_{22}\overline{\theta}dx - 4\int \partial_2 u_2\partial_{22}\overline{\theta}\partial_{22}\widetilde{\theta}dx - 2\int \partial_2 u_2\partial_{22}\widetilde{\theta}\partial_{22}\widetilde{\theta}dx
\end{aligned}
$$

$$:= J_{34321} + J_{34322} + J_{34323}. \tag{3.39}$$

By $\nabla \cdot u = 0$ and Lemma 2.1, the first term J_{34321} is clearly zero,

$$J_{34321} = -2\int \partial_2 u_2 \partial_{22}\overline{\theta}\partial_{22}\overline{\theta}dx = 2\int \partial_1 \widetilde{u_1}\partial_{22}\overline{\theta}\partial_{22}\overline{\theta}dx = 0. \tag{3.40}$$

Applying Lemmas 2.4 and 2.5 and Young's inequality,

$$\begin{aligned} J_{34322} &:= -4\int \partial_2 u_2 \partial_{22}\overline{\theta}\partial_{22}\widetilde{\theta}dx \\ &\le c\|\partial_{22}\overline{\theta}\|_{L^2}\|\partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2 u_2\|_{L^2}^{\frac{1}{2}} \\ &\le c\|\partial_{22}\overline{\theta}\|_{L^2}\|\partial_1\partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2 u_2\|_{L^2}^{\frac{1}{2}} \\ &\le c\|\theta\|_{H^2}\|\partial_1\theta\|_{H^2}\|\partial_2 u\|_{H^2} \\ &\le c\|\theta\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\Big), \end{aligned} \tag{3.41}$$

$$\begin{aligned} J_{34323} &:= -4\int \partial_2 u_2 \partial_{22}\widetilde{\theta}\partial_{22}\widetilde{\theta}dx \\ &\le c\|\partial_{22}\overline{\theta}\|_{L^2}\|\partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2 u_2\|_{L^2}^{\frac{1}{2}} \\ &\le c\|\partial_{22}\overline{\theta}\|_{L^2}\|\partial_1\partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2 u_2\|_{L^2}^{\frac{1}{2}} \\ &\le c\|\theta\|_{H^2}\|\partial_1\theta\|_{H^2}\|\partial_2 u\|_{H^2} \\ &\le c\|\theta\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\Big). \end{aligned} \tag{3.42}$$

The bounds for J_{3432} in (3.40), (3.41) and (3.42) lead to,

$$J_{3432} \le c\|\theta\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\Big). \tag{3.43}$$

Combining (3.38) and (3.43) and inserting them in (3.37) we obtain

$$J_{343} \le c\|\theta\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\Big). \tag{3.44}$$

Inserting (3.32), (3.36) and (3.44) in (3.31) we get

$$J_{34} \le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2\Big). \tag{3.45}$$

Thus, by (3.15), (3.26), (3.30), (3.45), and (3.11),

$$J_3 \le c\|(u,\theta)\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2 + \|\partial_1\theta\|_{H^2}^2\Big). \tag{3.46}$$

Now we turn to the next term J_2. As we have explained in the introduction, we need the help of the extra regularization term

$$\int_0^t \|\partial_1 u_2\|_{L^2}^2\, d\tau. \tag{3.47}$$

In order to make efficient use of the anisotropic dissipation, we further write J_2 as

$$J_2 = -\int \partial_1 u_1\, (\partial_1\omega)^2\, dx - \int \partial_1 u_2\, \partial_1\omega\, \partial_2\omega\, dx$$

$$
\begin{aligned}
&- \int \partial_2 u_1 \, \partial_1 \omega \, \partial_2 \omega dx - \int \partial_2 u_2 \, (\partial_2 \omega)^2 dx \\
&= \int \partial_2 u_2 \, (\partial_1 \omega)^2 dx - \int \partial_1 u_2 \, \partial_1 \omega \, \partial_2 \omega \, dx \\
&\quad - \int \partial_2 u_1 \, \partial_1 \omega \, \partial_2 \omega \, dx - \int \partial_2 u_2 \, (\partial_2 \omega)^2 \, dx \\
&:= J_{21} + J_{22} + J_{23} + J_{24}. \qquad (3.48)
\end{aligned}
$$

The terms J_{21} through J_{24} can be bounded as follows. Due to $\nabla \cdot u = 0$, integration by parts and Lemmas 2.1 and 2.4,

$$
\begin{aligned}
J_{21} &:= - \int \partial_1 u_1 (\partial_1 \omega)^2 \, dx \\
&= \int \partial_2 u_2 (\partial_1 \omega)^2 \, dx \\
&= -2 \int \widetilde{u_2} \partial_1 \widetilde{\omega} \partial_2 \partial_1 \widetilde{\omega} \, dx \\
&\leq c \|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \widetilde{\omega}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \widetilde{\omega}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \widetilde{\omega}\|_{L^2} \\
&\leq c \|u\|_{H^2} \|\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 u\|_{H^2}^{\frac{3}{2}} \\
&\leq c \|u\|_{H^2} \Big(\|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u\|_{H^2}^2 \Big). \qquad (3.49)
\end{aligned}
$$

According to Lemmas 2.1 and 2.4,

$$
\begin{aligned}
J_{22} &:= - \int \partial_1 u_2 \partial_1 \omega \partial_2 \omega \, dx \\
&= - \int \partial_1 \widetilde{u_2} \partial_1 \widetilde{\omega} \partial_2 \omega \, dx \\
&\leq c \|\partial_1 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_1 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{\omega}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \widetilde{\omega}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \omega\|_{L^2} \\
&\leq c \|u\|_{H^2} \|\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 u\|_{H^2}^{\frac{3}{2}} \\
&\leq c \|u\|_{H^2} \Big(\|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u\|_{H^2}^2 \Big). \qquad (3.50)
\end{aligned}
$$

To bound J_{23}, we first use the orthogonal decomposition of u_1 and ω and Lemma 2.1, to write J_{23} as

$$
\begin{aligned}
J_{23} &:= - \int \partial_2 u_1 \partial_1 \omega \partial_2 \omega \, dx = - \int \partial_2 u_1 \partial_1 \widetilde{\omega} \partial_2 \omega \, dx \\
&= - \int \partial_2 \overline{u_1} \partial_1 \widetilde{\omega} \partial_2 \overline{\omega} \, dx - \int \partial_2 \overline{u_1} \partial_1 \widetilde{\omega} \partial_2 \widetilde{\omega} \, dx - \int \partial_2 \widetilde{u_1} \partial_1 \widetilde{\omega} \partial_2 \omega \, dx \\
&= J_{231} + J_{232} + J_{233}. \qquad (3.51)
\end{aligned}
$$

According to Lemma 2.1, the first term J_{231} is clearly zero,

$$
J_{231} := - \int \partial_2 \overline{u_1} \partial_1 \widetilde{\omega} \partial_2 \overline{\omega} \, dx = - \int_{\mathbb{R}} \partial_2 \overline{u_1} \partial_2 \overline{\omega} \int_{\mathbb{T}} \partial_1 \widetilde{\omega} \, dx_1 dx_2 = 0. \qquad (3.52)
$$

The terms J_{232} and J_{233} can be bounded directly. By Lemma 2.4,

$$\begin{aligned} J_{232} &:= -\int \partial_2\overline{u_1}\partial_1\widetilde{\omega}\partial_2\widetilde{\omega}\,dx \\ &\le c\|\partial_2\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_2\overline{u_1}\|_{L^2} \\ &\le c\|u\|_{H^2}\|\partial_2 u\|_{H^2}^2, \end{aligned} \tag{3.53}$$

$$\begin{aligned} J_{233} &:= -\int \partial_2\widetilde{u_1}\partial_1\widetilde{\omega}\partial_2\omega\,dx \\ &\le c\|\partial_2\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_2\omega\|_{L^2} \\ &\le c\|u\|_{H^2}\|\partial_2 u\|_{H^2}^2. \end{aligned} \tag{3.54}$$

Inserting these upper bounds in (3.51) yields

$$J_{23} \le c\|u\|_{H^2}\|\partial_2 u\|_{H^2}^2. \tag{3.55}$$

To deal with J_{24} we use the divergence-free condition of u, Lemma 2.1, and the inequality (2.3) in Lemma 2.5

$$\begin{aligned} J_{24} &:= -\int \partial_2 u_2(\partial_2\omega)^2\,dx \\ &= -\int \partial_1\widetilde{u_1}(\partial_2\overline{\omega}+\partial_2\widetilde{\omega})^2\,dx \\ &= -2\int \partial_1\widetilde{u_1}\partial_2\overline{\omega}\partial_2\widetilde{\omega}\,dx - 2\int \partial_1\widetilde{u_1}(\partial_2\widetilde{\omega})^2\,dx \\ &\le c\Big(\|\partial_2\overline{\omega}\|_{L^2}+\|\partial_2\widetilde{\omega}\|_{L^2}\Big)\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{\omega}\|_{L^2} \\ &\le c\|u\|_{H^2}\|\partial_2 u\|_{H^2}^2. \end{aligned} \tag{3.56}$$

Collecting the bounds for J_{21} through J_{24} obtained in (3.49), (3.50), (3.55) and (3.56), we obtain

$$J_2 \le c\|u\|_{H^2}\|\partial_2 u\|_{H^2}^2\,. \tag{3.57}$$

Inserting $J_1 = 0$, (3.46) and (3.57) in (3.10), we get

$$\begin{aligned} &\frac{d}{dt}(\|\Delta u\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2)+2\nu\|\partial_2\Delta u\|_{L^2}^2+2\eta\|\partial_1\Delta\theta\|_{L^2}^2 \\ &\le c\,\|(u,\theta)\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2\Big). \end{aligned} \tag{3.58}$$

Integrating (3.58) over the time interval $[0,t]$ yields

$$\begin{aligned} &\|\Delta u(t)\|_{L^2}^2+\|\Delta\theta(t)\|_{L^2}^2+2\nu\int_0^t\|\partial_2\Delta u\|_{L^2}^2 d\tau+2\eta\int_0^t\|\Delta\partial_1\theta\|_{L^2}^2 d\tau \\ &\le \|\Delta u_0\|_{L^2}^2+\|\Delta\theta_0\|_{L^2}^2+c\int_0^t\|(u,\theta)\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2\Big) \\ &\le E(0)+\,c\,E(t)^{\frac{3}{2}}. \end{aligned} \tag{3.59}$$

The next major step is to bound the last piece in $E(t)$ defined by (1.16), namely

$$\int_0^t \|g_0\partial_1 u_2\|_{L^2}^2\, d\tau. \tag{3.60}$$

Our strategy is to make use of the special structure of the equation for θ in (1.2) and replace $g_0\partial_1 u_2$ in (3.60) via the equation of θ,

$$g_0\partial_1 u_2 = -\partial_t\partial_1\theta - \partial_1(u\cdot\nabla\theta) + \eta\partial_{111}\theta. \tag{3.61}$$

Multiplying (3.61) with $g_0\partial_1 u_2$ and then integrating over $\mathbb{R}^2$ yields

$$\begin{aligned}\|g_0\partial_1 u_2\|_{L^2}^2 &= -g_0\int \partial_t\partial_1\theta\,\partial_1 u_2\,dx - g_0\int \partial_1 u_2\,\partial_1(u\cdot\nabla\theta)\,dx + g_0\eta\int \partial_1 u_2\,\partial_{111}\theta\,dx\\ &:= K_1 + K_2 + K_3.\end{aligned} \tag{3.62}$$

Even though the estimate of K_3 appears to be easy, the term with unfavorable derivative $\partial_1 u_2$ will be absorbed by the left-hand side,

$$|K_3| \le \eta\|g_0\partial_1 u_2\|_{L^2}\,\|\partial_{111}\theta\|_{L^2} \le \frac{1}{2}\|g_0\partial_1 u_2\|_{L^2}^2 + c\,\|\partial_1\theta\|_{H^2}^2. \tag{3.63}$$

We shift the time derivative in K_1,

$$K_1 = -g_0\frac{d}{dt}\int \partial_1\theta\,\partial_1 u_2\,dx + g_0\int \partial_1\theta\,\partial_1\partial_t u_2\,dx := K_{11} + K_{12}. \tag{3.64}$$

Invoking the equation for the second component of the velocity, we have

$$\begin{aligned}K_{12} &= -g_0\int \partial_1\partial_1\theta\,\partial_t u_2\,dx\\ &= -g_0\int \partial_{11}\theta(-(u\cdot\nabla u_2) - \partial_2 p + \nu\partial_{22}u_2 + g_0\theta)\,dx\\ &= g_0\int \partial_{11}\theta\,(u\cdot\nabla u_2)\,dx\; + g_0\int \partial_{11}\theta\;\partial_2 p\,dx\\ &\quad -g_0\nu\int \partial_{11}\theta\;\partial_{22}u_2\;dx\; - g_0^2\int \partial_{11}\theta\;\theta\;dx.\end{aligned}$$

We further replace the pressure term. Applying the divergence operator to the velocity equation yields

$$p = -\Delta^{-1}\nabla\cdot(u\cdot\nabla u) + g_0\Delta^{-1}\partial_2\theta.$$

Therefore,

$$\begin{aligned}K_{12} &= g_0\int \partial_{11}\theta\,(u\cdot\nabla u_2)\,dx\; + g_0\int \partial_{11}\theta\,(-\partial_2\Delta^{-1}\nabla\cdot(u\cdot\nabla u))\,dx\\ &\quad - g_0\nu\int \partial_{11}\theta\;\partial_{22}u_2\;dx\; - g_0^2\int \partial_{11}\theta\;\partial_{11}\Delta^{-1}\theta\;dx\\ &:= K_{121} + K_{122} + K_{123} + K_{124}.\end{aligned} \tag{3.65}$$

Due to Hölder's inequality and the fact that the double Riesz transform $\partial_{11}\Delta^{-1}$ is bounded on L^q for any $1 < q < \infty$ (see, e.g., [27]), we have

$$K_{124} := -g_0^2\int \partial_1\theta\,\partial_{11}\Delta^{-1}\partial_1\theta\;dx \le c\,\|\partial_1\theta\|_{L^2}\|\partial_{11}\Delta^{-1}\partial_1\theta\|_{L^2} \le c\,\|\partial_1\theta\|_{L^2}^2. \tag{3.66}$$

Thanks to Hölder's inequality,

$$K_{123} := -g_0\nu \int \partial_{11}\theta\, \partial_{22}u_2\, dx \le c\,\|\partial_{11}\theta\|_{L^2}\,\|\partial_{22}u_2\|_{L^2}. \tag{3.67}$$

By integration by parts, Hölder's inequality and the boundedness of the double Riesz transform,

$$\begin{aligned}
K_{122} &:= g_0 \int \partial_{11}\theta\,(-\partial_2\Delta^{-1}\nabla\cdot(u\cdot\nabla u))\, dx \\
&= g_0 \int \partial_1\theta\, \partial_{12}\Delta^{-1}\nabla\cdot(u\cdot\nabla u)\, dx \\
&\le c\|\partial_1\theta\|_{L^2}\,\|\Delta^{-1}\partial_{12}\nabla\cdot(u\cdot\nabla u)\|_{L^2} \\
&\le c\,\|\partial_1\theta\|_{L^2}\,\|\partial_2(u\cdot\nabla u)\|_{L^2} \\
&\le c\,\|\partial_1\theta\|_{L^2}\,\|\partial_2 u\cdot\nabla u + u\cdot\nabla\partial_2 u\|_{L^2} \\
&\le c\,\|\partial_1\theta\|_{L^2}\,(\,\|\partial_2 u\|_{L^4}\,\|\nabla u\|_{L^4} + \|u\|_\infty\|\nabla\partial_2 u\|_{L^2}) \\
&\le c\,\|\partial_1\theta\|_{L^2}\,\|\partial_2 u\|_{H^1}\,\|\nabla u\|_{H^1} + c\,\|\partial_1\theta\|_{L^2}\,\|u\|_{H^2}\|\nabla\partial_2 u\|_{L^2}.
\end{aligned} \tag{3.68}$$

To deal with K_{121}, we rewrite it as

$$\begin{aligned}
K_{121} &= g_0 \int \partial_{11}\theta(u_1\partial_1 u_2 + u_2\partial_2 u_2)dx \\
&= g_0 \int \partial_{11}\widetilde{\theta}\, u_1\, \partial_1 u_2\, dx + g_0 \int \partial_{11}\widetilde{\theta}\, u_2\, \partial_2 u_2\, dx \\
&= g_0 \int \partial_{11}\widetilde{\theta}\, \widetilde{u_1}\, \partial_1\widetilde{u_2}\, dx + g_0 \int \partial_{11}\widetilde{\theta}\, \overline{u_1}\, \partial_1\widetilde{u_2}\, dx + g_0 \int \partial_{11}\widetilde{\theta}\, u_2\, \partial_2 u_2\, dx \\
&:= K_{1211} + K_{1212} + K_{1213}.
\end{aligned} \tag{3.69}$$

By Lemma 2.4, the divergence-free condition of u and Lemma 2.5,

$$\begin{aligned}
K_{1211} &:= g_0 \int \partial_{11}\widetilde{\theta}\, \widetilde{u_1}\, \partial_1\widetilde{u_2}\, dx \\
&\le c\,\|\partial_{11}\theta\|_{L^2}^{\frac12}\|\partial_2\partial_{11}\theta\|_{L^2}\, \underbrace{\|\widetilde{u_1}\|_{L^2}^{\frac12}}_{\le\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}}\, \underbrace{\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}}_{=\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac12}}\|\partial_1\widetilde{u_2}\|_{L^2} \\
&\le c\,\|u\|_{H^2}\,\|\partial_2 u\|_{H^2}\,\|\partial_1\theta\|_{H^2} \\
&\le c\,\|u\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2\Big).
\end{aligned} \tag{3.70}$$

Due to Lemma 2.1, Hölder's inequality, Lemma 2.2 and then Young's inequality,

$$\begin{aligned}
K_{1212} &:= g_0 \int \partial_{11}\widetilde{\theta}\, \overline{u_1}\, \partial_1\widetilde{u_2}\, dx \\
&= g_0 \int_{\mathbb{R}} \overline{u_1}\Big(\int_{\mathbb{T}} \partial_{11}\widetilde{\theta}\partial_1\widetilde{u_2}dx_1\Big)dx_2 \\
&\le c\int_{\mathbb{R}} |\overline{u_1}|\|\partial_{11}\widetilde{\theta}\|_{L^2_{x_1}}\|\partial_1\widetilde{u_2}\|_{L^2_{x_1}}dx_2 \\
&\le c\|\overline{u_1}\|_{L^\infty_{x_2}}\|\partial_{11}\widetilde{\theta}\|_{L^2_{x_2}L^2_{x_1}}\|\partial_1\widetilde{u_2}\|_{L^2_{x_2}L^2_{x_1}}
\end{aligned}$$

$$\leq c\|\overline{u_1}\|_{H^1}\|\partial_{11}\widetilde{\theta}\|_{L^2}\|\partial_1\widetilde{u_2}\|_{L^2}$$
$$\leq c\|u\|_{H^2}\Big(\|\partial_1 u_2\|_{L^2}^2+\|\partial_1\theta\|_{H^2}^2\Big). \tag{3.71}$$

According to Lemma 2.4,

$$K_{1213}:=g_0\int \partial_{11}\widetilde{\theta}\; u_2\;\partial_2 u_2\;dx$$
$$\leq c\|\partial_{11}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{11}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\;\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|u_2\|_{L^2}$$
$$\leq c\|u\|_{H^2}\|\partial_2 u\|_{H^2}\|\partial_1\theta\|_{H^2}$$
$$\leq c\|u\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2\Big). \tag{3.72}$$

Inserting (3.70), (3.71) and (3.72) in (3.69) we get

$$K_{121}\leq c\|u\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2+\|\partial_1\theta\|_{H^2}^2\Big). \tag{3.73}$$

It then follows from (3.65), (3.66), (3.67), (3.68) and (3.73) that

$$|K_{12}|\leq\; c\|u\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2+\|\partial_1\theta\|_{H^2}^2\Big). \tag{3.74}$$

It remains to bound K_2. We first split it into four terms,

$$K_2:=-g_0\int \partial_1 u_2\,\partial_1(u\cdot\nabla\theta)\;dx$$
$$=-g_0\int \partial_1 u_2\,\partial_1 u_1\,\partial_1\theta\,dx-g_0\int \partial_1 u_2 u_1\partial_1\partial_1\theta\,dx$$
$$-\,g_0\int \partial_1 u_2\partial_1 u_2\partial_2\theta\,dx-g_0\int \partial_1 u_2 u_2\partial_1\partial_2\theta\,dx$$
$$:=K_{21}+K_{22}+K_{23}+K_{24}. \tag{3.75}$$

Due to $\partial_1\theta=\partial_1\widetilde{\theta}$, Lemma 2.4 and Young's inequality

$$K_{21}:=-g_0\int \partial_1 u_2\,\partial_1 u_1\,\partial_1\theta\,dx$$
$$=-g_0\int \partial_1 u_2\,\partial_1 u_1\,\partial_1\widetilde{\theta}\,dx$$
$$\leq c\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_1 u_2\|_{L^2}$$
$$\leq c\|u\|_{H^2}\|\partial_2 u\|_{H^2}\|\partial_1\theta\|_{H^2}$$
$$\leq c\|u\|_{H^2}\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2\Big). \tag{3.76}$$

Using Lemma 2.1 and invoking the decompositions $u=\overline{u}+\widetilde{u}$ we write K_{22} as

$$K_{22}:=-g_0\int \partial_1 u_2 u_1\partial_1\partial_1\theta\,dx$$
$$=g_0\int \partial_{11}\widetilde{\theta}\;\widetilde{u_1}\;\partial_1\widetilde{u_2}\;dx\;+g_0\int \partial_{11}\widetilde{\theta}\;\overline{u_1}\;\partial_1\widetilde{u_2}\;dx$$
$$:=K_{221}+K_{222}. \tag{3.77}$$

By Lemmas 2.4, 2.5 and the divergence-free condition of u,

$$\begin{aligned} K_{221} &:= g_0 \int \partial_{11}\widetilde{\theta}\, \widetilde{u_1}\, \partial_1 \widetilde{u_2}\, dx \\ &\le c\, \|\partial_{11}\theta\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_{11}\theta\|_{L^2} \underbrace{\|\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}}_{\le \|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}} \underbrace{\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}}_{=\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}} \|\partial_1\widetilde{u_2}\|_{L^2} \\ &\le c\, \|u\|_{H^2}\, \|\partial_2 u\|_{H^2}\, \|\partial_1\theta\|_{H^2} \\ &\le c\, \|u\|_{H^2} \Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2} \Big). \end{aligned} \tag{3.78}$$

To bound K_{222}, we first use Lemma 2.1, Hölder's inequality and then Lemma 2.2 to obtain

$$\begin{aligned} K_{222} &:= g_0 \int \partial_{11}\widetilde{\theta}\, \overline{u_1}\, \partial_1 \widetilde{u_2}\, dx \\ &= g_0 \int_{\mathbb{R}} \overline{u_1} \Big(\int_{\mathbb{T}} \partial_{11}\widetilde{\theta} \partial_1 \widetilde{u_2} dx_1 \Big) dx_2 \\ &\le c \int_{\mathbb{R}} |\overline{u_1}| \|\partial_{11}\widetilde{\theta}\|_{L^2_{x_1}} \|\partial_1 \widetilde{u_2}\|_{L^2_{x_1}} dx_2 \\ &\le c \|\overline{u_1}\|_{L^\infty_{x_2}} \|\partial_{11}\widetilde{\theta}\|_{L^2_{x_2}L^2_{x_1}} \|\partial_1\widetilde{u_2}\|_{L^2_{x_2}L^2_{x_1}} \\ &\le c \|\overline{u_1}\|_{H^1} \|\partial_{11}\widetilde{\theta}\|_{L^2} \|\partial_1\widetilde{u_2}\|_{L^2} \\ &\le c \|u\|_{H^2} \Big(\|\partial_1 u_2\|_{L^2}^2 + \|\partial_1\theta\|_{H^2}^2 \Big). \end{aligned} \tag{3.79}$$

Then (3.78), (3.79) and (3.77) together leads to

$$K_{22} \le c\|u\|_{H^2} \Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2 + \|\partial_1\theta\|_{H^2}^2 \Big). \tag{3.80}$$

By $\partial_1 u_2 = \partial_1 \widetilde{u_2}$ and $\theta = \widetilde{\theta} + \overline{\theta}$, we rewrite K_{23} as

$$\begin{aligned} K_{23} &:= -g_0 \int \partial_1 u_2 \partial_1 u_2 \partial_2 \theta\, dx \\ &= -g_0 \int \partial_1 \widetilde{u_2} \partial_1 \widetilde{u_2} \partial_2 \overline{\theta}\, dx - g_0 \int \partial_1 \widetilde{u_2} \partial_1 \widetilde{u_2} \partial_2 \widetilde{\theta}\, dx \\ &:= K_{231} + K_{232}. \end{aligned} \tag{3.81}$$

To estimate K_{231}, we make use of Lemma 2.1, Hölder's inequality and then Lemma 2.2 to get

$$\begin{aligned} K_{231} &:= g_0 \int \partial_1 \widetilde{u_2} \partial_1 \widetilde{u_2} \partial_2 \overline{\theta}\, dx \\ &= g_0 \int_{\mathbb{R}} \partial_2 \overline{\theta} \Big(\int_{\mathbb{T}} \partial_1 \widetilde{u_2} \partial_1 \widetilde{u_2} dx_1 \Big) dx_2 \\ &\le c \int_{\mathbb{R}} |\partial_2 \overline{\theta}| \|\partial_1 \widetilde{u_2}\|_{L^2_{x_1}} \|\partial_1 \widetilde{u_2}\|_{L^2_{x_1}} dx_2 \\ &\le c \|\partial_2 \overline{\theta}\|_{L^\infty_{x_2}} \|\partial_1 \widetilde{u_2}\|_{L^2_{x_2}L^2_{x_1}} \|\partial_1 \widetilde{u_2}\|_{L^2_{x_2}L^2_{x_1}} \end{aligned}$$

$$\leq c\|\partial_2\overline{\theta}\|_{H^1}\|\partial_1\widetilde{u_2}\|_{L^2}^2$$
$$\leq c\|\theta\|_{H^2}\|\partial_1 u_2\|_{L^2}^2. \tag{3.82}$$

Via Lemma 2.4,

$$\begin{aligned}K_{232} &:= g_0\int \partial_1\widetilde{u_2}\partial_1\widetilde{u_2}\partial_2\widetilde{\theta}\,dx\\ &\leq c\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_1 u_2\|_{L^2}\\ &\leq c\|\theta\|_{H^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|\partial_1\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1 u_2\|_{L^2}^{\frac{3}{2}}\\ &\leq c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2\Big).\end{aligned} \tag{3.83}$$

Inserting the bounds for K_{231} and K_{232} in (3.81), we find

$$K_{23}\leq c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2\Big). \tag{3.84}$$

The last term K_{24} can also be bounded due to the fact that $\overline{u_2}=0$, Lemmas 2.4 and 2.5

$$\begin{aligned}K_{24} &:= -g_0\int \partial_1 u_2 u_2\partial_1\partial_2\theta\,dx\\ &= -g_0\int \partial_1 u_2\widetilde{u_2}\partial_1\partial_2\theta\,dx\\ &\leq c\,\underbrace{\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}}_{\leq\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}}\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\theta\|_{L^2}\\ &\leq c\|\theta\|_{H^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|\partial_1\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1 u_2\|_{L^2}^{\frac{3}{2}}\\ &\leq c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2\Big).\end{aligned} \tag{3.85}$$

Inserting (3.76), (3.80), (3.84), (3.85), in (3.75) we obtain

$$K_2\leq c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2+\|\partial_2 u\|_{H^2}^2\Big). \tag{3.86}$$

Combining the bounds obtained above for K_1 through K_3 in (3.63), (3.64), (3.74) and (3.86) and inserting them in (3.62), we find

$$\begin{aligned}\frac{1}{2}\,\|g_0\partial_1 u_2\|_{L^2}^2 \leq\; & c\,\|\partial_1\theta\|_{H^2}^2 - g_0\frac{d}{dt}\int \partial_1\theta\,\partial_1 u_2\,dx\\ &+ c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2\Big).\end{aligned} \tag{3.87}$$

Integrating (3.87) over $[0,t]$ yields

$$\begin{aligned}\int_0^t \|g_0\partial_1 u_2\|_{L^2}^2\,d\tau \leq\; & c\int_0^t \|\partial_1\theta\|_{H^2}^2\,d\tau - 2g_0\int \partial_1\theta\,\partial_1 u_2\,dx + 2g_0\int \partial_1\theta_0\,\partial_1 u_{02}\,dx\\ &+ c\int_0^t \|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2+\|\partial_1 u_2\|_{L^2}^2\Big)\,d\tau\end{aligned}$$

$$\leq c \int_0^t \|\partial_1\theta\|_{H^2}^2 \, d\tau + c \int_0^t \|\partial_2 u\|_{H^2}^2 \, d\tau + c\,(\|u\|_{H^1}^2 + \|\theta\|_{H^1}^2) + c\,(\|u_0\|_{H^1}^2 + \|\theta_0\|_{H^1}^2) + c\,E(t)^{\frac{3}{2}}. \tag{3.88}$$

We then combine the H^1-bound in (3.9), the homogeneous H^2-bound in (3.59) and the bound for the extra regularization term in (3.88). When doing so, we need eliminate the quadratic terms on the right-hand side of (3.88) by the corresponding terms on the left-hand side, then it suffices to multiply both sides of (3.88) by a suitable small coefficient $\delta > 0$. Taking (3.9) + (3.59) + δ (3.88) gives

$$\begin{aligned} &\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 + 2\nu \int_0^t \|\partial_2 u\|_{H^2}^2 d\tau + 2\eta \int_0^t \|\partial_1\theta\|_{H^2}^2 d\tau + \delta \int_0^t \|g_0 \partial_1 u_2\|_{L^2}^2 \\ &\leq E(0) + c\,E(t)^{\frac{3}{2}} + c\,\delta\,(\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2) + c\,\delta\,(\|u_0\|_{H^2}^2 + \|\theta_0\|_{H^2}^2) \\ &\quad + c\,\delta \int_0^t \|\partial_2 u\|_{H^2}^2 d\tau + c\,\delta \int_0^t \|\partial_1\theta\|_{H^2}^2 d\tau + c\,\delta\,E(t)^{\frac{3}{2}}. \end{aligned} \tag{3.89}$$

If $\delta > 0$ is chosen to be sufficiently small, say

$$c\,\delta \leq \frac{1}{2}, \quad c\,\delta \leq \nu, \quad c\,\delta \leq \eta,$$

then (3.89) is reduced to

$$E(t) \leq C_1\,E(0) + C_2\,E(t)^{\frac{3}{2}}, \tag{3.90}$$

where C_1 and C_2 are positive constants. The proof of the desired stability result, is then completed by applying the bootstrapping argument on (3.90). In fact, if the initial data (u_0, θ_0), is sufficiently small, say,

$$E(0) = \|(u_0, \theta_0)\|_{H^2}^2 \leq \varepsilon^2 := \frac{1}{16 C_1 C_2^2}, \tag{3.91}$$

then (3.90) allows us to show that

$$\|(u(t), \theta(t))\|_{H^2}^2 \leq 2C_1\,\varepsilon^2.$$

To initiate the bootstrapping argument, we make the ansatz that, for $t \leq T$

$$E(t) \leq \frac{1}{4C_2^2}, \tag{3.92}$$

and we then show that $E(t)$ actually admits an even smaller bound by taking the initial H^2-norm $E(0)$ sufficiently small. In fact, Inserting (3.92) in (3.90) yields

$$\begin{aligned} E(t) &\leq C_1\,E(0) + C_2\,E(t)^{\frac{3}{2}} \\ &\leq C_1\,\varepsilon^2 + C_2\,\frac{1}{2C_2}\,E(t). \end{aligned}$$

That is,

$$\frac{1}{2}E(t) \leq C_1\,\varepsilon^2 \quad \text{or} \quad E(t) \leq 2\,C_1\,\frac{1}{16 C_1 C_2^2} = \frac{1}{8C_2^2} = 2C_1\,\epsilon^2, \quad \text{for all } t \leq T.$$

The bootstrapping argument then assesses that (3.92) holds for all time when $E(0)$ satisfies (3.91). This establishes the global stability.

Finally we establish the uniqueness of H^2-solutions to (1.2). Assume that $(u^{(1)}, p^{(1)}, \theta^{(1)})$ and $(u^{(2)}, p^{(2)}, \theta^{(2)})$ are two solutions of (1.2) with one of them in the H^2-regularity class say $(u^{(1)}, \theta^{(1)}) \in L^\infty(0, T; H^2)$. The difference between the two solutions (u^*, p^*, θ^*) with

$$u^* = u^{(2)} - u^{(1)}, \quad p^* = p^{(2)} - p^{(1)} \quad \text{and} \quad \theta^* = \theta^{(2)} - \theta^{(1)}$$

satisfies

$$\begin{cases} \partial_t u^* + u^{(2)} \cdot \nabla u^* + u^* \cdot \nabla u^{(1)} + \nabla p^* = \nu \partial_{22} u^* + g_0 \theta^* \mathbf{e}_2, \\ \partial_t \theta^* + u^{(2)} \cdot \nabla \theta^* + u^* \cdot \nabla \theta^{(1)} + g_0 {u_2}^* = \eta \partial_{11} \theta^*, \\ \nabla \cdot u^* = 0, \\ u^*(x, 0) = 0, \quad \theta^*(x, 0) = 0. \end{cases} \tag{3.93}$$

We estimate the difference (u^*, p^*, θ^*) in $L^2(\mathbb{R}^2)$. Taking the L^2-inner product of (3.93) with (u^*, θ^*) and applying the divergence-free condition, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|(u^*, \theta^*)\|_{L^2}^2 + \nu\|\partial_2 u^*\|_{L^2}^2 + \eta\|\partial_1 \theta^*\|_{L^2}^2 &= -\int u^* \cdot \nabla u^{(1)} \cdot u^* \, dx - \int u^* \cdot \nabla \theta^{(1)} \cdot \theta^* \, dx \\ &= I_1 + I_2. \end{aligned} \tag{3.94}$$

By Lemma 2.3 and the uniformly global bound for $\|u^{(1)}\|_{H^2}$,

$$\begin{aligned} I_1 &:= -\int u^* \cdot \nabla u^{(1)} \cdot u^* \, dx \\ &\le c \underbrace{\|\nabla u^{(1)}\|_{L^2}^{\frac{1}{2}} \Big(\|\nabla u^{(1)}\|_{L^2} + \|\partial_1 \nabla u^{(1)}\|_{L^2} \Big)^{\frac{1}{2}}}_{\le c} \|u^*\|_{L^2}^{\frac{1}{2}} \|\partial_2 u^*\|_{L^2}^{\frac{1}{2}} \|u^*\|_{L^2} \\ &\le c\|u^*\|_{L^2}^{\frac{3}{2}} \|\partial_2 u^*\|_{L^2}^{\frac{1}{2}} \\ &\le c\|u^*\|_{L^2}^2 + \frac{\nu}{4}\|\partial_2 u^*\|_{L^2}^2. \end{aligned} \tag{3.95}$$

By Lemma 2.3 and the uniformly global bound for $\|\theta^{(1)}\|_{H^2}$,

$$\begin{aligned} I_2 &:= -\int u^* \cdot \nabla \theta^{(1)} \cdot \theta^* \, dx \\ &\le c \underbrace{\|\nabla \theta^{(1)}\|_{L^2}^{\frac{1}{2}} \Big(\|\nabla \theta^{(1)}\|_{L^2} + \|\partial_1 \nabla \theta^{(1)}\|_{L^2} \Big)^{\frac{1}{2}}}_{\le c} \|u^*\|_{L^2}^{\frac{1}{2}} \|\partial_2 u^*\|_{L^2}^{\frac{1}{2}} \|\theta^*\|_{L^2} \\ &\le c\|u^*\|_{L^2}^{\frac{1}{2}} \|\partial_2 u^*\|_{L^2}^{\frac{1}{2}} \|\theta^*\|_{L^2} \\ &\le c\|\theta^*\|_{L^2} \Big(\|u^*\|_{L^2} + \|\partial_2 u^*\|_{L^2} \Big) \\ &\le c\|\theta^*\|_{L^2}^2 + c\|u^*\|_{L^2}^2 + \frac{\nu}{4}\|\partial_2 u^*\|_{L^2}^2. \end{aligned} \tag{3.96}$$

Putting the estimates (3.95) and (3.96) in (3.94) leads to

$$\frac{1}{2}\frac{d}{dt}\|(u^*, \theta^*)\|_{L^2}^2 + \nu\|\partial_2 u^*\|_{L^2}^2 + \eta\|\partial_1 \theta^*\|_{L^2}^2$$

$$\leq c\Big(\|u^*\|_{L^2}^2+\|\theta^*\|_{L^2}^2\Big)+\frac{\nu}{2}\|\partial_2 u^*\|_{L^2}^2$$

or

$$\frac{d}{dt}\|(u^*,\theta^*)\|_{L^2}^2+\nu\|\partial_2 u^*\|_{L^2}^2+\eta\|\partial_1\theta^*\|_{L^2}^2\leq c\|(u^*,\theta^*)\|_{L^2}^2. \tag{3.97}$$

Grönwall's inequality then implies,

$$\|u^*(t)\|_{L^2}=\|\theta^*(t)\|_{L^2}=0.$$

That is, these two solutions coincide. This completes the proof of Theorem 1.1. □

4. Decay Rates Result

This section is devoted to the proof the decay rates presented in Theorem 1.2.

Proof of Theorem 1.2. We first write the equations of $(\overline{u},\overline{\theta})$. Taking the average of (1.2),

$$\begin{cases}\partial_t\overline{u}+\overline{u\cdot\nabla\widetilde{u}}+\begin{pmatrix}0\\ \partial_2\overline{p}\end{pmatrix}=g_0\begin{pmatrix}0\\ \overline{\theta}\end{pmatrix}+\nu\partial_2^2\overline{u}\,,\\ \partial_t\overline{\theta}+\overline{u\cdot\nabla\widetilde{\theta}}=0,\end{cases} \tag{4.1}$$

where g_0 is a negative constant. Taking the difference of (1.2) and (4.1), we find

$$\begin{cases}\partial_t\widetilde{u}+\widetilde{u\cdot\nabla\widetilde{u}}+\widetilde{u_2}\partial_2\overline{u}-\nu\partial_2^2\widetilde{u}+\nabla\widetilde{p}=g_0\widetilde{\theta}e_2\,,\\ \partial_t\widetilde{\theta}+\widetilde{u\cdot\nabla\widetilde{\theta}}+\widetilde{u_2}\partial_2\overline{\theta}-\eta\partial_1^2\widetilde{\theta}+g_0\widetilde{u_2}=0.\end{cases} \tag{4.2}$$

Taking the L^2-inner product of $(\widetilde{u},\widetilde{\theta})$ with (4.2) yields,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\Big)+\nu\|\partial_2\widetilde{u}\|_{L^2}^2+\eta\|\partial_1\widetilde{\theta}\|_{L^2}^2\\ &=-\int\widetilde{u\cdot\nabla\widetilde{u}}\cdot\widetilde{u}dx-\int\widetilde{u_2}\partial_2\overline{u}\cdot\widetilde{u}dx-\int\widetilde{u\cdot\nabla\widetilde{\theta}}\cdot\widetilde{\theta}dx-\int\widetilde{u_2}\partial_2\overline{\theta}\cdot\widetilde{\theta}dx\\ &:=A_1+A_2+A_3+A_4.\end{aligned} \tag{4.3}$$

Now, we estimate A_1 through A_4. The first term A_1 is clearly zero due to $\nabla\cdot u=0$ and Lemma 2.1,

$$A_1:=-\int\widetilde{u\cdot\nabla\widetilde{u}}\cdot\widetilde{u}dx=\underbrace{-\int u\cdot\nabla\widetilde{u}\cdot\widetilde{u}dx}_{=0}+\underbrace{\int\overline{u\cdot\nabla\widetilde{u}}\cdot\widetilde{u}dx}_{=0}=0. \tag{4.4}$$

Similarly,

$$A_3:=\int\widetilde{u\cdot\nabla\widetilde{\theta}}\cdot\widetilde{\theta}dx=0. \tag{4.5}$$

To bound A_2 we first write it as,

$$\begin{aligned}A_2&:=-\int\widetilde{u_2}\partial_2\overline{u}\cdot\widetilde{u}dx\\ &:=-\int\widetilde{u_2}\partial_2\overline{u_1}\widetilde{u_1}dx-\int\widetilde{u_2}\partial_2\overline{u_2}\widetilde{u_2}dx\end{aligned}$$

$$:= A_{21} + A_{22}. \tag{4.6}$$

Due to the fact that $\overline{\widetilde{u_2}} = 0$ we have,

$$A_{22} = -\int \widetilde{u_2}\partial_2\overline{u_2}\widetilde{u_2}dx = 0. \tag{4.7}$$

Applying Lemmas 2.4 and 2.5, the divergence-free condition of u and then Young's inequality leads to

$$\begin{aligned}
A_{21} &:= -\int \widetilde{u_2}\partial_2\overline{u_1}\widetilde{u_1}dx \\
&\leq c\|\partial_2\overline{u}\|_{L^2}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \underbrace{\|\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}}_{\leq\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}=\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}} \underbrace{\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}}_{=\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}} \\
&\leq c\|u\|_{H^2}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u}\|_{L^2}^{\frac{3}{2}} \\
&\leq c\|u\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2 + \|\partial_2\widetilde{u}\|_{L^2}^2\Big).
\end{aligned} \tag{4.8}$$

Inserting (4.7) and (4.8) in (4.6) we get

$$A_2 \leq c\|u\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2 + \|\partial_2\widetilde{u}\|_{L^2}^2\Big). \tag{4.9}$$

The last term A_4 can be bounded via Lemma 2.1, Hölder's inequality, and Lemmas 2.2 and 2.5,

$$\begin{aligned}
A_4 &:= -\int \widetilde{u_2}\partial_2\overline{\theta}\cdot\widetilde{\theta}dx \\
&= -\int_{\mathbb{R}} \partial_2\overline{\theta}\Big(\int_{\mathbb{T}} \widetilde{\theta}\widetilde{u_2}dx_1\Big)dx_2 \\
&\leq c\|\partial_2\overline{\theta}\|_{L^\infty_{x_2}}\|\widetilde{u_2}\|_{L^2}\|\widetilde{\theta}\|_{L^2} \\
&\leq c\|\partial_2\overline{\theta}\|_{H^1}\|\widetilde{u_2}\|_{L^2}\|\partial_1\widetilde{\theta}\|_{L^2} \\
&\leq c\|\theta\|_{H^2}\|\widetilde{u_2}\|_{L^2}\|\partial_1\widetilde{\theta}\|_{L^2} \\
&\leq c\|\theta\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2 + \|\partial_1\widetilde{\theta}\|_{L^2}^2\Big).
\end{aligned} \tag{4.10}$$

Collecting the estimates obtained above for A_1 through A_4 leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\widetilde{u}\|_{L^2}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big) + \nu\|\partial_2\widetilde{u}\|_{L^2}^2 + \eta\|\partial_1\widetilde{\theta}\|_{L^2}^2 \\
&\qquad\leq c\|(u,\theta)\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2 + \|\partial_2\widetilde{u}\|_{L^2}^2 + \|\partial_1\widetilde{\theta}\|_{L^2}^2\Big).
\end{aligned} \tag{4.11}$$

Applying ∇ to (4.2) yields

$$\begin{cases}
\partial_t\nabla\widetilde{u} + \nabla\widetilde{(u\cdot\nabla u)} + \nabla(\widetilde{u_2}\partial_2\overline{u}) - \nu\partial_2^2\nabla\widetilde{u} + \nabla\nabla\widetilde{p} = g_0\nabla(\widetilde{\theta}e_2)\,, \\
\partial_t\nabla\widetilde{\theta} + \nabla\widetilde{(u\cdot\nabla\theta)} + \nabla(\widetilde{u_2}\partial_2\overline{\theta}) - \eta\partial_1^2\nabla\widetilde{\theta} + g_0\nabla\widetilde{u_2} = 0.
\end{cases} \tag{4.12}$$

Dotting (4.12) by $(\nabla\widetilde{u}, \nabla\widetilde{\theta})$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla\widetilde{u}(t)\|_{L^2}^2 + \|\nabla\widetilde{\theta}(t)\|_{L^2}^2\Big) + \nu\|\partial_2\nabla\widetilde{u}\|_{L^2}^2 + \eta\|\partial_1\nabla\widetilde{\theta}\|_{L^2}^2 \\
&= -\int \nabla(\widetilde{u\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{u}dx - \int \nabla(\widetilde{u}_2\partial_2\overline{u})\cdot\nabla\widetilde{u}dx \\
&\quad - \int \nabla(\widetilde{u\cdot\nabla\widetilde{\theta}})\cdot\nabla\widetilde{\theta}dx - \int \nabla(\widetilde{u}_2\partial_2\overline{\theta})\cdot\nabla\widetilde{\theta}dx \\
&:= B_1 + B_2 + B_3 + B_4. \qquad (4.13)
\end{aligned}$$

The terms B_1 through B_4 can be bounded as follows. We start with B_1. According to Lemma 2.1, we write B_1 explicitly into four pieces,

$$\begin{aligned}
B_1 &:= -\int \nabla(\widetilde{u\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{u}dx \\
&= -\int \nabla(u\cdot\nabla\widetilde{u})\cdot\nabla\widetilde{u}\,dx + \underbrace{\int \nabla(\overline{u\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{u}dx}_{=0} \\
&= -\int \partial_1u_1\partial_1\widetilde{u}\cdot\partial_1\widetilde{u}dx - \int \partial_1u_2\partial_2\widetilde{u}\cdot\partial_1\widetilde{u}dx \\
&\quad - \int \partial_2u_1\partial_1\widetilde{u}\cdot\partial_2\widetilde{u}dx - \int \partial_2u_2\partial_2\widetilde{u}\cdot\partial_2\widetilde{u}dx \\
&:= B_{11} + B_{12} + B_{13} + B_{14}. \qquad (4.14)
\end{aligned}$$

We write the first term B_{11} as

$$\begin{aligned}
B_{11} &:= -\int \partial_1u_1\partial_1\widetilde{u}\cdot\partial_1\widetilde{u}dx \\
&= -\int \partial_1\widetilde{u_1}\partial_1\widetilde{u_1}\partial_1\widetilde{u_1}dx - \int \partial_1\widetilde{u_1}\partial_1\widetilde{u_2}\partial_1\widetilde{u_2}dx \\
&:= B_{111} + B_{112}. \qquad (4.15)
\end{aligned}$$

By the divergence-free condition of u and Lemma 2.4

$$\begin{aligned}
B_{111} &:= -\int \partial_1\widetilde{u_1}\partial_1\widetilde{u_1}\partial_1\widetilde{u_1}dx \\
&= \int \partial_2\widetilde{u_2}\partial_2\widetilde{u_2}\partial_2\widetilde{u_2}dx \\
&\le c\|\partial_2\widetilde{u_2}\|_{L^2}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \\
&\le c\|u\|_{H^2}\|\partial_2\widetilde{u}\|_{H^1}^2. \qquad (4.16)
\end{aligned}$$

Due to $\nabla\cdot u = 0$, integration by parts and Lemma, 2.4

$$\begin{aligned}
B_{112} &:= -\int \partial_1\widetilde{u_1}\partial_1\widetilde{u_2}\partial_1\widetilde{u_2}dx \\
&= \int \partial_2\widetilde{u_2}\partial_1\widetilde{u_2}\partial_1\widetilde{u_2}dx
\end{aligned}$$

$$
\begin{aligned}
&= 2\int \widetilde{u_2}\partial_2\partial_1\widetilde{u_2}\partial_1\widetilde{u_2}dx\\
&\leq c\|\partial_2\partial_1\widetilde{u_2}\|_{L^2}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\\
&\leq c\|u\|_{H^2}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u}\|_{H^1}^{\frac{3}{2}}\\
&\leq c\|u\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2+\|\partial_2\widetilde{u}\|_{H^1}^2\Big). \qquad (4.17)
\end{aligned}
$$

Inserting the upper bound for B_{111} and B_{112} in (4.15) we get

$$B_{11}\leq c\|u\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2+\|\partial_2\widetilde{u}\|_{H^1}^2\Big). \qquad (4.18)$$

To deal with B_{12}, we write it first as

$$
\begin{aligned}
B_{12} &:= -\int \partial_1 u_2\partial_2\widetilde{u}\cdot\partial_1\widetilde{u}dx\\
&= -\int \partial_1\widetilde{u_2}\partial_2\widetilde{u_1}\partial_1\widetilde{u_1}dx-\int \partial_1\widetilde{u_2}\partial_2\widetilde{u_2}\partial_1\widetilde{u_2}dx\\
&:= B_{121}+B_{122}. \qquad (4.19)
\end{aligned}
$$

For B_{121}, we use the divergence-free condition of u and Lemma 2.4

$$
\begin{aligned}
B_{121} &:= -\int \partial_1\widetilde{u_2}\partial_2\widetilde{u_1}\partial_1\widetilde{u_1}dx=\int \partial_1\widetilde{u_2}\partial_2\widetilde{u_1}\partial_2\widetilde{u_2}dx\\
&\leq c\|\partial_1\widetilde{u_2}\|_{L^2}\|\partial_2\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\\
&\leq c\|u\|_{H^2}\|\partial_2\widetilde{u}\|_{H^1}^2. \qquad (4.20)
\end{aligned}
$$

The second piece B_{122} can be bounded using integrating by parts, Lemma 2.4 and then Young's inequality

$$
\begin{aligned}
B_{122} &:= -\int \partial_1\widetilde{u_2}\partial_2\widetilde{u_2}\partial_1\widetilde{u_2}dx\\
&= 2\int \partial_2\partial_1\widetilde{u_2}\widetilde{u_2}\partial_1\widetilde{u_2}dx\\
&\leq c\|\partial_2\partial_1\widetilde{u_2}\|_{L^2}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\\
&\leq c\|u\|_{H^2}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u}\|_{H^1}^{\frac{3}{2}}\\
&\leq c\|u\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2+\|\partial_2\widetilde{u}\|_{H^1}^2\Big) \qquad (4.21)
\end{aligned}
$$

Combining (4.20) and (4.21) and inserting them in (4.19) we obtain

$$B_{12}\leq c\|u\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2+\|\partial_2\widetilde{u}\|_{H^1}^2\Big). \qquad (4.22)$$

The term B_{13} is naturally divided into two integrals,

$$
\begin{aligned}
B_{13} &:= -\int \partial_2 u_1\partial_1\widetilde{u}\cdot\partial_2\widetilde{u}dx\\
&= -\int \partial_2 u_1\partial_1\widetilde{u_1}\partial_2\widetilde{u_1}dx-\int \partial_2 u_1\partial_1\widetilde{u_2}\partial_2\widetilde{u_2}dx
\end{aligned}
$$

$$:= B_{131} + B_{132}. \tag{4.23}$$

Due to $\nabla \cdot u = 0$ and Lemma 2.4,

$$\begin{aligned} B_{131} &:= -\int \partial_2 u_1 \partial_1 \widetilde{u_1} \partial_2 \widetilde{u_1} dx \\ &= \int \partial_2 u_1 \partial_2 \widetilde{u_2} \partial_2 \widetilde{u_1} dx \\ &\le c\|\partial_2 u_1\|_{L^2} \|\partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \\ &\le c\|u\|_{H^2} \|\partial_2 \widetilde{u}\|_{H^1}^2. \end{aligned} \tag{4.24}$$

Integrating by parts, making use of Lemma 2.4 and then Young's inequality

$$\begin{aligned} B_{132} &:= -\int \partial_2 u_1 \partial_1 \widetilde{u_2} \partial_2 \widetilde{u_2} dx \\ &= \int \partial_1 \partial_2 \widetilde{u_1} \widetilde{u_2} \partial_2 \widetilde{u_2} dx + \int \partial_2 u_1 \widetilde{u_2} \partial_1 \partial_2 \widetilde{u_2} dx \\ &\le c\|\partial_1 \partial_2 \widetilde{u_1}\|_{L^2} \|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \\ &\quad + c\|\partial_2 u_1\|_{L^2} \|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_1 \partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \\ &\le c\|u\|_{H^2} \|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u}\|_{H^1}^{\frac{3}{2}} \\ &\le c\|u\|_{H^2} \Big(\|\widetilde{u_2}\|_{L^2}^2 + \|\partial_2 \widetilde{u}\|_{H^1}^2 \Big). \end{aligned} \tag{4.25}$$

Inserting the estimates (4.24) and (4.25) in (4.23) we get

$$B_{13} \le c\|u\|_{H^2} \Big(\|\widetilde{u_2}\|_{L^2}^2 + \|\partial_2 \widetilde{u}\|_{H^1}^2 \Big). \tag{4.26}$$

The last term B_{14} can be bounded directly via Lemma 2.4,

$$\begin{aligned} B_{14} &:= -\int \partial_2 u_2 \partial_2 \widetilde{u} \cdot \partial_2 \widetilde{u} dx \\ &\le c\|\partial_2 u_2\|_{L^2} \|\partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\le c\|u\|_{H^2} \|\partial_2 \widetilde{u}\|_{H^1}^2. \end{aligned} \tag{4.27}$$

In view (4.14), collecting the upper bounds in (4.18), (4.22), (4.26) and (4.27) gives

$$B_1 \le c\|u\|_{H^2} \Big(\|\widetilde{u_2}\|_{L^2}^2 + \|\partial_2 \widetilde{u}\|_{H^1}^2 \Big). \tag{4.28}$$

The next term B_2 is naturally split into four parts,

$$\begin{aligned} B_2 &:= -\int \nabla(\widetilde{u_2} \partial_2 \overline{u}) \cdot \nabla \widetilde{u} dx \\ &= -\int \partial_1 \widetilde{u_2} \partial_2 \overline{u} \cdot \partial_1 \widetilde{u} dx - \int \partial_2 \widetilde{u_2} \partial_2 \overline{u} \cdot \partial_2 \widetilde{u} dx \\ &\quad - \int \widetilde{u_2} \partial_1 \partial_2 \overline{u} \cdot \partial_1 \widetilde{u} dx - \int \widetilde{u_2} \partial_2 \partial_2 \overline{u} \cdot \partial_2 \widetilde{u} dx \\ &:= B_{21} + B_{22} + B_{23} + B_{24}. \end{aligned} \tag{4.29}$$

We rewrite B_{21} as,

$$\begin{aligned} B_{21} &:= -\int \partial_1 \widetilde{u_2} \partial_2 \overline{u} \cdot \partial_1 \widetilde{u} dx \\ &= -\int \partial_1 \widetilde{u_2} \partial_2 \overline{u_1} \partial_1 \widetilde{u_1} dx - \int \partial_1 \widetilde{u_2} \partial_2 \overline{u_2} \partial_1 \widetilde{u_2} dx \\ &:= B_{211} + B_{212}. \end{aligned} \tag{4.30}$$

Clearly, due to $\overline{u_2} = 0$,

$$B_{212} := -\int \partial_1 \widetilde{u_2} \partial_2 \overline{u_2} \partial_1 \widetilde{u_2} dx = 0. \tag{4.31}$$

By the divergence-free condition of u, integration by parts, Lemma 2.1, Hölder's inequality and then Lemma 2.2

$$\begin{aligned} B_{211} &:= -\int \partial_1 \widetilde{u_2} \partial_2 \overline{u_1} \partial_1 \widetilde{u_1} dx \\ &= \int \partial_1 \widetilde{u_2} \partial_2 \overline{u_1} \partial_2 \widetilde{u_2} dx \\ &= \int \widetilde{u_2} \partial_2 \overline{u_1} \partial_1 \partial_2 \widetilde{u_2} dx \\ &= \int_{\mathbb{R}} \partial_2 \overline{u_1} \Big(\int_{\mathbb{T}} \widetilde{u_2} \partial_1 \partial_2 \widetilde{u_2} dx_1 \Big) dx_2 \\ &\leq c \|\partial_2 \overline{u_1}\|_{L^\infty_{x_2}} \|\widetilde{u_2}\|_{L^2} \|\partial_1 \partial_2 \widetilde{u_2}\|_{L^2} \\ &\leq c \|\partial_2 \overline{u_1}\|_{H^1} \|\widetilde{u_2}\|_{L^2} \|\partial_1 \partial_2 \widetilde{u_2}\|_{L^2} \\ &\leq c \|u\|_{H^2} \|\widetilde{u_2}\|_{L^2} \|\partial_2 \widetilde{u}\|_{H^1} \\ &\leq c \|\theta\|_{H^2} \Big(\|\widetilde{u_2}\|^2_{L^2} + \|\partial_2 \widetilde{u}\|^2_{H^1} \Big). \end{aligned} \tag{4.32}$$

It then follows from (4.31), (4.32) and (4.30) that

$$B_{21} \leq c \|u\|_{H^2} \Big(\|\partial_2 \widetilde{u}\|^2_{H^1} + \|\widetilde{u_2}\|^2_{L^2} \Big). \tag{4.33}$$

According to Lemma 2.4,

$$\begin{aligned} B_{22} &:= -\int \partial_2 \widetilde{u_2} \partial_2 \overline{u} \cdot \partial_2 \widetilde{u} dx \\ &\leq c \|\partial_2 \overline{u}\|_{L^2} \|\partial_2 \widetilde{u_2}\|^{\frac{1}{2}}_{L^2} \|\partial_1 \partial_2 \widetilde{u_2}\|^{\frac{1}{2}}_{L^2} \|\partial_2 \widetilde{u}\|^{\frac{1}{2}}_{L^2} \|\partial_2 \partial_2 \widetilde{u}\|^{\frac{1}{2}}_{L^2} \\ &\leq c \|u\|_{H^2} \|\partial_2 \widetilde{u}\|^2_{H^1}. \end{aligned} \tag{4.34}$$

Due to the definition of $\overline{u}$,

$$B_{23} := -\int \widetilde{u_2} \partial_1 \partial_2 \overline{u} \cdot \partial_1 \widetilde{u} dx = 0. \tag{4.35}$$

By Lemma 2.4 and Young's inequality,

$$B_{24} := -\int \widetilde{u_2} \partial_2 \partial_2 \overline{u} \cdot \partial_2 \widetilde{u} dx$$

$$\le c\|\partial_2\partial_2\overline{u}\|_{L^2}\|\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_2\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{u}\|_{L^2}^{\frac12}$$
$$\le c\|u\|_{H^2}\|\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_2\widetilde{u}\|_{H^1}^{\frac32}$$
$$\le c\|u\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2+\|\partial_2\widetilde{u}\|_{H^1}^2\Big). \tag{4.36}$$

Combining (4.33), (4.34), (4.35), (4.36) and (4.29), we obtain

$$B_2 \le c\|u\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2+\|\partial_2\widetilde{u}\|_{H^1}^2\Big). \tag{4.37}$$

To bound B_3, we first write $u=\overline{u}+\widetilde{u}$ and use Lemma 2.1

$$\begin{aligned}
B_3 &:= -\int \widetilde{\nabla(u\cdot\nabla\widetilde{\theta})}\cdot\nabla\widetilde{\theta}dx\\
&= -\int \nabla(u\cdot\nabla\widetilde{\theta})\cdot\nabla\widetilde{\theta}dx+\underbrace{\int\overline{\nabla(u\cdot\nabla\widetilde{\theta})}\cdot\nabla\widetilde{\theta}dx}_{=0}\\
&= -\int\partial_1\widetilde{\theta}\partial_1\widetilde{u_1}\partial_1\widetilde{\theta}dx-\int\partial_2\widetilde{\theta}\partial_1\widetilde{u_2}\partial_1\widetilde{\theta}dx\\
&\quad -\int\partial_1\widetilde{\theta}\partial_2u_1\partial_2\widetilde{\theta}dx-\int\partial_2\widetilde{\theta}\partial_2\widetilde{u_2}\partial_2\widetilde{\theta}dx\\
&:= B_{31}+B_{32}+B_{33}+B_{34}.
\end{aligned}\tag{4.38}$$

All terms in (4.38) can be bounded suitably. In fact, by Lemma 2.4,

$$\begin{aligned}
B_{31} &:= -\int\partial_1\widetilde{\theta}\partial_1\widetilde{u_1}\partial_1\widetilde{\theta}dx\\
&\le c\|\partial_1\widetilde{u_1}\|_{L^2}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\\
&\le c\|u\|_{H^2}\|\partial_1\widetilde{\theta}\|_{H^1}^2.
\end{aligned}\tag{4.39}$$

To deal with B_{32}, B_{33} and B_{34} we use Lemmas 2.4 and 2.5,

$$\begin{aligned}
B_{32} &:= -\int\partial_2\widetilde{\theta}\partial_1\widetilde{u_2}\partial_1\widetilde{\theta}dx\\
&\le c\|\partial_1\widetilde{u_2}\|_{L^2}\underbrace{\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}}_{\le\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\\
&\le c\|u\|_{H^2}\|\partial_1\widetilde{\theta}\|_{H^1}^2,
\end{aligned}\tag{4.40}$$

$$\begin{aligned}
B_{33} &:= -\int\partial_1\widetilde{\theta}\partial_2u_1\partial_2\widetilde{\theta}dx\\
&\le c\|\partial_2u_1\|_{L^2}\underbrace{\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}}_{\le\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\\
&\le c\|u\|_{H^2}\|\partial_1\widetilde{\theta}\|_{H^1}^2,
\end{aligned}\tag{4.41}$$

and

$$\begin{aligned} B_{34} &:= -\int \partial_2\widetilde{\theta}\partial_2\widetilde{u_2}\partial_2\widetilde{\theta}dx \\ &\leq c\|\partial_2\widetilde{\theta}\|_{L^2}\ \underbrace{\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}}_{\leq\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}}\ \|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \\ &\leq c\|\theta\|_{H^2}\|\partial_2\widetilde{u}\|_{H^1}\|\partial_1\widetilde{\theta}\|_{H^1} \\ &\leq c\|\theta\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big). \end{aligned} \tag{4.42}$$

Inserting the estimates (4.39), (4.40), (4.41) and (4.42) in (4.38) yields

$$B_3 \leq c\|(u,\theta)\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big). \tag{4.43}$$

It remains to bound B_4. Again, we split B_4 into four pieces,

$$\begin{aligned} B_4 &:= -\int \nabla(\widetilde{u_2}\partial_2\overline{\theta})\cdot\nabla\widetilde{\theta}dx \\ &= -\int \partial_1(\widetilde{u_2}\partial_2\overline{\theta})\cdot\partial_1\widetilde{\theta}dx - \int \partial_2(\widetilde{u_2}\partial_2\overline{\theta})\cdot\partial_2\widetilde{\theta}dx \\ &= -\int \partial_1\widetilde{u_2}\partial_2\overline{\theta}\partial_1\widetilde{\theta}dx - \int \widetilde{u_2}\partial_1\partial_2\overline{\theta}\partial_1\widetilde{\theta}dx \\ &\quad -\int \partial_2\widetilde{u_2}\partial_2\overline{\theta}\partial_2\widetilde{\theta}dx - \int \widetilde{u_2}\partial_2\partial_2\overline{\theta}\partial_2\widetilde{\theta}dx \\ &:= B_{41}+B_{42}+B_{43}+B_{44}. \end{aligned} \tag{4.44}$$

The terms above can be bounded as follows. Due to the definition of the horizontal average $\overline{\theta}$,

$$B_{42} := -\int \widetilde{u_2}\partial_1\partial_2\overline{\theta}\partial_1\widetilde{\theta}dx = 0. \tag{4.45}$$

To bound B_{41}, we use integration by parts, Lemma 2.1, Hölder's inequality and then Lemma 2.2

$$\begin{aligned} B_{41} &:= -\int \partial_1\widetilde{u_2}\partial_2\overline{\theta}\partial_1\widetilde{\theta}dx \\ &= -\int \widetilde{u_2}\partial_2\overline{\theta}\partial_1\partial_1\widetilde{\theta}dx \\ &= \int_{\mathbb{R}} \partial_2\overline{\theta}\Big(\int_{\mathbb{T}} \widetilde{u_2}\partial_1\partial_1\widetilde{\theta}dx_1\Big)dx_2 \\ &\leq c\|\partial_2\overline{\theta}\|_{L^\infty_{x_2}}\|\widetilde{u_2}\|_{L^2}\|\partial_1\partial_1\widetilde{\theta}\|_{L^2} \\ &\leq c\|\partial_2\overline{\theta}\|_{H^1}\|\widetilde{u_2}\|_{L^2}\|\partial_1\partial_1\widetilde{\theta}\|_{L^2} \\ &\leq c\|\theta\|_{H^2}\|\widetilde{u_2}\|_{L^2}\|\partial_1\widetilde{\theta}\|_{H^1} \\ &\leq c\|\theta\|_{H^2}\Big(\|\widetilde{u_2}\|_{L^2}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big). \end{aligned} \tag{4.46}$$

The other two terms B_{43}, B_{44} can be bounded via Lemmas 2.4 and 2.5,

$$\begin{aligned} B_{43} &:= -\int \partial_2\widetilde{u_2}\partial_2\overline{\theta}\partial_2\widetilde{\theta}dx \\ &\le c\|\partial_2\overline{\theta}\|_{L^2}\underbrace{\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}}_{\le\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\widetilde{u_2}\|_{L^2}\|\partial_2\partial_2\widetilde{u_2}\|_{L^2}^{\frac12} \\ &\le c\|\theta\|_{H^2}\|\partial_1\widetilde{\theta}\|_{H^1}\|\partial_2\widetilde{u}\|_{H^1} \\ &\le c\|\theta\|_{H^2}\Big(\|\partial_1\widetilde{\theta}\|_{H^1}^2+\|\partial_2\widetilde{u}\|_{H^1}^2\Big), \end{aligned} \tag{4.47}$$

$$\begin{aligned} B_{44} &:= -\int \widetilde{u_2}\partial_2\partial_2\overline{\theta}\partial_2\widetilde{\theta}dx \\ &\le c\|\partial_2\partial_2\overline{\theta}\|_{L^2}\underbrace{\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}}_{\le\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}\|\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac12} \\ &\le c\|\theta\|_{H^2}\|\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_2\widetilde{u}\|_{H^1}^{\frac12}\|\partial_1\widetilde{\theta}\|_{H^1} \\ &\le c\|\theta\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big). \end{aligned} \tag{4.48}$$

Inserting all the bounds obtained above for B_{41} through B_{44} in (4.44) gives

$$B_4 \le c\|(u,\theta)\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2\Big). \tag{4.49}$$

Combining (4.28), (4.37), (4.43) and (4.49) yields

$$\begin{aligned} &\frac12\frac{d}{dt}\Big(\|\nabla\widetilde{u}(t)\|_{L^2}^2+\|\nabla\widetilde{\theta}(t)\|_{L^2}^2\Big)+\nu\|\partial_2\nabla\widetilde{u}\|_{L^2}^2+\eta\|\partial_1\nabla\widetilde{\theta}\|_{L^2}^2 \\ &\qquad\le c\|(u,\theta)\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2\Big). \end{aligned} \tag{4.50}$$

In order to control the norm $\|\widetilde{u_2}\|_{L^2}$ appearing in (4.11) and (4.50), we need to add the following term,

$$-\frac{d}{dt}\Big(\delta(\widetilde{u_2},\widetilde{\theta})\Big)=-\delta(\partial_t\widetilde{u_2},\widetilde{\theta})-\delta(\widetilde{u_2},\partial_t\widetilde{\theta}),$$

where $\delta>0$ is a small constant to be fixed in the end of the proof. The inclusion of this term will generate an extra regularization term to help bound $\|\widetilde{u_2}\|_{L^2}$. Clearly, this stabilizing term comes from the interaction between $\widetilde{u}$ and $\widetilde{\theta}$. By Hölder's inequality, one easily sees that, for sufficiently small $\delta>0$,

$$\|(\widetilde{u},\widetilde{\theta})\|_{H^1}^2-\delta(\widetilde{u_2},\widetilde{\theta})\ge 0.$$

Considering to the first equation of (4.2) and using the fact that $\overline{u_2}=0$, we have

$$\partial_t\widetilde{u_2}+\widetilde{u\cdot\nabla u_2}+\underbrace{\widetilde{u_2}\partial_2\overline{u_2}}_{=0}-\nu\partial_2^2\widetilde{u_2}+\partial_2\widetilde{p}=g_0\widetilde{\theta}. \tag{4.51}$$

On the other hand, applying $\nabla\cdot$ to the first equation of (4.2), we get

$$\nabla\cdot(\widetilde{u\cdot\nabla u})+\nabla\cdot(\widetilde{u_2}\partial_2\overline{u})+\Delta\widetilde{p}=g_0\partial_2\widetilde{\theta}. \tag{4.52}$$

By (4.52), we can write

$$\widetilde{p} = -\Delta^{-1}\nabla\cdot(\widetilde{u\cdot\nabla u}) - \Delta^{-1}\nabla\cdot(\widetilde{u}_2\partial_2\overline{u}) + g_0\Delta^{-1}\partial_2\widetilde{\theta}.$$

Hence,

$$\partial_2\widetilde{p} = -\partial_2\Delta^{-1}\nabla\cdot(\widetilde{u\cdot\nabla u}) - \partial_2\Delta^{-1}\nabla\cdot(\widetilde{u}_2\partial_2\overline{u}) + g_0\partial_2\partial_2\Delta^{-1}\widetilde{\theta}. \tag{4.53}$$

Using (4.51) and the second equation of (4.2), we have

$$\begin{aligned}
-\delta\frac{d}{dt}(\widetilde{u}_2,\widetilde{\theta}) &= -\delta(\partial_t\widetilde{u}_2,\widetilde{\theta}) - \delta(\widetilde{u}_2,\partial_t\widetilde{\theta})\\
&= -\delta(g_0\widetilde{\theta} - \partial_2\widetilde{p} + \nu\partial_2^2\widetilde{u}_2 - \widetilde{u\cdot\nabla u}_2,\widetilde{\theta})\\
&\quad - \delta(\widetilde{u}_2, -g_0\widetilde{u}_2 + \eta\partial_1^2\widetilde{\theta} - \widetilde{u}_2\partial_2\overline{\theta} - \widetilde{u\cdot\nabla\theta})\\
&= -g_0\delta\|\widetilde{\theta}\|_{L^2}^2 + \int\partial_2\widetilde{p}\widetilde{\theta}dx - \delta\nu\int\partial_2^2\widetilde{u}_2\widetilde{\theta}dx + \delta\int\widetilde{u\cdot\nabla u}_2\widetilde{\theta}dx\\
&\quad + g_0\delta\|\widetilde{u}_2\|_{L^2}^2 - \delta\eta\int\partial_1^2\widetilde{\theta}\widetilde{u}_2dx + \delta\int\widetilde{u}_2\widetilde{u}_2\partial_2\overline{\theta}dx + \delta\int\widetilde{u\cdot\nabla\theta}\widetilde{u}_2dx\\
&:= N_1 + \cdots + N_8.
\end{aligned} \tag{4.54}$$

The terms N_1 through N_8 obey the following bounds. We start with N_2 and use (4.53) to rewrite it first as,

$$\begin{aligned}
N_2 &:= \delta\int\partial_2\widetilde{p}\widetilde{\theta}dx\\
&= -\delta\int\partial_2\Delta^{-1}\nabla\cdot(\widetilde{u\cdot\nabla u})\cdot\widetilde{\theta}dx - \delta\int\partial_2\Delta^{-1}\nabla\cdot(\widetilde{u}_2\partial_2\overline{u})\cdot\widetilde{\theta}dx\\
&\quad + g_0\delta\int\partial_2\partial_2\Delta^{-1}\widetilde{\theta}\cdot\widetilde{\theta}dx\\
&:= N_{21} + N_{22} + N_{23}.
\end{aligned} \tag{4.55}$$

By Lemma 2.1 and integration by parts we split N_{21} into three pieces

$$\begin{aligned}
N_{21} &:= -\delta\int\partial_2\Delta^{-1}\nabla\cdot(\widetilde{u\cdot\nabla u})\cdot\widetilde{\theta}dx\\
&= -\delta\int\partial_2\Delta^{-1}\nabla\cdot(u\cdot\nabla\widetilde{u})\cdot\widetilde{\theta}dx + \underbrace{\delta\int\partial_2\Delta^{-1}\nabla\cdot(\overline{u\cdot\nabla\widetilde{u}})\cdot\widetilde{\theta}dx}_{=0}\\
&= -\delta\int\partial_2\Delta^{-1}\partial_1(u_1\partial_1\widetilde{u})\cdot\widetilde{\theta}dx - \delta\int\partial_2\Delta^{-1}\partial_2(u_2\partial_2\widetilde{u})\cdot\widetilde{\theta}dx\\
&= -\delta\int(u_1\partial_1\widetilde{u})\cdot\partial_2\Delta^{-1}\partial_1\widetilde{\theta}dx - \delta\int(u_2\partial_2\widetilde{u})\cdot\partial_2\Delta^{-1}\partial_2\widetilde{\theta}dx\\
&= -\delta\int\partial_1\widetilde{u}_1\widetilde{u}\cdot\partial_2\Delta^{-1}\partial_1\widetilde{\theta}dx - \delta\int u_1\widetilde{u}\cdot\partial_1\partial_2\Delta^{-1}\partial_1\widetilde{\theta}dx\\
&\quad - \delta\int(u_2\partial_2\widetilde{u})\cdot\partial_2\Delta^{-1}\partial_2\widetilde{\theta}dx
\end{aligned} \tag{4.56}$$

$$= N_{211} + N_{212} + N_{213}. \tag{4.57}$$

Due to $\nabla \cdot u = 0$, Lemma 2.4 and the boundedness of the Riesz transform,

$$\begin{aligned}
N_{211} &= -\delta \int \partial_1 \widetilde{u_1} \widetilde{u} \cdot \partial_2 \Delta^{-1} \partial_1 \widetilde{\theta} dx \\
&= \delta \int \partial_2 \widetilde{u_2} \widetilde{u} \cdot \partial_2 \Delta^{-1} \partial_1 \widetilde{\theta} dx \\
&\le c \|\widetilde{u}\|_{L^2} \|\partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \Delta^{-1} \partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \Delta^{-1} \partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \\
&\le c \|u\|_{H^2} \|\partial_2 \widetilde{u}\|_{H^1} \|\widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \\
&\le c \|u\|_{H^2} \|\partial_2 \widetilde{u}\|_{H^1} \|\partial_1 \widetilde{\theta}\|_{H^1} \\
&\le c \|u\|_{H^2} \Big(\|\partial_2 \widetilde{u}\|_{H^1}^2 + \|\partial_1 \widetilde{\theta}\|_{H^1}^2 \Big).
\end{aligned} \tag{4.58}$$

According to Lemma 2.4, the boundedness of the Riesz transform, Lemma 2.5 and $\nabla \cdot u = 0$,

$$\begin{aligned}
N_{212} &= -\delta \int u_1 \widetilde{u} \cdot \partial_1 \partial_2 \Delta^{-1} \partial_1 \widetilde{\theta} dx \\
&\le c \|u_1\|_{L^2} \|\widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \Delta^{-1} \partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_1 \partial_2 \Delta^{-1} \partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \\
&\le c \|u\|_{H^2} \|\widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \\
&\le c \|u\|_{H^2} (\|\widetilde{u_1}\|_{L^2} + \|\widetilde{u_2}\|_{L^2})^{\frac{1}{2}} \|\partial_2 \widetilde{u}\|_{H^1}^{\frac{1}{2}} \|\partial_1 \widetilde{\theta}\|_{H^1} \\
&\le c \|u\|_{H^2} \Big(\underbrace{\|\widetilde{u_1}\|_{L^2}^2}_{\le \|\partial_1 \widetilde{u_1}\|_{L^2}^2 = \|\partial_2 \widetilde{u_2}\|_{L^2}^2} + \|\widetilde{u_2}\|_{L^2}^2 + \|\partial_2 \widetilde{u}\|_{H^1}^2 + \|\partial_1 \widetilde{\theta}\|_{H^1}^2 \Big) \\
&\le c \|u\|_{H^2} \Big(\|\widetilde{u_2}\|_{L^2}^2 + \|\partial_2 \widetilde{u}\|_{H^1}^2 + \|\partial_1 \widetilde{\theta}\|_{H^1}^2 \Big).
\end{aligned} \tag{4.59}$$

Applying Lemma 2.4, the boundedness of the Riesz transform and then Lemma 2.5,

$$\begin{aligned}
N_{213} &= -\delta \int (u_2 \partial_2 \widetilde{u}) \cdot \partial_2 \Delta^{-1} \partial_2 \widetilde{\theta} dx \\
&\le c \|u_2\|_{L^2} \|\partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \Delta^{-1} \partial_2 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \Delta^{-1} \partial_2 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \\
&\le c \|u\|_{H^2} \|\partial_2 \widetilde{u}\|_{H^1} \|\widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \\
&\le c \|u\|_{H^2} \|\partial_2 \widetilde{u}\|_{H^1} \|\partial_1 \widetilde{\theta}\|_{H^1} \\
&\le c \|u\|_{H^2} \Big(\|\partial_2 \widetilde{u}\|_{H^1}^2 + \|\partial_1 \widetilde{\theta}\|_{H^1}^2 \Big).
\end{aligned} \tag{4.60}$$

The bounds in (4.58), (4.59) and (4.60) lead to

$$N_{21} \le c \|u\|_{H^2} \Big(\|\widetilde{u_2}\|_{L^2}^2 + \|\partial_2 \widetilde{u}\|_{H^1}^2 + \|\partial_1 \widetilde{\theta}\|_{H^1}^2 \Big). \tag{4.61}$$

Now we turn to the next term N_{22}. By Hölder's inequality, the boundedness of the Riesz transform and Lemmas 2.1, 2.2 and 2.5

$$\begin{aligned} N_{22} &:= -\delta \int \partial_2 \Delta^{-1} \nabla \cdot (\widetilde{u_2} \partial_2 \overline{u}) \cdot \widetilde{\theta} dx \\ &\leq c\delta \|\partial_2 \Delta^{-1} \nabla \cdot (\widetilde{u_2} \partial_2 \overline{u})\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\ &\leq c\delta \|\widetilde{u_2} \partial_2 \overline{u}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\ &\leq c\delta \|\partial_2 \overline{u}\|_{L^\infty_{x_2}} \|\widetilde{u_2}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\ &\leq c\delta \|\partial_2 \overline{u}\|_{H^1} \|\widetilde{u_2}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\ &\leq c\delta \|u\|_{H^2} \|\widetilde{u_2}\|_{L^2} \|\partial_1 \widetilde{\theta}\|_{L^2} \\ &\leq c\delta \|u\|_{H^2} \Big(\|\widetilde{u_2}\|_{L^2}^2 + \|\partial_1 \widetilde{\theta}\|_{H^1}^2 \Big). \end{aligned} \tag{4.62}$$

For N_{23}, we integrate by parts, use Plancherel's theorem and then Lemma 2.5,

$$\begin{aligned} N_{23} &:= g_0 \delta \int \partial_2 \partial_2 \Delta^{-1} \widetilde{\theta} \cdot \widetilde{\theta} dx \\ &= g_0 \delta \int \partial_2 \Delta^{-\frac{1}{2}} \widetilde{\theta} \cdot \partial_2 \Delta^{-\frac{1}{2}} \widetilde{\theta} dx \\ &= g_0 \delta \|\partial_2 \Lambda^{-1} \widetilde{\theta}\|_{L^2}^2 \\ &= g_0 \delta \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \int_{\mathbb{R}} \frac{\xi_2^2}{k^2 + \xi_2^2} |\widehat{\widetilde{\theta}}(k, \xi_2)|^2 d\xi_2 \\ &\leq c\delta \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \int_{\mathbb{R}} \xi_2^2 |\widehat{\widetilde{\theta}}(k, \xi_2)|^2 d\xi_2 = c\delta \|\partial_2 \widetilde{\theta}\|_{L^2}^2 \leq c\delta \|\partial_1 \partial_2 \widetilde{\theta}\|_{L^2}^2 \leq c\delta \|\partial_1 \widetilde{\theta}\|_{H^1}^2, \end{aligned} \tag{4.63}$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$ and we have used the fact that the oscillation part has the horizontal mode equal to 0, or $\widehat{\widetilde{\theta}}(0, \xi_2) = 0$.

Collecting (4.57), (4.62), (4.63) and (4.55) yields

$$N_2 \leq c\delta \|(u, \theta)\|_{H^2} \Big(\|\partial_2 \widetilde{u}\|_{H^1}^2 + \|\widetilde{u_2}\|_{L^2}^2 + \|\partial_1 \widetilde{\theta}\|_{H^1}^2 \Big) + c\delta \|\partial_1 \widetilde{\theta}\|_{H^1}^2. \tag{4.64}$$

To deal with N_3 we use $\nabla \cdot u = 0$, integration by parts, Hölder's inequality and Lemma 2.5,

$$\begin{aligned} N_3 &:= -\delta\nu \int \partial_2^2 \widetilde{u_2} \widetilde{\theta} dx = \delta\nu \int \partial_2 \partial_1 \widetilde{u_1} \widetilde{\theta} dx \\ &= -\delta\nu \int \widetilde{u_1} \partial_2 \partial_1 \widetilde{\theta} dx \\ &\leq \delta\nu \|\widetilde{u_1}\|_{L^2} \|\partial_2 \partial_1 \widetilde{\theta}\|_{L^2} \\ &\leq c\delta \Big(\underbrace{\|\widetilde{u_1}\|_{L^2}^2}_{\leq \|\partial_1 \widetilde{u_1}\|_{L^2}^2 = \|\partial_2 \widetilde{u_2}\|_{L^2}^2} + \|\partial_2 \partial_1 \widetilde{\theta}\|_{L^2}^2 \Big) \end{aligned}$$

$$\leq c\delta\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big). \tag{4.65}$$

To estimate N_4, we make use of Lemma 2.1 and integration by parts, to write it as

$$\begin{aligned}
N_4 &:= \delta\int \widetilde{u\cdot\nabla u_2}\widetilde{\theta}dx\\
&= \delta\int u\cdot\nabla\widetilde{u_2}\widetilde{\theta}dx-\underbrace{\delta\int\overline{u\cdot\nabla\widetilde{u_2}}\widetilde{\theta}dx}_{=0}\\
&= \delta\int u\partial_1\widetilde{u_2}\widetilde{\theta}dx+\delta\int u\partial_2\widetilde{u_2}\widetilde{\theta}dx\\
&= -\delta\int\partial_1\widetilde{u}\widetilde{u_2}\widetilde{\theta}dx+\delta\int u\partial_2\widetilde{u_2}\widetilde{\theta}dx\\
&= N_{41}+N_{42}.
\end{aligned}\tag{4.66}$$

By Lemmas 2.4 and 2.5

$$\begin{aligned}
N_{41} &:= -\delta\int\partial_1\widetilde{u}\widetilde{u_2}\widetilde{\theta}dx\\
&\leq c\|\partial_1\widetilde{u}\|_{L^2}\|\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac12}\|\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\\
&\leq c\|u\|_{H^2}\|\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_2\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\widetilde{\theta}\|_{H^1}\\
&\leq c\|u\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2\Big).
\end{aligned}\tag{4.67}$$

Similarly,

$$\begin{aligned}
N_{42} &= \delta\int u\partial_2\widetilde{u_2}\widetilde{\theta}dx\\
&\leq c\|u\|_{L^2}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_2\partial_2\widetilde{u_2}\|_{L^2}^{\frac12}\|\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\\
&\leq c\|u\|_{H^2}\|\partial_2\widetilde{u}\|_{H^1}\|\partial_1\widetilde{\theta}\|_{H^1}\\
&\leq c\|u\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big).
\end{aligned}\tag{4.68}$$

Inserting (4.67) and (4.68) in (4.66) we find

$$N_4\leq c\delta\|u\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2\Big). \tag{4.69}$$

Clearly, the term N_5 can be bounded via Lemma 2.5,

$$N_5 := -g_0\delta\|\widetilde{\theta}\|_{L^2}^2\leq c\delta\|\partial_1\widetilde{\theta}\|_{L^2}^2\leq c\delta\|\partial_1\widetilde{\theta}\|_{H^1}^2. \tag{4.70}$$

Due to Hölder's inequality and Young's inequality,

$$\begin{aligned}
N_6 &:= -\delta\eta\int\partial_1^2\widetilde{\theta}\widetilde{u_2}dx\\
&\leq c\delta\|\partial_1^2\widetilde{\theta}\|_{L^2}\|\widetilde{u_2}\|_{L^2}\\
&\leq c\delta\|\partial_1\widetilde{\theta}\|_{H^1}\|\widetilde{u_2}\|_{L^2}
\end{aligned}$$

$$\leq c\delta\|\partial_1\widetilde{\theta}\|_{H^1}^2 + g_0\frac{\delta}{4}\|\widetilde{u_2}\|_{L^2}^2. \tag{4.71}$$

Using integration by parts and Lemma 2.4, we get

$$\begin{aligned} N_7 &:= \delta\int \widetilde{u_2}\widetilde{u_2}\partial_2\overline{\theta}dx = 2\delta\int \partial_2\widetilde{u_2}\widetilde{u_2}\overline{\theta}dx \\ &\leq c\delta\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\overline{\theta}\|_{L^2} \\ &\leq c\delta\|\partial_2\widetilde{u}\|_{L^2}^{\frac{3}{2}}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\theta\|_{H^2} \\ &\leq c\delta\|\theta\|_{H^2}(\|\partial_2\widetilde{u}\|_{H^1}^2 + \|\widetilde{u_2}\|_{L^2}^2). \end{aligned} \tag{4.72}$$

It remain to bound the last term N_8. We split it into three terms using Lemma 2.1,

$$\begin{aligned} N_8 &:= \delta\int \widetilde{u\cdot\nabla\theta}\widetilde{u_2}dx \\ &= \delta\int u\cdot\nabla\widetilde{\theta}\widetilde{u_2}dx - \underbrace{\delta\int \overline{u\cdot\nabla\widetilde{\theta}}\widetilde{u_2}dx}_{=0} \\ &= \delta\int \widetilde{u_1}\partial_1\widetilde{\theta}\widetilde{u_2}dx + \delta\int \overline{u_1}\partial_1\widetilde{\theta}\widetilde{u_2}dx + \delta\int u_2\partial_2\widetilde{\theta}\widetilde{u_2}dx \\ &:= N_{81} + N_{82} + N_{83}. \end{aligned} \tag{4.73}$$

Due to Lemmas 2.4, 2.5 and divergence-free condition of u, we have

$$\begin{aligned} N_{81} &:= \delta\int \widetilde{u_1}\partial_1\widetilde{\theta}\widetilde{u_2}dx \\ &\leq c\delta\|\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\theta}\|_{L^2} \\ &\leq c\delta\|\partial_2\widetilde{u}\|_{L^2}^{\frac{3}{2}}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\theta}\|_{L^2} \\ &\leq c\delta\|\theta\|_{H^2}(\|\partial_2\widetilde{u}\|_{H^1}^2 + \|\widetilde{u_2}\|_{L^2}^2). \end{aligned} \tag{4.74}$$

By Lemma 2.1, Hölder's inequality and Lemma 2.2,

$$\begin{aligned} N_{82} &:= \delta\int \overline{u_1}\partial_1\widetilde{\theta}\widetilde{u_2}dx \\ &\leq \delta\|\overline{u_1}\|_{L^\infty_{x_2}}\|\partial_1\widetilde{\theta}\widetilde{u_2}\|_{L^1} \\ &\leq c\delta\|\overline{u_1}\|_{L^\infty_{x_2}}\|\partial_1\widetilde{\theta}\|_{L^2}\|\widetilde{u_2}\|_{L^2} \\ &\leq c\delta\|u\|_{H^1}\|\partial_1\widetilde{\theta}\|_{L^2}\|\widetilde{u_2}\|_{L^2} \\ &\leq c\delta\|u\|_{H^2}\Big(\|\partial_1\widetilde{\theta}\|_{H^1}^2 + \|\widetilde{u_2}\|_{L^2}^2\Big). \end{aligned} \tag{4.75}$$

Due to $\overline{u_2} = 0$, integration by parts, Lemma 2.4 and Young's inequality

$$\begin{aligned} N_{83} &:= \delta\int u_2\partial_2\widetilde{\theta}\widetilde{u_2}dx = \delta\int \widetilde{u_2}\partial_2\widetilde{\theta}\widetilde{u_2}dx \\ &= 2\delta\int \partial_2\widetilde{u_2}\widetilde{\theta}\widetilde{u_2}dx \end{aligned}$$

$$\leq c\delta\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{\theta}\|_{L^2}$$
$$\leq c\delta\|\partial_2\widetilde{u}\|_{H^1}^{\frac{3}{2}}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\theta\|_{H^2}$$
$$\leq c\delta\|\theta\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2\Big). \tag{4.76}$$

Inserting (4.74), (4.75) and (4.76) in (4.73) leads to

$$N_8\leq c\delta\|(u,\theta)\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big). \tag{4.77}$$

In view of (4.54), combining (4.64), (4.65), (4.66), (4.70), (4.71), (4.72) and (4.77) we get

$$-\delta\frac{d}{dt}(\widetilde{u_2},\widetilde{\theta})\leq g_0\delta\|\widetilde{u_2}\|_{L^2}^2+c\delta\|(u,\theta)\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2\Big)$$
$$-g_0\frac{\delta}{4}\|\widetilde{u_2}\|_{L^2}^2+c\delta\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big). \tag{4.78}$$

It then follows from (4.11), (4.50) and (4.78) that

$$\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2+\|\widetilde{\theta}\|_{H^1}^2-\delta(\widetilde{u_2},\widetilde{\theta})\Big)+2\nu\|\partial_2\widetilde{u}\|_{H^1}^2+2\eta\|\partial_1\widetilde{\theta}\|_{H^1}^2$$
$$\leq c\|(u,\theta)\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2\Big)$$
$$+g_0\frac{3\delta}{4}\|\widetilde{u_2}\|_{L^2}^2+c\delta\|(u,\theta)\|_{H^2}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2\Big)$$
$$+c\delta\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big).$$

Now, by Theorem 1.1, if $\varepsilon>0$ is sufficiently small and $\|u_0\|_{L^2}+\|\theta_0\|_{L^2}\leq\varepsilon$, then $\|(u(t),\theta(t))\|_{H^2}\leq c\varepsilon$. Hence we have

$$\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2+\|\widetilde{\theta}\|_{H^1}^2-\delta(\widetilde{u_2},\widetilde{\theta})\Big)+2\nu\|\partial_2\widetilde{u}\|_{H^1}^2+2\eta\|\partial_1\widetilde{\theta}\|_{H^1}^2$$
$$\leq c\epsilon\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2\Big)$$
$$+g_0\frac{3\delta}{4}\|\widetilde{u_2}\|_{L^2}^2+c\delta\epsilon\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2\Big)$$
$$+c\delta\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big).$$

Choosing $\epsilon>0$ such that $c\epsilon\leq-g_0\min(\frac{1}{4},\frac{\delta}{4})$, we obtain

$$\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2+\|\widetilde{\theta}\|_{H^1}^2-\delta(\widetilde{u_2},\widetilde{\theta})\Big)+2\nu\|\partial_2\widetilde{u}\|_{H^1}^2+2\eta\|\partial_1\widetilde{\theta}\|_{H^1}^2$$
$$\leq\frac{\delta}{4}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big)-g_0\frac{\delta}{4}\|\widetilde{u_2}\|_{L^2}^2$$
$$+g_0\frac{3\delta}{4}\|\widetilde{u_2}\|_{L^2}^2-g_0\frac{\delta}{4}\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\widetilde{u_2}\|_{L^2}^2\Big)$$
$$+c\delta\Big(\|\partial_2\widetilde{u}\|_{H^1}^2+\|\partial_1\widetilde{\theta}\|_{H^1}^2\Big)$$

$$\leq g_0\frac{\delta}{4}\|\widetilde{u_2}\|_{L^2}^2 + c\delta\Big(\|\partial_2\widetilde{u}\|_{H^1}^2 + \|\partial_1\widetilde{\theta}\|_{H^1}^2\Big).$$

Choosing $\delta > 0$ such that $c\delta \leq \min(\nu, \eta, \frac{c}{2})$, we get

$$\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2 - \delta(\widetilde{u_2}, \widetilde{\theta})\Big) + \nu\|\partial_2\widetilde{u}\|_{H^1}^2 + \eta\|\partial_1\widetilde{\theta}\|_{H^1}^2 - g_0\frac{\delta}{4}\|\widetilde{u_2}\|_{L^2}^2 \leq 0. \quad (4.79)$$

Due to the above choice of δ, we have

$$\frac{1}{2}\Big(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2\Big) - \delta(\widetilde{u_2}, \widetilde{\theta}) \geq 0.$$

or

$$\frac{1}{2}(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2) \leq \|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2 - \delta(\widetilde{u_2}, \widetilde{\theta}) \leq \frac{3}{2}(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2).$$

For any $0 \leq s \leq t$, integrating (4.79) in time leads to

$$\frac{1}{2}(\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2) + \int_s^t (\nu\|\partial_2\widetilde{u}\|_{H^1}^2 + \eta\|\partial_1\widetilde{\theta}\|_{H^1}^2 - g_0\frac{\delta}{4}\|\widetilde{u_2}\|_{L^2}^2)\, d\tau$$
$$\leq \frac{3}{2}(\|\widetilde{u}(s)\|_{H^1}^2 + \|\widetilde{\theta}(s)\|_{H^1}^2).$$

Especially, for any $0 \leq s \leq t$,

$$\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2 \leq 3(\|\widetilde{u}(s)\|_{H^1}^2 + \|\widetilde{\theta}(s)\|_{H^1}^2) \quad (4.80)$$

and

$$\int_0^\infty (\nu\|\partial_2\widetilde{u}\|_{H^1}^2 + \eta\|\partial_1\widetilde{\theta}\|_{H^1}^2 - g_0\frac{\delta}{4}\|\widetilde{u_2}\|_{L^2}^2)\, d\tau \leq C < \infty.$$

Combining with the time integral bounds from Theorem 1.1,

$$\int_0^\infty \|\partial_2 u\|_{H^2}^2\, dt < \infty, \quad \int_0^\infty \|\partial_1 u_2\|_{L^2}^2\, dt < \infty \quad \textit{and} \quad \int_0^\infty \|\partial_1\theta\|_{H^2}^2\, dt < \infty,$$

we get

$$\int_0^\infty (\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2)\, dt < \infty. \quad (4.81)$$

Applying Lemma 2.6 to (4.80) and (4.81) yields

$$\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2 \leq c(1+t)^{-1},$$

and the asymptotic behavior, as $t \to \infty$,

$$t\,(\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2) \to 0.$$

This completes the proof of Theorem 1.2. □

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