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# A henselian preparation theorem

Laurent Moret-Bailly<sup>\*†‡</sup>

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## Abstract

We prove an analogue of the Weierstrass preparation theorem for henselian pairs, generalizing the local case recently proved by Bouthier and Česnavičius. As an application, we construct a henselian analogue of the resultant of  $p$ -adic series defined by Berger.

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## 1 Introduction

Let  $R$  be a ring (commutative, with unit). We denote by  $R\{t\}$  the henselization of the polynomial ring  $R[t]$  with respect to the ideal  $(t)$ : this is a subring of the power series ring  $R[[t]]$ . The aim of this work is to prove the following result:

**Theorem 1.1** (Henselian Preparation Theorem). *Let  $R$  be a ring,  $I$  an ideal of  $R$ . Assume that  $(R, I)$  is a henselian pair. Let  $d$  be a natural integer and let  $f$  be an element of  $R\{t\}$  which in  $R[[t]]$  has the form  $f = \sum_{i \geq 0} a_i t^i$ , where  $a_d \in R^\times$  and  $a_i \in I$  for  $i < d$ .*

*Then  $f$  can be written uniquely as  $f = (t^d + Q)v$  where  $v \in R\{t\}^\times$  and  $Q \in R[t]$  has degree  $< d$  and coefficients in  $I$ .*

See 3.4 for a more detailed statement, including the analogue of the Weierstrass division theorem.

The analogous statement where  $R\{t\}$  is replaced by  $R[[t]]$  and  $R$  is assumed  $I$ -adically complete and separated is due to O'Malley [6, 2.10]; the “classical” form of the latter is the special case where  $I$  is maximal, see for instance [2, §3, n° 8, prop. 5].

When  $R$  is *local henselian* with maximal ideal  $I$ , the above theorem was proved by Bouthier and Česnavičius in [3, 3.1.2], which inspired the present paper. The proof we give here is somewhat different and more direct: we do not use reduction to the noetherian case or the classical preparation theorem, but we work directly from the construction of  $R\{t\}$  as a filtered colimit of étale  $R[t]$ -algebras.<sup>1</sup>

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<sup>1</sup>Regrettably, there does not seem to be, at the moment, a general result covering all the above-mentioned variants (and more, such as rings of convergent series).

As an easy application, we define in Section 4 a notion of resultant in  $R\{\underline{t}\}$ , entirely similar to the resultant constructed by Berger [1] for  $p$ -adic formal power series.

**Notation and conventions.** All rings are commutative with unit; ring homomorphisms respect unit elements. The unit group of a ring  $R$  is denoted by  $R^\times$ , its Jacobson radical by  $\text{rad}(R)$ .

If  $x$  is a point of a scheme,  $\kappa(x)$  denotes its residue field.

Let  $Y$  be a closed subscheme of a scheme  $X$ . We say  $(X, Y)$  is a *Zariski pair* if  $X$  is the only open subscheme of  $X$  containing  $Y$ ; this condition only depends on the underlying spaces. If  $X = \text{Spec}(A)$  is affine and  $I \subset A$  is the ideal of  $Y$ , we say  $(A, I)$  is a Zariski pair if  $(X, Y)$  is a Zariski pair or, equivalently, if  $I \subset \text{rad}(A)$ . If  $(X, Y)$  is Zariski and  $X' \rightarrow X$  is a closed morphism, then  $(X', Y \times_X X')$  is Zariski.

If  $A$  is a ring and  $I \subset A$  an ideal, we denote by  $(A, I)^{\text{h}}$  or  $A^{\text{h}}$  the henselization of  $(A, I)$ . Recall that if  $A \rightarrow B$  is an integral morphism, then  $(B, IB)^{\text{h}} = (A, I)^{\text{h}} \otimes_A B$ . In particular, if  $(A, I)$  is henselian, so is  $(B, IB)$ .

## 2 Preliminary results

### 2.1 Structure of henselian series rings

Let  $R$  be a ring,  $\underline{t} = (t_1, \dots, t_n)$  a finite sequence of indeterminates <sup>2</sup> We denote by  $R\{\underline{t}\}$  the henselization of  $R[\underline{t}]$  at the ideal  $(t_1, \dots, t_n)$ ; it is an  $R[\underline{t}]$ -algebra with an isomorphism  $\varepsilon : R\{\underline{t}\}/(\underline{t}) \xrightarrow{\sim} R$ , and there is a natural injection  $R\{\underline{t}\} \hookrightarrow R[[\underline{t}]]$  making  $R[[\underline{t}]]$  the  $(\underline{t})$ -adic completion of  $R\{\underline{t}\}$ ; the image of  $f \in R\{\underline{t}\}$  in  $R[[\underline{t}]]$  will be denoted by  $f_{\text{for}}$ .

As a functor of  $R$ ,  $R\{\underline{t}\}$  is better behaved than  $R[[\underline{t}]]$ . In particular, it commutes with filtered colimits, and if  $I$  is any ideal of  $R$  we have  $R\{\underline{t}\}/IR\{\underline{t}\} \cong (R/I)\{\underline{t}\}$ .

For  $f \in R\{\underline{t}\}$  we have the equivalences:

$$f \in R\{\underline{t}\}^\times \Leftrightarrow f_{\text{for}} \in R[[\underline{t}]]^\times \Leftrightarrow \varepsilon(f) \in R^\times.$$

It follows that  $\text{rad}(R\{\underline{t}\})$  is generated by  $\text{rad}(R)$  and  $(\underline{t})$ . In particular, if  $(R, I)$  is a Zariski pair, so is  $(R\{\underline{t}\}, IR\{\underline{t}\} + (\underline{t}))$ .

Similarly, if  $(R, I)$  is a henselian pair, so is  $(R\{\underline{t}\}, IR\{\underline{t}\} + (\underline{t}))$ : to see this, view  $R$  as the quotient  $R\{\underline{t}\}/(\underline{t})$  and apply the transitivity property [8, 0DYD].

Classically,  $R\{\underline{t}\}$  can be constructed as the colimit of a filtered family  $(A_\lambda)_{\lambda \in L}$  of étale  $R[\underline{t}]$ -algebras, with compatible isomorphisms  $\varepsilon_\lambda : A_\lambda/(\underline{t})A_\lambda \xrightarrow{\sim} R$ . In particular, for all  $\lambda \in L$  and  $N \in \mathbb{N}$ , the natural morphism of  $R$ -algebras  $R[\underline{t}]/(\underline{t})^N \rightarrow A_\lambda/(\underline{t})^N A_\lambda$  is an isomorphism.

Each natural morphism  $\pi_\lambda : \text{Spec}(A_\lambda) \rightarrow \text{Spec}(R)$  is smooth of relative dimension  $n$ , and has a section  $s_\lambda$  deduced from  $\varepsilon_\lambda$ .

We say that an  $R$ -algebra  $A$  is *geometrically irreducible* if the natural morphism  $\text{Spec}(A) \rightarrow \text{Spec}(R)$  has geometrically irreducible fibers.

**Lemma 2.1.1.** *Let  $R$  and  $\underline{t} = (t_1, \dots, t_n)$  be as above. Then one can choose the system  $(A_\lambda)_{\lambda \in L}$  such that each  $A_\lambda$  is a geometrically irreducible  $R$ -algebra.*

<sup>2</sup>For the preparation theorem we only need the case  $n = 1$ . The case of an arbitrary set of indeterminates is left to the reader.

*Proof.* Starting with an arbitrary family  $(A_\lambda)_{\lambda \in L}$ , we may assume, by enlarging  $L$ , that for all  $\lambda \in L$  and  $f \in A_\lambda$  such that  $\varepsilon_\lambda(f) \in R^\times$ , the localized algebra  $A_\lambda[1/f]$  is still in the family. It suffices to show that, assuming this, the sub-system formed by the geometrically irreducible  $A_\lambda$ 's is cofinal. For each  $\lambda$ , let  $U_\lambda \subset \text{Spec}(A_\lambda)$  be the union of the connected components of the fibers of  $\pi_\lambda$  meeting the section  $s_\lambda$ . As  $\pi_\lambda$  is smooth,  $U_\lambda$  is open in  $\text{Spec}(A_\lambda)$  [4, (15.6.7)], and its fibers over  $\text{Spec}(R)$  are smooth and connected, with a rational point, hence geometrically irreducible. Since  $U_\lambda$  is open, there is  $f \in A_\lambda$  such that  $\text{Im}(s_\lambda) \subset \text{Spec}(A_\lambda[1/f]) \subset U_\lambda$  (in an affine scheme, every closed subset has a basis of principal open neighborhoods). The fibers of  $\text{Spec}(A_\lambda[1/f]) \rightarrow \text{Spec}(R)$  are nonempty and open in those of  $U_\lambda \rightarrow \text{Spec}(R)$  and therefore geometrically irreducible. This completes the proof.  $\square$

### 2.1.2 Evaluation

This section will not be used until Section 4.

Let us keep the notation of 2.1 and consider the category  $\text{Alg}_R^h$  of henselian pairs  $(A, J)$  where  $A$  is an  $R$ -algebra. Then  $(R\{\underline{t}\}, (\underline{t}))$  is an object of  $\text{Alg}_R^h$  corepresenting the set-valued functor  $(A, J) \mapsto \prod_{i=1}^n J$ . In particular, for an object  $(A, J)$  of  $\text{Alg}_R^h$  and a sequence  $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$  from  $J$ , we have a morphism ‘‘evaluation at  $\underline{\alpha}$ ’’ from  $R\{\underline{t}\}$  to  $A$  which we denote by  $f \mapsto f(\underline{\alpha})$ . One may construct it by noting that the morphism  $P \mapsto P(\underline{\alpha})$  from  $R[\underline{t}]$  to  $A$  maps the  $t_i$ 's into  $J$ , hence factors through  $R\{\underline{t}\}$  because  $(A, J)$  is henselian.

For given  $\underline{\alpha}$ , the element  $f(\underline{\alpha})$  is the sum in  $A$ , for the  $J$ -adic topology, of the series  $f_{\text{for}}(\underline{\alpha})$  obtained by substituting  $\underline{\alpha}$  for  $\underline{t}$ ; this property characterizes  $f(\underline{\alpha})$  if  $A$  is  $J$ -adically separated (but not in general).

The reader can check the following nice property, which will not be used here: if  $I$  is an ideal of  $R$  generated by  $n$  elements  $a_1, \dots, a_n$ , the evaluation morphism  $f \mapsto f(\underline{a})$  induces an isomorphism of  $R\{\underline{t}\}/(t_i - a_i)_{1 \leq i \leq n}$  with the henselization  $(R, I)^h$ .

## 2.2 Schemes over henselian pairs: a decomposition result

**Notation 2.2.1.** Let  $(R, I)$  be a henselian pair. Put  $S = \text{Spec}(R)$ ,  $\overline{R} = R/I$ , and  $\overline{S} = \text{Spec}(\overline{R})$ ; more generally, for each  $R$ -algebra  $A$ , (resp. each  $R$ -scheme  $X$ ) we shall put  $\overline{A} = A/IA$  (resp.  $\overline{X} = X \times_S \overline{S}$ ).

The following proposition is a variant of [7, XI, cor. 1 p. 119]:

**Proposition 2.2.2.** *With notation as above, let  $Z$  be a separated  $R$ -scheme of finite type. Assume that  $\overline{Z}$  is finite over  $\overline{R}$ .*

*Then there is a unique open and closed subscheme  $Z^f$  of  $Z$  which is finite over  $R$  and satisfies  $\overline{Z^f} = \overline{Z}$ . Moreover  $Z^f$  has the following properties:*

- (1) *The pair  $(Z^f, \overline{Z})$  is henselian.*
- (2)  *$Z^f$  is the smallest open subscheme of  $Z$  containing  $\overline{Z}$ .*
- (3) *Let  $T$  be an  $R$ -scheme and  $u : T \rightarrow Z$  an  $R$ -morphism. Assume that  $(T, \overline{T})$  is a Zariski pair. Then  $u$  factors through  $Z^f$ .*

*Proof.* Let us first assume the existence of  $Z^f$  and prove (1), (2) and (3). First, (1) is clear since  $(R, I)$  is henselian and  $Z^f$  is finite over  $R$ . In particular,  $(Z^f, \overline{Z})$  is a Zariski pair, and (2) follows because  $Z^f$  is open in  $Z$ . Now take  $u : T \rightarrow Z$  as in (3): then  $u^{-1}(Z^f)$  is a neighborhood of  $\overline{T}$  in  $T$ , hence equal to  $T$ , which proves (3).

Observe that (2), for instance, implies the uniqueness of  $Z^f$ . Now let us prove existence. First, consider the set  $Z'$  of points  $x \in Z$  isolated in their fiber above  $\text{Spec}(R)$ . Then  $Z'$  is open in  $Z$  [4, (13.1.4)] and, viewed as an open subscheme, it is quasifinite over  $\text{Spec}(R)$ ; in addition, we have  $\overline{Z'} = \overline{Z}$ . So it is clear that if  $Z'^f$  exists it is open in  $Z$ , and closed since it is finite over  $R$ , so we can take  $Z^f = Z'^f$ . Replacing  $Z$  by  $Z'$ , we can therefore assume  $Z$  quasifinite over  $R$ .

By Zariski's main theorem [5, (18.12.13)], there is an open immersion  $Z \hookrightarrow Z^c$ , where  $Z^c$  is a finite  $R$ -scheme. As  $\overline{Z}$  is finite over  $R$ , the induced open immersion  $\overline{Z} \hookrightarrow \overline{Z^c}$  is closed, so we have  $\overline{Z^c} = \overline{Z} \amalg Y$  for an open and closed subscheme  $Y$  of  $\overline{Z^c}$ . Since  $(R, I)$  is henselian and  $Z^c$  is finite over  $R$ , this decomposition is induced by a decomposition  $Z^c = Z^f \amalg Z_1^c$  of  $Z^c$ , where  $Z^f$  and  $Z_1^c$  are finite over  $R$  and  $\overline{Z^f} = \overline{Z}$ . In particular  $(Z^f, \overline{Z^f}) = (Z^f, \overline{Z})$  is a Zariski pair. Since  $Z \cap Z^f$  is open in  $Z^f$  and contains  $\overline{Z}$ , it is therefore equal to  $Z^f$  which means that  $Z^f \subset Z$  and  $Z = Z^f \amalg Z'$  with  $Z' := Z \cap Z_1^c$ . Thus, the desired conditions for  $Z^f$  are satisfied.  $\square$

**Remarks 2.2.3.** (1) Assertions (2) and (3) of 2.2.2 only use the existence of  $Z^f$  and the Zariski property for  $(R, I)$ .

(2) We see in particular that  $Z^f$  is the largest closed subscheme of  $Z$  which is finite over  $S$ . Moreover,  $Z^f$  is functorial in  $Z$ : if  $Y$  is a separated  $R$ -scheme of finite type with  $\overline{Y}$  finite over  $\overline{R}$ , every  $R$ -morphism  $Z \rightarrow Y$  sends  $Z^f$  to  $Y^f$ .

(3) Using more sophisticated tools, one can generalize 2.2.2 by replacing “finite” by “proper” in the conditions for  $\overline{Z}$  and  $Z^f$ . For the proof, the first step (reduction to the quasifinite case) is of course ignored. One uses Nagata compactification to choose an open immersion  $Z \hookrightarrow Z^c$  into a proper  $S$ -scheme  $p : Z^c \rightarrow S$ . Then by the properness of  $Z^c$  and the henselian property of  $(R, I)$ , we can apply [8, Tag 0A0C] to the sheaf  $(\mathbb{Z}/2\mathbb{Z})_{Z^c}$  to conclude that the idempotent defining  $\overline{Z}$  in  $\overline{Z^c}$  lifts to a unique idempotent on  $Z^c$ , which we take to define  $Z^f$ .

## 3 The preparation theorem

### 3.1 Notation and assumptions

We fix a ring  $R$  and an indeterminate  $t$ . We denote by  $\text{Alg}_{R[t]}^+$  the category of pairs  $(A, x)$  where  $A$  is an  $R[t]$ -algebra and  $x$  is an element of  $A$ .

We also fix an element  $f$  of  $R\{t\}$ , and we write

$$f_{\text{for}} = \sum_{i \geq 0} a_i t^i \in R[[t]] \quad (a_i \in R).$$

We assume that *the ideal generated by the  $a_i$ 's ( $i > 0$ ) is equal to  $R$* . Equivalently, for all  $\mathfrak{p} \in \text{Spec}(R)$ , the image of  $f$  in  $\kappa(\mathfrak{p})[[t]]$ , or in  $\kappa(\mathfrak{p})\{t\}$ , is not a constant.

Finally we denote by  $S$  the  $R[t]$ -algebra  $R\{t\}/(f)$ .

**Proposition 3.2.** *With the assumptions of 3.1, we also fix an indeterminate  $u$ .*

(1) *The object  $(R\{t\}, f)$  of  $\text{Alg}_{R[t]}^+$  is the filtered colimit of a system  $(A_\lambda, f_\lambda)_{\lambda \in L}$  with, for each  $\lambda \in L$ , the following properties:*

- (i) *The  $R[t]$ -algebra  $A_\lambda$  is étale and, for all  $n \in \mathbb{N}$ , the canonical morphism  $R[t]/(t^n) \rightarrow A_\lambda/t^n A_\lambda$  is an isomorphism.*
- (ii) *The canonical  $R$ -morphism  $R[u] \rightarrow A_\lambda$  mapping  $u$  to  $f_\lambda$  is flat and quasifinite.*

*In particular, the canonical  $R$ -morphism  $R[u] \rightarrow R\{t\}$  mapping  $u$  to  $f$  is flat, and  $f$  is a nonzerodivisor in  $R\{t\}$ .*

(2) *The  $R[t]$ -algebra  $S$  is the filtered colimit of a system  $(S_\lambda)_{\lambda \in L}$  with the following properties:*

- (i) *Each  $R$ -algebra  $S_\lambda$  is flat, of finite presentation and quasifinite, and the transition maps  $S_\lambda \rightarrow S_\mu$  ( $\lambda \leq \mu$ ) are étale. (In particular,  $S$  is flat over  $R$ .)*
- (ii) *For all  $n \in \mathbb{N}$  and  $\lambda \in L$ , the canonical morphism  $R[t]/(t^n) \rightarrow S_\lambda/t^n S_\lambda$  is surjective.*

*Proof.* Part (1) immediately implies part (2), with  $S_\lambda = A_\lambda/(f_\lambda)$  (the transition maps are étale due to the same property for the  $A_\lambda$ 's, which are étale over  $R[t]$ ).

To prove (1), write  $R\{t\} = \varinjlim_{\lambda \in L} A_\lambda$  as in Lemma 2.1.1, and call  $t_\lambda \in A_\lambda$  the canonical image of  $t$ . There exists  $\lambda_0 \in L$  and  $f_{\lambda_0} \in A_{\lambda_0}$  mapping to  $f$ ; we can restrict  $L$  to the indices  $\lambda \geq \lambda_0$  and, for each  $\lambda$ , denote by  $f_\lambda \in A_\lambda$  the image of  $f_{\lambda_0}$ . Clearly, we have  $(R\{t\}, f) = \varinjlim_{\lambda \in L} (A_\lambda, f_\lambda)$ . Part (1) (i) is obvious from the choice of  $(A_\lambda)_{\lambda \in L}$ .

Let us prove (1) (ii). For fixed  $\lambda$ , we can view  $f_\lambda$  as a morphism  $g_\lambda : X_\lambda := \text{Spec}(A_\lambda) \rightarrow \mathbb{A}_R^1 = \text{Spec}(R[u])$  of  $R$ -schemes. For  $s \in \text{Spec}(R)$ , the  $\kappa(s)$ -morphism  $g_{\lambda,s} : X_{\lambda,s} \rightarrow \mathbb{A}_{\kappa(s)}^1$  induced on the fibres is deduced from  $1 \otimes f_\lambda \in \kappa(s) \otimes_R A_\lambda$ , whose image in  $\kappa(s) \otimes_R R\{t\}$  is assumed nonconstant. So  $g_{\lambda,s}$  is not constant on  $X_{\lambda,s}$ , which is a smooth geometrically irreducible curve over  $\kappa(s)$ . It follows that  $g_{\lambda,s}$  is flat and quasifinite. Since  $X_\lambda$  and  $\mathbb{A}_R^1$  are smooth over  $\text{Spec}(R)$ , the ‘‘fiberwise flatness’’ criterion [4, (11.3.10)] shows that  $g_\lambda$  is flat. It is also quasifinite since it is affine of finite presentation with finite fibers. This completes the proof.  $\square$

**Definition 3.3.** *Let  $R$  be a ring,  $I$  an ideal of  $R$ ,  $t$  an indeterminate.*

*We say that a formal power series  $f = \sum_{i \geq 0} a_i t^i \in R[[t]]$  is  $I$ -normal if there is  $d \in \mathbb{N}$  such that  $a_d \in R^\times$  and  $a_i \in I$  for  $i < d$ . The integer  $d$  (unique if  $I \neq R$ ) is called the order of  $f$ .*

*We say that  $f$  is  $I$ -monic of order  $d$  if it is  $I$ -normal of order  $d$  and  $a_d = 1$ .*

*An element  $f$  of  $R\{t\}$  is  $I$ -normal ( $I$ -monic) of order  $d$  if  $f_{\text{for}} \in R[[t]]$  is.*

**Theorem 3.4.** *With notation and assumptions as in 3.1, let  $d \in \mathbb{N}$  be such that  $a_d \in R^\times$  and let  $I \subset R$  be the ideal generated by  $a_0, \dots, a_{d-1}$ . (Thus,  $f$  is  $I$ -normal of order  $d$ .)*

*Assume that  $(R, I)$  is a henselian pair. Then:*

- (1) *The images of  $1, t, \dots, t^{d-1}$  form a basis of the  $R$ -module  $S = R\{t\}/(f)$ .*
- (2) (Division theorem) *Every element of  $R\{t\}$  can be written uniquely in the form  $Bf + C$  where  $B \in R\{t\}$  and where  $C \in R[t]$  is a polynomial of degree  $< d$ .*
- (3) (Preparation theorem)  *$f$  can be written uniquely as  $f = (t^d + Q)v$  where  $v \in R\{t\}^\times$  and where  $Q \in R[t]$  has degree  $< d$  and coefficients in  $I$ . Moreover,  $v(0) \equiv a_d \pmod{I}$ .*

*Proof.* If  $d = 0$ , everything is trivial. We assume from now on that  $d > 0$ ; thus, the assumption of 3.1 is satisfied and, in particular, Proposition 3.2 applies to  $f$ .

Assertion (2) is an immediate consequence of (1), with uniqueness coming from the fact that  $f$  is a nonzerodivisor (3.2 (1)).

Let us prove that (2) implies (3). The relation in (3) can be rewritten as  $t^d = v^{-1}f - Q$ , so that uniqueness follows from the uniqueness part of (2); next, applying (2) to  $t^d$ , we find that  $t^d = Bf - Q$  where  $Q$  is a polynomial of degree  $< d$ . Reducing modulo  $I$  and comparing coefficients, we see that  $Q$  has coefficients in  $I$  and the constant term of  $B$  is in  $a_d + I$ , which gives (3) with  $v = B^{-1}$ .

It remains to prove (1). As in 2.2, we put  $\bar{A} = A/IA$  for every  $R$ -algebra  $A$ .

First we observe that the image  $\bar{f}$  of  $f$  in  $\bar{R}\{t\} \cong \bar{R}\{t\}$  is the product of  $t^d$  by a unit, so that  $\bar{S} \cong \bar{R}\{t\}/(t^d) \cong \bar{R}[t]/(t^d)$  which is  $\bar{R}$ -free with basis  $(1, t, \dots, t^{d-1})$ .

Let us write  $S$  as the colimit of a filtered system  $(S_\lambda)_{\lambda \in L}$  of  $R[t]$ -algebras with the properties of 3.2 (2). We have just seen that  $t^d$  vanishes in  $\bar{S}$ , so by changing  $L$  we may assume that  $t^d$  vanishes in  $\bar{S}_\lambda$  for all  $\lambda$ : thus,  $\bar{S}_\lambda = \overline{S_\lambda/t^d S_\lambda}$  hence, by 3.2 (2) (ii), it is a quotient of  $\bar{R}[t]/(t^d)$ . So we have morphisms of  $\bar{R}[t]$ -algebras  $\bar{R}[t]/(t^d) \rightarrow \bar{S}_\lambda \rightarrow \bar{S}$  where the first map is surjective and the composition is an isomorphism. We conclude that  $\bar{R}[t]/(t^d) \xrightarrow{\sim} \bar{S}_\lambda$  for all  $\lambda$ . In particular,  $\bar{S}_\lambda$  is finite over  $\bar{R}$ . As  $(R, I)$  is henselian, we may apply Proposition 2.2.2 and write  $S_\lambda = S_\lambda^f \times T_\lambda$ , where  $S_\lambda^f$  is *finite* over  $R$  and  $\bar{S}_\lambda^f = \bar{S}_\lambda$ . By functoriality (Remark 2.2.3), the quotients  $S_\lambda^f$  of the  $S_\lambda$ 's form an inductive system.

Since  $S$  is a quotient of  $R\{t\}$  and  $(R\{t\}, IR\{t\})$  is a Zariski pair, so is  $(S, IS)$ . Hence, for all  $\lambda$ , the canonical morphism  $S_\lambda \rightarrow S$  factors through  $S_\lambda^f$  by 2.2.2 (3), and finally  $S = \varinjlim_{\lambda \in L} S_\lambda^f$ .

Since, for each  $\lambda$ ,  $S_\lambda$  is a flat  $R$ -algebra of finite presentation, so is  $S_\lambda^f$ , which is in addition a finite  $R$ -module, hence locally free. As  $(1, t_\lambda, \dots, t_\lambda^{d-1})$  induces an  $\bar{R}$ -basis of  $\bar{S}_\lambda^f$ , and  $I \subset \text{rad}(R)$ , it follows easily that  $(1, t_\lambda, \dots, t_\lambda^{d-1})$  is an  $R$ -basis of  $S_\lambda^f$  for all  $\lambda$ , and part (1) follows.  $\square$

**Remark 3.4.1.** In Theorem 1.1 from the introduction,  $I$  denotes any ideal containing  $a_0, \dots, a_{d-1}$  and such that  $(R, I)$  is henselian. Clearly this implies that  $(R, (a_0, \dots, a_{d-1}))$  is henselian, so 1.1 follows from 3.4. Note that in the classical preparation theorem,  $I$  is the maximal ideal of the local ring  $R$ , while  $(a_0, \dots, a_{d-1})$  is not necessarily maximal; this is why the formulation of 1.1 is usually preferred in this case.

## 4 Application: a henselian resultant

If  $R$  is a ring,  $S$  a finite locally free  $R$ -algebra and  $x$  an element of  $S$ , we denote by  $N_{S/R}(x) \in R$  the norm of  $x$  in  $R$ , i.e. the determinant of multiplication by  $x$  in the  $R$ -module  $S$ .

**Definition 4.1.** Let  $(R, I)$  be a henselian pair. Let  $f \in R\{t\}$  be  $I$ -monic of order  $d$ . Denote by  $S$  the  $R$ -algebra  $R\{t\}/(f)$  (which is a free  $R$ -module of rank  $d$ , by 3.4 (1)).

For  $g \in R\{t\}$ , the (henselian) resultant of  $f$  and  $g$ , denoted by  $\text{Res}^h(f, g)$ , is the element of  $R$  defined by

$$\text{Res}^h(f, g) := N_{S/R}(g).$$

## 4.2 Properties of the resultant

We keep the notation and assumptions of 4.1, and we denote by  $P = t^d + Q$  the polynomial associated to  $f$  by 3.4 (3). The proofs of the following properties are easy and left to the reader.

**4.2.1. Functoriality:** Let  $\varphi : (R, I) \rightarrow (R', I')$  be a morphism of henselian pairs,  $f'$  et  $g'$  the images of  $f$  and  $g$  in  $R'\{t\}$ . Then  $\text{Res}^h(f', g') = \varphi(\text{Res}^h(f, g))$ .

**4.2.2.** By construction,  $\text{Res}^h(f, g)$  only depends on  $f$  via the  $R$ -algebra  $R\{t\}/(f)$ . In particular,  $\text{Res}^h(f, g) = \text{Res}^h(P, g)$ .

**4.2.3.**  $\text{Res}^h(f, g)$  only depends on  $g$  via its class modulo  $f$ ; in other words, we have  $\text{Res}^h(f, g + hf) = \text{Res}^h(f, g)$  for all  $h \in R\{t\}$ . Moreover,  $\text{Res}^h(f, g) \in R^\times$  if and only if the ideal  $(f, g) \subset R\{t\}$  equals  $R\{t\}$ . (More generally, see 4.2.8 below.)

**4.2.4. Special cases:** If  $\alpha \in R$ , we have  $\text{Res}^h(f, \alpha) = \alpha^d$  and  $\text{Res}^h(f, \alpha - t) = P(\alpha)$ .

If  $\alpha \in I$ , then  $\text{Res}^h(\alpha - t, g) = g(\alpha)$ , and  $\text{Res}^h(f, \alpha - t) = (1 + \varepsilon) f(\alpha)$  for some  $\varepsilon \in I$  by the second formula above (recall that  $f$  is  $I$ -monic).

**4.2.5. Multiplicativities:** If  $h \in R\{t\}$ , we have  $\text{Res}^h(f, gh) = \text{Res}^h(f, g) \text{Res}^h(f, h)$ ; if in addition  $h$  is  $I$ -monic of order  $m$ , then  $\text{Res}^h(fh, g) = \text{Res}^h(f, g) \text{Res}^h(h, g)$ . For the second equality, one may use the exact sequence

$$0 \longrightarrow R\{t\}/(h) \xrightarrow{\times f} R\{t\}/(fh) \longrightarrow R\{t\}/(f) \longrightarrow 0.$$

**4.2.6. Polynomials:** If  $f$  and  $g$  are in  $R[t]$ , with  $f$  monic of degree  $d$  (in the sense of polynomials), then  $\text{Res}^h(f, g)$  is the usual resultant. The condition on  $f$  is essential: for instance,  $\text{Res}^h(1 + \alpha t, g) = 1$  for all  $\alpha \in R$  and  $g \in R\{t\}$ . (In fact, for two possibly non-monic polynomials of respective degrees  $\leq d$  and  $\leq m$ , the definition of the classical resultant depends on the choice of  $d$  and  $m$ .)

**4.2.7. Weak symmetry:** Assuming that  $g$  is  $I$ -monic of order  $m$ , then  $\text{Res}^h(g, f) = (-1)^{md} (1 + \varepsilon) \text{Res}^h(f, g)$  for some  $\varepsilon \in I$ . To see this, reduce to the case of polynomials and apply 4.2.6.

**4.2.8. Elimination:** Let  $J \subset R\{t\}$  be the ideal generated by  $f$  and  $g$ . Then  $\text{Res}^h(f, g) \in J$  (thus it belongs to  $J \cap R$ ): indeed, in the free  $R$ -module  $S = R\{t\}/(f)$ , the image of multiplication by  $g$  contains  $\text{Res}^h(f, g) S$ .

Conversely, every  $\alpha \in J \cap R$  is a multiple of the class of  $g$  in  $S$  so, taking norms,  $\alpha^d$  is a multiple of  $\text{Res}^h(f, g)$  in  $R$ . In particular, we have in  $R$  the inclusions  $(\text{Res}^h(f, g)) \subset J \cap R \subset \sqrt{(\text{Res}^h(f, g))}$ . Geometrically, the closed subset  $V(\text{Res}^h(f, g)) \subset \text{Spec}(R)$  is the projection of  $V(f, g) \subset \text{Spec}(R\{t\})$ .

**4.2.9. Roots:** Let  $\varphi : R \rightarrow R'$  be a ring homomorphism, and let  $\alpha \in R'$  be a zero of  $P$  in  $R'$ . First, I claim that  $g(\alpha)$  makes sense in  $R'$  and is an element of  $R[\alpha] \subset R'$ . Indeed, the relation  $P(\alpha) = 0$  shows that (due to the form of  $P$ )  $\alpha^d \in IR[\alpha]$ , whence  $\alpha \in \sqrt{IR[\alpha]}$ . Since  $R[\alpha]$  is a finite  $R$ -module, the pair  $(R[\alpha], \sqrt{IR[\alpha]})$  is henselian, hence the claim.

Now assume that the image of  $P$  in  $R'[t]$  factors as  $\prod_{i=1}^d (t - \alpha_i)$ , where the  $\alpha_i$ 's are elements of  $R'$ . Then we have in  $R'$  the equality

$$\varphi(\text{Res}^h(f, g)) = \prod_{i=1}^d g(\alpha_i)$$



as follows from the above remark and properties 4.2.4 and 4.2.5 (applied in the ring  $R[\alpha_1, \dots, \alpha_d] \subset R'$ ).

Note that if we assume for simplicity that  $R = R'$  is a domain, then the  $\alpha_i$ 's are the zeros of  $f$  in  $\sqrt{I}$ .

**4.2.10. Power series:** Assume  $R$  is  $I$ -adically complete and separated. Then  $\text{Res}^h(f, g) = \text{Res}(f_{\text{for}}, g_{\text{for}})$  where  $\text{Res}$  denotes the resultant defined in [1].

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