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# Stochastic dynamic matching: A mixed graph-theory and linear-algebra approach

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## Abstract

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The stochastic dynamic matching problem has recently drawn attention in the stochastic-modeling community due to its numerous applications, ranging from supply-chain management to kidney exchange programs. In this paper, we consider a matching problem in which items of different classes arrive according to independent Poisson processes. Unmatched items are stored in a queue, and compatibility constraints are described by a simple graph on the classes, so that two items can be matched if their classes are neighbors in the graph. We analyze the efficiency of matching policies, not only in terms of system stability, but also in terms of matching rates between different classes.

Our results rely on the observation that, under any stable policy, the matching rates satisfy a conservation equation that equates the arrival and departure rates of each item class. Our main contributions are threefold. We first introduce a mapping between the dimension of the solution set of this conservation equation, the structure of the compatibility graph, and the existence of a stable policy. In particular, this allows us to derive a necessary and sufficient stability condition that is verifiable in polynomial time. Secondly, we describe the convex polytope of non-negative solutions of the conservation equation. When this polytope is reduced to a single point, we give a closed-form expression of the solution; in general, we characterize the vertices of this polytope using again the graph structure. Lastly, we show that greedy policies cannot, in general, achieve every point in the polytope. In contrast, non-greedy policies can reach any point of the interior of this polytope, and we give a condition for these policies to also reach the boundary of the polytope.

**2012 ACM Subject Classification** Mathematics of computing → Queueing theory; Mathematics of computing → Markov processes; Mathematics of computing → Matchings and factors

**Keywords and phrases** stochastic dynamic matching, graph theory, linear algebra, stability, matching rates, conservation equation

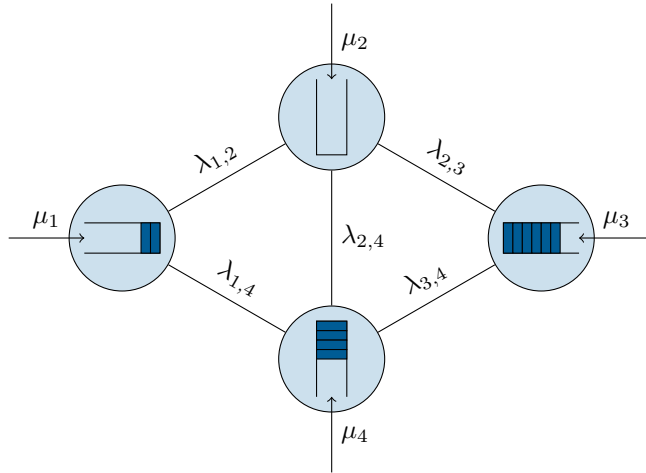
## 1 Introduction

The stochastic dynamic matching problem, in which items arrive at random instants to be matched with other items, has recently drawn a lot of attention in the stochastic-modeling community. This challenging control problem is indeed highly relevant in many applications, including supply-chain management, pairwise kidney exchange programs, and online marketplaces. In pairwise kidney exchange programs for example, each item represents a donor-receiver pair, and two pairs can be matched if the donor of each pair is compatible with the receiver of the other pair. In online marketplaces, items are typically divided into two categories, called demand and supply, and the objective is to maximize a certain long-term performance criteria by appropriately matching demand items with supply items.

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In this paper, we consider the following instance of the dynamic matching problem. Items of different classes arrive according to independent Poisson processes. Compatibility constraints between items are described by a simple graph on their classes, such that two items can be matched if their classes are neighbors in the graph. Unmatched items are stored in the queue of their class, and the matching policy decides which matches are performed and when. All in all, a stochastic matching model is described by a triplet  $(G, \mu, \Phi)$ , where  $G = (V, E)$  is the compatibility graph,  $\mu = (\mu_1, \mu_2, \dots, \mu_n)$  is the vector of per-class arrival rates, and  $\Phi$  is the matching policy. In Figure 1 for instance, there are four item classes numbered from 1 to 4; classes 2 and 4 are compatible with all classes, while classes 1 and 3 are only compatible with classes 2 and 4.



■ **Figure 1** Illustration of a matching model  $(G, \mu, \Phi)$  on the diamond graph.

We propose a unified approach to study two closely-related performance criteria, namely the stability and the matching rates along edges. Formally, a matching model  $(G, \mu, \Phi)$  is said to be stable if the associated continuous-time Markov chain is positive recurrent. Assuming that the matching model  $(G, \mu, \Phi)$  is stable, the matching rate  $\lambda_k$  along an edge  $k \in E$  with endpoints  $i, j \in V$  is the rate at which class- $i$  items and class- $j$  items are matched.

## 1.1 Contributions

The following observation is powerful despite its simplicity, and it is fundamental to all our results. Under any stable policy, the arrival rate of class- $i$  items is equal to the departure rate of these items, which in turn is equal to the sum of the matching rates along the edges that are incident to node  $i$ . In other words, the matching rates satisfy the following *conservation* equation:

$$\sum_{k \in E_i} \lambda_k = \mu_i, \quad i \in V,$$

where  $E_i \subseteq E$  is the set of edges that are incident to node  $i$ . In matrix form, this equation rewrites  $A\lambda = \mu$ , where  $A$  is the incidence matrix of the compatibility graph  $G$ . The solution set of this conservation equation is related to the structure of the graph  $G$  via the linear application  $y \in \mathbb{R}^m \mapsto Ay \in \mathbb{R}^n$ , where  $n$  is the number of nodes (or classes), and  $m$  is the number of edges. We say that the graph  $G$  is surjective (resp. injective, bijective) if the linear application  $y \in \mathbb{R}^m \mapsto Ay \in \mathbb{R}^n$  is surjective (resp. injective, bijective), and we give simple

equivalent conditions in terms of the graph structure (Definitions 2.4–2.7 and Proposition 2.8). Our main contributions are threefold, and all rely on these definitions.

We first prove that there exists a direct relation between the dimension of the affine space of the solutions of the conservation equation, the structure of the compatibility graph, and the existence of a stable policy. More specifically, we prove that a compatibility graph  $G$  is stabilizable (in the sense that there exist a vector  $\mu$  and a policy  $\Phi$  such that the matching model  $(G, \mu, \Phi)$  is stable) if and only if the graph  $G$  is surjective (Proposition 3.1). We then prove that a matching problem  $(G, \mu)$  is stabilizable (in the sense that there exists a policy  $\Phi$  such that the matching model  $(G, \mu, \Phi)$  is stable) if and only if the conservation equation has a solution with positive components (Proposition 3.2). In particular, this allows us to verify stabilizability in a time that is polynomial in the number of classes and edges.

We next describe the affine space of solutions of the conservation equation and the convex polytope of solutions with non-negative components. When this convex polytope is reduced to a single point, we derive a closed-form expression for the solution (Proposition 4.1). When this convex polytope is not reduced to a single point, we characterize its vertices, again using the graph structure. We prove in particular that a non-negative solution of the conservation equation is a vertex of the convex polytope if and only if the subgraph restricted to the support of this vector is injective (Proposition 5.7 and Corollary 5.8).

Lastly, we prove that greedy policies cannot, in general, achieve every point of this convex polytope (Propositions 6.2–6.4 and Corollary 6.5). A matching policy is said to be greedy if it never postpones a feasible match; equivalently, the set of unmatched item classes is always an independent set of the compatibility graph. In contrast, non-greedy policies can reach any point of the interior of this polytope (Proposition 6.12 and Corollary 6.13). They can also reach part or all the boundary of this polytope, depending on conditions that are again expressed in terms of the bijectivity of subgraphs of  $G$  (Proposition 6.10 and Corollary 6.11).

## 1.2 State of the art

We now review the relevant work related to (static or dynamic) matching problems.

### Non-bipartite or general stochastic matching

Our work is part of a broader research effort on the stochastic matching model that will be described in details in Section 2.1 [5, 6, 11, 14, 20, 23, 24]. Among these works, the following are particularly relevant because directly related to our results on stability. The paper [23] is the earliest work on this matching model. It derives several necessary and sufficient stability conditions that are instrumental in several of our results, in particular Propositions 3.1 and 3.2. This work also proves that the *match-the-longest* policy is maximally stable (in the sense that it is always stable if  $(G, \mu)$  is stabilizable), a result that is also applied in Proposition 3.2. The papers [14, 24] focus on the *first-come-first-matched* policy. In particular, [24] proves that the first-come-first-matched policy is maximally stable, and [14] provides a new sufficient stability condition we prove to be also necessary in Proposition 3.2.

Other variants of the model were studied recently, and an interesting future work would consist of generalizing our results to these variants. In particular, the paper [20] consider item abandonments, the paper [6] considers graphs with loops, and the papers [19, 25, 26] allow matches between several items by replacing the graph with a hypergraph.

The recent paper [5] is perhaps the closest to ours, and we provide a detailed discussion to highlight the relation with our paper. The equivalence of statements (ii) and (iv) in [5, Theorem 1] is synonym to the equivalence of statements 2 and 3 in Proposition 3.2. Our

proof is significantly shorter because it relies more heavily on existing results. [5, Theorem 4] is a special case of our observation at the beginning of Section 4 that the conservation equation has a unique solution if and only if the graph is bijective (and not surjective-only). Several other formulas derived in [5, Section 7] are special cases of the formulas derived in Proposition 4.1. The model in [5] is slightly more general because it consider graphs with loops, that is, an edge can have identical endpoints, but this paper does not adopt the mixed graph-theory and linear-algebra approach that supports most of our results.

### Bipartite stochastic matching

The first example of a stochastic matching model in the literature, which predated the model that we consider, is the bipartite matching model introduced in [12] and studied in [1, 2, 3, 8, 9, 10, 15]. In this model, the compatibility graph is bipartite, with two parts that correspond to supply and demand items, respectively. This bipartite matching model differs from our model by its arrival process: time is slotted and, during each time slot, one demand item *and* one supply item arrive. Several works have made contributions about stability [2, 8] and matching rates [2], and obtained results similar to those derived in the literature on our model. The bipartite nature of the graph simplifies some calculations, for instance by allowing the application of flow-maximization algorithms to calculate optimal matching rates.

### Static and fractional matching

The static matching problem, in which the nodes of the graph represent items (rather than classes), has been extensively studied in mathematics, computer science, and economy [22]. Although the questions raised in static and dynamic matching are often different, the conservation equation that we obtain is reminiscent of several results in static matching. For example, finding a maximum-cardinality matching in the graph  $G$  (that is, a maximum-cardinality set of edges without common endpoints) is equivalent to finding integers  $\lambda_k \in \{0, 1\}$  for each edge  $k \in E$  that maximize  $\sum_{k \in E} \lambda_k$  while satisfying the conservation equation with  $\mu_i = 1$  for each  $i \in V$ . The relaxation of this integer linear program leads to the so-called fractional matching problem, which has been studied in the literature [22, Section 7.2]. Therefore, the fractional matching polytope defined in [22, Section 7.5] is a special case of the convex polytope considered in Section 5.2, and our characterization of this convex polytope is a natural generalization of existing characterizations of the fractional polytope <sup>2</sup>.

## 1.3 Outline

The remainder of the paper is organized as follows. Section 2 gives a formal definition of the model and explains our approach. In particular, we introduce the conservation equation and define the notions of surjective, injective, and bijective graph that will be instrumental in the rest of the paper. In Section 3, we give necessary and sufficient stability conditions in terms of the solutions of the conservation equation. Sections 4 and 5 characterize the solution set of the conservation equations, and give a closed-form expression of the solution when it is unique. Lastly, in Section 6, we study the set of matching rates that are effectively achievable under a stable matching policy.

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<sup>2</sup> The fractional matching polytope is actually defined using non-strict inequalities rather than equalities. However, one can verify that both convex polytopes have the same non-zero vertices.

## 2 Model and approach

### 2.1 Dynamic matching

Consider a stochastic dynamical matching system in which items arrive at random times to be matched with other items. Each incoming item may be matched with any unmatched item of a compatible class; in this case, both items disappear immediately. Unmatched items are gathered in a waiting queue. In this paper, a stochastic matching model will be described by a triplet  $(G, \mu, \Phi)$ , where  $G$  is the compatibility graph,  $\mu$  is the vector of arrival rates, and  $\Phi$  is the matching policy. We now review each component in details. Notation is summarized Table 1, and an example is shown in Figure 1.

#### Compatibility graph $G$

Compatibility constraints between items are described by a graph  $G = (V, E)$ , called the *compatibility graph* of the model, that is simple (undirected and without loop). We let  $n$  denote the number of nodes and  $m$  the number of edges. The set of nodes is denoted by  $V = \{v_1, v_2, \dots, v_n\}$ , and each node corresponds to a class in the matching model. When there is no ambiguity, for ease of notation, we will refer to a class  $v_i$  using its index  $i$ . The set of edges is denoted by  $E = \{e_1, e_2, \dots, e_m\}$ . These edges represent compatibility constraints between item classes, in the sense that a class- $i$  item and a class- $j$  item can be matched with one another if and only if there is an edge with endpoints  $i$  and  $j$  in the graph. When there is no ambiguity, for ease of notation, we will refer to an edge  $e_k \in E$  with endpoints  $i, j \in V$  using either its index  $k$  or its set of endpoints  $\{i, j\}$ . In Figure 1 for instance, there are four item classes numbered from 1 to 4. Classes 2 and 4 are compatible with all classes, but classes 1 and 3 are only compatible with classes 2 and 4. The absence of loop means that an item of a given class cannot be matched with other items of the same class.

#### Vector $\mu$ of arrival rates

Class- $i$  items arrive according to an independent Poisson process with rate  $\mu_i > 0$ , for each  $i \in V$ . The vector of arrival rates is denoted by  $\mu = (\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{R}_{>0}^n$ . Scaling all components of  $\mu$  by the same positive constant is equivalent to changing the time unit, so we can renormalize  $\mu$  without changing the system dynamics. In this paper, we will often use the *unit* normalization in which  $\sum_{i \in V} \mu_i = 1$ . The couple  $(G, \mu)$  is called a *matching problem*.

#### Matching policy $\Phi$

The matching policy specifies, upon each item arrival, which matches are performed depending on the system state and the classes of incoming items. Observe that the nature of the information that the state needs to retain in order to obtain a continuous-time Markov chain depends on the policy that we consider. To overcome this difficulty, we assume that the state descriptor is an integral part of the definition of a policy.

Formally, a matching policy  $\Phi$  is a tuple  $(\mathcal{X}, \pi)$ , where  $\mathcal{X}$  is a countably infinite state space and  $\pi$  is a function that defines transitions. More specifically, for each  $x \in \mathcal{X}$  and  $i \in V$ ,  $\pi(\cdot|x, i) : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$  is a probability distribution on  $\mathcal{X}$ , so that  $\pi(y|x, i)$  is the probability that the state jumps from state  $x$  to state  $y$  upon the arrival of a class  $i$  item. We assume that the matching policy is such that only items with compatible classes are matched, and that matches occur only upon item arrivals.

We make two additional assumptions. First, we assume that the evolution of the system under policy  $\Phi$  defines a continuous-time Markov chain  $\{X(t), t \geq 0\}$ , with state space  $\mathcal{X}$ , that is irreducible. Second, we assume that the system state retains *at least* the size of each queue, in the sense that there exists a function  $h : \mathcal{X} \rightarrow \mathbb{N}^n$  such that, for each  $x \in \mathcal{X}$ , the  $i$ -th component of the vector  $h(x)$  gives the number of unmatched class- $i$  items in state  $x$ . This assumption guarantees that the intuitive notion of system stability is indeed captured by the positive recurrence of the Markov chain  $\{X(t), t \geq 0\}$ .

A policy is said to be *greedy* if an incoming item is matched as soon as it arrives if possible, that is when there is an unmatched item that is compatible. Equivalently, a policy is greedy if the set of unmatched items classes under this policy is always an independent set of the compatibility graph. Here are two examples of deterministic greedy policies:

- **Match-the-longest.** An incoming item is matched immediately with an item of the compatible class that has the longest queue, if any. The system state  $x = (x_1, x_2, \dots, x_n)$  gives the number of unmatched items of each class, and its state space is  $\mathcal{X} = \{x \in \mathbb{N}^n : x_i x_j = 0 \text{ if } i \in V_j\}$ , where  $V_j \subset V$  is the set of nodes adjacent to node  $j$ . The transition probabilities are as follows:
  - if there is  $j \in V_i$  such that  $x_j > 0$ , then  $\pi(x - e_{j^*} | x, i) = 1$ , where  $j^* = \operatorname{argmax}_{j \in V_i} (x_j, j)$  (we use the class index to tie-break queues of same size) and  $e_{j^*}$  is the  $n$ -dimensional vector with 1 in component  $j^*$  and 0 elsewhere;
  - otherwise,  $\pi(x + e_i | x, i) = 1$ .
- **First-come-first-matched.** An incoming item is matched immediately with the compatible item that has been waiting the longest, if any. The system state  $x = (x_1, x_2, \dots, x_p)$  gives the sequence of unmatched item classes, ordered by arrival times, so that  $x_1$  is the class of the oldest unmatched item. The state space  $\mathcal{X} \subset \{1, \dots, n\}^*$  is made of all finite sequences that do not contain items from compatible classes. In other words,  $\mathcal{X} = \bigcup_{\mathcal{I} \in \mathbb{I}} \mathcal{I}^*$ , where  $\mathbb{I}$  is the set of the independent sets of the graph  $G$ . The transition probabilities are as follows:
  - if there exists  $q \in \{1, 2, \dots, p\}$  such that  $x_q \in V_i$ , then  $\pi(y | x, i) = 1$  if  $y = (x_1, \dots, x_{q^*-1}, x_{q^*+1}, \dots, x_p)$ , where  $q^* = \min\{q \in \{1, 2, \dots, p\} : x_q \in V_i\}$ ;
  - otherwise,  $\pi(y | x, i) = 1$  if  $y = (x_1, x_2, \dots, x_p, i)$ .

A policy is said to be *semi-greedy* if can be seen as a greedy policy on a subgraph  $G' = (V, E')$  containing a subset  $E' \subseteq E$  of edges. Semi-greedy policies will be used in Section 6.

For the sake of simplicity, we assume that the state space  $\mathcal{X}$  contains a state  $\emptyset \in \mathcal{X}$  such that  $h(\emptyset) = 0^n$  (all queues are empty) and that  $h(x) \neq 0^n$ , for all  $x \neq \emptyset$ . We call  $\emptyset$  the *empty state*. An empty state always exists for greedy policies, as long as  $G$  has no isolated nodes: from any state with waiting items, if an item from a compatible class arrives, the number of waiting items is reduced by 1 and we can iterate until there is no waiting item. This assumption may not hold for an arbitrary policy (for example, it rules out a policy that does not allow any match as long as there are less than two items in the system). The uniqueness of the empty state is not necessary, all the results of the paper remain valid if there is a finite number of different states with no items in the queues.

On a broader note, our definition of a matching policy is intended to be general, but we still impose several limitations for ease of notation. For instance, our assumption that items can only be matched upon arrivals rules out policies that perform matches in batches at fixed time intervals. We believe that all our results can be extended to other policies if necessary on a case-by-cases basis.

In the remainder, we will identify the *matching model*  $(G, \mu, \Phi)$  with the continuous-time Markov chain  $\{X(t), t \geq 0\}$ . This allows us to define *stability*.

► **Definition 2.1** (Stability). A matching model  $(G, \mu, \Phi)$  is said to be stable if the associated Markov process  $\{X(t), t \geq 0\}$  is positive recurrent.

We also derive from stability the notion of *stabilizability*.

► **Definition 2.2** (Stabilizability). A compatibility graph  $G$  is said to be stabilizable if there exists a vector  $\mu \in \mathbb{R}_{\geq 0}^n$  of arrival rates and a matching policy  $\Phi$  such that the matching model  $(G, \mu, \Phi)$  is stable. A matching problem  $(G, \mu)$  is said to be stabilizable if there exists a matching policy  $\Phi$  such that the matching model  $(G, \mu, \Phi)$  is stable.

Another notion of importance is *maximal stability*.

► **Definition 2.3** (Maximal stability). A generic policy is a policy that can be applied to any matching problem  $(G, \mu)$ . Formally, it is a function that associates to each graph  $G$  a tuple  $(\mathcal{X}, \pi)$  adapted to  $G$  but with a slight abuse of notation we use the term to describe the produced policies as well. A generic policy  $\Phi$  is said to be maximally stable if whenever  $(G, \mu)$  is stabilizable,  $(G, \mu, \Phi(G))$  is stable.

The greedy policies *match-the-longest* and *first-come-first-matched* introduced above are obviously generic. They were proved to be maximally stable in [23, 24].

### Vector $\lambda$ of matching rates and conservation equation

If the matching model  $(G, \mu, \Phi)$  is stable, the *matching rate*  $\lambda_k$  along edge  $e_k = \{i, j\}$  is defined as the long-run average number of matches between a class- $i$  item and a class- $j$  item per time unit. Let  $\lambda(G, \mu, \phi) = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}_{\geq 0}^m$  denote the vector of matching rates associated to  $(G, \mu, \phi)$ . For ease of notation, we use in the rest of the paper  $\lambda(\Phi)$  or  $\lambda$ , instead of  $\lambda(G, \mu, \Phi)$ , if the underlying matching model is clear in the context. These matching rates satisfy the following *conservation equation*:

$$\sum_{k \in E_i} \lambda_k = \mu_i, \quad i \in V, \quad (1)$$

where  $E_i \subseteq E$  is the set of edges that are incident to node  $i$ , for each  $i \in V$ . In matrix form, this conservation equation writes

$$A\lambda = \mu, \quad (2)$$

where the  $n \times m$  matrix  $A = (a_{i,k})_{i \in V, k \in E}$  is the *incidence matrix* of the graph  $G$ , defined by  $a_{i,k} = 1$  if edge  $k$  is incident to node  $i$  and  $a_{i,k} = 0$  otherwise. The majority of the results presented here are deeply connected to the conservation equation (2). In particular, the last part of the paper focuses on the solutions of (2) and considers, for a given stabilizable problem  $(G, \mu)$ , the following sets:

$$\begin{aligned} \Lambda &= \{y \in \mathbb{R}^m : Ay = \mu\} \text{ (studied in Section 5.1).} \\ \Lambda_{\geq 0} &= \{y \in \mathbb{R}_{\geq 0}^m : Ay = \mu\} \text{ (studied in Section 5.2).} \\ \Lambda_{> 0} &= \{y \in \mathbb{R}_{> 0}^m : Ay = \mu\} \text{ (introduced in Section 6).} \\ \Lambda_{\mathcal{G}} &= \{\lambda(\Phi) : \Phi \text{ is a stable greedy policy}\} \text{ (studied in Section 6.1).} \\ \Lambda_{\mathcal{P}} &= \{\lambda(\Phi) : \Phi \text{ is a stable policy}\} \text{ (studied in Section 6.2).} \end{aligned}$$

Note that while it is not the main focus of the present work, long-term matching rates can usually be defined even if the matching model  $(G, \mu, \Phi)$  is unstable. In that case, the sum of matching rates along the edges incident to a node are at most equal to its arrival rate:  $A\lambda \leq \mu$ . Lemma 6.9 shows an example of an unstable policy that admits a limit matching rate.



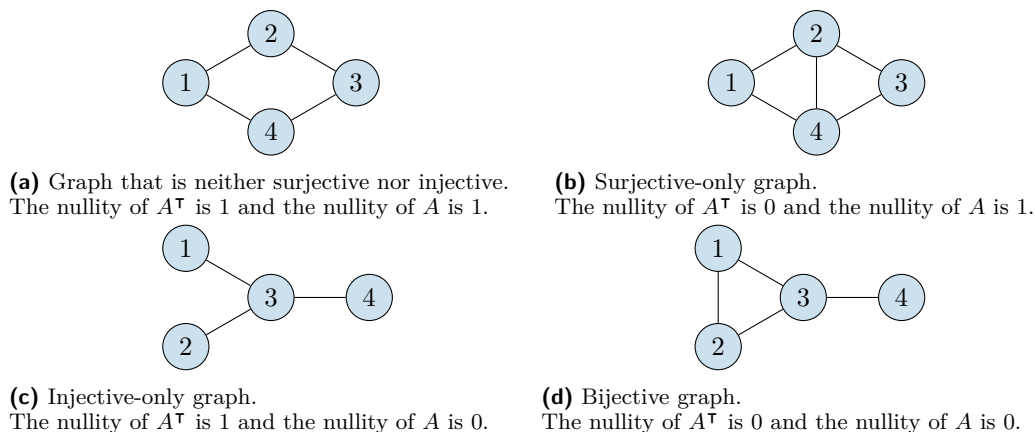
General notation	
$\mathbb{N}, \mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{R}_{> 0}$	Sets of non-negative integers, real numbers, non-negative real numbers, positive real numbers.
$\geq, \leq, >, <$	Component-wise comparison in $\mathbb{R}^n$ .
$ \mathcal{A} $	Cardinality of the set $\mathcal{A}$ .
Graph notation	
$G = (V, E)$	Simple graph $G$ with $ V  = n$ vertices and $ E  = m$ edges.
$v_i$	Vertex indexed by $i$ (denoted $i$ if there is not ambiguity).
$e_{i,j}, \{i, j\},$ or $e_k$	Edge between vertices $i$ and $j$ , indexed by $k$ .
$\mathbb{I}$	Family of independent sets of the graph $G$ .
$V_i \subseteq V$	Set of neighbors of node $v_i$ in the graph $G$ .
$V(\mathcal{I}) = \bigcup_{i \in \mathcal{I}} V_i$	Set of neighbors of the vertices indexed by the independent set $\mathcal{I} \in \mathbb{I}$ .
$d_{i,j}$	distance between nodes $i$ and $j$ .
$d_{i,k}$	distance between node $i$ and edge $k$ .
$K_\ell$	Complete graph of size $\ell \geq 3$ .
$C_\ell$	Cycle of size $\ell \geq 3$ .
$P_\ell$	Path of length $\ell \geq 0$ .
$KP_{\ell,r,p}$	Kayak paddle: two cycles $C_\ell$ and $C_r$ attached by a path $P_p$ .
Matching notation	
$\mu = (\mu_i)_{1 \leq i \leq n}$	Arrival rates of item classes.
$\Phi$	A matching policy.
$\lambda(\Phi) = (\lambda_k)_{1 \leq k \leq m}$ $= (\lambda_{i,j})_{\{i,j\} \in E}$	Matching rates of $\Phi$ along the edges (denoted $\lambda$ if there is not ambiguity).
$\Lambda_{\mathcal{P}}$	Set of matching rates achieved by stable policies.
$\Lambda_{\mathcal{G}}$	Set of matching rates achieved by stable greedy policies.
Linear algebra notation	
$x = (x_1, x_2, \dots, x_n)$	A vector in $\mathbb{R}^n$ . All vectors in $\mathbb{R}^n$ are column vectors.
$y = (y_1, y_2, \dots, y_m)$	A vector in $\mathbb{R}^m$ . All vectors in $\mathbb{R}^m$ are column vectors.
$A = (a_{i,k})_{i \in V, k \in E}$	Incidence matrix of the graph $G$ .
$A^\top = (a_{k,i})_{k \in E, i \in V}$	Transpose of the matrix $A$ .
$\ker(A) = \{y \in \mathbb{R}^m : Ay = 0\}$	Right kernel of the matrix $A$ . Its dimension is called the nullity of $A$ .
$\ker(A^\top) = \{x \in \mathbb{R}^n : A^\top x = 0\}$	Left kernel of the matrix $A$ . Its dimension is the nullity of $A^\top$ .
$d = m - n$	Dimension of the right kernel of the matrix $A$ if $G$ is surjective.
$\mathcal{B} = (b_1, \dots, b_d)$	Basis of the right kernel of the matrix $A$ if $G$ is surjective.
$\Lambda = \{y \in \mathbb{R}^m : Ay = \mu\}$	Affine space of the solutions of the conservation equation (2).
$\Lambda_{\geq 0} = \{y \in \mathbb{R}_{\geq 0}^m : Ay = \mu\}$	Polytope of non-negative solutions of (2).
$\Lambda_{> 0} = \{y \in \mathbb{R}_{> 0}^m : Ay = \mu\}$	Positive solutions of (2).

■ **Table 1** Table of notation

## 2.2 Surjectivity, injectivity, and bijectivity

Definitions 2.4–2.7 below introduce the notions of *surjectivity*, *injectivity*, and *bijectivity* of a graph. In a nutshell, a graph  $G$  is said to be surjective (resp. injective, bijective) if the linear application defined by its incidence matrix  $A$  is surjective (resp. injective, bijective). Interestingly, we find equivalent conditions in terms of the graph structure. As we will see

later, these notions are fundamental to study the stability of stochastic matching models and the associated matching rates. In particular, we will see in Section 3 that a compatibility graph  $G$  is stabilizable if and only if  $G$  is surjective in the sense of Definition 2.4. Later, in Sections 3 and 4, we will see that the matching rates in a matching problem  $(G, \mu)$  are independent of the matching policy  $\Phi$  (as long as the model is stable) if and only if  $G$  is bijective in the sense of Definition 2.6. Examples are shown in Figure 2.



■ **Figure 2** Examples of surjective and injective graphs.

► **Definition 2.4** (Surjective graph). Consider a simple graph  $G = (V, E)$  with  $n$  nodes and  $m$  edges. Let  $A$  denote the  $n \times m$  incidence matrix of  $G$ . The graph  $G$  is called surjective if one of the following equivalent conditions is satisfied:

1. The function  $y \in \mathbb{R}^m \mapsto Ay \in \mathbb{R}^n$  is surjective.
2. The equation  $Ay = \mu$  of unknown  $y \in \mathbb{R}^m$  has at least one solution for each  $\mu \in \mathbb{R}^n$ .
3. The left kernel of the matrix  $A$  is trivial.
4. Each connected component of the graph  $G$  is non-bipartite.

**Proof.** The equivalence of conditions 1, 2, and 3 is a well-known result in linear algebra. We prove that conditions 3 and 4 are equivalent. This proof is adapted from [16, Lemma 2.2.3].

Let  $a_1, a_2, \dots, a_n$  denote the rows of the matrix  $A$ , so that  $a_i = (a_{i,k})_{k \in \{1, \dots, m\}}$  where  $a_{i,k} = 1$  if node  $i$  is an endpoint of edge  $k$ , and  $a_{i,k} = 0$  otherwise. A vector  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  belongs to the left kernel of the matrix  $A$  if and only if

$$x_1 a_1 + x_2 a_2 + \dots + x_n a_n = 0.$$

For each  $k \in \{1, \dots, m\}$ , the  $k$ -th row of this equation reads  $x_j = -x_i$ , where  $i$  and  $j$  are the endpoints of edge  $k$ . An induction argument shows that, for every path  $i_1, i_2, \dots, i_k$  in the graph  $G$ , we have  $x_{i_p} = (-1)^p x_{i_1}$  for each  $p \in \{1, 2, \dots, k\}$ .

First assume that each connected component of  $G$  is non-bipartite, and let  $V'$  denote the set of nodes in a given connected component. Since this component is non-bipartite, there exists a cycle  $i_1, i_2, \dots, i_k, i_{p+1} = i_1$  consisting of an odd number  $p$  of nodes. The previous remark implies that  $x_{i_1} = 0$ , which in turn implies that  $x_i = 0$  for each  $i \in V'$ .

On the contrary, if there exists a connected component of  $G$  that is bipartite with parts  $V_+$  and  $V_-$ , then we build a non-zero vector by choosing  $x_i = 1$  for each  $i \in V_+$ ,  $x_i = -1$  for each  $i \in V_-$ , and  $x_i = 0$  for each  $i \in V \setminus (V_+ \cup V_-)$ . ◀

► **Definition 2.5** (Injective graph). *Consider a simple graph  $G = (V, E)$  with  $n$  nodes and  $m$  edges. Let  $A$  denote the  $n \times m$  incidence matrix of  $G$ . The graph  $G$  is called injective if one of the following equivalent conditions is satisfied:*

1. *The function  $y \in \mathbb{R}^m \mapsto Ay \in \mathbb{R}^n$  is injective.*
2. *The equation  $Ay = \mu$  of unknown  $y \in \mathbb{R}^m$  has at most one solution for each  $\mu \in \mathbb{R}^n$ .*
3. *The right kernel of the matrix  $A$  is trivial.*
4. *Each connected component of the graph  $G$  contains at most one odd cycle and no even cycle; in other words, each connected component of  $G$  is either a tree or a unicyclic graph with an odd cycle.*

**Proof.** The equivalence of conditions 1, 2, and 3 is a well-known result in linear algebra. We now prove that conditions 3 and 4 are equivalent.

Let us assume for now that the graph  $G$  is connected. We first remark that the proof of Definition 2.4 shows that the nullity of  $A^\top$  is 0 if  $G$  is non-bipartite and 1 if  $G$  is bipartite. We therefore distinguish two cases:

- If  $G$  is non-bipartite, the nullity of  $A^\top$  is 0. The rank-nullity theorem implies that the rank of  $A^\top$  is  $n$ , so that the rank of  $A$  is also  $n$ . A second application of the rank-nullity theorem implies that the nullity of  $A$  is  $m - n$ . In particular,  $\ker(A) = \{0\}$  if and only if  $m = n$ .
- If  $G$  is bipartite, the nullity of  $A^\top$  is 1, and we conclude similarly that the nullity of  $A$  is  $m - n + 1$ . In particular,  $\ker(A) = \{0\}$  if and only if  $m = n - 1$ .

All in all, we obtain that condition 3 is true if and only if either the graph  $G$  is non-bipartite and contains as many edges as nodes, or the graph  $G$  is bipartite and contains one less edge than it contains nodes. This, in turn, is equivalent to condition 4.

If the graph  $G$  is not connected, we can rewrite the matrix  $A$  as a bloc matrix in which each bloc corresponds to a connected component, and we can then use the previous argument to prove the equivalence for each connected component. ◀

► **Definition 2.6** (Bijective graph). *Consider a simple graph  $G = (V, E)$  with  $n$  nodes and  $m$  edges. Let  $A$  denote the  $n \times m$  incidence matrix of  $G$ . The graph  $G$  is called bijective if the following equivalent conditions are satisfied:*

1. *The function  $y \in \mathbb{R}^m \mapsto Ay \in \mathbb{R}^n$  is bijective.*
2. *The equation  $Ay = \mu$  of unknown  $y \in \mathbb{R}^m$  has exactly one solution for each  $\mu \in \mathbb{R}^n$ .*
3. *The matrix  $A$  is invertible.*
4. *Each connected component of the graph  $G$  contains one cycle and this cycle is odd.*

**Proof.** The function  $y \in \mathbb{R}^m \mapsto Ay \in \mathbb{R}^n$  is bijective if and only if it is both surjective and injective. Hence, the equivalence of conditions 1 to 4 follows directly from Definitions 2.4 and 2.5. ◀

► **Definition 2.7** (Surjective-only graph and injective-only graph). *An simple graph  $G$  is called surjective-only (resp. injective-only) if  $G$  is surjective but not injective (resp. injective but not surjective).*

The following proposition gives necessary conditions for surjectivity and injectivity in terms of the number of nodes and edges in the graph.

► **Proposition 2.8.** *Consider an undirected graph  $G = (V, E)$  with  $n$  nodes and  $m$  edges.*

1. *If  $G$  is surjective, then  $n \leq m$ .*
2. *If  $G$  is injective, then  $n \geq m$ .*
3. *If  $G$  is bijective, then  $n = m$ .*

4. If  $G$  is surjective, then  $G$  is also injective if and only if  $n = m$ .
5. If  $G$  is injective, then  $G$  is also surjective if and only if  $n = m$ .

**Proof.** These statements are again well-known results in linear algebra. ◀

### 3 Stability conditions

This section gives necessary and sufficient conditions under which a compatibility graph  $G$  or a matching problem  $(G, \mu)$  is *stabilizable* in the sense of Definition 2.2.

#### 3.1 Stabilizable graph

The following proposition gives necessary and sufficient conditions for a graph  $G$  to be stabilizable, in terms of either its structure or its incidence matrix.

► **Proposition 3.1.** *Let  $G$  be an undirected graph. The following conditions are equivalent:*

1. *The graph  $G$  is stabilizable.*
2. *The graph  $G$  is surjective.*

**Proof.** Equivalence between condition 1 in Proposition 3.1 and condition 4 in Definition 2.4 has been proved in [23, Theorem 1]. ◀

The equivalence between condition 1 in Proposition 3.1 and condition 4 in Definition 2.4 was already proved in [23] in the context of stochastic matching models. However, to the best of our knowledge, the equivalence between condition 1 in Proposition 3.1 and the other definitions of surjectivity introduced in Definition 2.4 has not been considered in the literature on stochastic matching models yet. As we will see later, this new characterization of the stabilizability of a graph  $G$  will be useful to analyze the matching rates.

Unless stated otherwise, in the rest of the paper, we assume that the graph  $G$  is surjective.

#### 3.2 Stabilizable arrival rates

We now turn to the stabilizability of a matching problem  $(G, \mu)$ . As recalled in Section 2.1, two examples of greedy policies that stabilize the model whenever this matching problem is stabilizable are the *match-the-longest* [23] and *first-come-first-matched* [24]. Proposition 3.2 provides necessary and sufficient conditions for the matching problem  $(G, \mu)$  to be stabilizable; condition 2 was already derived in [23], but condition 3 is new.

► **Proposition 3.2.** *Consider an undirected surjective graph  $G$  and a vector  $\mu \in \mathbb{R}_{>0}^n$ . The following conditions are equivalent:*

1. *The graph  $G$  and the vector  $\mu$  are stabilizable.*
2. *For each  $\mathcal{I} \in \mathbb{I}$ , we have  $\sum_{i \in \mathcal{I}} \mu_i < \sum_{i \in V(\mathcal{I})} \mu_i$ .*
3. *The conservation equation (2) has a solution  $\lambda \in \mathbb{R}_{>0}^m$  (i.e. with all components positive).*

**Proof.** Equivalence of conditions 1 and 2 follows from Lemma 1, Proposition 2, and Theorem 2 in [23]. We now prove that conditions 2 and 3 are equivalent. Condition 2 implies condition 3 because: (i) according to [23], under condition 2,  $(G, \mu, \Phi)$  is stable if  $\Phi$  is the *match-the-longest* policy, and (ii) the associated vector  $\lambda$  of matching rates satisfies condition 3 by ergodicity. That condition 3 implies condition 2 was proved in [14, Lemma 12]. ◀

One might imagine that the time complexity to verify condition 2 in Proposition 3.2 is exponential in the number  $n$  of classes in general, as the number of independent sets itself is exponential in  $n$ . Yet it was proved in [23, Proposition 1] that there exists an  $O(n^3)$ -time algorithm to verify this condition. Unfortunately, this verification is indirect in the sense that it requires constructing a second graph called the bipartite double cover of  $G$ . From this perspective, condition 3 gives a more direct way of verifying whether a graph  $G$  is stabilizable. We make a case disjunction, depending on whether the graph  $G$  (which we have assumed to be surjective) is surjective-only or bijective.

► **Remark 3.3.** If  $G$  is surjective, one can always find an explicit  $\mu$  such that  $(G, \mu)$  is stabilizable by choosing  $\mu = Ay$  for some  $y \in \mathbb{R}_{>0}^m$ . A simple example is  $y = (\beta, \dots, \beta)$  for some  $\beta > 0$ , which corresponds to having the components of  $\mu$  proportional to the degree of each node.

### Checking stability on bijective graphs

If the graph  $G$  is bijective, then the matrix  $A$  is invertible and the conservation equation has a unique solution  $A^{-1}\mu$ . This implies that the matching problem  $(G, \mu)$  is stabilizable if and only if all components of  $A^{-1}\mu$  are positive. The special case of bijective graphs will be investigated in details in Section 4, including a direct expression of  $A^{-1}\mu$ .

### Checking stability on surjective-only graphs

If  $G$  is surjective-only, the conservation equation has multiple solutions. To determine if one of these solutions is positive, it suffices to solve a linear optimization problem that maximizes the smallest component of a solution of (2). More specifically, in block-matrix notation, it suffices to solve the following linear optimization problem:

$$\begin{aligned} & \underset{z \in \mathbb{R}^{m+1}}{\text{Maximize}} && [0_{1 \times m} \quad 1] z, \\ & \text{Subject to} && [A \quad 0_{n \times 1}] z = \mu, \\ & && [I_{m \times m} \quad -I_{m \times 1}] z \geq 0_{m \times 1}, \end{aligned} \tag{3}$$

where  $0_{p \times q}$  is the  $p \times q$  zero matrix and  $I_{p \times p}$  is the  $p$ -dimensional identity matrix. Here, the first  $m$  components of the vector  $z$  are the components of a vector  $y \in \mathbb{R}^m$  that satisfies (2), and the last component of  $z$  is a lower bound of the components of this vector  $y$ . The equality constraint means that  $y$  satisfies (2), and the inequality constraint means that the last component of  $z$  is less than or equal to its other components. The quantity whose value is to be maximized is simply the last component of the vector  $z$ .

The optimization problem (3) always has solutions with finite components. Indeed, the set of vectors that satisfy the constraints of (3) contains at least one valid solution with real-valued components (this is again a consequence of the surjectivity of  $G$ ). We just need to consider an arbitrary solution  $y$  of the conservation equation (2) (see Section 5.1.1 for a concrete example using the Moore-Penrose inverse) and to let  $z_y = (y_1, y_2, \dots, y_m, \min(y))$ . Any solution better than  $z_y$  has all its components lower-bounded by  $\min(y)$  and upper-bounded by  $\max(\mu) - \min(0, (n-1) \min(y))$ . The latter bound is obtained by observing that, if edge  $k$  is incident to node  $i$  and if  $(y'_1, y'_2, \dots, y'_m, x')$  is a solution such that  $x' \geq \min(y)$ , then  $y'_k = \mu_i - \sum_{\ell \in E_i \setminus k} y'_\ell$ . We then use the inequalities  $\mu_i \leq \max(\mu)$  and  $\sum_{\ell \in E_i \setminus k} y'_\ell \geq \min(0, (n-1)x') \geq \min(0, (n-1) \min(y))$ . Therefore, the solutions better than  $z_y$  belong to a compact set of  $\mathbb{R}^{m+1}$ , which ensures the existence of an optimal solution with finite components.

This optimization problem is a textbook linear optimization problem. It can be solved with a time complexity that is polynomial in the number  $n$  of nodes and the number  $m$  of edges using many methods, for instance the interior-point-method [21].

Note that the linear optimization problem (3) has a solution with positive components if and only if the conservation equation (2) has a solution with positive components. According to Proposition 3.2, this is equivalent to saying that the matching problem  $(G, \mu)$  is stabilizable. Therefore, to verify if a matching problem  $(G, \mu)$  is stabilizable, it suffices to find a solution of the linear optimization problem (3) and to verify if all its components are positive.

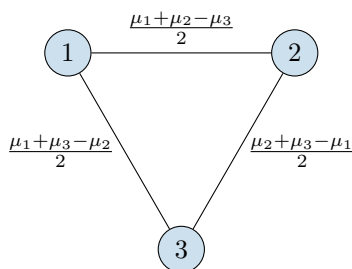
### 3.3 Early examples

We now illustrate Propositions 3.1 and 3.2 on a few toy examples that will also introduce useful notions for Sections 4–6.

#### 3.3.1 Bijective graphs

We first consider a compatibility graph  $G$  that is both surjective and injective. According to Definition 2.6, the conservation equation (2) has a unique solution for each vector  $\mu \in \mathbb{R}^n$  of arrival rates. Proposition 3.2 implies that the components of this solution are (strictly) positive if and only if the matching problem  $(G, \mu)$  is stabilizable. By Remark 3.3, one can always exhibit one vector  $\mu \in \mathbb{R}_{>0}^n$  of arrival rates that satisfies this condition.

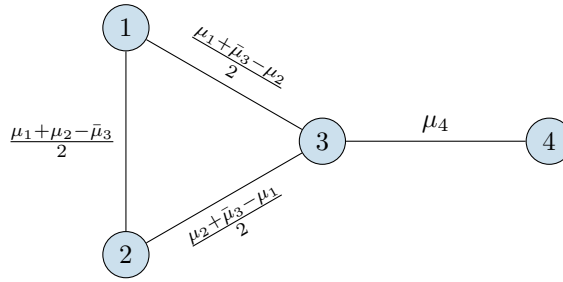
► **Example 3.4** (Triangle). If the graph  $G$  is a triangle graph  $\mathcal{C}_3$  and the vector  $\mu = (\mu_1, \mu_2, \mu_3)$  is given, the solution of the conservation equation (2) is unique and showed in Figure 3. According to condition 3 in Proposition 3.2,  $(G, \mu)$  is stabilizable if and only if all components of this solution are positive. One can verify that this is indeed equivalent to condition 2 in Proposition 3.2, which reads  $\mu_1 < \mu_2 + \mu_3$ ,  $\mu_2 < \mu_1 + \mu_3$ , and  $\mu_3 < \mu_1 + \mu_2$ . Note that an alternate way to express the condition is that  $(\mu_1, \mu_2, \mu_3)$  are the lengths of a non-degenerated triangle. Under these conditions, the matching model  $(G, \mu, \Phi)$  is stable under the unique greedy policy  $\Phi$  (this will be shown in Proposition 6.3).



■ **Figure 3** Matching rates in the triangle graph  $\mathcal{C}_3$ .

► **Example 3.5** (Paw graph). If  $G$  is a paw graph, the solution of the conservation equation (2) is again unique and showed in Figure 4.  $\bar{\mu}_3 = \mu_3 - \mu_4$  represents the remaining rate of class 3 after the needs of class 4 have been deduced. After this subtraction, the matching rates along edges  $\{1, 2\}$ ,  $\{1, 3\}$  and  $\{2, 3\}$  are as defined in the triangle graph of Figure 3.

Note that if positive matching rates guarantees the stability of greedy policies like *math-the-longest*, some greedy policies can be unstable. Lemma 6.8 will give a “recipe” that can be used to build an unstable greedy policy on a paw graph with stabilizable  $\mu$ .



■ **Figure 4** Matching rates in the paw graph.  $\bar{\mu}_3 = \mu_4 - \mu_3$  denotes the residual rate that class 3 can provide to classes 1 and 2.

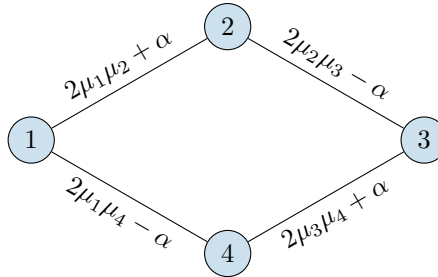
### 3.3.2 Bipartite graph (that is neither injective nor surjective)

► **Example 3.6** (Square graph). Figure 5 shows a square graph  $G = \mathcal{C}_4$ . This graph is not surjective because it is bipartite between  $\{1, 3\}$  (odd component) and  $\{2, 4\}$  (even component). Therefore, according to Proposition 3.1, this graph is not stabilizable. Yet, given a vector  $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$  of arrival rates, the conservation equation (2) may still have a solution with (strictly) positive components. This does not contradict Proposition 3.2, as the three statements in this proposition are equivalent only if the graph  $G$  is surjective. Assuming unit normalization, the conservation equation (2) has a solution if and only if

$$\mu_1 + \mu_3 = \mu_2 + \mu_4 = \frac{1}{2}. \quad (4)$$

If (4) is not satisfied, the difference between the numbers of unmatched items from the odd and even components evolves like a biased random walk on the integer number line  $\{\dots, -2, -1, 0, 1, 2, \dots\}$ . This implies that the underlying Markov process is transient, as the number of unmatched items in the component with the highest arrival rate grows linearly with time. On the other hand, if (4) is satisfied, then the random walk is unbiased, but the system is still unstable because the corresponding Markov chain is null recurrent. (Existing studies of matching in bipartite graphs usually solve this issue by coupling arrivals in both components [2, 8, 12] or by assuming that items have a finite patience time [20].)

If (4) is satisfied, the solutions of the conservation equation (2) can be described with a parameter  $\alpha$  as shown in Figure 5. The positive solutions correspond to values of  $\alpha$  such that  $-2 \min(\mu_1\mu_2, \mu_3\mu_4) < \alpha < 2 \min(\mu_2\mu_3, \mu_1\mu_4)$ .



■ **Figure 5** Matching rates in the square graph  $\mathcal{C}_4$  (not stabilizable), with the normalization  $\mu_1 + \mu_3 = \mu_2 + \mu_4 = \frac{1}{2}$ .

### 3.3.3 Surjective-only graphs

We finally consider compatibility graphs  $G$  that are surjective but not injective. In other words, the graph  $G$  is stabilizable and the conservation equation (2) has an infinite number of solutions. Whether or not these solutions are achievable by a matching policy will be discussed in Section 6.

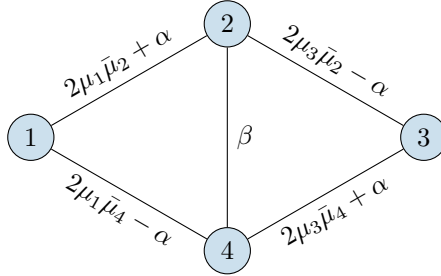
► **Example 3.7** (Diamond (double-fan) graph). Figure 6 shows a diamond graph, that is, a square graph with an additional edge between nodes 2 and 4. Compared to Example 3.6, this additional edge makes the graph non-bipartite, and therefore surjective, so that the graph is stabilizable according to Proposition 3.1. For ease of computation, we assume that the vector  $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$  of arrival rates is normalized so that  $\mu_1 + \mu_3 = \frac{1}{2}$ . According to condition 2 in Proposition 3.2, the matching problem  $(G, \mu)$  is stabilizable if and only if

$$\mu_2 < \mu_1 + \mu_3 + \mu_4, \quad \mu_4 < \mu_1 + \mu_2 + \mu_3, \quad \mu_1 + \mu_3 < \mu_2 + \mu_4$$

With  $\beta = \frac{1}{2}(\mu_2 + \mu_4 - \mu_1 - \mu_3) = \frac{1}{2}(\mu_2 + \mu_4) - \frac{1}{4}$ ,  $\bar{\mu}_2 = \mu_2 - \beta$ , and  $\bar{\mu}_4 = \mu_4 - \beta$ , these conditions rewrite:

$$\beta > 0, \quad \bar{\mu}_4 > 0, \quad \bar{\mu}_2 > 0. \quad (5)$$

If these inequalities are satisfied, the general solution of the conservation equation (2) can be described with a parameter  $\alpha$  as shown in Figure 6. In particular, the positive solutions correspond to values of  $\alpha$  such that  $-2 \min(\mu_1 \bar{\mu}_2, \mu_3 \bar{\mu}_4) < \alpha < 2 \min(\bar{\mu}_2 \mu_3, \mu_1 \bar{\mu}_4)$ .



■ **Figure 6** Matching rates in the diamond graph, with the normalization  $\mu_1 + \mu_3 = \frac{1}{2}$ .  $2\beta = \mu_2 + \mu_4 - \mu_1 - \mu_3 = \mu_2 + \mu_4 - \frac{1}{2}$  is the difference between the arrival rates of the even and odd components.  $\bar{\mu}_2 = \mu_2 - \beta$  and  $\bar{\mu}_4 = \mu_4 - \beta$  represent the residual rates that classes 2 and 4 can provide to classes 1 and 3.

Intuitively, compared to the square graph, stabilizable matching problems  $(G, \mu)$  have a positive difference of  $2\beta$  between the arrival rates of the even and odd components. This difference is absorbed by the central edge  $\{2, 4\}$ , which has matching rate  $\beta$ . After subtracting  $\beta$  from  $\mu_2$  and  $\mu_4$ , the solutions of the conservation equation (2) are exactly the same as in the square graph of Example 3.6.

Like Example 3.4 and unlike Example 3.5, the matching model  $(G, \mu, \Phi)$  is stable for every greedy policy  $\Phi$  provided that (5) is satisfied (this will be shown in Corollary A.2).

► **Example 3.8** (Kayak paddle graph). Figure 7 shows a kayak paddle  $G = KP_{3,3,1}$ , consisting of two triangles linked by an edge. According to Proposition 3.2, the matching problem  $(G, \mu)$  is stabilizable if and only if there exists  $\alpha > 0$  such that  $(\mu_1, \mu_2, \mu_3 - \alpha)$  and  $(\mu_4 - \alpha, \mu_5, \mu_6)$  are the vectors of arrival rates of two stabilizable triangle graphs  $\mathcal{C}_3$ .

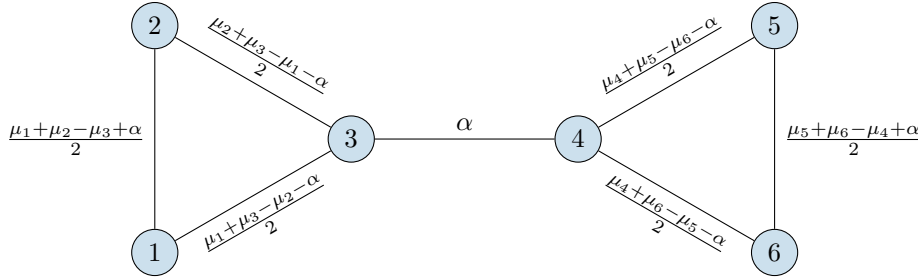


The solutions of the conservation equation (2) can be described by varying  $\alpha$  as shown in Figure 7. Assuming that the matching problem  $(G, \mu)$  is stabilizable, the solutions of the conservation equation (2) with positive components correspond to the values of  $\alpha$  such that

$$0 < \alpha < \min(\mu_3 - |\mu_2 - \mu_1|, \mu_4 - |\mu_5 - \mu_6|).$$

Intuitively, solutions with positive components have a positive matching rate  $\alpha$  along edge  $\{3, 4\}$ . After subtracting this rate from  $\mu_3$  and  $\mu_4$ , the subgraphs restricted to nodes 1, 2, and 3 and to nodes 4, 5, and 6, respectively, behave exactly like the triangle of Figure 3.

Like Example 3.5 and unlike Examples 3.4 and 3.7, the fact that  $(G, \mu)$  is stabilizable does not guarantee the stability of any greedy policy.



■ **Figure 7** Matching rates in the kayak paddle  $KP_{3,3,1}$ .

#### 4 Matching rates in bijective graphs

In the remainder, we will consider exclusively matching problems  $(G, \mu)$  that are stabilizable. According to Proposition 3.1 and Definition 2.4, this implies that the graph  $G$  is surjective, or equivalently, that each connected component of  $G$  is non-bipartite. According to statement 4 in Proposition 2.8 and Proposition 3.2, there are only two possible cases:

1. If  $n = m$ , then the graph  $G$  is also bijective. The conservation equation (2) has a unique solution given by  $\lambda = A^{-1}\mu$ . This solution, which has (strictly) positive components, is the matching rates achieved by any stable policy. Each connected component of the graph  $G$  is a unicyclic graph, and its unique cycle is odd.
2. If  $n < m$ , then the graph  $G$  is surjective-only. The conservation equation (2) has an infinite number of solutions, one of which has (strictly) positive components. Each connected component of the graph  $G$  is non-bipartite, and at least one of these connected components contains an even cycle or a pair of odd cycles.

Case 1 is studied in this section, while case 2 will be studied in Sections 5 and 6.

In Proposition 4.1 below, we give a simpler expression for the unique solution  $\lambda = A^{-1}\mu$  of the conservation equation (2) in terms of the arrival rate vector  $\mu$ , under the assumption that the graph  $G$  is bijective. We assume without loss of generality that the graph  $G$  is connected, as otherwise we can consider each connected component independently. Compared to the expression  $\lambda = A^{-1}\mu$ , the advantage of Proposition 4.1 is twofold: it does not require calculating a matrix inversion, and it highlights the monotonicity of the matching rates with respect to the arrival rates. This result will be illustrated in Examples 4.2 and 4.3.

► **Proposition 4.1.** *Consider a matching problem  $(G, \mu)$  with a compatibility graph  $G = (V, E)$  that is connected and bijective, and consider an edge  $k \in E$ .*

1. If edge  $k$  does not belong to the (unique odd) cycle of the graph  $G$ , then edge  $k$  separates the graph  $G$  into two parts, namely a tree and a unicyclic graph. If  $V_k \subset V$  denotes the set of nodes that belong to the tree (including one of the endpoints of edge  $k$ ), then the matching rate along edge  $k$  is given by

$$\lambda_k = \sum_{i \in V_k} (-1)^{d_{i,k}} \mu_i. \quad (6)$$

2. If edge  $k$  belongs to the (unique odd) cycle of the graph  $G$ , then the matching rate along edge  $k$  is given by

$$\lambda_k = \frac{1}{2} \left( \sum_{i \in V} (-1)^{d_{i,k}} \mu_i \right), \quad (7)$$

where  $d_{i,k}$  is the distance between node  $i$  and edge  $k$ , that is, the minimum distance between node  $i$  and an endpoint of edge  $k$ .

**Proof.** We first prove (6) for every edge  $k$  that does not belong to the cycle. As observed in the proposition, each edge  $k$  that does not belong to the cycle separates the graph into two parts, one of which is a tree with node set  $V_k$ ; the rooted tree associated with  $k$  is obtained by designating the corresponding endpoint of edge  $k$  as the root. We now prove (6) by induction on the height of this rooted tree. Equation (6) is true if the depth of this tree is zero. Indeed, in this case, the endpoint of edge  $k$  that belongs to the tree, say node  $i$ , has no other incident edge, so that applying (1) to node  $i$  yields  $\lambda_k = \mu_i$ , which is consistent with (6). Now consider an edge  $k$  so that the associated rooted tree has height  $h \geq 1$ . By applying (1) to the root  $i$  of this associated rooted tree, we obtain

$$\lambda_k = \mu_i - \sum_{\ell \in E_i \setminus \{k\}} \lambda_\ell.$$

The induction hypothesis guarantees that (6) is satisfied for every  $\ell \in E_i \setminus \{k\}$  (as the height of the associated rooted tree is at most  $h - 1$ ). The result for  $k$  then follows by observing that  $d_{j,k} = d_{j,\ell} + 1$  for each  $\ell \in E_i \setminus \{k\}$  and  $j \in V_\ell$  and that  $V_k = \{i\} \cup \left( \bigcup_{\ell \in E_i \setminus \{k\}} V_\ell \right)$  (all sets being disjoint).

We now prove (7) for each edge  $k$  that belongs to the cycle. Since the graph  $G$  is unicyclic, deleting edge  $k$  from  $G$  yields a (connected) tree, which can be seen as a bipartite graph. We let  $V_+$  denote set of nodes in the part that contains both endpoints of edge  $k$  (that both endpoints belong to the same part follows from the fact that the cycle is odd) and  $V_-$  the set of nodes in the other part. We obtain

$$\sum_{i \in V_+} \mu_i - \sum_{i \in V_-} \mu_i = \sum_{i \in V_+} \sum_{\ell \in E_i} \lambda_\ell - \sum_{i \in V_-} \sum_{\ell \in E_i} \lambda_\ell = 2\lambda_k.$$

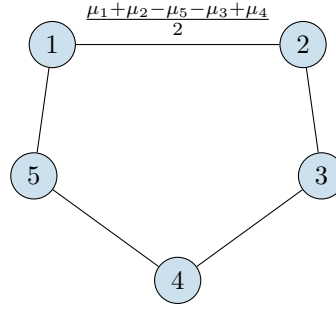
The first equality follows from (1). The second equality holds because each edge  $\ell \in E \setminus \{k\}$  has one endpoint in  $V_+$  and another in  $V_-$ , so that  $\lambda_\ell$  appears exactly once in the first nested sum and once in the second; on the contrary, since both endpoints of edge  $k$  belong to  $V_+$ ,  $\lambda_k$  appears twice in the first nested sum and zero times in the second. Equation (7) follows by observing that  $d_{i,k}$  is even if and only if  $i \in V_+$ .  $\blacktriangleleft$

We remark that only the parity of the distance between an edge and a node impacts the influence of that node on the edge rate. The actual distance does not. In other words, even in a very large (bijective) graphs, a node far away from an edge has the same (although possibly reversed) impact than an endpoint of that edge.

► **Example 4.2** (Cycle graph with 5 nodes). A cycle graph is the simplest bijective graph that we can consider, as it contains an odd cycle and no other edges. In the cycle graph of Figure 8, a direct application of statement 1 in Proposition 4.1 yields

$$\lambda_{1,2} = \frac{1}{2}(\mu_1 + \mu_2 - \mu_3 + \mu_4 - \mu_5).$$

Matching rates along other edges follow by symmetry. From the point of view of edge  $\{1, 2\}$ , we can partition nodes into two sets, namely  $\{1, 2, 4\}$  and  $\{3, 5\}$ . The former (resp. latter) set contains nodes at an even (resp. odd) distance of edge  $\{1, 2\}$ , and increasing the arrival rate of these nodes increases (resp. decreases) the matching rate along edge  $\{1, 2\}$ .



■ **Figure 8** Matching rates in the pentagon graph  $C_5$ . Only rate  $\lambda_{1,2}$  is shown for ease of display. The other rates are deduced by permutation.

► **Example 4.3** (Lying puppet). We now consider the graph of Figure 9. Edges  $\{1, 2\}$ ,  $\{1, 3\}$ , and  $\{2, 3\}$  belong to the cycle, and the other edges do not. According to Proposition 4.1, we have

$$\lambda_{1,2} = \frac{\mu_1 + \mu_2 - \bar{\mu}_3}{2}, \quad \lambda_{1,3} = \frac{\mu_1 - \mu_2 + \bar{\mu}_3}{2}, \quad \lambda_{2,3} = \frac{-\mu_1 + \mu_2 + \bar{\mu}_3}{2},$$

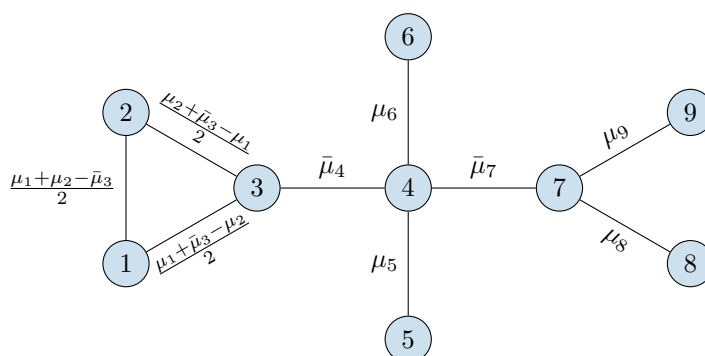
where  $\bar{\mu}_3 = \mu_3 - \lambda_{\{3,4\}}$ , and

$$\begin{aligned} \lambda_{4,5} &= \mu_5, & \lambda_{4,6} &= \mu_6, & \lambda_{7,8} &= \mu_8, \\ \lambda_{7,9} &= \mu_9, & \lambda_{4,7} &= \mu_7 - \lambda_{7,8} - \lambda_{7,9}, & \lambda_{3,4} &= \mu_4 - \lambda_{4,5} - \lambda_{4,6} - \lambda_{4,7}. \end{aligned}$$

This second set of equations can be obtained either by a direct application of (6) or by applying (1) recursively from the leaves. Indeed, applying (1) to nodes 5, 6, 8, and 9 gives directly the values of  $\lambda_{4,5}$ ,  $\lambda_{4,6}$ ,  $\lambda_{7,8}$ , and  $\lambda_{7,9}$ , then applying (1) to node 7 gives the value of  $\lambda_{4,7}$ , and finally applying (1) to node 4 gives the value of  $\lambda_{3,4}$ . The values of  $\lambda_{1,2}$ ,  $\lambda_{1,3}$ , and  $\lambda_{2,3}$  are similar to Example 3.5, where the arrival rate  $\mu_3$  is again replaced with the effective arrival rate  $\bar{\mu}_3$  from the point of view of classes 1 and 2.

## 5 Solution of the conservation equation in surjective-only graphs

Consider a stabilizable matching problem  $(G, \mu)$  with a surjective-only compatibility graph  $G$ . According to Definitions 2.4, 2.5, and 2.7, each connected component of the graph  $G$  is non-bipartite, and at least one of these connected components contains an even cycle or a pair of odd cycles. This means that the conservation equation (2) has an infinite number of solutions, and that at least one of these solutions has (strictly) positive components. This section describes these solutions. Section 5.1 characterizes the affine space of all real-valued



■ **Figure 9** Matching rates in a “lying puppet” graph with  $n = 9$  nodes and  $m = 9$  edges. The differences  $\bar{\mu}_7 = \mu_7 - \mu_8 - \mu_9$ ,  $\bar{\mu}_4 = \mu_4 - \mu_5 - \mu_6 - \bar{\mu}_7$ , and  $\bar{\mu}_3 = \mu_3 - \bar{\mu}_4$  are the residual rates that classes 7, 4, and 3 provide to their neighbors of lower index.

solutions of the conservation equation (2), while Section 5.2 focuses on the convex polytope made of the solutions with non-negative components. Whether or not these solutions are achievable by a matching policy will be discussed in Section 6.

Note that the results stated in this section are also applicable to bijective graphs but they are of little interest in that case, as the solution is unique and Section 4 already gives a closed-form expression for it. This is why we focus on surjective-only graphs.

## 5.1 Affine space of real-valued solutions

Let  $\Lambda$  denote the set of solutions of the conservation equation (2), with positive, zero, or negative components, that is

$$\Lambda = \{y \in \mathbb{R}^m : Ay = \mu\}. \quad (8)$$

In Section 5.1.1, we show that  $\Lambda$  is an affine space of dimension  $d = n - m$ . Section 5.1.2 gives a graph-based algorithm to construct a basis of this affine space.

### 5.1.1 Matrix solution

The following proposition characterizes the set of solutions of the conservation equation (2) using the incidence matrix  $A$ .

► **Proposition 5.1.** *Consider a stabilizable matching problem  $(G, \mu)$  with a surjective-only compatibility graph  $G$ . Let  $A$  denote the incidence matrix of  $G$ . The solution set  $\Lambda$  of the conservation equation (2) is the affine space obtained by translating the kernel of the matrix  $A$  by a particular solution  $y^\circ$  of the conservation equation (2), that is,*

$$\Lambda = \{y^\circ + y : y \in \ker(A)\}. \quad (9)$$

Furthermore, the dimension of  $\ker(A)$  and  $\Lambda$  is equal to  $d = m - n$ .

**Proof.** That the set  $\Lambda$  is of the form (9) is a well-known result in linear algebra, so we only need to prove that the nullity of  $A$  is  $d = m - n$ . Definition 2.4 about surjectivity implies that  $\text{rank}(A) = n$ . We conclude from the rank-nullity theorem that the nullity of  $A$  is  $d = m - n$ . ◀

### Pseudoinverse of the incidence matrix

A standard way to simultaneously find a particular solution  $y^\circ$  and characterize  $\ker(A)$  consists of using the pseudoinverse (or Moore-Penrose inverse) of the matrix  $A$ . Definition 2.4 about surjectivity implies that the rows of  $A$  are linearly independent, so that the pseudoinverse  $A^+$  of  $A$  has the following simple expression:

$$A^+ = A^\top B^{-1},$$

where the  $n \times n$  matrix  $B = AA^\top$  is invertible because  $\ker(B) = \ker(A^\top) = \{0\}$ . We can then describe a particular solution  $y^\perp$  and the kernel  $\ker(A)$  as follows:

$$y^\perp = A^+ \mu, \quad \ker(A) = \{(I_{m \times m} - A^+ A)y : y \in \mathbb{R}^m\},$$

where  $I_{m \times m}$  is the  $m$ -dimensional identity matrix. The vector  $A^+ \mu$  is the solution of the conservation equation (2) with the smallest Euclidean norm, and it is orthogonal to  $\ker(A)$ . In general, the components of this solution are not non-negative even if non-negative solutions exist. For example, in the diamond graph of Example 3.7, the matching problem  $(G, \mu)$  with  $\mu = (5, 5, 2, 2)$  is stabilizable, but the solution given by the pseudoinverse is  $y^\perp = (\frac{13}{4}, \frac{7}{4}, \frac{5}{4}, \frac{1}{2}, -\frac{1}{4})$ .

#### 5.1.2 Basis of the kernel of the incidence matrix

Characterizing the kernel of the matrix  $A$  using the pseudoinverse  $A^+$  is useful to make numerical calculations, but it does not give any insight into the relation between the structure of the graph  $G$  and that of the solution set  $\Lambda$ . In this section, we rewrite the affine space  $\Lambda$  as

$$\Lambda = \{y^\circ + \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_d b_d : \alpha_1, \alpha_2, \dots, \alpha_d \in \mathbb{R}\},$$

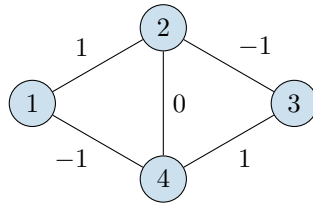
where  $\mathcal{B} = (b_1, b_2, \dots, b_d)$  is a basis of  $\ker(A)$ . We will now construct such a basis using the structure of the compatibility graph  $G$ .

A vector  $y \in \mathbb{R}^m$  belongs to  $\ker(A)$  if and only if  $Ay = 0$ , which means that  $\sum_{k \in E_i} y_k = 0$  for each  $i \in V$ . In other words, a vector  $y \in \mathbb{R}^m$  belongs to  $\ker(A)$  if and only if the sum of the components of  $y$  associated with the edges incident to the same node is zero. Using this observation, we first give examples of subgraphs that can be used to construct vectors that belong to  $\ker(A)$ , and then we give an algorithm that generates a basis  $\mathcal{B} = (b_1, b_2, \dots, b_d)$  of  $\ker(A)$ .

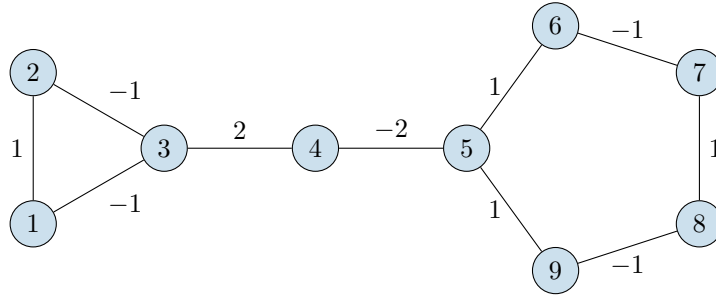
First observe that an even cycle, if it exists, defines a vector in  $\ker(A)$ : it suffices to assign alternatively the values  $+1$  and  $-1$  to the edges of this cycle and the value  $0$  to all other edges. In the diamond graph of Example 3.7 for instance, if edges are numbered in lexicographical order, then  $y = (1, -1, -1, 0, 1)$  is a vector of the unidimensional kernel, associated with the even cycle 1–2–3–4 (see Figure 10). Intuitively, even cycles can be used to move weights between “odd” and “even” edges without modifying the value of the product  $Ay$ . Actually, in this particular example, the only way to increase the matching rate along edges  $\{1, 2\}$  and  $\{3, 4\}$  is to reduce the matching rate along edges  $\{2, 3\}$  and  $\{1, 4\}$ , and conversely.

Apart from even cycles, other structures of interest are kayak paddles  $KP_{\ell, r, p}$  in which the lengths  $\ell$  and  $r$  of both cycles are odd. These graphs have a unidimensional kernel, and a base vector can be found by assigning properly the values  $+1$  and  $-1$  along the cycles and the values  $+2$  and  $-2$  along the path. Figure 11 shows such an assignment for  $KP_{3, 5, 2}$ .

Surprisingly, for any surjective graph  $G$ , one can build a basis of  $\ker(A)$  using only subgraphs of  $G$  that are even cycles and kayak paddles. This is what Algorithm 1 does.



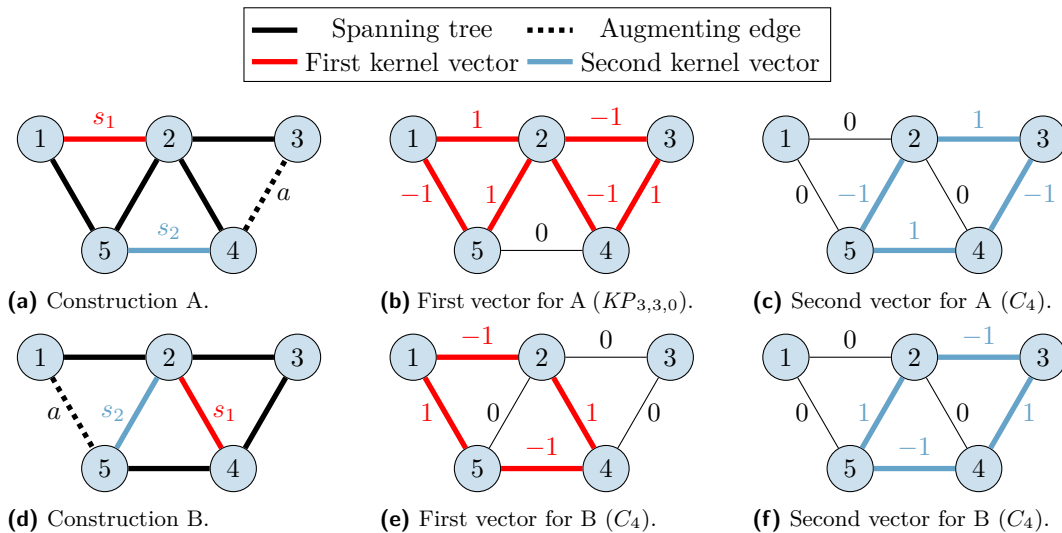
■ **Figure 10** Base vector of the kernel space of the diamond graph.



■ **Figure 11** Base vector of the kernel space of the kayak paddle  $KP_{3,5,2}$ .

It finds  $m - n$  distinct edges and associates to each one either an even-cycle base vector or a kayak-paddle base vector. We assume without loss of generality that the graph  $G$  is connected (in addition to being surjective-only). If not, we can apply the algorithm to each connected component separately, and then we embed the obtained vectors via zero padding.

Figures 12 and 13 show possible runs of Algorithm 1 on the triamond and codomino graphs, both of which have a two-dimensional kernel. Note that the basis is not unique and depends on our initial choice of the spanning tree  $\mathcal{T}$  and the augmenting edge  $a$  (see lines 2 and 3 in Algorithm 1). We now verify that Algorithm 1 terminates and yields the desired result.



■ **Figure 12** Two possible constructions of a kernel basis for the triamond graph.

**Data:** A connected surjective-only graph  $G = (V, E)$   
**Result:** A basis  $\mathcal{B}$  of the kernel of the incidence matrix  $A$  of  $G$

```

1  $\mathcal{B} \leftarrow \emptyset$ 
2  $\mathcal{T} \leftarrow$  the set of edges of a spanning tree of  $G$ 
3  $a \leftarrow$  an edge in  $E \setminus \mathcal{T}$  such that  $\mathcal{T} \cup \{a\}$  contains an odd cycle
4 for  $s \in E \setminus (\mathcal{T} \cup \{a\})$  do
5    $b \leftarrow 0_{m \times 1}$ 
6   if  $\mathcal{T} \cup \{a, s\}$  contains an even cycle  $C_\ell$  then
7      $c_1, \dots, c_\ell \leftarrow$  edges of  $C_\ell$ , numbered from an arbitrary starting point
8     for  $d \in \{1, \dots, \ell\}$  do
9        $k \leftarrow$  index of  $c_d$  in  $E$ 
10       $b_k = (-1)^d$ 
11   else
12      $\mathcal{T} \cup \{a, s\}$  contains a kayak paddle  $KP_{\ell, r, p}$  with  $\ell$  odd,  $r$  odd, and  $p \geq 0$ 
13      $v_i \leftarrow$  node connecting the kayak cycle  $C_\ell$  to the kayak central path
14      $v_j \leftarrow$  node connecting the kayak cycle  $C_r$  to the kayak central path
15      $c_1, \dots, c_\ell \leftarrow$  edges of  $C_\ell$ , numbered starting from node  $v_i$ 
16     for  $d \in \{1, \dots, \ell\}$  do
17        $k \leftarrow$  index of  $c_d$  in  $E$ 
18        $b_k = (-1)^d$ 
19      $c_1, \dots, c_p \leftarrow$  edges of the central path, numbered starting from node  $v_i$ 
20     for  $d \in \{1, \dots, p\}$  do
21        $k \leftarrow$  index of  $c_d$  in  $E$ 
22        $b_k = 2(-1)^{d+1}$ 
23      $c_1, \dots, c_r \leftarrow$  edges of  $C_r$ , numbered starting from node  $v_j$ 
24     for  $d \in \{1, \dots, r\}$  do
25        $k \leftarrow$  index of  $c_d$  in  $E$ 
26        $b_k = (-1)^{d+p+1}$ 
27    $\mathcal{B} \leftarrow \mathcal{B} \cup \{b\}$ 

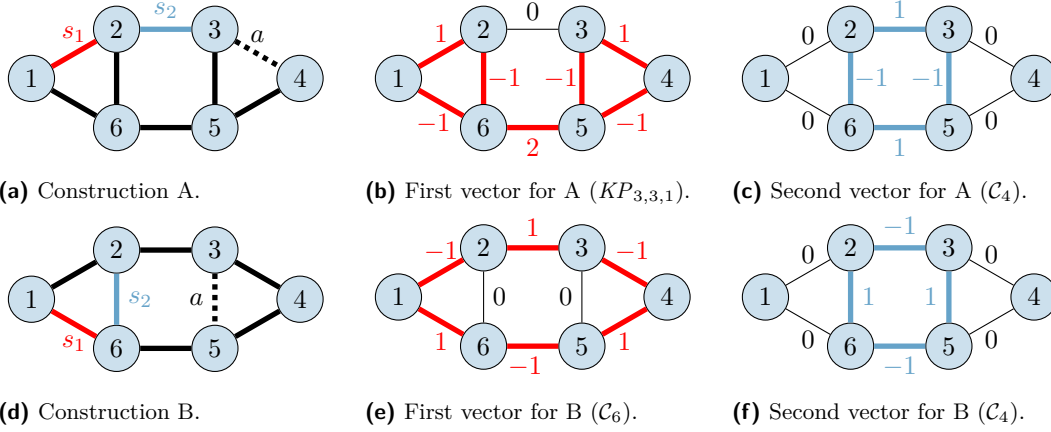
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■ **Algorithm 1** Construction of a basis of the kernel of the incidence matrix  $A$  of the graph  $G$ . This algorithm was initially introduced in [18, Section 3] to build a basis of the eigenspace associated with the eigenvalue  $-2$  of the adjacency matrix  $C$  of a line graph  $L$ .

► **Proposition 5.2.** *Algorithm 1 terminates and returns a basis of the kernel of the incidence matrix  $A$  of the compatibility graph  $G$ .*

**Proof.** See Appendix B. ◀

Importantly, given an edge  $k \in E$ , all solutions of the conservation equation (2) have the same value along edge  $k$  if and only if edge  $k$  does not belong to the support of any basis vector. According to Algorithm 1, this is equivalent to say that edge  $k$  belongs neither to an even cycle nor to a kayak paddle. In the diamond graph of Example 3.7 for instance, the edge  $\{2, 4\}$  is the only one that does not belong to the even cycle 1–2–3–4, and it is also the only one with a fixed value  $\beta$ . In general, if an edge  $k \in E$  satisfies this unicity condition, this implies that the matching rate along edge  $k$  in a stable matching model  $(G, \mu, \Phi)$  is independent of the policy  $\Phi$ . Note that there is no general relation between the number of edges with uniquely-defined matching rates and the dimensionality  $d$  of the affine space  $\Lambda$ .



■ **Figure 13** Two possible constructions of a kernel basis for the codomino graph.

## 5.2 Convex polytope of positive solutions

We now consider the set  $\Lambda_{\geq 0}$  of solutions of the conservation equation (2) that have non-negative components, that is,

$$\Lambda_{\geq 0} = \Lambda \cap \mathbb{R}_{\geq 0}^m = \{y \in \mathbb{R}^m : Ay = \mu, y \geq 0\}. \quad (10)$$

The set  $\Lambda_{\geq 0}$  is a  $d$ -dimensional convex polytope in  $\mathbb{R}^m$ , as it is the intersection of a  $d$ -dimensional affine space with the positive orthant  $\mathbb{R}_{\geq 0}^m$ , both of which are convex. The set  $\Lambda_{\geq 0}$  is neither empty nor restricted to the origin because  $(G, \mu)$  is stabilizable, which means that  $\Lambda$  contains a point with positive components. It is bounded because each  $y \in \Lambda_{\geq 0}$  satisfies  $0 \leq y_k \leq \min_{i \in V_k} (\mu_i)$ .

Since  $\Lambda_{\geq 0}$  is characterized by  $m$  inequalities (one per edge), it has at most  $m$  facets. It typically has fewer because some inequalities may be removed without changing  $\Lambda_{\geq 0}$  or have multiplicities. For example, in the diamond graph from Figure 6:

- $\{2, 4\}$  does not impact  $\Lambda_{\geq 0}$  (if  $y \in \Lambda_{\geq 0}$ ,  $y_{2,4} = \beta > 0$ );
- If  $\mu_1 \bar{\mu}_2 > \mu_3 \bar{\mu}_4$ ,  $\{1, 2\}$  does not impact  $\Lambda_{\geq 0}$  (if  $y \in \Lambda_{\geq 0}$ ,  $y_{1,2} > y_{3,4} \geq 0$ );
- If  $\mu_1 \bar{\mu}_2 = \mu_3 \bar{\mu}_4$ ,  $\{1, 2\}$  and  $\{3, 4\}$  yield the same inequality (if  $y \in \Lambda_{\geq 0}$ ,  $y_{1,2} = y_{3,4}$ ).

Let us formalize the different cases that may impact the number of facets and the properties of the vertices (extreme points of the polytope).

► **Definition 5.3** (Adapted from [4, 27]). *The inequality  $y_k \geq 0$  is irredundant when removing it changes  $\Lambda_{\geq 0}$ . It is tight if there exists  $y \in \Lambda_{\geq 0}$  such that  $y_k = 0$ , in which case we also say that it is tight for  $y$ . If all tight inequalities of the system  $y \geq 0$  are irredundant we say that the matching problem  $(G, \mu)$  is essential. Lastly, we say that  $\Lambda_{\geq 0}$  is simple when every vertex of  $\Lambda_{\geq 0}$  is contained in the minimal number of only  $d$  facets.*

### Coordinate space(s)

Given a solution  $y^\circ$  of the conservation equation (2) and a basis  $\mathcal{B} = (b_1, b_2, \dots, b_d)$  of  $\ker(A)$  (for example, obtained by applying Algorithm 1), there is a one-to-one mapping between a vector  $y = (y_1, y_2, \dots, y_m) \in \Lambda$  and the vector  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{R}^d$  giving the coordinates of  $y - y^\circ$  in the basis  $\mathcal{B}$ , via the linear equation

$$y = y^\circ + \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_d b_d.$$



This defines a linear isomorphism between the  $d$ -dimensional convex polytope  $\Lambda_{\geq 0}$  in  $\mathbb{R}_{\geq 0}^m$  and the  $d$ -dimensional convex polytope  $\Pi_{\geq 0}$  of coordinate vectors in  $\mathbb{R}^d$ , defined by

$$\Pi_{\geq 0} = \{ \alpha \in \mathbb{R}^d : y^\circ + \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_d b_d \geq 0 \}.$$

This alternative representation of the vectors of matching rates will be used extensively in Section 6. Since the sets  $\Lambda_{\geq 0}$  and  $\Pi_{\geq 0}$  are affinely isomorphic, we will often use the same letter  $\Lambda_{\geq 0}$  to describe both sets; which coordinate space we are actually using will be made clear by our choice of letters (either  $y$  or  $\alpha$ ). For instance, the solutions on Examples 3.7 and 3.8 are actually displayed in  $\alpha$ -coordinates. Having the possibility to use both  $y$  and  $\alpha$  representation often simplify things.

In general,  $\alpha$ -coordinates are convenient to show  $\Lambda_{\geq 0}$  and its vertices as they are  $d$ -dimensional instead of  $m$ -dimensional. Let us illustrate this on a few examples.

► **Example 5.4 (Essential matching problem).** We consider in Figure 14 a codomino graph with the vector of arrival rates  $\mu = (4, 5, 3, 2, 3, 5)$ . A particular solution of the conservation equation (2) is  $y^\circ = (2, 2, 1, 2, 1, 1, 1, 1) \in \mathbb{R}^8$ . For the basis of  $\ker(A)$ , we use the vectors  $b_1 = (-1, 1, 1, 0, -1, 0, 1, -1)$  and  $b_2 = (0, 0, -1, 1, 0, 1, 0, -1)$  calculated in construction B of Figure 13. The set  $\Lambda_{\geq 0}$  of vectors  $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ , for which  $y \geq 0$ , shown in Figure 14b, is characterized by the following tight inequalities:

$$-1 \leq \alpha_1 \leq 1, \quad \alpha_2 \geq -1, \quad \alpha_1 - \alpha_2 \geq -1, \quad \alpha_1 + \alpha_2 \leq 1.$$

These five inequalities are irredundant (each one corresponds to one of the five facets without multiplicity), so  $(G, \mu)$  is essential.

The vertices of this convex polytope are  $y_1 = (0, 1)$ ,  $y_2 = (-1, 0)$ ,  $y_3 = (1, 0)$ ,  $y_4 = (-1, -1)$ , and  $y_5 = (1, -1)$ . The corresponding vectors of matching rates are shown in Figures 14c–14g. Like all 2-dimensional polytopes,  $\Lambda_{\geq 0}$  is simple.

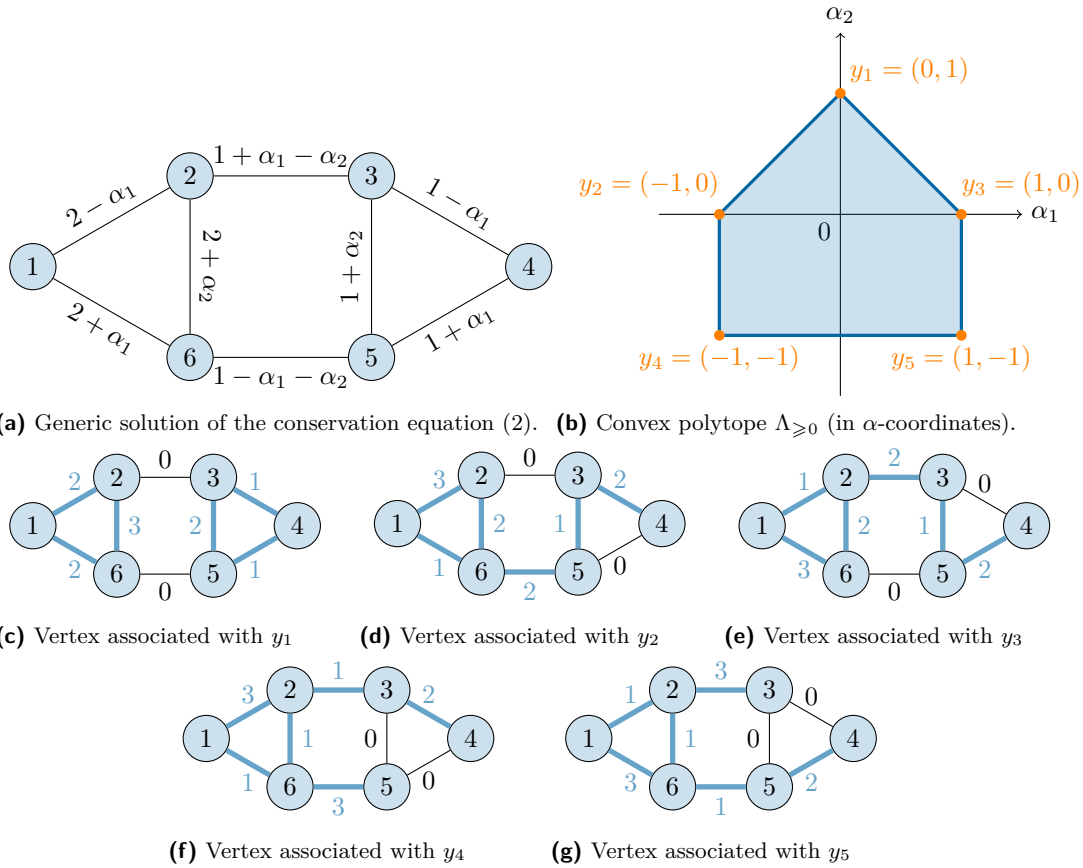
► **Example 5.5 (Non-essential matching problem).** We consider the same codomino graph as before with the same basis of  $\ker(A)$ , but with the vector of arrival rates  $\mu = (2, 4, 4, 2, 2, 2)$ . A particular solution of the conservation equation (2) is  $y^\circ = (1, 1, 2, 1, 1, 1, 1, 0) \in \mathbb{R}^8$ . The set  $\Lambda_{\geq 0}$  of vectors  $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ , for which  $y \geq 0$ , shown in Figure 15b, is characterized by the following tight inequalities:

$$-1 \leq \alpha_1 \leq 1, \quad \alpha_2 \geq -1, \quad \alpha_1 - \alpha_2 \geq -2, \quad \alpha_1 + \alpha_2 \leq 0.$$

Only the last inequality is irredundant:  $(G, \mu)$  is not essential. For example,  $\alpha_1 \leq 1$  actually stands for both  $y_{1,2} \geq 0$  and  $y_{3,4} \geq 0$ ; similarly,  $\alpha_1 - \alpha_2 \geq -2$  is only tight for one point and can be removed without affecting  $\Lambda_{\geq 0}$ . The vertices of this convex polytope are  $y_1 = (-1, 1)$ ,  $y_2 = (-1, -1)$ , and  $y_3 = (1, -1)$ . The corresponding vectors of matching rates are shown in Figures 15c–15e.

► **Example 5.6 (Non-simple polytope).** As a last example we exhibit an essential matching problem with a non-simple associated polytope. As non-simple polytopes implies  $d \geq 3$ , the example is a bit more complex than the previous ones. We consider the matching problem from Figure 16a. The arrival rate is  $\mu = (3, 3, 6, 3, 4, 4, 6, 3, 4, 4) \in \mathbb{R}^{10}$ . The particular solution and kernel basis are shown in the edges. The set  $\Lambda_{\geq 0}$  of vectors  $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ , for which  $y \geq 0$ , shown in Figure 16b, is an Egyptian pyramid characterized by the following tight inequalities:

$$\alpha_3 \geq 0, \quad 1 + \alpha_1 - \alpha_3 \geq 0, \quad 1 - \alpha_1 - \alpha_3 \geq 0, \quad 1 + \alpha_2 - \alpha_3 \geq 0, \quad 1 - \alpha_2 - \alpha_3 \geq 0.$$



■ **Figure 14** Vector of coordinates associated with the codomino graph of Figure 13. The vector of arrival rates is  $\mu = (4, 5, 3, 2, 3, 5) \in \mathbb{R}^6$ , a particular solution of the conservation equation (2) is  $y^\circ = (2, 2, 1, 2, 1, 1, 1, 1) \in \mathbb{R}^8$ , and the chosen base vectors for  $\ker(A)$  are  $b_1 = (-1, 1, 1, 0, -1, 0, 1, -1)$  and  $b_2 = (0, 0, -1, 1, 0, 1, 0, -1)$ .

These five inequalities are irredundant (each one corresponds to exactly one of the five facets), so  $(G, \mu)$  is essential.

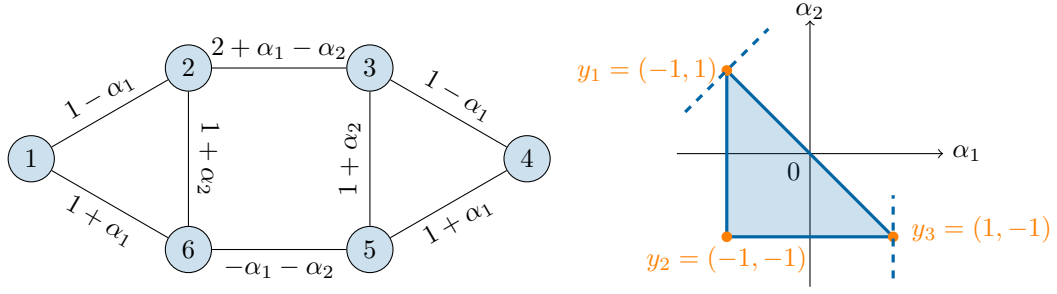
The vertices of this convex polytope are  $y_1 = (-1, -1, 0)$ ,  $y_2 = (1, -1, 0)$ ,  $y_3 = (1, 1, 0)$ ,  $y_4 = (-1, 1, 0)$ , and  $y_5 = (0, 0, 1)$ . The corresponding vectors of matching rates are shown in Figures 16c–16g.  $\Lambda_{\geq 0}$  is not simple because  $y_5$  (the “top” of the pyramid) is contained in 4 facets.

### Vertices of the convex polytope

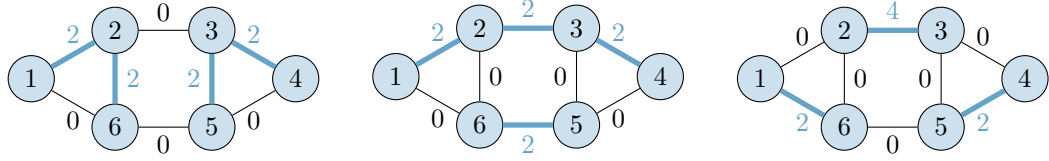
As vertices are extreme points of a polytope, if one wants to optimize a linear function of the matching rates, it is likely that the result leads us to a vertex of  $\Lambda_{\geq 0}$  (or at least a face) according to the simplex algorithm. Hence it is useful to better understand the structure of these vertices.

This is what we do now, again using the structure of the graph.

► **Proposition 5.7.** *Consider a vector  $y \in \Lambda_{\geq 0}$ . Let  $E' = \{k \in E : y_k > 0\}$  denote the support of  $y$ ,  $G(y) = (V, E')$  the subgraph of  $G$  obtained by deleting the edges that are not in  $E'$ , and  $A'$  the incidence matrix of  $G(y)$ . The following statements are equivalent:*



(a) Generic solution of the conservation equation (2). (b) Convex polytope  $\Lambda_{\geq 0}$  (in  $\alpha$ -coordinates).



(c) Vertex associated with  $y_1$  (d) Vertex associated with  $y_2$  (e) Vertex associated with  $y_3$

■ **Figure 15** Vector of coordinates associated with the codomino graph of Figure 13. The vector of arrival rates is  $\mu = (2, 4, 4, 2, 2, 2) \in \mathbb{R}^6$ , a particular solution of the conservation equation (2) is  $y^\circ = (1, 1, 2, 1, 1, 1, 0) \in \mathbb{R}^8$ , and the chosen base vectors for  $\ker(A)$  are  $b_1 = (-1, 1, 1, 0, -1, 0, 1, -1)$  and  $b_2 = (0, 0, -1, 1, 0, 1, 0, -1)$ .

1.  $y$  is a vertex of  $\Lambda_{\geq 0}$ .
2. The graph  $G(y)$  is injective.

**Proof.** We will use the fact that  $y$  is a vertex of  $\Lambda_{\geq 0}$  if and only if it cannot be expressed as a convex combination of points in  $\Lambda_{\geq 0} \setminus \{y\}$ . If  $y$  is not a vertex, there exists  $y_1, y_2 \in \Lambda_{\geq 0} \setminus \{y\}$  and  $0 < \alpha < 1$  such that  $y = \alpha y_1 + (1 - \alpha)y_2$ . The components of  $y_1$  and  $y_2$  are non-negative, which implies they are null on the edges where  $y$  is null. If  $y'$  and  $y'_1$  are the restrictions of  $y$  and  $y_1$  to  $|E'|$ , we have  $y' \neq y'_1$  and  $A'y' = A'y'_1 = \mu$ :  $G(y) = (V, E')$  is not injective.

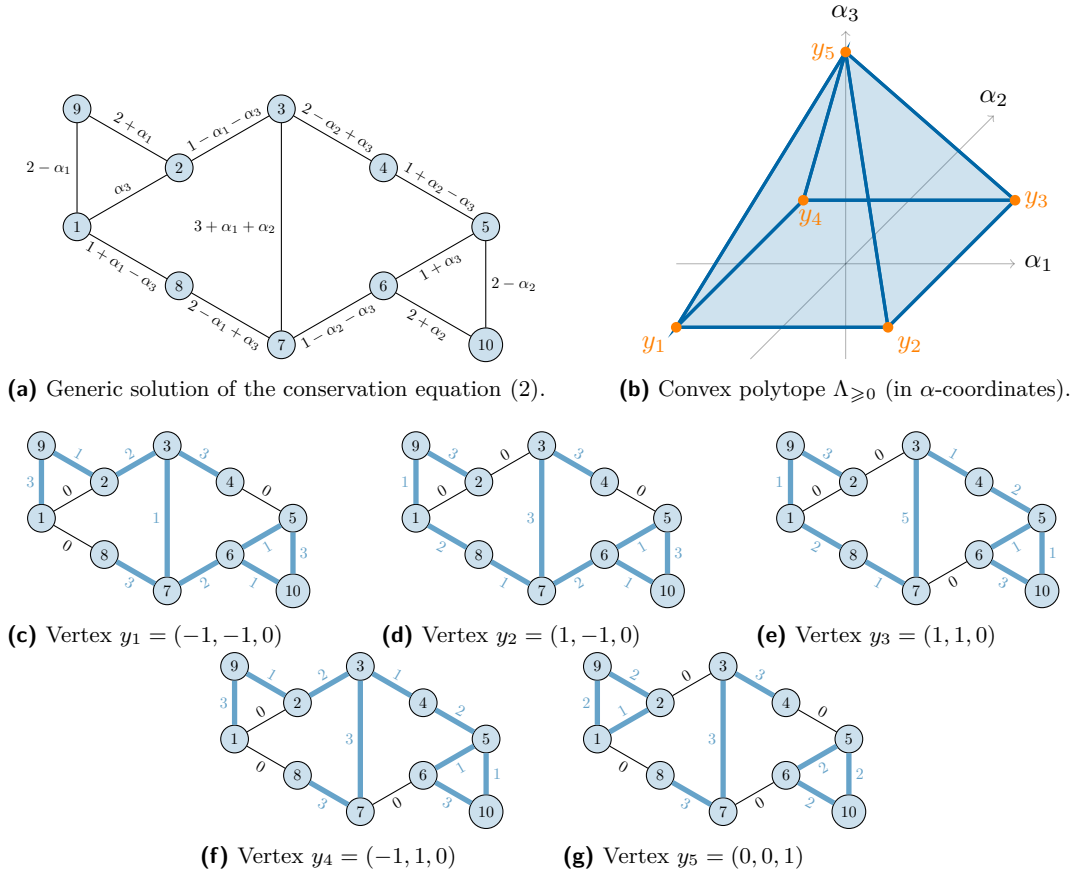
Conversely, if  $G(y)$  is not injective, there exists a non-zero vector  $z'$  on  $E'$  such that  $A'z' = 0$ . If we embed  $z'$  in  $E$  with zero-padding, we get a non-null vector  $z$  which is null on the edges where  $y$  is null and verifies  $Az = 0$ . For  $\varepsilon > 0$  small enough, both  $y - \varepsilon z$  and  $y + \varepsilon z$  belong to  $\Lambda_{\geq 0}$  and  $y = \frac{1}{2}(y - \varepsilon z) + \frac{1}{2}(y + \varepsilon z)$ :  $y$  is not a vertex. ◀

As explained in Corollary 5.8 below, this gives us an effective way of verifying whether an edge set  $E' \subseteq E$  may be the support of a vertex of  $\Lambda_{\geq 0}$ .

► **Corollary 5.8.** Consider a set  $E' \subseteq E$  of edges. Let  $p = |E'|$  denote the cardinality of  $E'$ ,  $G' = (V, E')$  the subgraph of  $G$  restricted to the edges in  $E'$ , and  $A'$  the incidence matrix of  $G'$ . If  $E'$  is the support of a vertex  $y$  of  $\Lambda_{\geq 0}$ , then the following statements are true:

1.  $E'$  contains as most as many edges as there are nodes in  $G$ , that is,  $p \leq n$ .
2. The equation  $A'y' = \mu$ , of unknown  $y' \in \mathbb{R}^p$ , has a unique solution. All components of this solution are positive, and this solution is the restriction of  $y$  to its positive components.
3. Each connected component of  $G'$  is either a tree or a unicyclic graph with an odd cycle.
4. For each connected component of  $G'$  that is a tree, we have  $\sum_{i \in V_+} \mu_i = \sum_{i \in V_-} \mu_i$ , where  $V_+$  and  $V_-$  are the parts of this bipartite graph.

**Proof.** We assume that  $E'$  is the support of a vertex of  $\Lambda_{\geq 0}$ . In other words, there exists a vector  $y \in \Lambda_{\geq 0}$  that satisfies the conditions of Proposition 5.7. We have:



■ **Figure 16** Matching problem that is essential with a non-simple associated  $\Lambda_{\geq 0}$ . The arrival rate is  $\mu = (3, 3, 6, 3, 4, 4, 6, 3, 4, 4) \in \mathbb{R}^{10}$ . A particular solution and the chosen base vectors for  $\ker(A)$  are implicitly shown on the edges of Figure 16a.

1. Statement 1 is a consequence of Proposition 5.7 and of statement 2 in Proposition 2.8.
2. Statements 2 and 3 follow from Proposition 5.7 and Definition 2.5.
3. Statement 4 is obtained by summing (1) over the nodes in  $V_+$  on the one hand, summing (1) over the nodes in  $V_-$  on the other hand, and verifying that the left-hand sides of both equations are equal to each other.



By a slight abuse of notation, we will say that a vertex  $y \in \Lambda_{\geq 0}$  is bijective (resp. injective-only) to express that  $G(y)$  is bijective (resp. injective-only). The following proposition characterizes bijective vertices.

► **Proposition 5.9.** *Let  $y$  be a vertex of  $\Lambda_{\geq 0}$ . The following statements are equivalent:*

1.  $y$  is bijective.
2. Exactly  $d$  inequalities are tight for  $y$ .

*In particular, all vertices of  $\Lambda_{\geq 0}$  are bijective if and only if  $(G, \mu)$  is essential and  $\Lambda_{\geq 0}$  is simple.*

**Proof.** We know from Proposition 5.7 that  $G(y) = (V, E')$  is injective. From Proposition 2.8, it is bijective if and only if  $|E'| = n$ . As the dimension of a vertex is 0 and the dimension of  $\Lambda_{\geq 0}$  is  $d = m - n$ ,  $y$  has at least  $d$  tight inequalities, each of them canceling one distinct edge.

Thus we have  $|E'| \leq m - d = n$ , with equality if and only if the number of tight inequalities is exactly  $d$ .

As for the last statement, if all vertices of  $\Lambda_{\geq 0}$  are tight for exactly  $d$  inequalities,  $\Lambda_{\geq 0}$  is simple by definition. Moreover, as any tight inequality is tight for at least one vertex, removing any inequality will impact one vertex. Reciprocally, if one vertex of  $\Lambda_{\geq 0}$  possesses more than  $d$  tight inequalities, either one of the inequality can be removed ( $(G, \mu)$  is not essential) or the vertex is contained in more than  $d$  facets ( $\Lambda_{\geq 0}$  is not simple). ◀

## 6 Matching rates in surjective-only graphs

Section 5 describes  $\Lambda_{\geq 0}$ , the set of non-negative solutions of the conservation equation (2). In this section, we investigate which of these solutions may, or may not, be achieved by a policy. There are many cases where this question matters:

- If an edge-dependent reward is earned each time a match is performed, the long-term reward is maximized by favoring edges with the maximal reward.
- In *chained matching*, the matches performed in the stable matching model  $(G, \mu, \Phi)$  form the (non-Poisson) arrival process of another matching model, with a compatibility graph  $G' = (V', E')$  with  $V' = E$ . In particular, the matching rate vector  $\lambda$  of the first model is the vector of arrival rates in the second model. Being able to control the vector  $\lambda$  can therefore help stabilize the second model.

Unless otherwise stated, we consider a stabilizable matching problem  $(G, \mu)$  with a surjective-only compatibility graph  $G$ , so that the set  $\Lambda_{\geq 0}$  of non-negative solutions of the conservation equation (2) is non-trivial.

Given a stable policy  $\Phi$  of  $(G, \mu)$ , let  $\lambda(\Phi)$  denote the matching rate of  $(G, \mu, \Phi)$ . We define the set of matching rates of stable policies (resp. of stable greedy policies) as follows:

- $\Lambda_{\mathcal{P}} = \{\lambda(\Phi) : \Phi \text{ is a stable policy of } (G, \mu)\}$ .
- $\Lambda_{\mathcal{G}} = \{\lambda(\Phi) : \Phi \text{ is a stable greedy policy of } (G, \mu)\}$ .

Section 6.1 focuses on  $\Lambda_{\mathcal{G}}$ . We show that  $\Lambda_{\mathcal{G}} \subseteq \Lambda_{>0}$ , where  $\Lambda_{>0}$  is the (non-empty) set of positive solutions of the conservation equation (2), and that in many cases the inclusion is strict. In other words, greedy policies are not very good at navigating inside  $\Lambda_{\geq 0}$ . Section 6.2 studies  $\Lambda_{\mathcal{P}}$ . We show that semi-greedy policies can be used to reach some faces of  $\Lambda_{\geq 0}$ . In particular, if all vertices of  $\Lambda_{\geq 0}$  are bijective, we have  $\Lambda_{\mathcal{P}} = \Lambda_{\geq 0}$ . Finally, we use a family of non-greedy policies to show that  $\Lambda_{>0} \subseteq \Lambda_{\mathcal{P}}$ .

The following result will be useful throughout this section.

► **Proposition 6.1.** *The sets  $\Lambda_{\mathcal{P}}$  and  $\Lambda_{\mathcal{G}}$  are convex.*

**Proof.** We first show the convexity of  $\Lambda_{\mathcal{P}}$ . Let  $\Phi_1 = (\mathcal{X}_1, \pi_1)$  and  $\Phi_2 = (\mathcal{X}_2, \pi_2)$  be two stable policies and  $0 < \beta < 1$  be a linear coefficient. To build a stable policy  $\Phi_\beta$  such that  $\lambda(\Phi_\beta) = \beta\lambda(\Phi_1) + (1 - \beta)\lambda(\Phi_2)$ , we combine  $\Phi_1$  and  $\Phi_2$  in proper proportions.

Let  $\varnothing_1 \in \mathcal{X}_1$  (resp.  $\varnothing_2 \in \mathcal{X}_2$ ) be the starting empty state of  $\Phi_1$  (resp.  $\Phi_2$ ). Let  $T_1$  (resp.  $T_2$ ) be the mean return time to  $\varnothing_1$  in  $(G, \mu, \Phi_1)$  (resp. to  $\varnothing_2$  in  $(G, \mu, \Phi_2)$ ). We construct the policy  $\Phi_\beta$  on the state space  $\mathcal{X}_1 \times \mathcal{X}_2$  with starting empty state  $(\varnothing_1, \varnothing_2)$ . On the arrival of an item of class  $i \in V$ :

- If the current state is  $(x_1, \varnothing_2)$  with  $x_1 \in \mathcal{X}_1 \setminus \{\varnothing_1\}$ , then we apply the policy  $\Phi_1$ , that is, we choose a new state  $y_1 \in \mathcal{X}_1$  according to  $\pi_1(\cdot | x_1, i)$  and move to  $(y_1, \varnothing_2)$ .
- If the current state is  $(\varnothing_1, x_2)$  with  $x_2 \in \mathcal{X}_2 \setminus \{\varnothing_2\}$ , then we apply the policy  $\Phi_2$ , that is, we choose a new state  $y_2 \in \mathcal{X}_2$  according to  $\pi_2(\cdot | x_2, i)$  and move to  $(\varnothing_1, y_2)$ .

- If the current state is  $(\emptyset_1, \emptyset_2)$ , then we apply  $\Phi_1$  with probability  $p_1 = \frac{\beta T_2}{\beta T_2 + (1-\beta)T_1}$  and  $\Phi_2$  otherwise. In other words, the next state is  $(y_1, \emptyset_2)$  with probability  $p_1$  and  $(\emptyset_1, y_2)$  with probability  $1 - p_1$ , where  $y_1$  is chosen according to  $\pi_1(\cdot | \emptyset_1, i)$  and  $y_2$  is chosen according to  $\pi_2(\cdot | \emptyset_2, i)$ .

In essence, every time the system is in state  $(\emptyset_1, \emptyset_2)$ ,  $\Phi_\beta$  selects either  $\Phi_1$  or  $\Phi_2$  and follows this policy until the next return to state  $(\emptyset_1, \emptyset_2)$ . The evolution of the system under the policy  $\Phi_\beta$  defines an irreducible continuous-time Markov chain with state space  $\mathcal{X}_1 \times \mathcal{X}_2$ . Importantly, if we focus on the time intervals when the policy  $\Phi_1$  (resp.  $\Phi_2$ ) is applied, the evolution is exactly the same as under policy  $\Phi_1$  (resp.  $\Phi_2$ ). This guarantees that the matching model  $(G, \mu, \Phi_\beta)$  is stable.

In addition, on the long run, the decisions to use  $\Phi_1$  or  $\Phi_2$  are in proportions with  $\beta T_2$  and  $(1 - \beta)T_1$ , which means that the time spent as  $\Phi_1$  or  $\Phi_2$  is in proportion with  $\beta$  and  $(1 - \beta)$ , respectively. We deduce that  $\lambda(\Phi_\beta) = \beta\lambda(\Phi_1) + (1 - \beta)\lambda(\Phi_2)$ . This proves that  $\Lambda_{\mathcal{P}}$  is convex.

For  $\Lambda_{\mathcal{G}}$ , it suffices to observe that, if  $\Phi_1$  and  $\Phi_2$  are greedy, so is  $\Phi_\beta$ . ◀

## 6.1 Greedy policies

► **Proposition 6.2.** *If the compatibility graph  $G$  is surjective-only and the matching problem  $(G, \mu)$  is stabilizable, then the set  $\Lambda_{\mathcal{G}}$  is non-empty and  $\Lambda_{\mathcal{G}} \subseteq \Lambda_{>0}$ .*

**Proof.** The set  $\Lambda_{\mathcal{G}}$  is non-empty because the greedy policies *match-the-longest* [23] and *first-come-first-matched* [24] are stable.

If  $G$  is surjective-only and stabilizable,  $\Lambda_{>0}$  is the (non-empty) set of positive solutions of the conservation equation (2). Let  $\Phi$  be a stable greedy strategy and  $\lambda_k$  the associated matching rate along edge  $e_k = \{i, j\}$ . Since the policy  $\Phi$  is greedy, a matching between classes  $i$  and  $j$  always occurs if the system is empty, then a class- $i$  item arrives, and a class- $j$  item arrives. Let  $p_\emptyset$  denote the probability that the matching model  $(G, \mu, \Phi)$  is empty. We know that  $p_\emptyset > 0$  because  $(G, \mu, \Phi)$  is stable. It follows that  $\lambda_k \geq p_\emptyset \mu_i \mu_j / \sum_{\ell \in V} \mu_\ell > 0$ . Since this is true for each  $k \in E$ , it follows that  $\lambda \in \Lambda_{>0}$ . ◀

► **Proposition 6.3.** *Consider the complete graph  $K_n$  for  $n \geq 3$ . All greedy policies are equivalent in the sense that they perform the same matches at the same time. Assuming unit normalization,  $(G, \mu)$  is stabilizable if and only if  $\mu_i < \frac{1}{2}$  for each  $i \in V$ , in which case the unique greedy policy is stable and its matching rates are given by*

$$\lambda_k = \mu_i p_j + \mu_j p_i, \quad k = \{i, j\} \in E, \quad (11)$$

with

$$p_i = \frac{\frac{\mu_i}{1-2\mu_i}}{1 + \sum_{\ell=1}^n \frac{\mu_\ell}{1-2\mu_\ell}}, \quad i \in V.$$

In particular, for each  $n > 3$ , we have  $\Lambda_{\mathcal{G}} \subsetneq \Lambda_{>0}$ .

**Proof.** First note that, with a complete compatibility graph  $K_n$  and a vector  $\mu$  of arrival rates normalized to 1, the matching problem  $(G, \mu)$  is stabilizable if and only if  $\mu_i < \frac{1}{2}$  for each  $i \in V$ . This is a direct consequence of Proposition 3.2.

In a complete graph, two non-empty classes can always be matched. As a result, a greedy policy can only reach states with at most one non-empty class. This means that an arriving item never needs to “decide” between two classes, so all greedy policies will have

the exact same behavior. So there is a unique greedy policy, which can be interpreted like *match-the-longest*. In particular, it is stable if and only if  $(G, \mu)$  is stabilizable [23].

We now prove Equation (11). First, we notice that for a given class  $i$ , the states where all classes but  $i$  are empty behave like a  $M/M/1$  queue with load  $\rho_i := \frac{\mu_i}{1-\mu_i}$ : it increases with intensity  $\mu_i$  (a new item  $i$  arrives) and decreases with intensity  $1 - \mu_i$  (an item  $j \neq i$  arrives and matches  $i$ ). If  $p_\emptyset$  denotes the probability of the empty state and  $p_i$  the probability that class  $i$  is not empty, it follows that

$$p_i = p_\emptyset \frac{\rho_i}{1 - \rho_i} = p_\emptyset \frac{\mu_i}{1 - 2\mu_i}.$$

As  $p_\emptyset + \sum_{i=1}^n p_i = 1$ , we deduce that  $p_\emptyset = \frac{1}{1 + \sum_{i=1}^n \frac{\mu_i}{1 - 2\mu_i}}$ , which determines  $p_i$  for  $1 \leq i \leq n$  as well.

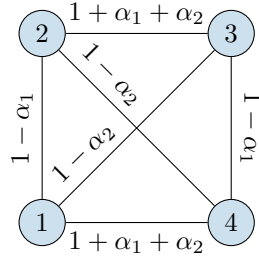
We conclude by observing that the cases where two distinct classes  $i$  and  $j$  are matched are exactly the following:

- An item  $i$  arrives while class  $j$  is non-empty, which happens with intensity  $\mu_i p_j$ ;
- An item  $j$  arrives while class  $i$  is non-empty, which happens with intensity  $\mu_j p_i$ .

◀

Proposition 6.3 essentially states that greedy algorithms do not allow any degree of freedom in the case of complete graphs, even if  $\Lambda_{\geq 0}$  has dimension  $m - n = n(n - 3)/2$ .

Figure 17 illustrates this result on a complete graph  $K_4$  in which all arrival rates are equal to 3. In  $\alpha$ -coordinates,  $\Lambda_{\geq 0}$  is defined by  $\alpha_1 \leq 1, \alpha_2 \leq 1, \alpha_1 + \alpha_2 \geq -1$ . Yet only the solution  $\alpha_1 = \alpha_2 = 0$  can be achieved by a greedy policy.



■ **Figure 17** Matching rates in  $K_4$  with homogeneous arrival rates ( $\mu_1 = \dots = \mu_4 = 3$ ).

The next propositions deal with a case where the set of solutions reachable with a greedy policy, while not reduced to a single vector, is still a strictly smaller subset than the set of positive solutions.

► **Proposition 6.4.** *Consider the diamond graph from Example 3.7 (remind that the normalization used is  $\mu_1 + \mu_3 = 1/2$ ).*

We define the following values:

$$\begin{aligned} q_{1,3} &= \frac{1}{4\beta}, & q_1 &= \frac{\mu_1}{\mu_3 + 2\beta}, & q_3 &= \frac{\mu_3}{\mu_1 + 2\beta}, \\ q_2 &= \frac{\bar{\mu}_2 + \beta}{2\bar{\mu}_4}, & q_4 &= \frac{\bar{\mu}_4 + \beta}{2\bar{\mu}_2}, & p_\emptyset &= \frac{1}{1 + q_{1,3} + q_2 + q_4}. \end{aligned}$$

$(G, \mu)$  is stable for all greedy policies. Moreover, the matching rates under any greedy policy verifies:

$$\begin{aligned} \lambda_{2,4} = \beta, \quad \lambda_{1,2} &> p_{\emptyset} \mu_1 q_2 + p_{\emptyset} \mu_2 q_1, \quad \lambda_{1,4} > p_{\emptyset} \mu_1 q_4 + p_{\emptyset} \mu_4 q_1, \\ \lambda_{3,2} &> p_{\emptyset} \mu_3 q_2 + p_{\emptyset} \mu_2 q_3, \quad \lambda_{3,4} > p_{\emptyset} \mu_3 q_4 + p_{\emptyset} \mu_4 q_3. \end{aligned} \quad (12)$$

In particular, if we let  $\beta$  grow to infinity without changing the values of  $\mu_1, \bar{\mu}_2, \mu_3,$  and  $\bar{\mu}_4,$  we have

$$\begin{aligned} \lim_{\beta \rightarrow +\infty} \lambda_{1,2} &= 2\mu_1 \bar{\mu}_2, \quad \lim_{\beta \rightarrow +\infty} \lambda_{1,4} = 2\mu_1 \bar{\mu}_4, \\ \lim_{\beta \rightarrow +\infty} \lambda_{3,2} &= 2\mu_3 \bar{\mu}_2, \quad \lim_{\beta \rightarrow +\infty} \lambda_{3,4} = 2\mu_3 \bar{\mu}_4. \end{aligned} \quad (13)$$

Propositions 6.3 and 6.4 essentially state that there are situations where no greedy policy can approach a border of  $\Lambda_{\geq 0}$  beyond a certain point. This means in particular that greedy policies may not be adapted to address rate optimization problems.

**Proof.** While for the complete graph, a greedy policy has no decision to take, there are two such cases for the diamond graph:

- An item 2 arrives while classes 1 and 3 are not empty;
- An item 4 arrives while classes 1 and 3 are not empty.

In all other situations, the decision of the policy is automatic. In particular, the evolution of the number of items in class 2, in class 4, and in classes (1 or 3) are independent of the greedy policy considered. We will leverage this to prove Equation (12). Let  $p_{\emptyset}$  be the probability of the empty state. We can partition the reachable states of the system into three  $M/M/1$  queues sharing the same empty state:

- The states where all classes but 2 are empty behave like a  $M/M/1$  queue with load  $\rho_2 = \frac{\bar{\mu}_2 + \beta}{1 + \beta - \bar{\mu}_2}$ . In particular, the probability that class 2 is not empty is  $p_{\emptyset} \frac{\rho_2}{1 - \rho_2} = p_{\emptyset} q_2$ .
- The states where all classes but 4 are empty behave like a  $M/M/1$  queue with load  $\rho_4 = \frac{\bar{\mu}_4 + \beta}{1 + \beta - \bar{\mu}_4}$ . In particular, the probability that class 4 is not empty is  $p_{\emptyset} \frac{\rho_4}{1 - \rho_4} = p_{\emptyset} q_4$ .
- The states where classes 2 and 4 are empty, partitioned by the total number of items in classes 1 and 3, behave like a  $M/M/1$  queue with load  $\rho_{1,3} = \frac{1}{1 + 4\beta}$ . In particular, the probability that class 1 or class 3 is not empty is  $p_{\emptyset} \frac{\rho_{1,3}}{1 - \rho_{1,3}} = p_{\emptyset} q_{1,3}$ .

It follows that the probability of the empty state is  $p_{\emptyset} = \frac{1}{1 + q_2 + q_4 + q_{1,3}}$ . Consider now the matchings between 1 and 2. There are two cases where these matchings occur:

- An item 1 arrives while 2 is not empty. This happens with intensity  $\mu_1 p_{\emptyset} q_2$ .
- An item 2 arrives and is matched with an item 1. The exact intensity depends on the policy but a strict lower bound is  $p_{\emptyset} q_1 \mu_2$  ( $p_{\emptyset} q_1$  is a strict lower bound of the probability that 1 is the only non-empty class; it comes from discarding the cases where 1 is not empty after one remaining item 3 has been removed; the remaining cases behave like a  $M/M/1$  queue with load  $\frac{\mu_1}{1 + 2 + 2\beta}$ ).

In other words,  $\lambda_{1,2} > p_{\emptyset} \mu_1 q_2 + p_{\emptyset} \mu_2 q_1$ . The lower bounds for  $\lambda_{1,4}$ ,  $\lambda_{3,2}$ , and  $\lambda_{3,4}$  are derived the same way. The equality  $\lambda_{2,4} = \beta$  has already been observed in Example 3.7. It comes from the fact that the edge  $\{2, 4\}$  does not belong to  $\ker(A)$  and has the same matching rate in all stable policies.

Lastly, we need to prove Equation (13). First observe that  $\lim_{\beta \rightarrow +\infty} p_{\emptyset} \mu_1 q_2 = 2\mu_1 \bar{\mu}_2$  and  $\lim_{\beta \rightarrow +\infty} p_{\emptyset} \mu_1 q_4 = 2\mu_1 \bar{\mu}_4$ , while  $\lim_{\beta \rightarrow +\infty} p_{\emptyset} \mu_2 q_1 = \lim_{\beta \rightarrow +\infty} p_{\emptyset} \mu_4 q_1 = 0$ . Hence if we add the two first inequalities in Equation (12) and take the limit we get

$$\lim_{\beta \rightarrow +\infty} \lambda_{1,2} + \lambda_{1,4} \geq 2\mu_1 \bar{\mu}_2 + 2\mu_1 \bar{\mu}_4 = \mu_1.$$



By conservation principle we always have  $\lambda_{1,2} + \lambda_{1,4} = \mu_1$ , which implies the two first lines in Equation (13). The two last lines are obtained the same way by considering  $\lambda_{3,2}$  and  $\lambda_{3,4}$ . ◀

One interpretation of Equation (13) is as follows: as the traffic intensity  $\beta$  between 2 and 4 grows, all other things being equal, the probability of having items 1 or 3 in the system goes to 0 as the only dominant cases are: 2 is not empty ( $p_2 \approx 2\bar{\mu}_2$ ); 4 is not empty ( $p_4 \approx 2\bar{\mu}_4$ ). This means that the probability that one particular policy has an actual decision to make goes to 0: all greedy policies tend to behave the same way.

The following re-states Proposition 6.4 in simpler terms.

► **Corollary 6.5.** *Consider the diamond graph from Example 3.7 and the associated parameter  $\alpha$ . In  $\alpha$ -coordinates,  $\Lambda_{\geq 0} = [-2 \min(\mu_1 \bar{\mu}_2, \mu_3 \bar{\mu}_4), 2 \min(\bar{\mu}_2 \mu_3, \mu_1 \bar{\mu}_4)]$ . Let  $\Phi_+$  be the policy that prioritizes edges  $\{1, 2\}$  and  $\{3, 4\}$  and  $\Phi_-$  be the policy that prioritizes edges  $\{2, 3\}$  and  $\{1, 4\}$ . Let  $\alpha_+$  (resp.  $\alpha_-$ ) be the value of  $\alpha$  that corresponds to the matching rate of  $\Phi_+$  (resp.  $\Phi_-$ ). Then we have, in  $\alpha$ -coordinates:*

- $-2 \min(\mu_1 \bar{\mu}_2, \mu_3 \bar{\mu}_4) < \alpha_- \leq \alpha_+ < 2 \min(\bar{\mu}_2 \mu_3, \mu_1 \bar{\mu}_4)$ ;
- $\Lambda_{\mathcal{G}} = [\alpha_-, \alpha_+] \subsetneq \Lambda_{>0}$ ;
- $\lim_{\beta \rightarrow +\infty} \alpha_- = \lim_{\beta \rightarrow +\infty} \alpha_+ = 0$ .

**Proof.** We only need to show that  $\Lambda_{\mathcal{G}} = [\alpha_-, \alpha_+]$ , as the rest is just straightforward rephrasing of Proposition 6.4 in terms of  $\alpha$ . To do that, we notice that the policy  $\Phi_+$  maximizes the number of matchings along  $\{1, 2\}$  and  $\{3, 4\}$ : if we consider a greedy policy  $\Phi$  and observe the evolution of a given arrival sequence under  $\Phi$  and  $\Phi_+$ , we can associate to each step where  $\Phi$  selects  $\{1, 2\}$  or  $\{3, 4\}$  and  $\Phi_+$  does not a past step where  $\Phi_+$  selects  $\{1, 2\}$  or  $\{3, 4\}$  and  $\Phi$  does not in a way that the selected past steps do not overlap. Using a similar reasoning with  $\Phi_-$  it follows that the parameter  $\alpha$  of any greedy algorithm is in  $[\alpha_-, \alpha_+]$ . The convexity of  $\Lambda_{\mathcal{G}}$  concludes the proof. ◀

To illustrate Corollary 6.5, we consider in Figure 18 a symmetric example where  $\mu_1 = \bar{\mu}_2 = \mu_3 = \bar{\mu}_4 = \frac{1}{4}$ . The Figure shows  $\Lambda_{\geq 0}$  and  $\Lambda_{\mathcal{G}}$  and compares it to the bounds and limits from Equations (12) and (13).

We observe, as expected, the convergence to 0 when  $\beta$  grows. We also notice that the bounds from Equation (12) are not tight when  $\beta$  is small, but that  $\Phi_+$  and  $\Phi_-$  seem to be able to be as close as we want to the borders of  $\Lambda_{\geq 0}$  when  $\beta$  is small enough (this does not contradict the fact that for any given  $\beta$ ,  $\Lambda_{\mathcal{G}}$  is strictly inside  $\Lambda_{\geq 0}$ ). The reason for the gap is that to get Equation (12), we discarded some cases where both 1 and 3 are present (cf the second case of the proof for  $\lambda_{1,2}$ ): these situations become not negligible when  $\beta$  is small.

► **Proposition 6.6.** *Consider the matching  $(G, \mu)$  depicted Figure 19.  $(G, \mu)$  is stabilizable and  $\Lambda_{\geq 0} = [0, 1]$  (expressed in  $\alpha$ -coordinates).*

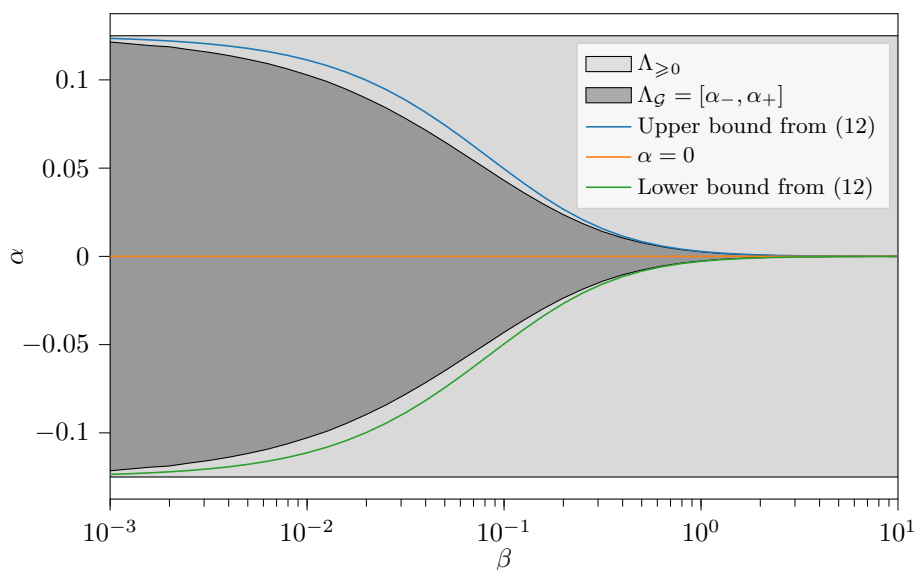
*For  $k \geq 0$ , let  $\Phi_k$  be the greedy matching policy defined by:*

- *if there are less than  $k$  items of type 4 in the system, choose by decreasing order of priority edges  $\{1, 3\}$ ,  $\{2, 3\}$ ,  $\{3, 4\}$ ,  $\{5, 6\}$ , or one of the other edges;*
- *otherwise choose by decreasing order of priority edges  $\{3, 4\}$ ,  $\{5, 6\}$ ,  $\{1, 3\}$ ,  $\{2, 3\}$ , or one of the other edges.*

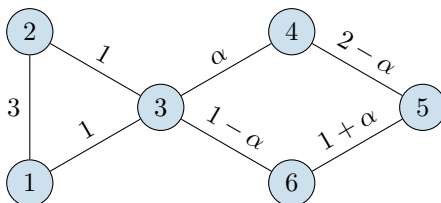
*We have:*

- $(G, \mu, \Phi_k)$  is stable for all  $k \geq 0$ ;
- if  $\alpha(k)$  denotes the coordinate of the matching rate of  $\Phi_k$ ,  $\lim_{k \rightarrow \infty} \alpha(k) = 1$ .

► **Corollary 6.7.** *For the matching  $(G, \mu)$  described in Figure 19, we have  $\Lambda_{\mathcal{G}} = ]0, 1[ = \Lambda_{>0}$ .*



■ **Figure 18** Evolution of  $\Lambda_G$  as a function of  $\beta$  for the diamond graph with  $\mu_1 = \bar{\mu}_2 = \mu_3 = \bar{\mu}_4 = \frac{1}{4}$ . Each point is computed by simulating  $10^8$  arrivals.  $\Lambda_{\geq 0}$  and other bounds are displayed for comparison.



■ **Figure 19** Fish graph with arrival rates  $\mu = (4, 4, 3, 2, 3, 2)$ . The solutions of Equation (2) are shown in  $\alpha$  coordinates.

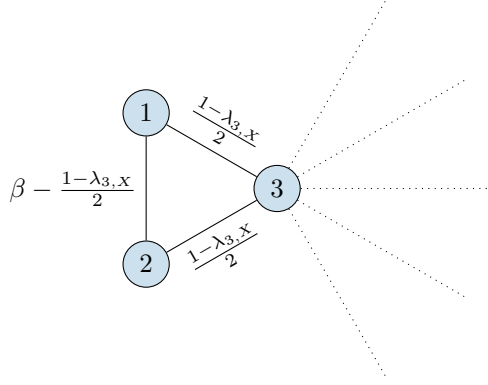
From Propositions 6.2–6.4 and Corollary 6.5, one could be tempted to conjecture that for any stabilizable, surjective-only,  $(G, \mu)$  there is a strict inclusion between  $\Lambda_G$  and  $\Lambda_{>0}$ . Proposition 6.6 and Corollary 6.7 state that this is not the case.

Note that  $\Phi_k$  is uniquely defined: all decisions that may occur in Figure 19 are uniquely determined. The fact that  $(G, \mu)$  is stabilizable is a direct consequence of the existence of positive solutions (e.g.  $\alpha = 1/2$ ). The rest of the proof of Proposition 6.6 is in three steps. First we show that the matching rate between 3 and (4 or 6) is limited when the priority of  $\{1, 3\}$  and  $(2, 3)$  is high.

► **Lemma 6.8.** *Let  $(G, \mu)$  be a matching problem that contains the triangle subgraph depicted in Figure 20 (the triangle is only connected to the rest of  $G$  through vertex 3). Let  $\Phi$  be a greedy policy where vertex 3 always prefer 1 or 2 over any other edge (the behavior of  $\Phi$  for other decisions can be arbitrary). If  $\beta > 1/2$ , we have:*

- The process that describes the number of items in vertex 1, 2, and 3 is positive recurrent;
- The total matching rate  $\lambda_{3,X}$  between 3 and nodes other than 1 and 2 is limited by  $\frac{1}{1+2\beta}$ .

**Proof.** Consider some  $\beta > 1/2$ . We first observe that the number of items in 3 is stable as it is controlled by  $2\beta > 1$  (vertices 1 and 2 are not linked to anyone but each other and 3). To



■ **Figure 20** Triangle sub-graph, with arrival rates  $\mu = (\beta, \beta, 1, \dots)$ . Node 3 is the unique bridge between the triangle and the rest of the graph.  $\lambda_{3,X} = \mu_3 - \lambda_{2,3} - \lambda_{1,3}$  denotes the sum of matching rates between 3 and the rest of the graph.

prove the stability of 1 and 2, we must show that  $\{1, 3\}$  and  $\{2, 3\}$  have positive rates, which is the case if  $\lambda_{3,X} < 1$ . To do that, we associate to each arrival sequence of items 1, 2, and 3 the following virtual policy  $\Psi$ : the items are matched as in  $\Phi$ , with the assumption that 3 always has a non-empty neighbor in the rest of the graph. Within  $\Psi$ , the rest of the graph acts like a “black hole” so whenever an item 3 arrives and 1 and 2 are empty, it is matched with some node from the rest of the graph. We can easily check that the number of matches  $\{3, X\}$  made under  $\Phi$  is always at most the number of matches  $\{3, X\}$  under  $\Psi$ . Under  $\Psi$ , 3 is always empty (all arrivals are instantly matched) and we can partition the state of queues in 1, 2, and 3 into two  $M/M/1$  queues sharing the same empty state, each one having load  $\rho := \frac{\beta}{1+\beta}$ . It follows that the probability of the empty state is

$$p_{\emptyset} = \frac{1}{1 + 2\frac{\rho}{1-\rho}} = \frac{1}{1 + 2\beta}.$$

The matching rate of  $\{3, X\}$  in  $\Psi$  is exactly  $p_{\emptyset}$  (arrival of an item 3 while 1 and 2 are empty), so the matching rate  $\lambda_{3,X}$  in the original policy  $\Phi$  checks

$$\lambda_{3,X} \leq \frac{1}{1 + 2\beta} < 1.$$

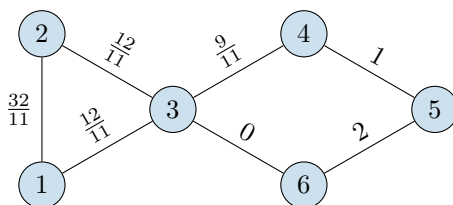
◀

The second step consists in producing a greedy policy that is unstable with respect to node 4 but stable for the other nodes.

► **Lemma 6.9.** *In the matching  $(G, \mu)$  depicted Figure 21, let  $\Phi_{\infty}$  be the greedy matching policy defined by: choose by decreasing order of priority edges  $\{1, 3\}$ ,  $\{2, 3\}$ ,  $\{3, 4\}$ ,  $\{5, 6\}$ , or one of the other edges.*

*We have:*

- $(G, \mu, \Phi_{\infty})$  is unstable;
- The process defined by the queue sizes of all vertices but 4 is positive recurrent;
- The matching rates achieved by the system are the ones from Figure 21 (conservation does not hold).



■ **Figure 21** Matching rates under the unstable policy  $\Phi_{+\infty}$  ( $\mu = (4, 4, 3, 2, 3, 2)$ ). Note that the unstable node 4 does not satisfy the conservation principle.

**Proof.** We first apply Lemma 6.8 on vertices 1 to 3, using a scaling of 3 and  $\beta = 4/3$ . It shows that 1, 2, and 3 are stable and that  $\lambda_{3,X}$  is at most  $3 \frac{1}{1+2\frac{4}{3}} = \frac{9}{11}$ .

We now focus on the even cycle 3, 4, 5, 6. The rate  $\mu_4 + \mu_6 = 4$  needs to be absorbed by 3 and 5 but the rate of 5 is 3 and the rate that 3 can allocate to the cycle is at most  $\frac{9}{11}$ . We deduce that the total number of items of class 4 or 6 is transient. However, 5 always chooses 6 if it can and  $\mu_5 > \mu_6$  so node 6 is stable. This means that 4 is the unique transient node and that node 5 is stable (it is actually always empty, like 3).

As 4 is not empty with probability 1,  $\Phi_\infty$  behaves like  $\Psi$  for nodes 1, 2, and 3. This gives  $\lambda_{1,2}$ ,  $\lambda_{1,3}$ ,  $\lambda_{2,3}$ , and  $\lambda_{3,X}$ . Because of the priority rule,  $\lambda_{3,4} = \lambda_{3,X}$  and  $\lambda_{3,6} = 0$ . The last matching rates are obtained by using the conservation principle on the (stable) nodes 6 and 5. ◀

We now can conclude.

**Proof of Proposition 6.6.** We first need to prove that  $\Phi_k$  is stable. The number of items in 4 is positive recurrent: as soon as it is greater than  $k$ , all arrival from 3 are matched to 4 so the departure rate becomes greater than the arrival rate ( $\mu_3 = 3 > \mu_4 = 2$ ). Stability of other vertices ensues. Call  $p_\emptyset(k)$  the probability that 4 is empty under  $\Phi_k$ . As  $\Phi_k$  behaves for 4 like the transient process of  $\Phi_\infty$  up to size  $k$ , we have  $\lim_{k \rightarrow \infty} p_\emptyset(k) = 0$ . A match  $\{3, 6\}$  can only occur when an item 3 or 6 arrives while vertex 4 is empty so we have  $0 \leq 1 - \alpha(k) \leq 5p_\emptyset(k)$ , which concludes the proof. ◀

**Proof of Corollary 6.7.** We proved that there is a family of greedy policies such that  $\alpha$  can be arbitrary close to 1. We just need to exhibit another family such that  $\alpha$  can be arbitrary close to 0 and use the convexity of  $\Lambda_G$  to conclude. For  $k \geq 0$ , let  $\Phi_{-k}$  be the greedy matching policy defined by:

- if there are less than  $k$  items of type 4 in the system, choose by decreasing order of priority edges  $\{1, 3\}$ ,  $\{2, 3\}$ ,  $\{3, 6\}$ ,  $\{4, 5\}$ , or one of the other edges;
- otherwise choose by decreasing order of priority edges  $\{3, 6\}$ ,  $\{4, 5\}$ ,  $\{1, 3\}$ ,  $\{2, 3\}$ , or one of the other edges.

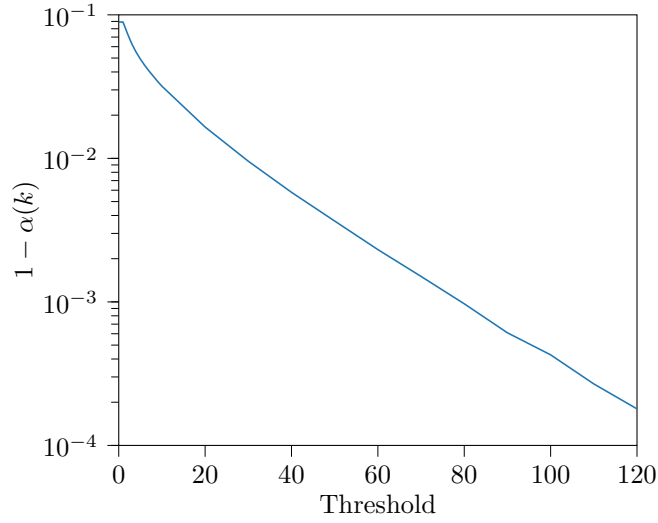
Using exactly the same proof than for Proposition 6.6 ( $\Phi_{-k}$  is  $\Phi_k$  after switching the labels 4 and 6), we get the result. ◀

Figure 22 shows values of  $\alpha(k)$  obtained by simulation. It confirms Proposition 6.6 and hints that the convergence is exponential with  $k$ .

## 6.2 Arbitrary policies

▶ **Proposition 6.10.** *Let  $y$  be vertex of  $\Lambda_{\geq 0}$ .*

- *If  $y$  is bijective,  $y \in \Lambda_P$ ;*



■ **Figure 22** Convergence of  $\alpha(k)$  to 1 as a function of the policy threshold  $k$ . Each point is computed by simulating  $10^9$  arrivals.

■ *If  $y$  is injective-only,  $y \notin \Lambda_{\mathcal{P}}$ .*

**Proof.** Let  $G(y) = (V, E')$  be the subgraph of  $G$  associated to  $y$ . The first part is straightforward by considering a semi-greedy *match-the-longest* policy on  $E'$ , denoted  $\Phi(y)$ .  $(G(y), \mu)$  is stabilizable and  $y$  is its unique matching rate.  $\Phi(y)$  behaves like a greedy *match-the-longest* policy on  $G(y)$ , so  $(G, \mu, \Phi(y))$  is stable and its matching rate is  $y$ . The second part is obtained by noticing that if  $y$  is injective-only,  $G(y)$  admits at least a bipartite component (more specifically a tree). The arrival drift between the two parts of the bipartite component makes impossible the existence of a stable policy. ◀

Proposition 6.10 essentially states that bijective vertices are easy to reach, while injective-only vertices are not due to a “bipartite curse”. This gives us a first partial characterization of  $\Lambda_{\mathcal{P}}$ .

► **Corollary 6.11.** *If a face  $F$  of  $\Lambda_{\geq 0}$  contains only bijective vertices,  $F \in \Lambda_{\mathcal{P}}$ . In particular,  $\Lambda_{\mathcal{P}} = \Lambda_{\geq 0}$  if and only if  $\Lambda_{\geq 0}$  is simple and  $(G, \mu)$  is essential.*

**Proof.** We just use the Proposition 6.1 (convexity of  $\Lambda_{\mathcal{P}}$ ) and Proposition 6.10. The last statement comes from Proposition 5.9. ◀

To go further, we propose to introduce a family of stable policies that is able to arbitrary approach a vertex of  $\Lambda_{\geq 0}$ .

► **Proposition 6.12.** *Let  $y$  be a vertex of  $\Lambda_{\geq 0}$ . For  $k \geq 0$ , consider the following policy, denoted  $\Phi_k(y)$ :*

- *If the size of the longest queue is less than  $k$ , apply a semi-greedy match-the-longest policy on  $G(y) = (V, E')$ ;*
  - *Otherwise, apply a greedy match-the-longest policy.*
- $\Phi_k(y)$  is stable and  $\lim_{k \rightarrow \infty} \lambda(\Phi_k(y)) = y$ .

**Sketch of proof.** In essence, the technique is similar to the one used for Proposition 6.6: we take an unstable policy that achieves the desired goal and make it stable by reverting to a

stable policy when the queue sizes become too large. If the threshold is high enough, most of the matchings will be made under the unstable policy.

We present here a sketch of proof for the hard case where  $G(y)$  is a tree (like in Example 3.7 when  $\mu_1\bar{\mu}_2 = \mu_3\bar{\mu}_4$ ). When  $G(y)$  has multiple connected components, bijective or injective-only, the proof needs to be adapted accordingly.

The stability of  $\Phi_k(y)$  comes from the fact that apart from a compact set of states (the states where the queue sizes are less than  $k$ ),  $\Phi_k(y)$  behaves like a stable policy.

Let  $p_\ell$  be the probability that the longest queue of the system has size  $\ell$ . If we look at  $\ell \geq k$ , the stability induces a negative drift, which means that we have  $\sum_{\ell \geq k} p_\ell \leq Cp_k$  for a constant  $C$  that does not depend on  $k$ . Conversely, for  $\ell \leq k$ , the size of the longest queue is mostly controlled by the drift of an unbiased random walk between the two parts of the bipartite graph, which means  $p_\ell \geq cp_k$  for another constant  $c$ . By combining the two, we get that the probability that the longest queue of the system has a size greater than  $k$  tend to 0 when  $k$  goes to infinity. As matchings outside  $E'$  only occur when the longest queue is greater than  $k$ , we conclude that the matching rate of an edge outside  $E'$  goes to 0, which by continuity of the conservation principle and injectivity of  $G(y)$  means that  $\lim_{k \rightarrow \infty} \lambda(\Phi_k(y)) = y$ . ◀

One drawback of Proposition 6.12 is that the average size of the longest queue will tend to grow with  $k$  when the vertex is injective-only. In other words, there exists a trade-off between approaching an injective-only vertex and the minimization of the waiting time. This issue is somehow similar to the threshold-based greedy policies we introduce for Proposition 6.6, where the vertices of  $\Lambda_{\geq 0}$  could be asymptotically reached by making the average queue size of a given node grow.

Note that in [25], a family of policies is introduced to optimize a reward function on edges. Their goal is similar to ours, with two main differences: first they aim at optimizing a reward function on matching rates but do not provide any description of the limit rate, while we show that the limit rate is a vertex of  $\Lambda_{\geq 0}$ ; second their family is a linear combination of a reward-based policy and a virtual queue policy, parameterized by a balance factor  $\beta$ , while we rely on semi-greedy policies with a threshold cut  $k$ . Yet, they also notice in their solution that the system has an average queue size that grows in  $1/\beta$ , which is the equivalent of our  $k$ .

► **Corollary 6.13.** *Any positive solution of the conservation equation (2) can be obtained by a stable policy. In other words,*

$$\Lambda_{>0} \subseteq \Lambda_{\mathcal{P}} \subseteq \Lambda_{\geq 0}.$$

**Proof.** Obviously  $\Lambda_{\mathcal{P}} \subseteq \Lambda_{\geq 0}$  because any stable policy must respect the conservation principle. From Proposition 6.12, we know that any vertex of  $\Lambda_{\geq 0}$  is part of the closure of  $\Lambda_{\mathcal{P}}$ , so by convexity of  $\Lambda_{\mathcal{P}}$  this closure is  $\Lambda_{\geq 0}$ . In particular,  $\Lambda_{\mathcal{P}}$  contains the interior of  $\Lambda_{\geq 0}$ , which is  $\Lambda_{>0}$ . ◀

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## References

- 1 Ivo Adan, Ana Bušić, Jean Mairesse, and Gideon Weiss. Reversibility and Further Properties of FCFS Infinite Bipartite Matching. *Mathematics of Operations Research*, 43(2):598–621, December 2017. Publisher: INFORMS. doi:10.1287/moor.2017.0874.
- 2 Ivo Adan and Gideon Weiss. Exact FCFS matching rates for two infinite multitype sequences. *Operations Research*, 60(2):475–489, 04 2012. Publisher: INFORMS. doi:10.1287/opre.1110.1027.

- 3 Angelos Aveklouris, Levi DeValve, and Amy R. Ward. Matching Impatient and Heterogeneous Demand and Supply. *arXiv:2102.02710 [cs, eess, math]*, December 2021. arXiv: 2102.02710. URL: <http://arxiv.org/abs/2102.02710>.
- 4 Egon Balas and William R. Pulleyblank. The perfectly matchable subgraph polytope of a bipartite graph. *Networks*, 13(4):495–516, 1983. doi:10.1002/net.3230130405.
- 5 Jocelyn Begeot, Irène Marcovici, and Pascal Moyal. Stability regions of systems with compatibilities, and ubiquitous measures on graphs. *arXiv:2111.13231 [math]*, November 2021. arXiv: 2111.13231. URL: <http://arxiv.org/abs/2111.13231>.
- 6 Jocelyn Begeot, Irène Marcovici, Pascal Moyal, and Youssef Rahme. A general stochastic matching model on multigraphs. *arXiv:2011.05169 [math]*, November 2020. arXiv: 2011.05169. URL: <http://arxiv.org/abs/2011.05169>.
- 7 Pierre Bremaud. *Markov Chains: Gibbs Fields, Monte Carlo Simulation, and Queues*. Texts in Applied Mathematics. Springer-Verlag, New York, 1999. URL: <https://www.springer.com/gp/book/9780387985091>, doi:10.1007/978-1-4757-3124-8.
- 8 Ana Bušić, Varun Gupta, and Jean Mairesse. Stability of the Bipartite Matching Model. *Advances in Applied Probability*, 45(2):351–378, June 2013. Publisher: Cambridge University Press. doi:10.1239/aap/1370870122.
- 9 Ana Bušić and Sean Meyn. Approximate optimality with bounded regret in dynamic matching models. *arXiv:1411.1044 [cs, math]*, June 2016. arXiv: 1411.1044. URL: <http://arxiv.org/abs/1411.1044>.
- 10 Arnaud Cadas, Ana Bušić, and Josu Doncel. Optimal control of dynamic bipartite matching models. In *Proceedings of the 12th EAI International Conference on Performance Evaluation Methodologies and Tools, VALUETOOLS 2019, Universitat de les Illes Balears, Palma de Mallorca, Spain, March 12-15, 2019*, pages 39–46. ACM, 2019. doi:10.1145/3306309.3306317.
- 11 Arnaud Cadas, Josu Doncel, Jean-Michel Fourneau, and Ana Bušić. Flexibility can hurt dynamic matching system performance. *arXiv:2009.10009 [cs, math]*, September 2020. arXiv: 2009.10009. URL: <http://arxiv.org/abs/2009.10009>.
- 12 René Caldentey, Edward H. Kaplan, and Gideon Weiss. FCFS infinite bipartite matching of servers and customers. *Advances in Applied Probability*, 41(3):695–730, 09 2009. Publisher: Cambridge University Press. doi:10.1239/aap/1253281061.
- 13 Céline Comte and Jaap Storm. Global and local stability in parallel service systems with redundancy.
- 14 Céline Comte. Stochastic non-bipartite matching models and order-independent loss queues. *Stochastic Models*, pages 1–36, October 2021. Publisher: Taylor & Francis. doi:10.1080/15326349.2021.1962352.
- 15 Céline Comte and Jan-Pieter Dorsman. Performance Evaluation of Stochastic Bipartite Matching Models. In *Performance Engineering and Stochastic Modeling*, Lecture Notes in Computer Science, pages 425–440. Springer International Publishing, 2021. doi:10.1007/978-3-030-91825-5\_26.
- 16 Dragoš Cvetkovic, Peter Rowlinson, and Slobodan Simic. *Spectral Generalizations of Line Graphs: On Graphs with Least Eigenvalue -2*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2004. doi:10.1017/CB09780511751752.
- 17 Reinhard Diestel. *Graph Theory*. Springer Publishing Company, Incorporated, 5th edition, August 2017.
- 18 Michael Doob. An interrelation between line graphs, eigenvalues, and matroids. *Journal of Combinatorial Theory, Series B*, 15(1):40–50, 08 1973. doi:10.1016/0095-8956(73)90030-0.
- 19 Itai Gurvich and Amy Ward. On the Dynamic Control of Matching Queues. *Stochastic Systems*, 4(2):479–523, October 2014. Publisher: INFORMS. doi:10.1287/13-SSY097.
- 20 Matthieu Jonckheere, Pascal Moyal, Claudia Ramírez, and Nahuel Soprano-Loto. Generalized max-weight policies in stochastic matching. *arXiv:2011.04535 [math]*, November 2020. arXiv: 2011.04535. URL: <http://arxiv.org/abs/2011.04535>.

- 21 N. Karmarkar. A new polynomial-time algorithm for linear programming. In *Proceedings of the sixteenth annual ACM symposium on Theory of computing*, STOC '84, pages 302–311, New York, NY, USA, December 1984. Association for Computing Machinery. doi: 10.1145/800057.808695.
- 22 László Lovász and M. D. Plummer. *Matching Theory*. American Mathematical Soc., 2009.
- 23 Jean Mairesse and Pascal Moyal. Stability of the stochastic matching model. *Journal of Applied Probability*, 53(4):1064–1077, 12 2016. Publisher: Cambridge University Press. doi:10.1017/jpr.2016.65.
- 24 Pascal Moyal, Ana Bušić, and Jean Mairesse. A product form for the general stochastic matching model. *Journal of Applied Probability*, 58(2):449–468, 06 2021. doi:10.1017/jpr.2020.100.
- 25 Mohammadreza Nazari and Alexander L. Stolyar. Reward maximization in general dynamic matching systems. *Queueing Systems*, 91(1):143–170, February 2019. doi:10.1007/s11134-018-9593-y.
- 26 Youssef Rahme and Pascal Moyal. A stochastic matching model on hypergraphs. *Advances in Applied Probability*, 53(4):951–980, December 2021. doi:10.1017/apr.2021.8.
- 27 Günter M. Ziegler. *Lectures on polytopes*. Springer-Verlag, New York, 1995. URL: [http://www.worldcat.org/search?qt=worldcat\\_org\\_all&q=9780387943657](http://www.worldcat.org/search?qt=worldcat_org_all&q=9780387943657).

## A Minimal stability region for greedy matching policies

The following result gives a sufficient stability condition for greedy matching policies. The proof relies on a linear Lyapunov function. This result can be seen as the counterpart of [8, Proposition 5.1] for non-bipartite matching models.

► **Proposition A.1.** *Consider an undirected, connected, non-bipartite graph  $G$  and a vector  $\mu \in \mathbb{R}_{>0}^n$  such that  $\sum_{i \in V} \mu_i = 1$  and*

$$\mu(V(\mathcal{I})) > \frac{1}{2}, \quad \mathcal{I} \in \mathbb{I}. \quad (14)$$

*Then the matching model  $(G, \mu, \Phi)$  is stable for every greedy matching policy  $\Phi$ .*

**Proof.** Consider a greedy matching policy  $\Phi$  and its continuous-time Markov chain  $\{X(t), t \geq 0\}$ , with state space  $\mathcal{X}$ . Let  $(X_k)_{k \in \mathbb{N}}$  be the discrete time chain, obtained by uniformization with rate 1. We consider the Lyapunov function  $L : \mathcal{X} \rightarrow \mathbb{R}$  defined by  $L(x) = \|h(x)\|_1$ , that is, the number of unmatched items in state  $x$ . For each  $k \in \mathbb{N}$  and  $x \in \mathcal{X}$ , we have

$$\mathbb{E}(L(X_{k+1}) \mid X_k = x) - L(x) = \mu(V \setminus V(\mathcal{I})) - \mu(V(\mathcal{I})) = -(2\mu(V(\mathcal{I})) - 1),$$

where  $\mathcal{I}$  is the set of unmatched customer classes in state  $x$ . Importantly, if  $L(x) > 0$ , then  $\mathcal{I}$  is an independent set of the graph  $G$  because the policy  $\Phi$  is greedy. It follows that  $\mathbb{E}(L(X_{k+1}) \mid X_k = x) - L(x) \leq -\varepsilon$  with  $\varepsilon = \min_{\mathcal{I} \in \mathbb{I}} (2\mu(V(\mathcal{I})) - 1)$ . Equation (14) implies that  $\varepsilon > 0$ . Using the Lyapunov-Foster theorem [7, Theorem 1.1], we conclude that the matching model  $(G, \mu, \Phi)$  is stable. ◀

In general, Equation (A.1) is more restrictive than Condition 2 in Proposition 3.2. Indeed, Equation (A.1) is equivalent to

$$\mu(V \setminus V(\mathcal{I})) < \mu(V(\mathcal{I})), \quad \mathcal{I} \in \mathbb{I},$$

which implies Condition 2 in Proposition 3.2 since  $\mathcal{I} \subseteq V \setminus V(\mathcal{I})$  for each  $\mathcal{I} \in \mathbb{I}$ .



► **Corollary A.2.** *For all graphs of diameter 1 (complete graphs, e.g.  $K_n, n \geq 3$ ), any greedy policy is maximally stable for non-bipartite complete graphs. This is also the case for the diamond graph (Example 3.7). On the other hand, Proposition A.1 is never satisfied if the graph possesses one leaf (node of degree 1) or has diameter greater than 2.*

**Proof.** The first part was already proved in Proposition 6.3 but Proposition A.1 gives an alternate proof: if  $G$  is a complete graph, then  $\mathcal{I} = V \setminus V(\mathcal{I})$  for each  $\mathcal{I} \in \mathbb{I}$  so Proposition 3.2 and Equation (14) are equivalent: if  $\mu$  is in the stability region, a greedy policy is stable.

For the diamond graph, we just check that the inequalities that characterize stability in Example 3.7 imply Equation (14).

On the other hand, if  $G$  has a leaf  $i$ , Equation (14) cannot hold for both  $i$  and its neighbor. Likewise, if two nodes are at distance 3 or more, Equation (14) cannot hold for these two nodes. ◀

## B Proof of Proposition 5.2

The algorithm is mainly based on the notion of *cycle space* of a graph, whose main notions we recall (cf [17, Section 1.9] for details).

A *spanning subgraph* of a graph  $G = (V, E)$  is a subgraph  $G = (V, E')$  with  $E' \subseteq E$ . the same set of nodes. A subgraph is *Eulerian* if every vertex has an even degree. In particular, if  $E'$  are the edges of a cycle of  $G$ , then  $(V, E')$  is Eulerian. The *cycle space* of  $G$  is the vector space made of all Eulerian spanning subgraphs of  $G$ , using symmetric difference for addition and the two-element field for scalar multiplication. Its dimension is  $m - n + 1$ .

**Proof that Algorithm 1 terminates.** We first prove the existence of edge  $a$  defined on line 3 of Algorithm 1. By definition of a spanning tree,  $\mathcal{T} \cup \{a\}$  contains a unique cycle for each  $a \in E \setminus \mathcal{T}$ . The  $m - n + 1$  cycles thus obtained are independent (each one contains a distinct  $a$ ) so they form a basis of the cycle space of  $G$ . Since the linear combination of even cycles cannot produce an odd cycle and  $G$  contains at least one (it is non-bipartite), one of the basis vector must be an odd cycle.

We now verify that, for each  $s \in E \setminus (\mathcal{T} \cup \{a\})$ ,  $\mathcal{T} \cup \{a, s\}$  contains either (i) an even cycle  $C_\ell$  or (ii) a kayak paddle  $KP_{\ell, r, p}$  with two odd cycles. By construction,  $\mathcal{T} \cup \{a\}$  contains a unique cycle  $C_r$ , which is odd, and  $\mathcal{T} \cup \{s\}$  contains a unique cycle  $C_\ell$ .  $\mathcal{T} \cup \{a, s\}$  contains both  $C_r$  and  $C_\ell$ . We now proceed by elimination:

- If  $C_\ell$  is even, then this is an even cycle included into  $\mathcal{T} \cup \{a, s\}$ , and we are therefore in case (i).
- If  $C_\ell$  is odd, and  $C_r$  and  $C_\ell$  have at least one edge in common, then the symmetric difference of  $C_r$  and  $C_\ell$  is an even cycle, and it is again included into  $\mathcal{T} \cup \{a, s\}$ , so we are again in case (i).
- If  $C_\ell$  is odd, and  $C_r$  and  $C_\ell$  have no edge in common, then we are in case (ii). ◀

**Proof that Algorithm 1 returns the correct result.** We finally prove that the vectors  $v$  generated (i) belong to the kernel of  $A$ , (ii) are independent, and (iii) there are  $m - n$  of them. Statement (iii) is straightforward because the cardinality of  $E \setminus (\mathcal{T} \cup \{a\})$  is  $m - n$ . To verify (i), it suffices to observe that  $Av = 0$ , e.g. that for each node the total of its edges weighted with  $v$  is 0:

- Any node not adjacent to the support of  $v$  has only null edges;
- Any cycle node apart from  $v_i$  and  $v_j$  has non-zero weights  $(-1, 1)$ ;
- Any path node apart from  $v_i$  and  $v_j$  has non-zero weights  $(-2, 2)$ ;

- If  $p > 0$ ,  $v_i$  has non-zeros weights  $(-1, -1, 2)$ ;  $v_j$  has non-zeros weights  $(-1, -1, 2)$  if  $p$  is odd or  $(1, 1, -2)$  if  $p$  is even;
- If  $p = 0$ ,  $v_i$  is  $v_j$ . It has non-zeros weights  $(-1, -1, 1, 1)$ .

Lastly, to verify (ii), we observe that, for each  $s \in E \setminus (\mathcal{T} \cup \{a\})$ ,  $s$  belongs to the cycle or kayak paddle used, so the vector constructed from edge  $s$  is the only one with a non-zero value in  $s$ . ◀