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Abstract. We give a classification of toric, Hermitian, Ricci flat, ALF Riemannian metrics in dimension 4, including metrics with conical singularities. The only smooth examples are on one hand the hyperKähler ALF metrics, on the other hand, the Kerr, Taub-NUT and Chen-Teo metrics. There are examples with conical singularities with infinitely many distinct topologies. We provide explicit formulas.

More than ten years ago Chen and Teo discovered a new family of Riemannian AF gravitational instantons [CT11], where AF (Asymptotically Flat) means that the end of the instanton is diffeomorphic to that of $\mathbb{R}^3 \times S^1$. The family depends on one parameter (up to scale). Before that, only one family was known: the Kerr family (also a one parameter family up to scale), among which one finds the well-known Schwarzschild metric.

Our work started from the observation by Steffen Aksteiner and Lars Andersson [AA21] that the Chen-Teo instantons are 'half algebraically special', or in more geometric terms that the selfdual Weyl tensor W^+ admits a double eigenvalue. Maybe up to taking a $\mathbb{Z}_2$ cover, this is equivalent to say that the metric is Hermitian, and more precisely is conformally Kähler. For a toric metric the problem can then be expressed in the setting of toric Kähler geometry, and the objects of interest are Bach flat extremal Kähler metrics with Poincaré behaviour at infinity (that is asymptotic to the product of a hyperbolic 2-cusp and a spherical 2-metric, see section 2), to be constructed from some given 'moment polytope' in the real plane. This polytope completely characterizes the instanton up to scale, and we will calculate it for all classical examples.

This perspective gave some hope to construct other ALF gravitational instantons. ALF metrics are metrics which at infinity look locally like the product $\mathbb{R}^3 \times S^1$ (see the precise definition in section 1). We give a complete classification of Hermitian, toric, Ricci flat, ALF metrics in terms of one real convex piecewise linear function on the real line. As a byproduct we obtain:

Theorem A. *Suppose* (X^4, g) *is a smooth Ricci flat, Hermitian, toric, ALF metric. If* g *is not Kähler, then* g *is a Kerr metric, a Chen-Teo metric, the Taub-NUT metric (with the orientation opposed to the hyperKähler orientation) or the Taub-bolt metric.*

If we replace the Hermitian condition by the half algebraically special condition, then some $\mathbb{Z}_2$ quotients can also occur.

The Kähler examples are actually hyperKähler gravitational instantons and are well-known: one recovers only multi-Taub-NUT metrics. The Taub-NUT metric itself is special since it is hyperKähler for one orientation, and conformally Kähler for the other orientation, hence we have put it in the list of the theorem.

Curiously our classification looks like a "non compact dual" of that of compact Hermitian Einstein metrics with positive Ricci which are not Kähler, see [LeB97, CLW08]: these are known to be $\mathbb{C}P^2$ (the Fubini-Study metric with the reverse orientation), its blow up at 1 point (the Page metric) or 2 points (the Chen-LeBrun-Weber metric). The analogy with the Taub-NUT, Kerr and Chen-Teo metrics is obvious, and is confirmed by the fact that their polytopes have the same number of edges. But we have not found any direct relation between both classifications. The Taub-bolt instanton does not fit in this correspondence but can be considered as a degeneration of Chen-Teo instantons.

Of course theorem A leaves open the fascinating question of the classification of all gravitational instantons, or at least of Hermitian gravitational instantons, see [AA21, conjecture 1]. Only the classification in the hyperKähler case is now known after the efforts of lots of mathematicians [Kro89, Min11, CC15, CC19, CC21a, SZ21].

Theorem A is a classification theorem which does not give new examples of ALF instantons (nevertheless it gives an alternative construction of the Chen-Teo instantons). But we have a general classification result (theorem 7.5) in terms of convex piecewise linear functions, which also allows to understand metrics with conical singularities. These singularities occur around 2-spheres (divisors from the complex viewpoint) which are fixed by the torus action. Our results are quite different from the recent classification results in [KL21]: on one hand our techniques are not limited to the AF case, on the other hand by restricting to the Hermitian case we obtain a full description in the sense that we can understand all geometric properties (smoothness, cone angles, etc.) from our piecewise linear functions. In particular we obtain:

Theorem B. *Suppose* (X^4, g) *is a Hermitian, toric, Ricci flat, ALF instanton with conical singularities of cone angles* $2\pi\alpha_i$ *around the fixed divisors* D_i. *Let* $x = D_j \cap D_{j'}$ *be a fixed point of the action. Then for each* $\alpha < \alpha_j + \alpha_{j'}$ *close to* $\alpha_j + \alpha_{j'}$, *the blowup of* X *at the point* x *admits a Hermitian, toric, Ricci flat, ALF instanton, with the same conical angles around the* D_i*'s, and a conical singularity of angle* $2\pi\alpha$ *around the additional divisor.*

This theorem implies that by successive blowups one can construct ALF instantons with conical singularities on an infinite family of topologically distinct manifolds. Of course it follows from the condition on α that we need larger and larger angles.

As an illustration of the theorem, one can start from a Kerr metric and blow up a point: one obtains a family of metrics with cone angle slightly

less than 4π on the exceptional divisor. It turns out that one can decrease the angle to the angle 2π to create a smooth instanton which is a Chen-Teo instanton. These metrics fit in the larger 5-parameter family of Chen and Teo in [CT15], but we have not studied them systematically, see nevertheless remark 8.4.

It can be seen that the geometric process for this blow up, for example in the case $\alpha_j = \alpha_{j'} = 1$, consists in gluing a ramified double cover of the Eguchi-Hanson metric on T^*S^2, hence with cone angle 4π around S^2.

Since the problem of constructing toric Hermitian gravitational instantons can be reinterpreted in terms of Kähler metrics, we are in the general setting of the Yau-Tian-Donaldson conjecture on extremal Kähler metrics. This is solved in the toric case [Don09, CC21b, CC21c, He19], but the resolution does not seem to cover exactly our case because of the Poincaré behaviour at infinity. Even with solutions, it is a hard problem to determine when the extremal metric is Bach flat. But we avoid this problem, because the Ricci flat equation (becoming the Bach flat equation on the Kähler metric) is so strong that an ansatz exists for the solutions in terms of an axisymmetric harmonic function on $\mathbb{R}^3$. This was discovered recently by Paul Tod [Tod20], and we give another derivation of his ansatz in this paper.

Applying the ansatz to our possible polytopes gives that all possible harmonic functions are classified in terms of a single convex piecewise linear real function on the real axis. We can then read the smoothness of the solution in terms of the usual theory of toric Kähler metrics ('Delzant polytopes') and proceed to a complete classification.

Overview of the paper. In section 1 we give a general definition of ALF metrics and introduce the Hermitian condition; in particular we find that a possible complex structure has only one possible behaviour at infinity (proposition 1.6). After recalling the toric machinery in section 2, we give our derivation of the Tod ansatz in section 3. From the behaviour of the Kähler metric at the boundary of the polytope, we deduce in sections 4 and 5 that the generating function U of Tod, which is an axisymmetric harmonic function on $\mathbb{R}^3$, is generated by a convex piecewise linear positive function on the real line (it is actually the Dirac density of ΔU along the real axis). We then study the metric just from the data of this convex piecewise linear function: the polytope is found in section 6, the regularity is studied in section 7 (theorem 7.5), where we also describe the classical examples from this point of view: these are the Kerr-Taub-bolt metrics [GP80], they all correspond to instantons with conical singularities, including on the divisors going to infinity. These ideas are applied to prove theorem A in section 8, which contains also a calculation of the polytopes of the Chen-Teo instantons. Section 9 contains the construction of conical instantons by blowup, leading to theorem B.

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1. ALF metrics and complex structures

Definition 1.1. A Riemannian manifold (M^4, g) is ALF if

- it has one end diffeomorphic to $(R, +\infty) \times L$ for some $R > 0$, where L is $S^1 \times S^2$, S^3, or a finite quotient;
- there is a triple (η, T, γ) defined on L, where η is a 1-form, T a vector field such that $\eta(T) = 1$ and $T \lrcorner d\eta = 0$, and γ is a T-invariant metric on the distribution $\ker \eta$;
- the transverse metric γ has constant curvature $+1$ (the transverse metric is the metric induced on local quotients by the action generated by T);
- the metric g has the behaviour

$$g = dr^2 + r^2\gamma + \eta^2 + h, \text{ with } |\nabla^k h| = O(r^{-1-k}), \tag{1}$$

where γ is extended to the whole TL by deciding that $\ker \gamma$ is generated by T; the covariant derivative ∇ and the norm are with respect to the asymptotic model $dr^2 + r^2\gamma + \eta^2$.

The AF case is when $L = S^2 \times S^1$, so that the end has the topology of $(\mathbb{R}^3 \setminus B(R)) \times S^1$ for some large R.

The hypothesis tells us that locally near a point of L the metric is $dr^2 + r^2 g_{S^2} + dt^2 + O(r^{-1})$, in particular the curvature is $O(r^{-3})$. If we allow conical singularities up to infinity, then we must generalize the definition to admit transverse metrics γ with conical singularities as well.

The volume growth is r^3, which corresponds to the fact that the direction of the vector field T collapses at infinity. The definition is designed so that the orbits of T do not need be closed. This does not occur in the hyperKähler case, but is the general case for purely Riemannian Ricci flat metrics (for example the Kerr metrics).

We now suppose that the metric is toric, that is there is an action of a 2-torus $\mathbb{T}^2$ on (M^4, g) which preserves g. We shall denote by $\mathfrak{t}^2$ its Lie algebra, so that $\mathbb{T}^2 = \mathfrak{t}^2/\mathbb{Z}^2$ for some lattice $\mathbb{Z}^2 \subset \mathfrak{t}^2 \simeq \mathbb{R}^2$. There is a compatibility condition between the ALF metric and the action:

Definition 1.2. A Riemannian manifold (M^4, g) is toric ALF if it is ALF, and the torus action near the end of M is an effective action on L which preserves (η, T, γ), and such that T is generated by the infinitesimal action of some element of $\mathfrak{t}^2$.

We now introduce the Hermitian condition:

Theorem 1.3 ([Der83, Prop. 5]). *Let* (M^4, g) *be an oriented Einstein manifold, whose selfdual Weyl tensor* W^+*, viewed as a trace-free symmetric endomorphism of* Λ^+M*, has a simple nontrivial eigenvalue* λ *and a double eigenvalue* $-\frac{\lambda}{2}$*. Then:*

1. W^+ *(and therefore* λ*) does not vanish on* M.
2. *Denote* J *an almost complex structure such that the selfdual 2-form* $\omega = g(J\cdot, \cdot)$ *is an eigenform of* W^+ *for the eigenvalue* λ*. Then* $(g_K := \lambda^{\frac{2}{3}} g, J, \omega_K = \lambda^{\frac{2}{3}}\omega)$ *is Kähler. In particular* J *is a complex structure.*

The condition on the Weyl tensor is known as half algebraically special, or half type D. We choose the orientation of M so that W^+ (rather than W^-) is degenerate.

Note the ambiguity $\pm J$ in the choice of the complex structure. As a result, one may have to pass to a 2-cover for J to exist globally. Our definition of Hermitian includes the global existence of J:

Definition 1.4. We say that (M, g) is Hermitian if M admits a complex structure J and a compatible Kähler metric g_K which is conformal to g.

Another equivalent way to state the condition on W^+ (when it is nonzero) is to say that (M^4, g) admits a twistor 2-form, that is a selfdual 2-form τ satisfying the equation (also known as the Killing-Yano equation)

$$\mathscr{T}^g(\tau)_X := \nabla_X \tau - \frac{1}{3}(X \lrcorner d\tau - X \wedge \delta\tau) = 0 \tag{2}$$

for all vector fields X, see [Pon92] and Lemma 2 in [ACG03].

One has $f^{-3}\mathscr{T}^{f^2 g}(f^3\tau) = \mathscr{T}^g(\tau)$, and it follows that if (M^4, g) is half algebraically special then $\tau = \lambda^{-\frac{1}{3}}\omega = \lambda^{-1}\omega_K$ is a twistor 2-form for g (and the converse is true).

Example 1.5. In $\mathbb{R}^3 \times S^1$ we have the twistor 2-form $\tau = r(-dr \wedge dt + r^2 \operatorname{vol}_{S^2})$.

The next proposition proves that Example 1.5 is the only possible asymptotic behaviour for non hyperKähler toric ALF instantons:

Proposition 1.6. *Let* (M^4, g) *be a toric Ricci flat ALF manifold, and* τ *a twistor 2-form. Suppose* W^+ *is nonzero. Then up to a nonzero multiplicative constant,* τ *has the asymptotic behaviour*

$$\tau = r(-dr \wedge \eta + r^2 \operatorname{vol}_\gamma) + O(1). \tag{3}$$

Observe that this formula implies that we have chosen the sign of η so that an oriented coframe is obtained from the forms $(-dr, \eta)$ and from an oriented coframe of $\ker \eta$.

Proof. We begin by solving the twistor equation for $\mathbb{T}^2$-invariant selfdual 2-forms on $\mathbb{R}^3 \times S^1$. We take radial coordinates (r, θ, ϕ) on $\mathbb{R}^3$ and an angular coordinate t on S^1, so the metric of $\mathbb{R}^3 \times S^1$ is

$$g_0 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + dt^2. \tag{4}$$

The torus is given by the angular variables (ϕ, t). We introduce the orthonormal coframe $\theta_1 = dr$, $\theta_2 = rd\theta$, $\theta_3 = r\sin\theta d\phi$, $\theta_4 = dt$. These are invariant under the torus action. We also use the basis of selfdual forms (hidden here is a choice of orientation of $\mathbb{R}^3 \times S^1$):

$$\omega_1 = -\theta_1\wedge\theta_4 + \theta_2\wedge\theta_3, \quad \omega_2 = \theta_1\wedge\theta_2 - \theta_3\wedge\theta_4, \quad \omega_3 = \theta_1\wedge\theta_3 + \theta_2\wedge\theta_4. \tag{5}$$

We take an invariant selfdual form $\tau = \sum_1^3 a_i(r,\theta)\omega_i$. The reader can check that the twistor equation (2) is given by the system of the following 8 equations:

$$\begin{aligned} r(a_1)_r &= a_1 - \cot(\theta)a_3 = a_1 - (a_3)_\theta \\ r(a_2)_r &= a_2 = 0 \\ r(a_3)_r &= a_3 + (a_1)_\theta = 0 \\ (a_2)_\theta &= \cot(\theta)a_2 = 0 \end{aligned} \tag{6}$$

It is immediate that the solutions are $a_1 = k_1\cos(\theta) + k_2 r$, $a_2 = 0$, $a_3 = k_1\sin(\theta)$ for some constants k_1 and k_2. So we have two solutions: $\tau_1 = r\omega_1$ and $\tau_2 = \cos(\theta)\omega_1 + \sin(\theta)\omega_3$. Actually $\tau_2 = -d(r\cos\theta)\wedge\theta_4 + d(r\sin\theta)\wedge\theta_3$ is parallel, since $r\cos(\theta)$ is the coordinate on the invariant axis of $\mathbb{R}^3$ and $d(r\sin\theta)\wedge\theta_3$ is the volume form in the orthogonal plane. In particular it is bounded, and we have proved (3) for the flat model $\mathbb{R}^3 \times S^1$.

The case of a general ALF metric can be deduced: starting from the behaviour (1), we write locally $\eta = dt + \alpha$ where $dt(T) = 1$ and α is a 1-form on the local quotient by the action of T. Then with respect to g one has $\alpha = O(r^{-1})$. Taking also local spherical coordinates (θ, ϕ) on this local quotient, such that ∂_θ is a second generator of $\mathbb{t}^2$ (the first being T), we can write locally the ALF metric as $g = g_0 + O(r^{-1})$, where g_0 is our flat metric (4) on $\mathbb{R}^3 \times S^1$. The twistor equation for g is the same as for g_0 with $O(r^{-1})$ perturbation, but more importantly the twistor operator $\mathscr{T}$ has the same symbol: this implies that the system (6) includes a set of ODE's for (a_1, a_2, a_3), that is $(\partial_r a_1, \partial_r a_2, \partial_r a_3)$ is completely determined in terms of (a_1, a_2, a_3) (without derivative). It follows that a solution τ for g is asymptotic to a solution for g_0 up to $O(r^{-1})$ terms. In particular, since $|\tau_1| = r$, for some constant c one has

$$\tau = c\tau_1 + O(1). \tag{7}$$

There remains to prove that $c \neq 0$. If $c = 0$, then we get instead $\tau = c'\tau_2 + O(r^{-1})$. As we have seen, up to a constant, we would then have that $\tau = \lambda^{-\frac{1}{3}}\omega$ with $|\omega|^2 = 2$ an eigenform of W^+ for the eigenvalue λ. Since by (7) we have $|\tau| = O(1)$ we would obtain $\lambda^{-1} = O(1)$. By (1) we have $|W| = O(r^{-3})$ so this is impossible, except if $\lambda \equiv 0$ that is if $W^+ \equiv 0$. □

A last fact we need on ALF instantons in the following topological result, which is probably well-known:

Lemma 1.7. *A non flat, Ricci flat, ALF manifold* (M, g) *has vanishing first De Rham cohomology group:* $H^1(M, \mathbb{R}) = 0$.

Proof. By the Bochner formula $dd^* + d^*d = \nabla^*\nabla + \mathrm{Ric}$ on 1-forms, and since $\mathrm{Ric} = 0$, the L^2-cohomology vanishes in degree 1. By [HHM04] this is the relative cohomology group $H^1(M, \partial M, \mathbb{R})$ (where the boundary $\partial M = L$ of definition 1.1 is taken in a topological sense). In the non AF case $H^1(L, \mathbb{R}) = 0$ and therefore $H^1(M, \mathbb{R}) = 0$. In the AF case $H^1(L, \mathbb{R}) = \mathbb{R}$ where the isomorphism is obtained by integration along the S^1 factor in $L = S^2 \times S^1$; therefore if $H^1(M, \mathbb{R}) \neq 0$ then the circle $S^1 \subset L$ has infinite order in the fundamental group of M and therefore will give a line in the universal cover $\tilde{M}$, from which we can produce a geodesic line, and therefore by the splitting theorem of Cheeger-Gromoll $(\tilde{M}, g)$ is a product $\mathbb{R} \times N^3$, so g is flat, which is a contradiction. □

2. Toric Kähler geometry

We start from a toric Ricci flat ALF manifold (M, g) which is Hermitian (definition 1.4) and has a nonzero W^+. We therefore have a complex structure J and a conformal metric $g_K = \lambda^{\frac{2}{3}} g$ which is Kähler. Moreover, from Proposition 1.6 we deduce that for some constant $k \neq 0$ we have $\lambda = \frac{k}{r^3} + O(\frac{1}{r^4})$. We change the conformal factor $\lambda^{\frac{2}{3}}$ to obtain the following asymptotic behaviour for the Kähler metric and the Kähler form as $r \to +\infty$:

$$g_K = (k^{-1}\lambda)^{\frac{2}{3}} g = \frac{dr^2 + \eta^2}{r^2} + \gamma + O_{g_K}(\frac{1}{r}), \tag{8}$$

$$\omega_K = (k^{-1}\lambda)\tau = -\frac{dr \wedge \eta}{r^2} + \mathrm{vol}_\gamma + O_{g_K}(\frac{1}{r}), \tag{9}$$

which means that the Kähler metric (and its complex structure) is locally asymptotic to the product of a Poincaré disk (curvature -1) with a round sphere (curvature $+1$).

One has $W^+_{g_K} = (k^{-1}\lambda)^{-\frac{2}{3}} W^+_g$ (seen as endomorphisms of Λ^+) and therefore the simple eigenvalue of $W^+_{g_K}$ is $k^{\frac{2}{3}}\lambda^{\frac{1}{3}}$. For a Kähler metric, this is also $\frac{\mathrm{Scal}}{6}$ so we have

$$\mathrm{Scal}_{g_K} = 6k^{\frac{2}{3}}\lambda^{\frac{1}{3}}. \tag{10}$$

In particular, $\mathrm{Scal}_{g_K} \to 0$ when $r \to \infty$, which is compatible with the previous asymptotics.

Recall that a function f on M gives rise to a Hamiltonian vector field X on M by $df = -X \lrcorner \omega_K$; the function f is the Hamiltonian potential, or moment, of X.

A Kähler metric is *extremal* if the scalar curvature is the Hamiltonian potential of a vector field which is also a Killing vector field of g_K.

Lemma 2.1. *If the ALF space (M^4, g) admits a twistor 2-form τ satisfying (3), then the following are equivalent:*

- *(M^4, g) is Ricci flat;*
- *$\frac{g}{|\tau|^3}$ is an extremal, Bach flat, Kähler metric.*

Proof. This lemma is well known, but it will be useful to recall the proof. Since $\mathrm{Ric}^g = 0$, after the conformal change we obtain

$$\mathrm{Ric}_0^{g_K} = -2\frac{(\nabla^{g_K} d\lambda^{\frac{1}{3}})_0}{\lambda^{\frac{1}{3}}}. \tag{11}$$

Since g_K is Kähler, it follows that $\nabla^{g_K} d\lambda^{\frac{1}{3}}$ is J-invariant and therefore the (real) vector field $\nabla^{g_K}\lambda^{\frac{1}{3}}$ is holomorphic. In particular, the Hamiltonian vector field $T = J\nabla^{g_K}(k^{-1}\lambda)^{\frac{1}{3}}$ is Killing with respect to g_K and therefore with respect to g as well.

Remark that for $\mathbb{R}^3 \times S^1$ we have $g_K = \frac{dr^2+dt^2}{r^2} + \gamma$ and $J\nabla_{g_K}(\frac{1}{r}) = -J\frac{\partial}{\partial r} = \frac{\partial}{\partial t}$, so the vector field T defined above coincides with the T in definition 1.1. Since any ALF metric is locally asymptotic to this model, in general it is still true that $T = J\nabla^{g_K}(k^{-1}\lambda)^{\frac{1}{3}}$ coincides with the T of section 1.

If x_1 is the moment of T with respect to ω_K ($dx_1 = -T\lrcorner\omega_K$), we have

$$x_1 = (k^{-1}\lambda)^{\frac{1}{3}}, \qquad g_K = x_1^2 g, \qquad \mathrm{Scal}_{g_K} = 6kx_1. \tag{12}$$

The scalar curvature is therefore the Hamiltonian potential of a Killing vector field, which means that g_K is extremal. It is also Bach flat since it is conformal to a Ricci flat metric.

The converse is well-known, see for example [CLW08]: since g_K is extremal and Bach flat, one obtains that g is Einstein, maybe with nonzero Einstein constant; the asymptotics (1) implies the decay $\mathrm{Ric} = O(r^{-3})$, so the Einstein constant has to vanish. □

At this point we develop the machinery of toric Kähler geometry for (M, g_K). The rest of of this section is devoted to recalling well-known facts of toric Kähler geometry (see for example the lecture notes [Apo22]), unless otherwise stated.

A toric Kähler manifold (M^4, g_K, ω_K) is a Kähler manifold with an effective action of a torus $\mathbb{T}^2$, which is Hamiltonian: this means that all the Killing fields induced by the infinitesimal action of the torus are Hamiltonian, this is automatic if $H^1(M, \mathbb{R}) = 0$, which is the case here by lemma 1.7.

The moment map $\mu : M \to (\mathbb{t}^2)^* = \mathbb{R}^2$ is defined for $A \in \mathbb{t}^2$, by (v_A being the vector field given by the infinitesimal action of A on M):

$$d\langle \mu, A\rangle = -v_A \lrcorner \omega_K.$$

In simpler terms, if we write in a basis (A_1, A_2) of $\mathbb{t}^2$, then $\mu = (\mu_1, \mu_2)$ with μ_i being a moment of the Hamiltonian vector field v_{A_i}.

The moment map μ has image a convex polytope P. Every edge of the polytope is the image of a divisor (a 2-sphere) fixed by an element of $\mathbb{t}^2$; the vertices are the fixed points of the action. If we take an integral basis of $\mathbb{t}^2$ (so we get an identification $\mathbb{T}^2 = \mathbb{t}^2/\mathbb{Z}^2 = \mathbb{R}^2/\mathbb{Z}^2$) and the vector fields $(\partial/\partial\theta_1, \partial/\partial\theta_2)$ on M generated by the infinitesimal action of this basis, then the corresponding moments (μ_1, μ_2) are affine coordinates on P. The Kähler

form takes the form (action-angle coordinates):

$$\omega_K = d\mu_1 \wedge d\theta_1 + d\mu_2 \wedge d\theta_2. \tag{13}$$

There is a convex function ψ on P, the symplectic potential, so that the metric takes the form

$$g_K = \sum_{i,j=1}^{2} (\psi_{\mu_i\mu_j} d\mu_i d\mu_j + \psi^{\mu_i\mu_j} d\theta_i d\theta_j), \tag{14}$$

where $\psi_{\mu_i\mu_j} = \frac{\partial^2\psi}{\partial\mu_i\partial\mu_j}$ and $(\psi^{\mu_i\mu_j}) = (\psi_{\mu_i\mu_j})^{-1}$. The scalar curvature is given by Abreu's formula,

$$\mathrm{Scal}_{g_K} = -\sum_{i,j=1}^{2} (\psi^{\mu_i\mu_j})_{\mu_i\mu_j}, \tag{15}$$

and the metric is extremal if and only if this is the Hamiltonian function of a Killing vector field, which here means that Scal_{g_K} is an affine function.

The complex structure is given above the interior of P by the holomorphic coordinates $z_j = \exp(\nu_j + i\theta_j)$ with $\nu_j = \psi_{\mu_j}$. It is useful to note that we always have a global holomorphic form above the interior of P:

$$\Omega = -4i\partial\theta_1 \wedge \partial\theta_2 = i(d\nu_1 + id\theta_1) \wedge (d\nu_2 + id\theta_2). \tag{16}$$

When M is smooth (see the Delzant condition below), the form Ω extends over the whole M as a meromorphic form with simple poles along the divisors. Note that the form does not depend on the choice of the basis of determinant 1 of $\mathbb{t}^2$ giving the θ_i's. The precise normalization in (16) will be used later.

For M to be smooth we need P to be a Delzant polytope:

- Each edge E must be rational, and in particular has an integral equation λ_E. This means:
  - λ_E is an affine function on $(\mathbb{t}^2)^*$ vanishing on E; so we can write $\lambda_E(x) = \langle v_E, x\rangle + c_E$ for some $v_E \in \mathbb{t}^2$ and $c_E \in \mathbb{R}$;
  - the 'normal' $v_E = d\lambda_E$ is a primitive element of $\mathbb{Z}^2 \subset \mathbb{t}^2$, and it is an inward normal (so that $\lambda_E > 0$ in the interior of P).
- At each vertex $E_1 \cap E_2$, the two inward normals v_{E_1}, v_{E_2} form an integral basis of $\mathbb{Z}^2 \subset \mathbb{t}^2$. In our case this does not apply to the edge at infinity E_∞ which can be irrational (this corresponds to the vector field T having non closed orbits).

Then the smoothness of the metric g_K is read on its symplectic potential:

- near an edge E one must have $\psi = \frac{1}{2}\lambda_E \log\lambda_E$+smooth function;
- near a vertex $E_1 \cap E_2$ one must have $\psi = \frac{1}{2}\sum_{j=1}^{2} \lambda_{E_j} \log\lambda_{E_j}$+smooth function.

A conical singularity of angle $2\pi\alpha_E$ around E is obtained by replacing λ_E by $\frac{1}{\alpha_E}\lambda_E$ in the above behaviour of ψ.

Finally, we give the idea to obtain a Poincaré behaviour (8) near the edge at infinity E_∞. This is not a standard fact in the literature, and in particular

there does not exist a nice regularity result so we will actually not use the proposed behaviour below, we will need a much weaker condition, see section 4.3. Choose an affine equation x_1 of E_∞, and suppose:

- near E_∞ one has $\psi = -\log x_1 + a x_1 \log x_1$+smooth function for some constant a: this does not depend on the choice of x_1, which can be irrational;
- near a vertex at infinity $\mathrm{E}_\infty \cap \mathrm{E}_j$ we need $\psi = -\log x_1 + a x_1 \log x_1 + \frac{1}{2}\lambda_{\mathrm{E}_j} \log \lambda_{\mathrm{E}_j}$+smooth function.

Then the corresponding metric has Poincaré behaviour near E_∞. (The leading term $-\log x_1$ gives in $\psi_{x_1x_1} dx_1^2 + \psi^{x_1x_1} d\theta_1^2$ a leading term $\frac{dx_1^2}{x_1^2} + x_1^2 d\theta_1^2$, which is a cusp). The function x_1 here is the same as in (12): it is the Hamiltonian function of a Killing vector field (which will be our T), and vanishes on the boundary at infinity.

3. The Tod ansatz

In this section, we present a quick derivation of the ansatz of Tod [Tod20]. In contrast with [Tod20], our approach initially relies on a Kähler ansatz due to LeBrun, but like in [Tod20] ends up with a celebrated Bäcklund transformation introduced by Ward [War90].

3.1. **The first ansatz.** We start from a Kähler structure $(g_{\mathrm{K}}, \mathrm{J}, \omega_{\mathrm{K}})$ with a Hamiltonian Killing vector field T. We can then apply the LeBrun ansatz:

Proposition 3.1 ([LeB91, §2]). *Suppose* T *is a Hamiltonian Killing vector field for the Kähler structure* $(g_{\mathrm{K}}, \mathrm{J}, \omega_{\mathrm{K}})$. *Then one can locally write* g_{K} *and* ω_{K} *under the form*

$$g_{\mathrm{K}} = \mathrm{W}\left(dx_1^2 + e^v(dx_2^2 + dx_3^2)\right) + \mathrm{W}^{-1}\eta^2 \tag{17}$$

$$\omega_{\mathrm{K}} = dx_1 \wedge \eta + \mathrm{W}e^v dx_2 \wedge dx_3, \tag{18}$$

where the 1-form η *and the functions* $v(x_1, x_2, x_3)$ *and* $\mathrm{W} = \mathrm{W}(x_1, x_2, x_3) = |\mathrm{T}|_{g_{\mathrm{K}}}^{-2}$ *satisfy* $\mathrm{T} \lrcorner \eta = 1$ *and*

$$d\eta = (\mathrm{W}e^v)_{x_1} dx_2 \wedge dx_3 + \mathrm{W}_{x_2} dx_3 \wedge dx_1 + \mathrm{W}_{x_3} dx_1 \wedge dx_2, \tag{19}$$

$$(\mathrm{W}e^v)_{x_1x_1} + \mathrm{W}_{x_2x_2} + \mathrm{W}_{x_3x_3} = 0. \tag{20}$$

The second equation is the compatibility condition $d(d\eta) = 0$.

Conversely, given a local coframe (dx_1, dx_2, dx_3, η) *(where the* x_i*'s are local functions), and functions* v, $\mathrm{W} > 0$ *of* (x_1, x_2, x_3) *satisfying (19)–(20), the formulas (17)–(18) provide a local Kähler structure with a Hamiltonian Killing vector field* T *given by* $\mathrm{T} \lrcorner dx_i = 0$ *and* $\mathrm{T} \lrcorner \eta = 1$.

Finally, in this setting the scalar curvature is given by the formula

$$\mathrm{Scal}_{g_{\mathrm{K}}} = -\frac{(e^v)_{x_1x_1} + v_{x_2x_2} + v_{x_3x_3}}{\mathrm{W}e^v}. \tag{21}$$

Remark 3.2. This is a priori a local statement, but there is a way to give to the coordinates and the functions an intrinsic interpretation, which we will use later to obtain a global description of the metric via this ansatz. Indeed

x_1 is a moment of the Hamiltonian vector field T. The coordinates x_2, x_3 and the function v are determined by the choice of a local, non-vanishing holomorphic T-invariant 2-form $\Omega = e^{-\frac{v}{2}}(\omega_2 + i\omega_3)$, where $(\omega_{\mathrm{K}}, \omega_2, \omega_3)$ form a direct (local) orthonormal frame of $\Lambda^+\mathrm{M}$ (with $|\omega_j|^2 = 2$) and $e^{-v} = \frac{1}{4}|\Omega|^2_{g_{\mathrm{K}}}$; Ω being holomorphic is closed, and so is $\mathrm{T}\lrcorner\Omega$, since Ω is T-invariant; we thus define x_2, x_3 by $\mathrm{T}\lrcorner\Omega = -(dx_2 + i dx_3)$, while ω_2, ω_3 are given by $\omega_2 = e^{\frac{v}{2}}(dx_2 \wedge \eta + \mathrm{W}\, dx_3 \wedge dx_1)$, $\omega_3 = e^{\frac{v}{2}}(dx_3 \wedge \eta + \mathrm{W}\, dx_1 \wedge dx_2)$; since $e^{-\frac{v}{2}}\omega_2$, $e^{-\frac{v}{2}}\omega_3$ are closed, as well as ω_{K}, (19), hence (20), readily follow.

We deduce

Proposition 3.3. *Under the same hypothesis as in proposition 3.1, suppose that the Kähler structure $(g_{\mathrm{K}}, \mathrm{J}, \omega_{\mathrm{K}})$ is locally given by (17)–(18). Then the conformal metric $g = x_1^{-2} g_{\mathrm{K}}$ is Ricci flat if and only if*

$$\mathrm{W} = \frac{2}{k x_1^3} - \frac{v_{x_1}}{k x_1^2} \tag{22}$$

and v satisfies the twisted Toda equation

$$(e^v)_{x_1 x_1} + v_{x_2 x_2} + v_{x_3 x_3} = -6k x_1 \mathrm{W} e^v. \tag{23}$$

Proof. Since $g = x_1^{-2} g_{\mathrm{K}}$ we have $\mathrm{Scal}_g = x_1^2 \mathrm{Scal}_{g_{\mathrm{K}}} - 6x_1 \Delta_{\mathrm{K}} x_1 - 12|dx_1|^2_{\mathrm{K}}$ and therefore

$$k x_1^3 - x_1 \Delta_{\mathrm{K}} x_1 - 2|dx_1|^2_{\mathrm{K}} = 0. \tag{24}$$

We calculate $\Delta_{\mathrm{K}} x_1$: one has $d^{\mathrm{C}} x_1 = \frac{\eta}{\mathrm{W}}$ so

$$dd^{\mathrm{C}} x_1 = \frac{d\eta}{\mathrm{W}} - \frac{d\mathrm{W} \wedge \eta}{\mathrm{W}^2}. \tag{25}$$

Using (19) we thus get $\Delta_{\mathrm{K}} x_1 = -\frac{v_{x_1}}{\mathrm{W}}$. Injecting this value in (24) we obtain (22). Equation (23) is just (20) with $\mathrm{Scal}_{g_{\mathrm{K}}} = 6k x_1$.

Conversely, if we have (22) and (23) it follows that $\mathrm{Scal}_{g_{\mathrm{K}}} = 6k x_1$ and $\mathrm{Scal}_g = 0$. Since $\mathrm{Ric}^g_0 = \mathrm{Ric}^{g_{\mathrm{K}}}_0 + 2x_1^{-1}(\nabla_{\mathrm{K}} dx_1)_0$, we infer that g is Ricci flat if $\rho^{g_{\mathrm{K}}}_0 + x_1^{-1}(dd^{\mathrm{C}} x_1)_0 = 0$, where the Ricci form $\rho^{g_{\mathrm{K}}}$ is the (1,1)-form $\mathrm{Ric}^{g_{\mathrm{K}}}(\mathrm{J}\cdot, \cdot)$. By remark 3.2 we have a holomorphic volume form Ω with $|\Omega|^2_{\mathrm{K}} = 4e^{-v}$ so $\rho^{g_{\mathrm{K}}} = -\frac{1}{2} dd^{\mathrm{C}} v$, so finally g is Ricci flat if

$$\frac{1}{2}(dd^{\mathrm{C}} v)_0 + \frac{(dd^{\mathrm{C}} x_1)_0}{x_1} = 0. \tag{26}$$

The calculation of (26) given (22) and (23) is left to the reader. □

One can reduce (23) into the Toda equation (29) below, which is (23) with a zero RHS, by setting:

$$\mathrm{V} = x_1^2 \mathrm{W}, \quad \xi = \frac{1}{x_1}, \quad u = v - 4 \log x_1. \tag{27}$$

We then get the following formulas, for a Ricci flat metric $g = x_1^{-2} g_K$:

$$\mathrm{V} = \frac{1}{k}(-2\xi + \xi^2 u_\xi), \tag{28}$$

$$(e^u)_{\xi\xi} + u_{x_2x_2} + u_{x_3x_3} = 0, \tag{29}$$

$$g = \frac{g_K}{x_1^2} = \mathrm{V}\big(d\xi^2 + e^u(dx_2^2 + dx_3^2)\big) + \mathrm{V}^{-1}\eta^2, \tag{30}$$

$$d\eta = -\xi^2(\xi^{-2}e^u\mathrm{V})_\xi dx_2 \wedge dx_3 - \mathrm{V}_{x_2} dx_3 \wedge d\xi - \mathrm{V}_{x_3} d\xi \wedge dx_2. \tag{31}$$

Again we have the converse: starting from a local function $u(\xi, x_2, x_3)$ satisfying (29), define V by (28), then if $u_\xi > 2\xi^{-1}$ (that is $\mathrm{V} > 0$) and the 1-form η satisfies (31), then the metric g of (30) is a Hermitian, Ricci flat metric, and x_1 is the Hamiltonian potential of a Killing vector field T.

Remark 3.4. Suppose we define (x_2, x_3) and v from a holomorphic form Ω as in remark 3.2. If we replace Ω by $a\Omega$ for some $a \in \mathbb{R}^*$, then (x_2, x_3) is transformed into (ax_2, ax_3) and v is into $v - \log(a^2)$; therefore u becomes $u - \log(a^2)$, the function V and the metrics g_K and g are unchanged.

3.2. **The second ansatz**. We are still in the setting of proposition 3.1: we describe a local Kähler metric with a Hamiltonian vector field T from the local coordinates x_1, x_2, x_3, the 1-form η and the equations (17)–(21). We also suppose that $g = x_1^{-2} g_K$ is Ricci flat so we also have the description (27)–(31). Now suppose additionally that this data does not depend on x_3. From (19)–(20) it now follows that $d\eta = dx_3 \wedge \alpha$ for a 1-form α satisfying $d\alpha = 0$, so we can write $\alpha = d\mathrm{F}$ and we obtain

$$\eta = dt - \mathrm{F}dx_3 \tag{32}$$

for some choice of the function t, which completes the coordinate system $\xi = \frac{1}{x_1}$, x_2, x_3, and $\mathrm{F} = \mathrm{F}(\xi, x_2)$. The Hamiltonian vector field T is simply $\frac{\partial}{\partial t}$, and we have another commuting Hamiltonian Killing vector field, $\frac{\partial}{\partial x_3}$, so we are now in the toric situation. The Toda equation (29) becomes

$$(e^u)_{\xi\xi} + u_{x_2x_2} = 0. \tag{33}$$

From (31) (28) and (33) it follows that

$$\mathrm{F}_\xi = -\mathrm{V}_{x_2} = -\frac{1}{k}\xi^2 u_{\xi x_2}, \qquad \mathrm{F}_{x_2} = \xi^2(\xi^{-2}e^u\mathrm{V})_\xi = -\frac{1}{k}\big(\xi^2 u_{x_2x_2} - 2e^u + 2\xi e^u u_\xi\big). \tag{34}$$

The compatibility equation is the Toda equation (33).

We now perform the Ward ansatz. Take standard coordinates (u_1, u_2, u_3) on $\mathbb{R}^3$ and define $\rho = \sqrt{u_1^2 + u_2^2}$ and $z = u_3$. The main statement is the following:

Proposition 3.5. *Suppose that* $\mathrm{U}(\rho, z)$ *is a harmonic axisymmetric function on a domain of* $\mathbb{R}^3$*, that is satisfies the equation*

$$\frac{1}{\rho}(\rho \mathrm{U}_\rho)_\rho + \mathrm{U}_{zz} = 0. \tag{35}$$

Define

$$\xi = \frac{1}{2}\rho U_\rho, \qquad x_2 = -\frac{1}{2}U_z. \tag{36}$$

Restrict to the domain where $\Delta := \frac{\rho}{4}(U_{zz}^2 + U_{\rho z}^2) > 0$. *Then* (ξ, x_2) *are local coordinates and the function* $u = \log \rho^2$ *is a solution to the reduced Toda equation (33).*

Conversely any solution to the reduced Toda equation (33) can be locally expressed in this way on the domain where the differential du *does not vanish.*

Notation. To simplify notation, we avoid the parenthesis in U_{zz}^2, but we always mean that we first take derivatives and then a power, so $U_{zz}^2 = (U_{zz})^2$, $u_{x_2}^2 = (u_{x_2})^2$, etc.

Proof. Start from a harmonic axisymmetric function U. We prove the proposition by direct calculation. We have the first derivatives

$$\xi_\rho = -\frac{1}{2}\rho U_{zz}, \quad \xi_z = \frac{1}{2}\rho U_{\rho z}, \quad (x_2)_\rho = -\frac{1}{2}U_{\rho z}, \quad (x_2)_z = -\frac{1}{2}U_{zz}. \tag{37}$$

On the domain where $\Delta > 0$ we invert this into

$$\rho_\xi = -\frac{U_{zz}}{2\Delta}, \quad \rho_{x_2} = -\frac{\rho U_{\rho z}}{2\Delta}, \quad z_\xi = \frac{U_{\rho z}}{2\Delta}, \quad z_{x_2} = -\frac{\rho U_{zz}}{2\Delta}. \tag{38}$$

A straightforward calculation using (38) leads to

$$(e^u)_{\xi\xi} + u_{x_2x_2} = -\frac{2\Delta_\rho}{\Delta^2} + \frac{U_{zz}^2}{2\Delta^2} + \frac{\rho U_{\rho zz}U_{zz}}{\Delta^2} + \frac{\rho U_{\rho\rho z}U_{\rho z}}{2\Delta^2} - \frac{\rho U_{zzz}U_{\rho z}}{2\Delta^2}. \tag{39}$$

But $\Delta_\rho = \frac{1}{4}(U_{\rho z}^2 + U_{zz}^2) + \frac{\rho}{2}(U_{zz}U_{\rho zz} + U_{\rho z}U_{\rho\rho z})$, so we get

$$\begin{aligned}(e^u)_{\xi\xi} + u_{x_2x_2} &= -\frac{U_{\rho z}}{2\Delta^2}(U_{\rho z} + \rho U_{\rho\rho z} + \rho U_{zzz}) \\ &= -\frac{U_{\rho z}}{2\Delta^2}(\rho U_{\rho\rho} + \rho U_{zz} + U_\rho)_z = 0.\end{aligned} \tag{40}$$

Therefore the function u is a solution of the reduced Toda equation.

Conversely, start from a local solution $u(\xi, x_2)$ of the reduced Toda equation. We define $\rho = e^{\frac{u}{2}}$ and recover locally z by setting

$$z_\xi = -\frac{1}{\rho}\rho_{x_2}, \quad z_{x_2} = \rho\rho_\xi. \tag{41}$$

The compatibility condition for this system is the reduced Toda equation (33). We have local coordinates (ρ, z) where $\delta := \rho\rho_\xi^2 + \frac{1}{\rho}\rho_{x_2}^2 > 0$ (which is equivalent to the nonvanishing of du). We calculate $\xi_\rho = \frac{z_{x_2}}{\delta}$, $\xi_z = -\frac{\rho_{x_2}}{\delta}$, $(x_2)_\rho = -\frac{z_\xi}{\delta}$ and $(x_2)_z = \frac{\rho_\xi}{\delta}$. It is now obvious that the compatibility condition $\xi_z = -\rho(x_2)_\rho$ for the system (36) is satisfied, so we get the function U, and the harmonic function equation (35) is $\frac{\xi_\rho}{\rho} - (x_2)_z = 0$ which is also true. Note that $\delta = \frac{1}{\Delta}$ so the two conditions $\delta > 0$ and $\Delta > 0$ correspond to each other. □

Corollary 3.6 (Tod ansatz). *Under the same notations as in proposition 3.5, from a local axisymmetric harmonic function* U *we deduce a local Hermitian toric Ricci flat metric given by the formulas*

$$g = e^{2\nu}(d\rho^2 + dz^2) + V\rho^2 dx_3^2 + V^{-1}(dt - F dx_3)^2, \tag{42}$$

where the functions ν, V *and* F *are given by:*

$$e^{2\nu} = \frac{1}{4} V\rho^2 (U_{\rho z}^2 + U_{zz}^2), \tag{43}$$

$$V = -\frac{1}{k}\left(\rho U_\rho + \frac{U_\rho^2 U_{zz}}{U_{\rho z}^2 + U_{zz}^2}\right), \tag{44}$$

$$F = -\frac{1}{k}\left(-\frac{\rho U_\rho^2 U_{\rho z}}{U_{\rho z}^2 + U_{zz}^2} + \rho^2 U_z + 2H\right). \tag{45}$$

The function $H = H(\rho, z)$ *is a conjugate of the harmonic function* U*: it is determined up to an additive constant by*

$$H_\rho = -\rho U_z, \qquad H_z = \rho U_\rho. \tag{46}$$

The formulas make sense on the domain where $V > 0$ *and* $U_{\rho z}^2 + U_{zz}^2 > 0$.

Conversely, suppose we have a local Hermitian toric Ricci flat metric, so it is written under the form (28)–(31), and if t *and* x_3 *are the toric variables, we can write additionally* η *under the form (32) and* u *satisfies the reduced Toda equation (33). Then the metric is given by the ansatz (42), with* U *being the harmonic function coming from proposition 3.5.*

We will call U the *generating function* of the metric g.

Remark 3.7. The parameter k is a scale parameter, the metric obtained for the value k is the same as the metric obtained for the value 1 with a scale factor $\frac{1}{k}$. It is nevertheless useful to keep it since it influences the scale of the Kähler metric $g_K = x_1^2 g$. Since we want g_K to be locally asymptotic to a product of a hyperbolic cusp with a round sphere, we will need to fix k accordingly.

Proof of Corollary 3.6. We first prove the converse, starting from a local Hermitian toric Ricci flat metric with t and x_3 the toric variables. It is a matter of putting everything together. From (37) we infer

$$d\xi^2 + e^u dx_2^2 = \frac{\rho^2}{4}(U_{zz}^2 + U_{\rho z}^2)(d\rho^2 + dz^2). \tag{47}$$

Then (42) follows from (30), with $e^{2\nu}$ given by (43); and (44) follows from (28) and (38). There remains to obtain the expression of F in terms of ρ, z, and here we follow a trick of Tod: from (34) it is natural to introduce a function H by

$$F = -\frac{1}{k}(\xi^2 u_{x_2} - 2x_2 e^u + 2H), \tag{48}$$

and H satisfies

$$\begin{aligned} H_\xi &= -\xi u_{x_2} + x_2 (e^u)_\xi = \frac{\rho}{2\Delta}(U_\rho U_{\rho z} + \rho U_z U_{zz}), \\ H_{x_2} &= \xi (e^u)_\xi + x_2 (e^u)_{x_2} = -\frac{\rho^2}{2\Delta}(U_\rho U_{zz} + U_z U_{\rho z}). \end{aligned} \tag{49}$$

From (37) again we then obtain (46) and (45).

In the other direction, starting from the axisymmetric harmonic function $U(\rho, z)$, we reconstruct from proposition 3.5 the variables ξ, x_2 and the function $u(\xi, x_2)$ satisfying the reduced Toda equation (33). The condition $u_\xi > 2\xi^{-1}$ is the same as the condition $V > 0$, and we can reconstruct all the conditions in (28)–(31), so we have a Hermitian, toric, Ricci flat metric, which by the above formulas coincides with the metric g of (42). □

Remark 3.8. In the toric case, (42) is the Harmark form of the metric which is well known in the physics literature, see for example [Har04]. In particular the 'horizontal' metric $\sum_{i,j=1}^{2} \psi_{\lambda_i \lambda_j} d\lambda_i d\lambda_j$ on P can be uniformized by the Riemann uniformization theorem, and (42) implies that (ρ, z) are coordinates which give this uniformization. The corollary is a priori a local statement, but we will see in section 4.4 that (ρ, z) are global coordinates on P giving the uniformization by the upper half-plane.

Remark 3.9. There is a homogeneity in the formulas (42)–(46): if we consider the transformation $h_a(\rho, z) = (|a|\rho, az + b)$ with $a \in \mathbb{R}^*$ and the real number $\lambda > 0$, then the harmonic function $\lambda h_a^* U$ leads to a metric $\lambda^3 H_{a,\lambda}^* g$, where $H_{a,\lambda}(\rho, z, t, x_3) = (|a|\rho, az + b, \frac{t}{\lambda^2}, \frac{x_3}{a\lambda})$.

If we transform the holomorphic form Ω into $a\Omega$ then we have seen in remark 3.4 that (x_2, x_3, u) are transformed into $(ax_2, ax_3, u - 2\log(a^2))$ with the metric unchanged, and in terms of the harmonic function U one calculates from proposition 3.5 that this corresponds to considering $h_{a,0}^* U$ instead of U, that is to make an homothety on the variables (ρ, z).

4. Boundary conditions

We now come back to our problem, with a toric extremal Kähler metric (g_K, J, ω_K), with polytope P, and conformal to a Ricci flat metric g. We analyze the boundary conditions on ∂P and deduce the boundary conditions for the harmonic function U. This will enable us to show that the constructions of the previous section can be made global.

In our toric setting, we have a global meromorphic 2-form Ω defined by (16), which is holomorphic above the interior of P. When we write Ω we can make the following choice of affine coordinates on P: we complete the moment $x_1 = \mu_1$ of T with another affine function μ_2 on P so that $(d\mu_1, d\mu_2)$ is a basis of determinant 1 of $\mathbb{t}^2$ (but not an integral basis in general because x_1 might be irrational). Then by remark 3.2 we extract from (16) a global solution of $T \lrcorner \Omega = -d(x_2 + ix_3)$ by $x_2 = \nu_2$ and $x_3 = \theta_2$ (so x_3 is an angular variable, maybe irrational), and the function $v(x_1, x_2)$ is defined by $e^{-v} = \frac{1}{4}|\Omega|^2_{g_K}$. As explained in section 3.2 we have $\eta = dt - F\, dx_3$

for $t = \theta_1$ and some function $F(x_1, x_2)$. Actually this is not the best choice and we will use in section 4.3 the freedom to replace Ω by $a\Omega$ for some real number a, see remark 3.9.

Of course we can pass to coordinates $(\xi = x_1^{-1}, x_2)$ and to the function u as in (27), which is now a solution of the reduced Toda equation (33). We would like to apply Tod's ansatz (proposition 3.5 and corollary 3.6): we have $\rho = e^{\frac{u}{2}}$ and z is defined by its differential (41), so they are globally defined functions on P, but it is not clear a priori that they are coordinates on P, so we can apply proposition 3.5 and corollary 3.6 only locally on the domain where $d\rho \neq 0$.

The plan of this section is the following: we are always in this setting and we determine equivalents of the various objects near the boundary of P. At the end we can prove in particular that (ρ, z) are global coordinates on P in proposition 4.1, so we have indeed a global description of our metrics.

4.1. **The case of a finite edge**. We begin with the case of an edge E of P with integral equation $\lambda_1 = \lambda_E$. Near an interior point of E we have $\psi = \frac{1}{2}\lambda_1 \log \lambda_1 + \text{smooth function}$. Completing λ_1 into a basis (λ_1, λ_2) we obtain the behaviour of the metric on P near E:

$$\sum_{i,j=1}^{2} \psi_{\lambda_i\lambda_j} d\lambda_i d\lambda_j \sim \frac{d\lambda_1^2}{2\lambda_1} + \psi_{\lambda_2\lambda_2} d\lambda_2^2, \tag{50}$$

and it follows from (16) and (27) that

$$e^{-v} = \frac{1}{4}|\Omega|^2 \sim \frac{\psi_{\lambda_2\lambda_2}}{2\lambda_1}, \quad \rho = e^{\frac{u}{2}} = \frac{e^{\frac{v}{2}}}{x_1^2} \sim \frac{1}{x_1^2}\sqrt{\frac{2\lambda_1}{\psi_{\lambda_2\lambda_2}}}. \tag{51}$$

Choosing the standard orientation of P such that if E and E′ are successive edges in the trigonometric order, then $(\lambda_E, \lambda_{E'})$ is oriented, and assuming that λ_2 was chosen so that (λ_1, λ_2) is oriented, we deduce

$$\begin{aligned} d\rho &\sim \frac{d\lambda_1}{x_1^2\sqrt{2\lambda_1\psi_{\lambda_2\lambda_2}}} + \sqrt{2\lambda_1}\Big(\frac{1}{x_1^2\sqrt{\psi_{\lambda_2\lambda_2}}}\Big)_{\lambda_2} d\lambda_2, \\ dz = -*_2 d\rho &\sim -\frac{d\lambda_2}{x_1^2} + \frac{1}{\sqrt{\psi_{\lambda_2\lambda_2}}}\Big(\frac{1}{x_1^2\sqrt{\psi_{\lambda_2\lambda_2}}}\Big)_{\lambda_2} d\lambda_1. \end{aligned} \tag{52}$$

The Hodge star in the second equation is the Hodge operator for the metric $\sum \psi_{\lambda_i\lambda_j} d\lambda_i d\lambda_j$ induced on P, see remark 3.8. It follows from this formula that the function z attains finite values on E. Since along the boundary ∂P we have

$$\frac{\partial z}{\partial \lambda_2} = -x_1^{-2} < 0, \tag{53}$$

we deduce that z is monotone on $\partial P \setminus E_\infty$, where E_∞ is the edge at infinity $(x_1 = 0)$. Actually if we go along $\partial P \setminus E_\infty$ in the trigonometric sense, the function z is increasing; moreover when we reach E_∞ we see that $z \to \pm\infty$, so z is actually increasing from $-\infty$ to $+\infty$ on $\partial P \setminus E_\infty$.

The equation (51) says that $\rho|_E = 0$. From (36) we have $\rho U_\rho = \frac{2}{x_1}$ and therefore near E we have

$$U \sim \frac{1}{x_1|_E} \log \rho^2. \tag{54}$$

From (53) it follows that $\frac{1}{x_1|_E}$ is actually an affine function of $z|_E$. It follows that there is a continuous piecewise linear function $f : \mathbb{R} \to \mathbb{R}_+^*$, which is the same as $\frac{1}{x_1}$ if we see z as a parameter of ∂P. Near each edge one has

$$U \sim f(z) \log \rho^2. \tag{55}$$

We will prove later the crucial observation that the slopes actually increase from -1 to $+1$. In particular the function f is convex.

4.2. **The case of a finite vertex**. This is similar to the previous case: if we have a vertex $E \cap E'$, where (E, E') are in trigonometric order, then we can take $\lambda_1 = \lambda_E$ as above and $\lambda_2 = \lambda_{E'}$. Then near the vertex one has $\psi = \frac{1}{2}\sum_{i=1}^2 \lambda_i \log \lambda_i$+smooth function. We have the same formulas as above except that now $\psi_{\lambda_2\lambda_2} = \frac{1}{2\lambda_2}$+smooth function. Then (51) and (52) give

$$\rho \sim \frac{2\sqrt{\lambda_1\lambda_2}}{x_1^2}, \quad dz \sim \frac{d\lambda_1 - d\lambda_2}{x_1^2}. \tag{56}$$

In particular the behaviour of z and U near the vertex remains the same as described above. Together with the previous case, it implies that the asymptotic of U in (55) is global near $\{\rho = 0\} = \partial P \setminus E_\infty$.

4.3. **The case of the edge at infinity**. We take coordinates $(\lambda_1 = x_1, \lambda_2)$, where λ_2 can be chosen to be $\lambda_2 = \lambda_E$ for the edge E going to infinity such that (x_1, λ_2) is a direct basis. Here we make the definitive choice of holomorphic form: say the basis (x_1, λ_2) has determinant a, then we choose the form $a\Omega$ because in the basis (x_1, λ_2) the form $a\Omega$ is given by the same formula (16). Since by remark 3.9 this amounts to an homothety on the variables (ρ, z) we observe that the behaviour (55) of U for some piecewise linear function f remains unchanged.

We have seen that our Kähler Poincaré type metric is locally asymptotic to the product of a hyperbolic cusp (symplectic potential $-\log x_1$) and a spherical metric (say, symplectic potential $\varpi(\lambda_2)$). Therefore we claim (and will prove at the end of this section) that the symplectic potential ψ satisfies

$$\psi(x_1, \lambda_2) = -\log x_1 + \varpi(\lambda_2) + \zeta \tag{57}$$

where $\varpi(\lambda_2)$ is the symplectic potential of a spherical metric, actually given by formula (60) below, and ζ is a function which vanishes on $x_1 = 0$, such that the second derivatives $f = x_1^2\zeta_{x_1x_1}, x_1\zeta_{x_1\lambda_2}$ or $\zeta_{\lambda_2\lambda_2}$ all satisfy

$$(x_1\partial_{x_1})^{k_1}\partial_{\lambda_2}^{k_2} f = O(x_1) \text{ for all } k_1, k_2 \geq 0. \tag{58}$$

Before proving this, we use formulas (57)–(58) to establish the behaviour of U at infinity. It follows from these formulas that (defining $\varpi^{\lambda_2\lambda_2} = \varpi^{-1}_{\lambda_2\lambda_2}$):

$$\sum_{i,j=1}^{2} \psi_{\lambda_i\lambda_j} d\lambda_i d\lambda_j \sim \frac{dx_1^2}{x_1^2} + \varpi_{\lambda_2\lambda_2} d\lambda_2^2, \qquad \rho \sim \frac{\sqrt{\varpi^{\lambda_2\lambda_2}}}{x_1}. \tag{59}$$

At this point, we note that because of equation (12) and Abreu's formula (15), we must have $\sum_{i,j=1}^{2} \psi^{\lambda_i\lambda_j}_{\lambda_i\lambda_j} = 0$ along $x_1 = 0$, which translates into $\varpi^{\lambda_2\lambda_2}_{\lambda_2\lambda_2} = -2$, that is $\varpi(\lambda_2)$ is the symplectic potential of a spherical metric, as we announced above. We have two edges E and E′ going to infinity, and it follows from the description of the behavior at $E \cap E_\infty$ and $E' \cap E_\infty$ that we must have $\varpi^{\lambda_2\lambda_2} = -\lambda_E^2 + 2\lambda_E = -\lambda_{E'}^2 + 2\lambda_{E'}$ along E_∞, which implies that $\lambda_{E'} = 2 - \lambda_E$ along E_∞ and that E_∞ has length 2 for λ_E (or $\lambda_{E'}$). At the end we deduce that, up to some global affine function, we have

$$\varpi(\lambda_2) = \frac{1}{2}\big(\lambda_2 \log \lambda_2 + (2-\lambda_2)\log(2-\lambda_2)\big). \tag{60}$$

Coming back to the function ρ, we have

$$d\rho \sim -\sqrt{\varpi^{\lambda_2\lambda_2}}\frac{dx_1}{x_1^2} + \frac{\varpi^{\lambda_2\lambda_2}_{\lambda_2}}{2\sqrt{\varpi^{\lambda_2\lambda_2}}}\frac{d\lambda_2}{x_1}, \quad dz \sim \frac{d\lambda_2}{x_1} + \frac{\varpi^{\lambda_2\lambda_2}_{\lambda_2}}{2}\frac{dx_1}{x_1^2}. \tag{61}$$

Observe that our equation $\varpi^{\lambda_2\lambda_2}_{\lambda_2\lambda_2} = -2$ is indeed the compatibility condition for the asymptotic terms of dz. Since $\varpi^{\lambda_2\lambda_2} = \lambda_2(2-\lambda_2)$ on E_∞ and $\varpi^{\lambda_2\lambda_2}_{\lambda_2} = 2(1-\lambda_2)$, we get from (59) and (61) the asymptotics when $x_1 \to 0$:

$$\rho \sim \frac{\sqrt{\lambda_2(2-\lambda_2)}}{x_1}, \qquad z \sim \frac{\lambda_2 - 1}{x_1}. \tag{62}$$

We obtain $R := \sqrt{\rho^2 + z^2} \sim x_1^{-1}$ and $\lambda_2 \sim \frac{R+z}{R}$, and therefore from (36)

$$\begin{aligned} \rho U_\rho &= \frac{2}{x_1} \sim 2R \\ U_z &= -2\psi_{\lambda_2} \sim -2\varpi_{\lambda_2} \sim \log\frac{2-\lambda_2}{\lambda_2} \sim \log\frac{R-z}{R+z}. \end{aligned} \tag{63}$$

Therefore we find the asymptotics of U at infinity, that is when $R \to +\infty$ (which is the same as $x_1 \to 0$):

$$U \sim U_0(\rho, z) := 2R + z\log\frac{R-z}{R+z}. \tag{64}$$

Note that on the edge E we have $\lambda_2 = 0$ and $z \to -\infty$, so we deduce $U \sim -z\log\rho^2$ when $\rho \to 0$ and $R \to \infty$; since we have (55) when $\rho \to 0$ with f piecewise linear, we deduce that the slope of f on the segment going to $-\infty$ is -1. Similarly, on the other edge E′ going to infinity we have $z \to +\infty$ and $U \sim z\log\rho^2$ when $\rho \to 0$ and $R \to \infty$, therefore the slope of f on the segment going to $+\infty$ is $+1$.

We now justify the behaviour (57)–(58). Since it seems that there is no regularity theory in the literature for extremal Kähler metrics with Poincaré behaviour, we instead obtain it directly from the definition (1)

of an ALF metric. We have $g_K = x_1^2 g$ with $x_1 = r^{-1}(1 + O(r^{-1}))$ and the corresponding bounds on the derivatives. Therefore

$$g_K = \frac{dx_1^2}{x_1^2} + x_1^2\eta^2 + \gamma + h, \quad |\nabla^k h|_{g_K} = O(x_1). \tag{65}$$

We deduce the bounds on the Hessian of ψ:

$$\sum_{i,j=1}^{2} \psi_{\lambda_i\lambda_j} d\lambda_i d\lambda_j = \frac{dx_1^2}{x_1^2} + \varpi_{\lambda_2\lambda_2} d\lambda_2^2 + \sum_{ij=1}^{2} A_{i,j} d\lambda_i d\lambda_j, \tag{66}$$

where $\varpi(\lambda_2)$ is the symplectic potential of a spherical metric (that is $\varpi^{\lambda_2\lambda_2}_{\lambda_2\lambda_2} = -2$, an equation that we have already seen from another viewpoint), and $x_1^2 A_{11}$, $x_1 A_{12}$ and A_{22} all satisfy $(x_1\partial_{x_1})^{k_1}\partial_{\lambda_2}^{k_2} f = O(x_1)$ for all $k_1, k_2 \geq 0$.

From the Hessian we deduce the symplectic potential itself: $\psi = -\log x_1 + \varpi(\lambda_2) + \zeta$ with $x_1^2\zeta_{x_1x_1}, x_1\zeta_{x_1\lambda_2}, \zeta_{\lambda_2\lambda_2} = O(x_1)$; and the same bound for all derivatives of type $(x_1\partial_{x_1})^{k_1}\partial_{\lambda_2}^{k_2}$, which proves the claim. It is plausible that one can obtain a better regularity from the extremal equation, probably an asymptotic development in the variable x_1; in our case (with the additional Bach equation) no regularity theory is necessary since at the end the metrics will be explicit.

4.4. **The uniformization**. From their definition, the functions (ρ, z) uniformize $\sum \psi_{\lambda_i\lambda_j} d\lambda_i d\lambda_j$. We have the following global statement:

Proposition 4.1. *The functions* (ρ, z) *give a diffeomorphism of the interior of* P *with the upper half plane.*

Proof. The metric γ induced on the polytope P can be uniformized by the upper half-plane $\mathbb{H}$. From the very definition of z by (52), the function $\varphi = z + i\rho$ is a holomorphic function on $\mathbb{H}$, with real values on the real axis. By Schwarz reflection principle, the function φ extends as a holomorphic function on $\mathbb{C}$. The point is to prove that it is an affine function.

The behaviour of γ at infinity (that is near E_∞) is given by (59), with $\varpi(\lambda_2) = \frac{1}{\lambda_2(2-\lambda_2)}$. Taking $\sin\theta = \lambda_2 - 1$ and $\xi = x_1^{-1}$, we therefore have

$$\sum_{i,j=1}^{2} \psi_{ij} d\lambda_i d\lambda_j \sim \frac{d\xi^2 + \xi^2 d\theta^2}{\xi^2}. \tag{67}$$

This is conformal to the hyperbolic metric $\frac{d\xi^2+\xi^2 d\theta^2}{\xi^2\cos^2\theta}$, with ξ being the radius in the upper half plane. Since $|\varphi| = R \sim \xi$, the function φ has linear growth at infinity, and is therefore affine. □

4.5. **The convexity of** f.

Lemma 4.2. *Suppose that* U *is a solution of equation (35) on the upper half-plane* $\rho > 0$.

- *If* $U_\rho > 0$ *when* $\rho \to 0$ *and when* $R = \sqrt{\rho^2 + z^2} \to \infty$, *then* $U_\rho > 0$ *everywhere.*

- *If* $U_{zz} < 0$ *when* $\rho \to 0$ *and when* $R = \sqrt{\rho^2 + z^2} \to \infty$, *then* $U_{zz} < 0$ *everywhere.*

Proof. Since U_{zz} satisfies the same equation (35), the second statement is an immediate application of the maximum principle. For the first statement observe that U_ρ satisfies $(U_\rho)_{\rho\rho} + \frac{1}{\rho}(U_\rho)_\rho + (U_\rho)_{zz} - \frac{1}{\rho^2}U_\rho = 0$, and the maximum principle can also be applied to this equation. □

Now come back to our setting of a toric extremal Kähler metric g_K, with polytope P, and conformal to a Ricci flat ALF metric g. From proposition 4.1 we know that (ρ, z) are global coordinates, we have a harmonic axisymmetric function U on $\mathbb{R}^3 \setminus \{\rho = 0\}$ and the metric g is globally described from U by corollary 3.6. The asymptotic (55) near $\partial P \setminus E_\infty$ implies that $U_\rho > 0$ when $\rho \to 0$. For the function V in (44) to be positive we therefore must have $U_{zz} < 0$ when $\rho \to 0$. On the other hand, when $R = \sqrt{\rho^2 + z^2} \to \infty$, the asymptotic function $U_0(\rho, z)$ given (64) also has $U_\rho > 0$ and $U_{zz} < 0$. Therefore we deduce from lemma 4.2 that $U_\rho > 0$ and $U_{zz} < 0$ everywhere. In particular for each ρ the function $z \mapsto U(\rho, z)$ is concave, and at the limit $\rho \to 0$ the limit $f(z) = \lim_{\rho\to 0} \frac{U(\rho,z)}{\log \rho^2}$ is convex. Together with the calculations of the two limit slopes in section 4.3 this gives:

Corollary 4.3. *Suppose we have a toric extremal Kähler metric with polytope* P, *conformal to an ALF Ricci flat metric* g. *Then above the interior of the polytope the metric* g *is generated from a harmonic axisymmetric function* U *on* $\mathbb{R}^3 \setminus \{\rho = 0\}$, *with asymptotic behaviour (55) and (66) on the axis* $\{\rho = 0\}$ *and at infinity. Moreover the piecewise linear function* $f(z)$ *in (55) is convex, and has its slopes increasing from* -1 *to* $+1$. □

Remark 4.4. It is useful to note that the whole section extends to the case where the metric has conical singularities around the fixed point sets. This is because we then have exactly the same asymptotic behaviours with the equations λ_E replaced by $\frac{\lambda_E}{\alpha_E}$, as explained in section 2.

5. The solutions

5.1. **The Taub-NUT generating function**. A significant role will be played by the following family of harmonic axisymmetric function parametrized by $n \geq 0$, namely, with $R := \sqrt{\rho^2 + z^2}$:

$$U_n(\rho, z) = 2R - 2z \log \frac{R + z}{\rho} + 2n \log \rho^2. \tag{68}$$

Notice that, when $\rho \to 0$, we have the behaviour (55) for the piecewise linear function

$$f_n(z) = 2n + |z|. \tag{69}$$

For $n > 0$, U_n is actually a generating function of the (self-dual) Taub-NUT metric on parameter n. Using corollary 3.6 we can reconstruct the metric. We normalize $k = 4n$: this will be understood in section 7.1, see (89). One

calculates the conjugate function $H_n(\rho, z) = z(R + 4n) + \rho^2 \log \frac{R+z}{\rho}$, then $V = e^{2v} = 1 + \frac{2n}{R}$ and $F = 2n\frac{z}{R}$. The formula (42) then gives

$$g_n = \Big(1 + \frac{2n}{R}\Big)\big(d\rho^2 + dz^2 + \rho^2 dx_3^2\big) + \frac{1}{1 + \frac{2n}{R}}\Big(dt - 2n\frac{z}{R}dx_3\Big)^2. \tag{70}$$

This is the Taub-NUT metric where the usual coordinates (R, θ) are related to (ρ, z) by $z = R\cos\theta$ and $\rho = R\sin\theta$. When the parameter n varies all these metrics are homothetic, so in this family there is only one metric up to scale. The function $U_0(\rho, z) = 2R(\rho, z) - 2z \log \frac{R(\rho,z)+z}{\rho}$ was already obtained in (64) as giving the asymptotics of any U: it is no longer a generating function but will play a prominent part in the expression of the general solution in the next subsection. Notice that its conjugate function is $H_0(\rho, z) = zR(\rho, z)+\rho^2 \log(R(\rho, z) + z)-\rho^2 \log \rho$, up to additive constant.

5.2. **Superposition**. Suppose that U is an axisymmetric harmonic function which satisfies $U_\rho > 0$ and $U_{zz} < 0$ everywhere (by lemma 4.2 it is sufficient to have this sign at the boundaries). In particular the condition $U_{\rho z}^2 + U_{zz}^2 > 0$ of corollary 3.6 is satisfied. The condition $V > 0$ in the same corollary is more difficult to achieve, but the following proposition shows that it is preserved when we superpose solutions:

Proposition 5.1. *Suppose that we have two solutions* U_1 *and* U_2 *of (35) which both satisfy* $U_{zz} < 0$, $U_\rho > 0$ *and*

$$\rho U_\rho + \frac{U_\rho^2 U_{zz}}{U_{\rho z}^2 + U_{zz}^2} < 0. \tag{71}$$

Then the same inequality remains true for any barycenter $a_1U_1+a_2U_2$ *(*$a_1, a_2 > 0$, $a_1 + a_2 = 1$*).*

Proof. The inequality to be proved for $U = a_1U_1 + a_2U_2$ is

$$U_{\rho z}^2 < -U_{zz}\Big(\frac{U_\rho}{\rho} + U_{zz}\Big). \tag{72}$$

One can check easily that if we have numbers $\alpha_i \in \mathbb{R}$, $\beta_i > 0$, $\gamma_i > 0$ $(i = 1, 2)$ such that $\alpha_i^2 \le \beta_i\gamma_i$, then one has $(a_1\alpha_1+a_2\alpha_2)^2 \le (a_1\beta_1+a_2\beta_2)(a_1\gamma_1+a_2\gamma_2)$. Applying to $U_{\rho z}$, $-U_{zz}$ and $\frac{U_\rho}{\rho} + U_{zz}$ we obtain the result. □

We can now describe all the solutions:

Corollary 5.2. *Suppose that* f *is a convex piecewise linear positive function on* $\mathbb{R}$*, with slopes* $-1 = f_0' < f_1' < \cdots < f_r' = 1$*, and angular points at* $z_1 < \cdots < z_r$*. It follows that*

$$f(z) = A + \sum_{i=1}^{r} a_i|z - z_i| \tag{73}$$

for real numbers $a_i = \frac{1}{2}(f_i' - f_{i-1}') > 0$ *satisfing* $\sum_1^r a_i = 1$.

Then the corresponding harmonic function

$$\mathrm{U} = \mathrm{A}\log\rho^2 + \sum_{i=1}^{r} a_i \mathrm{U}_0(\rho, z - z_i), \tag{74}$$

where U_0 *is defined in (64) or (68), satisfies the conditions* $\mathrm{V} > 0$ *and* $\mathrm{U}_{\rho z}^2 + \mathrm{U}_{zz}^2 > 0$ *of corollary 3.6 if and only if* $\mathrm{A} > 0$.

Proof. If the condition $\mathrm{A} > 0$ is satisfied, then $\mathrm{U}(\rho, z)$ is a barycenter of functions $\mathrm{U}_{n_i}(\rho, z - z_i)$ with $n_i > 0$ and the result follows from proposition 5.1.

Conversely, if $\mathrm{A} \leq 0$ one can show that the inequality (72) is not true: one calculates easily all the terms of the inequality, and making $\rho \to 0$ it would imply an inequality on the function $f(z)$ which is satisfied only for $\mathrm{A} > 0$. We leave the details of the calculation to the reader, since we will not really use it in our constructions. □

5.3. **Solutions**. We can now summarize what we have done. By corollary 4.3, any extremal Kähler Bach flat metric on a toric manifold with polytope P and scalar curvature vanishing on the edge at infinity, is generated via the ansatz of corollary 3.6 from an axisymmetric harmonic function $\mathrm{U}(\rho, z)$ with the boundary conditions:

$$\mathrm{U} \sim \begin{cases} f(z)\log\rho^2, & \rho \to 0, \\ \mathrm{U}_0(\rho, z), & \mathrm{R} \to \infty, \end{cases} \tag{75}$$

with U_0 defined in (64). Here $f : \mathbb{R} \to \mathbb{R}_+^*$ is a convex piecewise linear function, with slopes ± 1 at $\pm\infty$, and the first equivalent is valid only locally with respect to z. Conversely:

Proposition 5.3. *Given any convex, piecewise linear function* $f : \mathbb{R} \to \mathbb{R}_+^*$ *with slopes* ± 1 *at* $\pm\infty$*, and satisfying the condition* $\mathrm{A} > 0$ *when it is written under the form (73), there is an axisymmetric harmonic function* U *on* $\mathbb{R}^3$ *satisfying (75), which is unique up to the addition of an affine function of* z*. This function generates a Hermitian toric Ricci flat ALF metric via corollary 3.6 on* $\{\rho > 0\}$*.*

We say above $\rho > 0$ because for a general f as in the proposition, there is no reason to produce a metric which can be compactified into a smooth metric. This will be studied in the next section.

Observe that by remark 3.9, if the function $f(z)$ satisfies the hypothesis of the proposition, then for $a \neq 0$ the function $|a|^{-1} f(az + b)$ also does, and the two metrics generated by $f(z)$ and $|a|^{-1} f(az + b)$ are homothetic.

Proof. We first prove the uniqueness of U satisfying the asymptotics (75). If we have two such axisymmetric harmonic functions $\mathrm{U}_1, \mathrm{U}_2$, then $\mathrm{U}_1(\rho, z) - \mathrm{U}_2(\rho, z) = o(\log\rho)$ when $\rho \to 0$. By [HP70, Theorem 6.4] it follows that $\mathrm{U}_1 - \mathrm{U}_2$ extends smoothly across $\rho = 0$. Therefore $\mathrm{U}_1 - \mathrm{U}_2$ is a global harmonic function on $\mathbb{R}^3$. From the behaviour when $\mathrm{R} \to +\infty$ we deduce

that $U_1 - U_2 = O(R \log R)$ which implies that $U_1 - U_2$ is an affine function on $\mathbb{R}^3$. Since it is axisymmetric, it is an affine function of z alone.

Given the function f, the harmonic function U is given by (74), and corollary 5.2 says that we can apply corollary 3.6 to obtain a Hermitian toric Ricci flat metric. From the explicit formula on U we extract asymptotics on U when $R \to \infty$: one has $U \sim U_n$ with $n = \frac{A}{2}$, so the metric g at infinity approaches g_n given in (70) and is therefore ALF. □

6. The moment map

We now recover the polytope P from the generating function U, and so ultimately from the convex piecewise linear function f of proposition 5.3. We start from a Hermitian Ricci flat ALF metric generated by a function U as in corollary 3.6. By construction we also have the Kähler metric $g_K = x_1^2 g$. The Hamiltonian Killing vector fields ∂_t and ∂_{x_3} have moments x_1 and μ with respect to ω_K. These can be calculated from U by:

Proposition 6.1. *As functions of* ρ, z*, the moments* x_1, μ *are given by*

$$x_1 = \frac{2}{H_z}, \qquad \mu = -\frac{2}{k}\Big(z + \frac{\rho H_\rho - 2H}{H_z}\Big). \tag{76}$$

Proof. By (36) and (46) we have $x_1 = \frac{2}{\rho U_\rho} = \frac{2}{H_z}$. To obtain μ we calculate the Kähler form ω_K: from (18) we deduce

$$\omega_K = -\frac{d\xi}{\xi^2} \wedge (dt - F dx_3) + \frac{V e^u}{\xi^2} dx_2 \wedge dx_3. \tag{77}$$

Using (35) and (36) it follows that

$$\omega_K = \frac{2}{U_\rho^2}\left(\frac{1}{\rho}(U_{zz} d\rho - U_{\rho z} dz) \wedge (dt - F dx_3) - V(U_{\rho z} d\rho + U_{zz} dz) \wedge dx_3\right). \tag{78}$$

Calculating with (44)–(46) we obtain that $d\mu = -\partial_{x_3} \lrcorner \omega_K$ is given by the formula

$$d\mu = \frac{2}{k H_z^2}\Big(\big((\rho H_\rho - 2H)H_{\rho z} + \rho H_z H_{zz}\big) d\rho + \big(H_z^2 + (\rho H_\rho - 2H)H_{zz} - \rho H_z H_{\rho z}\big) dz\Big) \tag{79}$$

and the formula for μ follows. □

The image of the moment map is the polytope P. The (finite) boundary is obtained for $\rho = 0$, while the boundary at infinity is obtained for $R = \infty$, that is $x_1 = 0$. Suppose that f has $r+1$ different slopes $f_0' = -1 < f_1' < \cdots < f_r' = 1$ and denote $z_1 < \ldots < z_r$ the angular points of f, so that the slope of f on the segment $[z_i, z_{i+1}]$ is f_i' (we take $z_0 = -\infty$ and $z_{r+1} = +\infty$).

Proposition 6.2. *Suppose* U *is generated from a convex piecewise linear function* $f(z)$ *as in proposition 5.3. Then for* $i = 0, \ldots, r$*:*

- *On each segment* (z_i, z_{i+1}) *on which* f *is affine and non constant:*

– *the function* $\mathrm{F}(0,z)$ *is constant; we denote this constant* F_i*;*
– *one has*

$$x_1 = \frac{1}{f(z)}, \qquad \mu = -\frac{\mathrm{F}_i}{f(z)} + \frac{2}{k}\Big(\frac{f(z)}{f'(z)} - z\Big). \tag{80}$$

In particular $\mu + \mathrm{F}_i x_1$ *is constant on each segment of* f*, and it follows that a normal vector to* $\partial\mathrm{P}$ *along the corresponding edge is* $\partial_{x_3} + \mathrm{F}_i\partial_t$*.*

- *On a segment* $[z_i, z_{i+1}]$ *where* f *is constant, then* $x_1 = \frac{1}{f(z)}$ *is constant and a normal vector to* $\partial\mathrm{P}$ *is* ∂_t*.*

Proof. We calculate the asymptotics of F near $\rho = 0$. In the calculations below we take derivatives of equivalents, but all our assertions can be checked easily since U is given by the explicit formula (74). So when $\rho \to 0$ we have $\mathrm{U}(\rho, z) \sim f(z)\log\rho^2$ and it follows that on a segment (z_i, z_{i+1}) where $f'(z) \neq 0$:

$$\rho\mathrm{U}_\rho \sim 2f(z), \quad \mathrm{U}_z \sim f'(z)\log\rho^2, \quad \mathrm{U}_{\rho z} \sim \frac{2f'(z)}{\rho}. \tag{81}$$

One also calculates

$$\mathrm{U}_{zz} = -2\sum_{i=1}^{r} \frac{a_i}{\sqrt{\rho^2 + (z - z_i)^2}} \tag{82}$$

which in particular extends on $\rho = 0$ in the interior of the interval (z_i, z_{i+1}). It follows that

$$\mathrm{U}_{\rho z}^2 + \mathrm{U}_{zz}^2 \sim \frac{4f'(z)^2}{\rho^2}. \tag{83}$$

From (45) we then obtain that F also extends on $\rho = 0$ with

$$\mathrm{F}(0,z) = \frac{2}{k}\Big(\frac{f(z)^2}{f'(z)} - \mathrm{H}(0,z)\Big). \tag{84}$$

Since $\mathrm{H}_z = \rho\mathrm{U}_\rho = 2f(z)$ on $\rho = 0$, we obtain $\mathrm{F}(0,z)_z = 0$ so $\mathrm{F}(0,z)$ is constant on (z_i, z_{i+1}).

The formula (80) is then an immediate consequence of (76) and (46). The last assertion on a segment (z_i, z_{i+1}) where f is constant is tautological. □

Corollary 6.3. *Under the same hypotheses the moment polytope* P *of* g_{K} *has* $r+1$ *edges* E_i*, parametrized by the segments* $[z_i, z_{i+1}]$ *(*$i = 0...r$*) via the formula (80), and one additional edge at infinity* E_∞ *with equation* $x_1 = 0$*.*

7. Regularity

We now study the compactification of a metric coming from a generating function U. Since we have a toric Kähler metric, we can apply the toric theory: the compactification is a smooth manifold if the polytope satisfies the Delzant condition, that is two consecutive (finite) edges have primitive integral normals which form an integral basis of a lattice $\mathbb{Z}^2 \subset \mathbb{t}^2$. The

point here is that the integral structure, that is the lattice $\mathbb{Z}^2$, is not a priori given, since the Tod ansatz is local. In particular there is no reason why the coordinates (t, x_3) above would be nice angular coordinates: they parametrize $\mathbb{t}^2$ but we have to find the lattice inside.

We will treat the general case where we allow the metric g to have conical singularities along the divisor D_i which is the preimage by the moment map of the finite edge E_i of the polytope, say of angle $2\pi\alpha_i$ where $\alpha_i > 0$. Each D_i is a 2-sphere except for the two edges going to infinity where the point at infinity is missing.

7.1. **The regularity criterion**. We denote $f_i = f(z_i)$.

Lemma 7.1. *The metric g_{K} extends over the sphere D_i (outside the two fixed points) with a conical singularity of angle $2\pi\alpha_i$ if the vector v_i, normal to the edge E_i, defined by*

$$v_i = \begin{cases} \alpha_i f_i'(\partial_{x_3} + \mathrm{F}_i\partial_t) & \text{if } f_i' \neq 0, \\ \frac{2}{k}\alpha_i f_i^2 \partial_t & \text{if } f_i' = 0, \end{cases} \tag{85}$$

is primitive in $\mathbb{Z}^2$.

Proof. Suppose $f_i' \neq 0$. The formulas (81)–(82) imply that V extends over $\rho = 0$ along the segment (z_i, z_{i+1}) with

$$\mathrm{V}(0, z) = -\frac{1}{k}\big(2f(z) + \frac{f(z)^2}{f'(z)^2}\mathrm{U}_{zz}\big) \tag{86}$$

and similarly

$$e^{2\nu(0,z)} = f'(z)^2\mathrm{V}(0, z). \tag{87}$$

From formula (42) we see that g (or $g_{\mathrm{K}} = x_1^2 g$) can be compactified over the segment (z_i, z_{i+1}) if the vector field $\partial_{x_3} + \mathrm{F}_i\partial_t \in \ker(dt - \mathrm{F}_i dx_3)$ generates a circle, which will be contracted into a point at $\rho = 0$. Suppose this is the case, so we have an integral generator $\lambda(\partial_{x_3} + \mathrm{F}_i\partial_t)$ of the circle. Then the metric extends continuously with conical singularity of angle $2\pi\alpha_i$ if $e^{-\nu}\sqrt{\mathrm{V}}dx_3(\lambda(\partial_{x_3} + \mathrm{F}_i\partial_t)) = \alpha_i$, that is $\pm\frac{\lambda}{f_i'} = \alpha_i$. We can take $\lambda = \alpha_i f_i'$, and so we need $\alpha_i f_i'(\partial_{x_3} + \mathrm{F}_i\partial_t)$ to be an (integral) generator of a circle. If this is the case, then the standard regularity theory for toric extremal Kähler metrics tells us that the metric extends over the divisor D_i (in the conical case, this means that the symplectic potential has the behaviour described in section 2).

The case where $f_i' = 0$ is similar but the behaviour of the coefficients of g is different. One calculates that $\mathrm{U}_{\rho z} = \mathrm{O}(\rho)$ and it follows that

$$\mathrm{V} \sim -\frac{4}{k}\frac{f_i^2}{\rho^2\mathrm{U}_{zz}}. \tag{88}$$

Therefore the contracted direction is now generated by ∂_t. We need the integral generator $\lambda\partial_t$ of the circle to satisfy $e^{-\nu}\mathrm{V}^{-\frac{1}{2}}\lambda \sim \alpha_i\rho$, and this gives $\lambda = \frac{2}{k}f_i^2\alpha_i$. □

Since F is defined only up to an additive constant, we can always choose $F_0 = 0$. In the case $\alpha_0 = 1$ this implies that ∂_{x_3} is primitive and therefore μ is primitive and gives the normal of the first edge E_0 of the polytope, corresponding to the segment $(-\infty, z_1)$ of f. In the general case $\alpha_0 \neq 1$, that is when we have conical singularities, the function μ is still the weighted equation $\frac{1}{\alpha_0}\lambda_{E_0}$ used to describe the symplectic potential in (60). In all cases from (80) we see that the value of μ at the two fixed points of P at infinity is $\frac{2}{k}\lim_{z\to\pm\infty}(\frac{f(z)}{f'(z)} - z)$. In terms of formula (73) this is $\frac{2}{k}(\pm A - \sum_1^r a_i z_i)$. From the normalization (60) on the polytope, the length of the edge at infinity with respect to μ should be 2, which means that we want $\frac{4A}{k} = 2$, that is

$$k = 2A. \tag{89}$$

In particular this gives the value $k = 4n$ used for the Taub-NUT metric in section 5.1.

Remark 7.2. Formula (89) can be alternatively derived as follows. From the ALF condition (1), we infer that $\frac{\partial^2\psi}{\partial x_1 \partial x_1} \sim \frac{1}{x_1^2}$ at infinity, see (66), while $\frac{\partial^2\psi}{\partial x_1 \partial x_1} = \frac{\partial v_1}{\partial x_1} = W = \frac{V}{x_1^2}$, cf. Remark 3.2. It follows that $V \sim 1$ at infinity, while from (74) and (44) we easily infer $V \sim \frac{2A}{k}$ when R tends to infinity.

There remains to study the regularity at the fixed points. We apply the Delzant condition to deduce that the metric g_K (hence g) extends to a smooth compactification X^4 as a metric with conical singularities if for $i = 1, ..., r$ the basis (v_{i-1}, v_i) is an integral basis of $\mathbb{Z}^2$. We can compare v_{i-1} and v_i thanks to:

Proposition 7.3. *If $f'_{i-1} \neq 0$ and $f'_i \neq 0$, then*

$$F_i - F_{i-1} = \frac{2}{k} f_i^2 \Big(\frac{1}{f'_i} - \frac{1}{f'_{i-1}}\Big). \tag{90}$$

If $f'_i = 0$, then

$$F_{i+1} - F_{i-1} = \frac{2}{k}\left(f_i^2\Big(\frac{1}{f'_{i+1}} - \frac{1}{f'_{i-1}}\Big) - 2(z_{i+1} - z_i) f_i\right). \tag{91}$$

Proof. The first formula is an immediate consequence of (84), because $H(0, z)$ is continuous. The second formula is also a consequence, because $H(0, z)_z = 2f(z) = 2f_i$ on (z_i, z_{i+1}), so the variation of $H(0, z)$ on this interval is $2(z_{i+1} - z_i) f_i$. □

Remark 7.4. Using this proposition, formula (85) for the normals and the convexity of the polytope, one can give another proof of the convexity of $f(z)$ (corollary 4.3).

This gives our final classification result:

Theorem 7.5. *Suppose that we have a convex piecewise linear function $f(z)$ with slopes $-1 = f'_0 < \cdots < f'_r = 1$, singular points $z_1 < \cdots < z_r$, and satisfying the condition $A > 0$ of corollary 5.2. Define the constants F_i for*

all i's such that $f_i' \neq 0$ by $F_0 = 0$ and equations (90)–(91). Then the corresponding instanton extends to a smooth compactification X^4 and the metric g has conical singularities of angles $2\pi\alpha_0$,..., $2\pi\alpha_r$ along the divisors D_i if and only if the normals v_i to the edge E_i of the polytope defined by (85) satisfy the Delzant condition, that is each pair (v_{i-1}, v_i) can be deduced from (v_0, v_1) by a transformation in $GL(2, \mathbb{Z})$.

Conversely, all toric Hermitian Ricci flat ALF metrics, with conical singularities around the fixed point loci, are obtained by this construction.

Note that this theorem is really constructive: given f, Tod's generating function U is given by (74) and then the metric by (42).

Proof. The only fact which maybe remains to be proved is that the Ricci flat metric g is indeed ALF. One can see this from the Kähler formalism along the lines of section 4.3. Note in particular that since we have explicit formulas for U we have a much better control than the one stated there.

Alternatively we can also see directly the ALF nature of g from the formulas (42)–(46), since we know that $U(\rho, z) \sim U_0(\rho, z)$ given by (64) when $R \to \infty$ (and again U is actually explicit so we have a full asymptotic development). It is the same asymptotic behaviour for all these metrics, and in particular coincides with that of the Taub-NUT potential, and we calculated the corresponding metric (70). We deduce that we have

$$g \sim d\rho^2 + dz^2 + \rho^2 dx_3^2 + dt^2 = dr^2 + r^2(d\theta^2 + \sin^2(\theta) dx_3^2) + dt^2 \tag{92}$$

which is the common asymptotic behaviour of all ALF metrics. □

The condition on the function f in order to get a nice compactification is very strong. As we will see, it can be used in practice to understand the solutions.

Remark 7.6. It is not surprising that the Delzant condition on the normal vectors (v_i) is the same as the condition used in the physics literature to understand the rod structures which give rise to smooth solutions, see for example [CT10]. The data of the Delzant polytope is close but not identical to that of the rod structure.

7.2. **The basic examples ($r \leq 2$).** We have already seen the Taub-NUT example in section 5.1. We now describe all the examples where the polytope has three finite edges (and one edge at infinity). We recover in this way the family of the Kerr-Taub-bolt metrics. We only derive from the possible functions $f(z)$ all the possible solutions, but we leave to the reader the explicit calculation of the polytopes from the usual formulas for the Kerr-Taub-bolt metrics.

The piecewise linear function $f(z)$ (figure 1) has two singular points that we choose to be $z_1 = -b$ and $z_2 = b$. We write $f_1 = f(z_1) = b+a+m+n$ and $f_2 = f(z_2) = b-a+m+n$, where the parameters a, b, m and n are fixed by the condition $b^2 = a^2 + m^2 - n^2$ and $|n| \leq m$. The convention is consistent with the usual parameters of the Kerr-Taub-bolt metrics. We have a slope

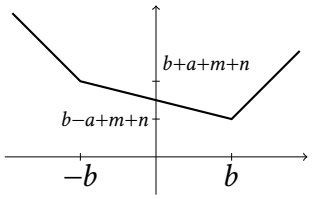

FIGURE 1. The function $f(z)$ for the Kerr-Taub-bolt metrics

$f_1' = -\frac{a}{b}$ on the middle segment. Note that the sign of a is not fixed, and figure 1 corresponds to the case $a > 0$. We now analyze the various cases.

7.2.1. *The Kerr-Taub-bolt metrics ($a \neq 0$, $n \neq 0$).* One obtains

$$f(z) = m + n + \frac{b-a}{2b}|z+b| + \frac{b+a}{2b}|z-b|, \tag{93}$$

in particular $k = 2(m+n)$. We have the vectors $v_0 = -(\partial_{x_3} + \mathrm{F}_0\partial_t)$, $v_1 = -\frac{a}{b}(\partial_{x_3} + \mathrm{F}_1\partial_t)$ and $v_2 = \partial_{x_3} + \mathrm{F}_2\partial_t$. We are looking for metrics with cone angles α_0, α_1 and α_2. Therefore we will get a smooth manifold if for some $\ell \in \mathbb{Z}$ one has

$$\alpha_0 v_0 \pm \alpha_2 v_2 = \ell\alpha_1 v_1. \tag{94}$$

We have the flexibility to vary $\alpha_1 \in \mathbb{R}_+^*$, so we can actually suppose $\ell \in \{-1, 0, 1\}$ (the other values of ℓ correspond to quotienting by $\mathbb{Z}_{|\ell|}$). The equation (94) gives the system

$$\begin{aligned} -\alpha_0 \pm \alpha_2 &= \ell\alpha_1 f_1', \\ -\alpha_0\mathrm{F}_0 \pm \alpha_2\mathrm{F}_2 &= \ell\alpha_1 f_1'\mathrm{F}_1. \end{aligned} \tag{95}$$

The second equation, together with the first one and (90), gives

$$\alpha_0 f_1^2(1 + \frac{1}{f_1'}) \pm \alpha_2 f_2^2(1 - \frac{1}{f_1'}) = 0. \tag{96}$$

This implies that $\pm$ is actually a $+$. Taking $\mathrm{F}_0 = 0$ (since F is defined up to a constant), one calculates $\mathrm{F}_1 = -\frac{1}{a}((b+m)^2 - (a+n)^2)$. We have two linear equations on α_0, α_2, and if $n \neq 0$ we calculate the solution:

$$\alpha_0 = \frac{\ell}{4nb}\big((b+m)^2 - (a-n)^2\big), \quad \alpha_2 = \frac{\ell}{4nb}\big((b+m)^2 - (a+n)^2\big). \tag{97}$$

In order to have positive angles, we need to take $\ell = \operatorname{sign} n$, hence the final formula

$$\alpha_0 = \frac{1}{4|n|b}\big((b+m)^2 - (a-n)^2\big), \quad \alpha_2 = \frac{1}{4|n|b}\big((b+m)^2 - (a+n)^2\big). \tag{98}$$

So we see that for any such function f, there is a choice of angles so that the metric extends to a smooth manifold, with the corresponding conical singularities. The polytope is now calculated from proposition 6.2: we

choose the integral basis of vectors $(v_1 = -\frac{a}{b}(\partial_{x_3} + \mathrm{F}_1\partial_t), -\alpha_0 v_0 = \alpha_0\partial_{x_3})$ which give us coordinates

$$x = -\frac{a}{b}(\frac{y}{\alpha_0} + \mathrm{F}_1 x_1), \qquad y = \alpha_0(\mu + \frac{a}{m+n}). \tag{99}$$

The translation on y is here to center the polytope, since from (80) one obtains $\lim_{z\to\pm\infty}\mu(z) = \pm 1 - \frac{a}{m+n}$.

We now calculate the vertices $\mathrm{V}_i = (x(z_i), y(z_i))$ of the polytope for $0 \leq i \leq 3$, where $z_0 = -\infty$ and $z_3 = +\infty$. We use formula (80). We have

$$\mathrm{V}_0 = (\frac{a}{b}, -\alpha_0), \qquad \mathrm{V}_3 = \big(-\frac{a}{b}, \alpha_0\big). \tag{100}$$

Then V_1 has the same y as V_0 (since ∂_y is a normal to E_0), with $x_1(z_1) = \frac{1}{f(z_1)} = \frac{1}{b+a+m+n}$. Therefore from $x(z_1) = -\frac{a}{b}(-1 + \mathrm{F}_1 x_1(z_1))$ one calculates

$$\mathrm{V}_1 = \big(\frac{b+m-n}{b}, -\alpha_0\big). \tag{101}$$

Similarly V_2 has the same x coordinate as V_1 (since ∂_x is a normal to E_2) and one obtains

$$\mathrm{V}_2 = \big(\frac{b+m-n}{b}, \alpha_0\frac{b+m-a-3n}{b+m-a+n}\big). \tag{102}$$

A normal to the last edge E_3 is $v_2 = -v_0 + \ell v_1 = -v_0 + \mathrm{sign}(n)v_1$, so the form of the polytope depends on the sign of n. The polytopes are represented on figure 2, for the same values of the parameters, except for the sign of n (note that this exchanges α_0 and α_2), with the angles for each edge.

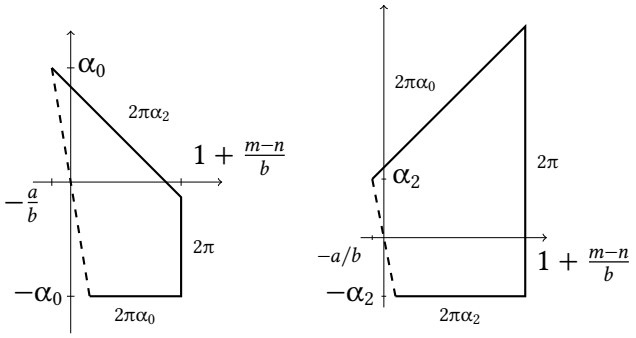

FIGURE 2. Kerr-Taub-bolt polytopes with the two orientations: $n > 0$ and $n < 0$

7.2.2. *The Taub-bolt metrics ($a = 0$, $n \neq 0$).* They are obtained at the limit $a \to 0$. In that case $\alpha_0 = \alpha_2$, so by renormalizing the angular variables by $\frac{1}{\alpha_0}$ we can interpret the solutions as being smooth along E_0 and E_2, and with conical singularity of angle $\frac{2\pi}{\alpha_0}$ along the 2-sphere corresponding to E_1. Writing $m = \lambda|n|$ we obtain $b = \sqrt{\lambda^2 - 1}|n|$ and an angle

$$\frac{2\pi}{\alpha_0} = \frac{4\pi}{\sqrt{\lambda^2 - 1} + \lambda} \tag{103}$$

which varies from 4π to 0 when λ varies from 1 to $+\infty$, and takes the value 2π for $m = \frac{5}{4}|n|$: this is the smooth Taub-bolt metric (with both orientations, depending on the sign of n). The limit $\lambda = 1$ (with $b = 0$ so the edge E_1 is removed) is the Taub-NUT metric, with the corresponding S^2 contracted to a point. One can check that a suitable rescaling when $\lambda \to 1$ converges to a ramified double cover of the Eguchi-Hanson metric (hence the limit angle 4π around the 2-sphere). So one can consider the metrics for $\lambda > 1$ small as obtained by blowing up the Taub-NUT metric at the fixed point and grafting a ramified double cover of the Eguchi-Hanson metric. This point of view will be explored systematically in section 9.

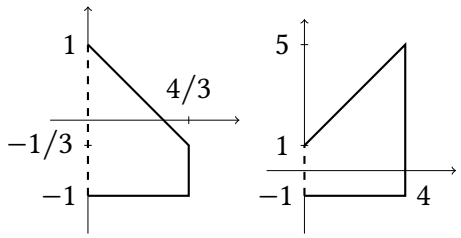

FIGURE 3. Taub-bolt polytopes (two orientations)

7.2.3. *The Kerr metrics ($n = 0$).* Via formula (90) this condition corresponds exactly to $F_2 = F_0$, which gives $v_2 = -v_0$ and therefore $\ell = 0$. The corresponding solutions are smooth and we obtain the Kerr family. We let the reader check that we obtain the polytopes on figure 4. The parameter $p = \frac{a}{b}$ varies in the interval $(-1, 1)$, with a symmetry between p and $-p$. For $p = 0$ one recovers the Schwarzschild metric; when $p \to 1$ the Kerr metric degenerates to the Taub-NUT metric.

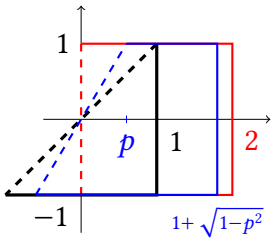

FIGURE 4. Taub-NUT, Schwarzschild (red) and Kerr (blue) polytopes

8. THE CLASSIFICATION OF SMOOTH HERMITIAN ALF INSTANTONS

8.1. **The constraints.** Smoothness imposes strong constraints on the convex piecewise linear function f from section 7.1. Suppose we have three consecutive segments of f, say with slopes $f'_{j-1} < f'_j < f'_{j+1}$, then the smoothness at the vertex corresponding to z_j is expressed by saying that

the two basis (v_{j-1}, v_j) and (v_j, v_{j+1}) of vectors defined by (85) generate the same lattice, that is satisfy

$$v_{j-1} + \varepsilon_j v_{j+1} = \ell_j v_j, \qquad \varepsilon_j = \pm 1, \ell_j \in \mathbb{Z}. \tag{104}$$

We get the system

$$\begin{aligned} f'_{j-1} + \varepsilon_j f'_{j+1} &= \ell_j f'_j, \\ f'_{j-1}\mathrm{F}_{j-1} + \varepsilon_j f'_{j+1}\mathrm{F}_{j+1} &= \ell_j f'_j \mathrm{F}_j. \end{aligned} \tag{105}$$

Using the first equation, the second equation becomes

$$f'_{j-1}(\mathrm{F}_j - \mathrm{F}_{j-1}) = \varepsilon_j f'_{j+1}(\mathrm{F}_{j+1} - \mathrm{F}_j). \tag{106}$$

Using (90) we obtain

$$f_{j+1}^2 = \varepsilon_j f_j^2 \frac{f'_j - f'_{j-1}}{f'_{j+1} - f'_j}. \tag{107}$$

Since $f'_{j-1} < f'_j < f'_{j+1}$ we deduce that $\varepsilon_j = +1$ and therefore the system (105) is now reduced to the equations

$$\begin{aligned} f'_{j-1} + f'_{j+1} &= \ell_j f'_j, \\ f_{j+1}^2 &= f_j^2 \frac{f'_j - f'_{j-1}}{f'_{j+1} - f'_j}. \end{aligned} \tag{108}$$

Using (85) in the case where f'_{j-1} or f'_{j+1} vanishes, the calculation leads to the same conclusion: $\varepsilon_j = +1$ and one has (108). If finally $f'_j = 0$, then the calculation gives $\varepsilon_j = +1$ and

$$\begin{aligned} f'_{j+1} &= -f'_{j-1}, \\ z_{j+1} - z_j &= \frac{2 + \ell_j}{2} \frac{f_j}{f'_{j+1}} \end{aligned} \tag{109}$$

Lemma 8.1. *Suppose that we have a smooth Hermitian ALF instanton generated by a convex piecewise linear function f on $\mathbb{R}$ as before, with $r+1$ different slopes $-1 = f'_0 < f'_1 < \cdots < f'_r = 1$. Then in three successive slopes $f'_{j-1} < f'_j < f'_{j+1}$ we must have $f'_{j-1} < 0$ and $f'_{j+1} > 0$. In particular $r \leq 3$.*

Proof. Indeed suppose for example that $f'_{j-1} < f'_j < f'_{j+1} \leq 0$. Then the first equation in system (108) implies that $\ell_j \geq 2$. But $f_{j+1} < f_j$ so the second equation (108) tells us that $f'_{j+1} - f'_j > f'_j - f'_{j-1}$, that is $f'_{j-1} + f'_{j+1} = \ell_j f'_j > 2f'_j$ which is a contradiction. □

8.2. **The case $r = 3$.** Suppose now that we have a smooth instanton with $n = 3$, so by lemma 8.1 we have $-1 = f'_0 < f'_1 < 0 < f'_2 < f'_3 = 1$. We will denote $p = -f'_1$ and $q = f'_2$ so that both are inside $(0, 1)$. This is illustrated in figure 5. Applying twice (108) we obtain the equations

$$\begin{aligned} -1 + q &= -\ell_1 p, \qquad -p + 1 = \ell_2 q, \\ (1-p)f_1^2 &= (p+q)f_2^2 = (1-q)f_3^2. \end{aligned} \tag{110}$$

From the first equation we see that $\ell_1, \ell_2 > 0$. On the other hand, since $f_2 < f_1$, by the second equation we have $1 - p < p + q$ which gives $2p >$

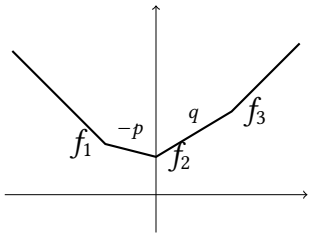

FIGURE 5. The function $f(z)$ for the Chen-Teo metrics

$1-q=\ell_1 p$. Finally we get $\ell_1=1$, and similarly $\ell_2=1$. In particular (104) now becomes $v_0+v_2=v_1$ and $v_1+v_3=v_2$, which imply $v_3=-v_0$, that is these are AF instantons.

Now the system (110) reduces to $p+q=1$ and

$$f_1=\frac{f_2}{\sqrt{q}},\qquad f_3=\frac{f_2}{\sqrt{p}}. \tag{111}$$

This completely determines f up to the transformation of f into $\frac{1}{a}f(az+b)$. Therefore we have constructed a 1-dimensional family, indexed by $p\in(0,1)$, which is exactly the Chen-Teo family. We have proved:

Theorem 8.2. *A smooth Hermitian, non hyperKähler, ALF gravitational instanton, is either:*

- *the Taub-NUT metric (with the orientation opposed to the hyperKähler orientation)*
- *the 1-parameter family of Kerr instantons*
- *the Taub-bolt metric (with respect to both orientations)*
- *the 1-parameter family of Chen-Teo instantons.*

□

Remark that this is a classification theorem, we do not need to compare our metrics with the explicit formulas for the Chen-Teo instantons: by uniqueness our 1-dimensional family has to be the Chen-Teo family. In particular we have given an independent construction of the Chen-Teo instantons. Nevertheless it is possible to compare directly our data (the function f) with the explicit formulas of Chen-Teo, but we do not need it here.

We can calculate the polytopes for the 1-parameter family of Chen-Teo instantons. We fix $p,q\in(0,1)$. We can choose $f_2=pq$ so that $f_1=p\sqrt{q}$ and $f_3=q\sqrt{p}$. Then we take $z_2=0$ and therefore $z_1=-\frac{f_1-f_2}{p}=q-\sqrt{q}$, similarly $z_3=\sqrt{p}-p$. We put f in the form (73):

$$\begin{aligned} &f(z)=\frac{1}{2}\left(k+q|z+\sqrt{q}-q|+|z|+p|z-\sqrt{p}+p|\right),\\ &\text{with } k=1-p^{\frac{3}{2}}-q^{\frac{3}{2}}. \end{aligned} \tag{112}$$

We choose $\mathrm{F}_0=0$ and therefore $\mathrm{F}_1=\frac{2}{k}p^2q(-\frac{1}{p}+1)=-\frac{2}{k}pq^2$. An integral basis of the lattice is $(v_1=-p(\partial_{x_3}+\mathrm{F}_1\partial_t),-v_0=\partial_{x_3})$ which gives

coordinates

$$x = -p(y + F_1 x_1), \quad y = \mu + \frac{1}{k}(p^{\frac{3}{2}} - p^2 - q^{\frac{3}{2}} + q^2), \tag{113}$$

where the constant added to μ in y is chosen so that $y|_{E_\infty}$ varies in $(-1, 1)$, see the definition of k in (89). This is chosen so that the first edge E_0 is horizontal and the second one E_1 is vertical. The form of the polytope is now given by its normals to the edges: $v_0, v_1, v_2 = v_1 - v_0, v_3 = v_2 - v_1 = -v_0$. Using the same method as in section 7.2, one can calculate the vertices $V_i = (x(z_i), y(z_i))$ of the polytope (with $z_0 = -\infty$ and $z_4 = +\infty$). Skipping the details of the calculation, we get:

$$\begin{aligned} &V_0 = (p, -1), && V_1 = (p(1 + \frac{2}{k}q^{\frac{3}{2}}), -1), \\ &V_2 = \big(p(1 + \frac{2}{k}q^{\frac{3}{2}}), -1 - \frac{2}{k}(q^{\frac{3}{2}} - q)\big), \\ &V_3 = (-p(1 - \frac{2}{k}\sqrt{pq}), 1), && V_4 = (-p, 1). \end{aligned} \tag{114}$$

There is a symetry: the polytope for (q, p) is obtained from that for (p, q) by the integral transformation

$$(x, y) \mapsto (x + y, -y). \tag{115}$$

This is somehow similar to the Kerr family with the symmetry between the parameter a and $-a$, with $a = 0$ begin the Schwarzschild metric. In our case, the polytope which admits a symmetry is obtained for $p = q = \frac{1}{2}$, and has vertices

$$(\frac{1}{2}, -1), \quad (1 + \frac{1}{\sqrt{2}}, -1), \quad (1 + \frac{1}{\sqrt{2}}, 0), \quad (\frac{1}{\sqrt{2}}, 1), \quad (-\frac{1}{2}, 1). \tag{116}$$

We represent on figure 6 the symmetric polytope (in red), then a polytope in the family obtained for $p = 0.1$ (in blue), and (dashed) the limit polytope when $p \to 0$ which is the polytope of the Taub-bolt metric. Of course the picture for $p \geq \frac{1}{2}$ is obtained by the symmetry (115).

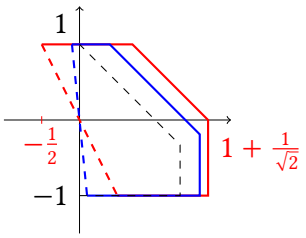

FIGURE 6. Polytopes of the Chen-Teo instantons

Remark 8.3. From the data of a polytope, it is possible to calculate abstractly (without solving the extremal metric problem) the scalar curvature (we need it to vanish on the edge at infinity) and the Calabi functional of the extremal Kähler metric. Bach flat metrics appear as critical points of

the Calabi functional when one varies the edges of the polytope without changing their normals, see [CLW08]. (In our problem the edges come with a measure which is the integral measure multiplied by the angle α; the Poincaré behaviour at infinity is obtained by taking a measure zero on the edge at infinity). So the polytopes which generate an instanton can a priori be characterized abstractly by algebraic equations, but these seem impossible to solve. Conversely it is possible to check that a given polytope, with given weights on the edges, can generate an instanton. We checked this on the Chen-Teo polytopes using Maple.

Remark 8.4. We have not tried here to recover the whole 5-parameter family of Chen and Teo in [CT15], which amounts for us to 4 parameters since we do not include scale. But one can remark that our piecewise linear functions with 3 singular points have exactly the same number of parameters: we can fix $f_2 = f(0)$, then there remain 4 free parameters, the two slopes p_1, p_2 and the two values f_1, f_3. So it looks likely that this gives the full Chen-Teo family. Using our methods it is then possible to say which ones are conical (probably a 3-parameter subfamily, since the three cone angles give two additional parameters).

9. Blowing up and cone angles

In this section we prove theorem B by describing a procedure which enables to construct new solutions from a given solution. Geometrically it is a blowing up of a fixed point of the torus, but we will see that there is a simple interpretation in terms of the piecewise linear function.

So suppose that we have a Hermitian, Ricci flat, ALF instanton, generated by a function $f(z)$ as before. We suppose that the instanton is defined on a smooth manifold but may have conical singularities along the fixed point sets. We shall describe a procedure to add a new segment to the function f. This is illustrated in figure 7, where we want to add a segment with slope p between two segments with slopes p_j and p_{j+1}. At the end the procedure will be slightly different but it is useful to start with this idea.

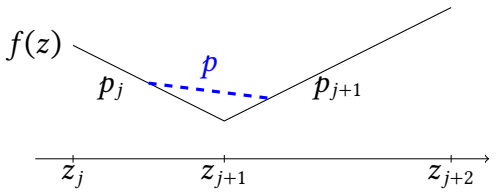

Figure 7. Blowing up

We denote $f_i = f(z_i)$, the slope of each segment is p_i and a normal vector to the corresponding edge E_i of the polytope is $v_i = p_i(\partial_{x_3} + F_i\partial_t)$ (for simplicity we suppose that all slopes are nonzero but the proof below can be adapted to this case). We have a conical singularity of angle $2\pi\alpha_i$ around the corresponding sphere, and from section 7.1 we know that since

the manifold is smooth we have the relation $\alpha_{i-1}v_{i-1}+\alpha_{i+1}v_{i+1}=\ell_i\alpha_i v_i$ for some integers ℓ_i (the sign in the equation is + because of the convexity of the polytope). We now fix a given j and we want to add a new normal $v=p(\partial_{x_3}+\mathrm{F}\partial_t)$ and an angle α. So we have

$$\begin{aligned}\alpha_{j-1}v_{j-1}+\alpha_{j+1}v_{j+1}&=\ell_j\alpha_j v_j\\ \alpha_j v_j+\alpha_{j+2}v_{j+2}&=\ell_{j+1}\alpha_{j+1}v_{j+1}\end{aligned}\tag{117}$$

To keep the smoothness we want the new data to satisfy:

$$\begin{aligned}\alpha_{j-1}v_{j-1}+\alpha v&=(\ell_j+1)\alpha_j v_j\\ \alpha_j v_j+\alpha_{j+1}v_{j+1}&=\alpha v\\ \alpha v+\alpha_{j+2}v_{j+2}&=(\ell_{j+1}+1)\alpha_{j+1}v_{j+1}\end{aligned}\tag{118}$$

The choice of the integer constants in the RHSs is justified by the fact that the first equation in (117) follows from the two first equations in (118), and the second equation follows from the two last equations in (118). In particular, given the system (117) our more complicated system (118) is equivalent to the single middle equation, that is to the system

$$\begin{aligned}\alpha_j p_j+\alpha_{j+1}p_{j+1}&=\alpha p,\\ \alpha_j p_j(\mathrm{F}_j-\mathrm{F})+\alpha_{j+1}p_{j+1}(\mathrm{F}_{j+1}-\mathrm{F})&=0.\end{aligned}\tag{119}$$

We deduce the values of p and F in terms of α:

$$p=\frac{\alpha_j p_j+\alpha_{j+1}p_{j+1}}{\alpha},\quad \mathrm{F}=\frac{\alpha_j p_j\mathrm{F}_j+\alpha_{j+1}p_{j+1}\mathrm{F}_{j+1}}{\alpha_j p_j+\alpha_{j+1}p_{j+1}}.\tag{120}$$

If $\alpha_j p_j+\alpha_{j+1}p_{j+1}=0$ then $p=0$ and we know that F is not relevant. Here we suppose $p\neq 0$ (one can obtain the case $p=0$ either by using (91) or by taking a limit $p\to 0$). Let us call f_- and f_+ the values of f at the boundary of the new segment. By (90) we must have

$$\mathrm{F}-\mathrm{F}_j=\frac{2}{k}f_-^2\Big(\frac{1}{p}-\frac{1}{p_j}\Big),\quad \mathrm{F}_{j+1}-\mathrm{F}=\frac{2}{k}f_+^2\Big(\frac{1}{p_{j+1}}-\frac{1}{p}\Big).\tag{121}$$

Writing $\mathrm{F}_{j+1}-\mathrm{F}_j=\frac{2}{k}f_{j+1}^2(\frac{1}{p_{j+1}}-\frac{1}{p_j})$, we finally deduce from (120)

$$f_-^2=f_{j+1}^2\frac{(p_{j+1}-p_j)\alpha_{j+1}}{\alpha_{j+1}p_{j+1}-(\alpha-\alpha_j)p_j},\quad f_+^2=f_{j+1}^2\frac{(p_{j+1}-p_j)\alpha_j}{(\alpha-\alpha_{j+1})p_{j+1}-\alpha_j p_j}.\tag{122}$$

Observe that for $\alpha=\alpha_j+\alpha_{j+1}$ the slope $p\in(p_j,p_{j+1})$ and $f_-=f_{j+1}=f_+$, so the procedure does not change anything (we added a segment of length 0). We can rewrite (122) as

$$f_{j+1}^2-f_-^2=f_{j+1}^2\frac{(\alpha_j+\alpha_{j+1}-\alpha)p_j}{\alpha(p-p_j)},\quad f_+^2-f_{j+1}^2=f_{j+1}^2\frac{(\alpha_j+\alpha_{j+1}-\alpha)p_{j+1}}{\alpha(p_{j+1}-p)}.\tag{123}$$

Now suppose that $\alpha<\alpha_j+\alpha_{j+1}$ but is close to $\alpha_j+\alpha_{j+1}$. Then $f_{j+1}-f_-$ has the sign of p_j and f_+-f_{j+1} that of p_{j+1}. So $f_\pm=f(z_\pm)$ for some $z_-\in(z_j,z_{j+1})$ and $z_+\in(z_{j+1},z_{j+2})$, both close to z_{j+1}. So it looks possible to modify f as in figure 7, except that to introduce a segment of slope p we would need $f_+-f_-=p(z_+-z_-)$ which does not follow from our formulas.

So we must do something slightly more complicated: we must modify globally the function f in order to match the new segment of slope p. For $\alpha < \alpha_j + \alpha_{j+1}$ close enough to $\alpha_j + \alpha_{j+1}$ we have the slopes $p_0 = -1 < p_1 < \cdots < p_j < p < p_{j+1} < \cdots < p_r = 1$, the constants $F_0, ..., F_j, F, F_{j+1}, ...$, the angles $\alpha_0, ..., \alpha_j, \alpha, \alpha_{j+1}, ...$ and the corresponding values of the piecewise linear function $f_1, ..., f_j, f_-, f_{j+1}, f_+, f_{j+2}, ...$ This is coherent in the sense that the sign of each $f_{i+1} - f_i$ is the sign of p_i, as we have just checked for the new values $f_\pm$. So there exists a convex piecewise linear function f_α with these slopes and these values at the singular points (f_α is unique up to translation in the z variable). From our construction the constraints of proposition 7.3 and the Delzant condition on the normals $\alpha_0 v_0, ..., \alpha_j v_j, \alpha v, \alpha_{j+1} v_{j+1}, ...$ are still satisfied, so we obtain a family of new instantons with the same normals and cone angles as the initial one, and one additional edge to the polytope with cone angle $2\pi\alpha$. Geometrically the addition of an edge to the polytope is a complex blowup. This finishes the proof of theorem B.

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