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Abstract: The S-matrix of the well-studied sinh-Gordon model possesses a remarkable strong/weak coupling duality $b \to 1/b$. Since there is no understanding nor evidence for such a duality based on the quantum action of the model, it should be questioned whether the properties of the model for $b > 1$ are simply obtained by analytic continuation of the weak coupling regime $0 < b < 1$. In this article we assert that the answer is no, and we develop a concrete and specific proposal for the properties when $b > 1$. Namely, we propose that in this region one needs to introduce a background charge $Q_\infty = b + 1/b - 2$ which differs from the Liouville background charge by the shift of -2. We propose that in this regime the model has non-trivial massless renormalization group flows between two different conformal field theories. This is in contrast to the weak coupling regime which is a theory of a single massive particle. Evidence for our proposal comes from higher order beta functions. We show how our proposal correctly reproduces the freezing transitions in the multi-fractal exponents of a Dirac fermion in $2+1$ dimensions in a random magnetic field, which provides a strong check since such transitions have several detailed features. We also point out a connection between a semi-classical version of this transition and the so-called Manning condensation phenomena in polyelectrolyte physics.

Keywords: Conformal Field Theory, Integrable Field Theories, Random Systems, Renormalization Group

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1 Introduction

The sinh-Gordon model is the simplest relativistic model in $1+1$ dimensions that is integrable. It can be defined by the action

$$\mathcal{S} = \int d^2x \left(\frac{1}{8\pi} \partial_\mu \phi \partial_\mu \phi + 2\mu \cosh(\sqrt{2} b \phi) \right), \tag{1.1}$$

with b is a real parameter and ϕ a real scalar field. The current understanding is that the spectrum consists of a single massive particle, with an S-matrix that is factorizable in terms of the two particle S-matrix [1], eq. (2.1) below. Based on this, a great deal is known about the model. As a partial list of references, let us mention the following. The form factors

have been computed [2, 3], which can be used to calculate correlation functions. Even finite temperature 1-point correlations are computable [4]. The thermodynamic Bethe ansatz (TBA) has also been investigated [5]; we will say more about this later.

In spite of this vast amount of known results concerning the sinh-Gordon model, one important aspect has essentially remained unanswered. Remarkably, the S-matrix satisfies the strong/weak coupling duality in that it is invariant under $b \to 1/b$. The most common viewpoint is that one first defines the theory for $0 < b < 1$ where one can trust perturbation theory around $b = 0$, and then one *defines* the theory for $1 < b < \infty$ using the duality. Since the form factors and TBA are invariant under $b \to 1/b$, from this perspective the theory for $b > 1$ is the same as the dual theory at $0 < b < 1$. However an important question arises. Since there is no indication of a $b \to 1/b$ duality based on the action (1.1) itself, the analytic continuation $b \to 1/b$ may actually not be valid. Relatively recently this issue was studied in much detail using a truncated Hilbert space approach [7] and indeed indications were found that this approach breaks down as b approaches the self-dual point $b = 1$. It was suggested there that for $b > 1$ the theory may actually be massless, however definite properties of such a theory remained unspecified and are still unknown. A similar breakdown of analyticity is known to occur in freezing phenomena, such as in disordered systems [8, 9], and was recently shown to be present in Coulomb gas systems [10] which bear analogies with the sinh-Gordon model. The freezing transition in the sinh-Gordon, in connection with disordered fermions, was also considered in [11, 12]. We will say more about this connection later.

The purpose of the present article is to make a definite proposal for the behavior of the sinh-Gordon model for $b > 1$ that is not a simple analytic continuation $b \to 1/b$ of the $0 < b < 1$ regime. Our specific proposal is easily described. For $0 < b < 1$ the theory can be defined as a perturbation of a free massless boson in the ultraviolet (UV), and the standard properties based on a single massive particle with the known S-matrix all apply. However at $b = 1$ and above, a background charge Q_∞ is spontaneously generated. This background charge is not the same as one would obtain if one views the sinh-Gordon theory as a perturbation of the Liouville conformal field theory (CFT), but is rather given by

$$Q_\infty = b + 1/b - 2 \tag{1.2}$$

which is a shift of the Liouville value by -2. Furthermore, the $b > 1$ regime is a massless phase, but is not conformally invariant. Rather the theory can be described by a massless renormalization group (RG) flow between two conformal field theories.

In section 3 we propose our main result (1.2) based on some rather simple criteria. However these simple arguments by themselves are not enough to understand the true nature of the theory beyond the self-dual point. Ultimately the properties of this theory should be tied to properties of the RG, since the non-zero background charge Q_∞ affects anomalous dimensions, etc. To this aim we study the RG for the sinh-Gordon model based on the beta functions proposed in [13] and understood in more detail in [14]. These beta functions are well suited to our purposes since they are ultimately based on the action (1.1) of the sinh-Gordon theory itself. More importantly, the physics of interest here concerns massless flows between conformal field theories, and it was shown in [14] that these proposed

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beta functions predict RG flows that agree precisely with exact results for massless flows in the so-called "imaginary" sine Gordon model [15, 16]. We will review this below. These beta functions also correctly predicted cyclic RG flows. This gives us some confidence in at least attempting to use these beta functions to explore the physics we are trying to understand. As we will show in section 4.1 there is a clear difference between the RG flows for $b < 1$ verses $b > 1$ since these beta functions do not have the symmetry $b \to 1/b$. Furthermore, we can argue based on these beta functions that $b > 1$ is a massless phase and provide support for our proposed Q_∞.

The beta functions in [13] are based on anisotropic current-current perturbations of a Wess-Zumino-Witten model at a level k, and the map to sine and sinh Gordon theory was made in [14], which only involves level $k = 1$. It needs to be mentioned that the beta functions in [13] are still conjectural. It was argued in [17] that there are corrections at 4-loops. One of us has also pointed out that there could be $1/k$ corrections to these beta functions for a different class of "flavor" anisotropic models [18]. Similar kinds of "all-orders" beta functions were considered in [19–22] using rather different gravitational methods. There it was also argued that there are higher $1/k$ corrections. On the other hand, as already stated above, the renormalization scheme used in [13] to obtain an all-orders beta function has already been shown to provide exact results for the kind of physics being explored here. In light of these statements, in this paper we will simply assume the beta functions in [13, 14] to be correct enough to capture the physics we are trying to understand and leave aside the issue of possible corrections and whether they affect our conclusions. Our analysis of these beta functions at this stage should be viewed as supportive, but not indisputable, evidence for our main proposal described above. In any case, irrespective of the present work, it is not at all understood how these proposed $1/k$ corrections can be reconciled with the correct exact predictions on massless flows in [14–16].

In section 5 we apply our proposal to the freezing transition of a Dirac fermion in $2 + 1$ dimensions in a random U(1) gauge field, namely a magnetic field. We first map the problem to the sinh-Gordon theory. Then using our proposed Q_∞, we compute in detail the multi-fractal exponents and their transitions, which can all be traced to the transition at $b = 1$ of the sinh-Gordon model. Our results agree with known results based on Derrida's random energy model or other random fermion models [8, 9, 11, 12, 23]. This provides rather strong evidence for our proposals concerning the transition in the sinh-Gordon model at the self-dual point.

By studying a simple semi-classical evaluation of one-point functions in the sinh-Gordon model, one can understand how the premises of a transition can be found in such an approximation, and how it is related to the well-known counter-ion Manning's condensation [24] in polyelectrolyte solutions. This semi-classical computation actually points towards a freezing transition in the spectrum of possible exponential operators, as a function of their weights for fixed value of the sinh-Gordon parameter b. We relegated these results to an appendix since the calculation is semi-classical and detailed properties are beyond its scope. However we found it instructive to include this analysis since it provides a simple intuitive picture for the transition.

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2 Sinh-Gordon conventions

Since there are several conventions in the literature, and factors of $1/4\pi$ and $\sqrt{2}$ are important here, we clearly define our conventions. In the standard understanding, almost certainly valid in the weak coupling regime $0 < b < 1$, the spectrum consists of a single particle of mass m_{phys} with two particle S-matrix

$$S(\theta) = \frac{\sinh\theta - i\sin\pi\gamma}{\sinh\theta + i\sin\pi\gamma}, \qquad \gamma \equiv b^2/(1+b^2). \tag{2.1}$$

Here θ is the difference of the usual rapidity parameterization of energy/momentum: $(E,p) = m_{\text{phys}}\,(\cosh\theta, \sinh\theta)$. This S-matrix is invariant under the duality $b \to 1/b$ which corresponds to $\gamma \to 1-\gamma$.

The free gaussian field when $\mu = 0$ can be decomposed as $\phi(z,\overline{z}) = \varphi(z) + \overline{\varphi}(\overline{z})$, where $z = x + iy$ and $\overline{z} = x - iy$. With the $1/8\pi$ in the action, the above fields have the canonical two point functions, $\langle\varphi(z)\varphi(w)\rangle = -\log(z-w)$, and similarly for $\overline{\varphi}$, and its Virasoro central charge is $c = 1$. It is most natural to view the cosh potential as a perturbation of the free gaussian field. Let $[[*]]$ denote the total scaling dimension of $*$ in mass units (for fields this is the sum of left and right conformal dimensions $\Delta + \overline{\Delta}$). One has

$$[[\cosh(\sqrt{2}b\phi)]] = -2b^2, \tag{2.2}$$

which is always relevant for real b. Thus $\mu = [\text{mass}]^{2+2b^2}$ for some mass parameter. One can take the latter as the physical mass of the single sinh-Gordon particle, such that

$$m_{\text{phys}} = F(b)\ \mu^{1/(2+2b^2)}. \tag{2.3}$$

The non-trivial function $F(b)$ was computed by Al. Zamolodchikov by comparing conformal perturbation theory with the thermodynamic Bethe ansatz (TBA) since the latter is expressed in terms of m_{phys} [25]. We will not need the explicit form of $F(b)$ which is somewhat complicated but only its value in the limit $b \to 1$.

We can now clearly address the issue we are proposing to resolve in this paper that was referred to in the Introduction. The S-matrix (2.1) is invariant under the strong/weak duality $b \to 1/b$. The self-dual point is $b = 1$. This, combined with the Z_2 symmetry $b \to -b$ naively suggests one need only solve the theory the region $0 \le b \le 1$, and then analytically continue the result to all b on the real line. First of all, there is no guarantee this analytic continuation is valid since it is not a symmetry of the lagrangian whatsoever. Moreover there is concrete evidence that some phenomenon is going on at $b = 1$ which as yet is not understood. One indication is that from the exact form of $F(b)$, one finds

$$\lim_{b\to 1} m_{\text{phys}} = \lim_{b\to 1} \frac{4\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)^2}\left(\frac{\pi\mu}{\Gamma(1-b^2)}\right)^{1/4} = 0, \tag{2.4}$$

which formally implies $m_{\text{phys}} = 0$ at $b = 1$. For $b > 1$ it would appear m_{phys} is complex. If the physical mass is zero at $b = 1$, then the S-matrix (2.1) does not make much sense since rapidity is not defined if $m_{\text{phys}} = 0$. More recently the sinh-Gordon model was studied

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starting from the action in a truncated Hilbert space approach and clear deviations from the TBA predictions were observed as $b \to 1$ [7]. On the other hand, for b not too large but still in the region $0 \leq b \leq 1$, results based on a truncated space of the free gaussian field work very well. Based on this one could conclude that the theory defined by the lagrangian for $b > 1$ has different properties than one would expect from an analytic continuation of $0 \leq b < 1$. Another piece of evidence for this transition comes from the analysis of random fermions done in [11, 12]. There, the existence of a freezing transition in the sinh-Gordon was conjectured in connection with glassy behavior of random Dirac fermions. Their analysis was based on using one-loop RG equations, in the form of the so-called KPP equations, and their traveling wave solutions. Up to now these works, though very interesting, have not provided a concrete indication of the properties of the theory for $b > 1$.

3 Sinh-Gordon theory with a background charge

3.1 Generalities

In this section we consider the sinh-Gordon model with a background charge Q_∞ at infinity. Formally one can deform the action as follows:

$$\mathcal{S} = \mathcal{S}_{\text{shG}} + Q_\infty \, \phi_\infty. \tag{3.1}$$

Alternatively one can couple the field ϕ to the curvature R, adding a term proportional to $Q_\infty \, R \, \phi$ to the lagrangian. Either way, in the unperturbed $\mu = 0$ conformal field theory, the effect is to modify the conformal stress tensor

$$T(z) = -\frac{1}{2}\partial_z\varphi\partial_z\varphi + \frac{Q_\infty}{\sqrt{2}} \, \partial_z^2\varphi, \tag{3.2}$$

and similarly for $\overline{T}(\overline{z})$. The Virasoro central charge is now

$$c = 1 + 6\, Q_\infty^2. \tag{3.3}$$

The main effect of non-zero Q_∞ is to change the scaling dimensions of operators in the free boson CFT:

$$[[e^{\sqrt{2}a\phi}]] = 2a(Q_\infty - a). \tag{3.4}$$

The two exponentials in the cosh now have different scaling dimensions, thus one should write

$$2\mu \, \cosh(\sqrt{2}b\phi) \to \mu_+ e^{\sqrt{2}b\phi} + \mu_- e^{-\sqrt{2}b\phi}. \tag{3.5}$$

Although the dimensions of $e^{\pm\sqrt{2}b\phi}$ differ, their sum adds up to $-4b^2$ for any Q_∞.

For a weight "a" in $e^{\sqrt{2}a\phi}$, let us define its dual $\widetilde{a}$:

$$\widetilde{a} \equiv Q_\infty - a, \tag{3.6}$$

and similarly for the sinh-Gordon coupling $\widetilde{b} \equiv Q_\infty - b$. Note that for zero Q_∞, $\widetilde{a} = -a$, including of course $\widetilde{b} = -b$. Thus with no background charge, the duality $b \to \widetilde{b}$ simply corresponds to the Z_2 symmetry of the action $b \to -b$.

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For the CFT with non-zero Q_∞, one has the duality that the dimension of $e^{\sqrt{2}a\phi}$ is invariant under $a \to \widetilde{a}$. Coulomb gas techniques indicate the equivalence

$$e^{\sqrt{2}a\phi} \simeq e^{\sqrt{2}(Q_\infty - a)\phi}, \tag{3.7}$$

in the CFT correlation functions. In the TBA this equivalence can be expressed in terms of so-called reflection amplitudes $R(a)$

$$e^{\sqrt{2}a\phi} = R(a)\, e^{\sqrt{2}(Q_\infty - a)\phi} \tag{3.8}$$

which are known for the Liouville case [27]. This reflection symmetry is known to be valid in the Liouville theory but only conjectural in the sinh-Gordon theory (see the appendix for a discussion of this point).

For any background charge Q_∞, the effective central charge c_{eff} of the TBA is the same if the particle is considered massive. The TBA equations based on the S-matrix (2.1) do not depend explicitly on Q_∞. However the effective UV central charge is $c_{\text{eff}} = c_{\text{vir}} - 12d_0$ where c_{vir} is the Virasoro central charge and d_0 is the ground state energy. Now $d(a) = 2a(Q_\infty - a)$ which has a maximum at $a = Q_\infty/2$ which corresponds to $d_0 = Q_\infty^2/2$. Thus $c_{\text{eff}} = 1 + 6Q_\infty^2 - 12(Q_\infty^2/2) = 1$, independently of Q_∞.

In principle, the sinh-Gordon model can be considered with any Q_∞. In this paper we will only consider two choices, the Liouville case and the choice described in the subsequent subsection. Let us consider the first.

3.2 Liouville case

This is the most natural choice besides the perturbation of the $Q_\infty = 0$ free massless boson. Many works indicate that the sinh-Gordon model may be viewed as a perturbation of the Liouville theory, in particular [26–28]. In this choice,

$$Q_\infty = b + 1/b, \qquad \Longrightarrow \quad [[e^{\sqrt{2}b\phi}]] = 2. \tag{3.9}$$

Namely the positive exponential is an exactly marginal operator, $[[\mu_+]] = 2$, and the additional $e^{-\sqrt{2}b\phi}$ is viewed as a relevant perturbation of the Liouville CFT. Although this may seem like an unnecessary complication, surprisingly it has been shown that the conformal perturbation theory of this model with non-zero Q_∞ can reproduce the perturbation theory with $Q_\infty = 0$ when $0 < b < 1$ [7].

For $Q_\infty = b + 1/b$, the duality (3.6) is $b \to \widetilde{b} = 1/b$. This indicates that this Liouville formulation of the sinh-Gordon model is unable to address the problem posed in this paper since this dual is the usual one that maps the region $0 \le b \le 1$ to $1 \le b \le \infty$. Thus it has nothing to say about any novel behavior for $b > 1$.

3.3 Freezing transition at the self-dual point

Our aim is to find a different choice of background charge Q_∞ that can define the sinh-Gordon model for $b > 1$, which is expected to have different properties. As discussed above, the model with $|b| \le 1$ appears to be well-defined as a perturbation of the free massless

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boson with zero Q_∞, or as a perturbed Liouville theory. Taking the simpler option, we assume there is no background charge in this region, i.e.

$$Q_\infty = 0, \qquad \text{for } \ 0 < \mathrm{b} < 1. \tag{3.10}$$

At $b = 1$ we introduce a non-zero Q_∞. It would be very interesting to understand what precise mechanism spontaneously generates this non-zero Q_∞, however we leave aside that question in this work.

The conditions we impose on Q_∞ for $|b| > 1$ are quite natural and are the following:

- Based on the symmetry of the S-matrix, we require Q_∞ to be self-dual, as for the Liouville case. This implies we can expand Q_∞ as a series in $b + 1/b$: $Q_\infty = \sum_{n=0}^{\infty} \alpha_n \, (b + 1/b)^n$. Since there are not enough constraints to fix all α_n, we assume only α_0 and α_1 are non-zero, as in the Liouville theory.

- For continuity with $|b| < 1$, we require the background charge $Q_\infty = 0$ at $b = 1$: $Q_\infty = \alpha_0 + \alpha_1(b + 1/b) = 0$ at $b = 1$, so that $\alpha_0 = -2\alpha_1$. This in turn implies that at the self-dual point $b = 1$, in the UV one has $c = 1$, as for $0 < b < 1$. Thus in the UV, the central charge c is continuous and only changes at the self-dual point $b = 1$.

- To fix α_1 we need a condition at $b = \infty$. We require that under the duality $b \to \widetilde{b} = Q_\infty - b$, the dual coupling constant $\widetilde{b}$ remains in the non-zero Q_∞ region $|\widetilde{b}| \geq 1$. For a fixed b, this amounts to $\widetilde{b} \leq -1$, which implies $\alpha_1 \leq b/(b-1)$. Requiring the above for all b, in particular $b = \infty$, leads to the minimal choice $\alpha_1 = 1$. Although at this stage this appears somewhat ad hoc, as we will see it leads to correct predictions for the random energy model.

In summary, we thus propose

$$Q_\infty = b + 1/b - 2, \qquad \text{for } \ \mathrm{b} > 1, \tag{3.11}$$

and zero otherwise. This is just a shift of the Liouville background charge by the integer -2. Note that as $b \to \infty$, Q_∞ is the same as for the Liouville case. Notice also the dichotomy: for $Q_\infty = 0$ and the Liouville choice $Q_\infty = b + 1/b$, $\widetilde{b} = -b$ and $1/b$ respectively, whereas for the above choice $\widetilde{b} = -2 + 1/b$, which equals $-b$ for $b = 1$.

With this choice of Q_∞ one has

$$[[e^{\sqrt{2}b\phi}]] = 2 - 4b, \qquad [[e^{-\sqrt{2}b\phi}]] = 4b - 2 - 4b^2. \tag{3.12}$$

Thus the dimensions of the parameters $\mu_\pm$ are

$$[[\mu_+]] = 4b, \qquad [[\mu_-]] = 4(b^2 - b + 1). \tag{3.13}$$

For $b \geq 1$, both operators $e^{\pm\sqrt{2}b\phi}$ are then relevant, even though they have different dimension. The ultra-violet limit is controlled by the highest dimension operator, namely the least relevant. This is the operator $e^{+\sqrt{2}b\phi}$. We thus propose that in this frozen phase

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$b > 1$ one effectively has $[[\cosh(\sqrt{2}b\phi)]] \sim 2 - 4b$. The term "frozen" is borrowed from the theory of disordered systems; see below.

In summary, we have proposed that

$$[[e^{\sqrt{2}b\phi}]] = \begin{cases} -2b^2, & \text{for } \ 0 < b < 1, \\ 2 - 4b, & \text{for } \ b > 1. \end{cases} \tag{3.14}$$

Furthermore, as we explained, we identify the above dimensions (3.14) as the effective scaling dimension of $\cosh(\sqrt{2}b\phi)$. This transition is induced by the generation of a background charge for $b > 1$. We conjecture that the sinh-Gordon model, which is well defined for $0 < b < 1$ with $Q_\infty = 0$, is actually ill-defined for $b > 1$ without background charge but well-defined with the background charge $Q_\infty = b + 1/b - 2$ as in eq. (3.11).

4 Massless renormalization group flows in the sine- and sinh-Gordon models

In the present context by "massless flows" we mean the following. Suppose an RG flow originates as a perturbation of an UV fixed point CFT by a relevant operator of dimension $\Gamma_{\rm UV} < 2$ and flows to a different non-trivial fixed point in the infrared (IR), necessarily arriving there via an irrelevant operator of dimension $\Gamma_{IR} > 2$. Generally in the flow to the IR, massive particles decouple, thus if the IR theory is non-trivial some massless particles must survive the flow. In the deep IR, the theory is approximated by the interactions of these massless degrees of freedom.

4.1 Higher order beta functions

The sinh-Gordon model can be viewed as a current-current perturbation of an SU(2) WZW model at level 1 with action,

$$\mathcal{S} = S_{\rm wzw} + \frac{1}{2\pi}\int d^2x \left(g_1\left[J^+\overline{J}^- + J^-\overline{J}^+\right] + g_2 J_3\overline{J}_3\right), \tag{4.1}$$

with $J^\pm = \frac{1}{\sqrt{2}}e^{\pm i\sqrt{2}\varphi}$ and $J_3 = \frac{i}{\sqrt{2}}\partial_z\varphi$, where $\varphi(z)$ is the z-dependent part of a free massless scalar field $\phi(z,\overline{z}) = \varphi(z) + \overline{\varphi}(\overline{z})$. The advantage of doing this is that current algebra Ward identities allow an easier approach to calculating higher order corrections to the beta functions since both couplings $g_{1,2}$ are marginal. The bosonized form of the action is now

$$\mathcal{S} = \frac{1}{4\pi}\int d^2x \left(\frac{1}{2}(\partial\phi)^2 + g_1\,\cosh(\sqrt{2}b\phi)\right). \tag{4.2}$$

The coupling b is a function of g_2 presented below, and can be real or imaginary corresponding to either sinh or sine Gordon phases [14]. For reasons that will become clear, let us postpone this identification for now since such an identification depends on Q_∞, and first describe the general properties of the flows based solely on the beta functions for g_1, g_2.

In our original treatment [14], g_1 was taken to be real. For several reasons in this section we present our conclusions for g_1 imaginary. One reason is that for our purposes we

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are interested in massless RG flows, and this case provides a point of comparison with the known exact results [15, 16]. The second is that for the map to sinh-Gordon for disordered systems, equation (5.9) below, g_1 is indeed imaginary. Thirdly, under the continuation $g_1 \to ig_1$, the poles in the beta functions in [14] no longer exist and one does not have to deal with continuing the flow through these poles. Fortunately as we will comment on at the end of this section, for the sinh-Gordon flows we are interested in the distinction between real and imaginary g_1 does not matter as far as the endpoints of the flows are concerned, even though the details of the RG trajectories do differ.

For the reasons just described above, we extend the results in [14] to $g_1 \to ig_1$, corresponding to imaginary μ in (1.1). Taking $g_1 \to ig_1$, the beta functions in [13] become the reasonably simple functions

$$\frac{dg_1}{d\log a} \equiv \beta_{g_1} = \frac{g_1(g_2 + g_1^2/4)}{(1 + g_1^2/16)(1 + g_2/4)}\,, \tag{4.3a}$$

$$\frac{dg_2}{d\log a} \equiv \beta_{g_2} = -\frac{g_1^2(1 - g_2/4)^2}{(1 + g_1^2/16)^2}\,, \tag{4.3b}$$

where a is a cut-off scale. With these conventions, the flow to the IR corresponds to $a \to \infty$. Flows with $g_1 < 0$ are just a mirror image of those with $g_1 > 0$ since the beta functions are invariant under $g_1 \to -g_1$, thus we will only discuss the case $g_1 > 0$.

The above beta functions have a remarkable strong/weak coupling duality. For both β_{g_1,g_2} (recall that beta functions transform as vector fields):

$$\beta(16/g_1, 16/g_2) = -\beta(g_1, g_2). \tag{4.4}$$

Also, again rather remarkably, there exists an RG invariant which allows us to map out basic features of the flows without explicitly solving the coupled differential equations based on the beta functions. Such an invariant $\mathcal{I}$ satisfies $\sum_g \beta_g \partial_g \mathcal{I} = 0$. One may check that the following $\mathcal{I}$ is an invariant [14]

$$\mathcal{I}(g_1, g_2) = \frac{g_1^2 + g_2^2}{(g_2 - 4)^2(g_1^2 + 16)}. \tag{4.5}$$

This invariant satisfies the strong-weak coupling duality of the beta functions (4.4),

$$\mathcal{I}(16/g_1, 16/g_2) = \mathcal{I}(g_1, g_2). \tag{4.6}$$

The line $g_1 = 0$ is a line of fixed points where both beta functions are zero. By computing the slope of the beta function near $g_1 = 0$, one can determine the dimension Γ_0 of the perturbation $\cosh(\sqrt{2}b\phi)$ there with the general formula

$$\beta(g) = (2 - \Gamma(g_c))(g - g_c) + \cdots\,, \tag{4.7}$$

near a critical point g_c. Since $\beta_{g_1} = [4g_2/(4 + g_2)]\, g_1$ near $g_1 = 0$, equating the slope with $2 - \Gamma_0(g_2)$ yields

$$\Gamma_0(g_2) \equiv \Gamma(g_1 = 0) = 2\left(\frac{4 - g_2}{4 + g_2}\right). \tag{4.8}$$

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Thus, the g_2 axis (at g_1 =0) is divided into several regions, where the perturbations are classified as relevant ($\Gamma_0 < 2$) or irrelevant ($\Gamma_0 > 2$). We thus identify three distinction regions at $g_1 = 0$:

$$\begin{aligned} &\text{– Relevant:} && -\infty < g_2 < -4; \\ &\text{– Irrelevant:} && -4 < g_2 < 0; \\ &\text{– Relevant:} && 0 < g_2 < \infty. \end{aligned}$$

For the relevant regions the flows *originate* at $g_1 = 0$ and flow toward increasing g_1. For the irrelevant region, the flows *terminate* at $g_1 = 0$ arriving from positive g_1.

As we will see, based on the invariant $\mathcal{I}$, many of the flows that originate at $g_1 = 0$ end up at $g_1 = \infty$. Not all however, depending on whether g_1 is real or imaginary, see below. But in the case of interest, namely the sinh-Gordon model, flows indeed start at $g_1 = 0$ in the UV and flow to $g_1 = \infty$ in the IR. Whereas the dimensions of the perturbations around $g_1 = 0$ are unambiguous as a function of g_2 since g_1 is an obvious line of fixed points (see $\Gamma_0(g_2)$ in (4.8)), the scaling dimension of the perturbation at $g_1 = \infty$ is less obvious. We propose the following identification. Based on the duality of the beta functions (4.4), flows at $g_1 = 0$ can be mapped into flows at $g_1 = \infty$, where formally if the beta functions are zero at $g_1 = 0$ they are also zero at $g_1 = \infty$ if one uses $(16/g_1, 16/g_2)$ as coordinates. However the minus sign in (4.4) implies the UV and IR are exchanged, since they are related by $a \to 1/a$. We propose that along the flow the dimension of the perturbation is given by the same functional form as $\Gamma_0(g_2)$ for g_2 as a function of the RG scale. To be more precise, we are assuming that the dimension of the perturbation $\Gamma(t)$, where $t = \log a$ is the RG time, satisfies $\Gamma(t) = \Gamma_0(g_2(t))$. Thus at $t = 0$, $\Gamma = \Gamma_0(g_2(0)) = \Gamma_0(g_2^{\rm UV}) = \Gamma_{\rm UV}$. On the other hand at $t = \infty$,

$$\Gamma = \Gamma_0(g_2(\infty)) \equiv \Gamma_{\rm IR}. \tag{4.9}$$

The flow of $g_2(t)$ thus implies a relation between $\Gamma_{\rm UV}$ and $\Gamma_{\rm IR}$.

For instance if a flow originates at $g_1 = 0$ from a relevant perturbation with $\Gamma_{\rm UV} = \Gamma_0(g_2^{\rm UV})$, and $g_2^{\rm UV}$ flows to $g_2^{\rm IR}$, then in the IR at $g_1 = \infty$ we identify the dimension as $\Gamma_{\rm IR}$ where

$$\Gamma_{\rm IR} = \Gamma_0(g_2^{\rm IR}) = 2\left(\frac{4 - g_2^{\rm IR}}{4 + g_2^{\rm IR}}\right), \quad \text{at } g_1 = \infty. \tag{4.10}$$

Let us make several remarks supporting our rather natural proposal (4.9) since it will be essential in the following:

- The equation (4.9) correctly predicts the exact relation between $\Gamma_{\rm UV}$ and $\Gamma_{\rm IR}$ for massless flows that both begin and end at $g_1 = 0$. These are the flows in [15, 16], see below.

- At $g_1 = \infty$, the coupling g_2 stops flowing, i.e. remains constant, as it does at $g_1 = 0$, which implies the dimensions $\Gamma_{\rm UV}$ and $\Gamma_{\rm IR}$ are constant there. One can see this as follows. One can eliminate g_1 and write the beta function in terms of g_2 and $\mathcal{I}$ only:

$$\beta_{g_2} = \frac{16\left(g_2^2 - 16\mathcal{I}(g_2 - 4)^2\right)\left(1 - \mathcal{I}(g_2 - 4)^2\right)}{(g_2 + 4)^2} \tag{4.11}$$

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Now one has

$$\mathcal{I}_0 \equiv \mathcal{I}(g_1 = 0, g_2) = \frac{g_2^2}{16(g_2-4)^2}, \qquad \mathcal{I}_\infty \equiv \mathcal{I}(g_1 = \infty, g_2) = \frac{1}{(g_2-4)^2}. \tag{4.12}$$

One sees that for both $\mathcal{I} = \mathcal{I}_0$ and $\mathcal{I} = \mathcal{I}_\infty$, the beta function $\beta_{g_2} = 0$, thus g_2 is a constant in RG time both at $g_1 = 0$ and $g_1 = \infty$. Alternatively, β_{g_2} vanishes either for $16\mathcal{I}(g_2-4)^2 = g_2^2$ or for $\mathcal{I}(g_2-4)^2 = 1$. The former corresponds to $g_1 = 0$, the latter to $g_1 = \infty$.

- Since under the duality $g_2 \to 16/g_2$ the IR and UV limits are exchanged due to (4.4), one should expect that $\Gamma_0(16/g_2) = -\Gamma_0(g_2)$, which is satisfied.
- When $g_2 = 4$, g_2 does not flow at all since $\beta_{g_2} = 0$ for all g_1. Thus it must be that $\Gamma_{\rm UV} = \Gamma_{\rm IR} = \Gamma_0(4) = 0$. As we will see below, this corresponds to the $b = 0$ point of the sinh-Gordon theory which is just a free *massive* boson.

Notice that at $g_1 \to \infty$ in the IR, the theory might either be the trivial massive theory, with all degrees of freedom frozen, or a non trivial theory, potentially a conformally invariant theory.

4.2 Identification of sinh and sine Gordon phases

It remains to identify where the above model with g_1, g_2 corresponds to the sinh-Gordon model. This identification clearly depends on whether we assume the presence of a background charge or not, and this fact will be important later. If we view the cosh potential as a perturbation of the free gaussian field with no background charge, then $\Gamma_0 = -2b^2$:

$$b^2 = \frac{(g_2-4)}{(g_2+4)}. \tag{4.13}$$

Again — at the price of repeating ourselves — this identification relies on a specifically chosen relation between the scaling dimension and the parameter b (which here assumes the absence of background charge). If this dimension is positive we view the potential as being in a sine-Gordon regime $g_1 \cos(\sqrt{2}\beta\phi)$ with $\beta = ib$. There are now four distinct regions:

- Relevant: $-\infty < g_2 < -4$, sinh-Gordon with $1 < b^2 < \infty$;
- Irrelevant: $-4 < g_2 < 0$, sine-Gordon with $1 < \beta^2 < \infty$;
- Relevant: $0 < g_2 < 4$, sine-Gordon with $0 < \beta^2 < 1$;
- Relevant: $4 < g_2 < \infty$, sinh-Gordon with $0 < b^2 < 1$.

The above regions are the same as those already identified in [14]. Note already that the regions $b < 1$ and $b > 1$ are clearly distinguished.

For $g_1 = 0$ in the UV, the duality $g_2 \to 16/g_2$ corresponds to $b \to ib$, i.e. maps from the sinh-Gordon to sine-Gordon regimes. On the other hand, the usual hypothetical sinh-Gordon duality $b \to 1/b$ corresponds to $g_2 \to -g_2$. However the latter is not a symmetry of the beta functions and indicates that the RG properties of $0 < b < 1$ verses $b > 1$ are indeed different. This is one of the main points of this paper which we will subsequently explore in more detail.

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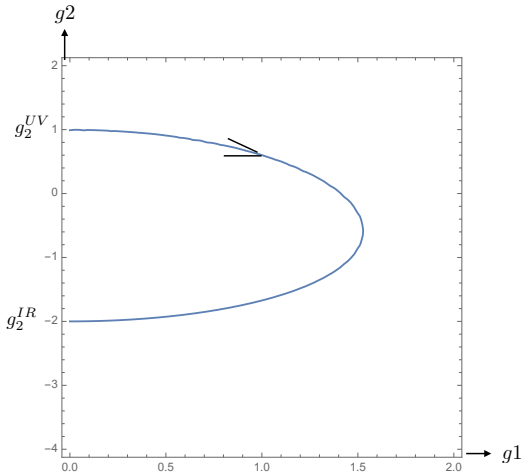

Figure 1. Massless flow in the imaginary sine-Gordon model from $g_2^{\rm UV} = 1$ to $g_2^{\rm IR} = -2$. This corresponds to a flow from $\beta_{\rm UV} = \sqrt{3/5}$ to $\beta_{\rm IR} = \sqrt{3}$. What is shown is a contour plot of the RG invariant $\mathcal{I}$.

4.3 Massless flows in the "imaginary" sine-Gordon model

Following the terminology in [15, 16] we refer to the sinh-Gordon action (1.1) with $\mu \propto g_1$ imaginary as the "imaginary" sine-Gordon model. In this case there are flows that both begin and end at $g_1 = 0$, indicating a massless flow between two different CFT's, both at $c = 1$, which differ in their radius of compactification β. Here since the flows both start and end at $g_1 = 0$, there is no ambiguity in determining anomalous dimensions in the UV nor the IR. This situation was already explained in [14] based on the beta functions above, however we review it here since it represents a prototype of the kinds of flows we will propose in the sinh-Gordon case.

In the sine-Gordon regime with small couplings g_1 and g_2, $\mathcal{I} \propto (g_1^2 + g_2^2)$, thus the RG flows are approximately circles. This implies that flows can both begin and end on the g_2 axis, which is a massless flow as defined above. Such flows are straightforward to analyze to all orders. For $g_1 = 0$, $\mathcal{I} = \mathcal{I}_0$ defined in (4.12). Since $\mathcal{I}$ is preserved along the flow, one must have

$$\frac{g_2^{\rm UV}}{g_2^{\rm UV} - 4} = -\frac{g_2^{\rm IR}}{g_2^{\rm IR} - 4} \quad \Longrightarrow \quad g_2^{\rm IR} = \frac{2g_2^{\rm UV}}{g_2^{\rm UV} - 2}. \tag{4.14}$$

In terms of β,

$$\beta_{\rm IR}^2 = \frac{\beta_{\rm UV}^2}{2\beta_{\rm UV}^2 - 1}, \tag{4.15}$$

which implies the dimensions of the perturbation in the UV verses IR are related as follows

$$\Gamma_{\rm IR} = \frac{\Gamma_{\rm UV}}{\Gamma_{\rm UV} - 1}. \tag{4.16}$$

For irrelevance in the IR, $\Gamma_{\rm IR} > 2$, requires $0 < g_2 < 4/3$ or equivalently $1/2 < \beta^2 < 1$, consistent with [15, 16]. A contour plot of such a flow is shown in figure 1. The existence of the flows and the relation (4.16) have been conjectured long ago in [15, 16]. The fact that

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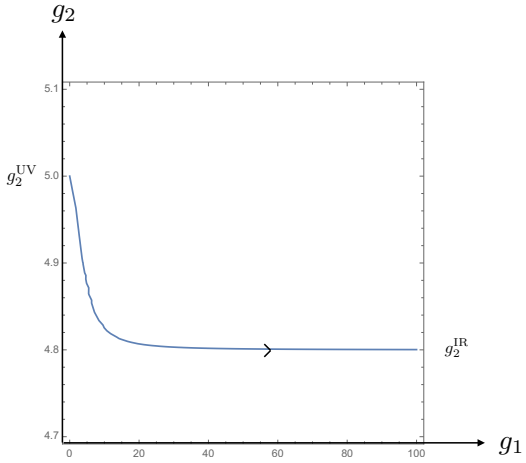

Figure 2. A massive flow in the imaginary sine-Gordon model from $g_2^{\rm UV} = 5$ to $g_2^{\rm IR} = 24/5$. This corresponds to a flow from $b_{\rm UV} = 1/3$ to $b_{\rm IR} = 1/\sqrt{11}$. What is shown is a contour plot of the RG invariant $\mathcal{I}$ where the x, y axes corresponds to g_1, g_2.

we recover them and the correct relation (4.16) provides further support for the effectiveness of the beta functions (4.3a) in understanding this kind of physics.

4.4 RG flows in the sinh-Gordon model: $b < 1$ verses $b > 1$

Here we consider flows in the different regimes $0 < b < 1$ and $1 < b < \infty$ for the sinh-Gordon with imaginary μ. As explained at the end of this section, the case of real μ is not very different. Recall these regimes correspond to $g_2 > 4$ and $g_2 < -4$ at $g_1 = 0$, respectively. For such large coupling g_2, constant $\mathcal{I}$ is not at all approximated by a circle as in the sine-Gordon case. All flows originating at $g_1 = 0$ end up at $g_1 = \infty$. As we now explain, there are two cases which have rather different behavior, and correspond precisely to $0 < b < 1$ verses $b > 1$. We need to relate g_2 in the UV and IR. Since $\mathcal{I}$ is preserved along the flow, one must have $\mathcal{I}_0(g_2^{\rm UV}) = \mathcal{I}_\infty(g_2^{\rm IR})$. A fortunate and promising result that has not been pre-programmed into the above beta functions is that the two cases correspond precisely to weak verses strong coupling:

- For $0 < b < 1$. The flows originating at $(g_1, g_2) = (0, g_2^{\rm UV} > 4)$ end up at $(\infty, g_2^{\rm IR}))$ where
$$\frac{g_2^{\rm UV}}{4(g_2^{\rm UV} - 4)} = \frac{1}{(g_2^{\rm IR} - 4)} \quad \Longrightarrow \quad g_2^{\rm IR} = \frac{8(g_2^{\rm UV} - 2)}{g_2^{\rm UV}}. \tag{4.17}$$
Expressing this in terms of the scaling dimensions:
$$\Gamma_{\rm IR} = \frac{\Gamma_{\rm UV}}{1 - \Gamma_{\rm UV}}\,. \tag{4.18}$$
Identifying the parameter b using the relation $\Gamma = -2b^2$ yields $b_{\rm IR}^2 = b_{\rm UV}^2/(1 + 2b_{\rm UV}^2)$. Both $b_{\rm UV}$ and $b_{\rm IR}$ then remain in the weak coupling region $0 < b < 1$. This should be a massive flow since $\Gamma_{\rm IR}$ still signifies a relevant perturbation, although with an imaginary coupling μ. One instance of it is shown in figure 2.

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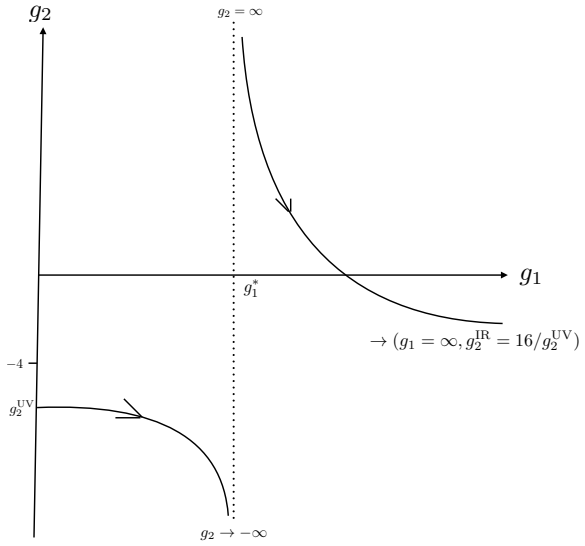

Figure 3. Sketch of the massless flow in the imaginary sinh-Gordon model from $g_2^{\rm UV} < -4$ to $g_2^{\rm IR} = 16/g_2^{\rm UV}$. The value g_1^* is given in the text: $g_1^* = 8\sqrt{4-2g_2}/|g_2|$.

- For $1 < b < \infty$. The flows originating at $(g_1, g_2) = (0, g_2^{\rm UV} < -4)$ end up at $(\infty, g_2^{\rm IR}))$ where there is just a change of sign in (4.17):

$$\frac{g_2^{\rm UV}}{4(g_2^{\rm UV}-4)} = -\frac{1}{(g_2^{\rm IR}-4)} \quad \Longrightarrow \quad g_2^{\rm IR} = \frac{16}{g_2^{\rm UV}}. \tag{4.19}$$

  Expressing this in terms of the dimensions Γ:

$$\Gamma_{\rm IR} = -\Gamma_{\rm UV}. \tag{4.20}$$

  Now in this case, $\Gamma_{\rm IR} > 2$, i.e. irrelevant, and this is thus a massless flow. This flow in the sinh-Gordon region is rather analogous to the massless flows in the sine-Gordon model described above, since in the IR they both end up in the irrelevant regime where $-4 < g_2^{\rm IR} < 0$. The details of the flow are however more intricate compared to the previous case. Rather the flows start at $(g_1, g_2) = (0, g_2^{\rm UV})$ and first flow to $g_2 = -\infty$. This occurs at $g_1 = g_1^*$ such that $\mathcal{I}(0, g_2) = \mathcal{I}(g_1^*, \pm\infty)$, that is $g_1^* = 8\sqrt{4-2g_2}/|g_2|$. Using the cylindrical topology proposed in [14] which identifies g_2 with $-g_2$ at $|g_2| = \infty$, the flow then continues from $g_2 = \infty$ to $g_1 = \infty$ but with a different $g_2 = g_2^{\rm IR}$, which is actually the dual of g_2. This implies the flow

$$(g_1, g_2) = (0, g_2) \quad \overset{UV \to IR}{\longrightarrow} \quad (\infty, 16/g_2). \tag{4.21}$$

  The self-dual point $b_{\rm UV} = 1$ flows to a marginally irrelevant perturbation in the IR, i.e. $\Gamma_{\rm IR} = 2^+$, which seems desirable if it is indeed a massless flow. Such flows are sketched in figure 3 and were verified numerically.

We now explain why the above RG flows cannot be properly interpreted if we stick to the relation $\Gamma = -2b^2$, and argue that they acquire a natural interpretation if we introduce the background charge $Q_\infty = b + 1/b - 2$. This is one of the main points of this paper.

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Let us present supporting arguments for the introduction of a background charge. If we continue to identify the scaling dimension Γ_0 with $-2b^2$ then the above relation (4.20) implies the peculiarity of $b_{\rm IR} = ib_{\rm UV}$, i.e. becoming imaginary. The flow to imaginary b seems unsatisfactory since it takes us out of the proper sinh-Gordon regime manifold and into the sine-Gordon one. This would correspond, roughly speaking, from a flow from a non-compact model to a compact one. We suggest that this problem arose since we identified the coupling b with the dimension $\Gamma_0 = -2b^2$ which assumed there was no background charge. Introduction of the background charge Q_∞ in (1.2) can resolve this issue.

The perturbative calculations that led to the above beta functions (4.3a) did not incorporate a background charge. However the flows do predict dimensions of operators $\Gamma_{\rm UV,IR}$ regardless of the free gaussian identification $\Gamma = -2b^2$. Incorporating a background charge should just modify this identification, while preserving the flows in (g_1, g_2). We can indeed modify this identification, but still must preserve the relation $\Gamma_{\rm IR} = -\Gamma_{\rm UV}$, since the latter is predicted by the beta functions regardless of the identification relating Γ and b.

We require that both the UV and IR are in the same regime of "b". Let us identify Γ_0 with the dimension proposed in section 3

$$\Gamma_0 = 2 - 4b \tag{4.22}$$

which was based on a background charge $Q_\infty = b + 1/b - 2$ in the region $b > 1$.

The identification (4.22) modifies the relation between $g_2^{\rm UV}$ and $b_{\rm UV}$, as $4b_{\rm UV} = 2 - \Gamma_0(g_2^{\rm UV})$, which then reads $b_{\rm UV} = g_2^{\rm UV}/(4 + g_2^{\rm UV})$. For $g_2^{\rm UV} < -4$, we still have $b_{\rm UV}$ in the strong coupling regime $b_{\rm UV} > 1$.

Then $\Gamma_{\rm IR} = -\Gamma_{\rm UV}$ in (4.20) implies the simple relation

$$b_{\rm IR} = 1 - b_{\rm UV}. \tag{4.23}$$

This has the desired property that the whole region $b_{\rm UV} > 1$ is mapped to $b_{\rm IR} < 0$ which excludes the usual sinh-Gordon region $0 < b < 1$. Importantly, note that $b_{\rm UV} = 1$ is mapped to $b_{\rm IR} = 0$, where $\Gamma_{\rm IR} = 2$, thus the $b_{\rm UV} = 1$ theory is marginally irrelevant in the IR, consistent with a massless flow.

One can also argue that the background charge must be $Q_\infty = \alpha_1(b + 1/b - 2)$ with $\alpha_1 = 1$ as follows. We identify $\Gamma = 2b(Q_\infty - b)$ based on above considerations. The RG flows predict $\Gamma_{\rm IR} = -\Gamma_{\rm UV}$, which is a complicated relation between $b_{\rm IR}$ and $b_{\rm UV}$ for generic value of α_1. One can check that unless $\alpha_1 = 1$, for $b_{\rm UV} > 1$, $b_{\rm IR}$ is generally complex. Only for $\alpha_1 = 1$ does one have the simple relation $b_{\rm IR} = 1 - b_{\rm UV}$.

4.5 Remarks on real verses imaginary μ

Let use make a few remarks concerning the g_1 real case originally considered in [14]. First of all, the massless flows for the imaginary sine-Gordon theory no longer exist, since for small $g_{1,2}$ the RG trajectories based on the RG invariant $\mathcal{I}$ are no longer approximately circles, but rather hyperbolas. However the flows that begin at $g_1 = 0$ and end up at $g_1 = \infty$ have the same endpoints, and the relations between $\Gamma_{\rm UV}$ and $\Gamma_{\rm IR}$ presented above remain the same. This can be seen from the fact that equations (4.12) are the same. However the

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detailed trajectories are different. One can easily see with contour plots of $\mathcal{I}$ for real verses imaginary g_1 that the topologies of the flows in figures 2 and 3 are essentially interchanged.

5 Application to the freezing transition in disordered systems

We now make contact with disordered systems and explain the relation between the above sinh-Gordon model, with imaginary coupling, and Dirac fermions in random gauge field. See also [11, 12]. This was actually our motivation when we started looking at this problem fifteen years ago, and left it aside for a short while.

5.1 Dirac fermions in a random U(1) gauge field

We consider two-component Dirac fermions in a random gauge field A_μ in two spatial dimensions (x, y) plus time. Defining complex spatial coordinates $z = (x + iy)/\sqrt{2}$ and $\overline{z} = (x - iy)/\sqrt{2}$, the model is defined by the random hermitian hamiltonian

$$H = \begin{pmatrix} 0 & -i\partial_{\overline{z}} + A_{\overline{z}} \\ -i\partial_z + A_z & 0 \end{pmatrix}. \tag{5.1}$$

The probability distribution will be specified below.

The Green functions, Fourier transformed in time to energy $\mathcal{E}$, are given by functional integrals with respect to the action

$$\mathcal{S} = i \int \frac{d^2x}{2\pi} \Psi^\dagger \left(H - \mathcal{E}\right) \Psi. \tag{5.2}$$

Introducing component fields as follows, $\Psi = \begin{pmatrix} \psi_+ \\ \overline{\psi}_+ \end{pmatrix}$ and $\Psi^\dagger = \left(\overline{\psi}_-, \psi_-\right)$, one finds

$$\mathcal{S}(\Psi, A) = \int \frac{d^2x}{2\pi} \left[\overline{\psi}_-(\partial_z - iA_z)\overline{\psi}_+ + \psi_-(\partial_{\overline{z}} - iA_{\overline{z}})\psi_+ + i\mathcal{E}\left(\overline{\psi}_-\psi_+ + \psi_-\overline{\psi}_+\right)\right]. \tag{5.3}$$

Disorder averaged correlation functions $\langle\mathcal{O}\rangle$ are then defined as functional integrals over A:

$$\overline{\langle\mathcal{O}\rangle} = \int DA\, P[A]\, \langle\mathcal{O}\rangle_A \tag{5.4}$$

where the probability distribution for A is taken to be gaussian:

$$P[A] = \exp\left(-\frac{1}{g}\int \frac{d^2x}{2\pi}\, A_z A_{\overline{z}}\right). \tag{5.5}$$

The coupling constant g is a measure of the strength of the disorder. In (5.4), $\langle\mathcal{O}\rangle_A$ is the correlation function in a given realization of the disorder:

$$\langle\mathcal{O}\rangle_A = \frac{1}{Z(A,\mathcal{E})} \int D\Psi\, e^{-S(\Psi,A)}\, \mathcal{O} \tag{5.6}$$

where $Z(A, \mathcal{E})$ is the partition function.

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5.2 Map to the sinh-Gordon model

It is convenient to parameterize the gauge field in terms of a scalar field η as follows:[1]

$$A_\mu = \frac{1}{2}\epsilon_{\mu\nu}\partial_\nu\eta, \qquad \Longrightarrow \qquad P[\eta] = \exp\left(-\frac{1}{4g}\int \frac{d^2x}{4\pi}\,(\partial_\mu\eta)^2\right). \tag{5.7}$$

The coupling of the fermions to the gauge field can then be removed by the chiral gauge transformation:

$$\overline{\psi}'_+ = e^{\eta/2}\,\overline{\psi}_+, \quad \overline{\psi}'_- = e^{-\eta/2}\,\overline{\psi}_-, \quad \psi'_- = e^{\eta/2}\,\psi_-, \quad \psi'_+ = e^{-\eta/2}\,\psi_+\,, \tag{5.8}$$

and the action becomes

$$\mathcal{S} = \int \frac{d^2x}{2\pi}\left[\overline{\psi}'_-\partial_z\overline{\psi}'_+ + \psi'_-\partial_{\overline{z}}\psi'_+ + i\mathcal{E}\left(e^{\eta}\,\overline{\psi}'_-\psi'_+ + e^{-\eta}\,\psi'_-\overline{\psi}'_+\right)\right]. \tag{5.9}$$

To make further progress, we first consider $\mathcal{E}$ to be very small, and later restore it as a perturbation. When $\mathcal{E} = 0$, the jacobian which arises in passing from Ψ to Ψ' in the functional integral precisely cancels the $1/Z$ factor in (5.6). This is easily seen by bosonizing the fermions Ψ with a single boson ϕ so that the action (5.3) becomes (when $\mathcal{E} = 0$):

$$\mathcal{S} = \int \frac{d^2x}{4\pi}\left(\frac{1}{2}\,(\partial\phi)^2 + i\partial\eta\partial\phi\right). \tag{5.10}$$

The functional integrals over Ψ and Ψ' are then simply related by the shift $\phi \to \phi' - i\eta$.

When $\mathcal{E} = 0$ the functional integrals over Ψ' can be done and do not introduce any new η dependence. To restore the $\mathcal{E}$ perturbation, we make a mean field approximation and replace the Ψ' fermion bilinears by their one-point functions in a finite geometry of size L. Since the fermions have dimension $1/2$, we have:

$$\langle\overline{\psi}'_-\psi'_+\rangle \sim \langle\psi'_-\overline{\psi}'_+\rangle \sim 1/L\,. \tag{5.11}$$

One is finally left with the functional integral over η. Rescaling $\eta = \sqrt{2g}\,\phi$, one finds the sinh-Gordon action

$$\mathcal{S}[\phi] = \int d^2x\left(\frac{1}{8\pi}\,(\partial\phi)^2 + 2\mu\cosh\left(\sqrt{2}\,b\,\phi\right)\right), \tag{5.12}$$

where

$$b = \sqrt{g}, \qquad \mu = i\,\frac{\mathcal{E}}{2\pi L}\,. \tag{5.13}$$

The density of states operator is the one that couples to $\mathcal{E}$, which we chose to normalize as follows:

$$\rho \equiv \frac{1}{L}\cosh\left(\sqrt{2}\,b\,\phi\right). \tag{5.14}$$

By definition one has $\overline{\langle\rho\rangle} = \int D\phi e^{-S[\phi]}\,\rho$.

[1] In $\mathbb{R}^2$, any gauge potential can be decomposed as $A_\mu = \partial_\mu\vartheta + \frac{1}{2}\epsilon_{\mu\nu}\partial_\nu\eta$. But the pure gauge part $A_\mu = \partial_\mu\vartheta$ can be gauged away in (5.3) and only the component $A_\mu = \frac{1}{2}\epsilon_{\mu\nu}\partial_\nu\eta$ matters.

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5.3 Multi-fractal density of states exponents

We first review some general standard definitions of exponents characterizing the density of states. Let $\rho(x)$ denote the density of states field operator. The physical density of states is its vacuum expectation value, i.e. the 1-point function denoted as $\langle\rho\rangle$, and depends on the realization of the disorder. Let $\overline{\langle\rho\rangle}$ denote the disorder averaged quantity. For a system of size L, one defines the fundamental exponent Γ_1 as

$$\overline{\langle\rho\rangle} \sim L^{-\Gamma_1}. \tag{5.15}$$

In other words, the exponent Γ_1 is just the anomalous dimension of the operator ρ in the disorder averaged theory.

Also of interest are multi-fractal exponents Γ_q defined as follows: Γ_q is defined as the anomalous dimension of the q-th moment of ρ:

$$\Gamma_q = [[\,\overline{\langle\rho\rangle^q}\,]]\,, \tag{5.16}$$

where we use the same notation as above, where $[[X]]$ denotes the scaling dimension of X in inverse length units.

Because it is related to the multi-fractal spectrum of the density $\langle\rho(x)\rangle$, or of the associated measure $\langle\rho(x)\rangle\, dx$, a related quantity that is often studied is the normalized ratio

$$P^{(q)} = \frac{\int d^2x\, \overline{\langle\rho(x)\rangle^q}}{\left(\int d^2x\, \overline{\langle\rho(x)\rangle}\right)^q}\,. \tag{5.17}$$

Simple scaling leads to

$$P^{(q)} \sim L^{-\tau(q)} \tag{5.18}$$

where

$$\tau(q) = \Gamma_q - q\Gamma_1 + 2(q-1). \tag{5.19}$$

Legendre transform of $\tau(q)$ gives access to the spectrum of multi-fractal dimensions of the density $\langle\rho(x)\rangle$.

5.4 Multi-fractal spectrum

Returning to our model of interest, using the mapping to the sinh-Gordon model and (5.14), we have

$$\Gamma_1(g) = 1 + \gamma(g)\,, \tag{5.20}$$

where the "1" comes from the $1/L$ in (5.14), and $\gamma(g)$ the scaling dimension of $\cosh(\sqrt{2g}\,\phi)$,

$$\gamma(g) = [[\cosh(\sqrt{2g}\,\phi)]]\,. \tag{5.21}$$

For higher q, since the leading term in ρ^q is $\cosh(q\sqrt{2g}\,\phi)/L^q$, one has

$$\Gamma_q(g) = q + [[\cosh(q\sqrt{2g}\,\phi)]]\,. \tag{5.22}$$

Since in the above equation the cosh-operator is related to ρ by $g \to q^2 g$, this immediately leads to the fundamental equation

$$\Gamma_q(g) = \Gamma_1(q^2 g) + q - 1. \tag{5.23}$$

The latter implies

$$\tau(q) = \Gamma_1(q^2 g) - q\Gamma_1(g) + 3(q-1). \tag{5.24}$$

Given (5.20), one sees that everything boils down to the dimension of the $\cosh(\sqrt{2}\, b\phi)$ operator in the sinh-Gordon theory. According to our proposal for the freezing transition in sinh-Gordon, we have $\Gamma_1(g) = 1 - 2g$ for $g < 1$ and $\Gamma_1(g) = 3 - 4\sqrt{g}$ for $g > 1$.

Transitions in the variable q are thereby related to transitions in $b = \sqrt{g}$. Using our proposal for a freezing transition in the sinh-Gordon model (3.14), the two transition points are then $b = g = 1$ and $q^2 g = q^2 b^2 = 1$. There are thus 4 distinct regimes. In terms of g and q, they are:

$$g < 1, \;\; q < 1/\sqrt{g}: \qquad \tau(q) = 2(q-1)(1-qg) \tag{5.25a}$$

$$g < 1, \;\; q > 1/\sqrt{g}: \qquad \tau(q) = 2q(1-\sqrt{g})^2 \tag{5.25b}$$

$$g > 1, \;\; q < 1/\sqrt{g}: \qquad \tau(q) = -2(1-q\sqrt{g})^2 \tag{5.25c}$$

$$g > 1, \;\; q > 1/\sqrt{g}: \qquad \tau(q) = 0. \tag{5.25d}$$

This agrees with known results [11, 12, 23].

6 Summary and discussion

We have presented a specific proposal for the behavior of the sinh-Gordon model above the self-dual point $b > 1$ that is quite different from the analytic continuation $b \to 1/b$ of the well-understood properties of the massive theory for $0 < b < 1$. The main properties of this theory is that unlike the $0 < b < 1$ region it has a non-zero background charge Q_∞ given in (1.2). The theory is massless but not conformally invariant, but rather is a relevant perturbation in the UV that flows to another CFT in the IR, arriving there via an irrelevant operator. We provided two supporting arguments. The first was based on the beta functions in [13, 14], which are ultimately based on perturbation theory for the sinh-Gordon action, and do not show a $b \to 1/b$ symmetry, and clearly predict different RG flows for $b < 1$ verses $b > 1$. The second is that our proposal correctly reproduces known exact results for a Dirac fermion in a random magnetic field, in particular all the transitions in the multi-fractal exponents.

If our proposal is indeed correct, it remains to determine the S-matrices for the massless flow when $b > 1$ along the lines formulated in [29]. This is beyond the original scope of this paper, however there are some natural guesses. Letting L and R signify left verses right movers as in [29], it is likely that the LL and RR S-matrices are $S_{\rm LL} = S_{\rm RR} = S_{\rm shG}$ where $S_{\rm shG}$ is the function of rapidity in (2.1). This would guarantee that in the IR, $c = 1$. It remains to specify left-right scattering $S_{\rm LR}$ which controls the UV. It is natural to consider $S_{\rm LR} = S_{\rm shG}$ here also, however there are clearly other possibilities to be explored, such

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as the very simplest possibility $S_{\rm LR}(\theta) = -\tanh\left(\frac{\theta}{2} - \frac{i\pi}{4}\right)$. Clearly more work needs to be done in this direction.

There are some natural questions that would be worthwhile to investigate to provide further support for our proposal. We can think of these:

- We should say that the validity of the beta functions we used in section 4.1 and our interpretation of scaling dimensions at $g_1 = \infty$, namely based on (4.9), could benefit from closer scrutiny, even though we showed how these beta functions can reproduce known exact results on massless flows in the sine-Gordon model [15, 16]. We refer to the Introduction for further remarks about this.

- Konik et al. [7] essentially showed that for the sinh-Gordon theory, perturbation theory of the Liouville theory and the free gaussian field agree in the weak coupling region $b < 1$. Can this analysis be extended to $b > 1$ with the different background charge proposed here?

- Can the semi-classical analysis in the appendix be extended to higher order in perturbation theory? It's unlikely this can fully confirm our exact proposal to all orders, but a few low orders could provide convincing evidence.

- Our suggestion in the last paragraph for the exact S-matrix clearly needs more investigation. A clear way to proceed is with the Thermodynamic Bethe Ansatz.

- It would be interesting to investigate the problem by completely different means, for instance from a lattice formulation of the sinh-Gordon model, or using continuous network tensor techniques adapted to field theory [30, 31]. Or, perhaps a rigorous probabilistic construction as in [32] is possible.

There are other possible applications of the freezing transition that our work may shed some light on. An obvious one is to more complicated disordered systems such as the quantum Hall transition. We also mention that it has been applied to extreme values of the Riemann zeta function [33, 34].

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A Semi-classical freezing and the Manning condensation

Let us imagine computing semi-classically the one-point function of an exponential operator in the sinh-Gordon theory. Restoring $\hbar$ so that the action becomes $\mathcal{S} \to \hbar^{-1}\mathcal{S}$, the one-point function of the operator $\exp(\sqrt{2}a\hbar^{-1}\phi)$, $a > 0$, located at the position x_0, is represented by the functional integral

$$\int D\phi\, e^{-\hbar^{-1}\left(\mathcal{S}-\sqrt{2}a\phi(x_0)\right)} . \tag{A.1}$$

In the semi-classical limit $\hbar \to 0$, the integral is dominated by the saddle point field configuration $\phi_{\rm cl}$, solutions of

$$-\frac{1}{4\pi}\Delta_x\phi_{\rm cl}(x) + 2\widehat{\mu}\,\sinh(\sqrt{2}b\phi_{\rm cl}(x)) = \sqrt{2}a\,\delta^{(2)}(x-x_0)\,, \tag{A.2}$$

where Δ_x is the Laplacian in 2D and $\delta^{(2)}(x-x_0)$ the Dirac measure at x_0 and $\widehat{\mu} = \sqrt{2}b\mu$. Equation (A.2) can be solved exactly using tau function techniques [35], but we do not need this explicit solution for the simple argument we now present. To take care of the δ-function source, we should have $\phi_{\rm cl}(x) \simeq -\sqrt{2}a\log|x-x_0|^2$ as $|x| \to x_0$. Thus we set $\phi_{\rm cl}(x) = -\sqrt{2}a\log|x-x_0|^2 + \varphi(x)$, with $\varphi(x)$ sub-leading near x_0. We take φ decreasing as a power law, so that

$$\phi_{\rm cl}(x) = -\sqrt{2}a\log|x-x_0|^2 + c_0 + c_1\,|x-x_0|^\sigma + \cdots\,, \tag{A.3}$$

with c_0, c_1 two constants and $\sigma > 0$ (so that φ is sub-leading as x approaches x_0) and where the dots refer to higher sub-leading terms near x_0. The exponent σ is found by matching the leading terms in $\Delta_x\phi_{\rm cl}$ and in $\sinh(\sqrt{2}b\phi_{\rm cl})$. This yields

$$|x-x_0|^{\sigma-2} \sim e^{-2ab\log|x-x_0|^2} = |x-x_0|^{-4ab} \quad \Longrightarrow \quad \sigma = 2(1-2ab)\,. \tag{A.4}$$

Since we should have $\sigma > 0$, this is possible only for $a < a_c = 1/2b$. For $a > a_c$, the operator $\exp(\sqrt{2}a\hbar^{-1}\phi)$ is actually screened such that its effective weight $a_{\rm eff}$ at large scale is a_c.

This semi-classical computation indicates the possibility of a freezing transition. For any fixed sinh-Gordon parameter b, the exponential operators $\exp(\sqrt{2}a\,\phi)$ are well-defined for $a < a_c$ only, for some critical value a_c, but they get frozen for $a > a_c$ to the critical exponential operator $\exp(\sqrt{2}a_c\,\phi)$ with critical weight a_c. In view of the symmetry relation (3.8), valid in Liouville theory, it is tempting to propose that $a_c = Q_\infty/2$. This is compatible with the semi-classical limit $a_c \simeq 1/2b$ for $b \to 0$.

This phenomena is known in the physics of polyelectrolyte solutions as the Manning condensation [24]. Imagine considering a positively charged polymer, say a DNA, immersed in a polyelectrolyte made of positive and negative charged ions, and ask what is the electrostatic potential for this system. If we imagine the polymer to be straight along the z-axis, then (A.2) is the Poisson-Boltzmann equation for this electrostatic problem in the 2D transverse directions. If the charge density of the polymer is too high, larger than a critical value a_c, it is screened by oppositely charged ions which occupy a cylindrical volume around the polymer of diameter r_c, so that the system formed by the polymer and these counter-ions behaves at a distance higher than r_c like a polymer of critical charge density a_c. This is the Manning's screening effect.

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