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# Exploring proof in elementary schools within Research-Situations for the Classroom – The case of polyominoes

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**Abstract.** Although there has been considerable interest in how students learn to prove, the elementary school is less studied than the other schooling levels. Several research in mathematics education points out the pressing need of research about proof and teacher's guidance in elementary school. This article proposes to implement a research situation coming from ongoing discrete mathematics involving polyominoes in two ordinary elementary classrooms. Such Research-Situations for the Classroom (RSCs) are designed and studied by mathematicians and researchers in education for twenty years in France. The aims for research with such an implementation which promotes an authentic mathematical activity are twofold: to identify what kinds of proving processes and arguments young students are able to produce within RSCs and to discuss the ways the teacher manages and supports students' proving processes in a RSC. A theoretical background is defined by taking three components into account: definitions of proof and arguments and their specificities in elementary school, characterization of RSCs, use of the Theory of Didactical Situations to implement and manage RSCs. The results of the experiment in both classrooms highlight the pragmatic arguments produced by the students, the obstacles overcome during the experiment, the teacher's guidance and the limits of the use of RSCs in ordinary elementary classrooms. Then, the article presents new perspectives for mathematics education regarding RSCs and their potentialities to foster the teaching and the learning of proof for lower grades and for teacher training.

**Keywords:** proof; Research-Situation for the Classroom; polyominoes; elementary school; discrete mathematics.

## 1. INTRODUCTION

In different countries, policy makers and curriculum frameworks are calling for the promotion of a “mathematical maturity and expertise throughout the elementary, middle and high school years” (NGA Center & CCSSO, 2020). In France, the recent Mathematics Strategy Plan (Villani & Torossian, 2018) launched by the Minister of Education faces the difficulties encountered by the teaching of mathematics pointed out by the national and international assessments. This plan asks to give proof its rightful place: students are expected to learn a “mathematical activity” from kindergarten to university, especially skills such as researching and reasoning (arguing, proving, communicating).

Obviously, the literature in mathematics education about proof has produced fundamental findings regarding the epistemological, philosophical and didactical aspects of proof (e.g. Hanna & De Villiers, 2012; Hanna, Jahnke & Pulte, 2010). However, elementary school is less studied than the other schooling levels and requires more research: the design and the support of proving tasks at elementary level remains complex for “non-expert” teachers and students who are new with proof (e.g. Stylianides, 2016). Mariotti et al. (2018)'s synthesis of international research about proof conclude about the need of new research to analyze “(...) the teacher's role in designing and managing didactic situations concerning argumentation and proof” (p.86).

Besides, in the last twenty years, new perspectives for the teaching and the learning of proof have also emerged from the research based on interviews and interactions with active

mathematicians (e.g. Burton 2004; Weber, 2008; Ouvrier-Bufferet, 2015ab). Some of them bring new kinds of situations for the classrooms designed and studied by mathematicians and didacticians for twenty years: *Research-Situations for the Classroom* (RSCs) emphasize the practice of a mathematical activity in the classrooms. They are in the line with a tradition of heuristics in nonroutine problem solving coming from Pólya (Mousoulides & Sriraman, 2020), Theory of Didactical Situation (TDS) and open problems (Brousseau & Gibel, 2005). They enrich the frame of Inquiry Based Mathematics Education (Artigue & Blomhøj, 2013) in the methodology of didactical engineering (Artigue, 1992), especially in discrete mathematics (Ouvrier-Bufferet, 2020) which remains fruitful for the development of mathematical thinking and brings a new beginning for students and teachers (Goldin, 2010).

Therefore, taking into consideration the international context of the promotion of proof and the pressing need of research about proof and teacher's guidance in elementary school, I address the following research questions: what kinds of proving processes and arguments are students able to produce within RSCs? How can the teacher manage and support students' processes in a RSC?

I will characterize the RSCs and propose a theoretical framework to study students' proving processes in the frame of the TDS (Brousseau, 1997) (§2). Then, I will present the chosen RSC (§3). My didactical choices, the implementation of the RSC and the analysis criteria will be presented (§4) before the results (§5). Finally, I will use these findings to open perspectives about RSCs, teachers' management and proof at the elementary level.

## 2. THEORETICAL BACKGROUND

### 2.1. Characterization of Research-Situations for the Classroom (RSCs)

In *Maths à Modeler* Research Federation, supported by the French Ministry of Research, mathematicians in discrete mathematics and mathematics education researchers build innovative RSCs from real problems coming from the ongoing mathematical research for twenty years (e.g. Grenier & Payan, 1999). With the help of TDS (§2.3), they design and implement them mainly in secondary and university levels, and in popularization events.

The aim of RSCs is to put students in the role of a mathematical researcher. RSCs have to fulfill the six following criteria (derived from Grenier & Payan, 1999, 2003; Ouvrier-Bufferet, 2009). Some of them are close to those of problem solving. Firstly, the research field should be "huge", have roots in mathematical ongoing research, and a large part of it should be accessible to the students. As a consequence, there are only criteria of local resolution (local ending criteria) and possibly no final ending: an answered question often leads to a new question. Secondly, the students should manage their research themselves: among the multiple *didactical variables*<sup>1</sup> that are raised by the problem, at least one of them is left to the students (*a research variable*), while the others can be set by the teacher. These variables are chosen depending on the didactical interest of questions that they can generate for the activity. Thirdly, the problem should be easily understood. Fourthly, there should be few notional traps to start the research. Initial strategies without pre-requisites exist. Note that both previous criteria contribute to make the *devolution*<sup>2</sup> of the problem easier. Fifthly, many

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<sup>1</sup> A *didactical variable* is a parameter of the situation, whose value can be set by the teacher and whose change will impact on the strategies and the behavior of the students.

<sup>2</sup> *Devolution* is "the part of the teacher's job which consists in getting the students to accept the risk of not knowing how to solve a problem" (Brousseau & Gibel, 2015, p. 23).

strategies may put the research forward and several developments are possible to enable the research process and the emergence of mathematical skills and knowledge. Sixthly, at the end of the research process, students are invited to share their results, e.g. through posters in their school or a presentation at university (in front of other school students, who are also there to present their results, and researchers from university). Research is an activity involving proof that requires a big amount of time, so if one expects students to do research, one must grant them enough time to search: a long-term situation is essential, and the teacher's management has to be defined (§2.3).

In RSCs, a specific *research contract* is implemented to avoid the bias of the usual didactical contract. The main idea is to encourage students' research processes and to foster a mathematical community, close to the sociomathematical norms where the criteria for acceptability of arguments are negotiated in the classroom (e.g. Yackel & Cobb, 1996; Stylianides, 2016): the social dimension of exploratory proving remains fundamental (e.g. Balacheff, 2010; Stylianides, 2016). In RSCs, the initial problem comes from the mathematical research, and nobody (neither the students nor the teacher) knows the solution. The teacher's reactions should not be an implicit indicator for the students of the value of their processes.

## 2.2. Characterizing students' proving process at elementary school

I will mainly use two complementary frameworks which both emphasize the role of the classroom community, as well as TDS: Stylianides' 2016 definition of proof and Balacheff's (1987, 1988, 2010) characterizations of arguments.

Stylianides (2016, p.13) defines proof as a *mathematical argument* with three *components*, accepted by or within the conceptual reach of the classroom community: 1) the set of accepted statements (true and available statements, without further justification), 2) the modes of argumentation (valid forms of reasoning), and 3) the modes of argument representation (communication with forms of expression). Stylianides' three components help recognize what counts as a proof in the classroom community. His research has demonstrated the interest of this definition for the elementary school.

Balacheff (1987, 1988, 2010) makes finer distinctions among empirical arguments than other researchers and then completes Stylianides' definition. For him, the word *proof* refers to an explanation accepted by a given community at a given time. Balacheff (1987) distinguishes *pragmatic* and *intellectual proofs* and uses the TDS, mainly the three dialectics of action, formulation and validation (see below §2.3).

*Pragmatic proofs* rely upon direct actions on examples, which is appropriate at the elementary level. In the pragmatic proofs, Balacheff (1988) defines two pragmatic arguments: the *naïve empiricism* and the *crucial experiment*. The *naïve empiricism* "consists of asserting the truth of a result after verifying several cases" (ibid. p.218) without justifying. The *crucial experiment* consists in taking explicitly into account the problem of the generalization of a result, by staking all on the outcome of a particular case "not too special": it grounds the students' conviction.

The *generic example* is a transitional stage in moving from pragmatic to intellectual proofs: it "involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of its class." (ibid. p.219). The generic example reflects an inability to express a justification in general terms. Stylianides (2016, p.17) defines the generic arguments (component 2 of his definition) "(...) as arguments that justify the truth of a statement about a set of cases by showing that the statement holds for a particular case (or example) in the

domain of the statement that possesses no special properties so that the reasoning used for that case can be applied to any other case.”

In the intellectual proof *thought experiment*, the action is internalized and detached from a particular representation, as a consequence of a decontextualization, a de-temporalization and a depersonalization process (Balacheff, 1988, p.217).

### 2.3. Implementing proving tasks and RSCs in elementary classrooms: the use of TDS

Regarding the teacher’s management in implementation of proving tasks, Stylianides (2016, p.158s) draws three conclusions. First, he points out the setting up when the teacher engages students in an “initial exploration of a mathematical situation before raising explicitly the issue of proof” in order to motivate the need for a proof. Second, Stylianides describes the teacher’s complex role in filtering students’ ideas and selecting some of them for discussion, encouraging the class to be a mathematical community (e.g. Martino & Maher, 1999; Maher et al., 2010; Yackel & Cobb, 1996) and questioning students who “do not naturally seek to build a justification or proof of the validity of a solution” (Martino & Maher, 1999, p.56). Stylianides’ third and last conclusion questions how teachers can help students overcome obstacles in their proving activity.

These conclusions highlight a crucial issue for research about proof at elementary school. In all the research, the teacher is a moderator but also a pivot orchestrating mathematical discussions (Bartolini Bussi, 1996; Boero et al. 2010), engaging students in an authentic mathematical activity (Lampert, 1990), encouraging the classroom to be a mathematical community (Yackel & Cobb, 1996). It coincides with the questions brought by the TDS (Brousseau, 1997) which is the common underlying background of Balacheff’s works and RSCs: what does the teacher do, say, or even *institutionalize* in proving activities at elementary level?

In Brousseau’s 1997 engineering, the didactical situation is divided into three phases (*action, formulation, validation*). The interactions between the student(s) and the environment (*milieu*) constitutes the *dialectic of action*. This production of a set of experiences and results should be rich enough to nourish the collective discussion in the next phase. The *dialectic of formulation* consists in “progressively establishing a shared language”, making “possible the explanation of actions and modes of action”. There is here no discussion about mathematical proof. During the *validation* phase, students “have to put forward propositions and to prove to an opponent that they are either true or false” (Brousseau, 1997, p.4). Here, the purpose of the debate to accept a proof is to determine a common system of validation for the (classroom) community (Balacheff, 1987). In a situation of validation, the students discuss, between peers, the validity of their strategies (its truth or its efficacy), their criteria and common decision’s tools: it “is much more a problem of comparison of evaluations, of rejection of proofs, than it is of searching for a mathematical proof” (ibid. p. 89).

Balacheff (2010) underscores that the shift from proof to validation is more important than we may have realized. Thus, one comes back to the crucial teacher’s role. Stylianides (2007, p.16) emphasizes the process of institutionalization suggesting a scenario which is “consistent with the conceptualization of the social dimension of proof”: the teacher ratifies students’ argument as a proof at the end of an episode. I will focus on local institutionalizations in RSCs, “necessary to shore up the practices and their use elsewhere” (Brousseau, 1997, p.18), bearing in mind the following definition: “in institutionalization, the teacher defines the relationship that can be allowed between the student’s “free” behavior or production and the cultural and scientific knowledge and the didactical project: she provides a way of “reading”

these activities and gives them a status.” (ibid. p.56). Indeed, in RSCs, there are only local ending criteria and possibly no final ending. So, in the context of ordinary classrooms, the teacher has to define an acceptable closure (and then an institutionalization) depending on students’ processes.

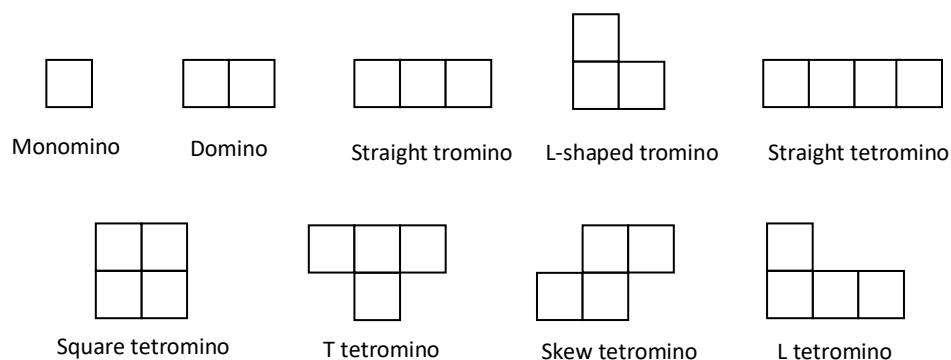
The implementation of RSCs should preserve the dialectics of action, formulation and validation (Grenier & Payan, 2003). After the devolution of the RSC, the action phase engages students in the exploration of the problem, often with manipulatives. The formulation phase has an explicit social dimension: it allows to make the actions explicit. Validation takes place when interactions with peers include explanations, refutations, and parts of proof.

### 3. A COMBINATORIAL GEOMETRY PROBLEM

I chose the mathematical field of combinatorial geometry because it is not part of the regular curriculum, following Maher et al. (2010, p.10) who underscore the importance for students “to come to the problem fresh”. The field of combinatorics, and polyominoes in combinatorial geometry in particular, is a “fascinating ‘enrichment’ material in school mathematics programs” (Golomb, 1966, p.14). It allows the acquisition of a ‘mathematical attitude’ for the 3-13 years old (Freudenthal, 1991).

#### 3.1. The mathematical problem

The *Hunting the beast!* problem checks the previous criteria of a RSC as I will demonstrate throughout Section 3. It is derived from an ongoing research problem from the field of combinatorial geometry and involves polyominoes (Fig. 1) popularized by the game Tetris and the Martin Gardner’s puzzles. A polyomino is a shape “made by connecting certain numbers of equal-sized squares, each joined together with at least 1 other square along an edge” (Golomb, 1966, p.19).



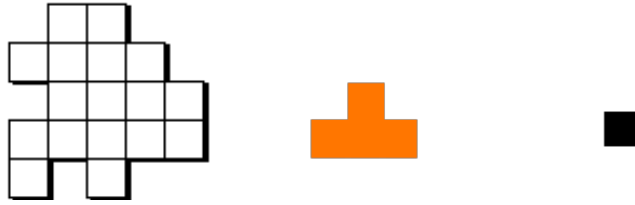
**Fig. 1** The simpler polyominoes

Here is the usual way to present the problem to students (Fig. 2) in *Maths à Modeler*:

#### *Hunting the beast!*

Your garden is a collection of adjacent squares and a beast is a collection of squares. Your objective is to prevent a beast from entering your garden. To do this, you can buy traps. A trap is represented by a single square that can be placed on any square of the garden. The question is: what is the minimum number of traps you need to place so that no beast can land on your garden?

These rules allow the beast to be rotated (90,180 or 270 degrees) or reflected (flipped over) at will.



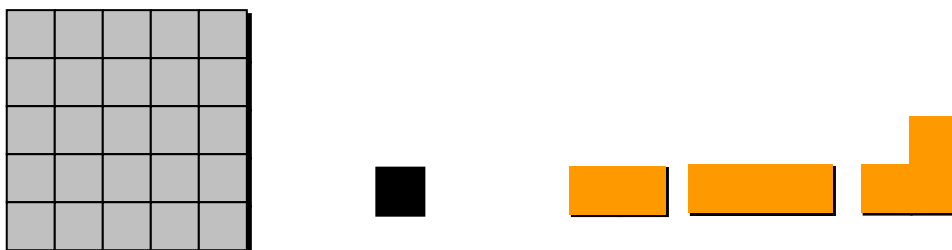
**Fig. 2** A garden, a beast and a trap

In this problem, placing a trap on each square is clearly not optimal. The difficulty thus lies in finding a configuration with the *smallest* number of traps. In the literature, this problem can be seen as a variation of the *Pentomino Exclusion Problem* introduced by Golomb (1966). In the latter problem, the garden is a  $k \times n$  rectangular board and the aim is to minimize the number of monominoes so that no pentomino can be placed on it. For both problems, it turns out that the computation of the minimum number of traps is NP-complete<sup>3</sup> in terms of algorithmic complexity. However, for some special cases of the board and the beast, results exist about the minimum number of traps (e.g. Dorbec, 2007; Gravier, Moncel & Payan, 2007; Gravier & Payan, 2001) and new integer programming models are still in progress (e.g. Kartak & Fabarisova, 2019).

### 3.2. Choices of the value of the didactical variables

The number of unit squares used to define the beast is a didactical variable (V1): naturally, hunting “small” beasts appears to be more accessible. The shape of the beast (V2) plays an important part: for example, hunting a rectangle (whatever its size) is a problem that is completely solvable through elementary considerations. On the contrary, the problems may become harder if the beast is a non-connected set of squares. The number of types of beasts (V3) that are simultaneously hunted complexifies the research: in the initial case (Fig. 3), we consider only one type of beast. In the *Pentomino Exclusion Problem*, all the polyominoes of size 5 are hunted, which is very complex. After having considered each type of polyomino separately, hunting all of them simultaneously is also an interesting work. The size and the topology of the board (V4) are also important: it may be relevant to let the students work on larger sizes, or also play on rectangular boards.

These didactical variables led me to choose the configuration of Fig. 3 for elementary schools in order to imply the devolution of the mathematical problem and then the effective search of optimal configurations and the formulation of arguments. It is also possible to open the problem with other values of the didactical variables and then to question in depth the generalization of the results.



**Fig. 3** A 5x5 garden, a trap, and three kinds of beasts

<sup>3</sup> i.e. there is no known algorithm that computes an optimal solution with a polynomial number of elementary operations.

### 3.3. How the problem may be addressed in mathematics (e.g. Golomb, 1966; Dorbec, 2007)

To solve the problems, it is **necessary** first to exhibit a way of placing a certain number of traps on the garden so as to exclude the given polyomino and then to prove that no fewer traps could have been used for the same purpose “by whatever combinatorial reasoning or tricks suggest themselves” (Golomb, 1966, p.42). Examining all possible cases (**proof by exhaustion**) is still possible but already too laborious. The manipulation of beasts (**tiling proof**) brings the **necessity proof**, accessible in primary schools, and questions the **sufficient proof**. The manipulation of the beasts may suggest using a tiling argument: it comes from the fact that if we fill all the garden with disjoint beasts, then we get a tiling. From the combinatorial optimization point of view, this covering problem has a dual packing problem (*duality*, Schrijver, 1986). No prerequisite in combinatorial optimization is required to understand the *weak duality* relationship that links the problem of placing the minimum number of traps so as to exclude all the beasts and the problem of placing the maximum number of non-overlapping beasts, namely that any upper bound of the first provides a lower bound of the second.

#### 3.3.1. To exclude the domino with 12 traps (tiling proof – necessity and sufficient proof)

A tiling technique proves that 12 nonoverlapping dominoes are necessary to cover the garden. To exclude the domino, there must be at least 1 trap in each of the 12 dominoes. Then, a minimum of 12 traps is necessary to exclude the domino: it is a lower bound. To exhibit a placement of 12 traps is sufficient to end the proof. Two configurations to exclude domino (Fig. 4) can be found by manipulating the traps, but 13 is not optimal.

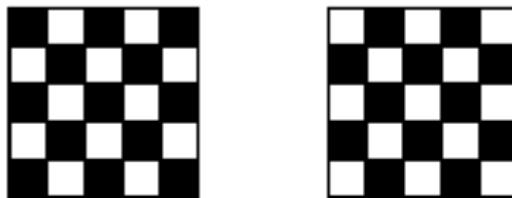


Fig. 4 Domino is excluded with 13 traps and 12 traps

#### 3.3.2. To exclude the straight tromino with 8 traps (tiling proof – necessity and sufficient proof)

The proof follows the same structure as for dominoes. The garden can be tiled with 8 straight trominoes. So, 8 traps are necessary to exclude the straight tromino. And 8 traps are also seen to be sufficient (Fig. 5)<sup>4</sup>. It ends the proof. We can generalize and prove the proposition (P): “If one covers the garden with  $n$  nonoverlapping beasts, then at least  $n$  traps are needed to exclude the beast.”

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<sup>4</sup> This kind of proof leads to a generalization for rectangular gardens  $n \times m$ , where whole part of  $nm/3$  is the optimal value (Dorbec, 2007).



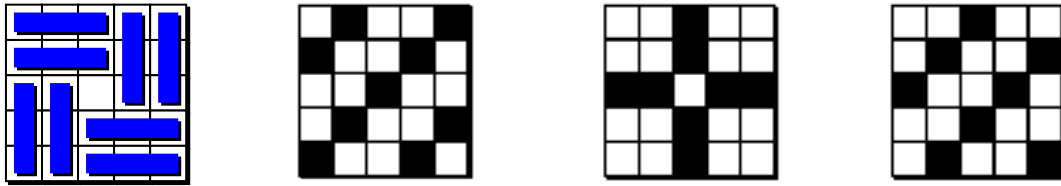


Fig. 5 Tiling with straight trominoes (left figure).

Tromino is excluded with 9 traps and 8 traps (middle and right figures)

Other very interesting proofs exist, using covering numbers or proof *by reductio ad absurdum* (see Appendix 2).

### 3.3.3. To exclude the L-shaped tromino with 10 traps (proof by exhaustion)

The use of proposition (P) is not enough here. Indeed, only 8 L-shaped trominoes cover the 5x5 garden, but 10 traps are needed to exclude the L-shaped tromino (Fig. 6).

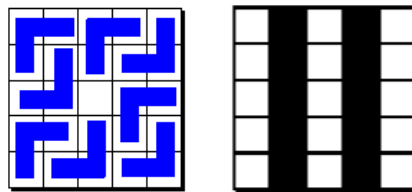


Fig. 6 Tiling with L-shaped trominoes. L-shaped tromino is excluded with 10 traps

Then, the optimal value is framed between 8 and 10. Such conclusions are frequent in combinatorics' research problems. Improving the tiling proof is required to prove that 10 traps are necessary. Remark that 2 traps are needed to keep the L-shaped tromino off a 2x2 garden (Fig. 7). Then, one trap is needed for the beast landed on this garden and one more trap on the black square in order to prevent another neighboring beast from coming on the 2x2 garden.



Fig. 7 Lower bound for hunting the L-shaped tromino

So, for each 2x2 garden, 2 traps are needed. When one structures the 5x5 garden with 2x2 squares (Fig. 7), four 2x2 squares "at worst" appear and two other beasts can come: then, 4 times 2 traps and 2 traps are required, so 10 traps in order to keep the beast off the 5x5 garden.

This problem can be solved by exhausting and examining all possible cases.

### 3.3.4. A common wrong reasoning

One common wrong reasoning is often used by students: "If I remove any trap, a beast can land in the garden. Therefore, my value is optimal." This only proves that the value proposed is locally minimal: it cannot be improved just by removing traps. Yet this does not prove it is optimal. To show that this reasoning is wrong, a counterexample built with another locally

minimal placement of the traps, with more traps, is enough (Fig. 8). It is unlikely that any student would think that the placement on the right (with 13 traps) is the best possible.

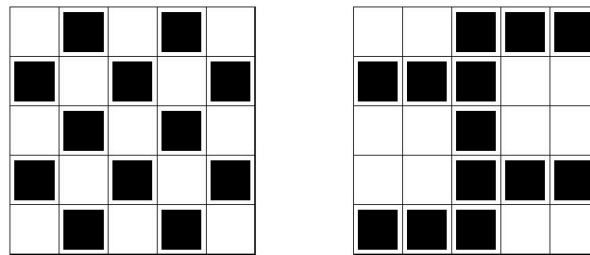


Fig. 8 Straight tromino is excluded with two locally minimal but non-optimal placements of the traps

#### 4. DESIGN AND ANALYSIS CRITERIA

##### 4.1. Implementing the RSC in ordinary elementary classrooms

###### Setting

I implemented the RSC during the yearly *French National Week of Mathematics* in two classrooms in parallel (28 4<sup>th</sup> graders and 24 students in dual-level 4<sup>th</sup> & 5<sup>th</sup> grades). Two volunteering experienced teachers opened their classrooms for this project. For each class, the teacher and a researcher in mathematics education (the author of this paper) were present. Students were divided into groups of 4 by the teachers.

###### Data

Each group of students had a “research notebook” to keep track of their research from one session to another. The researcher collected these notebooks at the end of the experiment. I also noted down the global research processes of each group during the sessions, and took pictures of students’ solutions and of the synthesis made in collaboration with the students on the board with posters. The teachers noted the researcher’s guidance made during synthesis. To preserve the ordinary classroom context, no recording was made.

###### Design and choices for implementation

The devolution of a real research contract was clarified to the students. The historical context of the discovery of polyominoes was quickly presented with the book of Golomb (1966), as well as some applications of tiling problems and the game *Tetris*. Then, the problem was explained with manipulatives (beasts and boards cut out of cardboard paper):

*Hunting the beast!* is an open mathematical problem coming from the mathematical research. The aim of our research is to help me to write an article. No one (neither mathematicians nor your teacher nor me) knows the general solution of this problem. We will work on particular cases during five sessions and write an article with your results for a journal for teachers.

The epistemological and linguistic basis for interaction and communication were the same for all the participants. The students had never dealt with proofs before. So, the researcher explained the aims of “justifying solutions” by “explaining and convincing the classroom that they had the minimum number of traps”, and called such justifications “proofs”. During the synthesis’ phases, the researcher asked the students to formulate their “ideas” (in the sense of Martino & Maher, 1999).

For the experiment, I chose to engage the students in an article-writing style as the way to share their results for two main reasons. First, it simplified the management of the synthesis’

phases (dialectic of formulation) and it legitimated them. Second, it engaged students in the writing, a specific skill in research: the students tried to formulate their statements and arguments without the use of the manipulatives. I am aware that writing is a difficult process of reconstruction. But, in the case of the polyominoes, I made the hypothesis that the article-writing style should not be an obstacle because the mathematical proofs partially come from the manipulation of the mathematical objects (§3).

The teacher and the researcher answered general questions about the guidelines, asked for justification or validation in order to convince the peers in the classroom. They encouraged the students to engage a dialectic of formulation and validation (see §2). They also filtered and highlighted the students' ideas which could be relevant to explore new beasts in the next sessions, but avoided any premature institutionalizations. Students had to be responsible of the validity of their own processes and results.

The five one-hour sessions followed the features of a RSC in TDS (Table 1).

Session	Management	TDS' frame
Session 1	Devolution of the problem and research with the dominoes	Devolution and action
Session 2	Synthesis on board and research with the straight trominoes	Formulation and action
Session 3	<ul style="list-style-type: none"> <li>- Pooling and confrontation of results</li> <li>- Synthesis and local institutionalization to enhance students' proving process: how to prove that configurations are optimal for domino and straight tromino?</li> </ul>	<ul style="list-style-type: none"> <li>- Dialectic of formulation and validation</li> <li>- Formulation and institutionalization (if required)</li> </ul>
Session 4	<ul style="list-style-type: none"> <li>- Research with the L-shaped tromino.</li> <li>- Confrontation of results and synthesis with a question: are the proofs of optimality the same for domino, straight tromino and L-shaped tromino?</li> <li>- The writing of students' results started there and the manipulatives were removed.</li> </ul>	<ul style="list-style-type: none"> <li>- Action</li> <li>- Dialectic of formulation and validation</li> <li>- Formulation</li> </ul>
Session 5	Writing of students' results.	Formulation

**Table 1** Presentation of the five sessions in the frame of TDS

Each group was in charge of the writing of a part of the article in session 5: presentation of the polyominoes' problem; optimal identified configurations; proofs accepted and validated by the classroom; conclusions. The mathematical results of the classrooms were compiled by the researcher after the experiment. During a sixth session, the choices of the presentation of the results of the students were made with both classes. The article was then written by the researcher and presented to the classes with an explanation of the upcoming peer-review and the editing processes<sup>5</sup>. The accepted article (Ouvrier-Bufferet et al., 2017) was published in a national journal for teachers' trainers.

<sup>5</sup> The students also held a stand with the *Hunting the beast!* game at the school party to challenge the parents.

## **4.2. Analysis criteria: Stylianides' components, Balacheff's arguments and the question of validation**

### **4.2.1. Modes of argument representation**

The names of the beasts (polyominoes) were introduced by the researcher. The manipulation of the polyominoes and their names can engage the students to a natural representation of them with a simple language (polyominoes or beasts, number of squares, shapes of polyominoes) and simple representations. Besides, the above-described proofs (§3) mostly mobilize natural language, including optimization vocabulary (at least, at most, frame). Note that no formalism was required in the studied cases; it is obvious that studying the general problem is a different perspective. Then, there is *a priori* a great closeness between the manipulation of the manipulatives and the explanation of configurations of traps, optimal configurations and arguments: I tested this hypothesis in the experiment.

### **4.2.2. Statements and arguments**

The function of proof in this RSC is mainly to decide what is the optimal configuration for each beast and to convince that 10 traps is the optimal configuration for the L-shaped tromino. Following the first and second Stylianides' components, the statements concern configurations and optimal configurations, boundaries and tiling. The mathematical proofs (§3) require several modes of argumentation: necessary and sufficient proof, trial and error, proof by exhaustion (tiling or placements of traps), proof by *reductio ad absurdum*. They also mobilize heuristics which enrich the ways of reasoning: duality (tiling), studying smallest cases, using row and columns arguments.

Let's consider Balacheff's categories of arguments. Three kinds of polyominoes were chosen in order to make the proving process evolve. Naïve empiricism is present with the manipulation of beasts and traps in action phases. Dealing with domino and tromino enabled the formulation of a generic argument: domino and tromino can be seen as generic examples and can lead to a generic argument linked to proposition (P) (§3.3.2). The simultaneous consideration of domino and tromino must bring out the necessary condition and the formulation of optimality ("at least"). The L-shaped tromino is a shift in the proving process. Its use is then justified in the experiment from the crucial experiment and the generic argument perspectives. Indeed, the crucial experiment can appear with questionings about the reusability and the generalization of arguments coming from the study of particular cases, mainly in the shift from domino and tromino to L-shaped tromino. The proof of a generic argument linked to (P) is also at stake. This analysis gives us clues for the teacher's guidance.

### **4.2.3. The question of validation (of optimal configurations)**

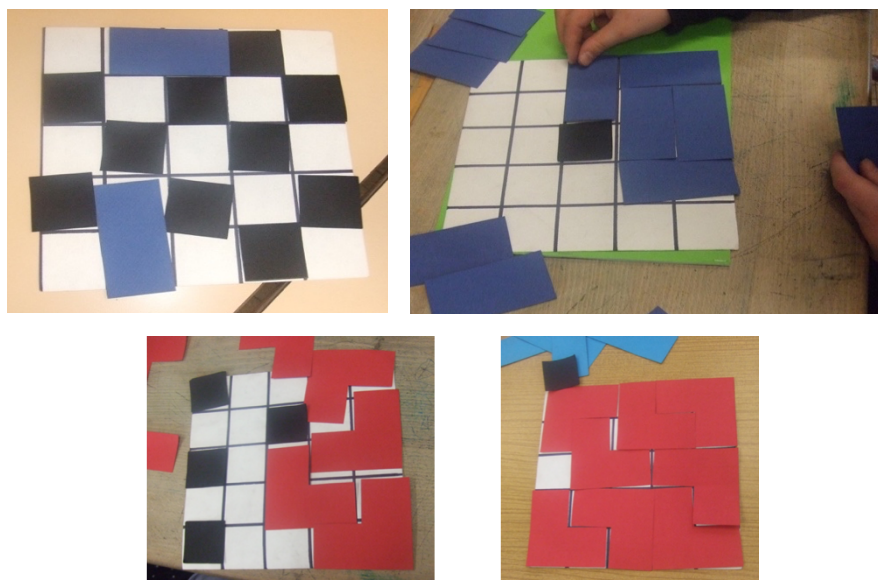
The relations between the polyominoes, the monominoes, their shapes and the board are fundamental to structure a proof: the description of the manipulation of the beasts and the traps is clearly a part of a proof. Besides, to exhibit a configuration of the traps has a particular function in the proof of optimality: it ends the proof if one has a lower bound. There is clearly an external validation of an optimal placement of the traps (it "works") in the polyominoes' problem. The use of the manipulatives reinforces the belief that one has an optimal configuration ("one cannot do better"). Then, how can the teacher or the *milieu* engage students to switch from a pragmatic approach of an optimal value to the question of the validity of arguments towards a proof? This shift is obviously complex at elementary school and required a specific teacher's role. Local institutionalizations can support this shift, taking into account students' processes.

## 5. RESULTS AND DISCUSSION

The posters are fully reproduced in Appendix 1: the evolution of each class's proving processes is highlighted. I propose below the results and an analysis of the students' processes following TDS, Stylianides' definition of proof and Balacheff's categories of arguments. I emphasize the teacher's guidance in a specific part.

### 5.1. Devolution of the problem, students' action and modes of representation

The RSC ensures a 'good' devolution. The tiling technique was in minority in both classrooms when dealing with the domino: it was probably induced by the instructions which talked about the minimal number of traps and students manipulated the traps in priority (Fig. 9). During the sessions, the students increasingly used the tiling technique and the search of the number of traps per beast, and then explore the dual problem which is accessible for young students (as demonstrated in §3).



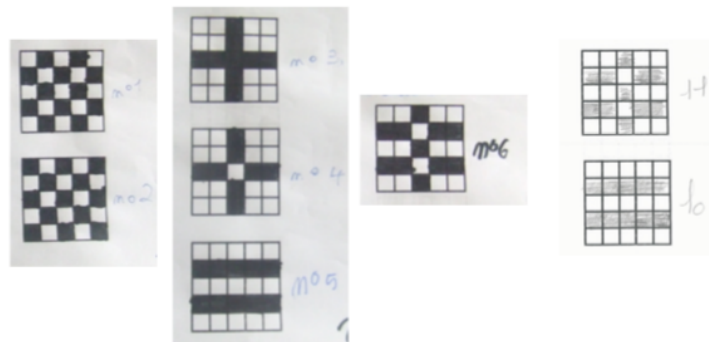
**Fig. 9** Two different considerations of the problem: to exclude the beasts one by one using traps one by one (left) and to tile the garden with beasts (right).

The vocabulary of the first column of Tables 2 and 3 (Appendix 1) shows the predominance of the manipulation and the focus made on the traps and the features of the garden (diagonals, edges) in the study of the domino. The dialectics of action and formulation provided by the designed situation engage the students to clearly detach themselves from the manipulation: when exploring the trominoes, their formulations did not include the manipulation of the traps anymore (Appendix 1). Moreover, when exploring the trominoes, the students asked "how many traps to keep a beast off the garden?"

The students did not have difficulties in manipulating the objects and in verbalizing the relations between them, in finding and representing placements of traps and optimal configurations in their research notebooks (Fig. 10). In agreement with our hypothesis, there was a great closeness for the students between the objects and their representations, and the dual problem is accessible.

## 5.2. Students' optimal configurations and arguments

The following excerpts of the students' arguments came from the research notebooks and the posters made during the synthesis' phases.



**Fig. 10** Configurations and optimal configurations validated by the students (dominoes, left; straight tromino, middle; L-shaped tromino, right)

### 5.2.1. Students' difficulties

Dealing with domino and tromino, the difficulty to understand the necessary condition still remained for the 4<sup>th</sup> graders whose arguments, formulated during action phases, refer to naïve empiricism: "To hunt 8 beasts, we need 8 traps. To hunt 9 beasts, we need 9 traps."

With the straight tromino, the students found one configuration with 9 traps and two with 8 traps (Fig. 5).

The 5<sup>th</sup> graders still used the division and that argument spread in all groups, giving the minimal number of traps: "We cannot use less than 8 traps because  $25/3=8.3$ ". But that did not take into account the shapes of the board and of the beasts. One group tiled the garden and explained "We can use 8 beasts at most, then 8 traps (1 trap for a beast)". During the writing session, these students had difficulties writing a proof because they mixed up several arguments linked to different mathematical proofs (i.e. necessity proof and proof on rows and columns):

We will prove that the solution with 8 traps is the best one. We have the idea that in order to hunt a beast, we need at least one trap. We also know that the best way to hunt the straight tromino is the cross. Explanation: we avoid the beast landing vertically, then we put traps horizontally. And vice-versa.

Generally speaking, the students' difficulties clearly lay in the understanding of the necessary condition and the identification of the generic nature of their arguments for dominoes and trominoes. The guidance of the situation has to take such obstacles into account to make students' processes evolve (§5.4).

### 5.2.2. Arguments for the optimal configuration for domino

All the students identified that a domino covered two squares, which was an invariant: "We have put one trap every two squares because the beast cover two squares" (4<sup>th</sup> graders). That led them to find the configurations with 13 and 12 traps (Fig. 4). The 4<sup>th</sup> graders said: "For one beast, one trap is required". Only one group tiled the garden and found 12 beasts. The necessary proof was not yet complete, and students' arguments referred to naïve empiricism: it is logical at the beginning of the experiment in the exploration of the problem (action phase).

Only one group explicitly said during a synthesis: “To keep the beast off the garden, at least one trap is required.” (§5.4).

The 5<sup>th</sup> graders used two more strategies. They tried to use the division, which was the notion introduced by the teacher the week before the experiment: “When we tile the garden with the beasts: 12 traps. The half of 25 is 12.5 and we cannot divide a square.” Some of them wrote “ $24/2=12$  then 12 traps.” That did not take into account the shapes of the board and of the beasts.

During the writing session, the importance of the locution “at least” came back (necessary and sufficient proof with a tiling technique §3.3.1): it was explicit during the confrontation of arguments with dominoes and trominoes. For the article, the 4<sup>th</sup> and the 5<sup>th</sup> graders wrote the following proofs which each involve a generic argument (linked to (P) (§3) for the 4<sup>th</sup> graders):

When we hunt the dominoes, we find a solution with 12 traps. It is now our best solution. We can tile the garden with 12 dominoes at most. We need at least one trap for a beast. Then for 12 dominoes, we need at least 12 traps. With the trominoes, one trap for a beast does not work, we need two traps for a beast.

To justify the solution with 12 traps, we use the Idea 4, i.e. we need at least one trap for one beast.

We see that 12 is an even number and 13 is an odd number. When we tile the garden with beasts, we find the solution with 12 traps.

The 5<sup>th</sup> graders using the division wrote a naïve empiricism argument: “ $25/2=12.5$  for the domino. It implies that we should have 12 traps.”

The common wrong reasoning described above (§3.3.5) did not come out at all.

### **5.2.3. The uncertainty with the L-shaped tromino: a crucial experiment**

The shift proposed by the research of an optimal configuration with the L-shaped tromino engaged the students in a crucial experiment in the sense of Balacheff: the students identified that some of their arguments are not always valid. There was a shift from action and formulation to validation. The tiling technique was at that stage well used and the students identified the generic argument used for domino and tromino. The students then tried to formulate a necessary condition and get closer to the necessity proof, also when they used the division (5<sup>th</sup> graders): “For the straight tromino, we divided:  $25/3 = 8.333\dots$  We learned that we needed at least 8 traps. For the L-shaped tromino, the division does not work because the shape is different. Then the division allows us to know how many traps at least we need”. The lack of time did not allow the researcher to engage students to deeply reflect on the validity domain of the use of a strategy based upon the division by changing the shape of the garden, engaging students in a crucial experiment.

It led to uncertainty about a mathematical claim: “To keep the L-shaped tromino off the garden, we need at least 8 traps: it is weird, we have 8 beasts and 10 traps.” (4<sup>th</sup> graders).

Then, several processes emerged going back to action: “For the solution with 10 traps, we have built 2 vertical lines to avoid the L-shaped trominoes to land. We also find that the L-shaped trominoes cover 2 lines and then we need to put traps every other line.” (4<sup>th</sup> graders).

The writing of a proof was more difficult. Two groups (one in each grade) mobilized an heuristic (Mousoulides & Sriraman, 2020): reasoning on the smallest garden possible. They identified a generic argument which allows the beginning of a proof: to keep a L-shaped tromino off a  $2 \times 2$  garden, two traps were required.

We have to use 2 traps for a beast on a  $2 \times 2$  garden. But sometimes, one trap is enough. We are convinced that 8 is impossible because there are 8 beasts and sometimes we need 2 traps. We have found a solution with 10 traps. The solution is between 8 and 10. We do not find a solution with 9 traps. We are sure that 10 is the best solution.

This proof was retained by the students for the article. A longer experiment would probably have led the students to an exhaustive list of configurations and then to a complete and valid mathematical proof.

### **5.3. Students' validation of optimal configurations**

The students clearly provided an external validation in action phases of optimal configurations "we cannot do better" in a pragmatic approach of truth (naïve empiricism). The phases of synthesis allowed a peer-validation, but the validity of arguments was not explicitly discussed by the students. As described in §5.4, the researcher's interventions enriched the students' *milieu*, placing them in the search of a generic argument, and then in the perspective of the validation. This is visible in the evolution of the students' posters (Appendix 1). Indeed, making a focus on the reusability of previous ideas led students to systematically test their ideas when dealing with a new beast: it engages them in a crucial experiment and in the search of generic arguments. The students invalidated some arguments because they were naïve or not reusable with another beasts. The difficulty facing the L-shaped tromino engaged the students to test every argument and to search new ones (Appendix 1, Table 2, Idea 9). They explored the mathematical situation with new heuristics when they studied a 2x2 garden (the smaller case to study) with the L-shaped tromino: one trap was not sufficient, at least one trap was required. The students understood then the necessity proof.

### **5.4. Researcher's interventions: discussion**

During the five sessions, students shared and discussed their ideas with peers. As described by Martino & Maher (1999), they indicated satisfaction with their configurations mainly obtained by trial-and-error methods: the intervention of the researcher was required to trigger students' justifications in a formulation phase. Their mathematical "ideas" were written on a poster on the board in an interactive discussion between the students and the researcher (the teacher stayed back in those phases).

In both classrooms, the researcher identified two main obstacles: the formulation of the necessity proof and the students' ways to organize their "ideas". She used the synthesis' phases (sessions 2 and 3) to emphasize two key-elements bearing in mind crucial experiment and generic arguments and she tested Stylianides' alternative scenario involving institutionalization (2007):

- During a formulation situation, the researcher reminded the students that one group had said: "To hunt a beast, we need at least one trap.". Then, the students rephrased a generic argument: "For each beast, domino or tromino, we need at least one trap.". This intervention supports the formulation of the necessity proof in Session 3. Indeed, the students did not identify its need in their initial reasoning. We can make the hypothesis that this formulation must be induced in the management of this RSC with 4<sup>th</sup> & 5<sup>th</sup> graders who have never dealt with proof. Nevertheless, this kind of local institutionalization did not imply an immediate understanding of a necessary condition. The students accepted that it was important but conceptualized it only when dealing with the L-shaped tromino (crucial experiment) and when they had to write a part of the article (formulation situation).
- The researcher explicitly explained to the students that they could reuse their "ideas" when they dealt with new beasts. It fosters a crucial experiment and a search of generic arguments. Then, during the synthesis' phases, the same ideas



were labelled with a same number by the researcher. The posters reflected the students' ideas validated by the classroom community at each step. Each idea was systematically tested when the students explored a new beast: the researcher explicitly engaged the students in a crucial experiment way of thinking and the search of a generic argument.

The choices of the beasts and the dialectic of formulation are fundamental to foster students' insight. The researcher chose to mediate the relationship between students' process and mathematical proofs in two main interventions because of the limited time of the experiment. A longer time experiment with the polyominoes allowing an in-depth work on the L-shaped tromino and various gardens could have enabled a finer work on the study of the validity domains of students' arguments, and consequently on proofs.

## 6. CONCLUSIONS AND PERSPECTIVES

In RSC *Hunting the beast!* the devolution of the mathematical problem and the search of optimal configurations was facilitated by the efficient proximity between the manipulatives and the mathematical concepts involved. The positive aspects of this experiment lie in the fact that students were able to manipulate, explore the problem, search for optimal configurations, identify invariants, and formulate arguments. The students were confronted with uncertainty: it is an introduction to a mathematical activity and to proof. The search for certainty came after the exploration of the mathematical problem and the writing of first solutions, especially with the L-shaped tromino. A proof for each case was required by the researcher in the perspective of the writing of an article, emphasizing another function of proof i.e. the communication of results. The students' modes of argumentation concerned necessary proof, trial and error, and proof by exhaustion. They are enriched by heuristics: duality (tiling proof), studying smallest cases, and using row and columns arguments (but without reaching the proof by *reductio ad absurdum*). Their arguments were pragmatic. The conceptual shift for the students between their exploration of the problem (action and formulation) and the validation of their statements was not reached: the external validation based on evidence and configurations' construction was predominant, perhaps due to the recreational aspect of the beasts. Two main obstacles were identifiable with the analysis through Stylianides and Balacheff's frameworks. Those obstacles were overcome with the researcher's local institutionalizations in the context of the whole ordinary classrooms: the identification of the necessity proof (the researcher emphasized the "at least" sentence proposed by one group) and the use of generic arguments (the researcher suggested students reusing their previous "ideas"). Those institutionalizations engaged students in crucial experiment and generic argument search. The conventional dimension of a mathematical proof is not yet institutionalized in grades 4 and 5, but the students themselves perceived the distance between their arguments and the writing of a proof. In addition, the work on arguments was supported by formulating and testing the students' "ideas" but would need to be developed further. All these elements lead to conclude to the accessibility of this RSC and point out its limits as an introduction to proof at elementary level. Indeed, and with no surprise considering previous works, the main point concerns the guidance of such a situation in order to help students to use their pragmatic arguments and to conceptualize them, reaching the question of their validity. This experiment was an isolated experience for these students: they were not used to doing research and proofs. It can be hypothesized that a wider

use of RSCs may enable students to identify the heuristics specific to the shift from formulation to validation.

At the didactic level, conceiving a progression to allow this transition between formulation and validation requires regular experiments over a long period of time in whole ordinary classrooms: the underlying open question is to know if RSCs can allow such learning of proof and how to enrich them with other types of situations. The didactical question is clearly: How to engage students in validating their statements? How to design a didactical situation that facilitates this shift (e.g. Pedemonte & Balacheff, 2016) and also the structural continuity between argumentation and proof at elementary school (e.g. Pedemonte, 2007)? How to organize the devolution of such situations to elementary teachers who have misconceptions about proof? Besides, the question about how to measure the appropriation and development of such skills appeared to be one of the most important ones: it concerns both students and 'non-expert' teachers who are not familiar with RSCs and with the guidance of open inquiry-oriented classrooms. I have not yet assessed if the mathematical skills developed in a RSC are reused in ordinary problem solving. The research

The current distance between mathematical research and the "rest of the world" is often mentioned. Today, trying to make people understand ongoing discovery in mathematics appears as a society problem. The RSCs developed by *Maths à Modeler* come from discrete mathematics, where results are graspable using little mathematical knowledge. Moreover, it brings students closer to current research problems by putting them in the position of a mathematical researcher: a gate is opened to explore new (a) didactical situations for the teaching and the learning of proof at elementary level and in teacher training.

### Data Availability

All relevant data are included in the article. The original data (pictures of students' writings and synthesis) are available on request from the author.

### Funding and/or 'Competing interests

No funds or grants was received for the submitted work.

No potential conflicts of interest.

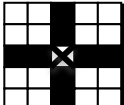
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## Appendix 1 – Posters

Synthesis sessions 2&3 (domino beast)	Synthesis sessions 3&4 (straight tromino beast)	Synthesis session 4 (L-shaped tromino beast)
Two solutions: 12 and 13 traps. 11 is impossible	Solutions with 8 traps: on the diagonals, cross-shaped. Solutions with 9 traps: on the diagonals, cross-shaped. Solution with 7 traps impossible.	One solution with 10 traps. Conjecture <sup>6</sup> : 10 is the best solution. One solution with 11 traps.
Idea 1: one can use the chess to color diagonals.	Idea 1: a solution with 8 traps on the diagonals.	Idea 1: one has 12 or 13 traps, it is too many <sup>7</sup> .
Idea 2: for one beast, one trap is required.		
Idea 3: if one tiles the garden, one has 12 beasts.	Idea 3: here, one can tile with 8 beasts.	Idea 3: if one tiles the garden, one has 8 beasts.
Idea 4: to keep the beast off the garden, AT LEAST <sup>8</sup> one trap is required.	Idea 4: at least one trap to keep a beast off the garden. When one has 8 beasts, at least 8 traps are required.	Idea 4: AT LEAST 8 traps are required.
Idea 5: to place the trap on each row.	Idea 8: (similar to Idea 5) To prevent the beasts from settling in a horizontal position, a column of traps is placed. And to prevent the beasts from settling in a vertical position, a row of traps is placed (in the middle). One then removes the square in the middle (which is useless). 	Ideas 5 – 8: the L-shaped tromino takes up two rows. One blocks every other row.

<sup>6</sup> This word was introduced by the researcher.

<sup>7</sup> Idea 1 is unanimously crossed out by the students.

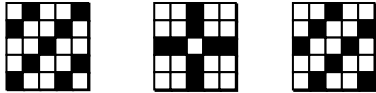


<sup>8</sup> A group of 4<sup>th</sup> graders tried to formulate this idea which was taken up and institutionalized by the researcher.

Idea 6: one places 1 trap every two squares.		Idea 6: one has 12 or 13 traps <sup>9</sup> .
Idea 7: odd or even number on the diagonals <sup>10</sup> ⇒ 11 is impossible.		
		Idea 9: to keep a beast off the garden, 2 traps are required. 

**Table 2** Synthesis carried out on the board with 4<sup>th</sup> graders

<sup>9</sup> Idea 6 is unanimously crossed out by the students.

<sup>10</sup> A group of 4<sup>th</sup> graders who formulated their argument as follows: "We found 12 and we are sure that 11 is impossible because 13 uses odd numbers (5 and 3) on the diagonals and 12 uses even numbers (4 and 2) on the diagonals, this means that 11 uses odd numbers 1 and 3 and that it makes a hole in the middle of 3 squares where we can place the domino beast. We also found out that 11 is impossible if you color every other square."

Synthesis sessions 2&3 (domino beast)	Synthesis sessions 3&4 (straight tromino beast)	Synthesis session 4 (L-shaped tromino beast)
	One solution with 9 traps and two solutions with 8 traps:  7 traps is impossible.	Solution with 10  Solution with 11  Which ideas work for the L-shaped tromino?
Idea 1: 11 is impossible.		
Idea 2: one tries to put as few traps around the edges as possible.		Idea 2 <sup>11</sup> : it does not allow to find the best solution (but it is a good idea to look for solutions)
Idea 3: 24: 2 = 12 then 12 traps (to be discussed).	Idea 3: 24:3 = 8.	
Idea 4: in the solution with 13 traps, one has 12 white squares; when changing the coloring, one has a solution with 12 traps.		Idea 4 <sup>12</sup> : no
Idea 5: one tiles the garden with 12 beasts, and one keeps the beasts one by one off the garden and one places 12 traps. When I want to hunt a beast, how many traps should I use?	Idea 5: one can place (at most) 8 beasts → 8 traps ( <u>one obstacle per beast</u> )	Idea 5: one trap per beast is not enough for the L-shaped tromino. Warning: one puts AT LEAST one obstacle to keep a beast off the garden.
Idea 6: one places a trap every two squares.	Idea 6: one places a trap every three squares.	Idea 6 <sup>13</sup> : no

<sup>11</sup> The aim was to put as few traps as possible on the edges.

<sup>12</sup> The idea was to invert the coloring of the squares to find a better solution from an existing solution (this idea was a good one for the domino beast to find the solution with 12 traps starting from the solution with 13 traps).

<sup>13</sup> To place a trap every three squares.


Idea 7: $25:2 = 12.5$ then 12 traps because the beast covers two squares.	Idea 7: $25:3 \approx 8,333\dots$ 8 traps.	Idea 7: the result of the division tells us the maximum number of beasts.
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**Table 3** Synthesis carried out on the board with 4<sup>th</sup> and 5<sup>th</sup> graders



**Appendix 2 - To exclude the straight tromino with 8 traps (proof by *reductio ad absurdum*)**

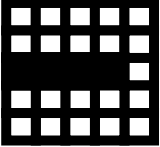
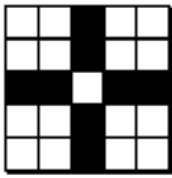
We have the following argument:

Argument (A): To exclude a beast of a single column of the garden, we need at least one trap. This trap must be on the middle square.	
The same is true for rows.	

**Table 4** Row and columns argument (A)

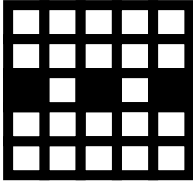
From (A), we conclude that at least 5 traps are needed on the 5x5 garden, and that the only possible solution with five traps would have all the traps placed on the middle row. Obviously, this is not a solution.

So, we know that a solution must have at least 6 traps.

In a 6 traps solution, 4 columns would contain 1 trap, and the remaining column 2 traps. By (A), these 4 columns must have their single trap placed on the middle row. Thus, the 4 other rows altogether contain at most 2 traps, which is not enough, by the same argument.	
Note that placing one trap on each of these rows, we may find the 8 traps cross-solution of Fig. 5.	

**Table 5** Row and columns argument – Proof by *reductio ad absurdum* for 6 traps

Then 6 traps are not enough, suppose a solution with only 7 traps exists.

In a 7 traps solution, at least 3 columns contain exactly 1 trap, placed on the middle row. Obviously, these three columns cannot be consecutive, or the beast would be able to settle on them. If two of these columns are consecutive, then every adjacent column must contain at least 4 traps to prevent the beast to settle there. 4 traps on a column plus at least one trap per column makes at least 8 traps, which is already too much. So, the columns with only one trap are non-adjacent: they are precisely the first, the third and the fifth column. By (A) on rows, this implies that each row, except the middle one, has exactly 1 trap. This is not possible since the middle column has a unique trap.	
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**Table 6** Row and columns argument – Proof by *reductio ad absurdum* for 7 traps

Therefore, 8 traps are required.

A frequent mistake is to forget the discussion on the 7 traps solution. However, the discussion for the 7 traps solution is enough for proving the whole result, since a solution with less than 7 traps must also contain at least 3 columns with exactly 1 trap.