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L-types for resource awareness: an implicit name approach

Silvia Ghilezan¹, Jelena Ivetić¹, Simona Kašterović¹, and Pierre Lescanne²

¹ Faculty of Technical Sciences, University of Novi Sad
Trg Dositeja Obradovića 6
Novi Sad, 21000, Serbia

`gsilvia@uns.ac.rs`, `jelenaivetic@uns.ac.rs`, `simona.k@uns.ac.rs`

² École Normale Supérieure de Lyon,
LIP (UMR 5668 CNRS ENS Lyon UCBL),
46 allée d'Italie, 69364 Lyon, France
`pierre.lescanne@ens-lyon.fr`

Abstract

Since the early work of Church on λI -calculus and Gentzen on structural rules, the control of variable use has gained an important role in computation and logic emerging different resource aware calculi and substructural logics with applications to programming language and compiler design. This paper presents the first formalization of variable control in calculi with implicit names based on de Bruijn indices. We design and implement three calculi: first, a restricted calculus with implicit names; then, a restricted calculus with implicit names and explicit substitution, and finally, an extended calculus with implicit names and resource control. We propose a novel concept of \mathcal{L} -types, which are used (a) to define terms and (b) to characterize certain classes of terms in each of the presented calculi. We have adopted to work simultaneously on the design and implementation in Haskell and Agda. The benefit of this strategy is twofold: dependent types enable to express and check properties of programs in the type system and constructive proofs of preservation enable a constructive evaluator for terms (programs).

Keywords: language design, functional programming, lambda calculus, de Bruijn index, type system, resource awareness, Agda, Haskell

1 Introduction

In computation, the control of variable use goes back to Church’s λI -calculus and restricted terms [18]. Likewise, in logic, the control of formula use is present in Gentzen’s structural rules [11] which enable a wide class of substructural logics [9]. In programming, the augmented ability to control the use of operations and objects has a wide range of applications which enable, among others, compiling functional languages without garbage collector and avoids memory leaking [32, 31]; inline expansion in compiler optimisations [6]; safe memory management [36]; controlled type discipline as a framework for resource-sensitive compilation [12]; the interpretation of linear formulae as session types that provides a purely logical account of session types [5]. At the core of all these phenomena is the Curry-Howard correspondence of formulae-as-types and proofs-as-terms.

Control: by restriction vs by extension There are several restricted classes of lambda terms, where the restrictions are due to the control of variable use. The best known among them are: λI -terms, aka *relevant* terms, where variables occur at least once; BCK λ -terms, aka *affine* terms, where variables occur at most once; BCI λ -terms, aka *linear* terms, where each variable occurs exactly once [18, 14]. E.g. the combinator K is not a λI -term and the combinator S is not a BCK λ -term. This “*control by restriction*” approach is widely present in substructural

logics [9], substructural type theory [36], linear logic [15, 27], among others. On the other hand, the control of variable use can be achieved by extending the language by operators meant to tightly encode the control. If a variable has to be reused, it will be explicitly duplicated, whereas if the variable is not needed, it will be explicitly erased. These two resource control operators, *duplication* and *erasure*, are extensions of the syntax of the λ -calculus which allow all λ -terms to become: relevant (only erasure is used), affine (only duplication is used) and linear (both erasure and duplication are used). The advantage of this “*control by extension*” approach is that all λ -terms can be encoded in the extended calculus. Hence, the extended calculi are equivalent, in computational power, to λ -calculus, which is not the case with the restricted calculi. This approach has been developed in different theoretical [2, 19] and applicative [32, 6] settings. From a proof theoretical perspective, such a simply typed extended λ -calculus has a Curry-Howard correspondence with intuitionistic logic with *Contraction* and *Thinning* structural rules [34], whereas a restricted λ -calculus corresponds to substructural logic [9, 30] such as relevant, affine or linear logic [14].

Names: explicit vs implicit The well-known lambda calculus is a calculus with explicit use of variables (names). On the other hand, the calculus with implicit names is de Bruijn notation of lambda calculus that avoids the explicit naming of variables by employing de Bruijn indices [8, 7, 26]. Each variable is replaced by a natural number which is the number of λ 's crossed in order to reach the binder of that variable. For instance in de Bruijn notation, the combinator $I \equiv \lambda x.x$ is $\lambda \underline{0}$, the combinator $K \equiv \lambda x.\lambda y.x$ is $\lambda \lambda \underline{1}$ and the combinator $S \equiv \lambda x.\lambda y.\lambda z.xz(yz)$ is $\lambda \lambda \lambda \underline{2} \underline{0} (\underline{1} \underline{0})$. The profound advantage of de Bruijn notation is that α -conversion, the renaming of bound variables, is not needed, which significantly facilitates implementation and also, in the case of an extended λ -calculus, simplifies the rules.

Foundations In this paper, we introduce both restricted and extended control of variable use in calculi with *implicit names*. This means that instead of (explicit) variables we use either de Bruijn indices, or novel \mathbb{R} -indices. Inspired by de Bruijn indices, \mathbb{R} -indices provide information about duplication of names. We design and implement three calculi. First, a restricted calculus with implicit names λ^{in} ; then, a restricted calculus with implicit names and explicit substitution λ_v^{in} , and finally, $\lambda_{\mathbb{R}}$, an extended calculus with implicit names and explicit duplication and erasure. In all introduced calculi, well-formed terms are defined as typeable terms in systems we call \mathcal{L} -types. The \mathcal{L} -type of a term represents the list of its free indices and is convenient for checking its well-formedness. In this paper, we characterize mostly closed linear terms and we show that, in the \mathcal{L} -type systems we proposed, typeability by the empty list is equivalent to closedness and linearity.

Implementation We worked simultaneously on the development of the \mathcal{L} -type calculi and on their implementation, because the implementation (in Haskell and Agda) shapes the development. Agda [21, 22] is a dependently typed programming language, which helps us to eliminate all possible errors and shows to be appropriate for the implementations of our calculi, because, thanks to its dependent types, it makes possible to express finely the features of λ^{in} , λ_v^{in} and $\lambda_{\mathbb{R}}$. The obstacles we encountered during the implementation pointed to definitions which were changed or adapted accordingly. Moreover implementation makes precise the side conditions necessary to make the \mathcal{L} -type systems deterministic.

The main contributions of this paper are:

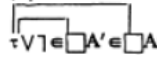


Figure 1: Bourbaki assembly in [3]

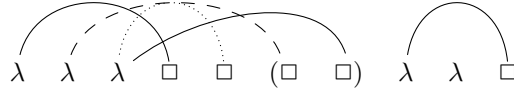


Figure 2: Bourbaki assembly of SK

- the use of de Bruijn indices (an old concept) and \mathbb{R} -indices (a new concept), because they simplify and ease the formalization,
- the novel concept of \mathcal{L} -types for characterizing resource awareness,
- a formal definition of linearity in a calculus with explicit substitution based on \mathcal{L} -types,
- a proof of \mathcal{L} -type preservation in variants of the λ -calculus,
- an implementation in Agda for a part of the formalism (see the [GitHub repository¹](#)) and in Haskell for the whole framework (see the [GitHub repository²](#)). This second repository contains also a version of this paper where the programs are better presented.

The rest of this paper is organized as follows. We first review the background on de Bruijn indices, lambda calculus with implicit names in Section 2. Next, we introduce a type system for restricted terms with implicit names and its implementation in Agda in Section 3. In Section 4, we introduce a type system for restricted terms with implicit names and explicit substitution and prove type preservation. In Section 5, we design an extended calculus with implicit names and resource control and implement it in Haskell and Agda. In Section 6, we discuss related work. Section 7 concludes the paper.

2 Terms with implicit names Λ

Our development relies on the paradigm of implicit names in formal calculi. We recall the notion of term with implicit names based on de Bruijn indices. Let us consider the (regular) lambda term SK and its three contractions

$$(\lambda x.\lambda y.\lambda z.xz(yz))(\lambda x.\lambda y.x) \rightarrow \lambda y.\lambda z.(\lambda x.\lambda y.x)z(yz) \rightarrow \lambda y.\lambda z.(\lambda y.z)(yz) \rightarrow \lambda y.\lambda z.z.$$

Assume that we want to represent those terms without using variables. Such a variable-free representation is called sometimes *Bourbaki assembly*, because this variable-free two dimensional representation of terms has been first used by Bourbaki [3] (see Figure 1) and has been called “assembly” [4, 17]. It resembles the picture in Figure 2 (we use here an infix notation for the binary operator “application” and Bourbaki uses prefix notations).

¹<https://github.com/PierreLescanne/Lambda-R>

²<https://github.com/PierreLescanne/LambdaCalculusWithDuplicationsAndErasures>

Later and independently, de Bruijn proposed an one dimension variable-free representation using natural numbers³, called since *de Bruijn indices*. Each variable is replaced by a natural number which is the number of λ 's crossed in order to reach the binder of that variable. For instance, $\lambda x.\lambda y.\lambda z.xz(yz)$ is replaced by $\lambda\lambda\lambda\underline{2}\underline{0}(\underline{1}\underline{0})$. Indeed, x is replaced by $\underline{2}$ because one crosses two λ 's to meet its binder, y is replaced by $\underline{1}$ because one crosses one λ to meet its binder and z is replaced by $\underline{0}$ because one crosses no λ to meet its binder.

The abstract syntax of terms with de Bruijn notation is the following:

$$t ::= \underline{n} \mid \lambda t \mid t t$$

where \underline{n} , associated with $n \in \mathbb{N}$, is an index. The set of all terms with de Bruijn notation will be denoted by Λ and it will be ranged over by t, s, \dots . We will call them terms or λ -terms without mentioning de Bruijn indices if there is no place for confusion.

Terms with de Bruijn notation are also called terms with *implicit names* since variables are implicit rather than explicit as in the regular λ -calculus. Using implicit names is convenient because terms with de Bruijn indices represent classes of α -conversion of terms with explicit variables. Moreover, β -reduction is easily described with de Bruijn indices because variable capture is avoided. We will see that they also enable simple descriptions of features connected with linearity, duplication and erasure that are otherwise described with cumbersome notations [19, 20, 13]. The formal definition of β -reduction will be given in the rest of the paper.

The above chain of contractions of the term SK is replaced by

$$(\lambda\lambda\lambda\underline{2}\underline{0}(\underline{1}\underline{0}))(\lambda\lambda\underline{1}) \rightarrow \lambda\lambda((\lambda\lambda\underline{1})\underline{0}(\underline{1}\underline{0})) \rightarrow \lambda\lambda(\lambda\underline{1}(\underline{1}\underline{0})) \rightarrow \lambda\lambda\underline{0}.$$

Figure 3 presents SK and its three contractions. It shows how de Bruijn indices are built from variables (aka explicit names), indicates the links between names and their binders and presents the chain of β -reductions in de Bruijn notation. Notice that in $\lambda\lambda(\lambda\underline{1}(\underline{1}\underline{0}))$ the same variable z is associated with two de Bruijn indices, $\underline{1}$ and $\underline{0}$ and that the same de Bruijn index $\underline{1}$ is associated with two variables, y and z . In the de Bruijn notation the value of an index associated with a variable depends on the context.

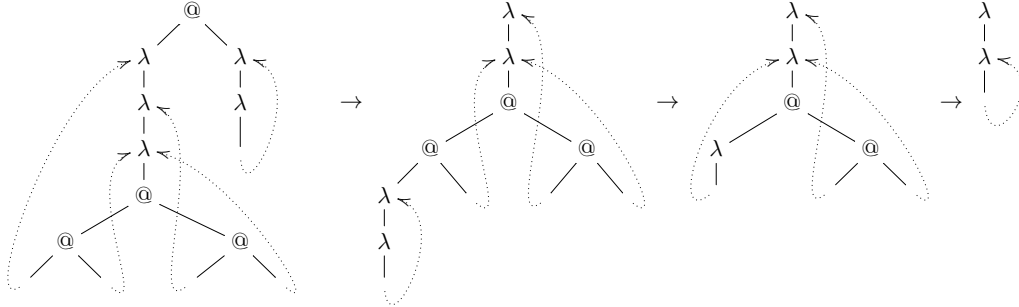
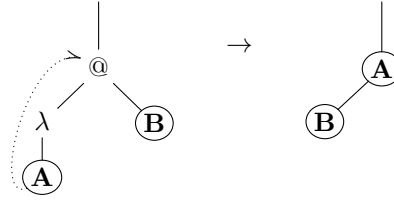


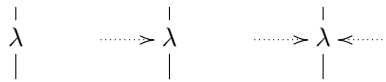
Figure 3: The term SK and three contractions

³This has been popularised by Curien [7]. Notice that de Bruijn and Curien make the indices to start at 1, but the last author proposed in [26] the indices to start at 0, a convention largely adopted since [29, 35].

The basic reduction considered here is



Three patterns are of interest in Figure 3:



The first pattern corresponds to a λ that binds no index, the second pattern corresponds to a λ that binds exactly one index and the third pattern corresponds to a λ that binds two indices. This later pattern is representative, but clearly, there are patterns with more bound indices (see Figure 8, page 27). We propose the control of variable by restricting the language to Λ^{in} in Section 3 and Λ_v^{in} in Section 4, and by extending the language to Λ_{R} in Section 5. In the new language Λ_{R} , extended with two new operators ∇ (duplicator) and \odot (erasure), terms are *linearised*, meaning that only patterns corresponding to a λ that binds exactly one index are present in the Bourbaki representation (see Figure 4). This recalls Lamping’s optimal calculus [23, 16, 1], but in Λ_{R} , we have an atomic substitution, whereas in Lamping’s calculus there is none; *fans* (a kind of duplicators) are propagated. However, the connection should be deepened.

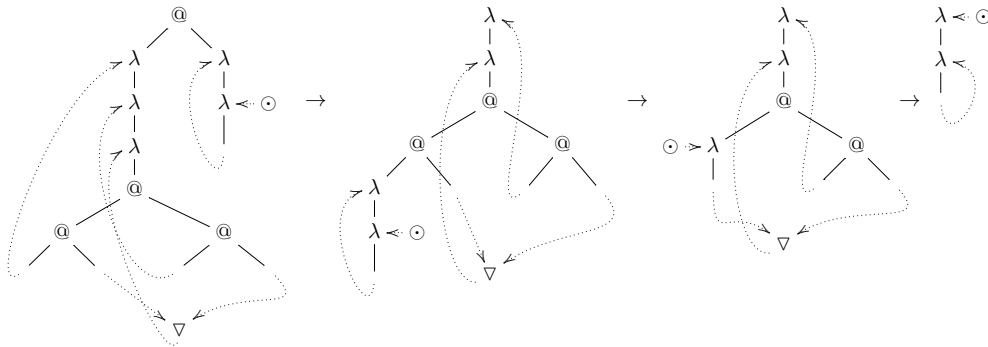


Figure 4: Terms with duplicators and erasures

3 Restricted terms Λ^{in}

In this section, we focus on *restricted terms* [18] with implicit names [8, 26]. We first define the concept of \mathcal{L} -types. Then we define a type system which assigns \mathcal{L} -types to terms with

implicit names and prove how this type system singles out closed linear terms (BCI λ -terms) with implicit names. The set of terms typeable with \mathcal{L} -types will be denoted by Λ^{in} .

3.1 \mathcal{L} -types for Λ^{in}

Lists of natural numbers are called \mathcal{L} -types for Λ^{in} .

Definition 1 (\mathcal{L} -types). *The abstract syntax of \mathcal{L} -types is given by*

$$\ell ::= [] \mid i :: \ell \quad \text{where } i \in \mathbb{N}$$

The empty list is $[]$ and the *cons* operation, $::$, puts an element in front of a list. We write⁴ the list made of $1 :: [3, 5]$ as $[1, 3, 5]$. A list is *affine* if its elements are not repeated. On lists, we define two operations: a binary operation *merge*, \ddagger , and a unary operation *decrement*, \downarrow .

Definition 2 (Merge). *The binary operation \ddagger which merges two lists is defined as follows:*

$$\begin{aligned} [] \ddagger \ell &= \ell \\ (i :: \ell) \ddagger [] &= i :: \ell \\ (i_1 :: \ell_1) \ddagger (i_2 :: \ell_2) &= \text{if } i_1 < i_2 \text{ then } i_1 :: (\ell_1 \ddagger (i_2 :: \ell_2)) \\ &\quad \text{if } i_1 > i_2 \text{ then } i_2 :: ((i_1 :: \ell_1) \ddagger \ell_2) \end{aligned}$$

Remark 1. *Note that \ddagger is **not total**. For instance if j occurs both in ℓ_1 and in ℓ_2 then $\ell_1 \ddagger \ell_2$ is not defined. Note that if two sorted lists are merged, the result is a sorted list.*

If all elements of a list are strictly positive, the list is said to be a strictly positive list. We define a unary operation \downarrow on empty and strictly positive lists. The result is either the empty list or the list where all indices of the initial list are decremented. By $List(\mathbb{N}^+)$ we will denote the set which contains both the empty list and all strictly positive lists.

Definition 3 (Decrement). *The unary operation \downarrow is defined as follows:*

$$\begin{aligned} \downarrow [] &= [] \\ \downarrow ((i+1) :: \ell) &= i :: \downarrow \ell \end{aligned}$$

We assume that the list $(i+1) :: \ell$ is strictly positive, thus the list ℓ is also strictly positive and $\downarrow \ell$ is defined.

The function \downarrow **fails** if the list contains a 0.

The type system that defines the set of restricted terms Λ^{in} is given as follows.

Definition 4 (Terms Λ^{in}). *Λ^{in} -terms are all λ -terms that can be typed by the following rules.*

$$\boxed{\begin{array}{ccc} \text{(ind)} \frac{}{\underline{i} : [i]} & \text{(abs)} \frac{t : 0 :: \ell}{\lambda t : \downarrow \ell} & \text{(app)} \frac{t_1 : \ell_1 \quad t_2 : \ell_2}{t_1 t_2 : \ell_1 \ddagger \ell_2} \end{array}}$$

A \mathcal{L} -type assigned to a term represents the list of natural numbers corresponding to its free implicit names. For instance, $\lambda \underline{0} \underline{5} \underline{2}$ has \mathcal{L} -type $[1, 4]$ since the \mathcal{L} -type of $\underline{0} \underline{5} \underline{2}$ is clearly $[0, 2, 5]$ and to obtain the \mathcal{L} -type of $\lambda \underline{0} \underline{5} \underline{2}$ one removes the $\underline{0}$ which is bound and one decrements the other indices. Moreover, it is a sorted list, as shown by the following Proposition 1.

⁴Beware ! The reader should not confuse lists as \mathcal{L} -types and lists of references.

Why no side condition? The type system has no side condition. If the function \downarrow fails the rule **(abs)** fails as well. Likewise, if the operator \ddagger fails the rule **(app)** fails as well. Thus the non determinism of the type system lies in the failures of the functions it uses. The other types systems of this paper have also no side condition for **(abs)** and **(app)**.

The reader will notice that in the Agda code, such failures are not allowed because all functions must terminate. Therefore something like *side condition* is implemented. Alas, this makes the code more complex. For instance \downarrow has two parameters, a list ℓ and a proof that ℓ is made of strictly positive naturals.

Proposition 1 (Sortedness of lists). *If $t : \ell$ then ℓ is sorted.*

Proof. $[i]$ is sorted and \ddagger and \downarrow preserve sortedness. \square

Example 1 (Typing terms).

$$\frac{\frac{\frac{\underline{1} : [1] \quad \underline{0} : [0]}{\underline{1\ 0} : [0, 1]}}{\lambda \underline{1\ 0} : [0]}}{\lambda \lambda \underline{1\ 0} : []}}{\frac{\frac{\underline{2} : [2] \quad \underline{0} : [0]}{\underline{2\ 0} : [0, 2]} \quad \frac{\frac{\underline{1} : [1] \quad \underline{0} : [0]}{\underline{1\ 0} : [0, 1]} \quad \underline{0} : [0]}{\lambda \underline{0} : []}}{\underline{2\ 0} (\underline{1\ 0}) :?} \quad \lambda \lambda \underline{0} :? \quad \triangle \quad \triangle}$$

Let us notice the following facts:

1. The term $\underline{2\ 0} (\underline{1\ 0})$ is not \mathcal{L} -typeable since there are two free occurrences of index $\underline{0}$. We cannot merge lists $[0, 2]$ and $[0, 1]$, thus $\underline{2\ 0} (\underline{1\ 0})$ does not belong to Λ^{in} .
2. The empty list $[]$ does not start with 0, thus $\lambda \lambda \underline{0}$ is not \mathcal{L} -typeable, i.e. it does not belong to Λ^{in} .

The set of terms of type ℓ is denoted by $\Lambda^{in}(\ell)$.

Proposition 2. *If $t : \ell$ then t is affine.*

Proof. Only in the rule **app** we merge two lists. In order to successfully apply \ddagger , the two lists of free indices must be disjoint and there cannot be more than one occurrence of an index in the application. Hence, if $t : \ell$ then in the term t each index occurs at most once, thus t is affine. \square

Notice that to be abstracted by λ , a term must contain once and only once the index $\underline{0}$ to be bound, therefore, terms of $\Lambda^{in}([])$ are linear.

Proposition 3. *$t : []$ iff t is closed and linear.*

Proof. In a term of $\Lambda^{in}([])$, the unique membership of each index is checked when the index is abstracted by λ .

\Rightarrow If a term is of type $[]$ there are no free indices, hence the term is closed. Furthermore, in a closed term of type $[]$, all the indices are abstracted, then the check for unique membership is made for all of them. Therefore all the indices occur once and only once, hence a closed term of type $[]$ is linear.

\Leftarrow Reciprocally, in a linear and closed term, since the term is closed the list of the free indices is empty, i.e. $t : []$. \square

\square

3.2 Reduction in Λ^{in}

Notice that we do not treat reduction in Λ^{in} or more precisely reduction using implicit substitution. We will treat fully reduction in the framework of explicit substitution in Λ_v^{in} (Section 4). Consequently the reader will find no β_{in} reduction and no statement of a theorem of type preservation. For a discussion the reader is invited to look at the last paragraph of Section 4.

3.3 Implementation of Λ^{in}

We implemented Λ^{in} in Agda [28]. The code may be found on [GitHub](#).

3.3.1 Plain Λ

To say how an Agda implementation looks like, consider the implementation of plain λ -terms.

```
data  $\Lambda$  : Set where
  dB : (k :  $\mathbb{N}$ )  $\rightarrow$   $\Lambda$ 
   $\_ \alpha$  :  $\Lambda \rightarrow \Lambda \rightarrow \Lambda$ 
   $\lambda$  : (t :  $\Lambda$ )  $\rightarrow$   $\Lambda$ 
```

3.3.2 Empty list or Sequence

We implemented sorted lists. For that, we use a specific implementation which we call **LIST**. In this, we make the empty list distinct from non empty lists, which are called **Sequence**'s. In other words,

```
LIST : Set
LIST =  $\top$   $\uplus$  Sequence
```

where \uplus is the direct sum in Agda. We have chosen to distinguish non empty lists from empty list, so that we can define **hd** as a total function. We define on **LIST**'s a predicate **sortedL** derived from a similar predicate on **Sequence**'s. On both data structures, we define a binary operator \ddagger that merges two lists and we prove that \ddagger preserves sortedness.

3.3.3 Decrement a list

We define \downarrow as a function on proofs **p**. Such a proof **p** has type $\ell \in -0 :: \text{LIST} - \mathbb{N}^+$, which means that **p** is a proof that ℓ starts with zero followed by strictly positive elements. Therefore providing the parameter ℓ to \downarrow is not necessary because it is a parameter of the type of **p** and decrementing the items of the list ℓ is safe.

3.3.4 Λ^{in}

Here is the definition of Λ^{in} :

```
data  $\Lambda_{in}$  : ( $\ell$  : LIST)  $\rightarrow$  Set where
  dB : (k :  $\mathbb{N}$ )  $\rightarrow$   $\Lambda_{in}$  [ k ]
   $\lambda$  :  $\ell$  : LIST  $\rightarrow$   $\Lambda_{in}$   $\ell$   $\rightarrow$  (p :  $\ell \in -0 :: \text{LIST} - \mathbb{N}^+$ )  $\rightarrow$   $\Lambda_{in}$  ( $\downarrow$  p)
   $\_ \alpha$  :  $\ell_1 \ell_2$  : LIST  $\rightarrow$   $\Lambda_{in}$   $\ell_1$   $\rightarrow$   $\Lambda_{in}$   $\ell_2$   $\rightarrow$   $\Lambda_{in}$  ( $\ell_1 \ddagger \ell_2$ )
```

3.3.5 Checking linearity in Λ

A predicate `is-lin` checks whether a plain term is a linear term of the given \mathcal{L} -type:

```
data is-lin : (t :  $\Lambda$ ) → (l : LIST) → Set where
  is-lin-dB : k :  $\mathbb{N}$  → is-lin (dB k) [ k ]
  is-lin- $\lambda$  : t :  $\Lambda$  → l : LIST → is-lin t l → (p : l ∈-0::LIST- $\mathbb{N}^+$ ) →
    is-lin ( $\lambda$  t) ( $\downarrow$  p)
  is-lin- $\boxtimes$  : t1 t2 :  $\Lambda$  → l1 l2 : LIST → is-lin t1 l1 → is-lin t2 l2 →
    is-lin (t1  $\boxtimes$  t2) (l1  $\ddagger$  l2)
```

A function

```
is-lin? : (t :  $\Lambda$ ) → (l : LIST) → Maybe (is-lin t l)
```

builds a proof that a given plain term is linear, if this is the case. This will be used in the implementation of the β -reduction, described in the next section.

4 Restricted terms with explicit substitution Λ_v^{in}

In this section, we focus on terms with implicit names and explicit substitution. First we modify the syntax of λv -calculus, a simple calculus with explicit substitution introduced by Lescanne in [25]. Then we define restricted terms Λ_v^{in} by typeability with \mathcal{L} -types and we prove \mathcal{L} -type preservation under reduction. The design of the language is inspired by [26].

The set of plain λv -terms, denoted by Λv , is given by the following syntax:

$$\begin{aligned} t &::= \underline{n} \mid \lambda t \mid tt \mid t[s] \\ s &::= t / \mid \uparrow(s) \mid \uparrow \end{aligned}$$

A term t can be a natural number n (i.e., a de Bruijn index), an abstraction, an application or a substituted term, where a substitution can be one of the following three: a slash $t/$, a lift $\uparrow(s)$ or a shift \uparrow . The rewriting rules of λv are given by the rules in Figure 5.

$(\lambda t_1) t_2$	$\xrightarrow{\lambda v}$	$t_1[t_2/]$	(B)
$(t_1 t_2)[s]$	$\xrightarrow{\lambda v}$	$(t_1[s]) (t_2[s])$	(App)
$(\lambda t)[s]$	$\xrightarrow{\lambda v}$	$\lambda(t[\uparrow(s)])$	(Lambda)
$\underline{0}[t/]$	$\xrightarrow{\lambda v}$	t	(FVar)
$\underline{n+1}[t/]$	$\xrightarrow{\lambda v}$	\underline{n}	(RVar)
$\underline{0}[\uparrow(s)]$	$\xrightarrow{\lambda v}$	$\underline{0}$	(FVarLift)
$\underline{n+1}[\uparrow(s)]$	$\xrightarrow{\lambda v}$	$\underline{n}[s][\uparrow]$	(RVarLift)
$\underline{n}[\uparrow]$	$\xrightarrow{\lambda v}$	$\underline{n+1}$	(VarShift)

Figure 5: The rewriting system for λv -calculus

In what follows $\xrightarrow{\lambda v+}$ is the transitive closure of the rewriting relation $\xrightarrow{\lambda v}$. In order to characterise linearity by a type system, we consider two kinds of objects: $[\uparrow^i(\uparrow)]$ is called an *updater* and abbreviated as $[[i]]$, $i = 0, 1, \dots$, whereas $[\uparrow^i(t/)]$ is called simply a *substitution*

and abbreviated as $\{t, i\}$, $i = 0, 1, \dots$. According to the introduced abbreviations, we propose an alternative syntax that will be used in the definition of terms Λ_v^{in} :

$$t ::= \underline{n} \mid \lambda t \mid tt \mid t[[i]] \mid t\{t, i\}$$

4.1 \mathcal{L} -types for Λ_v^{in}

Just like in the case of Λ^{in} , \mathcal{L} -types for Λ_v^{in} provide information on free indices of a Λ_v^{in} -term. In a declaration $t : \ell$, the type ℓ represents a sorted list of free indices of t . The operation “up”, denoted by \uparrow , increments all elements of a list.

Predicates

In order to ease list manipulation, we introduce predicates. Given a predicate p on naturals and a list ℓ , then $(p \mid \ell)$ is the list filtered by the predicate.

$$\begin{aligned} (p \mid []) &= [] \\ (p \mid i :: \ell) &= \text{if } p(i) \text{ then } i :: (p \mid \ell) \text{ else } (p \mid \ell) \end{aligned}$$

We will consider three predicates

$$\begin{aligned} < i &=^{def} \lambda k . k < i \\ > i &=^{def} \lambda k . k > i \\ \geq i &=^{def} \lambda k . k \geq i \end{aligned}$$

We can modify a predicate p into $p_{[i \leftarrow e]}$ which is p in which each free occurrence of i is replaced by e . For instance, given a predicate $p(i, j)$ (with two free variables i and j) and an expression e (with two free variables k_1 and k_2), one gets $p_{[i \leftarrow e]}(k_1, k_2, j)$ if and only if $p(e(k_1, k_2), j)$. Assume that predicates are made of

- constants,
- free variables,
- basic predicates $< i$, $> i$, and $\geq i$,
- connectors,
- functions on the naturals, like $\lambda k . k + 1$

we define $p_{[i \leftarrow e]}$ by structural induction as follows (we assume k is not the same variable as i):

- $(< exp)_{[i \leftarrow e]} =^{def} \lambda k . k < exp_{[i \leftarrow e]}$
- $(> exp)_{[i \leftarrow e]} =^{def} \lambda k . k > exp_{[i \leftarrow e]}$
- $(\geq exp)_{[i \leftarrow e]} =^{def} \lambda k . k \geq exp_{[i \leftarrow e]}$
- $(p \vee q)_{[i \leftarrow e]} =^{def} p_{[i \leftarrow e]} \vee q_{[i \leftarrow e]}$
- $(p \wedge q)_{[i \leftarrow e]} =^{def} p_{[i \leftarrow e]} \wedge q_{[i \leftarrow e]}$

The substitution in expressions over the naturals is done as usual, as the substitution in universal algebra.

Example 2 (Predicates).

- $(< 3 \mid [0, 2, 3, 4]) = [0, 2]$
- $(\geq 3 \mid [0, 2, 3, 4]) = [3, 4]$
- $(> 3 \mid [0, 2, 3, 4]) = [4]$
- $\uparrow(\geq 3 \mid [0, 2, 3, 4]) = \uparrow[3, 4] = [4, 5]$
- $\downarrow(\geq 3 \mid [0, 2, 3, 4]) = \downarrow[3, 4] = [2, 3]$
- $(< (i + 1))_{[i \leftarrow j+1]} = < ((j + 1) + 1) = < (j + 2)$
- $(\geq (i + 1))_{[i \leftarrow 0]} = \geq 1$

Now, we can prove the following auxiliary lemma, containing list related properties needed in the proof of type preservation.

Lemma 1. *Let ℓ, ℓ_1, ℓ_2 and ℓ_3 be sorted lists. The following equations hold, if all lists that appear in the equations are defined.*

- a) $\ell_1 \ddagger \ell_2 = \ell_2 \ddagger \ell_1$;
- b) $\ell_1 \ddagger (\ell_2 \ddagger \ell_3) = (\ell_1 \ddagger \ell_2) \ddagger \ell_3$;
- c) $(p \mid \ell_1) \ddagger (p \mid \ell_2) = (p \mid \ell_1 \ddagger \ell_2)$;
- d) $\uparrow \ell_1 \ddagger \uparrow \ell_2 = \uparrow (\ell_1 \ddagger \ell_2)$;
- e) $\downarrow \ell_1 \ddagger \downarrow \ell_2 = \downarrow (\ell_1 \ddagger \ell_2)$;
- f) $\uparrow (p \mid \ell) = (p_{[i \leftarrow i+1]} \mid \uparrow \ell)$;
- g) $\downarrow (p_{[i \leftarrow i+1]} \mid \ell) = (p \mid \downarrow \ell)$. (by definition)

Proof. **a)** Case $\square \ddagger \ell$ and $\ell \ddagger \square$ are by definition. Consider the case $(n_1 :: \ell_1) \ddagger (n_2 :: \ell_2)$ with $n_1 < n_2$:

$$\begin{aligned}
 (n_1 :: \ell_1) \ddagger (n_2 :: \ell_2) &= n_1 :: (\ell_1 \ddagger (n_2 :: \ell_2)) \quad (\text{by definition}) \\
 &= n_1 :: ((n_2 :: \ell_2) \ddagger \ell_1) \quad (\text{by induction}) \\
 &= (n_2 :: \ell_2) \ddagger (n_1 :: \ell_1) \quad (\text{by definition}).
 \end{aligned}$$

Case $n_2 < n_1$ is symmetric.

b) Cases where one of ℓ_1, ℓ_2 or ℓ_3 is \square are easy. For the general case, consider $n_1 < n_2 < n_3$. The other cases are on the same pattern.

$$\begin{aligned}
 (n_1 :: \ell_1) \ddagger ((n_2 :: \ell_2) \ddagger (n_3 :: \ell_3)) &= (n_1 :: \ell_1) \ddagger (n_2 :: (\ell_2 \ddagger (n_3 :: \ell_3))) \\
 &= n_1 :: (\ell_1 \ddagger (n_2 :: (\ell_2 \ddagger (n_3 :: \ell_3)))) \\
 &= n_1 :: (\ell_1 \ddagger ((n_2 :: \ell_2) \ddagger (n_3 :: \ell_3))) \\
 &= n_1 :: ((\ell_1 \ddagger (n_2 :: \ell_2)) \ddagger (n_3 :: \ell_3)) \\
 &= (n_1 :: (\ell_1 \ddagger (n_2 :: \ell_2))) \ddagger (n_3 :: \ell_3) \\
 &= ((n_1 :: \ell_1) \ddagger (n_2 :: \ell_2)) \ddagger (n_3 :: \ell_3)
 \end{aligned}$$

c) By case and structural induction

$$\begin{aligned}
(p \mid \square) \ddagger (p \mid \ell) &= \square \ddagger (p \mid \ell) = (p \mid \ell) = (p \mid \square \ddagger \ell) \\
(p \mid i :: \ell) \ddagger (p \mid \square) &= (p \mid i :: \ell) \ddagger \square \\
&= (p \mid i :: \ell) \\
&= (p \mid (i :: \ell) \ddagger \square)
\end{aligned}$$

General case and sub-case $i_1 < i_2$ and $\neg p(i_1)$:

$$\begin{aligned}
(p \mid i_1 :: \ell_1) \ddagger (p \mid i_2 :: \ell_2) &= (p \mid \ell_1) \ddagger (p \mid i_2 :: \ell_2) \\
&= (p \mid \ell_1 \ddagger (i_2 :: \ell_2)) = (p \mid i_1 :: (\ell_1 \ddagger (i_2 :: \ell_2))) \\
&= (p \mid (i_1 :: \ell_1) \ddagger (i_2 :: \ell_2))
\end{aligned}$$

by induction. Sub-case $i_1 < i_2$ and $p(i_1)$:

$$\begin{aligned}
(p \mid i_1 :: \ell_1) \ddagger (p \mid i_2 :: \ell_2) &= (i_1 :: (p \mid \ell_1)) \ddagger (p \mid i_2 :: \ell_2) \\
&= i_1 :: ((p \mid \ell_1) \ddagger (p \mid i_2 :: \ell_2)) \\
&= i_1 :: (p \mid \ell_1 \ddagger (i_2 :: \ell_2)) = (p \mid i_1 :: (\ell_1 \ddagger (i_2 :: \ell_2))) \\
&= (p \mid (i_1 :: \ell_1) \ddagger (i_2 :: \ell_2)).
\end{aligned}$$

The sub-cases $i_2 < i_1$ ($\neg p(i_2)$ and $p(i_2)$) are similar.

d) Also by case and induction

$$\begin{aligned}
(\uparrow \square) \ddagger (\uparrow \ell) &= \square \ddagger (\uparrow \ell) = \uparrow \ell = \uparrow (\square \ddagger \ell) \\
(\uparrow (i :: \ell)) \ddagger (\uparrow \square) &= (\uparrow (i :: \ell)) \ddagger \square = \uparrow (i :: \ell) = \uparrow ((i :: \ell) \ddagger \square)
\end{aligned}$$

case $i_1 < i_2$ (hence $(i_1 + 1) < (i_2 + 1)$):

$$\begin{aligned}
(\uparrow (i_1 :: \ell_1)) \ddagger (\uparrow (i_2 :: \ell_2)) &= ((i_1 + 1) :: \uparrow \ell_1) \ddagger ((i_2 + 1) :: \uparrow \ell_2) \\
&= (i_1 + 1) :: ((\uparrow \ell_1) \ddagger ((i_2 + 1) :: \uparrow \ell_2)) \\
&= (i_1 + 1) :: ((\uparrow \ell_1) \ddagger (\uparrow (i_2 :: \ell_2))) \\
&= (i_1 + 1) :: \uparrow (\ell_1 \ddagger (i_2 :: \ell_2)) \\
&= \uparrow (i_1 :: (\ell_1 \ddagger (i_2 :: \ell_2))) = \uparrow ((i_1 :: \ell_1) \ddagger (i_2 :: \ell_2))
\end{aligned}$$

Case $i_2 < i_1$ is similar.

e) This proof is similar to the proof of d).

f) Also by case and induction

$$\uparrow (p \mid \square) = \uparrow \square = \square = (p_{[i \leftarrow i+1]} \mid \square) = (p_{[i \leftarrow i+1]} \mid \uparrow \square)$$

Sub-case $\neg p(i)$, hence $\neg p_{[i \leftarrow i+1]}(i+1)$

$$\uparrow (p \mid i :: \ell) = \uparrow (p \mid \ell) = (p_{[i \leftarrow i+1]} \mid \uparrow \ell) = (p_{[i \leftarrow i+1]} \mid \uparrow (i :: \ell))$$

Sub-case $p(i)$, hence $p_{[i \leftarrow i+1]}(i+1)$

$$\begin{aligned}
\uparrow (p \mid i :: \ell) &= \uparrow (i :: (p \mid \ell)) = (i+1) :: \uparrow (p \mid \ell) \\
&= (i+1) :: (p_{[i \leftarrow i+1]} \mid \uparrow \ell) = (p_{[i \leftarrow i+1]} \mid \uparrow (i :: \ell))
\end{aligned}$$

g) Works like f).

□

Definition 5 (Terms Λ_v^{in}). Λ_v^{in} -terms are all plain λv -terms that can be \mathcal{L} -typed by the following rules.

$$\begin{array}{c}
 \text{(ind)} \frac{}{\underline{n} : [n]} \qquad \text{(abs)} \frac{t : 0 :: \ell}{\lambda t : \downarrow \ell} \\
 \text{(app)} \frac{t_1 : \ell_1 \quad t_2 : \ell_2}{t_1 t_2 : \ell_1 \ddagger \ell_2} \qquad \text{(upd)} \frac{t : \ell}{t[[i]] : (< i \mid \ell) \ddagger \uparrow (\geq i \mid \ell)} \\
 \text{(sub}_\in) \frac{t_1 : \ell_1 \quad t_2 : \ell_2}{t_1\{t_2, i\} : ((< i \mid \ell_1) \ddagger \downarrow (> i \mid \ell_1)) \ddagger \uparrow^i \ell_2} \quad i \in \ell_1 \\
 \text{(sub}_\notin) \frac{t_1 : \ell_1 \quad t_2 : \ell_2}{t_1\{t_2, i\} : ((< i \mid \ell_1) \ddagger \downarrow (> i \mid \ell_1)) \ddagger \uparrow^i \ell_2} \quad i \notin \ell_1
 \end{array}$$

About rule (sub_\notin) we may notice that we assume $t_2 : \ell_2$ although this assumption is not used in the consequence.⁵ This guarantees the well-formedness of t_2 .

Like for Λ^{in} , we notice that in the typing tree of a \mathcal{L} -typed closed term, we meet only sorted lists with unique occurrence of free indices. Since the notion of linearity of λv -terms is not easily formalised, we propose to use Λ_v^{in} - \mathcal{L} -typeability as the definition of linearity of closed λv -terms.

Definition 6. A closed λv -term t is linear if $t : []$.

4.2 Reduction of Λ_v^{in}

The rewriting system for λ_v^{in} -calculus, derived from the computationally equivalent rewriting system of λv , is given by the rules in Figure 6.

Theorem 1 (\mathcal{L} -type preservation). If $t : \ell$ and $t \xrightarrow{\lambda_v^{in}} t'$, then $t' : \ell$.

Proof. Assume that t matches the left-hand side of one of the rules of λ_v^{in} .

$$B_{in} : (\lambda t_1) t_2 \xrightarrow{\lambda_v^{in}} t_1\{t_2, 0\}$$

Left-hand side and right-hand side of the rule can be typed as follows:

$$\frac{\frac{t_1 : 0 :: \ell_1}{\lambda t_1 : \downarrow \ell_1} \quad t_2 : \ell_2}{(\lambda t_1) t_2 : \downarrow \ell_1 \ddagger \ell_2} \qquad \frac{t_1 : 0 :: \ell_1 \quad t_2 : \ell_2}{t_1\{t_2, 0\} : (< 0 \mid \ell_1) \ddagger \downarrow (> 0 \mid \ell_1) \ddagger \uparrow^0 \ell_2}$$

Successfully typing the left-hand side means $\downarrow \ell_1 \cap \ell_2 = []$. If this is the case, then $((< 0 \mid \ell_1) \ddagger \downarrow (> 0 \mid \ell_1)) \cap \uparrow^0 \ell_2 = []$ holds, so the right-hand side can be successfully typed.

The equality $\downarrow \ell_1 \ddagger \ell_2 = (< 0 \mid \ell_1) \ddagger \downarrow (> 0 \mid \ell_1) \ddagger \uparrow^0 \ell_2$ comes from

⁵A similar situation occurs with intersection types for explicit substitution [24], with the rule **(drop)**.

$(\lambda t_1) t_2$	$\xrightarrow{\lambda_v^{in}}$	$t_1 \{t_2, 0\}$	(B_{in})
$(t_1 t_2)[[i]]$	$\xrightarrow{\lambda_v^{in}}$	$t_1 [[i]] t_2 [[i]]$	(App_{\square})
$(t_1 t_2)\{t_3, i\}$	$\xrightarrow{\lambda_v^{in}}$	$t_1 \{t_3, i\} t_2 \{t_3, i\}$	$(App_{\{\}})$
$(\lambda t)[[i]]$	$\xrightarrow{\lambda_v^{in}}$	$\lambda(t[[i+1]])$	$(Lambda_{\square})$
$(\lambda t_1)\{t_2, i\}$	$\xrightarrow{\lambda_v^{in}}$	$\lambda(t_1 \{t_2, i+1\})$	$(Lambda_{\{\}})$
$\underline{0}\{t, 0\}$	$\xrightarrow{\lambda_v^{in}}$	t	$(FVar_{\{\}})$
$\underline{n+1}\{t, 0\}$	$\xrightarrow{\lambda_v^{in}}$	\underline{n}	$(RVar_{\{\}})$
$\underline{0}\{t, i+1\}$	$\xrightarrow{\lambda_v^{in}}$	$\underline{0}$	$(FVarLift_{\{\}})$
$\underline{n+1}\{t, i+1\}$	$\xrightarrow{\lambda_v^{in}}$	$\underline{n}\{t, i\}[[0]]$	$(RVarLift_{\{\}})$
$\underline{0}[[i+1]]$	$\xrightarrow{\lambda_v^{in}}$	$\underline{0}$	$(FVarLift_{\square})$
$\underline{n+1}[[i+1]]$	$\xrightarrow{\lambda_v^{in}}$	$\underline{n}[[i]][[0]]$	$(RVarLift_{\square})$
$\underline{n}[[0]]$	$\xrightarrow{\lambda_v^{in}}$	$\underline{n+1}$	$(VarShift_{\square})$

Figure 6: The rewriting system for λ_v^{in} -calculus

- $(< 0 \mid \ell_1) = \square$
- $(> 0 \mid \ell_1) = \ell_1$
- $\uparrow^0 \ell_2 = \ell_2$

$$App_{\square} : (t_1 t_2)[[i]] \xrightarrow{\lambda_v^{in}} t_1 [[i]] t_2 [[i]]$$

For the left-hand side of the rule we get

$$\frac{\frac{t_1 : \ell_1 \quad t_2 : \ell_2}{t_1 t_2 : \ell_1 \ddagger \ell_2}}{(t_1 t_2)[[i]] : (< i \mid \ell_1 \ddagger \ell_2) \ddagger \uparrow (\geq i \mid \ell_1 \ddagger \ell_2)}$$

For the right-hand side we get

$$\frac{\frac{t_1 : \ell_1}{t_1 [[i]] : (< i \mid \ell_1) \ddagger \uparrow (\geq i \mid \ell_1)} \quad \frac{t_2 : \ell_2}{t_2 [[i]] : (< i \mid \ell_2) \ddagger \uparrow (\geq i \mid \ell_2)}}{(t_1 [[i]]) (t_2 [[i]]) : ((< i \mid \ell_1) \ddagger \uparrow (\geq i \mid \ell_1)) \ddagger ((< i \mid \ell_2) \ddagger \uparrow (\geq i \mid \ell_2))}$$

Typing the left-hand side, means $\ell_1 \cap \ell_2 = \square$. As a consequence, $((< i \mid \ell_1) \ddagger \uparrow (\geq i \mid \ell_1)) \cap ((< i \mid \ell_2) \ddagger \uparrow (\geq i \mid \ell_2)) = \square$ holds, so the right-hand side can be successfully typed. From Lemma 1, we conclude that

$$\begin{aligned} (< i \mid \ell_1 \ddagger \ell_2) \ddagger \uparrow (\geq i \mid \ell_1 \ddagger \ell_2) &= ((< i \mid \ell_1) \ddagger (< i \mid \ell_2)) \ddagger \uparrow (\geq i \mid \ell_1) \ddagger \uparrow (\geq i \mid \ell_2) \\ &= ((< i \mid \ell_1) \ddagger \uparrow (\geq i \mid \ell_1)) \ddagger ((< i \mid \ell_2) \ddagger \uparrow (\geq i \mid \ell_2)) \end{aligned}$$

$$\text{App}_{\{\}} : (t_1 t_2)\{t_3, i\} \xrightarrow{\lambda_v^{in}} t_1\{t_3, i\}t_2\{t_3, i\}$$

For the left-hand side, with $i \in \ell_1$ we get

$$\frac{\frac{t_1 : \ell_1 \quad t_2 : \ell_2}{t_1 t_2 : \ell_1 \ddagger \ell_2} \quad t_3 : \ell_3}{(t_1 t_2)\{t_3, i\} : (< i | \ell_1 \ddagger \ell_2) \ddagger \downarrow (> i | \ell_1 \ddagger \ell_2) \ddagger \uparrow^i \ell_3}^{i \in \ell_1}$$

If the right-hand side is successfully typed, then $\ell_1 \cap \ell_2 = \square$, and since $i \in \ell_1$, then $i \notin \ell_2$. For the right-hand side, we get

$$\frac{\frac{t_1 : \ell_1 \quad t_3 : \ell_3}{t_1\{t_3, i\} : (< i | \ell_1) \ddagger \downarrow (> i | \ell_1) \ddagger \uparrow^i \ell_3}^{i \in \ell_1} \quad \frac{t_2 : \ell_2 \quad t_3 : \ell_3}{t_2\{t_3, i\} : (< i | \ell_2) \ddagger \downarrow (> i | \ell_2)}^{i \notin \ell_2}}{(t_1\{t_3, i\}) (t_2\{t_3, i\}) : ((< i | \ell_1) \ddagger \downarrow (> i | \ell_1)) \ddagger \uparrow^i \ell_3 \ddagger ((< i | \ell_2) \ddagger \downarrow (> i | \ell_2))}$$

Like the previous cases, it is straightforward to show that whenever the left-hand side is typeable, the right-hand side is typeable as well.

Here also, from Lemma 1, we get

$$(< i | \ell_1 \ddagger \ell_2) \ddagger \downarrow (> i | \ell_1 \ddagger \ell_2) \ddagger \uparrow^i \ell_3 = ((< i | \ell_1) \ddagger \downarrow (> i | \ell_1)) \ddagger \uparrow^i \ell_3 \ddagger ((< i | \ell_2) \ddagger \downarrow (> i | \ell_2)).$$

The case $i \notin \ell_1, i \in \ell_2$ is similar. Let us look now at case $i \notin \ell_1 \ddagger \ell_2$. For the left-hand side we have

$$\frac{\frac{t_1 : \ell_1 \quad t_2 : \ell_2}{t_1 t_2 : \ell_1 \ddagger \ell_2} \quad t_3 : \ell_3}{(t_1 t_2)\{t_3, i\} : (< i | \ell_1 \ddagger \ell_2) \ddagger \downarrow (> i | \ell_1 \ddagger \ell_2)}^{i \notin \ell_1 \ddagger \ell_2}$$

For the right-hand side we have

$$\frac{\frac{t_1 : \ell_1 \quad t_3 : \ell_3}{t_1\{t_3, i\} : (< i | \ell_1) \ddagger \downarrow (> i | \ell_1)}^{i \notin \ell_1} \quad \frac{t_2 : \ell_2 \quad t_3 : \ell_3}{t_2\{t_3, i\} : (< i | \ell_2) \ddagger \downarrow (> i | \ell_2)}^{i \notin \ell_2}}{(t_1\{t_3, i\}) t_2\{t_3, i\} : ((< i | \ell_1) \ddagger \downarrow (> i | \ell_1)) \ddagger ((< i | \ell_2) \ddagger \downarrow (> i | \ell_2))}$$

Whenever $\ell_1 \cap \ell_2 = \square$ holds and we can type the left-hand side of the rule, $((< i | \ell_1) \ddagger \downarrow (> i | \ell_1)) \cap ((< i | \ell_2) \ddagger \downarrow (> i | \ell_2)) = \square$ holds and the right-hand side of the rule can be typed.

From Lemma 1 we conclude that we obtain equal types for both left-hand side and right-hand side of the rule.

$$\text{Lambda}_{\square} : (\lambda t)[i] \xrightarrow{\lambda_v^{in}} \lambda(t[i+1])$$

Left-hand and right-hand sides of the rule can be typed as follows:

$$\frac{\frac{t : 0 :: \ell}{\lambda t : \downarrow \ell}}{(\lambda t)[i] : (< i | \downarrow \ell) \ddagger \uparrow (\geq i | \downarrow \ell)} \quad \frac{\frac{t : 0 :: \ell}{t[i+1] : 0 :: (< i+1 | \ell) \ddagger \uparrow (\geq i+1 | \ell)}}{\lambda(t[i+1]) : \downarrow ((< i+1 | \ell) \ddagger \uparrow (\geq i+1 | \ell))}$$

The equality $(\langle i \mid \downarrow \ell \rangle \ddagger \uparrow (\geq i \mid \downarrow \ell)) = \downarrow ((\langle i + 1 \mid \ell \rangle \ddagger \uparrow (\geq i + 1 \mid \ell)))$ is a consequence of Lemma 1 d) e) and g).

$$\text{Lambda}_{\{\}} : (\lambda t_1)\{t_2, i\} \xrightarrow{\lambda_v^{in}} \lambda(t_1\{t_2, i + 1\})$$

First, we consider case $i + 1 \notin \ell_1$ (with the same calculation as Lambda_{\square}):

$$\frac{\frac{t_1 : 0 :: \ell_1}{\lambda t_1 : \downarrow \ell_1} \quad t_2 : \ell_2}{(\lambda t_1)\{t_2, i\} : (\langle i \mid \downarrow \ell_1 \rangle \ddagger \downarrow (\rangle i \mid \downarrow \ell_1))}^{i \notin \ell_1}$$

$$\frac{t_1 : 0 :: \ell_1 \quad t_2 : \ell_2}{t_1\{t_2, i + 1\} : 0 :: ((\langle i + 1 \mid \ell_1 \rangle \ddagger \downarrow (\rangle i + 1 \mid \ell_1))}^{i + 1 \notin \ell_1}$$

$$\lambda(t_1\{t_2, i + 1\}) : \downarrow ((\langle i + 1 \mid \ell_1 \rangle \ddagger \downarrow (\rangle i + 1 \mid \ell_1))$$

From Lemma 1 we can conclude that the type of the term on the left-hand side of the rule and the type of the term on the right hand-side of the rule are equal.

Next, let us look at the case $i + 1 \in \ell_1$

$$\frac{\frac{t_1 : 0 :: \ell_1}{\lambda t_1 : \downarrow \ell_1} \quad t_2 : \ell_2}{(\lambda t_1)\{t_2, i\} : (\langle i \mid \downarrow \ell_1 \rangle \ddagger \downarrow (\rangle i \mid \downarrow \ell_1) \ddagger \uparrow^i \ell_2)}^{i \in (\downarrow \ell_1)}$$

$$\frac{t_1 : 0 :: \ell_1 \quad t_2 : \ell_2}{t_1\{t_2, i + 1\} : 0 :: (\langle i + 1 \mid \ell_1 \rangle \ddagger \downarrow (\rangle i + 1 \mid \ell_1) \ddagger \uparrow^{i+1} \ell_2)}^{i + 1 \in 0 :: \ell_1}$$

$$\lambda(t_1\{t_2, i + 1\}) : \downarrow ((\langle i + 1 \mid \ell_1 \rangle \ddagger \downarrow (\rangle i + 1 \mid \ell_1) \ddagger \uparrow^i \ell_2)$$

$$\text{FVar}_{\{\}} : \underline{0}\{t, 0\} \xrightarrow{\lambda_v^{in}} t$$

$$\frac{\underline{0} : [0] \quad t : \ell}{\underline{0}\{t, 0\} : (\langle 0 \mid [0] \rangle \ddagger (\rangle 0 \mid [0]) \ddagger \uparrow^0 \ell)}^{0 \in [0]} \quad t : \ell$$

The equality $(\langle 0 \mid [0] \rangle \ddagger (\rangle 0 \mid [0]) \ddagger \uparrow^0 \ell = \ell$ comes from the fact that $(\langle 0 \mid [0] \rangle \ddagger (\rangle 0 \mid [0]) = []$.

$$\text{RVar}_{\{\}} : \underline{n + 1}\{t, 0\} \xrightarrow{\lambda_v^{in}} \underline{n}$$

$$\frac{\underline{n + 1} : [n + 1] \quad t : \ell}{\underline{n + 1}\{t, 0\} : (\langle 0 \mid [n + 1] \rangle \ddagger \downarrow (\rangle 0 \mid [n + 1])}^{0 \notin [n + 1]} \quad \underline{n} : [n]$$

The equality of the types comes from the fact that $(\langle 0 \mid [n + 1] \rangle = []$ and $\downarrow (\rangle 0 \mid [n + 1]) = [n]$.

$$\text{FVarLift}_{\{\}} : \underline{0}\{t, i + 1\} \xrightarrow{\lambda_v^{in}} \underline{0}$$

$$\frac{\underline{0} : [0] \quad t : \ell}{\underline{0}\{t, i + 1\} : (\langle i + 1 \mid [0] \rangle \ddagger \downarrow (\rangle i + 1 \mid [0] \rangle))^{i+1 \notin [0]}} \quad \underline{0} : [0]$$

The equality of the types comes from $(\langle i + 1 \mid [0] \rangle) = [0]$ and $\downarrow (\rangle i + 1 \mid [0] \rangle) = []$.

$$\text{RVarLift}_{\{\}} : \underline{n + 1}\{t, i + 1\} \xrightarrow{\lambda_v^{in}} \underline{n}\{t, i\}[[0]]$$

We will consider three cases, depending on the numbers i and n . First, we consider the case where $i < n$

$$\frac{\underline{n + 1} : [n + 1] \quad t : \ell}{\underline{n + 1}\{t, i + 1\} : (\langle i + 1 \mid [n + 1] \rangle \ddagger \downarrow (\rangle i + 1 \mid [n + 1] \rangle))^{i+1 \notin [n+1]}}$$

$$\frac{\underline{n} : [n] \quad t : \ell}{\underline{n}\{t, i\} : \downarrow [n]}^{i \notin [n]}$$

$$\underline{n}\{t, i\}[[0]] : [n]$$

Since $i < n$, we have $i + 1 < n + 1$, and it holds that $(\langle i + 1 \mid [n + 1] \rangle) = []$ and $\downarrow (\rangle i + 1 \mid [n + 1] \rangle) = \downarrow [n + 1] = [n]$, so the types are equal.

Next, we consider the case where $i = n$.

$$\frac{\underline{n + 1} : [n + 1] \quad t : \ell}{\underline{n + 1}\{t, i + 1\} : ((\langle i + 1 \mid [n + 1] \rangle \ddagger \downarrow (\rangle i + 1 \mid [n + 1] \rangle)) \ddagger \uparrow^{i+1} \ell)^{i+1 \in [n+1]}}$$

$$\frac{\underline{n} : [n] \quad t : \ell}{\underline{n}\{t, i\} : \uparrow^i \ell}^{i \in [n]}$$

$$\underline{n}\{t, i\}[[0]] : \uparrow^{i+1} \ell$$

From $i = n$, we obtain $i + 1 = n + 1$, and it follows that $(\langle i + 1 \mid [n + 1] \rangle \ddagger \downarrow (\rangle i + 1 \mid [n + 1] \rangle)) = []$, hence the types are equal.

Finally, we consider the case $i > n$.

$$\frac{\underline{n + 1} : [n + 1] \quad t : \ell}{\underline{n + 1}\{t, i + 1\} : (\langle i + 1 \mid [n + 1] \rangle \ddagger \downarrow (\rangle i + 1 \mid [n + 1] \rangle))^{i+1 \notin [n+1]}}$$

$$\frac{\underline{n} : [n] \quad t : \ell}{\underline{n}\{t, i\} : [n]}^{i \notin [n]}$$

$$\underline{n}\{t, i\}[[0]] : [n + 1]$$

Since $i > n$, we have $i + 1 > n + 1$, and it follows that $\downarrow (\rangle i + 1 \mid [n + 1] \rangle) = []$. Hence, the types are equal.

We see that in all three cases we have typed both the term on the left-hand side and the term on the right-hand side of the rule with the same type.

$$\text{FVarLift}_{\square} : \underline{0}[[i+1]] \xrightarrow{\lambda_v^{in}} \underline{0}$$

Left-hand side and right-hand side of the rule can be typed as follows:

$$\frac{\underline{0} : [0]}{\underline{0}[[i+1]] : (\lt i+1 \mid [0]) \ddagger \uparrow (\geq i+1 \mid [0])} \quad \underline{0} : [0]$$

From Lemma 1 we have $(\geq i+1 \mid [0]) = []$, thus $\uparrow (\geq i+1 \mid [0]) = []$. From the latter and $(\lt i+1 \mid [0]) = [0]$ we obtain $((\lt i+1 \mid [0]) \ddagger \uparrow (\geq i+1 \mid [0])) = [0]$.

$$\text{RVarLift}_{\square} : \underline{n+1}[[i+1]] \xrightarrow{\lambda_v^{in}} \underline{n}[[i]][[0]]$$

Left-hand side and right-hand side of the rule can be typed as follows:

$$\frac{\underline{n+1} : [n+1]}{\underline{n+1}[[i+1]] : (\lt i+1 \mid [n+1]) \ddagger \uparrow (\geq i+1 \mid [n+1])}$$

$$\frac{\frac{\underline{n} : [n]}{\underline{n}[[i]] : (\lt i \mid [n]) \ddagger \uparrow (\geq i \mid [n])}}{\underline{n}[[i]][[0]] : \uparrow ((\lt i \mid [n]) \ddagger \uparrow (\geq i \mid [n]))}}$$

From Lemma 1 we get

$$(\lt i+1 \mid [n+1]) \ddagger \uparrow (\geq i+1 \mid [n+1]) = \uparrow ((\lt i \mid [n]) \ddagger \uparrow (\geq i \mid [n]))$$

$$\text{VarShift}_{\square} : \underline{n}[[0]] \xrightarrow{\lambda_v^{in}} \underline{n+1}$$

Left-hand side and right-hand side of the rule can be typed as follows:

$$\frac{\underline{n} : [n]}{\underline{n}[[0]] : (\lt 0 \mid [n]) \ddagger \uparrow (\geq 0 \mid [n])} \quad \underline{n+1} : [n+1]$$

Since we have that

- $(\lt 0 \mid [n]) = []$,
- $(\geq 0 \mid [n]) = [n]$, and
- $\uparrow [n] = [n+1]$,

it follows that $((\lt 0 \mid [n]) \ddagger \uparrow (\geq 0 \mid [n])) = [n+1]$.

□

Let us notice that this constructive proof of preservation enables a constructive evaluator for terms in Λ_v^{in} . Indeed Theorem 1 and Definition 6 entail correctness of λ_v^{in} . Correctness says that terms stay well-formed (linear) by reduction. In other words, reduction preserves \mathcal{L} -typeability.

Corollary 1 (Preservation of linearity). *If a λ_v^{in} -term t is linear and $t \xrightarrow{\lambda_v^{in}} t'$ then t' is linear.*

4.3 Toward an implementation of Λ_v^{in}

Full implementation in `Agda` of the λ_v^{in} -calculus is in progress. In this, *merge* is a total function as required by a `Agda` (See discussion in Section 5.6) and the quite involved \mathcal{L} -type system of Λ_v^{in} is not yet implemented.

For the `Agda` implementation of β -reduction in Λ^{in} we use the plain Λv calculus of explicit substitution, that is with no test of linearity, i.e., no test of a type resembling \mathcal{L} -type:

```

data  $\Lambda v$  : Set
data  $\Sigma v$  : Set

data  $\Lambda v$  where
  dB : ( $k$  :  $\mathbb{N}$ )  $\rightarrow$   $\Lambda v$ 
   $\_ \alpha \_$  :  $\Lambda v \rightarrow \Lambda v \rightarrow \Lambda v$ 
   $\lambda$  :  $\Lambda v \rightarrow \Lambda v$ 
   $\_ [-] \_$  :  $\Lambda v \rightarrow \Sigma v \rightarrow \Lambda v$ 

data  $\Sigma v$  where
   $\_ / \_$  :  $\Lambda v \rightarrow \Sigma v$ 
   $\uparrow \_$  :  $\Sigma v \rightarrow \Sigma v$ 
   $\uparrow$  :  $\Sigma v$ 

```

Therefore β -reduction uses a sequence of translations. The implementation works as follows.

1. We read a Λ^{in} -term as a plain λ -term $\Lambda in \rightarrow \Lambda$;
2. We translate a plain λ -term as a λv -term $\Lambda \rightarrow \Lambda v$;
3. We reduce the λv -term to its normal form, which exists since the term is linear hence strongly normalising `norm Λv` ;
4. We translate the λv -term as a plain term, which is simple since the normal form contains no closure $\Lambda v \rightarrow \Lambda$;
5. We read back by `$\Lambda \rightarrow$ Maybe Λin` the plain lambda term as a maybe Λin -term. This function relies on `is-lin?`⁶. The term we obtain is a priori a linear lambda-term, but since there is no guarantee of its linearity, we can produce only a term of type `Maybe Λin` .

```

norm  $\Lambda in$  :  $\mathbb{N} \rightarrow \Lambda in [] \rightarrow$  Maybe ( $\Lambda in []$ )
norm  $\Lambda in$  k t with ( $\Lambda v \rightarrow \Lambda$  (norm  $\Lambda v$  k ( $\Lambda \rightarrow \Lambda v$  ( $\Lambda in \rightarrow \Lambda$  t))))
... | just t' =  $\Lambda \rightarrow$ Maybe  $\Lambda in$  t'
... | nothing = nothing

```

To summarise, in the current `Agda` implementation, β -reduction requires five steps, namely:

$$\Lambda in \rightarrow \Lambda \rightarrow \Lambda v \xrightarrow{\text{norm}} \Lambda v \rightarrow \Lambda \rightarrow \text{Maybe } \Lambda in.$$

This implementation may serve as the reference for the strong normalization in Λ^{in} , but is not satisfactory since the normalization is not done fully on linear terms. When the implementation of Λ_v^{in} in `Agda` will be completed, we will use a design with three steps $\Lambda^{in} \rightarrow \Lambda_v^{in} \xrightarrow{\text{norm}} \Lambda_v^{in} \rightarrow \Lambda^{in}$.

⁶A full `is-lin?` cannot be typed in `Agda`. Therefore we implemented only a restricted version which is enough to treat most of examples.

5 Extended terms with resource control $\Lambda_{\mathbb{R}}$

In this section, we obtain full resource control by extending the language with explicit operators performing erasure and duplication on terms. The goal is to design a language capable to linearise all λ -terms. We adapt \mathcal{L} -types and use them to define terms $\Lambda_{\mathbb{R}}$ and to characterise linear terms in $\Lambda_{\mathbb{R}}$.

The abstract syntax of plain terms with resources and implicit names is generated by the following grammar:

$$t, s ::= (\underline{n}, \alpha) \mid \lambda t \mid t s \mid (\underline{n}, \alpha) \odot t \mid (\underline{n}, \alpha) \nabla t$$

where (\underline{n}, α) is an \mathbb{R} -index, \odot denotes the erasure of index in a term, and ∇ denotes the duplication of index in a term.

\mathbb{R} -indices

An \mathbb{R} -index is the pair (\underline{n}, α) , where n is a natural number and α is a string of booleans. For convenience, we will use the following abbreviations: $0 \equiv \text{false}$ and $1 \equiv \text{true}$. They are written $^\circ$ and 1 in `Agda` because they avoid clashes and they are available on keyboards. Therefore α will be a string of 0's and 1's. Whether 0 and 1 refer to natural numbers or to booleans will be easily distinguished; so we consider that using those notations will introduce no confusion. In (\underline{n}, α) \underline{n} corresponds to an index in Λ and α represents duplications of the index. The empty string of booleans, corresponding to absence of duplications, is denoted by ε . For instance, if $(\underline{n}, \varepsilon)$ is duplicated, it is represented by $(\underline{n}, 0)$ and $(\underline{n}, 1)$; if it is triplicated, it can be represented by $(\underline{n}, 0)$, $(\underline{n}, 10)$ and $(\underline{n}, 11)$ (or by $(\underline{n}, 00)$, $(\underline{n}, 01)$ and $(\underline{n}, 1)$).

In the following example and in Subsection 5.1 we introduce informally notions corresponding to $\Lambda_{\mathbb{R}}$ -terms, which will be formally defined in Subsection 5.2.

Example 3.

- The term $\lambda x.y$ is represented in $\Lambda_{\mathbb{R}}$ by the term $\lambda(\underline{0}, \varepsilon) \odot (\underline{1}, \varepsilon)$.
- The term $\lambda x.(x(\lambda y.xy))$ is represented in $\Lambda_{\mathbb{R}}$ by the term $\lambda((\underline{0}, \varepsilon) \nabla((\underline{0}, 0) (\lambda(\underline{1}, 1) (\underline{0}, \varepsilon))))$.
- The linear term $\lambda x.\lambda y.x y$ is represented in $\Lambda_{\mathbb{R}}$ by the term $\lambda \lambda(\underline{1}, \varepsilon) (\underline{0}, \varepsilon)$, that has neither ∇ nor \odot , since it is linear and needs no resource control.
- Term $\lambda x.x x x$ is discussed in Example 6.

Several more examples of $\Lambda_{\mathbb{R}}$ -terms will be elaborated in the following subsection.

5.1 A bestiary of $\Lambda_{\mathbb{R}}$ -terms

In this section, we examine basic and well known terms.

The term **I**

$$\mathbf{I} = \lambda(\underline{0}, \varepsilon).$$

This corresponds to the term $\lambda x.x$ in the lambda-calculus with explicit names. $(\underline{0}, \varepsilon)$ means that there is no λ between the \mathbb{R} -index $(\underline{0}, \varepsilon)$ and its binder and that there is no duplication.

The term K

$$K = \lambda\lambda(\underline{0}, \varepsilon) \odot (\underline{1}, \varepsilon).$$

In lambda-calculus, K is written $\lambda x.\lambda y.x$. In Λ , K is written $\lambda\lambda\underline{1}$. The index $\underline{0}$ does not occur in $\underline{1}$, but since we want Λ_{\odot} -terms to be linear, we make it to occur anyway, thus we write $(\underline{0}, \varepsilon) \odot (\underline{1}, \varepsilon)$. Notice that ε is the second component of all the \mathbb{R} -indices since there is no duplication. Recall that $\text{readback}(\lambda\lambda(\underline{0}, \varepsilon) \odot (\underline{1}, \varepsilon)) = \lambda\lambda\underline{1}$. This term can be read back using the notations of [13]

$$\lambda x.\lambda y.y \odot x$$

or using the notations of [19]

$$\lambda x.\lambda y.\mathcal{W}_y(x).$$

The term S

$$S = \lambda\lambda\lambda(\underline{0}, \varepsilon) \nabla((\underline{2}, \varepsilon)(\underline{0}, 0)((\underline{1}, \varepsilon)(\underline{0}, 1)))$$

In lambda-calculus, S is written $\lambda x.\lambda y.\lambda z.xz(yz)$ and in Λ , S is written $\lambda\lambda\lambda(20(10))$. We notice the double occurrence of z in lambda-calculus and of $\underline{0}$ in Λ . Therefore a duplication is necessary. From the \mathbb{R} -index $(\underline{0}, \varepsilon)$ it creates two indices $(\underline{0}, 0)$ and $(\underline{0}, 1)$. Where the second component 0 is the string of length 1 made of 0 alone and the second component 1 is the string of length 1 made of 1 alone. This term can be read back using the notations of [13]

$$\lambda x.\lambda y.\lambda z.(z \binom{z_0}{z_1} x z_0 (y z_1))$$

or using the notations of [19]

$$\lambda x.\lambda y.\lambda z.(C_z^{z_0|z_1}(x z_0 (y z_1))).$$

The term 5

$$5 = \lambda\lambda((\underline{1}, \varepsilon) \nabla(\underline{1}, 0) \nabla(\underline{1}, 00) \nabla(\underline{1}, 000) \nabla((\underline{1}, 0000)((\underline{1}, 0001)((\underline{1}, 001)((\underline{1}, 01)((\underline{1}, 1)(\underline{0}, \varepsilon))))))$$

5 represents the Church numeral 5. Recall that in lambda-calculus, 5 is written $\lambda f.\lambda x.(f(f(f(f f x))))$ and in Λ , $\lambda\lambda(1(1(1(1(10)))))$. Since $\underline{1}$ is repeated five times, we need four duplications. If we compute 5 other ways, we can get other forms. For instance, as the result of $3 + 2$:

$$\lambda\lambda(((\underline{1}, \varepsilon) \nabla(\underline{1}, 0) \nabla(\underline{1}, 00) \nabla((\underline{1}, 000)((\underline{1}, 001)((\underline{1}, 01)(\underline{1}, 1) \nabla((\underline{1}, 10)((\underline{1}, 11)(\underline{0}, \varepsilon))))))))$$

or as the result of $2 + 3$:

$$\lambda\lambda((\underline{1}, \varepsilon) \nabla(\underline{1}, 0) \nabla((\underline{1}, 00)((\underline{1}, 01)(\underline{1}, 1) \nabla(\underline{1}, 10) \nabla((\underline{1}, 100)((\underline{1}, 101)((\underline{1}, 11)(\underline{0}, \varepsilon))))))))$$

or as the result of $3 + 1 + 1$:

$$\lambda\lambda(\underline{1}, \varepsilon) \nabla(\underline{1}, 0) \nabla(\underline{1}, 00) \nabla((\underline{1}, 000)(\underline{1}, 001) \nabla((\underline{1}, 0010)((\underline{1}, 0011)((\underline{1}, 01)((\underline{1}, 1)(\underline{0}, \varepsilon))))))$$

The four above forms have the same **readback**, namely $\lambda\lambda(1(1(1(1(10)))))$. The translation **readback** will be defined in Section 5.4.

The terms ff and tt

The \mathbb{R} -term ff (i.e., the boolean false) is $\lambda((\underline{0}, \varepsilon) \odot \lambda(\underline{0}, \varepsilon))$ and the \mathbb{R} -term tt (i.e., the boolean true, that is also the combinator \mathbf{K}) is $\lambda(\lambda(\underline{0}, \varepsilon) \odot (\underline{1}, \varepsilon))$.

The Curry fixpoint combinator

The Curry fixpoint combinator \mathbf{Y} is:

$$\mathbf{Y} = \lambda(\underline{0}, \varepsilon) \nabla((\lambda((\underline{1}, 0) (\underline{0}, \varepsilon) \nabla((\underline{0}, 0) (\underline{0}, 1)))) (\lambda((\underline{1}, 1) (\underline{0}, \varepsilon) \nabla((\underline{0}, 0) (\underline{0}, 1))))))$$

and in notations of [13]:

$$\lambda x.(x \langle x_1^{x_0} \lambda y.(x_0 (y \langle y_1^{y_0} y_0 y_1)) \lambda y.(x_1 (y \langle y_1^{y_0} y_0 y_1))$$

or using the notations of [19]

$$\lambda x.(C_x^{x_0|x_1} (\lambda y.(x_0 (C_y^{y_0|y_1} (y_0 y_1))) \lambda y.(x_1 (C_y^{y_0|y_1} (y_0 y_1))))$$

5.2 \mathcal{L} -types for $\Lambda_{\mathbb{R}}$

Lists of \mathbb{R} -indices are called \mathcal{L} -types for $\Lambda_{\mathbb{R}}$.

Definition 7 (\mathcal{L} -types for $\Lambda_{\mathbb{R}}$). *The abstract syntax of \mathcal{L} -types for $\Lambda_{\mathbb{R}}$ is given by*

$$\ell ::= [] \mid (\underline{n}, \alpha) :: \ell$$

where (\underline{n}, α) is an \mathbb{R} -index.

Operations \ddagger and \downarrow are defined in Section 3 for lists on \mathbb{N} . Here we apply \ddagger to lists of \mathbb{R} -indices. For that, we have to define an order on the set of all \mathbb{R} -indices. We define first an order on strings of booleans.

Definition 8 (Order on strings of booleans). *An order $<_L$ on strings of booleans is defined as*

$$\frac{}{0 :: \ell <_L 1 :: \ell} \quad \frac{}{\varepsilon <_L b :: \ell} \quad \frac{\ell_1 <_L \ell_2}{b :: \ell_1 <_L b :: \ell_2}$$

In other words, $<_L$ is the lexicographic extension on lists of the order $0 < 1$.

Definition 9 (Order on \mathbb{R} -indices). *An order $<^{\mathbb{R}}$ on \mathbb{R} -indices is defined as*

$$\frac{n_1 < n_2}{(\underline{n}_1, \alpha_1) <^{\mathbb{R}} (\underline{n}_2, \alpha_2)} \quad \frac{\alpha_1 <_L \alpha_2}{(\underline{n}, \alpha_1) <^{\mathbb{R}} (\underline{n}, \alpha_2)}$$

In other words, $<^{\mathbb{R}}$ is the lexicographic product $< \times <_L$ of the orders $<$, on the naturals and $<_L$ on strings of booleans. By $\leq^{\mathbb{R}}$ we denote the relation $<^{\mathbb{R}}$ or $=$ and the relation $\leq^{\mathbb{R}}$ is total.

Definition 10 (Merge). *A binary operation which merges two lists of \mathbb{R} -indices is defined as follows:*

$$\begin{aligned} [] \ddagger \ell &= \ell \\ ((\underline{n}, \alpha) :: \ell) \ddagger [] &= (\underline{n}, \alpha) :: \ell \\ ((\underline{n}_1, \alpha_1) :: \ell_1) \ddagger ((\underline{n}_2, \alpha_2) :: \ell_2) &= \mathbf{if} (\underline{n}_1, \alpha_1) <^{\mathbb{R}} (\underline{n}_2, \alpha_2) \mathbf{then} (\underline{n}_1, \alpha_1) :: (\ell_1 \ddagger ((\underline{n}_2, \alpha_2) :: \ell_2)) \\ &\quad \mathbf{if} (\underline{n}_2, \alpha_2) <^{\mathbb{R}} (\underline{n}_1, \alpha_1) \mathbf{then} (\underline{n}_2, \alpha_2) :: (((\underline{n}_1, \alpha_1) :: \ell_1) \ddagger \ell_2) \end{aligned}$$

Remark 2. *The function \ddagger is not total.*

If a list ℓ is an empty list or it contains only indices with strictly positive first component, we write $\ell \in List^+$.

Definition 11 (Decrement). *Given a list ℓ , assume that we have a proof that $\ell \in List^+$, we can define operation \downarrow on this list:*

$$\begin{aligned} \downarrow [] &= [] \\ \downarrow ((\underline{n}+1, \alpha) :: \ell) &= (\underline{n}, \alpha) :: \downarrow \ell \end{aligned}$$

All properties proved in Lemma 1 hold also for the lists of \mathbb{R} -indices. We omit the proof, due to the lack of space and the fact that it is analogous to the proof of Lemma 1. By means of \mathcal{L} -typeability, we single out meaningful (well-formed) plain terms with resources and implicit names.

Definition 12 ($\Lambda_{\mathbb{R}}$). *$\Lambda_{\mathbb{R}}$ -terms are all plain terms with resources and implicit names that can be \mathcal{L} -typed by the following rules.*

$$\begin{array}{c} \text{(ind)} \frac{}{(\underline{n}, \alpha) : [(\underline{n}, \alpha)]} \quad \text{(abs)} \frac{t : (\underline{0}, \varepsilon) :: \ell}{\lambda t : \downarrow \ell} \quad \text{(app)} \frac{t_1 : \ell_1 \quad t_2 : \ell_2}{t_1 t_2 : \ell_1 \ddagger \ell_2} \\ \text{(era)} \frac{t : \ell}{(\underline{n}, \alpha) \odot t : [(\underline{n}, \alpha)] \ddagger \ell} \quad \text{(dup)} \frac{t : \ell \ddagger [(\underline{n}, \alpha 0), (\underline{n}, \alpha 1)]}{(\underline{n}, \alpha) \nabla t : [(\underline{n}, \alpha)] \ddagger \ell} \end{array}$$

The following example illustrates the Definition 12 by \mathcal{L} -typing the $\Lambda_{\mathbb{R}}$ -term SK.

Example 4.

$$\begin{array}{c} \frac{(\underline{2}, \varepsilon) : [(\underline{2}, \varepsilon)] \quad (\underline{0}, 0) : [(\underline{0}, 0)]}{(\underline{2}, \varepsilon) (\underline{0}, 0) : [(\underline{0}, 0), (\underline{2}, \varepsilon)]} \quad \frac{(\underline{1}, \varepsilon) : [(\underline{1}, \varepsilon)] \quad (\underline{0}, 1) : [(\underline{0}, 1)]}{(\underline{1}, \varepsilon) (\underline{0}, 1) : [(\underline{0}, 1), (\underline{1}, \varepsilon)]} \\ \frac{(\underline{2}, \varepsilon) (\underline{0}, 0) ((\underline{1}, \varepsilon) (\underline{0}, 1)) : [(\underline{0}, 0), (\underline{0}, 1), (\underline{1}, \varepsilon), (\underline{2}, \varepsilon)]}{(\underline{0}, \varepsilon) \nabla ((\underline{2}, \varepsilon) (\underline{0}, 0) ((\underline{1}, \varepsilon) (\underline{0}, 1))) : [(\underline{0}, \varepsilon), (\underline{1}, \varepsilon), (\underline{2}, \varepsilon)]} \quad \frac{(\underline{1}, \varepsilon) : [(\underline{1}, \varepsilon)]}{(\underline{0}, \varepsilon) \odot (\underline{1}, \varepsilon) : [(\underline{0}, \varepsilon), (\underline{1}, \varepsilon)]} \\ \frac{\lambda((\underline{0}, \varepsilon) \nabla ((\underline{2}, \varepsilon) (\underline{0}, 0) ((\underline{1}, \varepsilon) (\underline{0}, 1)))) : [(\underline{0}, \varepsilon), (\underline{1}, \varepsilon)]}{\lambda\lambda((\underline{0}, \varepsilon) \nabla ((\underline{2}, \varepsilon) (\underline{0}, 0) ((\underline{1}, \varepsilon) (\underline{0}, 1)))) : [(\underline{0}, \varepsilon)]} \quad \frac{(\underline{0}, \varepsilon) \odot (\underline{1}, \varepsilon) : [(\underline{0}, \varepsilon), (\underline{1}, \varepsilon)]}{\lambda(\underline{0}, \varepsilon) \odot (\underline{1}, \varepsilon) : [(\underline{0}, \varepsilon)]} \\ \frac{\lambda\lambda\lambda((\underline{0}, \varepsilon) \nabla ((\underline{2}, \varepsilon) (\underline{0}, 0) ((\underline{1}, \varepsilon) (\underline{0}, 1)))) : []}{\lambda\lambda\lambda((\underline{0}, \varepsilon) \nabla ((\underline{2}, \varepsilon) (\underline{0}, 0) ((\underline{1}, \varepsilon) (\underline{0}, 1)))) (\lambda\lambda(\underline{0}, \varepsilon) \odot (\underline{1}, \varepsilon)) : []} \end{array}$$

Notice that we abstract with λ (see Definition 12) only \mathbb{R} -index of the form $(\underline{0}, \varepsilon)$. Further, the definition of \ddagger ensures that in an \mathcal{L} -typed term an index can occur at most once (Theorem 4). The other binder, namely duplication, binds two indices of the form $(\underline{n}, \alpha 0)$ and $(\underline{n}, \alpha 1)$ and produces a new index (\underline{n}, α) . Closed terms are terms in which each \mathbb{R} -index is bound. We are mostly interested in linear and closed terms $\Lambda_{\mathbb{R}}$, i.e., terms in which all \mathbb{R} -indices are bound and occur once and only once.

Proposition 4 (Affineness). *If $t : \ell$ then t is affine.*

Proof. There are three rules where we merge lists, **app**, **era** and **dup**. In all these rules there is a function \ddagger defined only if it is applied in disjoint lists, hence it can not happen that there are two occurrences of an index in a typed term. As a consequence we have that if $t : \ell$, then each index occurs at most once in the term t , that is term t is affine. \square

Proposition 5 (Closedness). *If $t : []$ then t is closed.*

Proof. If $t : \ell$, then ℓ is the set of free \mathbb{R} -indices in the term. Therefore, if ℓ is empty then t has no free \mathbb{R} -index and t is closed. \square

Proposition 6 (Linearity). *If $t : []$ then t is linear.*

Proof. Actually there are two rules which eliminate \mathbb{R} -indices, namely **abs** and **dup**. But when **dup** eliminates two indices $(\underline{n}, \alpha 0)$ and $(\underline{n}, \alpha 1)$, it introduces (\underline{n}, α) . Therefore if a term is closed, all the \mathbb{R} -indices are checked for linearity when abstracted by λ . Therefore, if a term has no free \mathbb{R} -index, it is linear. \square

5.3 Reduction in $\Lambda_{\mathbb{R}}$

We define rewriting rules for normal forms w.r.t. \odot and ∇ and we prove type preservation. Consequently, linearity is preserved. Those rules are inspired by [13]. Basically, we propagate ∇ in the term and pull \odot out of the term.

First, we define replacement of an index in a term. By $t [(\underline{n}, \alpha) \leftarrow (\underline{m}, \beta)]$ we denote a term obtained from term t by replacing recursively the index (\underline{n}, α) by (\underline{m}, β) .

Definition 13 (Replacement). *Let us call $\text{cond}(\alpha, \delta, n, k)$ the condition*

$$n \neq k \vee \forall \gamma \in \{0, 1\}^* \delta \neq \alpha\gamma.$$

Notice that this can be written also

$$n \neq k \vee \neg(\alpha \text{ prefix } \delta).$$

Replacement $t [(\underline{n}, \alpha) \leftarrow (\underline{m}, \beta)]$ is defined as:

$$\begin{aligned} (\underline{n}, \alpha\gamma) [(\underline{n}, \alpha) \leftarrow (\underline{m}, \beta)] &= (\underline{m}, \beta\gamma) \\ (\underline{k}, \delta) [(\underline{n}, \alpha) \leftarrow (\underline{m}, \beta)] &= (\underline{k}, \delta) && \text{if } \text{cond}(\alpha, \delta, n, k) \\ (t_1 t_2) [(\underline{n}, \alpha) \leftarrow (\underline{m}, \beta)] &= t_1 [(\underline{n}, \alpha) \leftarrow (\underline{m}, \beta)] t_2 [(\underline{n}, \alpha) \leftarrow (\underline{m}, \beta)] \\ \lambda t [(\underline{n}, \alpha) \leftarrow (\underline{m}, \beta)] &= \lambda(t [(\underline{n+1}, \alpha) \leftarrow (\underline{m+1}, \beta)]) \\ ((\underline{k}, \delta) * t) [(\underline{n}, \alpha) \leftarrow (\underline{m}, \beta)] &= (\underline{k}, \delta) [(\underline{n}, \alpha) \leftarrow (\underline{m}, \beta)] * t [(\underline{n}, \alpha) \leftarrow (\underline{m}, \beta)], \quad * \in \{\odot, \nabla\} \end{aligned}$$

The rewriting system for $\lambda_{\mathbb{R}}$ -calculus is given by the rules in Figure 7.

Theorem 2 (\mathcal{L} -type preservation). *If $t : \ell$ and $t \rightarrow t'$, then $t' : \ell$.*

Proof. Assume that t matches the left-hand side of one of the rules in Figure 7. We consider the following two cases.

$$(\lambda - \odot) : \lambda(\underline{n+1}, \alpha) \odot t \rightarrow (\underline{n}, \alpha) \odot \lambda t$$

Rule $\lambda - \odot$ preserves type. Indeed

$\lambda(\underline{n+1}, \alpha) \odot t$	\rightarrow	$(\underline{n}, \alpha) \odot \lambda t$	$(\lambda - \odot)$
$(\underline{n}, \alpha) \nabla(\lambda t)$	\rightarrow	$\lambda(\underline{n+1}, \alpha) \nabla t$	$(\nabla - \lambda)$
$((\underline{n}, \alpha) \odot t_1) t_2$	\rightarrow	$(\underline{n}, \alpha) \odot (t_1 t_2)$	$(\mathbf{AppL} - \odot)$
$t_1 ((\underline{n}, \alpha) \odot t_2)$	\rightarrow	$(\underline{n}, \alpha) \odot (t_1 t_2)$	$(\mathbf{AppR} - \odot)$
$(\underline{n}, \alpha) \nabla(t_1 t_2)$	\rightarrow	$((\underline{n}, \alpha) \nabla t_1) t_2,$	$(\mathbf{AppL} - \nabla)$
$(\underline{n}, \alpha) \nabla(t_1 t_2)$	\rightarrow	$t_1 ((\underline{n}, \alpha) \nabla t_2),$	$(\mathbf{AppR} - \nabla)$
		$\mathbf{if}(\underline{n}, \alpha 0) \in t_1 \wedge (\underline{n}, \alpha 1) \in t_1$	
		$\mathbf{if}(\underline{n}, \alpha 0) \in t_2 \wedge (\underline{n}, \alpha 1) \in t_2$	
$(\underline{n}, \alpha) \odot (\underline{m}, \beta) \odot t$	\rightarrow	$(\underline{m}, \beta) \odot (\underline{n}, \alpha) \odot t, n < m$	$(\odot - \odot)$
$(\underline{n}, \alpha) \nabla(\underline{n}, \alpha 1) \odot t$	\rightarrow	$t \downarrow (\underline{n}, \alpha 0) \leftarrow (\underline{n}, \alpha)$	$(\odot - \nabla_1)$
$(\underline{n}, \alpha) \nabla(\underline{n}, \alpha 0) \odot t$	\rightarrow	$t \downarrow (\underline{n}, \alpha 1) \leftarrow (\underline{n}, \alpha)$	$(\odot - \nabla_0)$
$(\underline{n}, \alpha) \nabla(\underline{m}, \beta) \odot t$	\rightarrow	$(\underline{m}, \beta) \odot (\underline{n}, \alpha) \nabla t,$	$(\odot - \nabla)$
		$\mathbf{if} n \neq m \vee (\beta \neq \alpha 0 \wedge \beta \neq \alpha 1)$	
$(\underline{n}, \alpha) \nabla((\underline{n}, \alpha 1) \nabla t)$	\rightarrow	$(\underline{n}, \alpha) \nabla(\underline{n}, \alpha 0) \nabla(t \downarrow (\underline{n}, \alpha 0) \leftarrow (\underline{n}, \alpha 0)) \downarrow (\underline{n}, \alpha 1) \leftarrow (\underline{n}, \alpha 1)$	$(\nabla - \nabla_1)$
$(\underline{n}, \alpha) \nabla((\underline{m}, \beta) \nabla t)$	\rightarrow	$(\underline{m}, \beta) \nabla((\underline{n}, \alpha) \nabla t),$	$(\nabla - \nabla_2)$
		$\mathbf{if} n < m$	

Figure 7: The rewriting system for λ_{\oplus} -calculus

$$\frac{t : (\underline{0}, \varepsilon) :: \ell}{\frac{\lambda(\underline{n+1}, \alpha) \odot t : [(\underline{n+1}, \alpha)] \ddagger (\underline{0}, \varepsilon) :: \ell}{\lambda(\underline{n+1}, \alpha) \odot t : [(\underline{n}, \alpha)] \ddagger (\downarrow \ell)}} \qquad \frac{t : (\underline{0}, \varepsilon) :: \ell}{\frac{\lambda t : \downarrow \ell}{(\underline{n}, \alpha) \odot \lambda t : [(\underline{n}, \alpha)] \ddagger (\downarrow \ell)}}$$

Both the term on the left-hand side and the term on the right-hand side of the rule are typed with the same type.

$$(\nabla - \lambda) : (\underline{n}, \alpha) \nabla(\lambda t) \rightarrow \lambda(\underline{n+1}, \alpha) \nabla t$$

Rule $\nabla - \lambda$ preserves type. Indeed

$$\frac{t : [(\underline{n+1}, \alpha 0), (\underline{n+1}, \alpha 1)] \ddagger ((\underline{0}, \varepsilon) :: \ell)}{\frac{\lambda t : [(\underline{n}, \alpha 0), (\underline{n}, \alpha 1)] \ddagger \downarrow \ell}{(\underline{n}, \alpha) \nabla(\lambda t) : [(\underline{n}, \alpha)] \ddagger (\downarrow \ell)}} \qquad \frac{t : [(\underline{n+1}, \alpha 0), (\underline{n+1}, \alpha 1)] \ddagger ((\underline{0}, \varepsilon) :: \ell)}{\frac{(\underline{n+1}, \alpha) \nabla t : [(\underline{n+1}, \alpha)] \ddagger ((\underline{0}, \varepsilon) :: \ell)}{\lambda(\underline{n+1}, \alpha) \nabla t : [(\underline{n}, \alpha)] \ddagger (\downarrow \ell)}}$$

Both the term on the left-hand side and the term on the right-hand side of the rule are typed with the same type.

Proving that other rules preserve \mathcal{L} -type is straightforward. □

Example 5 $(\nabla - \nabla)$.

$$\lambda((\underline{0}, \varepsilon) \nabla(\underline{0}, 1) \nabla((\underline{0}, 0) (\underline{0}, 10) (\underline{0}, 11))) \rightarrow \lambda((\underline{0}, \varepsilon) \nabla(\underline{0}, 0) \nabla((\underline{0}, 00) (\underline{0}, 01) (\underline{0}, 1)))$$

See Figure 8.

Corollary 2 (Preservation of linearity). *If t is linear and $t \rightarrow t'$ then t' is also linear.*

5.4 Correspondence with Λ

In order to establish a correspondence between the introduced system $\Lambda_{\mathbb{R}}$ and the well-known system Λ , we follow the approach used in Section 4.3 and we define two translations: $\text{read} : \Lambda \rightarrow \Lambda_{\mathbb{R}}$ and $\text{readback} : \Lambda_{\mathbb{R}} \rightarrow \Lambda$.

Definition 14 (*read*). $\text{read} : \Lambda \rightarrow \Lambda_{\mathbb{R}}$

- $\text{read } \underline{n} = (\underline{n}, \varepsilon)$
- $\text{read } (\lambda t) = \text{let } u = \text{read } t$
 in if $(\underline{0}, \varepsilon) \in u$ then λu
 else $\lambda(\underline{0}, \varepsilon) \odot u$
- $\text{read}(t_1 t_2) = \bigvee_{(\underline{k}, \gamma) \in t_1^* \cap t_2^*} (\underline{k}, \gamma) (\text{rename } 0 (t_1^* \cap t_2^*)(\text{read } t_1) \text{rename } 1 (t_1^* \cap t_2^*)(\text{read } t_2))$

where

- *rename 0* ℓ replaces every \mathbb{R} -index of the form (\underline{n}, α) in the list ℓ of \mathbb{R} -indices by the corresponding \mathbb{R} -index of the form $(\underline{n}, \alpha 0)$ and similarly *rename 1* replaces all \mathbb{R} -index of the form (\underline{n}, α) in a the list by the corresponding \mathbb{R} -index of the form $(\underline{n}, \alpha 1)$.
- $t_1^* \cap t_2^*$ is a short notation for the list of \mathbb{R} -indices that occur both in $\text{read}(t_1)$ and in $\text{read}(t_2)$.

read is the formalization of the translations presented in Example 3.

Definition 15 (*readback*). $\text{readback} : \Lambda_{\mathbb{R}} \rightarrow \Lambda$

- $\text{readback } (\underline{n}, \alpha) = \underline{n}$
- $\text{readback } (\lambda t) = \lambda(\text{readback } t)$
- $\text{readback } (t_1 t_2) = (\text{readback } t_1) (\text{readback } t_2)$
- $\text{readback } ((\underline{n}, \alpha) \odot t) = \text{readback } t$
- $\text{readback } ((\underline{n}, \alpha) \nabla t) = \text{readback } t$.

Proposition 7 (Correctness of read). $\lambda t. \text{readback} (\text{read } t) : \Lambda \rightarrow \Lambda$ is the identity on Λ . In other words,

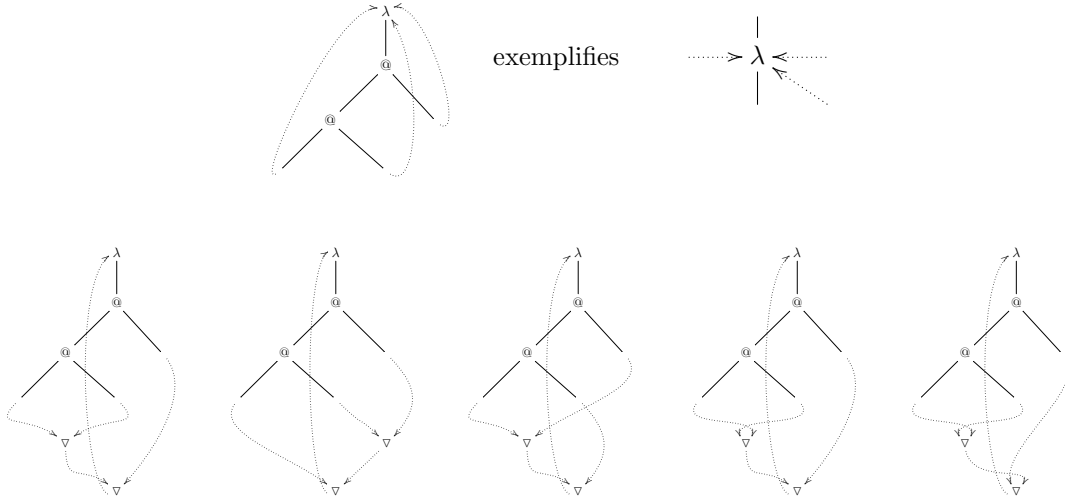
$$\text{readback} (\text{read } t) = t.$$

The function $\lambda t. \text{read} (\text{readback } t) : \Lambda_{\mathbb{R}} \rightarrow \Lambda_{\mathbb{R}}$ is an interesting function which associates with a term t another term with a somewhat standard disposition of \odot and ∇ , which we call *standardisation* of the term.

Notice that the same non linear λ -term may correspond to several $\Lambda_{\mathbb{R}}$ -term. For instance, this is the case for term $\lambda((\underline{0}\underline{0})\underline{0})$ (a $\Lambda_{\mathbb{R}}$ instance of $\lambda x.xxx$) illustrated by the following example and pictured in Figure 8.

Example 6. Consider the term $\lambda(\underline{0}, \varepsilon) \nabla(\underline{0}, 1) \nabla((\underline{0}, 0)(\underline{0}, 10))(\underline{0}, 11)$.

$$\text{readback}(\lambda(\underline{0}, \varepsilon) \nabla(\underline{0}, 1) \nabla((\underline{0}, 0)(\underline{0}, 10))(\underline{0}, 11)) = \lambda(\underline{0}\underline{0})\underline{0}$$

Figure 8: $\lambda((\underline{0}\underline{0})\underline{0})$ and antecedents by readback as terms with two duplications

but

$$\text{read}(\lambda(\underline{0}\underline{0})\underline{0}) = \lambda(\underline{0}, \varepsilon) \nabla(\underline{0}, 0) \nabla(((\underline{0}, 00)(\underline{0}, 01))(\underline{0}, 1))$$

Hence

$$\text{read} \circ \text{readback}(\lambda(\underline{0}, \varepsilon) \nabla(\underline{0}, 1) \nabla(\underline{0}, 0)(\underline{0}, 10)(\underline{0}, 11))) = \lambda(\underline{0}, \varepsilon) \nabla(\underline{0}, 0) \nabla((\underline{0}, 00)(\underline{0}, 01))(\underline{0}, 1)$$

The reader may notice that, in both terms, the first duplication is $(\underline{0}, \varepsilon) \nabla_-$. But the reader may also notice that the second duplication is $(\underline{0}, 1) \nabla_-$ in the first term and $(\underline{0}, 0) \nabla_-$ in the second term. So they are not the same. Choosing $(\underline{0}, 0) \nabla_-$ over $(\underline{0}, 1) \nabla_-$ is somewhat canonical. This corresponds to choosing the leftmost diagram in Figure 8. The fourth diagram corresponds to

$$\lambda(\underline{0}, \varepsilon) \nabla(\underline{0}, 0) \nabla(\underline{0}, 01)(\underline{0}, 00)(\underline{0}, 1)$$

and the fifth diagram corresponds to

$$\lambda(\underline{0}, \varepsilon) \nabla(\underline{0}, 1) \nabla(\underline{0}, 11)(\underline{0}, 10)(\underline{0}, 0).$$

We let the reader write the $\Lambda_{\mathbb{R}}$ term corresponding to the third diagram of Figure 8. There are 12 ways to write the term $\lambda(\underline{0}\underline{0})\underline{0}$ in $\Lambda_{\mathbb{R}}$ and to draw corresponding diagrams. The reader may devise the omitted cases.

5.5 Implementation of $\Lambda_{\mathbb{R}}$ in Haskell

We implemented the whole $\lambda_{\mathbb{R}}$ in Haskell, where the data type for $\Lambda_{\mathbb{R}}$ is as follows:

```
data RTerm = App RTerm RTerm
           | Abs RTerm
           | Ind Int [Bool]
           | Era Int [Bool] RTerm
           | Dup Int [Bool] RTerm
```

We give here a flavor of the implementation. We have defined functions `read` and `readback`. As presented in the previous section `readback` is relatively easy to define, by just forgetting duplications and erasures. Function `read`, denoted by `readLR`, is defined in Haskell as follows:

```
-- Given a list of indices and a term,
-- dupTheIndices applies all the duplications of that list to that term
dupTheIndices :: [(Int,[Bool])] -> RTerm -> RTerm
dupTheIndices [] t = t
dupTheIndices ((i,alpha):l) t = Dup i alpha (dupTheIndices l t)

-- 'consR' is a function used in 'readLR'
-- given a boolean and an index, put the boolean (0 or 1)
-- in front of all the alpha parts associated with the index
consR :: Bool -> Int -> RTerm -> RTerm
consR b i (App t1 t2) = App (consR b i t1) (consR b i t2)
consR b i (Abs t) = Abs (consR b (i+1) t)
consR b i (Ind j beta) = if i==j
                        then Ind j (b:beta)
                        else Ind j beta
consR b i (Era j beta t) = if i==j
                        then Era j (b:beta) (consR b i t)
                        else Era j beta (consR b i t)
consR b i (Dup j beta t) = if i==j
                        then Dup j (b:beta) (consR b i t)
                        else Dup j beta (consR b i t)
```

`indOf` is a function that extracts the indices of a term; `?` is an infix operator which returns a boolean, `i ? t` returns `True` if and only if `i` occurs in `t`.

```
readLR :: Term -> RTerm
readLR (Ap t1 t2) =
  let rt1 = readLR t1
      rt2 = readLR t2
      indToIndR i = (i,[])
      commonInd = sort (indOf t1 'intersect' indOf t2)
      pt1 = foldl (.) id (map (consR False) commonInd) rt1
      pt2 = foldl (.) id (map (consR True) commonInd) rt2
  in dupTheIndices (map indToIndR commonInd) (App pt1 pt2)
readLR (Ab t) = if 0 ? t then Abs (readLR t) else Abs (Era 0 [] (readLR t))
readLR (In i) = Ind i []
```

We also present the Haskell code for test of linearity and closedness:

```

-- (iL t) returns the list of free (R)-de Bruijn indices of t
-- if all the binders of the term binds one and only one (R)-index.
remove :: Eq a => a -> [a] -> Maybe [a]
remove _ [] = Nothing
remove x (y:l) = if x == y then Just l
                  else case (remove x l) of
                    Nothing -> Nothing
                    Just l' -> Just (y:l')

iL :: RTerm -> Maybe [(Int,[Bool])]
iL (Ind n alpha) = Just [(n,alpha)]
iL (Abs t) =
  case iL t of
    Nothing -> Nothing
    Just u -> case remove (0,[]) u of
      Nothing -> Nothing
      Just u' -> case find ((==) 0).fst) u' of
        Just _ -> Nothing
        Nothing -> Just $ map ((i,a)->(i-1,a)) u

iL (App t1 t2) =
  case iL t1 of
    Nothing -> Nothing
    Just u1 -> case iL t2 of
      Nothing -> Nothing
      Just u2 -> if null (u1 'intersect' u2)
                  then Just(u1 ++ u2)
                  else Nothing

iL (Era n alpha t) = case iL t of
  Nothing -> Nothing
  Just u -> Just ((n,alpha):u)

iL (Dup n alpha t) =
  case iL t of
    Nothing -> Nothing
    Just u -> if (n,alpha++[False]) 'elem' u &&
                 (n,alpha++[True]) 'elem' u
               then Just ((n,alpha):(delete (n,alpha++[False]) (delete (n,alpha++[True]) u)))
               else Nothing

-- is linear in the sense that all the binders bound one and only one index.
isLinearAndClosed t = case iL t of
  Nothing -> False
  Just u -> u == []

```

The β -reduction of $\lambda_{\mathbb{R}}$ -terms is in [GitHub](#).

5.6 Implementation of $\Lambda_{\mathbb{R}}$ in Agda

When presenting a theory in a paper (i.e., in English) and in a dependent type functional programming language (i.e., in Agda) the aims differ. Wherever in English one wants to minimise the number of concepts and to rely on intuition, in Agda, one wants to minimise the size of the description and overall the size of the proof of correctness that each term has to carry as part of its code. See for instance, term `S` in Section 5.6.3. As mentioned in Section 3.1: in the English text, typing rules are non deterministic, by using partial functions which may fail, wherever in Agda, the implementation of the same rules is made deterministic and terminating by the use of added parameters, which represent side conditions of the rules and which are proofs of the applicability of the functions. More precisely, in the English text and in Agda, there are two operators \downarrow and \ddagger . Wherever in the paper \downarrow and \ddagger are not total, i.e., \downarrow is defined only on list with strictly positive elements (Definition 3) and \ddagger is defined on pair of disjoint lists (Definition 2 and the following remark), Agda requires its functions to be total and therefore, in Agda, \downarrow and \ddagger must be total. Therefore, despite the same language is defined, the presentations in both approaches are largely different. Moreover since the implementation in Agda of the reduction should translate the \mathcal{L} -types of $\Lambda_{\mathbb{R}}$ (Section 5.2), it should be rather elaborated and as a matter of fact we are not able right now to provide in Agda an implementation for the reduction as we have done in Haskell. However we attach more confidence to the Agda code than to the Haskell code.

5.6.1 The data type $\Lambda_{\mathbb{R}}$

Let us now present the implementation in Agda of the syntax of what corresponds to $\Lambda_{\mathbb{R}}$ fully.

```
data  $\Lambda_{\mathbb{R}}$  : List (N × List Bool) → Set where
   $\mathbb{R}$ ind : (i : N) → (α : List Bool) →  $\Lambda_{\mathbb{R}}$  [ (i , α) ]
   $\mathbb{R}$ α :  $l_1$   $l_2$  : List (N × List Bool) →  $\Lambda_{\mathbb{R}}$   $l_1$  →  $\Lambda_{\mathbb{R}}$   $l_2$  →  $\Lambda_{\mathbb{R}}$  ( $l_1$   $\ddagger$   $l_2$ )
  λ :  $l$  : List (N × List Bool) →  $\Lambda_{\mathbb{R}}$  ((0 , ε) ::  $l$ ) → (p :  $\succ 0$   $l$ ) →  $\Lambda_{\mathbb{R}}$  (map-1 $\succ 0$   $l$  p)
   $\mathbb{R}$ ⊖ :  $l$  : List (N × List Bool) → (iα : N × List Bool) →  $\Lambda_{\mathbb{R}}$   $l$  →  $\Lambda_{\mathbb{R}}$  (iα  $\searrow$   $l$ )
   $\mathbb{R}$ _for_∇_ : ( $l$  : List (N × List Bool)) → (iα : N × List Bool) →
     $\Lambda_{\mathbb{R}}$  ((proj1 iα , proj2 iα ::r °)  $\searrow$  ((proj1 iα , proj2 iα ::r 1)  $\searrow$   $l$ )) →
     $\Lambda_{\mathbb{R}}$  (iα  $\searrow$   $l$ )
```

5.6.2 The components of $\Lambda_{\mathbb{R}}$

$\Lambda_{\mathbb{R}}$ requires *operators on lists*:

ε is the notation we have chosen for the empty list of Bool, for reason of simplicity.

$\succ 0$ tests whether a list contains only positive items (more precisely that the first components of pairs (i, α) are positive). In the program `p : $\succ 0$ l` means that `p` is a proof that the list l contains only \mathbb{R} -indices whose first components are positive.

`map-1 $\succ 0$` takes a list and a proof that this list contains only strictly positive elements and decrements all the items of lists. Thus we know that decrementing the elements of a list is safe, since it can only be applied to a list of strictly positive elements.

\searrow inserts, at the right place, an item in a sorted list.

Besides,

- ∇ has a third argument, occurring before the keyword `for`. This argument, which is a list of \mathbb{R} -indices, provides `Agda` a hint to deal with the constraint, that `Agda` would be otherwise unable to solve. This hint is a list in which the \mathbb{R} -index will be inserted by \searrow to produce the type of the result.
- \circ `and`¹ are shorthands for `Bool` values, which will be otherwise cumbersome using 0 or 1 and would clash.

The *five constructors* of $\Lambda_{\mathbb{R}}$ tell how to build a term of type $\Lambda_{\mathbb{R}}\ell$ where ℓ is determined by the context. If the parameter of $\Lambda_{\mathbb{R}}$ is the empty list ε , this means that the term is closed, like in the examples in the next section. The five constructors are:

`@ind` takes a natural `i` and a list α of booleans and produces a term which is the index (i, α) . This term has the type $\Lambda_{\mathbb{R}}$ with parameter the singleton $[(i, \alpha)]$, which means that it has a unique free \mathbb{R} -index, that is (i, α) .

\boxtimes is the binary operator that builds an application of a term t_1 to a term t_2 . If t_1 has type $\Lambda_{\mathbb{R}}\ell_1$ and t_2 has type $\Lambda_{\mathbb{R}}\ell_2$, then $t_1 \boxtimes t_2$ has type $\Lambda_{\mathbb{R}}(\ell_1 \ddagger \ell_2)$. Beware that \boxtimes does not guarantee that $t_1 \boxtimes t_2$ is affine, linear or anything else. It just collects the \mathbb{R} -indices. The test of “linearity” will be made by λ .

λ plays a key role. It takes a term t of type $\Lambda_{\mathbb{R}}((0, \varepsilon) :: \ell)$, then a proof that all the elements of ℓ are positive (i.e., non zero). This means that the ordered list of the \mathbb{R} -indices of t starts with $(0, \varepsilon)$ and is followed by a list of elements which are not $(0, \alpha)$. This way, one is sure that the \mathbb{R} -index which is abstracted by λ and which is by definition $(0, \varepsilon)$ occurs only once and has no clone of the form $(0, \alpha)$. Hence for this \mathbb{R} -index the linearity is guaranteed, as will be the linearity of all the \mathbb{R} -indices in a closed term (a term in which all the \mathbb{R} -indices are abstracted). Moreover, in the spirit of de Bruijn indices, the definition insures that λt is of type $\Lambda_{\mathbb{R}}(\text{map-1}\succ 0 \ell p)$. The proof `p` makes `map-1` \succ `0` to work properly.

\odot is the simplest operator of $\Lambda_{\mathbb{R}}$. It should not be seen as an *eraser* since we want it to work the other way around: it takes a term in which the \mathbb{R} -index `i` α (a shorthand for (i, α)) does not occur possibly, and produces a term in which it occurs.

`_for_` ∇ is a three places operator. Like ∇ it takes a \mathbb{R} -index and a term. The term contains the \mathbb{R} -index, but also other \mathbb{R} -indices in a list ℓ . For the system, the list ℓ is hard to guess, thus ℓ is provided to the system as a third argument, before the `for`. Therefore `for (i, α) ∇ t` has type $\Lambda_{\mathbb{R}}\ell'$ where ℓ' is ℓ in which (i, α) is inserted at the right place and `t` has type $\Lambda_{\mathbb{R}}\ell''$ where ℓ'' is ℓ in which both (i, α°) and (i, α^\dagger) are inserted at the right place. α° is α followed by \circ and α^\dagger is α followed by \dagger . One sees the *duplication* of the \mathbb{R} -index (i, α) from `for (i, α) ∇ t` to `t`.

5.6.3 Implementing the bestiary in Agda

Let us consider terms from Section 5.1 coded in our `Agda` implementation.

I is written:

$$\begin{aligned} \mathbf{I} &: \Lambda_{\mathbb{R}} \varepsilon \\ \mathbf{I} &= \lambda (\mathbb{R}\text{ind } 0 \ \varepsilon) \succ_0 \varepsilon \end{aligned}$$

The type declaration says that **i** is a closed term. $\succ_0 \varepsilon$ is a proof defined elsewhere which says that the empty list ε has all its elements strictly positive, which is more or less trivial since ε has no element!

K is written

$$\begin{aligned} \mathbf{K} &: \Lambda_{\mathbb{R}} \varepsilon \\ \mathbf{K} &= \lambda (\lambda ((0, \varepsilon) \odot (\mathbb{R}\text{ind } 1 \ \varepsilon)) (\succ_0 :: \mathbf{z} \prec_s \succ_0 \varepsilon)) \succ_0 \varepsilon \end{aligned}$$

K is yet another closed term. $\mathbf{z} \prec_s$ is a proof that $0 \prec 1$ and $\succ_0 :: \mathbf{z} \prec_s \succ_0 \varepsilon$ is a proof that the singleton $[(1, \varepsilon)]$ is made of \mathbb{R} -indices that are strictly positive. Indeed $\succ_0 ::$ combines a proof that **1** is greater than **0** and a proof that the empty list contains only positive elements to produce that proof.

S is written:

$$\begin{aligned} \mathbf{S} &: \Lambda_{\mathbb{R}} \varepsilon \\ \mathbf{S} &= \lambda (\lambda (\lambda (((1, \varepsilon) :: [(2, \varepsilon)]) \text{ for } (0, \varepsilon) \nabla \\ &\quad (((\mathbb{R}\text{ind } 2 \ \varepsilon) \boxtimes (\mathbb{R}\text{ind } 0 \ [^\circ]))) \boxtimes ((\mathbb{R}\text{ind } 1 \ \varepsilon) \boxtimes (\mathbb{R}\text{ind } 0 \ [^\dagger]))) \\ &\quad (\succ_0 :: \mathbf{z} \prec_s (\succ_0 :: \mathbf{z} \prec_s \succ_0 \varepsilon))) (\succ_0 :: \mathbf{z} \prec_s \succ_0 \varepsilon)) \succ_0 \varepsilon \end{aligned}$$

Term **S** uses proofs introduced previously and shows a usage of `_for_∇_` specifically with the introduction of the list $(1, \varepsilon) :: [(2, \varepsilon)]$ as a first argument.

two. In Section 5.1 we represented the term **5**. Here let us be more modest and let us represent the term **2**:

$$\begin{aligned} \mathbf{two} &: \Lambda_{\mathbb{R}} \varepsilon \\ \mathbf{two} &= \lambda (\lambda ([(0, \varepsilon)] \text{ for } (1, \varepsilon) \nabla ((\mathbb{R}\text{ind } 1 \ [^\circ]) \boxtimes ((\mathbb{R}\text{ind } 1 \ [^\dagger]) \boxtimes (\mathbb{R}\text{ind } 0 \ \varepsilon)))) \\ &\quad (\succ_0 :: \mathbf{z} \prec_s \succ_0 \varepsilon)) \succ_0 \varepsilon \end{aligned}$$

One sees the same proofs $\succ_0 \varepsilon$ and $\succ_0 :: \mathbf{z} \prec_s \succ_0 \varepsilon$ and that $(1, \varepsilon)$ is duplicated.

6 Discussion and Related work

Compared to languages with explicit names, like λlxr [19] or the language of [13], $\lambda_{\mathbb{R}}$ is a much *simpler* calculus, because, we can tell exactly how the \mathbb{R} -indices are duplicated, since we have a tight control on the way those indices are built. As consequences, there are fewer basic rules and a simple implementation is possible. For instance, if we consider a rough quantitative aspect, the calculus of [19] has 19 rules and 6 congruences, the system of [13] has 18 rules (9 basic rules

and 8 rules for substitution) and 4 congruences, whereas our system $\lambda_{\mathbb{R}}$ has 12 rules and no congruences.

The \mathcal{L} -types of our system address a notion of *correctness* which is somewhat orthogonal to this of classic types (say simple types or higher order types). A term is well \mathcal{L} -typed if it is linear and we prove, thanks to \mathcal{L} -type preservation, that linearity is preserved by reduction. The two notions of types are orthogonal in the sense that classic types say something about the result (the term is a natural or a boolean, for instance) whereas \mathcal{L} -types say something about the internal features of the terms (the term is linear). Since we do not characterize the “result” of a computation, but only the structure of the term, there is no notion of “progress” associated with \mathcal{L} -types, there is only a notion of “preservation” (terms stay linear along their reduction, i.e., \mathcal{L} -type is preserved). However, it is possible to introduce other standard type systems, such as simple types, intersection types or system F, to further characterize computational properties of \mathcal{L} -typed terms. As the continuation of the presented research, we intent to explore such a hierarchy of type systems for the $\lambda_{\mathbb{R}}$ calculus.

The calculus $\lambda_{\mathbb{R}}$ has connection with the *differential λ -calculus* of Erhard and Regnier [10] where the *fan* ∇ is a non commutative differential operator (similar to their D) and the *black-hole* \odot corresponds to an empty iteration of ∇ (like D^0). Therefore $\lambda_{\mathbb{R}}$ can be considered as a *non commutative differential λ -calculus*, where iterations are no more done on natural numbers, but on lists of `Bool`. These observations merit to be deepened.

The concepts of this paper are implemented in `Agda` and cover topics not present in the textbook *Programming Language Foundations in Agda* [35] that can be fitted into it. Especially the use of explicit substitution (Section 4.3) is a nice and short way to implement strong normalization in the λ -calculus.

The reader may have noticed that we focus on closed terms. This is not due to the fact that open terms are, for us, of no interest but this is because if we would check linearity for open terms as well, terms in `Agda` which are yet large, since they Carry their proofs of linearity at each abstraction would be otherwise intractable. Paul Tarau and Valeria de Pavia address a similar problem ([33] Section 4.3) in their attempt to generate closed linear lambda terms. Anyway in functional programming, programs of interest are those with no free (undeclared) variable.

7 Conclusion

This paper introduced three new calculi with implicit names dealing with linearity, i.e., the property that bound variables occur once and only once. The first calculus is the most straightforward, since it is just the λ -calculus with unique occurrence of each variable (BCI λ -terms). The second calculus addresses an abstract implementation of β -reduction through explicit substitution. The third calculus is a calculus of resource with explicit duplication and explicit erasure. For each of those calculi, we introduce a specific \mathcal{L} -type system, which is used (a) to define terms and (b) to characterise linearity of closed terms. Those types represent lists of free implicit names: de Bruin indices for the first two calculi, and new \mathbb{R} -indices for the calculus with resource control.

Moreover for these three calculi, we propose an implementation in a language with dependent types, namely `Agda`. The features of those calculi are not easily represented by dependent type constraints. For instance, the \mathcal{L} -type system for Λ_v^{in} is not trivial as is its `Agda` implementation and for a plain λ -term, it is not easy to build its corresponding $\lambda_{\mathbb{R}}$ -term in the calculus of resource. For these reasons, the implementation of β reduction of $\Lambda_{\mathbb{R}}$ is not done. However a whole code in `Haskell` exists where constraints are not as deeply checked as they are in the `Agda`

code. Completing the Agda code is ongoing, but working on the Agda code played a main role in the design of the calculi and of their \mathcal{L} -types and thanks to those \mathcal{L} -types, we were able to describe and characterise λ_v^{in} -terms (linear terms of explicit substitution).

To summarize, we have introduced four concepts: \mathcal{L} -types, implicit names (de Bruijn indices and \mathbb{R} -indices), explicit duplication and explicit erasure, and three calculi: λ^{in} , λ_v^{in} and $\lambda_{\mathbb{R}}$.

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