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# Morphisms Cohomology and Deformations of Hom-algebras

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## Abstract

The purpose of this paper is to study deformation theory of Hom-associative algebra morphisms and Hom-Lie algebra morphisms. We introduce a suitable cohomology and discuss Infinitesimal deformations, equivalent deformations and obstructions. Moreover, we provide some examples.

## Introduction

The first instance of Hom-type algebras appeared in Physics literature when looking for quantum deformations of some algebras of vector fields, like Witt and Virasoro algebras, in connection with oscillator algebras. A quantum deformation consists of replacing the usual derivation by a  $\sigma$ -derivation. The main examples use Jackson derivation, it turns out that the obtained algebras no longer satisfy Jacobi identity but a modified version involving a homomorphism. These algebras were called Hom-Lie algebras and studied by Hartwig, Larsson and Silvestrov in [10] and [11]. Hom-associative algebras play the role of associative algebras in the Hom-Lie setting. They were introduced by the last author and Silvestrov in [12], where it is shown that the commutator bracket defined by the multiplication in a Hom-associative algebra leads naturally to a Hom-Lie algebra. The adjoint functor was considered by D. Yau [17]. Usually, we call these type of algebras Hom-algebras because of the homomorphism involving in their structure.

The original deformation theory was developed by Gerstenhaber for ring and algebras using formal power series in [7]. It is closely related to Hochschild cohomology. Then,

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it was extended to Lie algebras, using Chevalley-Eilenberg cohomology, by Nijenhuis and Richardson [15]. Notice that a more general setting was considered by Fialowski and her collaborators in [2]. Deformation theory of associative algebra morphisms have been studied by Gerstenhaber and Schack in a series of papers [8, 9, 6]. Deformations of Lie algebra morphisms have been considered by Nijenhuis and Richardson in [15], and more recently by Frégier [3], see also [4, 5]. Cohomology and Deformations of Hom-associative algebras and Hom-Lie algebras were studied first in [12] then completed in [1].

The purpose of this paper is to provide first a Hochschild cohomology of Hom-associative algebras and a Chevalley-Eilenberg cohomology of Hom-Lie algebras which are compatible with Hom-algebra morphisms and then study their deformations. We aim to generalize the cohomology theory associated to deformations of Lie algebra morphisms given by Frégier for Lie algebras in [3] and that introduced by Gerstenhaber and Schack in [6] for associative algebra morphisms. Moreover, we generalize the algebra valued cohomology theory in [1] to any bimodule.

The paper is organized as follows. In Section 1, we review some basic definitions about Hom-algebras and Hom-type Hochschild cohomology (resp. Hom-type Chevalley-Eilenberg cohomology). In Section 2, we introduce a cohomology complex with values in an adjoint bimodule related to deformations of multiplicative Hom-associative algebras and give an explicit formula for the coboundary operator. In Section 3, we construct a cohomology complex of Hom-algebra morphisms  $\phi : \mathcal{A} \rightarrow \mathcal{B}$  for both Hom-associative algebras and Hom-Lie algebras. We define the module of  $n$ -cochains of a morphism  $\phi$  by  $C^n(\phi, \phi) = C^n(\mathcal{A}, \mathcal{A}) \otimes C^n(\mathcal{B}, \mathcal{B}) \otimes C^{n-1}(\mathcal{A}, \mathcal{B})$  and give the coboundary operator formula related to these triples. The corresponding cohomology is denoted by  $H^*(\mathcal{A}, \mathcal{B})$ . Moreover, we provide some examples. Section 4 deals with deformations of Hom-associative algebra morphisms. We generalize Gerstenhaber and Schack Theorems given in [8]. We have observed that the infinitesimal is a 2-cocycle in the deformation complex of the Hom-algebra morphism. Also, it is shown that if  $H^2(\mathcal{A}, \mathcal{B}) = 0$ , then every formal deformation is equivalent to a trivial deformation. Furthermore, we prove that the obstruction to extend a deformation of order  $N$  to a deformation of order  $N + 1$  is a 3-cocycle. One can derive as a consequence that if  $H^3(\mathcal{A}, \mathcal{B}) = 0$  then any infinitesimal deformation can be extended. The paper is ended with a section where we explicitly compute for several examples a cohomology of Hom-Lie algebra morphisms and discuss some deformations.

## 1 Preliminaries

In this section, we recall some basic definitions and summarize the Hom-type Hochschild cohomology and the Hom-type Chevalley-Eilenberg cohomology. We refer to a Hom-algebra as a triple consisting of a  $\mathbb{K}$ -vector space or a module together with a bilinear map (a multiplication) and a linear map. We assume that  $\mathbb{K}$  is an algebraically closed field of characteristic 0, even if most of the result are valid for any field. These Hom-algebras aim to generalize classical algebraic structures and the main feature is that the identities defining the structures are twisted by homomorphisms. In the sequel, we will write  $\otimes$  for

$\otimes_{\mathbb{K}}, \mathcal{A}^{\otimes n}$  for the  $n$ -fold tensor product  $\mathcal{A} \otimes \cdots \otimes \mathcal{A}$  and  $\mathcal{A}^{\times n}$  for  $\mathcal{A} \times \cdots \times \mathcal{A}$ .

**Definition 1.1.** A Hom-associative algebra over  $\mathcal{A}$  is a triple  $(\mathcal{A}, \mu, \alpha)$  consisting of a  $\mathbb{K}$ -vector space  $\mathcal{A}$ , a bilinear map  $\mu : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  and a linear map  $\alpha : \mathcal{A} \rightarrow \mathcal{A}$  satisfying

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)), \quad \text{for all } x, y, z \in \mathcal{A} \quad (\text{Hom-associativity}).$$

A Hom-associative algebra is called multiplicative if  $\alpha$  is an algebra morphism.

**Definition 1.2.** A Hom-Lie algebra is a triple  $(\mathcal{L}, [\cdot, \cdot], \alpha)$  consisting of a  $\mathbb{K}$ -vector space  $\mathcal{L}$ , a bilinear map  $[\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$  and a linear map  $\alpha : \mathcal{L} \rightarrow \mathcal{L}$  satisfying

$$[x, y] = -[y, x] \quad \text{for all } x, y \in \mathcal{L} \quad (\text{skew-symmetry}),$$

$$\bigcirc_{x,y,z} [\alpha(x), [y, z]] = 0 \quad \text{for all } x, y, z \in \mathcal{L} \quad (\text{Hom-Jacobi identity}),$$

where  $\bigcirc_{x,y,z}$  denotes summation over the cyclic permutation on  $x, y, z$ .

**Definition 1.3.** Let  $(\mathcal{A}, \mu, \alpha)$  and  $(\mathcal{A}', \mu', \alpha')$  (resp.  $(\mathcal{L}, [\cdot, \cdot], \alpha)$  and  $(\mathcal{L}', [\cdot, \cdot]', \alpha')$ ) be two Hom-associative (resp. Hom-Lie) algebras. A linear map  $\phi : \mathcal{A} \rightarrow \mathcal{A}'$  (resp.  $\phi : \mathcal{L} \rightarrow \mathcal{L}'$ ) is a Hom-associative (resp. Hom-Lie) algebra morphism if

$$\mu' \circ (\phi \otimes \phi) = \phi \circ \mu \quad (\text{resp. } [\cdot, \cdot]' \circ (\phi \otimes \phi) = \phi \circ [\cdot, \cdot]) \quad \text{and} \quad \phi \circ \alpha = \alpha' \circ \phi.$$

**Theorem 1.4** ([18]). Let  $A = (A, \mu)$  be an associative algebra (resp. a Lie algebra) and  $\alpha : A \rightarrow A$  be an algebra morphism with respect to  $\mu$ , i.e.  $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$ . Then  $A_\alpha = (A, \mu_\alpha = \alpha \circ \mu, \alpha)$  is a Hom-associative algebra (resp. a Hom-Lie algebra).

This theorem provides an easy way to deform a usual associative algebra (resp. Lie algebra) to a Hom-associative algebra (resp. Hom-Lie algebra). It could be also generalized to provide a new Hom-associative algebra (resp. Hom-Lie algebra) from a given Hom-associative algebra (resp. Hom-Lie algebra) along an algebra morphism.

## 1.1 Representations of Hom-associative algebras

**Definition 1.5.** Let  $(\mathcal{A}, \mu, \alpha)$  be a Hom-associative algebra. A (left)  $\mathcal{A}$ -module is a triple  $(M, f, \gamma)$  where  $M$  is  $\mathbb{K}$ -vector space and  $f : M \rightarrow M$ ,  $\gamma : \mathcal{A} \otimes M \rightarrow M$ , are  $\mathbb{K}$ -linear maps, such that the following identity is satisfied

$$\gamma \circ (\mu \otimes f) = \gamma \circ (\alpha \otimes \gamma).$$

**Proposition 1.6** (Left adjoint  $\mathcal{A}$ -module). Let  $(\mathcal{A}, \mu, \alpha)$  and  $(\mathcal{A}', \mu', \alpha')$  be two Hom-associative algebras and  $\phi : \mathcal{A} \rightarrow \mathcal{A}'$  be a Hom-associative algebra morphism.

We consider the triple  $(M, f, \gamma)$ , where  $M = \mathcal{A}'$ ,  $\gamma = \rho_l = \mu'(\phi \otimes id)$  and  $f = \alpha'$ . Then  $M$  is an  $\mathcal{A}$ -module called adjoint representation of  $\mathcal{A}$  induced by the Hom-associative algebra morphism  $\phi$ .

*Proof.* Indeed, for  $\rho_l : \mathcal{A} \times \mathcal{A}' \rightarrow \mathcal{A}'$  such that  $(a, a') \rightarrow \mu'(\phi(a), a')$ , we prove that the following diagram commutes

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}' & \xrightarrow{\mu \otimes \alpha'} & \mathcal{A} \otimes \mathcal{A}' \\ \alpha \otimes \rho_l \downarrow & & \downarrow \rho_l \\ \mathcal{A} \otimes \mathcal{A}' & \xrightarrow{\rho_l} & \mathcal{A}' \end{array}$$

□

**Remark 1.1.** Similarly, we define the right adjoint module by considering the triple  $(\mathcal{A}', \rho_r = \mu'(id \otimes \phi), \alpha')$ . These  $\mathcal{A}$ -modules are also called adjoint representations.

A bimodule structure is given by left and right actions with maps  $\rho_r$  and  $\rho_l$  that satisfy the following additional condition  $\rho_r(\rho_l(x, y), \alpha(z)) = \rho_l(\alpha(x), \rho_r(y, z))$ . Actually, left and right modules are special cases of bimodules, one may set  $\rho_r = 0$  (resp.  $\rho_l = 0$ ).

## 1.2 Representation of Hom-Lie algebras

Let  $(\mathcal{L}, [\cdot, \cdot], \alpha)$  be a Hom-Lie algebra and  $\beta \in \mathfrak{gl}(V)$  be an arbitrary linear self map on  $V$ , where  $V$  is an arbitrary vector space. We denote a left action of  $\mathcal{L}$  on  $V$  by the following bracket  $[\cdot, \cdot]_V : \mathcal{L} \times V \rightarrow V$  such that  $(g, V) \rightarrow [g, v]_V$ .

**Definition 1.7.** A triple  $(V, [\cdot, \cdot]_V, \beta)$  is called a left Hom-module on the Hom-Lie algebra  $\mathcal{L}$  or  $\mathcal{L}$ -Hom-module  $V$ , with respect to  $\beta \in \mathfrak{gl}(V)$  if it satisfies  $[\alpha(u), \beta(v)]_V = \beta([u, v]_V)$  and

$$[[u, v], \beta(z)]_V = [\alpha(u), [v, z]_V]_V - [\alpha(v), [u, z]_V]_V. \quad (1)$$

We say that  $(V, [\cdot, \cdot]_V, \beta)$  is a representation of  $\mathcal{L}$ .

**Proposition 1.8.** Let  $(\mathcal{L}, [\cdot, \cdot], \alpha)$  and  $(\mathcal{L}', [\cdot, \cdot]', \alpha')$  be two Hom-Lie algebras and  $\phi : \mathcal{L} \rightarrow \mathcal{L}'$  be a Hom-Lie algebra morphism. Let  $(\Pi, \beta, \rho_l)$  be a triple where  $\Pi = \mathcal{L}'$ ,  $\beta = \alpha'$ ,  $\rho_l = [\cdot, \cdot]_V = [\phi, id]'$ . Then,  $\Pi$  is a left  $\mathcal{L}$ -module called left adjoint representation of  $\mathcal{L}$  via  $\phi$ .

**Remark 1.2** (Coadjoint representation of  $\mathcal{L}$ ). Let  $(\mathcal{L}, [\cdot, \cdot], \alpha)$  be a Hom-Lie algebra and  $(V, [\cdot, \cdot]_V, \beta)$  be a representation of  $\mathcal{L}$ . Let  $V^*$  be the dual vector space of  $V$ . We define a bilinear map  $[\cdot, \cdot]_{V^*} : \mathcal{L} \times V^* \rightarrow V^*$  by  $[x, f]_{V^*}(v) = -f([x, v]_V)$ ,  $\forall x \in \mathcal{L}, f \in V^*$  and  $v \in V$ . Let  $f \in V^*$ ,  $x, y \in \mathcal{L}$  and  $v \in V$ , we compute the right hand side of the identity (1).

$$\begin{aligned} [\alpha(x), [y, f]_{V^*}]_{V^*} - [\alpha(y), [x, f]_{V^*}]_{V^*}(v) &= -[y, f]([ \alpha(x), v ]_V) + [x, f]([ \alpha(y), v ]_V) \\ &= f([y, [ \alpha(x), v ]_V]_V) - f([x, [ \alpha(y), v ]_V]_V). \end{aligned}$$

On the other hand, we set  $\beta^* = {}^t \beta$  then the left hand side of (1) gives

$$[[x, y], \beta^*(f)]_{V^*}(v) = -\beta^*(f)([x, y], v)_V = -{}^t \beta(f)([x, y], v)_V = -f \circ \beta([x, y], v)_V.$$

**Proposition 1.9.** Let  $(\mathcal{L}, [\cdot, \cdot], \alpha)$  be a Hom-Lie algebra and  $(V, [\cdot, \cdot]_V, \beta)$  be a representation of  $\mathcal{L}$ . The triple  $(V^*, [\cdot, \cdot]_{V^*}, \beta^*)$ , where  $[x, f]_{V^*}(v) = -f([x, v]_V)$ ,  $\forall x \in \mathcal{L}, f \in V^*, v \in V$ , defines a representation of the Hom-Lie algebra  $(\mathcal{L}, [\cdot, \cdot], \alpha)$  if and only if

$$[[x, y], \beta(v)]_V = [x, [\alpha(y), v]_V]_V - [y, [\alpha(x), v]_V]_V.$$

### 1.3 Hom-type Hochschild Cohomology

Let  $(\mathcal{A}, \mu, \alpha)$  be a multiplicative Hom-associative algebra. We call a  $n$ -cochain of  $\mathcal{A}$ , with coefficients in itself, any  $n$ -linear map from  $\mathcal{A}^{\times n}$  to  $\mathcal{A}$ . We denote by  $C_{Hom}^n(\mathcal{A}, \mathcal{A})$  the space of  $n$ -cochains defined to be the set of elements  $f \in Hom(\mathcal{A}^{\times n}, \mathcal{A})$ , satisfying

$$\alpha \circ f(x_0, \dots, x_{n-1}) = f(\alpha(x_0), \dots, \alpha(x_1), \dots, \alpha(x_{n-1})) \text{ for all } x_0, \dots, x_{n-1} \in \mathcal{A}.$$

For  $n = 0$ ,  $C_{Hom}^0(\mathcal{A}, \mathcal{A}) = \mathcal{A}$ . Let  $f \in C_{Hom}^n(\mathcal{A}, \mathcal{A})$ , define  $\delta_{Hom}^n f \in C_{Hom}^{n+1}(\mathcal{A}, \mathcal{A})$  by

$$\begin{aligned} \delta_{Hom}^n \varphi(x_0, \dots, x_n) &= \mu(\alpha^{n-1}(x_0), \varphi(x_1, x_2, \dots, x_n)) \\ &+ \sum_{k=1}^n (-1)^k \varphi(\alpha(x_0), \alpha(x_1), \dots, \alpha(x_{k-2}), \mu(x_{k-1}, x_k), \alpha(x_{k+1}), \dots, \alpha(x_n)) \\ &+ (-1)^{n+1} \mu(\varphi(x_0, \dots, x_{n-1}), \alpha^{n-1}(x_n)). \end{aligned} \quad (2)$$

So  $(C_{Hom}^*(\mathcal{A}, \mathcal{A}) = \bigoplus_{n \geq 0} C_{Hom}^n(\mathcal{A}, \mathcal{A}), \delta_{Hom})$  is a cochain complex defining a so called Hom-type Hochschild complex of  $\mathcal{A}$  with coefficients in itself.

More generally, let  $M$  be a vector space and  $\beta : M \rightarrow M$  be a linear map. We denote by  $C_{\alpha, \beta}(\mathcal{A}, M)$  the space of all  $(n+1)$ -linear maps  $\varphi : \mathcal{A}^{\times n+1} \rightarrow M$ , satisfying

$$\beta(\varphi(x_0, \dots, x_n)) = \varphi(\alpha(x_0), \dots, \alpha(x_n)) \text{ for all } x_0, \dots, x_n \in \mathcal{A}.$$

Let  $M$  be a  $\mathcal{A}$ -bimodule. For  $\psi \in C_{\alpha, \beta}^b(\mathcal{A}, M)$  and  $\varphi \in C_{\alpha}^a(\mathcal{A}, \mathcal{A})$ , where  $a \geq 0, b \geq 0$ , we define  $j_{\varphi}^{\alpha} \psi \in C_{\alpha}^{a+b+1}(\mathcal{A}, M)$  to be the composition product given by the operator

$$\begin{aligned} j_{\varphi}^{\alpha}(\psi)(x_0, \dots, x_{a+b}) &= \\ \sum_{k=0}^b (-1)^{ak} \psi(\alpha^a(x_0), \dots, \alpha^a(x_{k-1}), \varphi(x_k, \dots, x_{k+a}), \alpha^a(x_{a+k+1}), \dots, \alpha^a(x_{a+b})). \end{aligned} \quad (3)$$

When  $M = \mathcal{A}$ , we set,

$$[\varphi, \psi]_{\alpha}^{\Delta} = j_{\psi}^{\alpha}(\varphi) - (-1)^{ab} j_{\varphi}^{\alpha}(\psi).$$

The bracket  $[\varphi, \psi]_{\alpha}^{\Delta}$  is called Gerstenhaber bracket. It satisfies  $j_{[\varphi, \psi]_{\alpha}^{\Delta}} = [j_{\varphi}^{\alpha}, j_{\psi}^{\alpha}]$ . Moreover, if  $(\mathcal{A}, \mu, \alpha)$  is a Hom-algebra, then  $[\mu, \mu]_{\alpha}^{\Delta} = 0$  if and only if  $(\mathcal{A}, \mu, \alpha)$  is a Hom-associative algebra.

**Remark 1.3.** [1]. The pair  $(C_{\alpha}(\mathcal{A}, \mathcal{A}), [\cdot, \cdot]_{\alpha}^{\Delta})$  is a graded Lie algebra.

### 1.4 Hom-type Chevalley-Eilenberg Cohomology

**Definition 1.10.** Let  $(\mathcal{L}, [\cdot, \cdot], \alpha)$  be a Hom-Lie algebra. We call a  $n$ -cochain of the Hom-Lie algebra  $\mathcal{L}$ , with coefficients in itself, any  $n$ -linear alternating map  $f : \mathcal{L}^{\times n} \rightarrow \mathcal{L}$  satisfying

$$\alpha \circ f(x_0, \dots, x_{n-1}) = f(\alpha(x_0), \dots, \alpha(x_1), \dots, \alpha(x_{n-1})) \text{ for all } x_0, \dots, x_{n-1} \in \mathcal{L}.$$

The space of  $n$ -cochains is denoted by  $C_{HL}^n(\mathcal{L}, \mathcal{L})$ . We set  $C_{HL}(\mathcal{L}, \mathcal{L}) = \oplus_{n \geq 0} C_{HL}^n(\mathcal{L}, \mathcal{L})$ . Let  $\varphi \in C_{HL}^n(\mathcal{L}, \mathcal{L})$ , define  $\delta^n \varphi \in C_{HL}^{n+1}(\mathcal{L}, \mathcal{L})$  by

$$\begin{aligned} \delta_{HL}^n \varphi(x_0, \dots, x_n) &= \sum_{i=0}^n (-1)^i [\alpha^{n-1}(x_i), \varphi(x_0, \dots, \widehat{x_i}, \dots, x_n)] \\ &+ \sum_{0 \leq i < j \leq n+1} (-1)^{i+j} \varphi([x_i, x_j], \alpha(x_0), \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, \alpha(x_n)). \end{aligned} \quad (4)$$

Then, we have a cohomology complex  $(C_{HL}^*(\mathcal{L}, \mathcal{L}), \delta)$  which we call Hom-type Chevalley-Eilenberg complex.

More generally, let  $A$  be a vector space and  $\alpha : A \rightarrow A$  be a linear map. Let  $\varphi : A^{\times(n+1)} \rightarrow A$  be a  $(n+1)$ -linear alternating map satisfying for all  $x_0, \dots, x_n \in A$ ,  $\alpha(\varphi(x_0, \dots, x_n)) = \varphi(\alpha(x_0), \dots, \alpha(x_n))$ . We denote their set by  $\tilde{C}_\alpha^n(A, A)$  and

$$\tilde{C}_\alpha(A, A) = \bigoplus_{n \geq -1} \tilde{C}_\alpha^n(A, A).$$

We define the alternator  $\lambda : C_\alpha(A, A) \rightarrow \tilde{C}_\alpha(A, A)$  by

$$(\lambda\varphi)(x_0, \dots, x_a) = \frac{1}{(a+1)!} \sum_{\sigma \in \mathcal{S}_{a+1}} \varepsilon(\sigma) \varphi(x_{\sigma(0)}, \dots, x_{\sigma(a)}) \text{ for } \varphi \in C_\alpha^a(A, A),$$

where  $\mathcal{S}_{a+1}$  is a permutation group and  $\varepsilon(\sigma)$  is the signature of the permutation  $\sigma$ .

We define an operator and a bracket for  $\varphi \in \tilde{C}_\alpha^a(A, A)$ ,  $\psi \in \tilde{C}_\alpha^b(A, A)$  by

$$[\varphi, \psi]_\alpha^\wedge := i_\varphi^\alpha(\psi) - (-1)^{ab} i_\psi^\alpha(\varphi) \text{ where } i_\varphi(\psi) := \frac{(a+b+1)!}{(a+1)!(b+1)!} \lambda(j_\varphi^\alpha \psi).$$

Thus  $i_\varphi^\alpha(\psi) \in \tilde{C}_\alpha^{a+b+1}$ . The bracket  $[\varphi, \psi]_\alpha^\wedge$  is called Nijenhuis-Richardson bracket.

**Remark 1.4.** [1]. The pair  $(\tilde{C}_\alpha(A, A), [\cdot, \cdot]_\alpha^\wedge)$  is a graded Lie algebra.

## 2 Cohomology of Hom-algebras with values in an adjoint bimodule

The first and the second cohomology groups of Hom-associative algebras and Hom-Lie algebras were introduced in [14]. An  $\mathcal{A}$ -valued cohomology complex were introduced for multiplicative Hom-algebras in [1]. The purpose of this section is to construct a cochain complex cohomology for multiplicative Hom-associative algebras (resp. Hom-Lie algebras) with values in any  $\mathcal{A}$ -bimodule  $M$  (resp.  $\mathcal{L}$ -left module  $\Pi$ ).

### 2.1 Cohomology of Hom-associative algebras with values in an adjoint $\mathcal{A}$ -bimodule

We construct a cochain complex  $C_{\alpha, \alpha'}^*(\mathcal{A}, M)$  that defines a Hom-type Hochschild cohomology for multiplicative Hom-associative algebras in an adjoint  $\mathcal{A}$ -bimodule  $M$ . Let

$(\mathcal{A}, \mu, \alpha)$  and  $(\mathcal{A}', \mu', \alpha')$  be two Hom-associative algebras over  $\mathbb{K}$  and  $\phi : \mathcal{A} \rightarrow \mathcal{A}'$  be a Hom-associative algebra morphism. Let  $M = (\mathcal{A}', \rho_l, \rho_r)$  be a  $\mathcal{A}$ -bimodule, where  $\rho_l$  and  $\rho_r$  are defined in Proposition 1.6 (resp. Remark 1.1). Regard  $\mathcal{A}'$  as a  $\mathcal{A}$ -bimodule via the adjoint representation of  $\mathcal{A}$  induced by  $\phi$ .

The set of  $n$ -cochains on  $\mathcal{A}$  with values in a  $\mathcal{A}$ -bimodule  $M$ , is defined to be the set of  $n$ -linear maps which are compatible with  $\alpha$  and  $\alpha'$  in the sense that  $\alpha' \circ f = f \circ \alpha^{\otimes n}$ , i.e.

$$\alpha' \circ f(u_1, \dots, u_n) = f(\alpha(u_1), \dots, \alpha(u_n)) \quad \text{for all } u_1, \dots, u_n \in \mathcal{A}.$$

We denote by  $C_{Hom}^n(\mathcal{A}, M)$  the set of  $n$ -linear maps from  $\mathcal{A}$  to  $M$  and by  $C_{\alpha, \alpha'}^n(\mathcal{A}, M)$  the set of  $n$ -Hom-cochains:

$$C_{\alpha, \alpha'}^n(\mathcal{A}, M) = \{f \in C_{Hom}^n(\mathcal{A}, M) : \alpha' \circ f = f \circ \alpha^{\otimes n}\}.$$

For  $n = 0$ , we have  $C_{\alpha, \alpha'}^0(\mathcal{A}, M) = M$ .

**Definition 2.1.** We call, for  $n \geq 1$ ,  $n$ -coboundary operator associated to the triple  $(\mathcal{A}, M, \phi)$ , the linear map  $\delta_{Hom}^n : C_{\alpha, \alpha'}^n(\mathcal{A}, M) \rightarrow C_{\alpha, \alpha'}^{n+1}(\mathcal{A}, M)$  defined by

$$\begin{aligned} \delta_{Hom}^n \varphi(x_0, x_1, \dots, x_n) &= \mu'(\phi(\alpha^{n-1}(x_0))), \varphi(x_1, x_2, \dots, x_n)) \\ &+ \sum_{k=1}^n (-1)^k \varphi(\alpha(x_0), \alpha(x_1), \dots, \alpha(x_{k-2}), \mu(x_{k-1}, x_k), \alpha(x_{k+1}), \dots, \alpha(x_n)) \\ &+ (-1)^{n+1} \mu'(\varphi(x_0, \dots, x_{n-1}), \phi(\alpha^{n-1}(x_n))). \end{aligned} \quad (5)$$

**Lemma 2.2.** Let  $D_i : C_{\alpha, \alpha'}^n(\mathcal{A}, M) \rightarrow C_{\alpha, \alpha'}^{n+1}(\mathcal{A}, M)$  be linear operators defined for  $\varphi \in C_{\alpha, \alpha'}^n(\mathcal{A}, M)$  and  $x_0, x_1, \dots, x_n \in \mathcal{A}$  by

$$\begin{aligned} D_0^n \varphi(x_0, x_1, \dots, x_n) &= -\mu'(\phi(\alpha^{n-1}(x_0)), \varphi(x_1, \dots, x_n)) + \varphi(\mu(x_0, x_1), \alpha(x_2), \dots, \alpha(x_n)), \\ D_i^n \varphi(x_0, x_1, \dots, x_n) &= \varphi(\alpha(x_0), \dots, \mu(x_i, x_{i+1}), \dots, \alpha(x_n)) \quad \text{for } 1 \leq i \leq n-2, \\ D_{n-1}^n \varphi(x_0, \dots, x_n) &= \varphi(\alpha(x_0), \dots, \alpha(x_{n-2}), \mu(x_{n-1}, x_n)) - \mu'(\varphi(x_0, \dots, x_{n-1}), \phi(\alpha^{n-1}(x_n))), \\ D_i^n \varphi &= 0 \quad \text{for } i \geq n. \end{aligned}$$

$$\text{Then } D_i^{n+1} D_j^n = D_j^{n+1} D_{i-1}^n \text{ for } 0 \leq j < i \leq n, \quad \text{and} \quad \delta_{Hom}^n = \sum_{i=0}^n (-1)^{i+1} D_i^n.$$

**Proposition 2.3.** Let  $(\mathcal{A}, \mu, \alpha)$  be a Hom-associative algebra and  $\delta_{Hom}^n : C_{\alpha, \alpha'}^n(\mathcal{A}, M) \rightarrow C_{\alpha, \alpha'}^{n+1}(\mathcal{A}, M)$  be the operator defined in (5). Then,  $\delta_{Hom}^{n+1} \circ \delta_{Hom}^n = 0$  for  $n \geq 1$ .

*Proof.* Indeed

$$\begin{aligned} \delta_{Hom}^{n+1} \circ \delta_{Hom}^n &= \sum_{0 \leq i, j \leq n} (-1)^{i+j} D_i^{n+1} D_j^n = \sum_{0 \leq j < i \leq n} (-1)^{i+j} D_i^{n+1} D_j^n + \sum_{0 \leq i \leq j \leq n} (-1)^{i+j} D_i^{n+1} D_j^n \\ &= \sum_{0 \leq j < i \leq n} (-1)^{i+j} D_j^{n+1} D_{i-1}^n + \sum_{0 \leq i \leq j \leq n} (-1)^{i+j} D_i^{n+1} D_j^n \\ &= \sum_{0 \leq j \leq k \leq n} (-1)^{k+j+1} D_j^{n+1} D_k^n + \sum_{0 \leq i \leq j \leq n} (-1)^{i+j} D_i^{n+1} D_j^n = 0. \end{aligned}$$

□



**Lemma 2.4.** *With respect to the above notation, for  $\varphi \in C_{\alpha, \alpha'}^n(\mathcal{A}, M)$  and by the multiplicative property of these algebras, we have  $\delta_{Hom}^n \varphi \circ \alpha^{\otimes n+1} = \alpha' \circ \delta_{Hom}^n \varphi$ . Thus, the map  $\delta^n : C_{\alpha, \alpha'}^n(\mathcal{A}, M) \rightarrow C_{\alpha, \alpha'}^{n+1}(\mathcal{A}, M)$  is well defined.*

**Remark 2.1.** *General case: Let  $\mathcal{A}$  and  $\mathcal{A}'$  be two Hom-associative algebras. Let  $M = (\mathcal{A}', \rho_l, \rho_r)$  be a  $\mathcal{A}$ -bimodule, where  $\rho_l$  and  $\rho_r$  are left  $\mathcal{A}$ -module and right  $\mathcal{A}$ -module respectively. For  $\varphi \in C_{\alpha, \alpha'}^n(\mathcal{A}, M)$ , we set*

$$\begin{aligned} \delta_{Hom}^n \varphi(x_0, x_1, \dots, x_n) &= \rho_l(\alpha^{n-1}(x_0), \varphi(x_1, x_2, \dots, x_n)) \\ &+ \sum_{k=1}^n (-1)^k \varphi(\alpha(x_0), \alpha(x_1), \dots, \alpha(x_{k-2}), \mu(x_{k-1}, x_k), \alpha(x_{k+1}), \dots, \alpha(x_n)) \\ &+ (-1)^{n+1} \rho_r(\varphi(x_0, \dots, x_{n-1}), \alpha^{n-1}(x_n)). \end{aligned}$$

Then  $\delta_{Hom}^{n+1} \circ \delta_{Hom}^n = 0$ .

The proof is similar to that of Lemma 2.2. and for  $D_i^{n+1} \circ D_j^n = D_j^{n+1} \circ D_{i-1}^n$ , we use the compatibility between the left  $\mathcal{A}$ -module and the right  $\mathcal{A}$ -module and the multiplicativity of the algebras.

**Remark 2.2.** *In the particular case where  $M = \mathcal{A}$  and  $\rho_l, \rho_r = \mu$ , the Hom-associative algebra is a  $\mathcal{A}$ -bimodule over itself and  $\delta_{Hom}^n$  is the same as in (2).*

The space of  $n$ -cocycles is  $Z_{Hom}^n(\mathcal{A}, M) = \{\varphi \in C_{\alpha, \alpha'}^n(\mathcal{A}, M) : \delta_{Hom}^n \varphi = 0\}$ , and the space of  $n$ -coboundaries is  $B_{Hom}^n(\mathcal{A}, M) = \{\psi = \delta_{Hom}^{n-1} \varphi : \varphi \in C_{\alpha, \alpha'}^{n-1}(\mathcal{A}, M)\}$ . Obviously  $B_{Hom}^n(\mathcal{A}, M) \subset Z_{Hom}^n(\mathcal{A}, M)$ . We call the  $n^{th}$  Hochschild cohomology group of the Hom-associative algebra  $\mathcal{A}$  with values in an adjoint  $\mathcal{A}$ -bimodule the quotient  $H_{Hom}^n(\mathcal{A}, M) = \frac{Z_{Hom}^n(\mathcal{A}, M)}{B_{Hom}^n(\mathcal{A}, M)}$ .

## 2.2 Cohomology complex of multiplicative Hom-Lie algebras with values in a left $\mathcal{L}$ -module

Now, we construct a cochain complex  $C_{HL}^*(\mathcal{L}, \Pi)$  that defines a Chevalley-Eilenberg cohomology for multiplicative Hom-Lie algebras with values in a left  $\mathcal{L}$ -module  $\Pi$ .

Let  $\mathcal{L}$  and  $\mathcal{L}'$  be two Hom-Lie algebras and  $\phi : \mathcal{L} \rightarrow \mathcal{L}'$  be a Hom-Lie morphism. Regard  $\mathcal{L}$  as a representation  $\Pi$  of  $\mathcal{L}$  via  $\phi$  defined by (1.8). The set  $C_{HL}^k(\mathcal{L}, \Pi)$  of  $k$ -cochains on  $\mathcal{L}$  with values in  $\Pi$  is the set of skewsymmetric  $\mathbb{K}$ -linear maps from  $\mathcal{L}^{\times k}$  to  $\Pi$  :

$$C_{HL}^k(\mathcal{L}; \Pi) = \{f : \wedge^k \mathcal{L} \rightarrow \Pi \text{ linear map}\}.$$

A  $k$ -Hom-cochain on  $\mathcal{L}$  with values in  $\Pi$  is defined to be a  $k$ -cochain  $f \in C_{HL}^k(\mathcal{L}, \Pi)$  such that it is compatible with  $\alpha$  and  $\alpha'$  in the sense that  $\alpha' \circ f = f \circ \alpha^{\otimes k}$ , i.e.

$$\alpha'(f(u_1, \dots, u_k)) = f(\alpha(u_1), \dots, \alpha(u_k)) \quad \text{for all } u_1, \dots, u_k \in \mathcal{L}. \quad (6)$$

We denote by  $\tilde{C}_{\alpha, \alpha'}^k(\mathcal{L}, \Pi)$  the set of  $k$ -Hom-cochains:

$$\tilde{C}_{\alpha, \alpha'}^k(\mathcal{L}, \Pi) = \{f \in C_{HL}^k(\mathcal{L}, \Pi) : \alpha' \circ f = f \circ \alpha\}.$$

For  $k = 0$ , we have  $\tilde{C}_{\alpha, \alpha'}^0(\mathcal{L}, \Pi) = \Pi$ .

**Definition 2.5.** Let  $(\mathcal{L}, [\cdot, \cdot], \alpha)$  and  $(\mathcal{L}, [\cdot, \cdot]', \alpha')$  be two Hom-Lie algebras. Let  $\phi : \mathcal{L} \rightarrow \mathcal{L}'$  be a Hom-Lie algebra morphism. Regard  $\mathcal{L}'$  as a representation of  $\mathcal{L}$  via  $\phi$  wherever appropriate. We call, for  $n \geq 1$ ,  $n$ -coboundary operator associated to the triple  $(\mathcal{L}, \Pi, \phi)$  the linear map  $\delta_{HL}^n : C_{HL}^n(\mathcal{L}, \Pi) \rightarrow C_{HL}^{n+1}(\mathcal{L}, \Pi)$  defined by

$$\begin{aligned} \delta_{HL}^n \varphi(x_0, \dots, x_n) &= \sum_{i=0}^n (-1)^i [\phi(\alpha^{n-1}(x_i)), \varphi(x_0, \dots, \widehat{x_i}, \dots, x_n)]' \\ &+ \sum_{0 \leq i < j \leq n} (-1)^{i+j} \varphi([x_i, x_j], \alpha(x_0), \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, \alpha(x_n)). \end{aligned} \quad (7)$$

We show that  $(\tilde{C}_{\alpha, \alpha'}^*(\mathcal{L}, \Pi), \delta_{HL})$  is a cochain complex. The corresponding cohomology denoted by  $H_{HL}^*(\mathcal{L}, \Pi)$ , is called the cohomology of the Hom-Lie algebra  $\mathcal{L}$  with coefficients in the representation  $\Pi$ .

**Lemma 2.6.** With respect to the above notation, for any  $f \in \tilde{C}_{\alpha, \alpha'}^n(\mathcal{L}, \Pi)$ , we have

$$\delta_{HL}^n(f) \circ \alpha^{\otimes n+1} = \alpha' \circ \delta_{HL}^n(f).$$

Thus  $\delta^n : \tilde{C}_{\alpha, \alpha'}^n(\mathcal{L}, \Pi) \rightarrow \tilde{C}_{\alpha, \alpha'}^{n+1}(\mathcal{L}, \Pi)$  is well defined.

*Proof.* Let  $f \in \tilde{C}_{\alpha, \alpha'}^n(\mathcal{L}, \Pi)$  and  $(x_0, \dots, x_n) \in \mathcal{L}^{n+1}$ , then

$$\begin{aligned} \delta_{HL}^n f \circ \alpha(x_0, \dots, x_n) &= \delta_{HL}^n f(\alpha(x_0), \dots, \alpha(x_n)) \\ &= \sum_{i=0}^n (-1)^i [\phi(\alpha^n(x_i)), f(\alpha(x_0), \dots, \widehat{x_i}, \dots, \alpha(x_n))] \\ &+ \sum_{0 \leq i < j \leq n} (-1)^{i+j} f([\alpha(x_i), \alpha(x_j)], \alpha^2(x_0), \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, \alpha^2(x_n)) \\ &= \sum_{i=0}^n (-1)^i [\phi(\alpha^n(x_i)), f \circ \alpha(x_0, \dots, \widehat{x_i}, \dots, x_n)]' \\ &+ \sum_{0 \leq i < j \leq n} (-1)^{i+j} f \circ \alpha([x_i, x_j], \alpha(x_0), \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, \alpha(x_n)) \\ &= \sum_{i=0}^n (-1)^i \alpha'([\phi(\alpha^{n-1}(x_i)), f(x_0, \dots, \widehat{x_i}, \dots, x_n)]') \\ &+ \sum_{0 \leq i < j \leq n} (-1)^{i+j} \alpha' \circ f([x_i, x_j], \alpha(x_0), \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, \alpha(x_n)) \\ &= \alpha' \circ \delta_{HL}^n(f)(x_0, \dots, x_n). \end{aligned}$$

□

**Theorem 2.7.** We have  $\delta_{HL}^{n+1} \circ \delta_{HL}^n = 0$ .

*Proof.*

$$\begin{aligned} \delta_{HL}^{n+1} \circ \delta_{HL}^n \varphi(x_0, \dots, x_{n+1}) &= \sum_{i=0}^{n+1} (-1)^i [\phi(\alpha^n(x_i)), \delta_{HL}^n \varphi(x_0, \dots, \widehat{x_i}, \dots, x_{n+1})]' \\ &+ \sum_{0 \leq i < j \leq n+1} (-1)^{i+j} \delta_{HL}^n \varphi([x_i, x_j], \alpha(x_0), \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, \alpha(x_{n+1})) \end{aligned}$$

with

$$\begin{aligned} & \sum_{i=0}^{n+1} (-1)^i [\phi(\alpha^n(x_i)), \delta_{HL}^n \varphi(x_0, \dots, \widehat{x}_i, \dots, x_{n+1})]' \\ &= [\phi(\alpha^n(x_i)), \sum_{p=0}^{i-1} (-1)^p [\phi(\alpha^{n-1}(x_p)), \varphi(x_0, \dots, \widehat{x}_p, \dots, \widehat{x}_i, \dots, x_{n+1})]]' \end{aligned} \quad (8)$$

$$+ [\phi(\alpha^n(x_i)), \sum_{p=i+1}^{n+1} (-1)^{p-1} [\phi(\alpha^{n-1}(x_p)), \varphi(x_0, \dots, \widehat{x}_i, \dots, \widehat{x}_p, \dots, x_{n+1})]]' \quad (9)$$

$$+ [\phi(\alpha^n(x_i)), \sum_{i < p < q} (-1)^{p+q} \varphi([x_p, x_q], \alpha(x_0), \dots, \widehat{x}_i, \dots, \widehat{x}_p, \dots, \widehat{x}_q, \dots, \alpha(x_{n+1}))]' \quad (10)$$

$$+ [\phi(\alpha^n(x_i)), \sum_{p < i < q} (-1)^{p+q+1} \varphi([x_p, x_q], \alpha(x_0), \dots, \widehat{x}_p, \dots, \widehat{x}_i, \dots, \widehat{x}_q, \dots, \alpha(x_{n+1}))]' \quad (11)$$

$$+ [\phi(\alpha^n(x_i)), \sum_{p < q < i} (-1)^{p+q} \varphi([x_p, x_q], \alpha(x_1), \dots, \widehat{x}_p, \dots, \widehat{x}_q, \dots, \widehat{x}_i, \dots, \alpha(x_{n+1}))]' \quad (12)$$

and

$$\delta_{HL}^n \varphi([x_i, x_j], \alpha(x_0), \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, \alpha(x_{n+1}))$$

$$= [\phi(\alpha^{n-1}([x_i, x_j])), \varphi(\alpha(x_0), \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, \alpha(x_{n+1}))] \quad (13)$$

$$+ \sum_{p=0}^{i-1} (-1)^{p-1} [\phi(\alpha^n(x_p)), \varphi([x_i, x_j], \alpha(x_0), \dots, \widehat{x}_p, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, \alpha(x_{n+1}))]' \quad (14)$$

$$+ \sum_{p=i+1}^{j-1} (-1)^p [\phi(\alpha^n(x_p)), \varphi([x_i, x_j], \alpha(x_0), \dots, \widehat{x}_i, \dots, \widehat{x}_p, \dots, \widehat{x}_j, \dots, \alpha(x_{n+1}))]' \quad (15)$$

$$+ \sum_{p=j+1}^{n+1} (-1)^{p-1} [\phi(\alpha^n(x_p)), \varphi([x_i, x_j], \alpha(x_0), \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, \widehat{x}_p, \dots, \alpha(x_{n+1}))]' \quad (16)$$

$$+ \sum_{p=1}^{i-1} (-1)^p \varphi([x_i, x_j], \alpha(x_p), \alpha^2(x_0), \dots, \widehat{x}_p, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, \alpha^2(x_{n+1})) \quad (17)$$

$$+ \sum_{p=i+1}^{j-1} (-1)^{p-1} \varphi([x_i, x_j], \alpha(x_p), \alpha^2(x_0), \dots, \widehat{x}_i, \dots, \widehat{x}_p, \dots, \widehat{x}_j, \dots, \alpha^2(x_{n+1})) \quad (18)$$

$$+ \sum_{p=j+1}^{n+1} (-1)^p \varphi([x_i, x_j], \alpha(x_p), \alpha^2(x_0), \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, \widehat{x}_p, \dots, \alpha^2(x_{n+1})) \quad (19)$$

$$+ \sum_{p < q < i < j} (-1)^{p+q-1} \varphi([\alpha(x_p), \alpha(x_q)], \alpha([x_i, x_j]), \alpha^2(x_0), \dots, \widehat{x_{i,j,p,q}}, \dots, \alpha^2(x_{n+1})) \quad (20)$$

$$+ \sum_{i < p < q < j} (-1)^{p+q} \varphi([\alpha(x_p), \alpha(x_q)], \alpha([x_i, x_j]), \alpha^2(x_0), \dots, \widehat{x_{i,j,p,q}}, \dots, \alpha^2(x_{n+1})) \quad (21)$$

$$+ \sum_{i < j < p < q} (-1)^{p+q} \varphi([\alpha(x_p), \alpha(x_q)], \alpha([x_i, x_j]), \alpha^2(x_0), \dots, \widehat{x_{i,j,p,q}}, \dots, \alpha^2(x_{n+1})) \quad (22)$$

$$+ \sum_{i < p < j < q} (-1)^{p+q-1} \varphi([\alpha(x_p), \alpha(x_q)], \alpha([x_i, x_j]), \alpha^2(x_0), \dots, \widehat{x_{i,j,p,q}}, \dots, \alpha^2(x_{n+1})) \quad (23)$$

$$+ \sum_{p < i < q < j} (-1)^{p+q} \varphi([\alpha(x_p), \alpha(x_q)], \alpha([x_i, x_j]), \alpha^2(x_0), \dots, \widehat{x_{i,j,p,q}}, \dots, \alpha^2(x_{n+1})) \quad (24)$$

$$+ \sum_{p < i < j < q} (-1)^{p+q-1} \varphi([\alpha(x_p), \alpha(x_q)], \alpha([x_i, x_j]), \alpha^2(x_0), \dots, \widehat{x_{i,j,p,q}}, \dots, \alpha^2(x_{n+1})). \quad (25)$$

By the fact that  $\alpha$  is an algebra morphism, we get

$$\sum_{0 \leq i < j \leq n} (-1)^{i+j} [(20) + (22) + (21) + (25) + (24) + (23)] = 0.$$

Using Hom-Jacobi identity, the fact that  $\phi \circ [\cdot, \cdot] = [\cdot, \cdot]' \circ (\phi \otimes \phi)$  and (6), we obtain

$$\sum_{i=0}^{n+1} (-1)^i ((8) + (9)) + \sum_{0 \leq i < j \leq n+1} (-1)^{i+j} (13) = 0.$$

Hom-Jacobi identity leads also to  $\sum_{0 \leq i < j \leq n+1} (-1)^{i+j} ((19) + (18) + (17)) = 0$ , and simple calculation to the following equality

$$\sum_{i=0}^{n+1} (-1)^i ((10) + (11) + (12)) = - \sum_{0 \leq i < j \leq n+1} (-1)^{i+j} ((14) + (15) + (16)).$$

□

**Definition 2.8.** *The space of  $n$ -cocycles is defined by*

$$Z_{HL}^n(\mathcal{L}, \Pi) = \{\varphi \in \tilde{C}_{\alpha, \alpha'}^n(\mathcal{L}, \Pi) : \delta_{HL}^n \varphi = 0\},$$

*and the space of  $n$ -coboundaries is defined by*

$$B_{HL}^n(\mathcal{L}, \Pi) = \{\psi = \delta_{HL}^{n-1} \varphi : \varphi \in \tilde{C}_{\alpha, \alpha'}^{n-1}(\mathcal{L}, \Pi)\}.$$

One has  $B_{HL}^n(\mathcal{L}, \Pi) \subset Z_{HL}^n(\mathcal{L}, \Pi)$ . Then, we call the  $n^{\text{th}}$  cohomology group of the Hom-Lie algebra  $\mathfrak{g}$  with coefficients in  $\Pi$ , the quotient  $H_{HL}^n(\mathcal{L}, \Pi) = \frac{Z_{HL}^n(\mathcal{L}, \Pi)}{B_{HL}^n(\mathcal{L}, \Pi)}$ .

### 3 Cohomology Complex of Hom-algebra morphisms

In this section, we construct the module  $C^n(\phi, \phi)$  of  $n$ -cochains of the Hom-algebra morphism  $\phi$  and we give an explicit formula for the coboundary operator.

#### 3.1 Cohomology Complex of Hom-associative algebra morphisms

The original cohomology theory associated to deformation of associative algebra morphism was introduced by M. Gerstenhaber in [6]. In this section, we will discuss this theory for Hom-associative algebra morphisms.

Let  $\mathcal{A}, \mathcal{B}$  be two Hom-associative algebras and  $\phi : \mathcal{A} \rightarrow \mathcal{B}$  be a Hom-associative algebra

morphism. Regard  $\mathcal{B}$  as a representation of  $\mathcal{A}$  via  $\phi$  wherever appropriate. Define the module of  $n$ -cochains of  $\phi$  by

$$C_{Hom}^n(\phi, \phi) = C_{Hom}^n(\mathcal{A}, \mathcal{A}) \times C_{Hom}^n(\mathcal{B}, \mathcal{B}) \times C_{\alpha, \alpha'}^{n-1}(\mathcal{A}, \mathcal{B}).$$

The coboundary operator  $\delta^n : C_{Hom}^n(\phi, \phi) \rightarrow C_{Hom}^{n+1}(\phi, \phi)$  is defined by

$$\delta^n(\varphi_1, \varphi_2, \varphi_3) = (\delta_{Hom}^n \varphi_1, \delta_{Hom}^n \varphi_2, \phi \circ \varphi_1 - \varphi_2 \circ \phi^{\otimes n} - \delta_{Hom}^{n-1} \varphi_3),$$

where  $\delta_{Hom}^n \varphi_1$  and  $\delta_{Hom}^n \varphi_2$  are defined by (2) and  $\delta_{Hom}^n \varphi_3$  is defined by (5).

**Theorem 3.1.** *We have  $\delta^{n+1} \circ \delta^n = 0$ . Hence  $(C_{Hom}^*(\phi, \phi), \delta^n)$  is a cochain complex.*

*Proof.* The most-right component of  $(\delta^{n+1} \circ \delta^n)(\varphi_1, \varphi_2, \varphi_3)$  is  $\phi \circ (\delta_{Hom}^n \varphi_1) - (\delta_{Hom}^n \varphi_2) \circ \phi - \delta_{Hom}^n(\phi \circ \varphi_1 - \varphi_2 \circ \phi^{\otimes n} - \delta_{Hom}^{n-1} \varphi_3) = \phi \circ (\delta_{Hom}^n \varphi_1) - (\delta_{Hom}^n \varphi_2) \circ \phi^{\otimes n+1} - \delta_{Hom}^n(\phi \circ \varphi_1) - \delta_{Hom}^n(\varphi_2 \circ \phi^{\otimes n})$ . To finish the proof, one checks that  $\phi \circ (\delta_{Hom}^n \varphi_1) = \delta_{Hom}^n(\phi \circ \varphi_1)$  and  $(\delta_{Hom}^n \varphi_2) \circ \phi^{\otimes n+1} = \delta_{Hom}^n(\varphi_2 \circ \phi^{\otimes n})$ . Indeed,  $\phi \circ \varphi_1$  is defined as follows:  $(\phi \circ \varphi_1)(x_0, \dots, x_n) = \phi \circ (\varphi_1(x_0, \dots, x_n))$  and  $\varphi_2 \circ \phi^{\otimes n}$  as  $\varphi_2 \circ \phi(x_0, \dots, x_n) = \varphi_2 \circ (\phi(x_1), \dots, \phi(x_n))$ .  $\square$

**Proposition 3.2.** *If  $H_{Hom}^n(\mathcal{A}, \mathcal{A})$ ,  $H_{Hom}^n(\mathcal{B}, \mathcal{B})$  and  $H_{\alpha, \alpha'}^{n-1}(\mathcal{A}, \mathcal{B})$  are all trivial then so is  $H_{Hom}^n(\phi, \phi)$ .*

The proof is similar to that of Proposition 3.3 in [16].

We call the  $n^{th}$  Hochschild cohomology group of the Hom-associative algebra morphism  $\phi : \mathcal{A} \rightarrow \mathcal{B}$ ,

$$H_{Hom}^n(\phi, \phi) = H_{Hom}^n(\mathcal{A}, \mathcal{A}) \times H_{Hom}^n(\mathcal{B}, \mathcal{B}) \times H_{\alpha, \alpha'}^{n-1}(\mathcal{A}, \mathcal{B}).$$

The corresponding cohomology modules of the cochain complex  $(C_{Hom}^*(\phi, \phi), \delta^n)$  are denoted by  $H_{Hom}^n(\phi, \phi) := H_{Hom}^n(C_{Hom}^*(\phi, \phi), \delta)$ .

The next task is to define a composition product on cochains of Hom-associative algebra morphisms. Regard  $\mathcal{B}$  as a  $\mathcal{A}$ -bimodule and let  $C^p(\mathcal{A}, \mathcal{B})$  be the vector space of all  $p$ -linear maps of  $\mathcal{A}$  into  $\mathcal{B}$ . Set  $C^p = 0$  if  $p < 0$  and let  $C(\mathcal{A}, \mathcal{B}) = \bigoplus_{p \in \mathbb{Z}} C^p(\mathcal{A}, \mathcal{B})$ .

**Definition 3.3.** *Let  $\phi : \mathcal{A} \rightarrow \mathcal{B}$  be a Hom-associative algebra morphism,  $\varphi_1 \in C^a(\mathcal{B}, \mathcal{B})$  and  $\varphi_2 \in C^b(\mathcal{A}, \mathcal{B})$ . We define the composition product  $\varphi_1 \bar{\circ} \varphi_2 \in C^{a+b}(\mathcal{B}, \mathcal{B})$  by*

$$\varphi_1 \bar{\circ} \varphi_2 = \sum_{i=0}^b (-1)^{i(a-1)} \varphi_1(\phi(x_1), \dots, \phi(x_{i-1}), \varphi_2(x_i, \dots, x_{i+b-1}), \phi(x_{i+b}), \dots, \phi(x_{a+b})). \quad (26)$$

**Definition 3.4.** *Define the homogenous derivation  $D : C(\mathcal{A}, \mathcal{B}) \rightarrow C(\mathcal{A}, \mathcal{B})$  of degree 1 as follows. Let  $\varphi \in C^n(\mathcal{A}, \mathcal{B})$ , we set*

$$D\varphi(x_0, \dots, x_n) = \sum_{i=1}^n (-1)^i \varphi(\alpha(x_0), \dots, \mu_{\mathcal{A}}(x_{i-1}, x_i), \dots, \alpha(x_n)).$$

**Definition 3.5.** The cup product is a bilinear map  $\smile: C(\mathcal{A}, \mathcal{B}) \times C(\mathcal{A}, \mathcal{B}) \rightarrow C(\mathcal{A}, \mathcal{B})$  defined by  $\varphi \smile \psi(x_0, \dots, x_{a+b-1}) = \mu_B(\varphi(x_0, \dots, x_{a-1}), \psi(x_a, \dots, x_{a+b-1}))$  for all  $\varphi \in C^a(\mathcal{A}, \mathcal{B})$ ,  $\psi \in C^b(\mathcal{A}, \mathcal{B})$  and for all  $x_0, \dots, x_{a+b-1} \in \mathcal{A}$ .

**Proposition 3.6.** Let  $[\varphi, \psi]^\smile = \varphi \smile \psi - (-1)^{ab} \psi \smile \varphi$ . Then, the pair  $(C(\mathcal{A}, \mathcal{B}), [\cdot, \cdot]^\smile)$  defines a graded Lie algebra.

**Example 3.1.** We consider a 3-dimensional Hom-associative algebra  $A$  defined in [13], with respect to a basis  $\{e_1, e_2, e_3\}$ , by the multiplication  $\mu_A$  and the linear map  $\alpha_A$  such that

$$\begin{aligned} \mu_A(e_1, e_1) &= ae_1, & \mu_A(e_2, e_2) &= ae_2, \\ \mu_A(e_1, e_2) &= \mu_A(e_2, e_1) = ae_2, & \mu_A(e_2, e_3) &= be_3, \\ \mu_A(e_1, e_3) &= \mu_A(e_3, e_1) = be_3, & \mu_A(e_3, e_2) &= \mu_A(e_3, e_3) = 0, \\ \alpha_A(e_1) &= ae_1, \alpha_A(e_2) &= ae_2, \alpha_A(e_3) &= be_3, \end{aligned}$$

where  $a, b, c$  are parameters.

We consider also a 2-dimensional Hom-associative algebra  $B$  defined, with respect to a basis  $\{f_1, f_2\}$ , by the multiplication  $\mu_B$  and the linear map  $\alpha_B$  such that

$$\begin{aligned} \mu_B(f_1, f_1) &= f_1, \quad \mu_B(f_i, f_j) = f_2 \text{ for } \{i, j\} \notin \{1, 1\}, \\ \alpha_B(f_1) &= \beta f_1 - \beta f_2, \quad \alpha_B(f_2) = 0, \end{aligned}$$

where  $\beta$  is a parameter.

Let  $\phi: \mathcal{A} \rightarrow \mathcal{B}$  be a Hom-associative algebra morphism. It is wholly determined by a set of structure constants  $\lambda_{i,j}$ , such that  $\phi(e_j) = \sum_{i=1}^3 \lambda_{i,j} f_i$ . It turns out that it is defined as

$$\phi(e_1) = f_1 - f_2, \quad \phi(e_2) = f_1 - f_2, \quad \phi(e_3) = 0,$$

with  $a = \beta = 1$ .

In the following, we compute the second cohomology spaces  $H_{Hom}^2(\mathcal{A}, \mathcal{A})$  and  $H_{Hom}^2(\mathcal{B}, \mathcal{B})$ . Let  $\psi \in Z_{Hom}^2(\mathcal{A}, \mathcal{A})$ . The 2-cocycle  $\psi: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$  is a linear map satisfying  $\delta^2 \psi(e_i, e_j, e_k) = 0$  and  $\psi(\alpha(e_i), \alpha(e_j)) = \alpha(\psi(e_i, e_j))$ , for  $1 \leq i, j, k \leq 3$ . Therefore, with  $a = 1$ , we get the following 2-cocycles

$$\left\{ \begin{array}{l} \psi(e_1, e_1) = x_1 e_1 + (\frac{x_2}{b} - x_1) e_2 \\ \psi(e_2, e_1) = \frac{x_2}{b} e_2 \\ \psi(e_3, e_1) = b x_1 e_3 \end{array} \right\} \quad \left\{ \begin{array}{l} \psi(e_1, e_2) = \frac{x_2}{b} e_2 \\ \psi(e_2, e_2) = x_3 e_1 + x_4 e_2 \\ \psi(e_3, e_2) = -b x_3 e_3 \end{array} \right\} \quad \left\{ \begin{array}{l} \psi(e_1, e_3) = x_2 e_3 \\ \psi(e_2, e_3) = b(x_3 + x_4) e_3 e_3 \\ \psi(e_3, e_3) = 0 \end{array} \right.$$

We prove that all the 2-cocycles are 2-coboundaries. Let  $\varphi_A \in B_{Hom}^2(\mathcal{A}, \mathcal{A})$ , then there is a 1-cochain  $f \in C_{Hom}^1(\mathcal{A}, \mathcal{A}) = Hom(\mathcal{A}, \mathcal{A})$  such that  $\varphi_A = \delta_A f$  and  $f(\alpha(e_i)) = \alpha(f(e_i))$ . Set  $f(e_1) = x_1 e_1 + y_1 e_2$ ;  $f(e_2) = x_2 e_1 + y_2 e_2$ ;  $f(e_3) = z_3 e_3$  and  $\delta_A f(e_i, e_j) = \mu_A(f(e_i), e_j) + \mu_A(e_i, f(e_j)) - f(\mu(e_i, e_j))$ . Therefore

$$\begin{aligned} \delta_A f(e_1, e_1) &= x_1 e_1 + y_1 e_2, \delta_A f(e_1, e_2) = (x_1 + y_1) e_2, \delta_A f(e_1, e_3) = b(x_1 + y_1) e_3, \\ \delta_A f(e_2, e_1) &= (x_1 + y_1) e_2, \delta_A f(e_2, e_2) = -x_2 e_1 + (2x_2 + y_2) e_2, \delta_A f(e_2, e_3) = b(x_2 + y_2) e_3, \\ \delta_A f(e_3, e_1) &= b x_1 e_3, \delta_A f(e_3, e_2) = b x_2 e_3, \delta_A f(e_3, e_3) = 0. \end{aligned}$$

Using the fact  $\varphi_A(\alpha(e_i), \alpha(e_j)) = \alpha(\varphi_A(e_i, e_j))$ , we obtain

$$\begin{cases} \varphi_A(e_1, e_1) = x_1 e_1 + y_1 e_2 \\ \varphi_A(e_2, e_1) = (x_1 + y_1) e_2 \\ \varphi_A(e_3, e_1) = b x_1 e_3 \end{cases} \quad \begin{cases} \varphi_A(e_1, e_2) = (x_1 + y_1) e_2 \\ \varphi_A(e_2, e_2) = -x_2 e_1 + (2x_2 + y_2) e_2 \\ \varphi_A(e_3, e_2) = b x_2 e_3 \end{cases} \quad \begin{cases} \varphi_A(e_1, e_3) = b(x_1 + y_1) e_3 \\ \varphi_A(e_2, e_3) = b(x_2 + y_2) e_3 \\ \varphi_A(e_3, e_3) = 0 \end{cases}$$

Therefore  $H_{Hom}^2(\mathcal{A}, \mathcal{A}) = 0$ . Now, we do similar computations for  $\mathcal{B}$ . We get

$$B_{Hom}^2(\mathcal{B}, \mathcal{B}) = \{\psi = \delta_B^1 \varphi_2 \mid \psi(f_1, f_1) = c f_1 + d f_2, \psi(f_i, f_j) = (c + d) f_2, (i, j) \neq (1, 1)\},$$

$$Z_{Hom}^2(\mathcal{B}, \mathcal{B}) = \{\psi \mid \psi(f_1, f_1) = c f_1 + d f_2, \psi(f_i, f_j) = z f_2, (i, j) \neq (1, 1)\},$$

$$H_{Hom}^2(\mathcal{B}, \mathcal{B}) = \{\psi \mid \psi(f_1, f_1) = c f_1 + d f_2, \psi(f_i, f_j) = z f_2, z \neq (c + d), (i, j) \neq (1, 1)\},$$

$$Z_{Hom}^1(\mathcal{A}, \mathcal{B}) = \{\phi \in C_{Hom}^1(\mathcal{A}, \mathcal{B}) \mid \phi \circ \psi_1 - \psi_2 \circ \phi - \delta^1 \phi = 0 \mid \psi_1 \in Z_{Hom}^2(\mathcal{A}, \mathcal{A}), \psi_2 \in Z_{Hom}^2(\mathcal{B}, \mathcal{B})\}.$$

Let  $\phi_1 \in H_{Hom}^1(\mathcal{A}, \mathcal{B})$ , it turns out that it is defined by

$$\phi_1(e_1) = (\frac{x_2}{b} - c) f_1 + (c - \frac{x_2}{b}) f_2; \quad \phi_1(e_2) = (x_3 + x_4 - c) f_1 + (c - x_3 - x_4) f_2; \quad \phi_1(e_3) = 0,$$

with  $c + d = z$  and  $x_3 + x_4 = c$ .

In summary, we have

$$H_{Hom}^2(\mathcal{A}, \mathcal{A}) = \{0\},$$

$$H_{Hom}^2(\mathcal{B}, \mathcal{B}) = \{\psi \mid \psi(f_1, f_1) = c f_1 + d f_2, \psi(f_i, f_j) = z f_2, z \neq (c + d), (i, j) \neq (1, 1)\}$$

$$H_{Hom}^1(\mathcal{A}, \mathcal{B}) = \{\phi_1 \mid \phi_1(e_1) = (\frac{x_2}{b} - c) f_1 + (c - \frac{x_2}{b}) f_2,$$

$$\phi_1(e_2) = (x_3 + x_4 - c) f_1 + (c - x_3 - x_4) f_2; \quad \phi_1(e_3) = 0\}.$$

### 3.2 Cohomology complex of Hom-Lie algebra morphisms

The original cohomology theory of Lie algebra morphisms was developed by Frégier in [3]. In this section, we will generalize this theory to Hom-Lie algebras. We adopt the same notations as in in [3]. Consider the product  $\diamond$  defined for  $\lambda \in \tilde{C}_{\alpha, \alpha'}^n(\mathcal{L}', \mathcal{L}')$  and  $\phi \in Hom(\mathcal{L}, \mathcal{L}')$ , by  $\lambda \diamond \phi \in C_{HL}^n(\mathcal{L}, \mathcal{L}')$  by  $\lambda \diamond \phi(x_1, \dots, x_n) = \lambda(\phi(x_1), \dots, \phi(x_n))$ , where  $x_1, \dots, x_n \in \mathcal{L}$ .

Let  $\phi : \mathcal{L} \rightarrow \mathcal{L}'$  be a Hom-Lie algebra morphism. Regard  $\mathcal{L}'$  as a representation of  $\mathcal{L}$  via  $\phi$  wherever appropriate. Define the module of  $n$ -cochains of  $\phi$  by

$$C_{HL}^n(\phi, \phi) = C_{HL}^n(\mathcal{L}, \mathcal{L}) \times C_{HL}^n(\mathcal{L}', \mathcal{L}') \times \tilde{C}_{\alpha, \alpha'}^{n-1}(\mathcal{L}, \mathcal{L}').$$

The coboundary operator  $\delta^n(\phi, \phi) : C_{HL}^n(\phi, \phi) \rightarrow C_{HL}^{n+1}(\phi, \phi)$  is defined by

$$\delta^n(\varphi_1, \varphi_2, \varphi_3) = (\delta_{HL}^n \varphi_1, \delta_{HL}^n \varphi_2, \delta_{HL}^{n-1} \varphi_3 + (-1)^{n-1} (\phi \circ \varphi_1 - \varphi_2 \diamond \phi)),$$

where  $\delta^n \varphi_1$  and  $\delta^n \varphi_2$  are given by formula (4) and  $\delta^n \varphi_3$  by formula (7).

**Lemma 3.7.** *We have  $\delta^{n+1} \circ \delta^n = 0$ . Hence  $(C_{HL}^n(\phi, \phi), \delta^n)$  is a cohomology complex.*

*Proof.* The most right component of  $(\delta^{n+1} \circ \delta^n)(\varphi_1, \varphi_1, \varphi_3)$  is  $(-1)^{n-1} \delta_{HL}^n[\phi \circ \varphi_1 - \varphi_2 \diamond \phi] + (-1)^n[\phi \circ \delta_{HL}^n(\varphi_1) - \delta_{HL}^n(\varphi_2) \diamond \phi]$ . To finish the proof, one checks that  $\delta_{HL}^n(\phi \circ \varphi_1)(x_0, \dots, x_n) = -\phi \circ \delta_{HL}^n(\varphi_1)(x_0, \dots, x_n)$  and  $\delta_{HL}^n(\varphi_2 \diamond \phi) = -\delta_{HL}^n(\varphi_2) \diamond \phi$ . Indeed,  $\phi \circ \varphi_1$  is defined as  $(\phi \circ \varphi_1)(x_0, \dots, x_n) = \phi \circ (\varphi_1(x_0, \dots, x_n))$  and  $\varphi_2 \circ \phi^{\otimes n}$  as  $\varphi_2 \diamond \phi(x_0, \dots, x_n) = \varphi_2 \circ (\phi(x_1), \dots, \phi(x_n))$ .  $\square$

**Proposition 3.8.** *The corresponding cohomology modules of the cochain complex  $(C_{HL}^*(\phi, \phi), \delta^n)$  are denoted by*

$$H_{HL}^n(\phi, \phi) := H_{HL}^n(C_{HL}^*(\phi, \phi), \delta).$$

*If  $H_{HL}^n(\mathcal{L}, \mathcal{L})$ ,  $H_{HL}^n(\mathcal{L}', \mathcal{L}')$  and  $H_{HL}^{n-1}(\mathcal{L}, \mathcal{L}')$  are all trivial then so is  $H_{HL}^n(\phi, \phi)$ .*

The proof is similar to that of Proposition 3.3 in [16].

We call  $H_{HL}^n(\phi, \phi)$  the  $n^{th}$  Chevalley-Eilenberg cohomology group of the Hom-Lie algebra morphism  $\phi$ .

Now, we define a composition product that lead to a structure of a graded Lie algebra on cochains.

Regard  $\mathcal{L}'$  as an  $\mathcal{L}$ -bimodule. Let  $C^p(\mathcal{L}, \mathcal{L}')$  be the vector space of all skew-symmetric  $p$ -linear maps from  $\mathcal{L}$  to  $\mathcal{L}'$ . Set  $C^p(\mathcal{L}, \mathcal{L}') = 0$  if  $p < 0$  and let  $C = \bigoplus_{p \in \mathbb{Z}} C^p(\mathcal{L}, \mathcal{L}')$ .

**Definition 3.9.** *We define a product on  $C$ , denoted by  $[\cdot, \cdot]^\smile$  as follows. Let  $\varphi \in C^p(\mathcal{L}, \mathcal{L}')$  and  $\psi \in C^q(\mathcal{L}, \mathcal{L}')$ , then  $[\varphi, \psi]^\smile \in C^{p+q}(\mathcal{L}, \mathcal{L}')$  is given by*

$$[\varphi, \psi]^\smile(x_0, \dots, x_{p+q-1}) = \sum_{\sigma \in S_{p+q}} \varepsilon(\sigma) [\varphi(x_{\sigma(0)}, \dots, x_{\sigma(p-1)}), \psi(x_{\sigma(p)}, \dots, x_{\sigma(p+q-1)})]'$$

This product defines on  $C$  a structure of a graded Lie algebra.

**Definition 3.10.** *Define the homogenous derivation  $D : C \rightarrow C$  of degree 1. For  $\varphi \in C^n(\mathcal{A}, \mathcal{B})$ , we set*

$$D\varphi(x_0, \dots, x_n) = \sum_{i < j} (-1)^{i+j} \varphi([x_i, x_j], \alpha(x_0), \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, \alpha(x_n)).$$

**Example 3.2.**

*We consider the 4-dimensional Hom-Lie algebras  $(\mathfrak{g}_1, [\cdot, \cdot]_1, \alpha_1)$  and  $(\mathfrak{g}_2, [\cdot, \cdot]_2, \alpha_2)$  defined, with respect to the basis  $(e_i)_{1 \leq i \leq 4}$  and  $(f_i)_{1 \leq i \leq 4}$  respectively, by*

$$\left\{ \begin{array}{l} [e_1, e_2]_1 = be_4 \\ [e_3, e_4]_1 = de_2 \end{array} \right\} ; \left\{ \begin{array}{l} \alpha_1(e_1) = e_3 + ae_4 \\ \alpha_1(e_2) = be_4 \\ \alpha_1(e_3) = e_1 + ce_2 \\ \alpha_1(e_4) = de_2 \end{array} \right\} ; [f_1, f_2]_2 = df_4 ; \left\{ \begin{array}{l} \alpha_2(f_1) = af_1 + bf_2 + f_3 + cf_4 \\ \alpha_2(f_2) = df_4 \\ \alpha_2(f_3) = ef_1 + \frac{be}{a}f_2 \\ \alpha_2(f_4) = 0 \end{array} \right\}.$$



We compute now the second cohomology space for the first Hom-Lie algebra. The 2-cocycle space  $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1)$  is 7-dimensional and is generated by

$$\left\{ \begin{array}{l} \psi_1(e_1, e_2) = \frac{-1+bd}{c+ad} e_2 \\ \psi_1(e_1, e_3) = e_2 + \frac{-ad-bcd}{d(c+ad)} f_4 \\ \psi_1(e_2, e_3) = \frac{-1+bd}{c+ad} e_2 \\ \psi_1(e_1, e_4) = \frac{1-bd}{c+ad} e_4 \\ \psi_1(e_2, e_4) = 0 \\ \psi_1(e_3, e_4) = \frac{-1+bd}{c+ad} e_4 \end{array} \right\} \quad \left\{ \begin{array}{l} \psi_2(e_1, e_2) = \frac{-a-bc}{c+ad} e_2 \\ \psi_2(e_1, e_3) = \frac{bc^2-a^2d}{d(c+ad)} e_4 \\ \psi_2(e_2, e_3) = \frac{-a-bc}{c+ad} e_2 - \frac{b}{d} e_4 \\ \psi_2(e_1, e_4) = e_2 + \frac{a+bc}{c+ad} e_4 \\ \psi_2(e_2, e_4) = 0 \\ \psi_2(e_3, e_4) = \frac{-a-bc}{c+ad} e_4 \end{array} \right\} \quad \left\{ \begin{array}{l} \psi_3(e_1, e_2) = e_4 \\ \psi_3(e_1, e_3) = 0 \\ \psi_3(e_2, e_3) = 0 \\ \psi_3(e_1, e_4) = 0 \\ \psi_3(e_2, e_4) = 0 \\ \psi_3(e_3, e_4) = \frac{d}{b} e_2 \end{array} \right\}$$

Furthermore  $H_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1) = 0$ .

For the second Hom-Lie algebra,  $Z_{HL}^2(\mathfrak{g}_2, \mathfrak{g}_2)$  is 9-dimensional and is generated by

$$\left\{ \begin{array}{l} \psi_1(f_1, f_2) = \frac{1-a}{e} f_2 \\ \psi_1(f_1, f_3) = -\frac{b}{a} f_2 + \frac{ac+db}{ae} e_4 \\ \psi_1(f_2, f_3) = f_2 \\ \psi_1(f_1, f_4) = 0 \\ \psi_1(f_2, f_4) = -\frac{a}{be} f_4 \\ \psi_1(f_3, f_4) = \frac{1}{e} f_4 \end{array} \right\} \quad \left\{ \begin{array}{l} \psi_2(f_1, f_2) = 0 \\ \psi_2(f_1, f_3) = 0 \\ \psi_2(f_2, f_3) = 0 \\ \psi_2(f_1, f_4) = f_4 \\ \psi_2(f_2, f_4) = -\frac{a}{b} f_4 \\ \psi_2(f_3, f_4) = 0 \end{array} \right\} \quad \left\{ \begin{array}{l} \psi_3(f_1, f_2) = f_4 \\ \psi_3(f_1, f_3) = 0 \\ \psi_3(f_2, f_3) = 0 \\ \psi_3(f_1, f_4) = 0 \\ \psi_3(f_2, f_4) = 0 \\ \psi_3(f_3, f_4) = 0 \end{array} \right\} \quad \left\{ \begin{array}{l} \psi_4(f_1, f_2) = 0 \\ \psi_4(f_1, f_3) = -\frac{b}{a} f_4 \\ \psi_4(f_2, f_3) = f_4 \\ \psi_4(f_1, f_4) = 0 \\ \psi_4(f_2, f_4) = 0 \\ \psi_4(f_3, f_4) = 0 \end{array} \right\}$$

One distinguishes the case where  $e = 1 - a$ ,  $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1)$  is 9-dimensional and generated by

$$\left\{ \begin{array}{l} \varphi_1(f_1, f_2) = 0 \\ \varphi_1(f_1, f_3) = f_2 + \frac{b}{a} f_2 + \frac{1}{a-1} f_3 \\ \varphi_1(f_2, f_3) = \frac{bd+ac}{b(1-a)} f_4 \\ \varphi_1(f_1, f_4) = 0 \\ \varphi_1(f_2, f_4) = 0 \\ \varphi_1(f_3, f_4) = 0 \end{array} \right\} \quad \left\{ \begin{array}{l} \varphi_2(f_1, f_2) = 0 \\ \varphi_2(f_1, f_3) = 0 \\ \varphi_2(f_2, f_3) = \\ \varphi_2(f_1, f_4) = f_4 \\ \varphi_2(f_2, f_4) = -\frac{a}{b} f_4 \\ \varphi_2(f_3, f_4) = 0 \end{array} \right\} \quad \left\{ \begin{array}{l} \varphi_3(f_1, f_2) = f_2 \\ \varphi_3(f_1, f_3) = -\frac{b}{a} f_2 \\ \varphi_3(f_2, f_3) = f_2 + \frac{bd-ac}{b(1-a)} f_4 \\ \varphi_3(f_1, f_4) = 0 \\ \varphi_3(f_2, f_4) = \frac{-2a^2+a}{b(a-1)} f_4 \\ \varphi_3(f_3, f_4) = -\frac{1}{a-1} \end{array} \right\}$$

$$\left\{ \begin{array}{l} \varphi_4(f_1, f_2) = f_4 \\ \varphi_4(f_1, f_3) = 0 \\ \varphi_4(f_2, f_3) = \\ \varphi_4(f_1, f_4) = 0 \\ \varphi_4(f_2, f_4) = 0 \\ \varphi_4(f_3, f_4) = 0 \end{array} \right\} \quad \left\{ \begin{array}{l} \varphi_5(f_1, f_2) = 0 \\ \varphi_5(f_1, f_3) = f_4 \\ \varphi_5(f_2, f_3) = \frac{a-a^2}{b(1-a)} f_4 \\ \varphi_5(f_1, f_4) = 0 \\ \varphi_5(f_2, f_4) = 0 \\ \varphi_5(f_3, f_4) = 0 \end{array} \right\}$$

Therefore

$$\dim H_{HL}^2(\mathfrak{g}_2, \mathfrak{g}_2) = \begin{cases} 7 & \text{if } e \neq 1 - a \quad \text{and generated by } \psi_1, \psi_2 \\ 6 & \text{if } e = 1 - a \quad \text{and generated by } \varphi_1, \varphi_2, \varphi_3. \end{cases}$$

One refers to  $a_1, b_1, c_1, d_1$  for the parameters of  $\alpha_1$  and  $a_2, b_2, c_2, d_2$  for that of  $\alpha_2$ .

Now, let us construct the Hom-Lie algebra morphisms. Straightforward calculations lead to the following examples:

If  $e = 1 + a_2$

$$\left\{ \begin{array}{l} \phi(e_1) = \lambda_{1,1} f_1 + \frac{b_2 \lambda_{11}}{a_2} f_2 - \lambda_{11} f_3 + \frac{(-a_2 c_2 - b_2 d_2) \lambda_{11}}{a_2} f_4 \\ \phi(e_2) = 0 \\ \phi(e_3) = -\lambda_{11} f_1 - \frac{\lambda_{11} b_2}{a_2} f_2 + \lambda_{11} f_3 + \frac{(a_2 c_2 + b_2 d_2) \lambda_{11}}{a_2} f_4 \\ \phi(e_4) = 0 \end{array} \right.$$

If  $e = 1 - a_2$

$$\begin{cases} \phi(e_1) = \lambda_{1,1}f_1 + \frac{b_2\lambda_{11}}{a_2}f_2 + \lambda_{11}f_3 + \frac{(a_2c_2+b_2d_2)\lambda_{11}}{a_2}f_4 \\ \phi(e_2) = 0 \\ \phi(e_3) = \lambda_{11}f_1 + \frac{\lambda_{11}b_2}{a_2}f_2 + \lambda_{11}f_3 + \frac{(a_2c_2+b_2d_2)\lambda_{11}}{a_2}f_4 \\ \phi(e_4) = 0 \end{cases}$$

Under the condition  $a_2 = b_2$ , the first space of cocycles  $Z_{HL}^1(\mathfrak{g}_1, \mathfrak{g}_2)$  related to the second morphisms family is generated by

$$\begin{cases} \phi_1(e_1) = \kappa f_1 + \kappa f_2 + \kappa f_3 + (c_2 + d_2)\kappa f_4 \\ \phi_1(e_2) = 0 \\ \phi_1(e_3) = \kappa f_1 + \kappa f_2 + \kappa f_3 + (c_2 + d_2)\kappa f_4 \\ \phi_1(e_4) = 0 \end{cases}$$

where  $\kappa$  is a parameter.

Then  $H_{HL}^1(\mathfrak{g}_1, \mathfrak{g}_2)$  is 3-dimensional.

## 4 Deformations of Hom-associative algebra morphisms

In this section, we study one-parameter formal deformations of Hom-algebra morphisms using the approach introduced by Gerstenhaber. Recall that the main idea is to change the scalar field  $\mathbb{K}$  to a formal power series ring  $\mathbb{K}[[t]]$ , in one variable  $t$ , and the main results provide cohomological interpretations.

Let  $A[[t]]$  be the set of formal power series whose coefficients are elements of the vector space  $A$ , ( $A[[t]]$  is obtained by extending the coefficients domain of  $A$  from  $\mathbb{K}$  to  $\mathbb{K}[[t]]$ ).

### 4.1 Deformation of Hom-associative algebra morphisms

First, we recall the definition and the main results for Hom-associative algebras, involving Hom-type Hochschild cohomology.

**Definition 4.1.** A one-parameter formal deformation of a Hom-associative algebra  $(\mathcal{A}, \mu_0, \alpha)$  is a Hom-associative  $\mathbb{K}[[t]]$ -algebra  $(\mathcal{A}[[t]], \mu_t, \alpha)$ , where  $\mu_t = \sum_{i \geq 0}^N \mu_i t^i$ , which is a  $\mathbb{K}[[t]]$ -bilinear map satisfying the condition  $\mu_t \circ (\mu_t \otimes \alpha) = \mu_t \circ (\alpha \otimes \mu_t)$ .

The deformation is said to be of order  $N$  if  $\mu_t = \sum_{i \geq 0}^N \mu_i t^i$  and infinitesimal if  $N = 1$ . We have the following properties:

1. If the second Hom-type Hochschild cohomology group,  $H_{Hom}^2(\mathcal{A}, \mathcal{A})$  vanishes, then the algebra  $\mathcal{A}$  is rigid, in the sens that every deformation is equivalent to a trivial deformation.
2. On the other hand, if  $\mu_t = \sum_i \mu_i t^i$  is a deformation of  $\mathcal{A}$  and  $\mathcal{A}$  is rigid, then the 2-cocycle  $\mu_1$  is always a 2-coboundary of the Hom-type Hochschild cohomology.

3. The third cohomology group is the space of obstructions to extensions of a deformation of order  $N$  to a deformation of order  $N + 1$ .

Now, we discuss deformations of Hom-associative algebra morphisms. Let  $(\mathcal{A}, \mu_A, \alpha)$  and  $(\mathcal{B}, \mu_B, \beta)$  be two Hom-associative algebras.

**Definition 4.2.** Let  $\phi : \mathcal{A} \rightarrow \mathcal{B}$  be a Hom-associative algebra morphism. A deformation of  $\phi$  is given by a triple  $\Theta_t = (\mu_{A,t}, \mu_{B,t}, \phi_t)$  where

- $\mu_{A,t} = \sum_{n \geq 0} \mu_{A,n} t^n$  is a deformation of  $\mathcal{A}$ ,
- $\mu_{B,t} = \sum_{n \geq 0} \mu_{B,n} t^n$  is a deformation of  $\mathcal{B}$ ,
- $\phi_t : \mathcal{A}[[t]] \rightarrow \mathcal{B}[[t]]$  is a Hom-associative algebra morphism of the form  $\phi_t = \sum_{n \geq 0} \phi_n t^n$ , where each  $\phi_n : \mathcal{A} \rightarrow \mathcal{B}$  is a  $\mathbb{K}$ -linear map and  $\phi_0 = \phi$ .

**Proposition 4.3.** The linear coefficient  $\theta_1 = (\mu_{A,1}, \mu_{B,1}, \phi_1)$ , called the infinitesimal of the deformation  $\Theta_t$  of  $\phi$ , is a 2-cocycle in  $C_{Hom}^2(\phi, \phi)$ .

*Proof.* The proof is similar to the that one for dialgebras in [16] and for associative algebras in [7].

Let  $\mu_{A,t}$  (resp.  $\mu_{B,t}$ ) be a formal deformation of the Hom-associative algebra  $\mathcal{A}$  (resp.  $\mathcal{B}$ ) which is given by a  $\mathbb{K}[[t]]$ -bilinear map

$$\mu_{A,t} : \mathcal{A}[[t]] \times \mathcal{A}[[t]] \rightarrow \mathcal{A}[[t]] \quad (\text{resp. } \mu_{B,t} : \mathcal{B}[[t]] \times \mathcal{B}[[t]] \rightarrow \mathcal{B}[[t]])$$

of the form  $\mu_{*,t} = \sum_{n \geq 0} \mu_{*,n} t^n$ , where each  $\mu_{*,n}$  extended to be  $\mathbb{K}[[t]]$ -bilinear map, such that for  $x, y, z \in \mathcal{A}$  (resp.  $\mathcal{B}$ ) the following condition holds

$$\mu_{*,t}(\mu_{*,t}(x, y), \alpha(z)) = \mu_{*,t}(\alpha(x), \mu_{*,t}(y, z)), \quad (27)$$

where  $*$   $\in \{1, 2\}$  and refers to either the multiplication in  $\mathcal{A}$  or  $\mathcal{B}$ . The identity (27) is called deformation equation of the Hom-associative algebra and may be written as

$$\sum_{i \geq 0, j \geq 0} t^{i+j} (\mu_{*,i}(\mu_{*,j}(x, y), \alpha(z)) - \mu_{*,i}(\alpha(x), \mu_{*,j}(y, z))) = 0,$$

which is equivalent to the following infinite system of equations

$$\sum_{i=0}^s (\mu_{*,i}(\mu_{*,s-i}(x, y), \alpha(z)) - \mu_{*,i}(\alpha(x), \mu_{*,s-i}(y, z))) = 0, \quad \text{for } s = 0, 1, 2, \dots$$

**Definition 4.4.** We call  $\alpha$ -associator the map

$$Hom(\mathcal{A}^{\times 2}, \mathcal{A}) \times Hom(\mathcal{A}^{\times 2}, \mathcal{A}) \rightarrow Hom(\mathcal{A}^{\times 3}, \mathcal{A}), \quad (\mu_i, \mu_j) \mapsto \mu_i \circ_\alpha \mu_j,$$

defined for all  $x, y, z \in \mathcal{A}$  by  $\mu_i \circ_\alpha \mu_j(x, y, z) = \mu_i(\alpha(x), \mu_j(y, z)) - \mu_i(\mu_j(x, y), \alpha(z))$ .

By using  $\alpha_k$ -associator, the deformation equation may be written as

$$\sum_{i=0}^s \mu_{*,i} \circ_{\alpha} \mu_{*,s-i} = 0, \quad \text{for } s = 0, 1, 2, \dots \quad (28)$$

For  $s = 1$ , we have  $\mu_{*,0} \circ_{\alpha} \mu_{*,1} + \mu_{*,1} \circ_{\alpha} \mu_{*,0} = 0$  which is equivalent to  $\delta_{Hom}^2 \mu_{*,1} = 0$ . It turns out that  $\mu_{*,1}$  is always a 2-cocycle.

For  $s \geq 2$ , the identity (28) is equivalent to :  $\delta_{Hom}^2 \mu_{*,s} = -\sum_{p+q=s, p>0, q>0} \mu_{*,p} \circ_{\alpha} \mu_{*,q}$ . Let  $\phi$  be a Hom-associative algebra morphism, we have the following deformation equation

$$\phi_t(\mu_{A,t}(x, y)) = \mu_{B,t}(\phi_t(x), \phi_t(y)),$$

which may be written  $\sum_{i,j \geq 0} t^{i+j} \phi_i(\mu_{A,j}(x, y)) = \sum_{i,j,k} t^{i+j+k} \mu_{B,k}(\phi_i(x), \phi_j(y))$ . It is equivalent to the following infinite system of equations

$$\sum_{i=0}^s \phi_i(\mu_{A,s-i}(x, y)) = \sum_{i,j \geq 0; i+j \leq s} \mu_{B,i}(\phi_j(x), \phi_{s-i-j}(y)), \quad \text{for } s = 0, 1, 2, \dots$$

For  $s = 1$ , we have

$$\phi_0 \circ \mu_{A,1}(x, y) - \mu_{B,1}(\phi_0(x), \phi_0(y)) + \phi_1 \circ \mu_{A,0}(x, y) - \mu_{B,0}(\phi_1(x), \phi_0(y)) - \mu_{B,0}(\phi_0(x), \phi_1(y)) = 0.$$

It is equivalent to

$$\phi \circ \mu_{A,1} - \mu_{B,1} \circ (\phi \times \phi) - \delta^1 \phi_1 = 0.$$

This means that the 2-cochain  $\phi \circ \mu_{A,1} - \mu_{B,1} \circ (\phi \otimes \phi) - \delta^1 \phi_1 \in C_{Hom}^2(\mathcal{A}, \mathcal{B})$  is equal to 0. More generally, if  $\theta_i = 0$  for  $1 \leq i \leq n$ , then  $\theta_{n+1}$  is a 2-cocycle. Indeed, we have the following identity  $\sum_{p+q=s} \mu_{*,p} \circ_{\alpha} \mu_{*,q} = 0$ .

For  $s = n + 1$ , we have

$$\mu_{*,n+1} \circ_{\alpha} \mu_{*,0} + \mu_{*,0} \circ_{\alpha} \mu_{*,n+1} + \sum_{p+q=n+1, p>0, q>0} \mu_{*,p} \circ_{\alpha} \mu_{*,q} = 0.$$

It is equivalent to

$$\delta^2 \mu_{*,n+1} = - \sum_{p+q=n+1, p>0, q>0} \mu_{*,p} \circ_{\alpha} \mu_{*,q}.$$

As well, from the deformation equation of  $\phi$ , we have  $\phi \circ \mu_{A,n+1} - \mu_{B,n+1} \circ (\phi \otimes \phi) - \delta^1 \phi_{n+1} = 0$ . This is equivalent to say that the 2-cochain  $\phi \circ \mu_{A,n+1} - \mu_{B,n+1} \circ (\phi \otimes \phi) - \delta^1 \phi_{n+1} \in C_{Hom}^2(\mathcal{A}, \mathcal{B})$  is equal to 0.  $\square$

#### 4.1.1 Equivalent deformations

Let  $\mu_{A,t}$  and  $\tilde{\mu}_{A,t}$  be two formal deformations of a Hom-associative algebra  $\mathcal{A}$ . A *formal automorphism*  $\psi_t : \mathcal{A}[[t]] \rightarrow \mathcal{A}[[t]]$  is a power series  $\psi_t = \sum_{n \geq 0} \psi_n t^n$  in which each  $\psi_n \in \text{End}(\mathcal{A})$  and  $\psi_0 = \text{Id}_{\mathcal{A}}$  such that  $\psi_t(\mu_{t,A}(x, y)) = \tilde{\mu}_{t,A}(\psi_t(x), \psi_t(y))$  and  $\psi_t \circ \alpha = \tilde{\alpha} \circ \psi_t$  for all  $x, y \in \mathcal{A}$ . Two deformations  $\mu_{A,t}$  and  $\tilde{\mu}_{A,t}$  are said *equivalent* if and only if there exists a formal automorphism which transforms  $\mu_{A,t}$  to  $\tilde{\mu}_{A,t}$ ,  $\mu_{A,t} \rightarrow \tilde{\mu}_{A,t}$ .

**Definition 4.5.** Let  $\Theta_t = (\mu_{A,t}, \mu_{B,t}, \phi_t)$  and  $\tilde{\Theta} = (\tilde{\mu}_{A,t}, \tilde{\mu}_{B,t}, \tilde{\phi}_t)$  be two deformations of a Hom-associative algebra morphism  $\phi : \mathcal{A} \rightarrow \mathcal{B}$ .

A formal automorphism  $\phi_t : \Theta_t \rightarrow \tilde{\Theta}_t$  is a pair  $(\psi_{A,t}, \psi_{B,t})$ , where  $\psi_{A,t} : \mathcal{A}[[t]] \rightarrow \mathcal{A}[[t]]$  and  $\psi_{B,t} : \mathcal{B}[[t]] \rightarrow \mathcal{B}[[t]]$  are formal automorphisms such that  $\tilde{\phi}_t = \psi_{B,t} \phi_t \psi_{A,t}^{-1}$ .

Two deformations  $\Theta_t$  and  $\tilde{\Theta}_t$  are equivalent if and only if there exists a formal automorphism such that  $\theta_t \rightarrow \tilde{\theta}_t$ .

Given a deformation  $\Theta_t$  and a pair of power series  $\psi_t = (\psi_{A,t} = \sum_n \psi_{A,n} t^n, \psi_{B,t} = \sum_n \psi_{B,n} t^n)$ , one can define a deformation  $\tilde{\Theta}_t$  which is automatically equivalent to  $\Theta_t$ .

**Proposition 4.6.** [1] If  $\mu_{A,t}$  and  $\tilde{\mu}_{A,t}$  are equivalent deformations of  $\mathcal{A}$  given by the automorphism  $\phi_{A,t} : \mathcal{A}[[t]] \rightarrow \mathcal{A}[[t]]$ , then the infinitesimals of  $\mu_{A,t}$  and  $\tilde{\mu}_{A,t}$  belong to the same cohomology class with respect to the cohomology of the algebra  $\mathcal{A}$ .

**Theorem 4.1.** The infinitesimal of a deformation  $\Theta_t$  of  $\phi$  is a 2-cocycle in  $C_{Hom}^2(\phi, \phi)$  whose cohomology class is determined by the equivalence class of the first term of  $\Theta_t$ .

*Proof.* In view of Proposition 4.3, it remains to show that if  $\psi_t : \Theta_t \rightarrow \Theta_t$  is a formal automorphism, then the 2-cocycle  $\theta_1$  and  $\tilde{\theta}_1$  differ by a 2-coboundary. Write  $\psi_t = (\psi_{A,t}, \psi_{B,t})$  and  $\tilde{\Theta}_t = (\tilde{\mu}_{A,t}, \tilde{\mu}_{B,t}, \tilde{\phi}_t)$ . In view of Proposition 4.6, we have

$$\delta^1 \psi_{*,1} = \mu_{*,1} - \tilde{\mu}_{*,1} \in C_{Hom}^2(*, *) \quad (29)$$

for  $*$  denoting either  $A$  or  $B$ . To finish the proof, we develop both sides of  $\tilde{\phi}_t = \psi_{B,t} \phi_t \psi_{A,t}^{-1}$  and collecting the coefficients of  $t^n$  yield for  $n = 1$  the equality

$$\phi_1 - \tilde{\phi}_1 = \phi \psi_{A,1} - \psi_{B,1} \phi. \quad (30)$$

It follows that a 1-cochain  $\alpha = (\psi_{A,1}, \psi_{B,1}, 0) \in C_{Hom}^1(\phi, \phi)$  satisfies  $\delta^1 \alpha = \theta_1 - \tilde{\theta}_1$ .  $\square$

**Definition 4.7.** Let  $(\mathcal{A}, \mu, \alpha)$  be a Hom-associative algebra and  $\theta_1 = (\mu_{A,1}, \mu_{B,1}, \phi_1)$  be an element of  $Z_{Hom}^2(\phi, \phi)$ . The 2-cocycle  $\theta_1$  is said to be integrable if there exists a family  $(\mu_{A,t} = \sum_n \mu_{A,n} t^n, \mu_{B,t} = \sum_n \mu_{B,n} t^n, \phi_t = \sum_n \phi_n t^n)$  defining a formal deformation  $\Theta_t$  of  $\phi$ .

According to (30) and (29) identities, the integrability of  $\theta_1$  depends only on its cohomology class. Thus, we have the following result.

**Theorem 4.2.** Let  $(\mathcal{A}, \mu_A, \alpha)$  and  $(\mathcal{B}, \mu_B, \beta)$  be two Hom-associative algebras and  $\Theta_t = (\mu_{t,A}, \mu_{t,B}, \phi_t)$  be a deformation of a Hom-associative algebra morphism  $\phi : \mathcal{A} \rightarrow \mathcal{B}$ . Then, there exists an equivalent deformation  $\tilde{\Theta}_t = (\tilde{\mu}_{A,t}, \tilde{\mu}_{B,t}, \tilde{\phi}_t)$  such that  $\tilde{\theta}_1 \in Z_{Hom}^2(\phi, \phi)$  and  $\tilde{\theta}_1 \notin B_{Hom}^2(\phi, \phi)$ . Hence, if  $H_{Hom}^2(\phi, \phi) = 0$  then every formal deformation is equivalent to a trivial deformation.

*Proof.* Define a pair of power series  $\psi_t = (\psi_{A,t}, \psi_{B,t})$ . According to Definition 4.5, we define an equivalent deformation  $\tilde{\Theta}_t = (\tilde{\mu}_{A,t}, \tilde{\mu}_{B,t}, \tilde{\phi}_t) = \sum_n \tilde{\theta}_n t^n$ .

We have  $\mu_{*,1} \in Z_{Hom}^2(*, *)$  and also  $\mu_{*,1} - \tilde{\mu}_{*,1} \in Z_{Hom}^2(*, *)$  for  $*$   $\in \{\mathcal{A}, \mathcal{B}\}$ . Moreover  $\phi_1 \in Z_{Hom}^1(\mathcal{A}, \mathcal{B})$  leads to  $\phi_1 - \tilde{\phi}_1 \in Z_{Hom}^1(\mathcal{A}, \mathcal{B})$ . If  $\theta_1 \in B_{Hom}^2(\phi, \phi)$  then  $\theta_1 - \tilde{\theta}_1 = \delta^1 \varphi$  for  $\varphi \in C_{Hom}^1(\phi, \phi)$ .  $\square$

**Theorem 4.3.** *Let  $\Theta_t = \sum_{i \geq 0} \theta_i t^i$  be a deformation of a Hom-associative algebra morphism, in which  $\theta_i = 0$  for  $i = 1, \dots, n$  and  $\theta_{n+1}$  is a coboundary in  $C_{Hom}^2(\phi, \phi)$ , then there exists a deformation  $\tilde{\Theta}_t$  equivalent to  $\Theta_t$  and a formal automorphism  $\psi_t : \Theta_t \rightarrow \tilde{\Theta}_t$  such that  $\tilde{\theta}_i = 0$ , for  $i = 1, \dots, n+1$ .*

*Proof.* The proof is similar to that in [16].  $\square$

A Hom-associative algebras morphism for which every formal deformation is equivalent to a trivial deformation  $(\mu_{A,0}, \mu_{B,0}, \psi)$  is said to be analytically rigid. The vanishing of the second cohomology group ( $H_{Hom}^2(\phi, \phi) = 0$ ) gives a sufficient criterion for rigidity.

#### 4.1.2 Obstructions

A deformation of order  $N$  of  $\phi$  is a triple,  $\Theta_t = (\mu_{A,t}, \mu_{B,t}, \phi_t)$  satisfying  $\phi_t(\mu_{A,t}(a, b)) = \mu_{B,t}(\phi_t(a), \phi_t(b))$  or equivalently  $\sum_{i=0}^n \phi_i(\mu_{A,n-i}(a, b)) = \sum_{i+j+k=n} \mu_{B,i}(\phi_j(a), \phi_k(b))$ , for  $n \leq N$ .

Given a deformation  $\Theta_t$  of order  $N$ , it is said extended to order  $N+1$  if and only if there exists a 2-cochain  $\theta_{N+1} = (\mu_{A,N+1}, \mu_{B,N+1}, \phi_{N+1}) \in C_{Hom}^2(\phi, \phi)$  such that  $\bar{\Theta}_t = \Theta_t + t^{N+1}\theta_{N+1}$  is a deformation of order  $N+1$ .  $\bar{\Theta}_t$  is called an order  $N+1$  extension of  $\Theta_t$ .

Let  $\mu_{A,t} = \sum_{i=0}^{N+1} \mu_i t^i$  be a deformation of  $\mathcal{A}$ , the primary obstruction is

$$- \sum_{p+q=N+1, p>0, q>0} \mu_p \circ_\alpha \mu_q$$

which is a 3-cocycle.

The analogous primary obstruction to an infinitesimal deformation of a morphism  $\phi : \mathcal{A} \rightarrow \mathcal{B}$  is obtained by calculating the second order term in the deformation equation of  $\phi$ . Then

$$\begin{aligned} & \mu_{B,2}(\phi(a), \phi(b)) + \mu_{B,1}(\phi_1(a), \phi(b)) + \mu_{B,1}(\phi(a), \phi_1(b)) \\ & + \mu_{B,0}(\phi_1(a), \phi_1(b)) + \mu_{B,0}(\phi_2(a), \phi(b)) + \mu_{B,0}(\phi(b), \phi_2(b)) \\ & = \phi_2 \mu_{A,0}(a, b) + \phi_1 \mu_{1,A}(a, b) + \phi \mu_{A,2}(a, b) \end{aligned}$$

giving

$$\mu_{B,1} \bar{\circ} \phi_1 - \phi_1 \circ \mu_{A,1} + \phi_1 \smile \phi_1 = \phi \mu_{A,2}(a, b) - \mu_{B,2}(\phi(a), \phi(b)) - \delta_{Hom}^1 \phi_2.$$

Since  $\mu_{*,1} \circ_\alpha \mu_{*,1} = -\delta_{Hom}^2 \mu_{*,2}$ , for  $*$  in  $\{\mathcal{A}, \mathcal{B}\}$ , we have

$$(\mu_{A,1} \bar{\circ} \mu_{A,1}, \mu_{B,1} \bar{\circ} \mu_{B,1}, \mu_{1,B} \hat{\circ} \phi_1 - \phi_1 \circ \mu_{A,1} + \phi_1 \smile \phi_1) = \delta^2(\mu_{A,2}, \mu_{B,2}, \phi_2),$$

where  $\bar{\circ}$  is defined in (3) and  $\hat{\circ}$  in (26).

Set  $\mathcal{O}b_{\mathcal{A}} = \sum_{\substack{p+q=N+1 \\ p>0, q>0}} \frac{1}{2} [\mu_{A,p}, \mu_{A,q}]_{\alpha}^{\Delta}$  for the obstruction of a deformation of a Hom-associative

algebra  $\mathcal{A}$ ,  $\mathcal{O}b_{\mathcal{B}} = \sum_{\substack{p+q=N+1 \\ p>0, q>0}} \frac{1}{2}[\mu_{B,p}, \mu_{B,q}]_{\alpha}^{\Delta}$  for the obstruction of a deformation of Hom-associative algebra  $\mathcal{B}$  and

$$\mathcal{O}b_{\phi} = \sum_{\substack{p+q=N+1 \\ p>0, q>0}} \mu_{B,p} \bar{\circ} \phi_q - \sum_{\substack{p+q=N+1 \\ p>0, q>0}} \phi_p \circ \mu_{A,q} + \sum_{\substack{p+q=N+1 \\ p>0, q>0}} \phi_p \smile \phi_q + \sum_{\substack{p+q=N+1 \\ p>0, q>0, k>0}} \mu_{B,p} \circ (\phi_q, \phi_k)$$

for the obstruction of the extension of the Hom-associative algebra morphism  $\phi$ .

**Theorem 4.8.** *Let  $(\mathcal{A}, \mu_{A,0}, \alpha_A)$  and  $(\mathcal{B}, \mu_{B,0}, \alpha_B)$  be two Hom-associative algebras. Let  $\phi : \mathcal{A} \rightarrow \mathcal{B}$  be a Hom-associative algebra morphism and  $\Theta_t = (\mu_{A,t}, \mu_{B,t}, \phi_t)$  be an order  $k$  one-parameter formal deformation of  $\phi$ .*

*Then  $\mathcal{O}b = (\mathcal{O}b_{\mathcal{A}}, \mathcal{O}b_{\mathcal{B}}, \mathcal{O}b_{\phi}) \in Z_{Hom}^3(\phi, \phi)$ . Therefore the deformation extends to a deformation of order  $k+1$  if and only if  $\mathcal{O}b_{\phi}$  is a coboundary.*

*Proof.* We show that the 3-cochain  $(\mathcal{O}b_{\mathcal{A}}, \mathcal{O}b_{\mathcal{B}}, \mathcal{O}b_{\phi})$  is a cocycle, i.e.  $\delta^3(\mathcal{O}b_{\mathcal{A}}, \mathcal{O}b_{\mathcal{B}}, \mathcal{O}b_{\phi}) = 0$ , i.e.  $(\delta^3 \mathcal{O}b_{\mathcal{A}}, \delta^3 \mathcal{O}b_{\mathcal{B}}, \phi \mathcal{O}b_{\mathcal{A}} - \mathcal{O}b_{\mathcal{B}} \phi - \delta^2 \mathcal{O}b_{\phi}) = 0$ . One has already that  $\delta^3 \mathcal{O}b_{\mathcal{A}} = 0$  and  $\delta^3 \mathcal{O}b_{\mathcal{B}} = 0$  then it remains to show that  $\phi \mathcal{O}b_{\mathcal{A}} - \mathcal{O}b_{\mathcal{B}} \phi - \delta^2 \mathcal{O}b_{\phi} = 0$ .

$$\begin{aligned} \phi \mathcal{O}b_{\mathcal{A}} &= \phi \left( \sum_{p+q=N+1, p>0, q>0} \mu_{A,p} \circ_{\alpha} \mu_{A,q} \right) \\ &= \phi \left( \sum_{p+q=N+1, p>0, q>0} \mu_{A,p} \circ (\mu_{A,q} \otimes \alpha) - \mu_{A,p} \circ (\alpha_A \otimes \mu_{A,q}) \right) \\ &= \left( \sum_{p+q=N+1, p>0, q>0} (\phi \circ \mu_{A,p}) \circ (\mu_{A,q} \otimes \alpha_A) - (\phi \circ \mu_{A,p}) \circ (\alpha_A \otimes \mu_{A,q}) \right), \end{aligned}$$

$$\begin{aligned} \mathcal{O}b_{\mathcal{B}} \phi &= \sum_{p+q=N+1, p>0, q>0} (\mu_{B,p} \circ (\mu_{B,q} \otimes \alpha_B) - \mu_{B,p} \circ (\alpha_B \otimes \mu_{B,q})) \phi \\ &= \sum_{p+q=N+1, p>0, q>0} \mu_{B,p} \circ (\mu_{B,q} \circ (\phi \circ \phi) \otimes \alpha_B \circ \phi) \\ &\quad - \sum_{p+q=N+1, p>0, q>0} \mu_{B,p} \circ (\alpha_B \circ \phi \otimes \mu_{B,q} \circ (\phi \otimes \phi)), \end{aligned}$$

$$\begin{aligned} \delta^2(\phi_p \circ \mu_{A,q}) &= \mu_{B,0} \circ (\phi \circ \alpha_A \otimes \phi_p \circ \mu_{A,q}) - \phi_p \circ \mu_{A,q} \circ (\mu_{A,0} \otimes \alpha_A) \\ &\quad + \phi_p \circ \mu_{A,q} \circ (\alpha_A \otimes \mu_{A,0}) - \mu_{B,0} \circ (\phi_p \circ \mu_{A,q} \otimes \phi \circ \alpha_A) \\ &= \mu_{B,0} \circ (\phi \otimes \phi_p) \circ (\alpha_A \otimes \mu_{A,q}) - \phi_p \circ \mu_{A,q} \circ (\mu_{A,0} \otimes \alpha_A) \\ &\quad + \phi_p \circ \mu_{A,q} \circ (\alpha_A \otimes \mu_{A,0}) - \mu_{B,0} \circ (\phi_p \otimes \phi) \circ (\mu_{A,q} \otimes \alpha_A), \end{aligned}$$

$$\begin{aligned}
\delta^2(\phi_p \smile \phi_q) &= \mu_{B,0} \circ (\phi \circ \alpha_A \otimes \phi_p \smile \phi_q) - \phi_p \smile \phi_q \circ (\mu_{A,0} \otimes \alpha_A) \\
&+ \phi_p \smile \phi_q \circ (\alpha_A \otimes \mu_{A,0}) - \mu_{B,0} \circ (\phi_p \smile \phi_q \otimes \phi \circ \alpha_A) \\
&= \mu_{B,0} \circ (\phi \circ \alpha_A \otimes \mu_{B,0} \circ (\phi_p \otimes \phi_q)) - \mu_{B,0} \circ (\phi_p \otimes \phi_q) \circ (\mu_{A,0} \otimes \alpha_A) \\
&+ \mu_{B,0} \circ (\phi_p \otimes \phi_q) \circ (\alpha_A \otimes \mu_{A,0}) - \mu_{B,0} \circ (\mu_{B,0} \circ (\phi_p \otimes \phi_q) \otimes \phi \circ \alpha_A) \\
&= \mu_{B,0} \circ (\phi \otimes \mu_{B,0}) \circ (\alpha_A \otimes \phi_p \otimes \phi_q) - \mu_{B,0} \circ (\phi_p \otimes \phi_q) \circ (\mu_{A,0} \otimes \alpha_A) \\
&+ \mu_{B,0} \circ (\phi_p \otimes \phi_q) \circ (\alpha_A \otimes \mu_{A,0}) - \mu_{B,0} \circ (\mu_{B,0} \otimes \phi) \circ (\phi_p \otimes \phi_q \otimes \alpha_A),
\end{aligned}$$

$$\begin{aligned}
\delta^2(\mu_{B,p} \overline{\phi}_q) &= \mu_{B,0} \circ (\phi \circ \alpha_A \otimes \mu_{B,p} \overline{\phi}_q) - \mu_{B,p} \overline{\phi}_q \circ (\mu_{A,0} \otimes \alpha_A) \\
&+ \mu_{B,p} \overline{\phi}_q \circ (\alpha_A \otimes \mu_{A,0}) - \mu_{B,0} \circ (\mu_{B,p} \overline{\phi}_q \otimes \psi \circ \alpha_A) \\
&= \mu_{B,0} \circ (\phi \circ \alpha_A \otimes \mu_{B,p} \circ (\phi_q \otimes \phi)) + \mu_{B,0} \circ (\phi \circ \alpha_A \otimes \mu_{B,p} \circ (\phi \otimes \phi_q)) \\
&- \mu_{B,p} \circ (\phi_q \otimes \phi) \circ (\mu_{A,0} \otimes \alpha_A) - \mu_{B,p} \circ (\phi \otimes \phi_q \circ (\mu_{A,0} \otimes \alpha_A)) \\
&+ \mu_{B,p} \circ (\phi_q \otimes \phi) \circ (\alpha_A \otimes \mu_{A,0}) + \mu_{B,p} \circ (\phi \otimes \phi_q) \circ (\alpha_A \otimes \mu_{A,0}) \\
&- \mu_{B,0} \circ (\mu_{B,p} \circ (\phi_q \otimes \phi) \otimes \phi \circ \alpha_A) - \mu_{B,0} \circ (\mu_{B,p} \circ (\phi \otimes \phi_q) \otimes \phi \circ \alpha_A) \\
&= \mu_{B,0} \circ (\phi \otimes \mu_{B,p}) \circ (\alpha_A \otimes \phi_q \otimes \phi) + \mu_{B,0} \circ (\phi \otimes \mu_{B,p}) \circ (\alpha_A \otimes \phi \otimes \phi_q) \\
&- \mu_{B,p} \circ (\phi_q \otimes \phi) \circ (\mu_{A,0} \otimes \alpha_A) - \mu_{B,p} \circ (\phi \otimes \phi_q) \circ (\mu_{A,0} \otimes \alpha_A) \\
&+ \mu_{B,p} \circ (\phi_q \otimes \phi) \circ (\alpha_A \otimes \mu_{A,0}) + \mu_{B,p} \circ (\phi \otimes \phi_q) \circ (\alpha_A \otimes \mu_{A,0}) \\
&- \mu_{B,0} \circ (\mu_{B,p} \otimes \phi) \circ (\phi_q \otimes \phi \otimes \alpha_A) - \mu_{B,0} \circ (\mu_{B,p} \otimes \phi) \circ (\phi \otimes \phi_q \otimes \alpha_A),
\end{aligned}$$

$$\begin{aligned}
\delta^2(\mu_{B,p} \circ (\phi_p \otimes \phi_q)) &= \mu_{B,0} \circ (\phi \circ \alpha_A \otimes \mu_{B,p} \circ (\phi_p \otimes \phi_q)) - \mu_{B,p} \circ (\phi_p \otimes \phi_q) \circ (\mu_{A,0} \otimes \alpha_A) \\
&+ \mu_{B,p} \circ (\phi_p \otimes \phi_q) \circ (\alpha_A \otimes \mu_{A,0}) - \mu_{B,0} \circ (\mu_{B,p} \circ (\phi_p \otimes \phi_q) \otimes \phi \circ \alpha_A),
\end{aligned}$$

$$\phi \mathcal{O}b_A - \mathcal{O}b_B \phi - \delta^2(\mathcal{O}b(\phi)) = 0.$$

The proof is similar to that of Lemma 6.2 in [16] by setting

$$\Sigma' = \sum_{\substack{i+j=N+1 \\ i,j>0 \\ k=0}} + \sum_{\substack{i+k=N+1 \\ i,k>0 \\ j=0}} + \sum_{\substack{j+k=N+1 \\ j,k>0 \\ i=0}} \sum_{\substack{i+k+j=N+1 \\ i,k,j>0}}$$

and

$$\widetilde{\Sigma} = \sum_{\substack{\alpha+\beta+\gamma+\lambda+\mu=N+1 \\ 1 \leq \alpha+\beta+\gamma \leq N \\ \alpha,\beta,\gamma,\lambda,\mu \geq 0}} + \sum_{\substack{\alpha+\beta=N+1 \\ \gamma=\lambda=\mu=0}} + \sum_{\substack{\alpha+\gamma=N+1 \\ \beta=\lambda=\mu=0}} + \sum_{\substack{\beta+\gamma=N+1 \\ \alpha=\lambda=\mu=0}}.$$

By Hom-associativity, we get

$$\widetilde{\Sigma} [\mu_{B,\lambda}(\phi_\alpha \circ \alpha \otimes \mu_\mu(\phi_\beta \otimes \phi_\gamma)) - \mu_{B,\lambda}(\mu_{B,\mu}(\phi_\alpha \otimes \phi_\beta) \otimes \phi_\gamma \circ \alpha)] = 0.$$

One has moreover

$$\delta^2(\mu_{A,N+1}, \mu_{B,N+1}, \phi_{N+1}) = \sum_{\substack{p+q=N+1 \\ p>0, q>0}} (\phi \mathcal{O}b_A - \mathcal{O}b_B \phi, \mathcal{O}b_\phi).$$



Then, the order  $N$  formal deformation extends to an order  $N + 1$  formal deformation whenever

$\sum_{\substack{p+q=N+1 \\ p>0, q>0}} (\mu_{A,p} \circ_\alpha \mu_{A,q}, \mu_{B,p} \circ_\alpha \mu_{B,q}, \mu_{B,p} \bar{\circ} \phi_q - \phi_p \hat{\circ} \mu_{A,q} + \phi_q \smile \phi_p + \sum_{k>0} \mu_{B,p} \circ (\phi_q \otimes \phi_k))$  is a coboundary.

Setting

$$\begin{aligned} & - \sum' \mu_{B,p} \circ (\phi_p \otimes \phi_k) \circ (\mu_{A,0} \otimes \alpha_A) = - \sum_{\substack{p,q>0 \\ k=0}} \mu_{B,p} \circ (\phi_q \circ \mu_{A,0} \otimes \phi \circ \alpha_A) \\ & - \sum_{\substack{p,k>0 \\ q=0}} \mu_{B,p} \circ (\phi \circ \mu_{A,0} \otimes \phi_k \circ \alpha_A) - \sum_{\substack{q,k>0 \\ p=0}} \mu_{B,0} \circ (\phi_q \circ \mu_{A,0} \otimes \phi_k \circ \alpha_A) \\ & - \sum_{p,q,k>0} \mu_{B,p} \circ (\phi_p \circ \mu_{A,0} \otimes \phi_k \circ \alpha_A) \end{aligned}$$

and as well decomposing the sum on the right hand side of  $\sum' \mu_{B,p} \circ (\phi_q \circ \mu_{A,0} \otimes \phi_k \circ \alpha_A)$  as follows:

$$\sum' = \sum_{\substack{p,q>0 \\ k=0}} + \sum_{\substack{p,k>0 \\ q=0}} + \sum_{\substack{q,k>0 \\ p=0}} + \sum_{p,q,k>0},$$

we can write

$$\sum' \mu_{B,p} \circ (\phi_q \circ \mu_{A,0} \otimes \phi_k \circ \alpha_A) = \sum_{\substack{\alpha+\beta+\gamma=q \\ \alpha,\beta,\gamma \geq 0}} \mu_{B,i} \circ (\mu_{\beta,\alpha} \circ (\phi_\beta \otimes \phi_\gamma) \otimes \phi_k \circ \alpha_A) \quad (31)$$

$$+ \sum_{\substack{\lambda+\mu=q \\ 1 \leq \mu \leq q}} \mu_{B,i} (\phi_\lambda \circ \mu_{A,\mu} \otimes \phi_k \circ \alpha_A) \quad (32)$$

with

$$\begin{aligned} (31) &= - \sum_{\substack{i+\alpha+\beta+\gamma+k=N+1 \\ k=0 \\ i,\beta+\alpha+\gamma>0}} \mu_{B,i} \circ (\mu_{B,\alpha} \circ (\phi_\beta \otimes \phi_\gamma) \otimes \phi \circ \alpha_A) \\ &- \sum_{\substack{\alpha=\beta=\gamma=0 \\ i,k>0 \\ i+k=N+1}} \mu_{B,i} \circ (\mu_{B,0} \circ (\phi \otimes \phi) \otimes \phi_k \circ \alpha_A) \\ &- \sum_{\substack{k,\alpha+\beta+\gamma>0 \\ i=0}} \mu_{B,0} \circ (\mu_{B,\alpha} \circ (\phi_\beta \otimes \phi_\gamma) \otimes \phi_k \circ \alpha_A) \\ &- \sum_{i,\alpha+\beta+\gamma,k>0} \mu_{B,i} \circ (\mu_{B,\alpha} \circ (\phi_\beta \otimes \phi_\gamma) \otimes \phi_k \circ \alpha_A) \end{aligned} \quad (33)$$

The sum on the right hand side of (32) write  $\sum_{\lambda+\mu=q}^I = \sum_{\substack{i,\mu>0 \\ i+\lambda+\mu=N+1 \\ k=0}} + \sum_{\substack{\lambda+\mu+k=N+1 \\ \mu,k>0,\lambda\geq 0}} + \sum_{\substack{i,\mu,k>0 \\ i+\lambda+\mu+k=N+1}}$

and

$$(33) = - \sum_{\substack{i,\alpha>0 \\ i+\alpha=N+1}} \mu_{B,i} \circ (\mu_{B,\alpha} \circ (\phi \otimes \phi) \otimes \phi \circ \alpha_A) \quad (34)$$

$$- \sum_{\substack{i+\alpha+\beta+\gamma=N+1 \\ i,\beta+\gamma>0 \\ \alpha,\beta,\gamma\geq 0}} \mu_{B,i} \circ (\mu_{B,\alpha} \circ (\phi_\beta \otimes \phi_\gamma) \otimes \phi \circ \alpha_A).$$

Setting

$$\begin{aligned} \sum_{i,j,k}^I \mu_{B,i} \circ (\phi_j \circ \alpha_A \otimes \phi_k \circ \mu_{A,0}) &= \sum_{\substack{j+k=N+1 \\ j,k>0}} \mu_{B,0} \circ (\phi_j \circ \alpha_A \otimes \phi_k \circ \mu_{A,0}) \\ &+ \sum_{\substack{i+k=N+1 \\ i,k>0}} \mu_{B,i} \circ (\phi \circ \alpha_A \otimes \phi_k \circ \mu_{A,0}) + \sum_{\substack{i+j=N+1 \\ i,j>0}} \mu_{B,i} \circ (\phi_j \circ \alpha_A \otimes \phi \circ \mu_{A,0}) \\ &+ \sum_{\substack{i,j,k>0 \\ i+j+k=N+1}} \mu_{B,i} \circ (\phi_j \circ \alpha_A \otimes \phi_k \circ \mu_{A,0}), \end{aligned}$$

equally we have

$$\sum_{i+j+k=N+1}^I \mu_{B,i} \circ (\phi_j \circ \alpha_A \otimes \phi_k \circ \mu_{A,0}) = \sum_{i,j,\alpha,\beta,\gamma}^I \mu_{B,i} \circ (\phi_j \circ \alpha_A \otimes \mu_{B,\alpha} \circ (\phi_\beta \otimes \phi_\gamma)) \quad (35)$$

$$- \sum_{\substack{\lambda+\mu=k \\ 1\leq\mu\leq k}}^I \mu_{B,i} \circ (\phi_j \circ \alpha_A \otimes \phi_\lambda \circ \mu_{A,\mu}) \quad (36)$$

We can write (35) as

$$(35) = \sum_{\substack{i+\alpha+\beta+\gamma=N+1 \\ i,\alpha+\beta+\gamma>0 \\ \alpha,\beta,\gamma\geq 0}} \mu_{B,i} \circ (\phi \circ \alpha_A \otimes \mu_{B,\alpha} \circ (\phi_\beta \otimes \phi_\gamma)) \quad (37)$$

$$+ \sum_{\substack{i=0 \\ \alpha+\beta+\gamma+j=N+1 \\ j,\alpha+\beta+\gamma>0,\alpha,\beta,\gamma\geq 0}} \mu_{B,0} \circ (\phi_j \circ \alpha_A \otimes \mu_{B,\alpha} \circ (\phi_\beta \otimes \phi_\gamma))$$

$$+ \sum_{\substack{i,j>0 \\ \alpha=\beta=\gamma=0 \\ i+j=N+1}} \mu_{B,i} \circ (\phi_j \circ \alpha_A \otimes \mu_{B,0} \circ (\phi \otimes \phi))$$

$$+ \sum_{\substack{i+j+\alpha+\beta+\gamma=N+1 \\ i,j,\alpha+\beta+\gamma>0}} \mu_{B,i} \circ (\phi_j \circ \alpha_A \otimes \mu_{B,\alpha} \circ (\phi_\beta \otimes \phi_\gamma)).$$

The sum on the right hand side of (36) is  $\sum_{\substack{\lambda+\mu=k \\ 1 \leq \mu \leq k}}' = \sum_{\substack{j+\lambda+\mu=N+1 \\ i=0 \\ j, \mu > 0, \lambda \geq 0}} + \sum_{\substack{i+\lambda+\mu=N+1, j=0 \\ i, \mu > 0 \\ \lambda+\mu=k \\ \lambda \geq 0}} + \sum_{\substack{\mu, i, j > 0 \\ \lambda+\mu=k, \lambda \geq 0}}$

$$\begin{aligned}
(37) &= \sum_{\substack{i+\alpha+\beta+\gamma=N+1 \\ i, \alpha+\beta+\gamma > 0 \\ \alpha, \beta, \gamma \geq 0}} \mu_{B,i}(\phi \circ \alpha_A \otimes \mu_{B,\alpha} \circ (\phi_\beta \otimes \phi_\gamma)) \\
&= \sum_{\substack{i+\alpha=N+1 \\ i, \alpha > 0}} \mu_{B,i} \circ (\phi \circ \alpha_A \otimes \mu_{B,\alpha} \circ (\phi \otimes \phi)) \\
&\quad + \sum_{\substack{i+\alpha+\beta+\gamma=N+1 \\ i, \beta+\gamma > 0 \\ \alpha, \beta, \gamma \geq 0}} \mu_{B,i} \circ (\phi \circ \alpha_A \otimes \mu_{A,\alpha} \circ (\phi_\beta \otimes \phi_\gamma)).
\end{aligned} \tag{38}$$

The sum (34)+(38) is equal to  $\mathcal{O}b_B \circ \phi$ . In other hand, we have

$$\begin{aligned}
\sum_{p+q=N+1} \phi_p \circ \mu_{A,q} \circ (\mu_{A,0} \otimes \alpha_A) &= \sum_{\substack{p+q=N+1 \\ p, q > 0}} \phi_i \circ \mu_{A,q} \circ (\alpha_A \otimes \mu_{A,0}) \\
&\quad + \sum_{\substack{i+j+k=N+1 \\ i, k > 0, j \geq 0}} \phi_i \mu_{A,j} \circ (\alpha_A \otimes \mu_{A,k})
\end{aligned} \tag{39}$$

$$- \sum_{\substack{p+j+k=N+1 \\ p, k > 0, j \geq 0}} \phi_p \circ \mu_{A,j} \circ (\mu_{A,k} \otimes \alpha_A). \tag{40}$$

The first sum of the right hand side vanishes with the second last sum of  $\delta^2(\phi_p \circ \mu_{A,q})$ .

$$(39) = \sum_{k=1}^N \left[ \sum_{\substack{i+j=N+1-k \\ i, j \geq 0}} \phi_i \mu_{A,j} \circ (\alpha_A \otimes \mu_{A,k}) \right] \tag{41}$$

$$- \phi \left[ \sum_{\substack{j+k=N+1 \\ j, k > 0}} \mu_{A,j} \circ (\alpha_A \otimes \mu_{A,k}) \right]. \tag{42}$$

$$(40) = \sum_{k=1}^N \left[ \sum_{i+j=N+1-k} \phi_i \mu_{A,j} (\mu_{A,k} \otimes \alpha_A) \right] \tag{43}$$

$$+ \phi \left[ \sum_{\substack{j+k=N+1 \\ j, k > 0}} \mu_{A,j} \circ (\mu_{A,k} \otimes \alpha_A) \right]. \tag{44}$$

The sum (42)+(44) is equal to  $\phi \circ \mathcal{O}b_A$ . Furthermore

$$\begin{aligned}
(41) &= \sum_{k=1}^N \left[ \sum_{\substack{\beta+\alpha+\gamma+k=N+1 \\ \alpha, \beta, \gamma \geq 0}} \mu_{B, \alpha}(\phi_\beta \circ \alpha_A \otimes \phi_\gamma \circ \mu_{A, k}) \right] \\
&= \sum_{\substack{\gamma+k=N+1 \\ \gamma, k > 0}} \mu_{B, 0}(\phi \circ \alpha_A \otimes \phi_\gamma \circ \mu_{A, k}) \tag{45}
\end{aligned}$$

$$+ \sum_{\substack{\lambda+\mu=k \\ 1 \leq \mu \leq q \\ \lambda \geq 0}} \mu_{B, i}(\phi_j \circ \alpha_A \otimes \phi_\lambda \circ \mu_{A, \mu}). \tag{46}$$

The term (45) vanishes with the first sum of  $\delta^2(\phi_p \mu_{A, q})$  and (46) vanishes with (36). Also

$$\begin{aligned}
(43) &= \sum_{k=1}^N \left[ \sum_{i+j=N+1-k} \phi_i \mu_{A, j}(\mu_{A, k} \otimes \alpha_A) \right] = - \sum_{\substack{k+i+j=N+1 \\ k > 0}} \phi_i \mu_{A, j}(\mu_{A, k} \otimes \alpha_A) \\
&= - \sum_{\substack{\beta+\gamma+k+\alpha=N+1 \\ \alpha, \beta, \gamma \geq 0, k > 0}} \mu_{B, \alpha}(\phi_\beta \circ \mu_{A, k} \otimes \phi_\gamma \circ \alpha_A) \\
&= - \sum_{\substack{\beta+k=N+1 \\ k > 0}} \mu_{B, 0}(\phi_\beta \circ \mu_{A, k} \otimes \phi \circ \alpha_A) - \sum_{\substack{\lambda+\mu=j \\ 1 \leq \mu \leq j}} \mu_{B, i}(\phi_\lambda \circ \mu_{A, \mu} \otimes \phi_k \circ \alpha_A).
\end{aligned}$$

The second last sum vanishes with the last sum of  $\delta^2(\phi_p \mu_{A, q})$  and the last sum vanishes with (32).

The remaining terms can be written in the form of one sum as follows

$$\widetilde{\sum} [\mu_{B, \lambda}(\phi_\alpha \circ \alpha \otimes \mu_\mu(\phi_\beta \otimes \phi_\gamma)) - \mu_{B, \lambda}(\mu_{B, \mu}(\phi_\alpha \otimes \phi_\beta) \cdot \otimes \phi_\gamma \circ \alpha)]$$

By Hom-associativity, it equals 0.  $\square$

**Corollary 4.4.** *If  $H_{Hom}^3(\phi, \phi) = 0$ , then every infinitesimal deformation can be extended to a formal deformation of larger order.*

## 4.2 Deformations of Hom-Lie algebra morphisms

In this section, we discuss deformations of Hom-Lie algebra morphisms.

### 4.2.1 Infinitesimal Deformations

**Definition 4.9.** *Let  $(\mathcal{L}, [\cdot, \cdot], \alpha)$  be a Hom-Lie algebra. A one-parameter formal Hom-Lie deformation of  $\mathcal{L}$  is given by a  $\mathbb{K}[[t]]$ -bilinear map  $[\cdot, \cdot]_t : \mathcal{L}[[t]] \times \mathcal{L}[[t]] \rightarrow \mathcal{L}[[t]]$  of the form*

$$[\cdot, \cdot]_t = \sum_{i \geq 0} t^i [\cdot, \cdot]_i$$

where each  $[\cdot, \cdot]_i$  is a bilinear map  $[\cdot, \cdot]_i : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$  (extended to  $\mathbb{K}[[t]]$ -bilinear map),  $[\cdot, \cdot] = [\cdot, \cdot]_0$  and satisfying the following conditions

$$[x, y]_t = -[y, x]_t \text{ for all } x, y \in \mathcal{L} \quad (\text{skew-symmetry}),$$

$$\odot_{x,y,z} [\alpha(x), [y, z]_t]_t = 0 \quad (\text{Hom-Jacobi identity})$$

**Definition 4.10.** Let  $\phi : \mathcal{L} \rightarrow \mathcal{L}'$  be a Hom-Lie algebra morphism. A deformation of  $\phi$  is to a triple  $\Theta_t = ([\cdot, \cdot]_t; [\cdot, \cdot]'_t; \phi_t)$  in which :

- $[\cdot, \cdot]_t = \sum_{i \geq 0} t^i [\cdot, \cdot]_i$  is a deformation of  $\mathcal{L}$ ,
- $[\cdot, \cdot]'_t = \sum_{i \geq 0} t^i [\cdot, \cdot]'_i$  is a deformation of  $\mathcal{L}'$ ,
- $\phi_t : \mathcal{L}[[t]] \rightarrow \mathcal{L}'[[t]]$  is a Hom-Lie algebra morphism of the form  $\phi_t = \sum_{n \geq 0} \phi_n t^n$  where each  $\phi_n : \mathcal{L} \rightarrow \mathcal{L}'$  is a  $\mathbb{K}$ -linear map and  $\phi_0 = \phi$ .

**Proposition 4.11.** The linear coefficient,  $\theta_1 = ([\cdot, \cdot]_1, [\cdot, \cdot]'_1, \phi_1)$ , which is called the infinitesimal of the deformation  $\Theta_t$ , is a 2-cocycle in  $C_{HL}^2(\phi, \phi)$ .

*Proof.* Let  $\phi$  be a Hom-Lie algebra morphism, we have the following deformation equation

$$[\phi_t(x), \phi_t(y)]'_t = \phi_t([x, y]_t),$$

which may be written  $\sum_{i,j \geq 0} t^{i+j} \phi_i([x, y]_j) = \sum_{i,j,k \geq 0} t^{i+j+k} [\phi_i(x), \phi_j(y)]'_k$ . It is equivalent to the following infinite system of equations

$$\sum_{i=0}^s \phi_i([x, y]_{s-i}(x, y) = \sum_{i,j \geq 0; i+j \leq s} [\phi_i(x), \phi_j(y)]'_{s-i-j}, \quad \text{for } s = 0, 1, 2, \dots$$

For  $s = 1$ , we have

$$\phi([x, y]_1) - [\phi(x), \phi(y)]'_1 - (-\phi_1([x, y]_0) + [\phi_1(x), \phi(y)]'_0 + [\phi(x), \phi_1(y)]'_0) = 0.$$

This is equivalent to the 2-cochain  $\delta^1 \phi_1(x, y) - \phi([x, y]_1) + [\phi(x), \phi(y)]'_1$  equals 0.  $\square$

#### 4.2.2 Equivalent Deformations

Let  $(\mathcal{L}, [\cdot, \cdot], \alpha)$  be a multiplicative Hom-Lie algebra. Let  $\mathcal{L}_t = (\mathcal{L}, [\cdot, \cdot]_t, \alpha)$  and  $\mathcal{L}'_t = (\mathcal{L}, [\cdot, \cdot]'_t, \alpha)$  be two deformations of  $\mathcal{L}$ , where  $[\cdot, \cdot]_t = \sum_{i \geq 0} t^i [\cdot, \cdot]_i$  and  $[\cdot, \cdot]'_t = \sum_{i \geq 0} t^i [\cdot, \cdot]'_i$  with  $[\cdot, \cdot]_0 = [\cdot, \cdot]'_0 = [\cdot, \cdot]$ . We say that  $\mathcal{L}_t$  and  $\mathcal{L}'_t$  are equivalent if there exists a formal automorphism  $\psi_t : \mathcal{L}[[t]] \rightarrow \mathcal{L}[[t]]$ , that may be written in the form  $\psi_t = \sum_{i \geq 0} \psi_i t^i$  where  $\psi_i \in \text{End}(\mathcal{L})$  and  $\psi_0 = \text{id}$ , such that  $\psi_t([x, y]_t) = [\psi_t(x), \psi_t(y)]'_t$ .

A deformation  $\mathcal{L}_t$  is said to be trivial if and only if  $\mathcal{L}_t$  is equivalent to  $\mathcal{L}$ .

**Definition 4.12.** Let  $(\mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ ,  $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}}, \beta)$  be two Hom-Lie algebras and  $\phi : \mathcal{L} \rightarrow \mathcal{G}$  be a Hom-Lie algebra morphism. Let  $\Theta_t = ([\cdot, \cdot]_{\mathcal{L},t}, [\cdot, \cdot]_{\mathcal{G},t}, \phi_t)$  and  $\tilde{\Theta} = ([\cdot, \cdot]'_{\mathcal{L},t}, [\cdot, \cdot]'_{\mathcal{G},t}, \tilde{\phi}_t)$  be two deformations of a Hom-Lie algebra morphism  $\phi$ .

A formal automorphism  $\psi_t : \Theta_t \rightarrow \tilde{\Theta}_t$  is a pair  $(\psi_{\mathcal{L},t}, \psi_{\mathcal{G},t})$ , where  $\psi_{\mathcal{L},t} : \mathcal{L}[[t]] \rightarrow \mathcal{L}[[t]]$  and  $\psi_{\mathcal{G},t} : \mathcal{G}[[t]] \rightarrow \mathcal{G}[[t]]$  are formal automorphisms, such that  $\tilde{\phi}_t = \psi_{\mathcal{L},t} \circ \phi_t \circ \psi_{\mathcal{G},t}^{-1}$ .

- Two deformation  $\Theta_t$  and  $\tilde{\Theta}_t$  are equivalent if and only if there exists a formal automorphism that transforms  $\theta_t$  in to  $\tilde{\theta}_t$ ,  $\theta_t \rightarrow \tilde{\theta}_t$ .
- Given a deformation  $\Theta_t$  and a pair of power series  $\psi_t = (\psi_{\mathcal{L},t} = \sum_n \psi_{\mathcal{L},n} t^n, \psi_{\mathcal{G},t} = \sum_n \psi_{\mathcal{G},n} t^n)$ , one can define a deformation  $\tilde{\Theta}_t$ . The deformation  $\tilde{\Theta}_t$  is automatically equivalent to  $\Theta_t$ .

**Proposition 4.13.** If  $[\cdot, \cdot]_t$  and  $[\cdot, \cdot]'_t$  are equivalent deformations of  $\mathcal{L}$  given by the automorphism  $\psi_t : \mathcal{L}[[t]] \rightarrow \mathcal{L}[[t]]$ , the infinitesimals of  $[\cdot, \cdot]_t$  and  $[\cdot, \cdot]'_t$  belong to the same cohomology class.

**Theorem 4.5.** The infinitesimal of a deformation  $\Theta_t$  of  $\phi$  is a 2-cocycle in  $C_{HL}^2(\phi, \phi)$  whose cohomology class is determined by the equivalence class of the first term of  $\Theta_t$ .

*Proof.* Same proof as for Theorem 4.1. □

**Definition 4.14.** Let  $(\mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$  and  $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}}, \beta)$  be two Hom-Lie algebras, and  $\phi_1$  be an element of  $Z_{HL}^1(\mathcal{L}, \mathcal{G})$ , the 1-cocycle is said to be integrable if there exists a family  $(\phi_t)_{t \geq 0}$  such that  $\phi_t = \sum_{i \geq 0} t^i \phi_i$  defines a formal deformation  $\phi_t$  of  $\phi$ .

**Theorem 4.6.** Let  $(\mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$  and  $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}}, \beta)$  be two Hom-Lie algebras. Let  $\Theta_t = ([\cdot, \cdot]_{\mathcal{L},t}, [\cdot, \cdot]_{\mathcal{G},t}, \phi_t)$  be a deformation of a Hom-Lie algebra morphism  $\phi$ . Then there exists an equivalent deformation  $\tilde{\Theta}_t = ([\cdot, \cdot]'_{\mathcal{L},t}, [\cdot, \cdot]_{\mathcal{G},t}, \tilde{\phi})$  such that  $\tilde{\theta}_1 \in Z_{HL}^2(\phi, \phi)$  and  $\tilde{\theta}_1 \notin B_{HL}^2(\phi, \phi)$ . Hence, if  $H_{HL}^2(\phi, \phi) = 0$  then every formal deformation is equivalent to a trivial deformation.

### 4.2.3 Obstructions

A deformation of order  $N$  of  $\phi$  is a triple,  $\Theta_t = ([\cdot, \cdot]_{\mathcal{L},t}, [\cdot, \cdot]_{\mathcal{G},t}, \phi_t)$  satisfying  $\phi_t([x, y]_{\mathcal{L},t}) = [\phi_t(x), \phi_t(y)]_{\mathcal{G},t}$  or equivalently  $\sum_{i=0}^N \phi_i([x, y]_{\mathcal{L},N-i}) = \sum_{i+j+k=N} [\phi_i(x), \phi_j(y)]_{\mathcal{G},k}$ .

Given a deformation  $\Theta_t$  of order  $N$ , it extends to a deformation of order  $N + 1$  if and only if there exists a 2-cochain  $\theta_{N+1} = ([\cdot, \cdot]_{\mathcal{L},N+1}, [\cdot, \cdot]_{\mathcal{G},N+1}, \phi_{N+1}) \in C_{HL}^2(\phi, \phi)$  such that  $\bar{\Theta}_t = \Theta_t + t^{N+1} \theta_{N+1}$  is a deformation of order  $N + 1$ . Then  $\bar{\Theta}_t$  is said to be an extension of  $\Theta_t$  of order  $N + 1$ .

Let  $\mathcal{O}b_{\mathcal{L}} = \frac{1}{2} \sum_{p+q=N+1, p>0, q>0} [[\cdot, \cdot]_{\mathcal{L},p}, [\cdot, \cdot]_{\mathcal{L},q}]_{\alpha}^{\wedge}$  (resp.  $\mathcal{O}b_{\mathcal{G}} = \frac{1}{2} \sum_{p+q=N+1, p>0, q>0} [[\cdot, \cdot]_{\mathcal{G},p}, [\cdot, \cdot]_{\mathcal{G},q}]_{\alpha}^{\wedge}$ )

be the obstruction of a deformation of the Hom-Lie algebra  $\mathcal{L}$  (resp.  $\mathcal{G}$ ) and

$\mathcal{O}b_{\phi} = \sum_{i=0}^{N+1} \phi_i([x, y]_{\mathcal{L},N+1-i}) - \sum' [\phi_i(x), \phi_j(y)]_{\mathcal{G},k}$  be the obstruction of the extension of

the Hom-Lie algebra morphism  $\phi$ , where

$$\Sigma' = \sum_{\substack{i+j=N+1 \\ i,j>0 \\ k=0}} + \sum_{\substack{i+k=N+1 \\ i,k>0 \\ j=0}} + \sum_{\substack{j+k=N+1 \\ j,k>0 \\ i=0}} + \sum_{\substack{i+k+j=N+1 \\ i,k,j>0}}.$$

**Theorem 4.15.** *Let  $(\mathcal{L}, [\cdot, \cdot]_{\mathcal{L},0}, \alpha)$  and  $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G},0}, \beta)$  be two Hom-Lie algebras and  $\phi : \mathcal{L} \rightarrow \mathcal{G}$  be a Hom-Lie algebra morphism. Let  $\Theta_t = ([\cdot, \cdot]_{\mathcal{L},t}, [\cdot, \cdot]_{\mathcal{G},t}, \phi_t)$  be an order  $k$  one-parameter formal deformation of  $\phi$ . Then  $\mathcal{O}b = (\mathcal{O}b_{\mathcal{A}}, \mathcal{O}b_{\mathcal{B}}, \mathcal{O}b_{\phi}) \in Z_{HL}^3(\phi, \phi)$ .*

*Therefore, the deformation extends to a deformation of order  $k+1$  if and only if  $\mathcal{O}b$  is a coboundary.*

*Proof.* We must show that  $(\mathcal{O}b_{\mathcal{L}}, \mathcal{O}b_{\mathcal{G}}, \mathcal{O}b_{\phi})$  is a cocycle in  $C_{HL}^3(\phi, \phi)$  that is  $\delta^3(\mathcal{O}b_{\mathcal{L}}, \mathcal{O}b_{\mathcal{G}}, \mathcal{O}b_{\phi}) = 0$  i.e.  $(\delta^3 \mathcal{O}b_{\mathcal{L}}, \delta^3 \mathcal{O}b_{\mathcal{G}}, \delta^2 \mathcal{O}b_{\phi} + \phi \mathcal{O}b_{\mathcal{L}} - \mathcal{O}b_{\mathcal{G}} \phi) = 0$ . One has  $\delta^3 \mathcal{O}b_{\mathcal{L}} = 0$  and  $\delta^3 \mathcal{O}b_{\mathcal{G}} = 0$ , then it remains to show that  $\delta^2 \mathcal{O}b_{\phi} + \phi \mathcal{O}b_{\mathcal{L}} - \mathcal{O}b_{\mathcal{G}} \phi = 0$ .

We have

$$\left( - \sum_{p+q=s, p,q>0} \circ_{x,y,z} [\alpha(x), [y, z]_p]_q = \frac{1}{2} \sum_{p+q=s, p>0, q>0} [[\cdot, \cdot]_p, [\cdot, \cdot]_q]_{\alpha}^{\wedge}(x, y, z) \right)$$

$$\begin{aligned} \delta^2 \left( \sum_{i=1}^N \phi_i([\cdot, \cdot]_{\mathcal{L}, N+1-i} - \sum' [\phi_i, \phi_j]_{\mathcal{G}, k}) \right) (x, y, z) = & -[\phi(\alpha(x)), \sum' [\phi_i(y), \phi_j(z)]_{\mathcal{G}, k}]_{\mathcal{G}, 0} \\ & + [\phi(\alpha(y)), \sum' [\phi_i(x), \phi_j(z)]_{\mathcal{G}, k}]_{\mathcal{G}, 0} - [\phi(\alpha(z)), \sum' [\phi_i(x), \phi_j(y)]_{\mathcal{G}, k}]_{\mathcal{G}, 0} \\ & + \sum' [\phi_i([x, y]_{\mathcal{L}, 0}), \phi_j(\alpha(z))]_{\mathcal{G}, k} \end{aligned} \quad (47)$$

$$- \sum' [\phi_i([x, z]_{\mathcal{L}, 0}), \phi_j(\alpha(y))]_{\mathcal{G}, k} \quad (48)$$

$$+ \sum' [\phi_i([y, z]_{\mathcal{L}, 0}), \phi_j(\alpha(x))]_{\mathcal{G}, k} \quad (49)$$

$$+ [\phi(\alpha(x)), \sum_{i=1}^N \phi_i([y, z]_{\mathcal{L}, N+1-i})]_{\mathcal{G}, 0} \quad (50)$$

$$- [\phi(\alpha(y)), \sum_{i=1}^N \phi_i([x, z]_{\mathcal{L}, N+1-i})]_{\mathcal{G}, 0} \quad (51)$$

$$+ [\phi(\alpha(z)), \sum_{i=1}^N \phi_i([x, y]_{\mathcal{L}, N+1-i})]_{\mathcal{G}, 0} \quad (52)$$

$$- \sum_{i=1}^N \phi_i([x, y]_{\mathcal{L}, 0}, \alpha(z)]_{\mathcal{L}, N+1-i} \quad (53)$$

$$+ \sum_{i=1}^N \phi_i([x, z]_{\mathcal{L}, 0}, \alpha(y)]_{\mathcal{L}, N+1-i} \quad (54)$$

$$- \sum_{i=1}^N \phi_i([y, z]_{\mathcal{L}, 0}, \alpha(x)]_{\mathcal{L}, N+1-i}. \quad (55)$$

Using the fact that

$$\phi_i([x, y]_{\mathcal{L}, 0}) = \sum_{\substack{\beta+\alpha+\gamma=i \\ \alpha, \beta, \gamma>0}} [\phi_{\beta}(x), \phi_{\gamma}(z)]_{\mathcal{G}, \alpha} - \sum_{\substack{\lambda+\mu=i \\ 1 \leq \mu \leq i}} \phi_{\lambda}([x, z]_{\mathcal{L}, \mu}),$$

then (48) becomes

$$- \sum' [\phi_i([x, z]_{\mathcal{L}, 0}), \phi_j(\alpha(y))]_{\mathcal{G}, k} = - \sum'_{\substack{\alpha+\beta+\gamma=i \\ \alpha, \beta, \gamma \geq 0}} [[\phi_{\beta}(x), \phi_{\gamma}(z)]_{\mathcal{G}, \alpha}, \phi_j(\alpha(y))]_{\mathcal{G}, k} \quad (56)$$

$$+ \sum'_{\substack{\lambda+\mu=i \\ 1 \leq \mu \leq i}} [\phi_{\lambda}([x, z]_{\mathcal{L}, \mu}), \phi_j(\alpha(y))]_{\mathcal{G}, k}, \quad (57)$$

where the sum on the right hand side of (56) is

$$\sum'_{\substack{\alpha+\beta+\gamma=i \\ \alpha,\beta,\gamma \geq 0}} = \sum_{\substack{\alpha+\beta+\gamma,k>0 \\ \alpha,\beta,\gamma \geq 0 \\ j=0}} + \sum_{\substack{\alpha+\beta+\gamma,j>0 \\ \alpha,\beta,\gamma \geq 0 \\ k=0}} + \sum_{\substack{\alpha=\beta=\gamma=0 \\ j,k>0}} + \sum_{\substack{\alpha+\beta+\gamma,k,j>0 \\ \alpha,\beta,\gamma \geq 0}} \quad (58)$$

and (57) is of the form

$$\sum'_{\substack{\lambda+\mu=i \\ 1 \leq \mu \leq i}} = \sum_{\substack{\lambda+\mu+k=N+1 \\ k,\mu>0,\lambda \geq 0 \\ j=0}} + \sum_{\substack{\lambda+\mu+j=N+1 \\ j,\mu>0,\lambda \geq 0 \\ k=0}} + \sum_{\substack{\lambda+\mu+j+k=N+1 \\ j,\mu,k>0,\lambda \geq 0}}.$$

The first term of (58) can be written as

$$\begin{aligned} - \sum_{\substack{\alpha+\beta+\gamma,k>0 \\ \alpha,\beta,\gamma \geq 0 \\ j=0}} [[\phi_\beta(x), \phi_\gamma(z)]_{\mathcal{G},\alpha}, \phi(\alpha(y))]_{\mathcal{G},k} &= - \sum_{\substack{k+\alpha=N+1 \\ k,\alpha>0}} [[\phi(x), \phi(z)]_{\mathcal{G},\alpha}, \phi(\alpha(y))]_{\mathcal{G},k} \quad (59) \\ &- \sum_{\substack{\beta+\gamma+\alpha+k=N+1 \\ k,\beta+\gamma>0 \\ \alpha,\beta,\gamma \geq 0}} [[\phi_\beta(x), \phi_\gamma(z)]_{\mathcal{G},\alpha}, \phi(\alpha(y))]_{\mathcal{G},k}. \end{aligned}$$

The term (59) vanishes with the second term of  $-\mathcal{O}b_{\mathcal{G}} \circ \phi$ . We may write (47) as follows

$$\sum' [\phi_i([x, y]_{\mathcal{L},0}), \phi_j(\alpha(z))]_{\mathcal{G},k} = \sum'_{\substack{\alpha+\beta+\gamma=i \\ \alpha,\beta,\gamma \geq 0}} [[\phi_\beta(x), \phi_\gamma(y)]_{\mathcal{G},\alpha}, \phi_j(\alpha(z))]_{\mathcal{G},k} \quad (60)$$

$$- \sum'_{\substack{\lambda+\mu=i \\ 1 \leq \mu \leq i}} [\phi_\lambda([x, y]_{\mathcal{L},\mu}), \phi_j(\alpha(z))]_{\mathcal{G},k}, \quad (61)$$

where the sum on the right hand side of (60) is

$$\sum'_{\substack{\alpha+\beta+\gamma=i \\ \alpha,\beta,\gamma \geq 0}} = \sum_{\substack{\alpha+\beta+\gamma,k>0 \\ \alpha,\beta,\gamma \geq 0 \\ j=0}} + \sum_{\substack{\alpha+\beta+\gamma,j>0 \\ \alpha,\beta,\gamma \geq 0 \\ k=0}} + \sum_{\substack{\alpha=\beta=\gamma=0 \\ j,k>0}} + \sum_{\substack{\alpha+\beta+\gamma,k,j>0 \\ \alpha,\beta,\gamma \geq 0}}. \quad (62)$$

The first term of (62) can be written as

$$\begin{aligned} \sum_{\substack{\alpha+\beta+\gamma,k>0 \\ \alpha,\beta,\gamma \geq 0 \\ j=0}} [[\phi_\beta(x), \phi_\gamma(y)]_{\mathcal{G},\alpha}, \phi(\alpha(z))]_{\mathcal{G},k} &= \sum_{\substack{k+\alpha=N+1 \\ k,\alpha>0}} [[\phi(x), \phi(y)]_{\mathcal{G},\alpha}, \phi(\alpha(z))]_{\mathcal{G},k} \quad (63) \\ &+ \sum_{\substack{\beta+\gamma+\alpha+k=N+1 \\ k,\beta+\gamma>0 \\ \alpha,\beta,\gamma \geq 0}} [[\phi_\beta(x), \phi_\gamma(y)]_{\mathcal{G},\alpha}, \phi(\alpha(z))]_{\mathcal{G},k}. \end{aligned}$$



The term (63) vanishes with the first term of  $-\mathcal{O}b_{\mathcal{G}} \circ \phi$ . We can write (49) as follows

$$\sum' [\phi_i([y, z]_{\mathcal{L}, 0}), \phi_j(\alpha(x))]_{\mathcal{G}, k} = \sum'_{\substack{\alpha+\beta+\gamma=i \\ \alpha, \beta, \gamma \geq 0}} [[\phi_\beta(y), \phi_\gamma(z)]_{\mathcal{G}, \alpha}, \phi_j(\alpha(x))]_{\mathcal{G}, k} \quad (64)$$

$$- \sum'_{\substack{\lambda+\mu=i \\ 1 \leq \mu \leq i}} [\phi_\lambda([y, z]_{\mathcal{L}, \mu}), \phi_j(\alpha(x))]_{\mathcal{G}, k}, \quad (65)$$

where the sum on the right hand side of (64) is

$$\sum' = \sum_{\substack{\alpha+\beta+\gamma=i \\ \alpha, \beta, \gamma \geq 0}} + \sum_{\substack{\alpha+\beta+\gamma, k > 0 \\ \alpha, \beta, \gamma \geq 0 \\ j=0}} + \sum_{\substack{\alpha+\beta+\gamma, j > 0 \\ \alpha, \beta, \gamma \geq 0 \\ k=0}} + \sum_{\substack{\alpha=\beta=\gamma=0 \\ j, k > 0}} + \sum_{\substack{\alpha+\beta+\gamma, k, j > 0 \\ \alpha, \beta, \gamma \geq 0}}. \quad (66)$$

The first term of (66) can be written as

$$\begin{aligned} \sum_{\substack{\alpha+\beta+\gamma, k > 0 \\ \alpha, \beta, \gamma \geq 0 \\ j=0}} [[\phi_\beta(y), \phi_\gamma(z)]_{\mathcal{G}, \alpha}, \phi(\alpha(x))]_{\mathcal{G}, k} &= \sum_{\substack{k+\alpha=N+1 \\ k, \alpha > 0}} [[\phi(y), \phi(z)]_{\mathcal{G}, \alpha}, \phi(\alpha(x))]_{\mathcal{G}, k} \quad (67) \\ &+ \sum_{\substack{\beta+\gamma+\alpha+k=N+1 \\ k, \beta+\gamma > 0 \\ \alpha, \beta, \gamma \geq 0}} [[\phi_\beta(y), \phi_\gamma(z)]_{\mathcal{G}, \alpha}, \phi(\alpha(x))]_{\mathcal{G}, k}. \end{aligned}$$

The term (67) vanishes with the third term of  $-\mathcal{O}b_{\mathcal{G}} \circ \phi$ .

On the other hand, one uses the fact that

$$\circlearrowleft_{x, y, z} \left( \sum_{i=0}^N [[x, z]_{\mathcal{L}, 0}, \alpha(y)]_{\mathcal{L}, N+1-i} + \sum_{j=0}^N [[x, z]_{\mathcal{L}, N+1-j}, \alpha(y)]_{\mathcal{L}, j} \right) = 0,$$

then

$$\begin{aligned} (53) + (54) + (55) &= - \sum_{i=1}^N \phi_i([x, y]_{\mathcal{L}, 0}, \alpha(z)]_{\mathcal{L}, N+1-i}) \\ &+ \sum_{i=1}^N \phi_i([x, z]_{\mathcal{L}, 0}, \alpha(y)]_{\mathcal{L}, N+1-i}) - \sum_{i=1}^N \phi_i([y, z]_{\mathcal{L}, 0}, \alpha(x)]_{\mathcal{L}, N+1-i}) \\ &= - \sum_{i=1}^N \circlearrowleft_{x, y, z} (\phi_i([x, y]_{\mathcal{L}, 0}, \alpha(z)]_{\mathcal{L}, N+1-i})) = \sum_{\substack{i+j+k=N+1 \\ i, j > 0, k \geq 0}} \circlearrowleft_{x, y, z} \phi_i([x, y]_{\mathcal{L}, j}, \alpha(z)]_{\mathcal{L}, k}) \\ &= \sum_{j=1}^N \circlearrowleft_{x, y, z} \left( \sum_{\substack{i+k=N+1-j \\ i, k \geq 0}} \phi_i([x, y]_{\mathcal{L}, j}, \alpha(z)]_{\mathcal{L}, k}) \right) \quad (68) \end{aligned}$$

$$- \phi \left( \sum_{\substack{j+k=N+1 \\ j, k > 0}} \circlearrowleft_{x, y, z} [[x, y]_{\mathcal{L}, j}, \alpha(z)]_{\mathcal{L}, k} \right). \quad (69)$$

The term (69) vanishes with  $\phi \circ \mathcal{O}b_{\mathcal{L}}$ . The term (68) may be written as

$$\begin{aligned}
& \circlearrowleft_{x,y,z} \sum_{j=1}^N \left( \sum_{\substack{i+k=N+1-j \\ i,k \geq 0}} \phi_i([x,y]_j, \alpha(z))_{\mathcal{L},k} \right) \\
&= \sum_{j=1}^N \circlearrowleft_{x,y,z} \left( \sum_{\substack{\beta+\gamma+\alpha=N+1-j \\ \alpha,\beta,\gamma \geq 0}} [\phi_\beta, ([x,y]_j), \phi_\gamma(\alpha(z))]_{\mathcal{G},\alpha} \right) \\
&= \circlearrowleft_{x,y,z} \sum_{i=1}^N [\phi_i([x,y]_{\mathcal{L},N+1-i}), \phi(\alpha(z))]_{\mathcal{G},0}
\end{aligned} \tag{70}$$

$$+ \circlearrowleft_{x,y,z} \sum_{\substack{\lambda+\mu+j+\alpha=N+1 \\ \lambda+\mu=j \\ 1 \leq \mu \leq j}} [\phi_\lambda([x,y]_{\mathcal{L},\mu}), \phi_k(\alpha(z))]_{\mathcal{G},\alpha}. \tag{71}$$

The term (71) may be written as

$$\begin{aligned}
& \circlearrowleft_{x,y,z} \sum_{\substack{\lambda+\mu+j+\alpha=N+1 \\ \lambda+\mu=j \\ 1 \leq \mu \leq j}} [\phi_\lambda([x,y]_{\mathcal{L},\mu}), \phi_k(\alpha(z))]_{\mathcal{G},\alpha} = \sum_{\substack{\lambda+\mu+j+\alpha=N+1 \\ \lambda+\mu=j \\ 1 \leq \mu \leq j}} [\phi_\lambda([x,y]_{\mathcal{L},\mu}), \phi_k(\alpha(z))]_{\mathcal{G},\alpha} \\
& + \sum_{\substack{\lambda+\mu+j+\alpha=N+1 \\ \lambda+\mu=j \\ 1 \leq \mu \leq j}} [\phi_\lambda([z,x]_{\mathcal{L},\mu}), \phi_k(\alpha(y))]_{\mathcal{G},\alpha} + \sum_{\substack{\lambda+\mu+j+\alpha=N+1 \\ \lambda+\mu=j \\ 1 \leq \mu \leq j}} [\phi_\lambda([y,z]_{\mathcal{L},\mu}), \phi_k(\alpha(x))]_{\mathcal{G},\alpha} \\
& = -((57) + (61) + (65)).
\end{aligned}$$

Then (71) + (57) + (61) + (65) = 0 and (70) may be written as

$$\begin{aligned}
& \circlearrowleft_{x,y,z} \sum_{i=1}^N [\phi_i([x,y]_{\mathcal{L},N+1-i}), \phi(\alpha(z))]_{\mathcal{G},0} = \sum_{i=1}^N [\phi_i([y,z]_{\mathcal{L},N+1-i}), \phi(\alpha(x))]_{\mathcal{G},0} \\
& + \sum_{i=1}^N [\phi_i([z,x]_{\mathcal{L},N+1-i}), \phi(\alpha(y))]_{\mathcal{G},0} + \sum_{i=1}^N [\phi_i([x,y]_{\mathcal{L},N+1-i}), \phi(\alpha(z))]_{\mathcal{G},0} \\
& = -((50) + (51) + (52)).
\end{aligned}$$

Then (70) + ((50) + (51) + (52)) = 0.

The remaining components of  $\delta^2 \mathcal{O}b_\phi + \phi \mathcal{O}b_{\mathcal{L}} - \mathcal{O}b_{\mathcal{G}} \phi$  can be written as

$$\circlearrowleft_{\phi_\beta, \phi_\gamma, \phi_j} \widetilde{\sum} [[\phi_\beta(x), \phi_\gamma(y)]_{\mathcal{G},\alpha}, \beta(\phi_j(z))]_{\mathcal{G},k} = 0,$$

$$\text{where } \widetilde{\sum} = \sum_{\substack{j+\beta=N+1 \\ \beta,j>0}} + \sum_{\substack{j+\gamma=N+1 \\ \gamma,j>0}} + \sum_{\substack{\gamma+\beta=N+1 \\ \beta,\gamma>0}} + \sum_{\substack{j+\beta+\gamma=N+1 \\ \beta,j,\gamma>0}} + \sum_{\substack{j+\beta+\gamma+j+\alpha=N+1 \\ 1 \leq \beta+\gamma+j \leq N \\ \alpha,\beta,\gamma,j,k \geq 0}}.$$

Then  $\delta^2 \mathcal{O}b_\phi + \phi \mathcal{O}b_{\mathcal{L}} - \mathcal{O}b_{\mathcal{G}} \phi = 0$ .

One has moreover  $\delta^2([\cdot, \cdot]_{\mathcal{L},N+1}, [\cdot, \cdot]_{\mathcal{G},N+1}, \phi_{N+1}) = (\mathcal{O}b_{\mathcal{L}}, \mathcal{O}b_{\mathcal{G}}, \mathcal{O}b_\phi)$ . Then, the order  $N$  formal deformation extends to an order  $N+1$  formal deformation whenever  $(\mathcal{O}b_{\mathcal{L}}, \mathcal{O}b_{\mathcal{G}}, \mathcal{O}b_\phi)$  is a coboundary in  $C_{HL}^3(\phi, \phi)$ .  $\square$

## 5 Example

We compute in this section a cohomology of a given Hom-Lie algebra morphism and discuss some deformations. Let  $\mathfrak{g}_1 = (\mathfrak{g}_1, [\cdot, \cdot]_1, \alpha_1)$  and  $\mathfrak{g}_2 = (\mathfrak{g}_2, [\cdot, \cdot]_2, \alpha_2)$  be two Hom-Lie algebras defined with respect to the basis  $\{e_1, e_2, e_3\}$  (resp.  $\{f_1, f_2, f_3\}$ ) by

$$\mathfrak{g}_1 : \begin{cases} [e_1, e_2]_1 = e_3 \\ [e_2, e_3]_1 = 0 \\ [e_1, e_3]_1 = 0 \end{cases} \quad \alpha_1 = \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_1 p_2 \end{pmatrix}, \quad \mathfrak{g}_2 : \begin{cases} [f_1, f_2]_2 = f_1 + f_3 \\ [f_2, f_3]_2 = f_2 \\ [f_1, f_3]_2 = f_1 + 2f_3 \end{cases} \quad \alpha_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

where  $p_1, p_2, a, b, c, d$  are parameters.

Let  $\phi_{1,2} : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  be a Hom-Lie algebra morphism. The morphism  $\phi_{1,2}$  is wholly determined by a set of structure constants  $\lambda_{i,j}$ . It turns out that we have the following cases

$$\text{if } p_1 = p_2 = 2 \quad \begin{cases} \phi_{1,2}^1(e_1) = \lambda_{2,1}f_2 + \lambda_{3,1}f_3 \\ \phi_{1,2}^1(e_2) = \lambda_{2,2}f_2 + \frac{\lambda_{2,2}\lambda_{3,1}}{\lambda_{2,1}}f_3 \\ \phi_{1,2}^1(e_3) = 0 \end{cases}$$

and

$$\text{if } p_1 = 2 \text{ and } p_2 = 0 \quad \begin{cases} \phi_{1,2}^2(e_1) = \lambda_{2,1}f_2 + \lambda_{3,1}f_3 \\ \phi_{1,2}^2(e_2) = 0 \\ \phi_{1,2}^2(e_3) = 0 \end{cases}$$

We come now to the computation of  $\mathfrak{g}_1$  cohomology. A 2-cochain is given by a triple  $(\psi, \varphi, \rho)$ , where  $\psi : \mathfrak{g}_1 \times \mathfrak{g}_1 \rightarrow \mathfrak{g}_1$ ,  $\varphi : \mathfrak{g}_2 \times \mathfrak{g}_2 \rightarrow \mathfrak{g}_2$ ,  $\rho : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ . The 2-cochains of the Hom-Lie algebras  $\mathfrak{g}_1$  are defined by  $\psi_i, i = 1, \dots, 8$ ,

$$\begin{cases} \psi_1(e_1, e_2) = a_1 e_2 \\ \psi_1(e_2, e_3) = 0 \\ \psi_1(e_1, e_3) = 0 \end{cases} \quad \begin{cases} \psi_2(e_1, e_2) = a_2 e_3 \\ \psi_2(e_2, e_3) = 0 \\ \psi_2(e_1, e_3) = 0 \end{cases} \quad \begin{cases} \psi_3(e_1, e_2) = 0 \\ \psi_3(e_2, e_3) = a_3 e_3 \\ \psi_3(e_1, e_3) = 0 \end{cases} \quad \begin{cases} \psi_4(e_1, e_2) = 0 \\ \psi_4(e_2, e_3) = 0 \\ \psi_4(e_1, e_3) = a_4 e_2 \end{cases}$$

$$\begin{cases} \psi_5(e_1, e_2) = 0 \\ \psi_5(e_2, e_3) = 0 \\ \psi_5(e_1, e_3) = a_5 e_3 \end{cases} \quad \begin{cases} \psi_6(e_1, e_2) = a_6 e_1 \\ \psi_6(e_2, e_3) = 0 \\ \psi_6(e_1, e_3) = 0 \end{cases} \quad \begin{cases} \psi_7(e_1, e_2) = 0 \\ \psi_7(e_2, e_3) = a_7 e_1 \\ \psi_7(e_1, e_3) = 0 \end{cases} \quad \begin{cases} \psi_8(e_1, e_2) = 0 \\ \psi_8(e_2, e_3) = -\frac{p_2}{p_1} a_8 e_2 \\ \psi_8(e_1, e_3) = a_8 e_1 \end{cases}$$

where  $a_1, \dots, a_8$  are parameters.

We obtain the following results

1. If  $p_1 = 0$  then  $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1) = \langle \psi_2, \psi_3, \psi_5, \psi_6, \psi_7 \rangle$ . Hence,  $\dim Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1) = 5$ .
2. If  $p_1 = 1$  and  $p_2 \notin \{-1, 0, 1\}$ ,  $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1) = \langle \psi_1, \psi_2, \psi_4, \psi_5 \rangle$ , with dimension 4.
  - (a) If  $p_2 = 1$ ,  $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1)$  is generated in addition by  $\{\psi_3, \psi_6, \psi_7, \psi_8\}$ .
  - (b) If  $p_2 = 0$ ,  $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1)$  is generated in addition by  $\{\psi_3\}$ .
  - (c) If  $p_2 = -1$ ,  $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1)$  is generated in addition by  $\{\psi_7\}$ .
3. If  $p_1 = -1$  and  $p_2 \notin \{-1, 0, 1\}$  then  $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1) = \langle \psi_2, \psi_4 \rangle$ , with dimension 2.
  - (a) If  $p_2 = 1$ ,  $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1)$  is generated in addition by  $\{\psi_3, \psi_6, \psi_7\}$ .

- (b) If  $p_2 = 0$ ,  $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1)$  is generated in addition by  $\{\psi_1, \psi_3, \psi_5\}$ .
- (c) If  $p_2 = -1$ ,  $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1)$  is generated in addition by  $\{\psi_7, \psi_8\}$ .
- 4. If  $p_1 \notin \{-1, 0, 1\}$  and  $p_2 = \frac{1}{p_1}$ , then  $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1) = \langle \psi_2, \psi_8 \rangle$ , with dimension 2.
  - (a) If  $p_2 = 1$ ,  $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1)$  is generated by  $\{\psi_2, \psi_3, \psi_6, \psi_7\}$ .
  - (b) If  $p_2 = 0$ ,  $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1)$  is generated by  $\{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5\}$ .
  - (c) If  $p_2 = -1$ ,  $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1)$  is generated by  $\{\psi_2, \psi_7\}$ .
- 5. If  $p_1 \neq \{-1, 0, 1\}$  and  $p_2 \neq \{-1, 0, 1\}$ , such that  $p_2 \neq \frac{1}{p_1}$ , then  $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1) = \langle \psi_2 \rangle$ .

All cochains are not coboundaries except  $\psi_2$ . Therefore, in summary, we have

$$\dim H_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1) = \begin{cases} 4 & \text{if } p_1 = 0 \text{ with } a_3 \neq a_6 \\ 3 & \text{if } p_1 = 1 \text{ and } p_2 \notin \{-1, 0, 1\} \text{ with } a_1 \neq -a_5. \\ 7 & \text{if } p_1 = 1 \text{ and } p_2 = 1 \text{ with } a_1 \neq -a_5 \text{ and } a_3 \neq a_6. \\ 4 & \text{if } p_1 = 1 \text{ and } p_2 = 0 \text{ with } a_1 \neq -a_5. \\ 4 & \text{if } p_2 = -1 \text{ and } p_2 = -1 \text{ with } a_1 \neq -a_5. \\ 1 & \text{if } p_1 = -1 \text{ and } p_2 \notin \{-1, 0, 1\} \\ 4 & \text{if } p_1 = -1 \text{ and } p_2 = 1 \text{ with } a_3 \neq a_6. \\ 4 & \text{if } p_1 = -1 \text{ and } p_2 = 0. \\ 3 & \text{if } p_1 = -1 \text{ and } p_2 = -1. \\ 1 & \text{if } p_1 \notin \{-1, 0, 1\} \text{ and } p_2 = \frac{1}{p_1}. \\ 3 & \text{if } p_1 \notin \{-1, 0, 1\} \text{ and } p_2 = 1 \text{ with } a_3 \neq a_6. \\ 4 & \text{if } p_1 \notin \{-1, 0, 1\} \text{ and } p_2 = 0 \text{ with } a_1 \neq -a_5. \\ 1 & \text{if } p_1 \notin \{-1, 0, 1\} \text{ and } p_2 = -1. \\ 0 & \text{if } p_1 \notin \{-1, 0, 1\} \text{ and } p_2 \notin \{-1, 0, 1\}. \end{cases}$$

Now, we consider the second Lie algebra  $\mathfrak{g}_2$ . The 2-cocycles are defined as

$$\begin{cases} \varphi(f_1, f_2) = k_1 f_2 + k_2 f_3 \\ \varphi(f_2, f_3) = 0 \\ \varphi(f_1, f_3) = k_3 f_2 - k_1 f_3. \end{cases}$$

Therefore,  $\dim H_{HL}^2(\mathfrak{g}_2, \mathfrak{g}_2) = 1$  and it is generated by

$$\begin{cases} \varphi_2(f_1, f_2) = f_3 \\ \varphi_2(f_2, f_3) = 0 \\ \varphi_2(f_1, f_3) = 0. \end{cases}$$

Now, we consider the third component. Set  $\rho(e_i) = \sum_{k=1}^3 a_{k,i} f_k$ , we start with the first morphism  $\phi_{1,2}^1$ . There is only one 1-cocycle corresponding to  $\psi = \psi_2$  and  $\varphi$ , and it is given by

$$\rho = \begin{cases} \rho(e_1) = \left(-\frac{a_{3,2}\lambda_{2,1}^2}{\lambda_{2,2}\lambda_{3,1}} + \frac{a_{2,2}\lambda_{2,1}}{\lambda_{2,2}} + \frac{a_{3,1}\lambda_{2,1}}{\lambda_{3,1}}\right)f_2 + a_{3,1}f_3 \\ \rho(e_2) = a_{2,2}f_2 + a_{3,2}f_3 \\ \rho(e_3) = 0. \end{cases}$$

Therefore  $H_{HL}^1(\mathfrak{g}_1, \mathfrak{g}_2)$  is 6-dimensional.  
For the second morphism  $\phi_{1,2}^2$ , we have

$$\rho = \begin{cases} \rho(e_1) &= a_{2,1}f_2 + a_{3,1}f_3 \\ \rho(e_2) &= 0 \\ \rho(e_3) &= 0. \end{cases}$$

Therefore  $H_{HL}^1(\mathfrak{g}_1, \mathfrak{g}_2)$  is 2-dimensional.

Now, we provide examples of deformations. For  $\mathfrak{g}_1$  we consider the deformed brackets  $[\ , \ ]'_1$ .

$$[e_1, e_2]'_1 = e_3; \ [e_2, e_3]'_1 = 0; \ [e_1, e_3]'_1 = wte_2,$$

where  $w$  is a parameter, or

$$[e_1, e_2]'_1 = e_3 + te_2; \ [e_2, e_3]'_1 = 0; \ [e_1, e_3]'_1 = te_3.$$

Let  $[\ , \ ]'_2$  be a deformation of  $\mathfrak{g}_2$  defined by

$$[f_1, f_2]'_2 = af_1 + (b + tk_2)f_3; \ [f_2, f_3]'_2 = cf_2; \ [f_2, f_3]'_2 = df_1 + 2af_3$$

and  $\tilde{\phi}$  a deformation of the second morphism given by

$$\tilde{\phi}(e_1) = (\lambda_{2,1} + a_{2,1}t)f_2 + (\lambda_{3,1} + a_{3,1}t)f_3; \ \tilde{\phi}(e_2) = 0; \ \tilde{\phi}(e_3) = 0.$$

Then  $([\ , \ ]'_1, [\ , \ ]'_2, \tilde{\phi})$  is a deformation of  $\phi_{1,2}^2$ .

Let  $\tilde{\phi}$  be a deformation of the first morphism, where

$$\begin{aligned} \tilde{\phi}(e_1) &= (\lambda_{21} + (-\frac{a_{3,2}\lambda_{2,1}^2}{\lambda_{2,2}\lambda_{3,1}} + \frac{a_{22}\lambda_{2,1}}{\lambda_{2,2}} + \frac{a_{3,1}\lambda_{2,1}}{\lambda_{3,1}})t)f_2 + (\lambda_{3,1} + ta_{3,1})f_3 \\ \tilde{\phi}(e_2) &= (\lambda_{2,2} + ta_{2,2})f_2 + (\frac{\lambda_{2,2}\lambda_{3,1}}{\lambda_{2,1}} + ta_{3,2})f_3 \\ \tilde{\phi}(e_3) &= 0 \end{aligned}$$

Then  $([\ , \ ]_1, [\ , \ ]'_2, \tilde{\phi})$  is a deformation of  $\phi_{1,2}^1$ .

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