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# On Hom-Lie superbialgebras

Mohamed Fadous, Sami Mabrouk, Abdenacer Makhlouf

## Abstract

The purpose of this paper is to generalize to  $\mathbb{Z}_2$ -graded case the study of Hom-Lie bialgebras which were discussed first by D. Yau, then by C. Bai and Y. Sheng. We provide different ways for constructing Hom-Lie superbialgebras. Also we define Matched pairs, Manin supertriples and discuss their relationships. Moreover, we study coboundary and triangular Hom-Lie bialgebras, as well as infinitesimal deformations of the cobracket.

**Keywords :** Hom-Lie superalgebra, coboundary Hom-Lie bialgebra, triangular Hom-Lie bialgebra

**2010 Mathematics Subject Classification :** 17B62, 17B37.

## Introduction

Lie bialgebras were introduced by V. Drinfel'd in [7, 8], they are infinitesimal versions of compatible Poisson structures on Lie groups and maybe viewed as the Lie-theoretic case of a bialgebra. He raised various problems related to quantum groups and quantization. The study of quasi-triangular quantum groups involves the solutions of the quantum Yang-Baxter equations. In the classical limit, the solutions of the classical Yang-Baxter equations provide examples of Lie bialgebras. Since then a huge research activity was dedicated to these kind of algebraic structures.

The aim of this paper is to define and study Hom-Lie superbialgebras which are Hom-type generalization of Lie superbialgebras. Hom-Lie superbialgebras are Hom-Lie superalgebras provided with a cobracket and a compatibility condition. Motivated by examples of  $q$ -deformations of algebras of vector fields, J. Hartwig, D. Larsson, and S. Silvestrov introduced the notion of Hom-Lie algebra in [12], as a generalization of Lie algebras where the Jacobi condition is twisted by a Homomorphism. The graded case of Hom-Lie algebras were studied first by F. Ammar and the last author in [1], while Hom-Lie bialgebras were discussed by D. Yau, then by C. Bai and Y. Sheng. Recently L. Cai and Y. Sheng presented a slightly different approach of Hom-Lie bialgebras called purely Hom-Lie bialgebras in [5].

The paper is organized as follows, in the first section we provide the relevant definitions and some properties about Hom-Lie superbialgebras. Moreover, we give some key constructions and a classification of 3-dimensional Hom-Lie superbialgebras with 2-dimensional even part. In Section 2, We define Matched pairs and Manin supertriples, then we establish their relationships with Hom-Lie superbialgebras. We construct a Hom-Lie superalgebra structure on the direct sum of two Hom-Lie superalgebras  $(\mathfrak{g}, [\cdot, \cdot], \phi)$  and  $(\mathfrak{g}', [\cdot, \cdot]', \phi')$ , such that  $\mathfrak{g}$  is a  $\mathfrak{g}'$ -module and  $\mathfrak{g}'$  is a  $\mathfrak{g}$ -module, also we construct a Hom-Lie superbialgebra structure on the direct sum  $\mathfrak{g} \oplus \mathfrak{g}^*$  where  $\mathfrak{g}^*$  is the dual superspace of  $\mathfrak{g}$ . Section 3 is dedicated to coboundary Hom-Lie superbialgebras and quasi-triangular Hom-Lie superbialgebras. We show how a coboundary or quasi-triangular Hom-Lie superbialgebra can be constructed from a Hom-Lie superalgebra and an  $r$ -matrix. In the last section, we study perturbation of cobrackets in Hom-Lie superbialgebras, following Drinfel'd's perturbation theory of quasi-Hopf algebras. We describe Hom-Lie superbialgebras obtained by infinitesimal deformations of the cobracket.

## 1 Basics and Classification of Hom-Lie superbialgebras

In this section, we introduce and study Hom-Lie superbialgebras, which are Hom-type version of Lie superbialgebras, see [7, 8]. We extend to graded case the definition of Hom-Lie bialgebra introduced in [27]. We show that the dual of a finite dimensional Hom-Lie superbialgebra is also a Hom-Lie superbialgebra (Theorem 1.18), generalizing the self-dual property of Hom-Lie bialgebras.

First, let us start by fixing some definitions and notations. Let  $\mathfrak{L} = \mathfrak{L}_{\bar{0}} \oplus \mathfrak{L}_{\bar{1}}$  be a  $\mathbb{Z}_2$ -graded vector space over an arbitrary field  $\mathbb{K}$  of characteristic 0. In the sequel, we will consider only element which are  $\mathbb{Z}_2$ -homogeneous. For  $x \in \mathfrak{L}$ , we denote by  $|x| \in \mathbb{Z}_2$  its parity, i.e.,  $x \in \mathfrak{L}_{|x|}$ . We denote by  $\tau$  the super-twist map of  $\mathfrak{L} \otimes \mathfrak{L}$  namely,  $\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x$  for  $x, y \in \mathfrak{L}$ . The super-cyclic map  $\xi$  permutes the coordinates of  $\mathfrak{L} \otimes \mathfrak{L} \otimes \mathfrak{L}$ , it is defined as

$$\xi = (\mathbf{1} \otimes \tau) \cdot (\tau \otimes \mathbf{1}) : x \otimes y \otimes z \mapsto (-1)^{|x|(|y|+|z|)} y \otimes z \otimes x,$$

for  $x, y, z \in \mathfrak{L}$ , where  $\mathbf{1}$  is the identity map on  $\mathfrak{L}$ . We denote by  $\mathfrak{L}^* = \text{Hom}(\mathfrak{L}, \mathbb{K})$  the linear dual of  $\mathfrak{L}$ . For  $\phi \in \mathfrak{L}^*$  and  $x \in \mathfrak{L}$ , we often use the adjoint notation  $\langle \phi, x \rangle$  for  $\phi(x) \in \mathbb{K}$ .

For a linear map  $\Delta : \mathfrak{L} \rightarrow \mathfrak{L} \otimes \mathfrak{L}$  (comultiplication), we use Sweedler's notation  $\Delta(x) = \sum_{(x)} x_1 \otimes x_2$  for  $x \in \mathfrak{L}$ . We will often omit the summation sign  $\sum_{(x)}$  to make it simple. The parity  $|r|$  of  $r \in \mathfrak{L}^{\otimes 2}$  is defined as follows : since we assume  $r$  homogenous, there exists  $|r| \in \mathbb{Z}_2$ , such that  $r$  can be written as  $r = \sum r_1 \otimes r_2 \in \mathfrak{L}^{\otimes 2}$ ,  $r_1, r_2$  are homogenous elements with  $|r| = |r_1| + |r_2|$ .

**Definition 1.1.** ([1]). A *Hom-Lie superalgebra* is a triple  $(\mathfrak{L}, [\cdot, \cdot], \alpha)$  consisting of a superspace  $\mathfrak{L}$ , an even bilinear map  $[\cdot, \cdot] : \mathfrak{L} \times \mathfrak{L} \rightarrow \mathfrak{L}$  and an even superspace homomorphism  $\alpha : \mathfrak{L} \rightarrow \mathfrak{L}$  satisfying

$$[x, y] = -(-1)^{|x||y|} [y, x], \quad (1.1)$$

$$(-1)^{|x||z|} [\alpha(x), [y, z]] + (-1)^{|z||y|} [\alpha(z), [x, y]] + (-1)^{|y||x|} [\alpha(y), [z, x]] = 0 \quad (1.2)$$

for all homogeneous elements  $x, y, z$  in  $\mathfrak{L}$ .

It is multiplicative if, in addition  $\alpha \circ [\cdot, \cdot] = [\cdot, \cdot] \circ \alpha^{\otimes 2}$ , (i.e.,  $\alpha([x, y]) = [\alpha(x), \alpha(y)]$ ,  $\forall x, y \in \mathfrak{L}$ ).

**Definition 1.2.** ([13], [19]). A *Hom-Lie supercoalgebra* is a triple  $(\mathfrak{L}, \Delta, \alpha)$  consisting of a superspace  $\mathfrak{L}$ , an even superspace homomorphism  $\alpha : \mathfrak{L} \rightarrow \mathfrak{L}$  and a linear map  $\Delta : \mathfrak{L} \rightarrow \mathfrak{L} \otimes \mathfrak{L}$  (the cobracket) such that

$$\Delta(\mathfrak{L}^i) \subset \sum_{i=j+k} \mathfrak{L}^j \otimes \mathfrak{L}^k \text{ for } i \in \mathbb{Z}_2, \quad (1.3)$$

$$\text{Im} \Delta \subset \text{Im}(1 \otimes 1 - \tau) \text{ i.e. } \Delta \text{ is skew-supersymmetric,} \quad (1.4)$$

$$(1 \otimes 1 \otimes 1 + \xi + \xi^2) \circ (\alpha \otimes \Delta) \circ \Delta = 0 : \mathfrak{L} \rightarrow \mathfrak{L} \otimes \mathfrak{L} \otimes \mathfrak{L}. \quad (1.5)$$

If, in addition,  $\Delta \circ \alpha = \alpha^{\otimes 2} \circ \Delta$ , then  $\mathfrak{L}$  is called co-multiplicative.

A Hom-Lie supercoalgebra with  $\alpha = Id$  is exactly a Lie supercoalgebra [13].

*Remark 1.3.* Let  $\text{Im} \Delta \subset \text{Im}(1 \otimes 1 - \tau) \subset \ker(1 \otimes 1 + \tau)$ , then  $(1 \otimes 1 + \tau) \circ \Delta = 0$  i.e.  $\Delta$  is skew-supersymmetric ([24]).

**Definition 1.4.** (1) For an element  $x$  in a Hom-Lie superalgebra  $(\mathfrak{L}, [\cdot, \cdot], \alpha)$  and  $n \geq 2$ , define the adjoint map  $ad_x : \mathfrak{L}^{\otimes n} \rightarrow \mathfrak{L}^{\otimes n}$  by

$$ad_x(y_1 \otimes \cdots \otimes y_n) = \sum_{i=1}^n (-1)^{|x|(|y_1|+|y_2|+\cdots+|y_{i-1}|)} \alpha(y_1) \otimes \cdots \otimes \alpha(y_{i-1}) \otimes [x, y_i] \otimes \alpha(y_{i+1}) \cdots \otimes \alpha(y_n). \quad (1.6)$$

For  $n = 2$ ,  $ad_x(y_1 \otimes y_2) = [x, y_1] \otimes \alpha(y_2) + (-1)^{|x||y_1|} \alpha(y_1) \otimes [x, y_2]$ .

Conversely, given  $\gamma = y_1 \otimes \cdots \otimes y_n$ , we define the map  $ad(\gamma) : \mathfrak{L} \rightarrow \mathfrak{L}^{\otimes n}$  by  $ad(\gamma)(x) = ad_x(\gamma)$ , for  $x \in \mathfrak{L}$ .

(2) For an element  $x$  in a Hom-Lie superalgebra  $(\mathfrak{L}, [\cdot, \cdot], \alpha)$  and  $n \geq 2$ , define the adjoint map

$ad_{\alpha(x)} : \mathfrak{L}^{\otimes n} \rightarrow \mathfrak{L}^{\otimes n}$  by

$$ad_{\alpha(x)}(y_1 \otimes \cdots \otimes y_n) = \sum_{i=1}^n (-1)^{|x|(|y_1|+|y_2|+\cdots+|y_{i-1}|)} \alpha(y_1) \otimes \cdots \otimes \alpha(y_{i-1}) \otimes [\alpha(x), y_i] \otimes \alpha(y_{i+1}) \cdots \otimes \alpha(y_n). \quad (1.7)$$

**Definition 1.5.** A *(multiplicative) Hom-Lie superbialgebra* is a quadruple  $(\mathfrak{L}, [\cdot, \cdot], \Delta, \alpha)$  such that

1.  $(\mathfrak{L}, [\cdot, \cdot], \alpha)$  is a (multiplicative) Hom-Lie superalgebra.
2.  $(\mathfrak{L}, \Delta, \alpha)$  is a (co-multiplicative) Hom-Lie supercoalgebra.

3. The following compatibility condition holds for all  $x, y \in \mathfrak{L}$  :

$$\Delta([x, y]) = ad_{\alpha(x)}(\Delta(y)) - (-1)^{|x||y|} ad_{\alpha(y)}(\Delta(x)). \quad (1.8)$$

**Definition 1.6.** The map  $f : \mathfrak{L} \rightarrow \mathfrak{L}'$  is called even (resp. odd) map if  $f(\mathfrak{L}_i) \subset \mathfrak{L}'_i$  (resp.  $f(\mathfrak{L}_i) \subset \mathfrak{L}'_{i+1}$ ), for  $i = 0, 1$ . A morphism of Hom-Lie superbialgebras is an even linear map such that

$$\alpha \circ f = f \circ \alpha, \quad f \circ [\cdot, \cdot] = [\cdot, \cdot] \circ f^{\otimes 2} \quad \text{and} \quad \Delta \circ f = f^{\otimes 2} \circ \Delta.$$

An isomorphism of Hom-Lie superbialgebras is an invertible morphism of Hom-Lie superbialgebras. Two Hom-Lie superbialgebras are said to be isomorphic if there exists an isomorphism between them.

*Remark 1.7.* A Hom-Lie superbialgebra with  $\alpha = Id$  is exactly a Lie superbialgebra, as defined in [13, 7, 8].

*Remark 1.8.* The compatibility condition (1.8) is, in fact, a cocycle condition in Hom-Lie superalgebra cohomology [20], just as it is the case in a Lie superbialgebra with Lie superalgebra cohomology [8]. Indeed, we can regard  $\mathfrak{L}^{\otimes 2}$  as an  $\mathfrak{L}$ -module via the  $\alpha$ -twisted adjoint action (1.8):

$$\begin{aligned} x \cdot (y_1 \otimes y_2) &= ad_{\alpha(x)}(y_1 \otimes y_2) \\ &= [\alpha(x), y_1] \otimes \alpha(y_2) + (-1)^{|x||y_1|} \alpha(y_1) \otimes [\alpha(x), y_2], \end{aligned} \quad (1.9)$$

for  $x \in \mathfrak{L}$  and  $y_1 \otimes y_2 \in \mathfrak{L}^{\otimes 2}$ .

Then we can think of the cobracket  $\Delta : \mathfrak{L} \rightarrow \mathfrak{L} \otimes \mathfrak{L}$  as a 1-cochain  $\Delta \in C^1(\mathfrak{L}, \mathfrak{L}^{\otimes 2})$ . Here  $C^1(\mathfrak{L}, \mathfrak{L}^{\otimes 2})$  is defined as the linear super-subspace of  $\text{Hom}(\mathfrak{L}, \mathfrak{L}^{\otimes 2})$  consisting of maps that commute with  $\alpha$ . Generalizing [20] to include coefficients in  $\mathfrak{L}^{\otimes 2}$ , the differential on  $\Delta$  is given by

$$(\delta_{HL}^1 \Delta)(x, y) = \Delta([x, y]) - x \cdot \Delta(y) + (-1)^{|x||y|} y \cdot \Delta(x) = \Delta([x, y]) - ad_{\alpha(x)}(\Delta(y)) + (-1)^{|x||y|} ad_{\alpha(y)}(\Delta(x)). \quad (1.10)$$

Therefore, (1.8) says exactly that  $\Delta \in C^1(\mathfrak{L}, \mathfrak{L}^{\otimes 2})$  is a 1-cocycle.

**Example 1.9.** Classification of 2-dimensional Hom-Lie superbialgebras with 1-dimensional even part.

Let  $\mathfrak{L} = \mathfrak{L}_0 \oplus \mathfrak{L}_1$  be a 2-dimensional superspace where  $\mathfrak{L}_0$  is generated by  $e_1$  and  $\mathfrak{L}_1$  is generated by  $e_2$ . The triple  $(\mathfrak{L}, [\cdot, \cdot], \alpha)$  is a Hom-Lie superalgebra when  $[e_1, e_1] = 0$ ,  $[e_1, e_2] = be_2$  and  $[e_2, e_2] = ce_1$  with  $\alpha(e_1) = a_1 e_1$ ,  $\alpha(e_2) = a_2 e_2$  and  $a_2 bc = 0$ , where  $b, c, a_1, a_2$  are parameters in  $\mathbb{K}$ . The triple  $(\mathfrak{L}, \Delta, \alpha)$  is a Hom-Lie supercoalgebra if  $\Delta(e_1) = 0$  and  $\Delta(e_2) = d(e_1 \otimes e_2 - e_2 \otimes e_1)$ , where  $d \in \mathbb{K}$ . The triple  $(\mathfrak{L}, [\cdot, \cdot], \Delta, \alpha)$  is a Hom-Lie superbialgebras if  $a_1 = 1$  or  $a_1 = -1$  and  $a_2 bd = 0$ .

*Remark 1.10.* Recently Cai and Sheng introduced a different notion of Hom-Lie bialgebras called *purely Hom-Lie bialgebra*, see [5], which we can extend to the super case as follows. Let  $(\mathfrak{L}, [\cdot, \cdot], \alpha)$ , where  $\alpha$  is invertible, and  $(\mathfrak{L}^*, [\cdot, \cdot], (\alpha^{-1})^*)$  be two Hom-Lie superalgebras. The pair  $(\mathfrak{L}, \mathfrak{L}^*)$  is a purely Hom-Lie superbialgebra if holds the compatibility condition

$$\Delta([x, y]) = ad_{\alpha^{-1}(x)}(\Delta(y)) - (-1)^{|x||y|} ad_{\alpha^{-1}(y)}(\Delta(x)). \quad (1.11)$$

Notice that this condition is different from condition (1.8). Purely Hom-Lie bialgebras in a graded case will be studied in a forthcoming paper.

**Classification of 3-dimensional Hom-Lie superbialgebras with 2-dimensional even part.**

Let  $\mathfrak{L} = \mathfrak{L}_0 \oplus \mathfrak{L}_1$  be a 3-dimensional superspace where  $\mathfrak{L}_0$  is generated by  $e_1, e_2$  and  $\mathfrak{L}_1$  is generated by  $e_3$ . We aim to construct Hom-Lie superbialgebras. We set for the linear map  $\alpha$

$$\alpha(e_1) = a_1 e_1 + a_2 e_2, \quad \alpha(e_2) = a_3 e_1 + a_4 e_2, \quad \alpha(e_3) = a_5 e_3,$$

where  $a_1, a_2, a_3, a_4, a_5$  are parameters in  $\mathbb{K}$ .

The structure of the bracket  $[\cdot, \cdot]$  is of the form

$$[e_3, e_3] = b_1 e_1 + b_2 e_2, \quad [e_1, e_2] = b_3 e_1 + b_4 e_2, \quad [e_1, e_3] = b_5 e_3, \quad [e_2, e_3] = b_6 e_3,$$

where  $b_1, b_2, b_3, b_4, b_5, b_6$  are parameters in  $\mathbb{K}$ .  
The structure of the cobracket  $\Delta$  is of the form

$$\begin{aligned}\Delta(e_1) &= c_1(e_1 \otimes e_2 - e_2 \otimes e_1) + c_5 e_3 \otimes e_3, & \Delta(e_2) &= c_2(e_1 \otimes e_2 - e_2 \otimes e_1) + c_6 e_3 \otimes e_3, \\ \Delta(e_3) &= c_3(e_1 \otimes e_3 - e_3 \otimes e_1) + c_4(e_2 \otimes e_3 - e_3 \otimes e_2),\end{aligned}$$

where  $c_1, c_2, c_3, c_4$  are parameters in  $\mathbb{K}$ .

In the sequel, we will consider the linear map under the Jordan forms with respect to a suitable basis. We split the calculations in two cases. First, we deal with diagonal case and then with Jordan case.

1) **Diagonal case** : We consider the linear map  $\alpha$ , where with respect to a suitable basis the matrix is of the form:  $\alpha = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_4 & 0 \\ 0 & 0 & a_5 \end{pmatrix}$ , which corresponds to  $a_2 = 0, a_3 = 0$ , and the eigenvalues are pairwise non-equal.

We obtained, when the eigenvalues are nonzero, the following corresponding (multiplicative) Hom-Lie superbialgebras :

Linear map	bracket	cobracket
$\alpha(e_1) = e_1, \alpha(e_2) = a_4 e_2, \alpha(e_3) = -e_3$	$[e_3, e_3] = 0, [e_1, e_3] = b_5 e_3$ or $(b_5 = 0),$ $[e_1, e_2] = b_4 e_2, [e_2, e_3] = 0$	$\Delta(e_1) = c_5 e_3 \otimes e_3,$ $\Delta(e_2) = 0,$ $\Delta(e_3) = 0$
$\alpha(e_1) = e_1, \alpha(e_2) = a_4 e_2, \alpha(e_3) = a_5 e_3$	All bracket are zero	$\Delta(e_1) = 0,$ $\Delta(e_2) = c_2(e_1 \otimes e_2 - e_2 \otimes e_1),$ $\Delta(e_3) = c_3(e_1 \otimes e_3 - e_3 \otimes e_1)$
$\alpha(e_1) = e_1, \alpha(e_2) = a_4 e_2, \alpha(e_3) = -e_3$	$[e_3, e_3] = b_1 e_1,$ $[e_1, e_2] = [e_1, e_3] = [e_2, e_3] = 0$	$\Delta(e_1) = 0,$ $\Delta(e_2) = c_2(e_1 \otimes e_2 - e_2 \otimes e_1),$ $\Delta(e_3) = c_3(e_1 \otimes e_3 - e_3 \otimes e_1)$
$\alpha(e_1) = a_1 e_1, \alpha(e_2) = e_2, \alpha(e_3) = -e_3$	$[e_3, e_3] = [e_1, e_3] = 0, [e_1, e_2] = b_3 e_1$ $[e_2, e_3] = b_6 e_3$ or $(b_6 = 0$ and $b_3 = 0)$	$\Delta(e_1) = 0,$ $\Delta(e_2) = c_6 e_3 \otimes e_3,$ $\Delta(e_3) = 0$
$\alpha(e_1) = a_1 e_1, \alpha(e_2) = e_2, \alpha(e_3) = -e_3$	$[e_1, e_2] = [e_1, e_3] = [e_2, e_3] = 0,$ $[e_3, e_3] = b_2 e_2$ or $(b_2 = 0)$	$\Delta(e_1) = c_1(e_1 \otimes e_2 - e_2 \otimes e_1),$ $\Delta(e_2) = 0,$ $\Delta(e_3) = c_4(e_2 \otimes e_3 - e_3 \otimes e_2)$
$\alpha(e_1) = e_1, \alpha(e_2) = a_4 e_2, \alpha(e_3) = a_5 e_3$	$[e_3, e_3] = [e_1, e_3] = [e_2, e_3] = 0,$ $[e_1, e_2] = b_4 e_2$	$\Delta(e_1) = 0,$ $\Delta(e_2) = c_2(e_1 \otimes e_2 - e_2 \otimes e_1),$ $\Delta(e_3) = 0$
$\alpha(e_1) = e_1, \alpha(e_2) = a_4 e_2, \alpha(e_3) = a_5 e_3$	$[e_3, e_3] = [e_2, e_3] = 0$ $[e_1, e_2] = b_4 e_2, [e_1, e_3] = b_5 e_3$	All cobracket are zero
$\alpha(e_1) = a_1 e_1, \alpha(e_2) = a_4 e_2, \alpha(e_3) = a_5 e_3$ $a_5 = \pm\sqrt{a_1}$ or $(\alpha(e_1) = e_1$ and $\alpha(e_3) = -e_3)$	All bracket are zero	$\Delta(e_1) = c_5 e_3 \otimes e_3,$ $\Delta(e_2) = 0,$ $\Delta(e_3) = 0$
$\alpha(e_1) = a_1 e_1, \alpha(e_2) = e_2, \alpha(e_3) = a_5 e_3$	$[e_3, e_3] = [e_1, e_3] = 0$ $[e_1, e_2] = b_3 e_1, [e_2, e_3] = b_6 e_3$	All cobracket are zero
$\alpha(e_1) = e_1, \alpha(e_2) = a_4 e_2, \alpha(e_3) = a_5 e_3$ $(a_5 = \pm\sqrt{a_4})$	$[e_3, e_3] = b_2 e_2, [e_1, e_2] = b_4 e_2,$ $(b_5 = \frac{a_5 b_4}{2a_4}, c_3 = \frac{a_5 c_2}{2a_4}, c_6 = -\frac{b_4 c_2}{b_2})$ $[e_2, e_3] = 0, [e_1, e_3] = b_5 e_3,$	$\Delta(e_1) = 0, \Delta(e_2) = c_2(e_1 \otimes e_2 - e_2 \otimes e_1) + c_6 e_3 \otimes e_3,$ $\Delta(e_3) = c_3(e_1 \otimes e_3 - e_3 \otimes e_1)$
$\alpha(e_1) = a_1 e_1, \alpha(e_2) = e_2, \alpha(e_3) = a_5 e_3$ $(a_5 = \pm\sqrt{a_1})$	$[e_3, e_3] = b_1 e_1, [e_1, e_2] = b_3 e_1,$ $(b_6 = -\frac{a_5 b_3}{2a_1}, c_4 = -\frac{a_5 c_1}{2a_1}, c_5 = -\frac{b_3 c_1}{b_1})$ $[e_1, e_3] = 0, [e_2, e_3] = b_6 e_3,$	$\Delta(e_1) = c_1(e_1 \otimes e_2 - e_2 \otimes e_1) + c_5 e_3 \otimes e_3, \Delta(e_2) = 0,$ $\Delta(e_3) = c_4(e_2 \otimes e_3 - e_3 \otimes e_2)$
$\alpha(e_1) = a_1 e_1, \alpha(e_2) = e_2, \alpha(e_3) = a_5 e_3$ $(a_5 = \pm\sqrt{a_1})$	$[e_3, e_3] = 0, [e_1, e_2] = b_3 e_1,$ $(b_6 = -\frac{a_5 b_3}{2a_1})$ $[e_1, e_3] = 0, [e_2, e_3] = b_6 e_3,$	$\Delta(e_1) = c_5 e_3 \otimes e_3,$ $\Delta(e_2) = 0,$ $\Delta(e_3) = 0,$
$\alpha(e_1) = a_1 e_1, \alpha(e_2) = e_2, \alpha(e_3) = a_5 e_3$ $(a_5 = \pm\sqrt{a_1}, c_4 = -\frac{a_5 c_1}{2a_1})$	All bracket are zero	$\Delta(e_1) = c_1(e_1 \otimes e_2 - e_2 \otimes e_1) + c_5 e_3 \otimes e_3, \Delta(e_2) = 0,$ $\Delta(e_3) = c_4(e_2 \otimes e_3 - e_3 \otimes e_2)$

$\alpha(e_1) = e_1, \alpha(e_2) = a_4e_2, \alpha(e_3) = a_5e_3$ ( $a_5 = \pm\sqrt{a_4}, c_3 = \frac{a_5c_2}{2a_4}$ )	All bracket are zero	$\Delta(e_1) = 0, \Delta(e_2) = c_2(e_1 \otimes e_2 - e_2 \otimes e_1) + c_6e_3 \otimes e_3,$ $\Delta(e_3) = c_3(e_1 \otimes e_3 - e_3 \otimes e_1)$
$\alpha(e_1) = e_1, \alpha(e_2) = a_4e_2, \alpha(e_3) = a_5e_3$ ( $a_5 = \pm\sqrt{a_4}$ )	All bracket are zero	$\Delta(e_1) = 0,$ $\Delta(e_2) = c_2(e_1 \otimes e_2 - e_2 \otimes e_1),$ $\Delta(e_3) = c_3(e_1 \otimes e_3 - e_3 \otimes e_1)$
$\alpha(e_1) = e_1, \alpha(e_2) = a_4e_2, \alpha(e_3) = a_5e_3$ ( $a_5 = \pm\sqrt{a_4}$ )	$[e_1, e_2] = b_4e_2,$ $[e_3, e_3] = [e_1, e_3] = [e_2, e_3] = 0$	$\Delta(e_1) = 0,$ $\Delta(e_2) = c_2(e_1 \otimes e_2 - e_2 \otimes e_1),$ $\Delta(e_3) = 0$
$\alpha(e_1) = e_1, \alpha(e_2) = a_4e_2, \alpha(e_3) = a_5e_3$ ( $a_5 = \pm\sqrt{a_4}$ )	$[e_1, e_2] = b_4e_2, [e_1, e_3] = b_5e_3,$ ( $b_5 = \frac{a_5b_4}{2a_4}$ ) $[e_2, e_3] = [e_3, e_3] = 0,$	$\Delta(e_1) = 0,$ $\Delta(e_2) = c_6e_3 \otimes e_3,$ $\Delta(e_3) = 0$
$\alpha(e_1) = a_1e_1, \alpha(e_2) = a_4e_2, \alpha(e_3) = a_5e_3$	All bracket are zero	All cobracket are zero
$\alpha(e_1) = a_1e_1, \alpha(e_2) = e_2, \alpha(e_3) = a_5e_3$ ( $a_5 = \pm\sqrt{a_1}$ )	$[e_1, e_2] = b_3e_1,$ $[e_1, e_3] = [e_2, e_3] = [e_3, e_3] = 0$	$\Delta(e_1) = c_1(e_1 \otimes e_2 - e_2 \otimes e_1),$ $\Delta(e_2) = 0,$ $\Delta(e_3) = 0$
$\alpha(e_1) = a_1e_1, \alpha(e_2) = a_4e_2, \alpha(e_3) = a_5e_3$ ( $a_5 = \pm\sqrt{a_1}$ )	$[e_3, e_3] = b_1e_1,$ $[e_1, e_2] = [e_1, e_3] = [e_2, e_3] = 0$	All cobracket are zero
$\alpha(e_1) = a_1e_1, \alpha(e_2) = a_4e_2, \alpha(e_3) = a_5e_3$ ( $a_5 = \pm\sqrt{a_4}$ )	$[e_3, e_3] = b_2e_2,$ $[e_1, e_2] = [e_1, e_3] = [e_2, e_3] = 0$	All cobracket are zero
$\alpha(e_1) = e_1, \alpha(e_2) = a_4e_2, \alpha(e_3) = -e_3$	All bracket are zero	$\Delta(e_1) = 0,$ $\Delta(e_2) = 0,$ $\Delta(e_3) = c_3(e_1 \otimes e_3 - e_3 \otimes e_1)$
$\alpha(e_1) = a_1e_1, \alpha(e_2) = e_2, \alpha(e_3) = -e_3$	All bracket are zero	$\Delta(e_1) = 0,$ $\Delta(e_2) = 0,$ $\Delta(e_3) = c_4(e_2 \otimes e_3 - e_3 \otimes e_2)$
$\alpha(e_1) = e_1, \alpha(e_2) = a_4e_2, \alpha(e_3) = -e_3$	$[e_1, e_2] = b_4e_2,$ $[e_1, e_3] = [e_2, e_3] = [e_3, e_3] = 0$	$\Delta(e_1) = 0,$ $\Delta(e_2) = c_2(e_1 \otimes e_2 - e_2 \otimes e_1),$ $\Delta(e_3) = 0$

**2) Jordan case :** Now, we consider the linear map  $\alpha$  where the corresponding matrix is of the form

$$\alpha = \begin{pmatrix} a_1 & 1 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_5 \end{pmatrix}, \text{ that is } a_2 = 1, a_3 = 0, a_4 = a_1.$$

We obtained the following corresponding (multiplicative) Hom-Lie superbialgebras :

Linear map	bracket	cobracket
$\alpha(e_1) = e_2, \alpha(e_2) = \alpha(e_3) = 0$	$[e_3, e_3] = b_2e_2, [e_1, e_2] = b_4e_2,$ $[e_1, e_3] = [e_2, e_3] = 0$	$\Delta(e_1) = c_1(e_1 \otimes e_2 - e_2 \otimes e_1) + c_5e_3 \otimes e_3, \Delta(e_2) = 0,$ $\Delta(e_3) = c_4(e_2 \otimes e_3 - e_3 \otimes e_2)$
$\alpha(e_1) = e_2, \alpha(e_2) = \alpha(e_3) = 0$	$[e_3, e_3] = b_2e_2, [e_1, e_2] = b_4e_2,$ $[e_1, e_3] = b_5e_3, [e_2, e_3] = 0, \text{ or } (b_5 = 0)$	$\Delta(e_1) = c_1(e_1 \otimes e_2 - e_2 \otimes e_1) + c_5e_3 \otimes e_3, \Delta(e_2) = 0,$ $\Delta(e_3) = 0$
$\alpha(e_1) = e_1 + e_2, \alpha(e_2) = e_2, \alpha(e_3) = 0$	$[e_3, e_3] = [e_2, e_3] = 0,$ $[e_1, e_2] = b_4e_2, [e_1, e_3] = b_5e_3 \text{ or } (b_5 = 0)$	$\Delta(e_1) = c_1(e_1 \otimes e_2 - e_2 \otimes e_1),$ $\Delta(e_2) = 0,$ $\Delta(e_3) = c_4(e_2 \otimes e_3 - e_3 \otimes e_2)$
$\alpha(e_1) = e_2, \alpha(e_2) = 0, \alpha(e_3) = a_5e_3,$ ( $a_5 \neq 0$ )	$[e_3, e_3] = [e_1, e_3] = [e_2, e_3] = 0,$ $[e_1, e_2] = b_4e_2$	$\Delta(e_1) = c_1(e_1 \otimes e_2 - e_2 \otimes e_1),$ $\Delta(e_2) = \Delta(e_3) = 0$
$\alpha(e_1) = e_1 + e_2, \alpha(e_2) = e_2, \alpha(e_3) = a_5e_3$	All bracket are zero	$\Delta(e_1) = c_1(e_1 \otimes e_2 - e_2 \otimes e_1),$ $\Delta(e_2) = 0,$ $\Delta(e_3) = c_4(e_2 \otimes e_3 - e_3 \otimes e_2)$
$\alpha(e_1) = e_1 + e_2, \alpha(e_2) = e_2, \alpha(e_3) = a_5e_3$	$[e_1, e_2] = b_4e_2, [e_1, e_3] = b_5e_3,$ $[e_3, e_3] = [e_2, e_3] = 0$	$\Delta(e_1) = c_1(e_1 \otimes e_2 - e_2 \otimes e_1),$ $\Delta(e_2) = 0, \text{ for } c_4 = \frac{b_5c_1}{b_4}$ $\Delta(e_3) = c_4(e_2 \otimes e_3 - e_3 \otimes e_2)$
$\alpha(e_1) = e_1 + e_2, \alpha(e_2) = e_2, \alpha(e_3) = a_5e_3$ ( $a_5 = \pm 1, c_4 = -\frac{a_5c_1}{2}$ )	All bracket are zero	$\Delta(e_1) = c_1(e_1 \otimes e_2 - e_2 \otimes e_1) + c_5e_3 \otimes e_3, \Delta(e_2) = 0,$ $\Delta(e_3) = c_4(e_2 \otimes e_3 - e_3 \otimes e_2)$

$\alpha(e_1) = e_1 + e_2, \alpha(e_2) = e_2, \alpha(e_3) = a_5 e_3$ ( $a_5 = \pm 1, c_4 = -\frac{a_5 c_1}{2}$ )	$[e_3, e_3] = [e_2, e_3] = 0,$ ( $b_5 = -\frac{a_5 b_4}{2}$ ) $[e_1, e_2] = b_4 e_2, [e_1, e_3] = b_5 e_3,$	$\Delta(e_1) = c_1(e_1 \otimes e_2 - e_2 \otimes e_1)$ $+ c_5 e_3 \otimes e_3, \Delta(e_2) = 0,$ $\Delta(e_3) = c_4(e_2 \otimes e_3 - e_3 \otimes e_2)$
$\alpha(e_1) = e_1 + e_2, \alpha(e_2) = e_2, \alpha(e_3) = a_5 e_3$ ( $a_5 = \pm 1$ )	$[e_1, e_2] = [e_1, e_3] = [e_2, e_3] = 0,$  $[e_3, e_3] = b_2 e_2$	$\Delta(e_1) = c_1(e_1 \otimes e_2 - e_2 \otimes e_1),$ $\Delta(e_2) = 0,$ $\Delta(e_3) = c_4(e_2 \otimes e_3 - e_3 \otimes e_2)$
$\alpha(e_1) = -e_1 + e_2, \alpha(e_2) = -e_2, \alpha(e_3) = 0,$	$[e_3, e_3] = [e_1, e_2] = [e_2, e_3] = 0,$ $[e_1, e_3] = b_5 e_3, \text{ or } (b_5 = 0)$	$\Delta(e_1) = \Delta(e_2) = 0,$ $\Delta(e_3) = c_4(e_2 \otimes e_3 - e_3 \otimes e_2)$
$\alpha(e_1) = a_1 e_1 + e_2, \alpha(e_2) = a_4 e_2, \alpha(e_3) = a_5 e_3$ ( $a_1 \neq 0, a_5 \neq 0$ )	All bracket are zero	All cobracket are zero
$\alpha(e_1) = e_1 + e_2, \alpha(e_2) = e_2, \alpha(e_3) = a_5 e_3$ ( $a_5 = \pm 1$ )	$[e_3, e_3] = [e_2, e_3] = 0,$  $[e_1, e_3] = b_5 e_3, [e_1, e_2] = b_4 e_2, \text{ or } (b_4 = 0)$	$\Delta(e_1) = c_5 e_3 \otimes e_3,$  $\Delta(e_2) = \Delta(e_3) = 0$
$\alpha(e_1) = e_1 + e_2, \alpha(e_2) = e_2, \alpha(e_3) = a_5 e_3$ ( $a_5 = \pm 1$ )	$[e_3, e_3] = b_2 e_2, [e_1, e_2] = b_4 e_2,$ ( $b_5 = \frac{a_5 b_4}{2}, c_5 = \frac{-b_4 c_1 + 2a_5 b_4 c_4}{2b_2}, c_4 = \pm \frac{c_1}{2}$ ) $[e_1, e_3] = b_5 e_3, [e_2, e_3] = 0,$	$\Delta(e_1) = c_1(e_1 \otimes e_2 - e_2 \otimes e_1)$ $+ c_5 e_3 \otimes e_3, \Delta(e_2) = 0,$ $\Delta(e_3) = c_4(e_2 \otimes e_3 - e_3 \otimes e_2)$
$\alpha(e_1) = a_1 e_1 + e_2, \alpha(e_2) = a_4 e_2, \alpha(e_3) = 0$ ( $a_1 = a_4$ )	All bracket are zero	$\Delta(e_1) = \Delta(e_2) = 0,$ $\Delta(e_3) = c_4(e_2 \otimes e_3 - e_3 \otimes e_2)$
$\alpha(e_1) = e_1 + e_2, \alpha(e_2) = e_2, \alpha(e_3) = a_5 e_3$ ( $a_5 = \pm 1$ )	$[e_1, e_3] = b_5 e_3,$ $[e_3, e_3] = [e_1, e_2] = [e_2, e_3] = 0$	$\Delta(e_1) = \Delta(e_2) = 0,$ $\Delta(e_3) = c_4(e_2 \otimes e_3 - e_3 \otimes e_2)$
$\alpha(e_1) = a_1 e_1 + e_2, \alpha(e_2) = a_4 e_2, \alpha(e_3) = a_5 e_3$ ( $a_5 = \pm \sqrt{a_1}, a_1 = a_4$ )	$[e_3, e_3] = b_2 e_2,$ $[e_1, e_2] = [e_1, e_3] = [e_2, e_3] = 0$	All cobracket are zero
$\alpha(e_1) = a_1 e_1 + e_2, \alpha(e_2) = a_4 e_2, \alpha(e_3) = 0$ ( $a_1 = a_4$ )	$[e_1, e_3] = b_5 e_3,$ $[e_1, e_2] = [e_3, e_3] = [e_2, e_3] = 0$	All cobracket are zero

The following result shows that a Hom-Lie superbialgebra deforms into another Hom-Lie superbialgebra along any endomorphism.

**Theorem 1.11.** *Let  $(\mathfrak{L}, [\cdot, \cdot], \Delta, \alpha)$  be a Hom-Lie superbialgebra and an even map  $\beta : \mathfrak{L} \rightarrow \mathfrak{L}$  be a Hom-Lie superbialgebra morphism. Then*

$$\mathfrak{L}_\beta = (\mathfrak{L}, [\cdot, \cdot]_\beta = \beta \circ [\cdot, \cdot], \Delta_\beta = \Delta \circ \beta, \beta \alpha)$$

is also a Hom-Lie superbialgebra, which is multiplicative if  $\mathfrak{L}$  is.

*Proof.* It is immediate that  $[\cdot, \cdot]_\beta$  is skew-supersymmetric (1.1) because

$$[x, y]_\beta = \beta([x, y]) = \beta(-(-1)^{|x||y|}[y, x]) = -(-1)^{|x||y|}\beta([y, x]) = -(-1)^{|x||y|}[y, x]_\beta.$$

The Hom super-Jacobi identity holds in  $\mathfrak{L}_\beta$  because

$$(-1)^{|x||z|}[\beta \alpha(x), [y, z]_\beta]_\beta = (-1)^{|x||z|}\beta^2[\alpha(x), [y, z]] = -(-1)^{|z||y|}[\beta \alpha(z), [x, y]_\beta]_\beta - (-1)^{|y||x|}[\beta \alpha(y), [z, x]_\beta]_\beta,$$

i.e.,

$$(-1)^{|x||z|}[\beta \alpha(x), [y, z]_\beta]_\beta + (-1)^{|z||y|}[\beta \alpha(z), [x, y]_\beta]_\beta + (-1)^{|y||x|}[\beta \alpha(y), [z, x]_\beta]_\beta = 0.$$

$\Delta_\beta$  is skew-supersymmetric because

$$\tau \circ \Delta_\beta = \tau \circ \Delta \circ \beta = -\Delta \circ \beta = -\Delta_\beta.$$

Likewise, the Hom-super-co-jacobi identity holds in  $\mathfrak{L}_\beta$  because

$$(1 \otimes 1 \otimes 1 + \xi + \xi^2) \circ (\beta \alpha \otimes \Delta_\beta) \circ \Delta_\beta = (\beta^{\otimes 3})^2(1 \otimes 1 \otimes 1 + \xi + \xi^2) \circ (\alpha \otimes \Delta) \circ \Delta = 0.$$

To check the compatibility condition (1.8) in  $\mathfrak{L}_\beta$ , we compute as follows :

$$\begin{aligned}
\Delta_\beta([x, y]_\beta) &= (\beta^{\otimes 2})^2 \Delta([x, y]) \\
&= (\beta^{\otimes 2})^2([\alpha(x), y_1] \otimes \alpha(y_2)) + (\beta^{\otimes 2})^2((-1)^{|x||y_1|} \alpha(y_1) \otimes [\alpha(x), y_2]) \\
&\quad - (\beta^{\otimes 2})^2((-1)^{|x||y|} [\alpha(y), x_1] \otimes \alpha(x_2)) - (\beta^{\otimes 2})^2((-1)^{|x||y|} (-1)^{|y||x_1|} \alpha(x_1) \otimes [\alpha(y), x_2]) \\
&= [\beta\alpha(x), \beta(y_1)]_\beta \otimes \beta\alpha(\beta(y_2)) + (-1)^{|x||y_1|} \beta\alpha(\beta(y_1)) \otimes [\beta\alpha(x), \beta(y_2)]_\beta \\
&\quad - (-1)^{|x||y|} [\beta\alpha(y), \beta(x_1)]_\beta \otimes \beta\alpha(\beta(x_2)) - (-1)^{|x||y|} (-1)^{|y||x_1|} \beta\alpha(\beta(x_1)) \otimes [\beta\alpha(y), \beta(x_2)]_\beta \\
&= [\beta\alpha(x), \beta(y_1)]_\beta \otimes \beta\alpha(\beta(y_2)) + (-1)^{|x||\beta(y_1)|} \beta\alpha(\beta(y_1)) \otimes [\beta\alpha(x), \beta(y_2)]_\beta \\
&\quad - (-1)^{|x||y|} [\beta\alpha(y), \beta(x_1)]_\beta \otimes \beta\alpha(\beta(x_2)) - (-1)^{|x||y|} (-1)^{|y||\beta(x_1)|} \beta\alpha(\beta(x_1)) \otimes [\beta\alpha(y), \beta(x_2)]_\beta \\
&= ad_{\beta\alpha(x)}(\Delta_\beta(y)) - (-1)^{|x||y|} ad_{\beta\alpha(y)}(\Delta_\beta(x)).
\end{aligned}$$

Because  $|\beta(x_1)| = |x_1|$ , and  $|\beta(y_1)| = |y_1|$  (i.e., an even map  $\beta$ ). We have shown that  $\mathfrak{L}_\beta$  is a Hom-Lie superbialgebra. The super-multiplicativity assertion is obvious.  $\square$

Now we discuss two special cases of Theorem 1.11. The next result says that one can obtain multiplicative Hom-Lie superbialgebras from Lie superbialgebras and their endomorphisms. A construction result of this form for Hom-type algebras was first given in [26].

**Corollary 1.12.** *Let  $(\mathfrak{L}, [\cdot, \cdot], \Delta)$  be a Lie superbialgebra and an even map  $\beta : \mathfrak{L} \rightarrow \mathfrak{L}$  be a Lie superbialgebra morphism. Then*

$$\mathfrak{L}_\beta = (\mathfrak{L}, [\cdot, \cdot]_\beta = \beta \circ [\cdot, \cdot], \Delta_\beta = \Delta \circ \beta, \beta)$$

*is a multiplicative Hom-Lie superbialgebra.*

*Proof.* This is the  $\alpha = Id$  special case of Theorem 1.11.  $\square$

The next result says that every multiplicative Hom-Lie superbialgebra gives rise to an infinite sequence of multiplicative Hom-Lie superbialgebras.

**Corollary 1.13.** *Let  $(\mathfrak{L}, [\cdot, \cdot], \Delta, \alpha)$  be a multiplicative Hom-Lie superbialgebra. Then*

$$\mathfrak{L}_{\alpha^n} = (\mathfrak{L}, [\cdot, \cdot]_{\alpha^n} = \alpha^n \circ [\cdot, \cdot], \Delta_{\alpha^n} = \Delta \circ \alpha^n, \alpha^{n+1})$$

*is also a multiplicative Hom-Lie superbialgebra for each integer  $n \geq 0$ .*

*Proof.* This is the  $\beta = \alpha^n$  special case of Theorem 1.11.  $\square$

Next we consider when Hom-Lie superbialgebra of the form  $\mathfrak{L}_\beta$ , as in Corollary 1.12, are isomorphic.

**Theorem 1.14.** *Let  $\mathfrak{g}$  and  $\mathfrak{h}$  be Lie superbialgebras. Let  $\alpha : \mathfrak{g} \rightarrow \mathfrak{g}$  and  $\beta : \mathfrak{h} \rightarrow \mathfrak{h}$  be Lie superbialgebra morphisms with  $\beta$  and  $\beta^{\otimes 2}$  injective. Then the following statements are equivalent :*

- 1) *The Hom-Lie superbialgebras  $\mathfrak{g}_\alpha$  and  $\mathfrak{h}_\beta$  as in Corollary 1.12, are isomorphic.*
- 2) *There exists a Lie superbialgebra isomorphism  $\gamma : \mathfrak{g} \rightarrow \mathfrak{h}$  such that  $\gamma\alpha = \beta\gamma$ .*

*Proof.* To show that the first statement implies the second statement, suppose that  $\gamma : \mathfrak{g}_\alpha \rightarrow \mathfrak{h}_\beta$  is an isomorphism of Hom-Lie superbialgebras. Then  $\gamma\alpha = \beta\gamma$  automatically.

To see that  $\gamma$  is a Lie superbialgebra isomorphism, first we check that it commutes with the Lie bracket. For any two elements  $x$  and  $y$  in  $\mathfrak{g}$ , we have

$$\beta\gamma[x, y] = \gamma\alpha[x, y] = \gamma([x, y]_\alpha) = [\gamma(x), \gamma(y)]_\beta = \beta[\gamma(x), \gamma(y)].$$

Since  $\beta$  is injective, we conclude that  $\gamma[x, y] = [\gamma(x), \gamma(y)]$ , i.e.,  $\gamma$  is a Lie superbialgebra isomorphism.

To check that  $\gamma$  commutes with the Lie cobrackets, we compute as follows:

$$\begin{aligned}
\beta^{\otimes 2}(\gamma^{\otimes 2}(\Delta(x))) &= (\beta\gamma)^{\otimes 2}(\Delta(x)) = (\gamma\alpha)^{\otimes 2}(\Delta(x)) = \gamma^{\otimes 2}(\alpha^{\otimes 2}(\Delta(x))) = \gamma^{\otimes 2}(\Delta_\alpha(x)) = \Delta_\beta(\gamma(x)) \\
&= \beta^{\otimes 2}(\Delta(\gamma(x))).
\end{aligned}$$

The injectivity of  $\beta^{\otimes 2}$  now implies that  $\gamma$  commutes with the Lie cobrackets. Therefore,  $\gamma$  is a Lie superbialgebra isomorphism. The other implication is proved by a similar argument, much of which is already given above.  $\square$



For a Lie superbialgebra  $\mathfrak{g}$ , let  $\text{Aut}(\mathfrak{g})$  be the group of Lie superbialgebra isomorphisms from  $\mathfrak{g}$  to  $\mathfrak{g}$ . In Theorem 1.14, restricting to the case  $\mathfrak{g} = \mathfrak{h}$  with  $\alpha$  and  $\beta$  both invertible, we obtain the following special case.

**Corollary 1.15.** *Let  $\mathfrak{g}$  be a Lie superbialgebra and  $\alpha, \beta \in \text{Aut}(\mathfrak{g})$ . Then the Hom-Lie superbialgebras  $\mathfrak{g}_\alpha$  and  $\mathfrak{g}_\beta$ , as in Corollary 1.12, are isomorphic if and only if  $\alpha$  and  $\beta$  are conjugate in  $\text{Aut}(\mathfrak{g})$ .*

Corollary 1.15 can be restated as follows.

**Corollary 1.16.** *Let  $\mathfrak{g}$  be a Lie superbialgebra. Then there is a bijection between the following two sets:*

- 1) *The set of isomorphism classes of Hom-Lie superbialgebras  $\mathfrak{g}_\alpha$  with  $\alpha$  invertible.*
- 2) *The set of conjugacy classes in the group  $\text{Aut}(\mathfrak{g})$ .*

The next result shows that finite dimensional Hom-Lie superbialgebras, like Lie superbialgebras, can be dualized. A proof of this self-dual property for the special of Lie bialgebras can be found in [16].

*Remark 1.17.* • If  $(\mathfrak{L}, \Delta, \alpha)$  is a Hom-Lie supercoalgebra, then  $(\mathfrak{L}^*, [\cdot, \cdot], \alpha)$  is a Hom-Lie superalgebra. Here  $[\cdot, \cdot]$  and  $\alpha$  in  $\mathfrak{L}^*$  are dual to  $\Delta$  and  $\alpha$ , respectively, in  $\mathfrak{L}$ .

• Conversely, if  $(\mathfrak{L}, [\cdot, \cdot], \alpha)$  is a finite dimensional Hom-Lie superalgebra, then  $(\mathfrak{L}^*, \Delta, \alpha)$  is a Hom-Lie supercoalgebra, where  $\Delta$  and  $\alpha$  in  $\mathfrak{L}^*$  are dual to  $[\cdot, \cdot]$  and  $\alpha$ , respectively, in  $\mathfrak{L}$ .

**Theorem 1.18.** *Let  $(\mathfrak{L}, [\cdot, \cdot], \Delta, \alpha)$  be a finite dimensional (multiplicative) Hom-Lie superbialgebra. Then its linear dual  $\mathfrak{L}^* = \text{Hom}(\mathfrak{L}, \mathbb{K})$  is also a (multiplicative) Hom-Lie superbialgebra with the dual structure maps:*

$$\alpha(\phi) = \phi \circ \alpha, \quad \langle [\phi, \psi], x \rangle = \langle \phi \otimes \psi, \Delta(x) \rangle, \quad \langle \Delta(\phi), x \otimes y \rangle = \langle \phi, [x, y] \rangle, \quad (1.12)$$

for  $x, y \in \mathfrak{L}$  and  $\phi, \psi \in \mathfrak{L}^*$ .

*Proof.* As we mentioned right after Remark 1.17,  $(\mathfrak{L}^*, [\cdot, \cdot], \alpha)$  is a Hom-Lie superalgebra, which is true even if  $\mathfrak{L}$  is not finite dimensional. Moreover,  $(\mathfrak{L}^*, \Delta, \alpha)$  is a Hom-Lie supercoalgebra, whose validity depends on the finite dimensionality of  $\mathfrak{L}$ . Thus, it remains to check the compatibility condition (1.8) between the bracket and the cobracket in  $\mathfrak{L}^*$ , i.e.,

$$\langle \Delta([\phi, \psi]), x \otimes y \rangle = \langle ad_{\alpha(\phi)}(\Delta(\psi)) - (-1)^{|\phi||\psi|} ad_{\alpha(\psi)}(\Delta(\phi)), x \otimes y \rangle \quad (1.13)$$

for  $x, y \in \mathfrak{L}$  and  $\phi, \psi \in \mathfrak{L}^*$ .

Using Definition 1.12, the compatibility condition (1.8) in  $\mathfrak{L}$ , we compute the left-hand side of (1.13) as follows:

$$\begin{aligned} \langle \Delta([\phi, \psi]), x \otimes y \rangle &= \langle [\phi, \psi], [x, y] \rangle \\ &= \langle \phi \otimes \psi, \Delta([x, y]) \rangle \\ &= \langle \phi \otimes \psi, ad_{\alpha(x)}(\Delta(y)) - (-1)^{|x||y|} ad_{\alpha(y)}(\Delta(x)) \rangle \\ &= \langle \phi \otimes \psi, [\alpha(x), y_1] \otimes \alpha(y_2) \rangle + (-1)^{|x||y_1|} \langle \phi \otimes \psi, \alpha(y_1) \otimes [\alpha(x), y_2] \rangle \\ &\quad - (-1)^{|x||y|} \langle \phi \otimes \psi, [\alpha(y), x_1] \otimes \alpha(x_2) \rangle - (-1)^{|x||y|} (-1)^{|y||x_1|} \langle \phi \otimes \psi, \alpha(x_1) \otimes [\alpha(y), x_2] \rangle \\ &= \langle \Delta(\phi) \otimes \psi, \alpha(x) \otimes y_1 \otimes \alpha(y_2) \rangle + (-1)^{|x||y_1|} \langle \phi \otimes \Delta(\psi), \alpha(y_1) \otimes \alpha(x) \otimes y_2 \rangle \\ &\quad - (-1)^{|x||y|} \langle \Delta(\phi) \otimes \psi, \alpha(y) \otimes x_1 \otimes \alpha(x_2) \rangle \\ &\quad - (-1)^{|x||y|} (-1)^{|y||x_1|} \langle \phi \otimes \Delta(\psi), \alpha(x_1) \otimes \alpha(y) \otimes x_2 \rangle \\ &= \langle \phi_1 \otimes \phi_2 \otimes \psi, \alpha(x) \otimes y_1 \otimes \alpha(y_2) \rangle + (-1)^{|x||y_1|} \langle \phi \otimes \psi_1 \otimes \psi_2, \alpha(y_1) \otimes \alpha(x) \otimes y_2 \rangle \\ &\quad - (-1)^{|x||y|} \langle \phi_1 \otimes \phi_2 \otimes \psi, \alpha(y) \otimes x_1 \otimes \alpha(x_2) \rangle \\ &\quad - (-1)^{|x||y|} (-1)^{|y||x_1|} \langle \phi \otimes \psi_1 \otimes \psi_2, \alpha(x_1) \otimes \alpha(y) \otimes x_2 \rangle \\ &= \langle \alpha(\phi_1) \otimes \phi_2 \otimes \alpha(\psi), x \otimes y_1 \otimes y_2 \rangle + (-1)^{|x||y_1|} \langle \alpha(\phi) \otimes \alpha(\psi_1) \otimes \psi_2, y_1 \otimes x \otimes y_2 \rangle \\ &\quad - (-1)^{|x||y|} \langle \alpha(\phi_1) \otimes \phi_2 \otimes \alpha(\psi), y \otimes x_1 \otimes x_2 \rangle \\ &\quad - (-1)^{|x||y|} (-1)^{|y||x_1|} \langle \alpha(\phi) \otimes \alpha(\psi_1) \otimes \psi_2, x_1 \otimes y \otimes x_2 \rangle. \end{aligned}$$

Using, in addition, the skew-supersymmetric of the bracket and the cobracket in  $\mathfrak{L}^*$ ,  
 $([\phi_1, \alpha(\psi)] = -(-1)^{|\phi_1||\psi|}[\alpha(\psi), \phi_1]$  and  $\phi_1 \otimes \phi_2 = -(-1)^{|\phi_1||\phi_2|}\phi_2 \otimes \phi_1$ ).

The above four terms become:

$$\begin{aligned} &= -(-1)^{|\phi||\psi|}(-1)^{|\psi||\phi_1|}\langle \alpha(\phi_1) \otimes [\alpha(\psi), \phi_2], x \otimes y \rangle + (-1)^{|\phi||\psi_1|}\langle \alpha(\psi_1) \otimes [\alpha(\phi), \psi_2], x \otimes y \rangle \\ &- (-1)^{|\phi||\psi|}\langle [\alpha(\psi), \phi_1] \otimes \alpha(\phi_2), x \otimes y \rangle + \langle [\alpha(\phi), \psi_1] \otimes \alpha(\psi_2), x \otimes y \rangle. \\ &= \langle ad_{\alpha(\phi)}(\Delta(\psi)) - (-1)^{|\phi||\psi|}ad_{\alpha(\psi)}(\Delta(\phi)), x \otimes y \rangle \end{aligned}$$

This is exactly the right-hand side of (1.13).  $\square$

## 2 Matched pairs, Hom-Lie superbialgebras and Manin supertriples

In this section, we introduce the notions of matched pair of Hom-Lie superalgebras and a Manin supertriple of Hom-Lie superalgebras. First, we recall the basics about representations of a Hom-Lie superalgebras.

**Definition 2.1.** ([2, 22, 23]) Let  $(\mathfrak{L}, [\cdot, \cdot], \alpha)$  be a Hom-Lie superalgebra and  $M = M_{\bar{0}} \oplus M_{\bar{1}}$  an arbitrary vector superspace. A representation of the Hom-Lie superalgebra with respect to  $A \in \mathfrak{gl}(M)$ , is an even linear map  $\rho : \mathfrak{L} \rightarrow End(M)$ , such that  $\rho(\mathfrak{L}_i)(M_j) \subset M_{i+j}$  where  $i, j \in \mathbb{Z}_2$ , and satisfying

$$\rho(\alpha(x)) \circ A = A \circ \rho(x) \quad (2.1)$$

and

$$\rho([x, y]) \circ A = \rho(\alpha(x)) \circ \rho(y) - (-1)^{|x||y|}\rho(\alpha(y)) \circ \rho(x). \quad (2.2)$$

for all homogeneous elements  $x, y \in \mathfrak{L}$ .

We denote a representation by  $(M, \rho, A)$ . It is straightforward to see that  $(\mathfrak{L}, ad, \alpha)$  is a representation, called the adjoint representation, see [4, 22].

Given a representation  $(M, \rho, A)$ , define  $\rho^* : \mathfrak{L} \rightarrow End(M^*)$  by

$$\langle \rho^*(x)(\xi), \nu \rangle = -(-1)^{|x||\xi|}\langle \xi, \rho(x)(\nu) \rangle, \quad (2.3)$$

$\forall x \in \mathfrak{L}, \xi \in M^*, \nu \in M$ . This representation is called admissible representation with respect to  $(M, \rho, A)$ .

Now, let  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$  and  $(\mathfrak{g}', [\cdot, \cdot]_{\mathfrak{g}'}, \alpha_{\mathfrak{g}'})$  be two multiplicative Hom-Lie superalgebras. Set  $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$  and  $\mathfrak{g}' = \mathfrak{g}'_{\bar{0}} \oplus \mathfrak{g}'_{\bar{1}}$ . Let  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}')$  and  $\rho' : \mathfrak{g}' \rightarrow \mathfrak{gl}(\mathfrak{g})$  be two linear maps. Define a skew-supersymmetric bracket  $[\cdot, \cdot]_{\tilde{G}} : \tilde{G} \times \tilde{G} \rightarrow \tilde{G}$ , where  $\tilde{G}$  is given by  $\tilde{G} = \tilde{G}_{\bar{0}} \oplus \tilde{G}_{\bar{1}}$ , where  $\tilde{G}_{\bar{0}} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}'_{\bar{0}}$  and  $\tilde{G}_{\bar{1}} = \mathfrak{g}_{\bar{1}} \oplus \mathfrak{g}'_{\bar{1}}$ . We set  $|(x, x')| = |x| = |x'|$  and  $|(y, y')| = |y| = |y'|$ , for all homogeneous elements  $x, y$  in  $\mathfrak{g}$  and  $x', y'$  in  $\mathfrak{g}'$ . Define  $\tilde{\alpha} : \tilde{G} \rightarrow \tilde{G}$  by

$$\tilde{\alpha}(x, x') = (\alpha_{\mathfrak{g}}(x), \alpha_{\mathfrak{g}'}(x')),$$

and the bracket  $\tilde{G}$  by

$$[(x, x'), (y, y')]_{\tilde{G}} = ([x, y]_{\mathfrak{g}} - (-1)^{|x||y|}\rho'(y')(x) + \rho'(x')(y), [x', y']_{\mathfrak{g}'} + \rho(x)(y') - (-1)^{|x||y|}\rho(y)(x')). \quad (2.4)$$

**Theorem 2.2.** *The triple  $(\tilde{G} = \mathfrak{g} \oplus \mathfrak{g}', [\cdot, \cdot]_{\tilde{G}}, \tilde{\alpha})$ , where  $\tilde{G}, [\cdot, \cdot]_{\tilde{G}}, \tilde{\alpha}$  are defined above is a multiplicative Hom-Lie superalgebra if and only if  $\rho$  and  $\rho'$  are representations of  $\mathfrak{g}$  and  $\mathfrak{g}'$  respectively and the following conditions are satisfied*

$$\begin{aligned} \rho(\alpha_{\mathfrak{g}}(z))([x', y']_{\mathfrak{g}'}) &= [\rho(z)(x'), \alpha_{\mathfrak{g}'}(y')]_{\mathfrak{g}'} + (-1)^{|x||z|}[\alpha_{\mathfrak{g}'}(x'), \rho(z)(y')]_{\mathfrak{g}'} \\ &+ (-1)^{|x||y|+|y||z|}\rho(\rho'(y')(z))(\alpha_{\mathfrak{g}'}(x')) - (-1)^{|x||z|}\rho(\rho'(x')(z))(\alpha_{\mathfrak{g}'}(y')). \end{aligned} \quad (2.5)$$

$$\begin{aligned} \rho'(\alpha_{\mathfrak{g}'}(z'))([x, y]_{\mathfrak{g}}) &= [\rho'(z')(x), \alpha_{\mathfrak{g}}(y)]_{\mathfrak{g}} + (-1)^{|x||z|}[\alpha_{\mathfrak{g}}(x), \rho'(z')(y)]_{\mathfrak{g}} \\ &+ (-1)^{|x||y|+|y||z|}\rho'(\rho(y)(z'))(\alpha_{\mathfrak{g}}(x)) - (-1)^{|x||z|}\rho'(\rho(x)(z'))(\alpha_{\mathfrak{g}}(y)). \end{aligned} \quad (2.6)$$

*Proof.* Assume  $(\tilde{G} = \mathfrak{g} \oplus \mathfrak{g}', [\cdot, \cdot]_{\tilde{G}}, \tilde{\alpha})$  is a multiplicative Hom-Lie superalgebra. The multiplicativity condition writes

$$\tilde{\alpha}([(x, x'), (y, y')]_{\tilde{G}}) = [\tilde{\alpha}(x, x'), \tilde{\alpha}(y, y')]_{\tilde{G}}. \quad (2.7)$$

Developing (2.7), leads to the first condition (2.1). Indeed

$$\begin{aligned} \tilde{\alpha}([(x, x'), (y, y')]_{\tilde{G}}) &= (\alpha_{\mathfrak{g}}([x, y]_{\mathfrak{g}}) - (-1)^{|x||y|} \alpha_{\mathfrak{g}}(\rho'(y')(x)) + \alpha_{\mathfrak{g}}(\rho'(x')(y)), \\ &\quad \alpha_{\mathfrak{g}'}([x', y']_{\mathfrak{g}'} + \alpha_{\mathfrak{g}'}(\rho(x)(y')) - (-1)^{|x||y|} \alpha_{\mathfrak{g}'}(\rho(y)(x')))). \end{aligned}$$

$$\begin{aligned} [\tilde{\alpha}(x, x'), \tilde{\alpha}(y, y')]_{\tilde{G}} &= ([\alpha_{\mathfrak{g}}(x), \alpha_{\mathfrak{g}}(y)]_{\mathfrak{g}} - (-1)^{|x||y|} \rho'(\alpha_{\mathfrak{g}'}(y'))(\alpha_{\mathfrak{g}}(x)) + \rho'(\alpha_{\mathfrak{g}'}(x'))(\alpha_{\mathfrak{g}}(y)), \\ &\quad [\alpha_{\mathfrak{g}'}(x'), \alpha_{\mathfrak{g}'}(y')]_{\mathfrak{g}'} + \rho(\alpha_{\mathfrak{g}}(x))(\alpha_{\mathfrak{g}'}(y')) - (-1)^{|x||y|} \rho(\alpha_{\mathfrak{g}}(y))(\alpha_{\mathfrak{g}'}(x')))). \end{aligned}$$

Since  $\alpha_{\mathfrak{g}}$  and  $\alpha_{\mathfrak{g}'}$  are multiplicative, which implies that

$$\rho(\alpha_{\mathfrak{g}}(x)) \circ \alpha_{\mathfrak{g}'} = \alpha_{\mathfrak{g}'} \circ \rho(x), \quad (2.8)$$

$$\rho'(\alpha_{\mathfrak{g}'}(x')) \circ \alpha_{\mathfrak{g}} = \alpha_{\mathfrak{g}} \circ \rho'(x'). \quad (2.9)$$

Developing the Hom-super-Jacobi identity (1.2), that for  $x, y, z \in \mathfrak{g}$  and  $x', y', z' \in \mathfrak{g}'$ ,

$$\begin{aligned} (-1)^{|(x, x')||z, z'|} [\tilde{\alpha}(x, x'), [(y, y'), (z, z')]_{\tilde{G}}]_{\tilde{G}} &+ (-1)^{|(y, y')||x, x'|} [\tilde{\alpha}(y, y'), [(z, z'), (x, x')]_{\tilde{G}}]_{\tilde{G}} \\ &+ (-1)^{|(z, z')||y, y'|} [\tilde{\alpha}(z, z'), [(x, x'), (y, y')]_{\tilde{G}}]_{\tilde{G}} = 0, \end{aligned}$$

leads to the second condition (2.2). Indeed,

$$\begin{aligned} \bullet (-1)^{|(x, x')||z, z'|} [\tilde{\alpha}(x, x'), [(y, y'), (z, z')]_{\tilde{G}}]_{\tilde{G}} &= (-1)^{|x||z|} ([\alpha_{\mathfrak{g}}(x), [y, z]_{\mathfrak{g}}]_{\mathfrak{g}} - (-1)^{|y||z|} [\alpha_{\mathfrak{g}}(x), \rho'(z')(y)]_{\mathfrak{g}} \\ &+ [\alpha_{\mathfrak{g}}(x), \rho'(y')(z)]_{\mathfrak{g}} - (-1)^{|x|(|y|+|z|)} \rho'([y', z']_{\mathfrak{g}'}) (\alpha_{\mathfrak{g}}(x)) - (-1)^{|x|(|y|+|z|)} \rho'(\rho(y)(z')) (\alpha_{\mathfrak{g}}(x)) \\ &+ (-1)^{|x|(|y|+|z|)} (-1)^{|y||z|} \rho'(\rho(z)(y')) (\alpha_{\mathfrak{g}}(x)) + \rho'(\alpha_{\mathfrak{g}'}(x')) ([y, z]_{\mathfrak{g}}) \\ &- (-1)^{|y||z|} \rho'(\alpha_{\mathfrak{g}'}(x')) (\rho'(z')(y)) + \rho'(\alpha_{\mathfrak{g}'}(x')) (\rho'(y')(z)), [\alpha_{\mathfrak{g}'}(x'), [y', z']_{\mathfrak{g}'}]_{\mathfrak{g}'} \\ &+ [\alpha_{\mathfrak{g}'}(x'), \rho(y)(z')]_{\mathfrak{g}'} - (-1)^{|y||z|} [\alpha_{\mathfrak{g}'}(x'), \rho(z)(y')]_{\mathfrak{g}'} + \rho(\alpha_{\mathfrak{g}}(x)) ([y', z']_{\mathfrak{g}'}) \\ &+ \rho(\alpha_{\mathfrak{g}}(x)) (\rho(y)(z')) - (-1)^{|y||z|} \rho(\alpha_{\mathfrak{g}}(x)) (\rho(z)(y')) - (-1)^{|x|(|y|+|z|)} \rho([y, z]_{\mathfrak{g}}) (\alpha_{\mathfrak{g}'}(x')) \\ &+ (-1)^{|x|(|y|+|z|)} (-1)^{|y||z|} \rho(\rho'(z')(y)) (\alpha_{\mathfrak{g}'}(x')) - (-1)^{|x|(|y|+|z|)} \rho(\rho'(y')(z)) (\alpha_{\mathfrak{g}'}(x')))). \end{aligned}$$

Similarly, we compute :

$$(-1)^{|(y, y')||x, x'|} [\tilde{\alpha}(y, y'), [(z, z'), (x, x')]_{\tilde{G}}]_{\tilde{G}} \text{ and } (-1)^{|(z, z')||y, y'|} [\tilde{\alpha}(z, z'), [(x, x'), (y, y')]_{\tilde{G}}]_{\tilde{G}}.$$

Setting  $(|x| = |x'|, |y| = |y'|, \text{ and } |z| = |z'|)$  give

$$\rho([x, y]_{\mathfrak{g}}) \circ \alpha_{\mathfrak{g}'} = \rho(\alpha_{\mathfrak{g}}(x)) \circ \rho(y) - (-1)^{|x||y|} \rho(\alpha_{\mathfrak{g}}(y)) \circ \rho(x), \quad (2.10)$$

$$\rho'([x', y']_{\mathfrak{g}'}) \circ \alpha_{\mathfrak{g}} = \rho'(\alpha_{\mathfrak{g}'}(x')) \circ \rho'(y') - (-1)^{|x'||y'|} \rho'(\alpha_{\mathfrak{g}'}(y')) \circ \rho'(x'), \quad (2.11)$$

which implies (2.5), and (2.6).

By Eqs. (2.8) and (2.10), we deduce that  $\rho$  is a representation of the Hom-Lie superalgebra  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$  on  $\mathfrak{g}'$  with respect to  $\alpha_{\mathfrak{g}'}$ . By Eqs. (2.9) and (2.11), we deduce that  $\rho'$  is a representation of the Hom-Lie superalgebra  $(\mathfrak{g}', [\cdot, \cdot]_{\mathfrak{g}'}, \alpha_{\mathfrak{g}'})$  on  $\mathfrak{g}$  with respect to  $\alpha_{\mathfrak{g}}$ .  $\square$

**Example 2.3** (See [2]). *Given a representation  $(V, [\cdot, \cdot]_V, \beta)$  of a Hom-Lie superalgebra  $(G, [\cdot, \cdot], \alpha)$ . Set  $\tilde{G} = G \oplus V$  and  $\tilde{G}_k = G_k \oplus V_k$ . If  $x \in G_i$  and  $v \in V_i$  ( $i \in \mathbb{Z}_2$ ), we denote  $|(x, v)| = |x|$ . Define a skew-supersymmetric bracket  $[\cdot, \cdot]_{\tilde{G}} : \wedge^2(G \oplus V) \rightarrow G \oplus V$  by*

$$[(x, \mu), (y, v)]_{\tilde{G}} = ([x, y], [x, v]_V - (-1)^{|x||y|} [y, \mu]_V).$$

*Define  $\tilde{\alpha} : G \oplus V \rightarrow G \oplus V$  by  $\tilde{\alpha}(x, v) = (\alpha(x), \beta(v))$ . Then  $(G \oplus V, [-, -]_{\tilde{G}}, \tilde{\alpha})$  is a Hom-Lie superalgebra, which we call the semi-direct product of the Hom-Lie superalgebra  $(G, [\cdot, \cdot], \alpha)$  by  $V$ .*

**Definition 2.4.** For any  $x \in \mathfrak{g}$ , define  $ad_x : \wedge^n \mathfrak{g} \rightarrow \wedge^n \mathfrak{g}$  by  $ad_x P = [x, P]_{\mathfrak{g}}$ . Its dual map  $ad^* : \wedge^n \mathfrak{g}^* \rightarrow \wedge^n \mathfrak{g}^*$  is defined as

$$\langle ad_x(y), \xi \rangle = -(-1)^{|x||y|} \langle y, ad_x^*(\xi) \rangle, \quad (2.12)$$

for all  $x, y \in \mathfrak{g}$ , and  $\xi \in \mathfrak{g}^*$ , where the pairing  $\langle \cdot, \cdot \rangle$  is supersymmetric, i.e.  $\langle x, \xi \rangle = (-1)^{|x||\xi|} \langle \xi, x \rangle$ , for  $x \in \mathfrak{g}$ , and  $\xi \in \mathfrak{g}^*$ . More precisely, for any  $\xi_1, \dots, \xi_n \in \mathfrak{g}^*$ , we have

$$ad_x^*(\xi_1 \wedge \dots \wedge \xi_n) = \sum_{i=1}^n (-1)^{|x|(|\xi_1| + |\xi_2| + \dots + |\xi_{i-1}|)} \alpha_{\mathfrak{g}}^*(\xi_1) \wedge \dots \wedge ad_x^*(\xi_i) \wedge \dots \wedge \alpha_{\mathfrak{g}}^*(\xi_n). \quad (2.13)$$

**Definition 2.5.** A *matched pairs of Hom-Lie superalgebras*, which we denote by  $(\mathfrak{g}, \mathfrak{g}', \rho, \rho')$ , consists of two Hom-Lie superalgebras  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$  and  $(\mathfrak{g}', [\cdot, \cdot]_{\mathfrak{g}'}, \alpha_{\mathfrak{g}'})$ , together with representations  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}')$  and  $\rho' : \mathfrak{g}' \rightarrow \mathfrak{gl}(\mathfrak{g})$  with respect to  $\alpha_{\mathfrak{g}'}$  and  $\alpha_{\mathfrak{g}}$  respectively, such that the compatibility conditions (2.5) and (2.6) are satisfied.

In the following, we concentrate on the case that  $\mathfrak{g}'$  is  $\mathfrak{g}^*$ , the dual space of  $\mathfrak{g}$ , and  $\alpha_{\mathfrak{g}'} = \alpha_{\mathfrak{g}}^*$ ,  $\rho = ad^*$ ,  $\rho' = \mathfrak{ad}^*$ , where  $\mathfrak{ad}^*$  is the dual map of  $\mathfrak{ad}$ . Notice that  $ad$  and  $\mathfrak{ad}$  are the adjoint representations associated to Hom-Lie superalgebras  $\mathfrak{g}$  and  $\mathfrak{g}^*$  respectively. Let  $x, y, z$  be elements in  $\mathfrak{g}$  and  $\xi, \eta$  elements in  $\mathfrak{g}^*$ . For a Hom-Lie superalgebra  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$  (resp.  $(\mathfrak{g}^*, [\cdot, \cdot]_{\mathfrak{g}^*}, \alpha_{\mathfrak{g}^*})$ ), let  $\Delta^* : \mathfrak{g}^* \rightarrow \wedge^2 \mathfrak{g}^*$  (resp.  $\Delta : \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$ ) be the dual map of  $[\cdot, \cdot]_{\mathfrak{g}} : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$  (resp.  $[\cdot, \cdot]_{\mathfrak{g}^*} : \wedge^2 \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ ), i.e.

$$\langle \Delta^*(\xi), x \wedge y \rangle = \langle \xi, [x, y]_{\mathfrak{g}} \rangle, \quad \langle \Delta(x), \xi \wedge \eta \rangle = \langle x, [\xi, \eta]_{\mathfrak{g}^*} \rangle.$$

A Hom-Lie superalgebra  $(\mathfrak{L}, [\cdot, \cdot], \alpha)$  is called *admissible* if its adjoint representation is admissible, that is  $[(Id - \alpha^2)(x), \alpha(y)] = 0$ . In particular, if a Hom-Lie superalgebra  $(\mathfrak{L}, [\cdot, \cdot], \alpha)$  is *involutive*, that is  $\alpha^2 = Id$ , then it is admissible.

**Proposition 2.6.** A pair of admissible Hom-Lie superalgebras  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$  and  $(\mathfrak{g}^*, [\cdot, \cdot]_{\mathfrak{g}^*}, \alpha_{\mathfrak{g}^*})$  determines a Hom-Lie superbialgebra  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \Delta, \alpha_{\mathfrak{g}})$ , where  $\Delta$  is the dual operation of  $[\cdot, \cdot]_{\mathfrak{g}^*}$  if

$$\langle \Delta([x, y]_{\mathfrak{g}}), \alpha_{\mathfrak{g}}^*(\xi) \wedge \eta \rangle = \langle ad_{\alpha_{\mathfrak{g}}(x)}(\Delta(y)), \alpha_{\mathfrak{g}}^*(\xi) \wedge \eta \rangle - (-1)^{|x||y|} \langle ad_{\alpha_{\mathfrak{g}}(y)}(\Delta(x)), \alpha_{\mathfrak{g}}^*(\xi) \wedge \eta \rangle, \quad (2.14)$$

$$\langle \Delta^*([\xi, \eta]_{\mathfrak{g}^*}), \alpha_{\mathfrak{g}}(x) \wedge y \rangle = \langle \mathfrak{ad}_{\alpha_{\mathfrak{g}}^*(\xi)}(\Delta^*(\eta)), \alpha_{\mathfrak{g}}(x) \wedge y \rangle - (-1)^{|\xi||\eta|} \langle \mathfrak{ad}_{\alpha_{\mathfrak{g}}^*(\eta)}(\Delta^*(\xi)), \alpha_{\mathfrak{g}}(x) \wedge y \rangle. \quad (2.15)$$

*Remark 2.7.* Following Bai and Sheng, we may denote this Hom-Lie superbialgebra by  $(\mathfrak{g}, \mathfrak{g}^*)$ .

**Theorem 2.8.** Let  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$  and  $(\mathfrak{g}^*, [\cdot, \cdot]_{\mathfrak{g}^*}, \alpha_{\mathfrak{g}^*})$  be a pair of admissible Hom-Lie superalgebra. Then  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \Delta, \alpha_{\mathfrak{g}})$ , where  $\Delta$  is the dual of  $[\cdot, \cdot]_{\mathfrak{g}^*}$ , is a Hom-Lie superbialgebra if and only if  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$  and  $(\mathfrak{g}^*, [\cdot, \cdot]_{\mathfrak{g}^*}, \alpha_{\mathfrak{g}^*})$  is a matched pairs of Hom-Lie superalgebras, i.e.  $(\mathfrak{g} \oplus \mathfrak{g}^*, [\cdot, \cdot]_{\tilde{\mathfrak{G}}}, \alpha_{\mathfrak{g}} \oplus \alpha_{\mathfrak{g}^*})$  is a multiplicative Hom-Lie superalgebra, where  $[\cdot, \cdot]_{\tilde{\mathfrak{G}}}$  is given by Eq. (2.4), in which  $\rho = ad^*$  and  $\rho' = \mathfrak{ad}^*$ .

*Proof.* By Theorem 2.2, two admissible Hom-Lie superalgebras  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$  and  $(\mathfrak{g}^*, [\cdot, \cdot]_{\mathfrak{g}^*}, \alpha_{\mathfrak{g}^*})$  form a matched pair of Hom-Lie superalgebras if and only if

$$ad_{\alpha_{\mathfrak{g}}(z)}^*([\xi, \eta]_{\mathfrak{g}^*}) = [ad_z^*(\xi), \alpha_{\mathfrak{g}}^*(\eta)]_{\mathfrak{g}^*} + (-1)^{|\xi||z|} [\alpha_{\mathfrak{g}}^*(\xi), ad_z^*(\eta)]_{\mathfrak{g}^*} + (-1)^{|\xi||\eta| + |\eta||z|} ad_{\mathfrak{ad}_{\xi}^*(z)}^*(\alpha_{\mathfrak{g}}^*(\xi)) - (-1)^{|\xi||z|} ad_{\mathfrak{ad}_{\xi}^*(z)}^*(\alpha_{\mathfrak{g}}^*(\eta)). \quad (2.16)$$

$$\mathfrak{ad}_{\alpha_{\mathfrak{g}}^*(\xi)}^*([x, y]_{\mathfrak{g}}) = [\mathfrak{ad}_{\xi}^*(x), \alpha_{\mathfrak{g}}(y)]_{\mathfrak{g}} + (-1)^{|x||\xi|} [\alpha_{\mathfrak{g}}(x), \mathfrak{ad}_{\xi}^*(y)]_{\mathfrak{g}} + (-1)^{|x||y| + |y||\xi|} \mathfrak{ad}_{ad_y^*(\xi)}^*(\alpha_{\mathfrak{g}}(x)) - (-1)^{|x||\xi|} \mathfrak{ad}_{ad_x^*(\xi)}^*(\alpha_{\mathfrak{g}}(y)). \quad (2.17)$$

Since  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$  and  $(\mathfrak{g}^*, [\cdot, \cdot]_{\mathfrak{g}^*}, \alpha_{\mathfrak{g}^*})$  are admissible Hom-Lie superalgebras i.e.  $(\alpha_{\mathfrak{g}}^2 = Id, \text{ and } (\alpha_{\mathfrak{g}^*}^2 = Id)$ , see Lemma 2.9 in [23]. By Eq. (2.17), we get

$$\begin{aligned} 0 &= \langle -\mathfrak{ad}_{\alpha_{\mathfrak{g}}^*(\xi)}^*([x, y]_{\mathfrak{g}}) + [\mathfrak{ad}_{\xi}^*(x), \alpha_{\mathfrak{g}}(y)]_{\mathfrak{g}} + (-1)^{|x||\xi|} [\alpha_{\mathfrak{g}}(x), \mathfrak{ad}_{\xi}^*(y)]_{\mathfrak{g}} + (-1)^{|x||y| + |y||\xi|} \mathfrak{ad}_{ad_y^*(\xi)}^*(\alpha_{\mathfrak{g}}(x)) \\ &\quad - (-1)^{|x||\xi|} \mathfrak{ad}_{ad_x^*(\xi)}^*(\alpha_{\mathfrak{g}}(y)), \eta \rangle \\ &= (-1)^{|\xi|(|x| + |y|)} \langle [x, y]_{\mathfrak{g}}, [\alpha_{\mathfrak{g}}^*(\xi), \eta]_{\mathfrak{g}^*} \rangle - (-1)^{|y|(|x| + |\xi|)} \langle ad_{\alpha_{\mathfrak{g}}(y)}(\mathfrak{ad}_{\xi}^*(x)), \eta \rangle + (-1)^{|x||\xi|} \langle ad_{\alpha_{\mathfrak{g}}(x)}(\mathfrak{ad}_{\xi}^*(y)), \eta \rangle \\ &\quad - (-1)^{|\xi|(|x| + |y|)} \langle \alpha_{\mathfrak{g}}(x), [ad_y^*(\xi), \eta]_{\mathfrak{g}^*} \rangle + (-1)^{|x||\xi|} (-1)^{|y|(|x| + |\xi|)} \langle \alpha_{\mathfrak{g}}(y), [ad_x^*(\xi), \eta]_{\mathfrak{g}^*} \rangle \\ &= (-1)^{|\xi|(|x| + |y|)} \langle [x, y]_{\mathfrak{g}}, [\alpha_{\mathfrak{g}}^*(\xi), \eta]_{\mathfrak{g}^*} \rangle - (-1)^{|x||\xi|} \langle x, [\xi, ad_{\alpha_{\mathfrak{g}}(y)}^*(\eta)]_{\mathfrak{g}^*} \rangle + (-1)^{|y|(|x| + |\xi|)} \langle y, [\xi, ad_{\alpha_{\mathfrak{g}}(x)}^*(\eta)]_{\mathfrak{g}^*} \rangle \\ &\quad - (-1)^{|\xi|(|x| + |y|)} \langle x, [\alpha_{\mathfrak{g}}^*(ad_y^*(\xi)), \alpha_{\mathfrak{g}}^*(\eta)]_{\mathfrak{g}^*} \rangle + (-1)^{|x||\xi|} (-1)^{|y|(|x| + |\xi|)} \langle y, [\alpha_{\mathfrak{g}}^*(ad_x^*(\xi)), \alpha_{\mathfrak{g}}^*(\eta)]_{\mathfrak{g}^*} \rangle, \end{aligned}$$

$$\begin{aligned}
&= (-1)^{|\xi|(|x|+|y|)} \langle [x, y]_{\mathfrak{g}}, [\alpha_{\mathfrak{g}}^*(\xi), \eta]_{\mathfrak{g}^*} \rangle - (-1)^{|x||\xi|} \langle x, [(\alpha_{\mathfrak{g}}^*)^2(\xi), ad_{\alpha_{\mathfrak{g}}^*(y)}^*(\eta)]_{\mathfrak{g}^*} \rangle \\
&+ (-1)^{|y|(|x|+|\xi|)} \langle y, [(\alpha_{\mathfrak{g}}^*)^2(\xi), ad_{\alpha_{\mathfrak{g}}^*(x)}^*(\eta)]_{\mathfrak{g}^*} \rangle - (-1)^{|\xi|(|x|+|y|)} \langle x, [ad_{\alpha_{\mathfrak{g}}^*(y)}^*(\alpha_{\mathfrak{g}}^*(\xi)), \alpha_{\mathfrak{g}}^*(\eta)]_{\mathfrak{g}^*} \rangle \\
&+ (-1)^{|x||\xi|} (-1)^{|y|(|x|+|\xi|)} \langle y, [ad_{\alpha_{\mathfrak{g}}^*(x)}^*(\alpha_{\mathfrak{g}}^*(\xi)), \alpha_{\mathfrak{g}}^*(\eta)]_{\mathfrak{g}^*} \rangle \\
&= (-1)^{|\xi|(|x|+|y|)} \langle \Delta([x, y]_{\mathfrak{g}}), \alpha_{\mathfrak{g}}^*(\xi) \wedge \eta \rangle - (-1)^{|x||\xi|} \langle \Delta(x), (\alpha_{\mathfrak{g}}^*)^2(\xi) \wedge ad_{\alpha_{\mathfrak{g}}^*(y)}^*(\eta) \rangle \\
&+ (-1)^{|y|(|x|+|\xi|)} \langle \Delta(y), (\alpha_{\mathfrak{g}}^*)^2(\xi) \wedge ad_{\alpha_{\mathfrak{g}}^*(x)}^*(\eta) \rangle - (-1)^{|\xi|(|x|+|y|)} \langle \Delta(x), ad_{\alpha_{\mathfrak{g}}^*(y)}^*(\alpha_{\mathfrak{g}}^*(\xi)) \wedge \alpha_{\mathfrak{g}}^*(\eta) \rangle \\
&+ (-1)^{|x||\xi|} (-1)^{|y|(|x|+|\xi|)} \langle \Delta(y), ad_{\alpha_{\mathfrak{g}}^*(x)}^*(\alpha_{\mathfrak{g}}^*(\xi)) \wedge \alpha_{\mathfrak{g}}^*(\eta) \rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\langle \Delta([x, y]_{\mathfrak{g}}), \alpha_{\mathfrak{g}}^*(\xi) \wedge \eta \rangle &= -(-1)^{|x||y|} \langle \Delta(y), ad_{\alpha_{\mathfrak{g}}^*(x)}^*(\alpha_{\mathfrak{g}}^*(\xi)) \wedge \alpha_{\mathfrak{g}}^*(\eta) \rangle + (-1)^{|x||\xi|} \langle \alpha_{\mathfrak{g}}^*(\xi)^2 \wedge ad_{\alpha_{\mathfrak{g}}^*(x)}^*(\eta) \rangle \\
&+ \langle \Delta(x), ad_{\alpha_{\mathfrak{g}}^*(y)}^*(\alpha_{\mathfrak{g}}^*(\xi)) \wedge \alpha_{\mathfrak{g}}^*(\eta) \rangle + (-1)^{|y||\xi|} \langle \alpha_{\mathfrak{g}}^*(\xi)^2 \wedge ad_{\alpha_{\mathfrak{g}}^*(y)}^*(\eta) \rangle \\
&= -(-1)^{|x||y|} \langle \Delta(y), ad_{\alpha_{\mathfrak{g}}^*(x)}^*(\alpha_{\mathfrak{g}}^*(\xi) \wedge \eta) \rangle + \langle \Delta(x), ad_{\alpha_{\mathfrak{g}}^*(y)}^*(\alpha_{\mathfrak{g}}^*(\xi) \wedge \eta) \rangle \\
&= \langle ad_{\alpha_{\mathfrak{g}}^*(x)}(\Delta(y)), \alpha_{\mathfrak{g}}^*(\xi) \wedge \eta \rangle - (-1)^{|x||y|} \langle ad_{\alpha_{\mathfrak{g}}^*(y)}(\Delta(x)), \alpha_{\mathfrak{g}}^*(\xi) \wedge \eta \rangle,
\end{aligned}$$

which is exactly Eq. (2.14). Similarly, one deduces that Eq. (2.16) is equivalent to Eq. (2.15).  $\square$

Let  $V$  be a superspace and  $\langle \cdot, \cdot \rangle : V^* \times V \rightarrow \mathbb{K}$  be the canonical pairing. Then we identify  $V$  with  $V^*$  by the pairing  $\langle x, \xi \rangle = (-1)^{|x||\xi|} \langle \xi, x \rangle$ ,  $x \in V$  and  $\xi \in V^*$ . On the other hand, we shall say that a bilinear form  $(\cdot) : V \times V \rightarrow \mathbb{K}$  is supersymmetric if  $(v|\omega) = (-1)^{|v||\omega|} (\omega|v)$ .

**Definition 2.9.** A *Manin supertriple of Hom-Lie superalgebras* is a triple of Hom-Lie superalgebras  $(\mathfrak{M}, \mathfrak{g}, \mathfrak{g}')$  together with a nondegenerate supersymmetric bilinear form  $S$  on  $\mathfrak{M}$  such that

1.  $S$  is invariant, i.e. for any  $x, y, z \in \mathfrak{M}$ , we have

$$S([x, y]_{\mathfrak{M}}, z) = S(x, [y, z]_{\mathfrak{M}}), \quad (2.18)$$

$$S(\phi_{\mathfrak{M}}(x), y) = S(x, \phi_{\mathfrak{M}}(y)). \quad (2.19)$$

2.  $\mathfrak{g}$  and  $\mathfrak{g}'$  are isotropic Hom-Lie sub-superalgebra of  $\mathfrak{M}$ , such that  $\mathfrak{M} = \mathfrak{g} \oplus \mathfrak{g}'$  as vector superspace.

**Proposition 2.10.** Let  $(\mathfrak{g}, \mathfrak{g}^*)$  be a Hom-Lie superbialgebra in the sense of Proposition 2.6. Then  $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$  is a Manin supertriple of Hom-Lie superalgebras.

*Proof.* Let  $(\mathfrak{g}, \mathfrak{g}^*)$  be a Hom-Lie superbialgebra in the sense of Proposition 2.6, i.e.  $\mathfrak{g}$  and  $\mathfrak{g}^*$  are admissible Hom-Lie superalgebras such that Eqs. (2.14), (2.15) are satisfied. By Theorem 2.8, we know that  $(\mathfrak{g} \oplus \mathfrak{g}^*, [\cdot, \cdot]_{\mathfrak{g} \oplus \mathfrak{g}^*}, \alpha_{\mathfrak{g}} \oplus \alpha_{\mathfrak{g}^*})$  is a Hom-Lie superalgebra, where  $[-, -]_{\mathfrak{g} \oplus \mathfrak{g}^*}$  is given by

$$[x + \xi, y + \eta]_{\mathfrak{g} \oplus \mathfrak{g}^*} = [x, y]_{\mathfrak{g}} + [\xi, \eta]_{\mathfrak{g}^*} + ad_x^*(\eta) - (-1)^{|x||y|} ad_y^*(\xi) + \mathfrak{a}d_{\xi}^*(y) - (-1)^{|x||y|} \mathfrak{a}d_{\eta}^*(x), \quad (2.20)$$

for all homogeneous elements  $x, y$  and  $\xi, \eta$  in  $\mathfrak{g}$  and  $\mathfrak{g}^*$  respectively. From the construction above we have  $|x| = |\xi|$  and  $|y| = |\eta|$ .

Furthermore, there is an obvious supersymmetric bilinear form on  $\mathfrak{g} \oplus \mathfrak{g}^*$ :

$$S(x + \xi, y + \eta) = \langle x, \eta \rangle + \langle \xi, y \rangle = \langle x, \eta \rangle + (-1)^{|\xi||y|} \langle y, \xi \rangle. \quad (2.21)$$

It's straightforward, using the supersymmetry and  $\langle ad_x(y), \xi \rangle = -(-1)^{|x||y|} \langle y, ad_x^*(\xi) \rangle$ , with  $|x| = |\xi|$  and  $|y| = |\eta|$ , that Eqs. (2.18) and (2.19) are satisfied, i.e. the bilinear form defined by Eq. (2.21) is invariant.  $\square$

Conversely, if  $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$  is a Manin supertriple of Hom-Lie superalgebras with the invariant bilinear form  $S$  given by Eq. (2.21), then for any  $x, y \in \mathfrak{g}$  and  $\xi, \eta \in \mathfrak{g}^*$ , we have the natural scalar product on  $\mathfrak{g} \oplus \mathfrak{g}^*$  defined by

$$S(x, y) = 0, \quad S(\xi, \eta) = 0, \quad S(x, \xi) = \langle x, \xi \rangle, \quad x, y \in \mathfrak{g}, \quad \xi, \eta \in \mathfrak{g}^*.$$

Due to the invariance of  $S$ , we have

$$\begin{aligned}
S([x, \xi]_{\mathfrak{g} \oplus \mathfrak{g}^*}, y) &= (-1)^{|y|(|x|+|\xi|)} S(y, [x, \xi]_{\mathfrak{g} \oplus \mathfrak{g}^*}) = (-1)^{|y|(|x|+|\xi|)} S([y, x]_{\mathfrak{g} \oplus \mathfrak{g}^*}, \xi) = (-1)^{|y|(|x|+|\xi|)} S([y, x]_{\mathfrak{g}}, \xi) \\
&= -(-1)^{|y||\xi|} \langle ad_x(y), \xi \rangle = (-1)^{|y|(|x|+|\xi|)} \langle y, ad_x^*(\xi) \rangle = \langle ad_x^*(\xi), y \rangle,
\end{aligned}$$

$S([x, \xi]_{\mathfrak{g} \oplus \mathfrak{g}^*}, \eta) = S(x, [\xi, \eta]_{\mathfrak{g} \oplus \mathfrak{g}^*}) = S(x, [\xi, \eta]_{\mathfrak{g}^*}) = \langle x, \mathfrak{ad}_\xi(\eta) \rangle = -(-1)^{|x||\xi|} \langle \mathfrak{ad}_\xi^*(x), \eta \rangle$ ,  
which implies that :

$$[x, \xi]_{\mathfrak{g} \oplus \mathfrak{g}^*} = \mathfrak{ad}_x^*(\xi) - (-1)^{|x||\xi|} \mathfrak{ad}_\xi^*(x),$$

that is, the Hom-Lie bracket on  $\mathfrak{g} \oplus \mathfrak{g}^*$  is given by Eq. (2.20). Therefore,  $(\mathfrak{g}, \mathfrak{g}^*, \mathfrak{ad}^*, \mathfrak{ad}^*)$  is a matched pair of Hom-Lie superalgebras and hence  $(\mathfrak{g}, \mathfrak{g}^*)$  is a Hom-Lie superbialgebra. Note that we deduce naturally that both  $\mathfrak{g}$  and  $\mathfrak{g}^*$  are admissible Hom-Lie superalgebras.

Summarizing the above study, Theorem 2.8 and Proposition 2.10, we have the following conclusion.

**Theorem 2.11.** *Let  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$  and  $(\mathfrak{g}^*, [\cdot, \cdot]_{\mathfrak{g}^*}, \alpha_{\mathfrak{g}^*})$  be two admissible Hom-Lie superalgebras. Then the following conditions are equivalent.*

1.  $(\mathfrak{g}, \mathfrak{g}^*)$  is a Hom-Lie superbialgebra in the sense of Proposition 2.6.
2.  $(\mathfrak{g}, \mathfrak{g}^*, \mathfrak{ad}^*, \mathfrak{ad}^*)$  is a matched pair of Hom-Lie superalgebras.
3.  $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$  is a Manin supertriple of Hom-Lie superalgebras with the invariant bilinear form (2.21).

### 3 Coboundary and quasi-triangular Hom-Lie superbialgebras

In this section, we define and study coboundary Hom-Lie superbialgebras and quasi-triangular Hom-Lie superbialgebras. Then we show how a coboundary or a quasi-triangular Hom-Lie superbialgebra can be constructed from a Hom-Lie superalgebra and an  $r$ -matrix.

**Definition 3.1.** A (multiplicative) coboundary Hom-Lie superbialgebra  $(\mathfrak{L}, [\cdot, \cdot], \Delta, \alpha, r)$  consists of a (multiplicative) Hom-Lie superbialgebra  $(\mathfrak{L}, [\cdot, \cdot], \Delta, \alpha)$  and an element  $r = \sum r_1 \otimes r_2 \in \mathfrak{L}^{\otimes 2}$  such that  $\alpha^{\otimes 2}(r) = r$  and

$$\Delta(x) = \mathfrak{ad}_x(r) = \sum [x, r_1] \otimes \alpha(r_2) + (-1)^{|x||r_1|} \alpha(r_1) \otimes [x, r_2] \quad (3.1)$$

for all  $x \in \mathfrak{L}$ .

*Remark 3.2.*

$$\Delta(x) = (-1)^{|x||r|} \left( \sum [x, r_1] \otimes \alpha(r_2) + (-1)^{|x||r_1|} \alpha(r_1) \otimes [x, r_2] \right) \quad (3.2)$$

for  $x \in \mathfrak{L}$ , where the parity  $|r|$  of  $r$  is defined as follows : since we assume  $r$  is homogenous, there exists  $|r| \in \mathbb{Z}_2$ , such that  $r$  can be written as  $r = \sum r_1 \otimes r_2 \in \mathfrak{L}^{\otimes 2}$ ,  $r_1, r_2$  are homogenous elements with  $|r| = |r_1| + |r_2|$ . (Note that equation (3.2) and (1.3) show that we have  $|r| = \bar{0}$ , namely  $|r_1| = |r_2|$ ). So we get (3.1).

**Definition 3.3.** The classical Yang-Baxter equation (CYBE):

$$c(r) = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0,$$

where  $r_{ij}$  are defined by

$$\begin{aligned} r_{12} &= \sum r_1 \otimes r_2 \otimes 1 = r \otimes 1, \\ r_{13} &= \sum r_1 \otimes 1 \otimes r_2 = (1 \otimes \tau)(r \otimes 1) = (\tau \otimes 1)(1 \otimes r), \\ r_{23} &= \sum 1 \otimes r_1 \otimes r_2 = 1 \otimes r \end{aligned} \quad (3.3)$$

and considered as elements in  $U(\mathfrak{L})$ , the universal enveloping algebra of a Lie superalgebra  $\mathfrak{L}$ . Elements (3.3) belongs to  $\mathfrak{L} \otimes \mathfrak{L} \otimes \mathfrak{L}$ .

**Definition 3.4.** The classical Hom-Yang-Baxter equation (CHYBE) in a Hom-Lie superalgebra  $(\mathfrak{L}, [\cdot, \cdot], \alpha)$  is

$$[[r, r]]^\alpha = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \quad (3.4)$$

for  $r \in \mathfrak{L} \otimes \mathfrak{L}$ . The three brackets in (3.4) are defined as

$$\begin{aligned} [r_{12}, r'_{13}] &= \sum (-1)^{|r'_1||r_2|} [r_1, r'_1] \otimes \alpha(r_2) \otimes \alpha(r'_2), \\ [r_{12}, r'_{23}] &= \sum \alpha(r_1) \otimes [r_2, r'_1] \otimes \alpha(r'_2), \\ [r_{13}, r'_{23}] &= \sum (-1)^{|r'_1||r_2|} \alpha(r_1) \otimes \alpha(r'_1) \otimes [r_2, r'_2], \end{aligned} \quad (3.5)$$

where  $r = \sum r_1 \otimes r_2$  and  $r' = \sum r'_1 \otimes r'_2 \in \mathfrak{L} \otimes \mathfrak{L}$ . If  $\alpha = Id$ , then the (CHYBE) reduces to the (CYBE).

**Definition 3.5.** A (multiplicative) quasi-triangular Hom-Lie superbialgebra is a (multiplicative) coboundary Hom-Lie superbialgebra in which  $r$  is a solution of the CHYBE (3.4). In these cases, we also write  $\Delta$  as  $ad(r)$ .

*Remark 3.6.* Note that we do not require  $r$  to be skew-supersymmetric in a coboundary Hom-Lie superbialgebra, whereas in [8]  $r$  is assumed to be skew-supersymmetric in a coboundary Lie bialgebra. We follow the convention in [16] and [27].

*Remark 3.7.* Condition (3.1) is a natural because from Remark 1.8 the compatibility condition (1.8) in Hom-Lie superbialgebra  $\mathfrak{L}$  says that the cobracket  $\Delta$  is a 1-cocycle in  $C^1(\mathfrak{L}, \mathfrak{L}^{\otimes 2})$ , where  $\mathfrak{L}$  acts on  $\mathfrak{L}^{\otimes 2}$  via the  $\alpha$ -twisted adjoint action (1.9). The simplest 1-cocycles are the 1-coboundaries, i.e, images of  $\delta_{HL}^0$ . We can define the Hom-Lie 0-cochains and 0 th differential as follows, extending the definition in [20]. Set  $C^0(\mathfrak{L}, \mathfrak{L}^{\otimes 2})$  as the subspace of  $\mathfrak{L}^{\otimes 2}$  consisting of elements that are fixed by  $\alpha^{\otimes 2}$ . Then we define the differential

$$\delta_{HL}^0 : C^0(\mathfrak{L}, \mathfrak{L}^{\otimes 2}) \rightarrow C^1(\mathfrak{L}, \mathfrak{L}^{\otimes 2})$$

by setting  $\delta_{HL}^0(r) = ad(r)$ , as in (1.6). It is not hard to check that, for  $r \in C^0(\mathfrak{L}, \mathfrak{L}^{\otimes 2})$ , we have  $\delta_{HL}^1(\delta_{HL}^0(r)) = 0$ , where  $\delta_{HL}^1$  is defined in (1.10). In fact, what this condition says is that

$$\begin{aligned} 0 &= \delta_{HL}^1(\delta_{HL}^0(r))(x, y) \\ &= \delta_{HL}^1(ad(r))(x, y) = ad_{[x, y]}(r) - ad_{\alpha(x)}(ad_y(r)) + (-1)^{|x||y|} ad_{\alpha(y)}(ad_x(r)) \end{aligned} \quad (3.6)$$

for all  $x, y \in \mathfrak{L}$ . We will prove (3.6) in Lemma 3.11 below. Thus, such an element  $\delta_{HL}^0(r) = ad(r)$  is a 1-coboundary, and hence a 1-cocycle. This fact makes  $ad(r)$  (with  $\alpha^{\otimes 2}(r) = r$ ) a natural candidate for the cobracket in a Hom-Lie superbialgebra and also justifies the name coboundary Hom-Lie superbialgebra.

The following result is the analogue of Theorem 1.11 for coboundary or quasi-triangular Hom-Lie superbialgebras. It says that coboundary or quasi-triangular Hom-Lie superbialgebras deform into other coboundary or quasi-triangular Hom-Lie superbialgebras via suitable endomorphisms.

**Theorem 3.8.** Let  $(\mathfrak{L}, [\cdot, \cdot], \Delta = ad(r), \alpha, r)$  be a coboundary Hom-Lie superbialgebra and an even map  $\beta : \mathfrak{L} \rightarrow \mathfrak{L}$  be a morphism such that  $\beta^{\otimes 2}(r) = r$ . Then  $\mathfrak{L}_\beta = (\mathfrak{L}, [\cdot, \cdot]_\beta = \beta \circ [\cdot, \cdot], \Delta_\beta = \Delta \circ \beta, \beta\alpha, r)$  is also a coboundary Hom-Lie superbialgebra, which is multiplicative if  $\mathfrak{L}$  is. Moreover, if  $\mathfrak{L}$  is quasi-triangular, then so is  $\mathfrak{L}_\beta$ .

*Proof.* By Theorem 1.11 we know that  $\mathfrak{L}_\beta$  is a Hom-Lie superbialgebra, multiplicative if  $\mathfrak{L}$  is. To check that  $\mathfrak{L}_\beta$  is coboundary, first note that  $(\beta\alpha)^{\otimes 2}(r) = \beta^{\otimes 2}\alpha^{\otimes 2}(r) = r$ .

To check the condition (3.1) in  $\mathfrak{L}_\beta$ , we compute as follows:

$$\begin{aligned} \Delta_\beta(x) &= \beta^{\otimes 2}(\Delta(x)) = \beta^{\otimes 2}([x, r_1] \otimes \alpha(r_2)) + \beta^{\otimes 2}((-1)^{|x||r_1|} \alpha(r_1) \otimes [x, r_2]) \\ &= [x, r_1]_\beta \otimes \beta\alpha(r_2) + (-1)^{|x||r_1|} \beta\alpha(r_1) \otimes [x, r_2]_\beta. \end{aligned}$$

The last expression above is  $ad_x(r)$  in  $\mathfrak{L}_\beta$ , which shows that  $\mathfrak{L}_\beta$  is coboundary.

Finally, suppose in addition that  $\mathfrak{L}$  is quasi-triangular, i.e.,  $r$  is a solution of the CHYBE in  $\mathfrak{L}$ . Using the notation in (3.4) we have:

$$0 = \beta^{\otimes 3}([r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]) = [r_{12}, r_{13}]_\beta + [r_{12}, r_{23}]_\beta + [r_{13}, r_{23}]_\beta,$$

where the last expression is defined in  $\mathfrak{L}_\beta$ . This shows that  $r$  is a solution of the CHYBE in  $\mathfrak{L}_\beta$ , so  $\mathfrak{L}_\beta$  is quasi-triangular.

The following result is the analogue of Corollary 1.12 for coboundary or quasi-triangular Hom-Lie superbialgebras. It says that these objects can be obtained by twisting coboundary or quasi-triangular Lie superbialgebras via suitable endomorphisms.  $\square$

**Corollary 3.9.** *Let  $(\mathfrak{L}, [\cdot, \cdot], \Delta, r)$  be a coboundary Lie superbialgebra and an even map  $\beta : \mathfrak{L} \rightarrow \mathfrak{L}$  be a Lie superalgebra morphism such that  $\beta^{\otimes 2}(r) = r$ . Then  $\mathfrak{L}_\beta = (\mathfrak{L}, [\cdot, \cdot]_\beta = \beta \circ [\cdot, \cdot], \Delta_\beta = \Delta \circ \beta, \beta, r)$  is a multiplicative coboundary Hom-Lie superbialgebra. If, in addition,  $\mathfrak{L}$  is a quasi-triangular Lie superbialgebra, then  $\mathfrak{L}_\beta$  is a multiplicative quasi-triangular Hom-Lie superbialgebra.*

*Proof.* This is the  $\alpha = Id$  special case of Theorem 3.8, provided that we can show that  $\Delta \circ \beta = \beta^{\otimes 2} \circ \Delta$ . We compute as follows:

$$\begin{aligned} \beta^{\otimes 2}(\Delta(x)) &= \beta^{\otimes 2}(ad_x(r)) = \beta^{\otimes 2}([x, r_1] \otimes r_2) + \beta^{\otimes 2}((-1)^{|x||r_1|} r_1 \otimes [x, r_2]) \\ &= [\beta(x), \beta(r_1)] \otimes \beta(r_2) + (-1)^{|x||r_1|} \beta(r_1) \otimes [\beta(x), \beta(r_2)] \\ &= [\beta(x), r_1] \otimes r_2 + (-1)^{|x||r_1|} r_1 \otimes [\beta(x), r_2] = ad_{\beta(x)}(r) = \Delta(\beta(x)). \end{aligned}$$

□

The next result says that every multiplicative coboundary or quasi-triangular Hom-Lie superbialgebra gives rise to an infinite sequence of multiplicative coboundary or quasi-triangular Hom-Lie superbialgebra. It is similar to Corollary 1.13.

**Corollary 3.10.** *Let  $(\mathfrak{L}, [\cdot, \cdot], \Delta, \alpha, r)$  be a multiplicative coboundary (resp. quasi-triangular) Hom-Lie superbialgebra. Then  $\mathfrak{L}_{\alpha^n} = (\mathfrak{L}, [\cdot, \cdot]_{\alpha^n} = \alpha^n \circ [\cdot, \cdot], \Delta_{\alpha^n} = \Delta \circ \alpha^n, \alpha^{n+1}, r)$  is also a multiplicative coboundary (resp. quasi-triangular) Hom-Lie superbialgebra for each integer  $n \geq 0$ .*

*Proof.* This is the  $\beta = \alpha^n$  special case of Theorem 3.8. □

In the following result, we describe some sufficient condition under which a Hom-Lie superalgebra becomes a coboundary Hom-Lie superbialgebra.

In what follows, for an element  $r = \sum r_1 \otimes r_2$ , we write  $r_{21}$  for  $\tau(r) = \sum (-1)^{|r_1||r_2|} r_2 \otimes r_1$ .

**Lemma 3.11.** *Let  $(\mathfrak{L}, [\cdot, \cdot], \alpha)$  be a multiplicative Hom-Lie superalgebra and  $r \in \mathfrak{L}^{\otimes 2}$  be an element such that  $\alpha^{\otimes 2}(r) = r$ , (1.3) and (3.2), ((3.2) and (1.3), i.e.,  $|r| = \bar{0}$ ). Then  $\Delta = ad(r) : \mathfrak{L} \rightarrow \mathfrak{L}^{\otimes 2}$  satisfies (1.8), i.e.,  $ad_{[x,y]}(r) = ad_{\alpha(x)}(ad_y(r)) - (-1)^{|x||y|} ad_{\alpha(y)}(ad_x(r))$  for  $x, y \in \mathfrak{L}$ .*

*Proof.* We will use  $\alpha^{\otimes 2}(r) = r$ , the skew-supersymmetry (1.1) and the Hom-Jacobi identity of (1.2) and  $\alpha([x, y]) = [\alpha(x), \alpha(y)]$ , (multiplicative) in the computation below. For  $x, y \in \mathfrak{L}$ , we have:

$$\begin{aligned} ad_{[x,y]}(r) &= [[x, y], r_1] \otimes \alpha(r_2) + (-1)^{|[x,y]||r_1|} \alpha(r_1) \otimes [[x, y], r_2] \\ &= [[x, y], \alpha(r_1)] \otimes \alpha^2(r_2) + (-1)^{|x||r_1|} (-1)^{|y||r_1|} \alpha^2(r_1) \otimes [[x, y], \alpha(r_2)] \\ &= ([\alpha(x), [y, r_1]] + (-1)^{|x||y|} (-1)^{|x||r_1|} [\alpha(y), [r_1, x]]) \otimes \alpha^2(r_2) \\ &\quad + (-1)^{|x||r_1|} (-1)^{|y||r_1|} \alpha^2(r_1) \otimes ([\alpha(x), [y, r_2]] + (-1)^{|x||y|} (-1)^{|x||r_2|} [\alpha(y), [r_2, x]]) \\ &= [\alpha(x), [y, r_1]] \otimes \alpha^2(r_2) + (-1)^{|x||y|} (-1)^{|x||r_1|} [\alpha(y), [r_1, x]] \otimes \alpha^2(r_2) \\ &\quad + (-1)^{|x||r_1|} (-1)^{|y||r_1|} \alpha^2(r_1) \otimes [\alpha(x), [y, r_2]] + (-1)^{|x||y|} (-1)^{|y||r_1|} \alpha^2(r_1) \otimes [\alpha(y), [r_2, x]] \\ &= [\alpha(x), [y, r_1]] \otimes \alpha^2(r_2) + (-1)^{|x||y|} (-1)^{|x||r_1|} \alpha([y, r_1]) \otimes [\alpha(x), \alpha(r_2)] \\ &\quad + (-1)^{|y||r_1|} [\alpha(x), \alpha(r_1)] \otimes \alpha([y, r_2]) + (-1)^{|x||r_1|} (-1)^{|y||r_1|} \alpha^2(r_1) \otimes [\alpha(x), [y, r_2]] \\ &\quad - (-1)^{|x||y|} [\alpha(y), [x, r_1]] \otimes \alpha^2(r_2) - (-1)^{|y||r_1|} \alpha([x, r_1]) \otimes [\alpha(y), \alpha(r_2)] \\ &\quad - (-1)^{|x||y|} (-1)^{|x||r_1|} [\alpha(y), \alpha(r_1)] \otimes \alpha([x, r_2]) - (-1)^{|x||y|} (-1)^{|y||r_1|+|x||r_1|} \alpha^2(r_1) \otimes [\alpha(y), [x, r_2]] \\ &= ad_{\alpha(x)}([y, r_1] \otimes \alpha(r_2) + (-1)^{|y||r_1|} \alpha(r_1) \otimes [y, r_2]) \\ &\quad - (-1)^{|x||y|} ad_{\alpha(y)}([x, r_1] \otimes \alpha(r_2) + (-1)^{|x||r_1|} \alpha(r_1) \otimes [x, r_2]) \\ &= ad_{\alpha(x)}(ad_y(r)) - (-1)^{|x||y|} ad_{\alpha(y)}(ad_x(r)). \end{aligned}$$

□

**Theorem 3.12.** *Let  $(\mathfrak{L}, [\cdot, \cdot], \alpha)$  be a multiplicative Hom-Lie superalgebra and  $r \in \mathfrak{L}^{\otimes 2}$  be an element such that  $\alpha^{\otimes 2}(r) = r$ ,  $r_{21} = -r$ , (1.3) and (3.2), ((3.2) and (1.3), i.e.,  $|r| = \bar{0}$ ), and*

$$\alpha^{\otimes 3}(ad_x([r, r]^\alpha)) = 0 \tag{3.7}$$



for all  $x \in \mathfrak{L}$ , where  $[[r, r]]^\alpha$  is defined in (3.4). Define  $\Delta : \mathfrak{L} \rightarrow \mathfrak{L}^{\otimes 2}$  as  $\Delta(x) = ad_x(r)$  as in (3.1). Then  $(\mathfrak{L}, [\cdot, \cdot], \Delta, \alpha, r)$  is a multiplicative coboundary Hom-Lie superbialgebra.

*Proof.* We will show the following statements:

1)  $\Delta = ad(r)$  commutes with  $\alpha$ .

2)  $\Delta$  is skew-supersymmetric.

3) The compatibility condition (1.8) holds.

4) The condition (3.7) is equivalent Hom-super-coJacobi identity of  $\Delta$  (1.5).

Write  $r = \sum r_1 \otimes r_2$ ,  $r' = \sum r'_1 \otimes r'_2$  and  $\tau(r) = \sum (-1)^{|r_1||r_2|} r_2 \otimes r_1$ ,  $\tau(r') = \sum (-1)^{|r'_1||r'_2|} r'_2 \otimes r'_1$ . To show that  $\Delta = ad(r)$  commutes with  $\alpha$ , pick an element  $x \in \mathfrak{L}$ , the summation sign will often be omitted in computation to simplify the typography. Using the definition  $\Delta = ad(r)$ ,  $\alpha([x, r_1]) = [\alpha(x), \alpha(r_1)]$ ,  $\alpha([x, r_2]) = [\alpha(x), \alpha(r_2)]$  and the assumption  $\alpha^{\otimes 2}(r) = r$  we have

$$\begin{aligned} \Delta(\alpha(x)) &= [\alpha(x), r_1] \otimes \alpha(r_2) + (-1)^{|x||r_1|} \alpha(r_1) \otimes [\alpha(x), r_2] \\ &= \alpha([x, r_1]) \otimes \alpha^2(r_2) + (-1)^{|x||r_1|} \alpha^2(r_1) \otimes \alpha([x, r_2]) = \alpha^{\otimes 2}(\Delta(x)). \end{aligned}$$

This shows that  $\Delta$  commutes with  $\alpha$ .

Now we show that  $\Delta = ad(r)$  is skew-supersymmetric. We have

$$\begin{aligned} \Delta(x) &= [x, r_1] \otimes \alpha(r_2) + (-1)^{|x||r_1|} \alpha(r_1) \otimes [x, r_2]. \\ \tau(\Delta(x)) &= (-1)^{|r_1||r_2|} ([x, r_2] \otimes \alpha(r_1) + (-1)^{|x||r_2|} \alpha(r_2) \otimes [x, r_1]). \end{aligned}$$

Then  $\Delta(x) + \tau(\Delta(x)) = ad_x(r + r_{21}) = ad_x(0) = 0$ , since

$$r + r_{21} = \sum (r_1 \otimes r_2 + (-1)^{|r_1||r_2|} r_2 \otimes r_1).$$

We will prove that the compatibility condition (1.8) holds in Lemma 3.11.

Finally, we show that the Hom-super-coJacobi identity (1.5) of  $\Delta = ad(r)$  is equivalent to (3.7). Let us unwrap the Hom-super-coJacobi identity. Fix an element  $x \in \mathfrak{L}$ , and let  $r' = \sum r'_1 \otimes r'_2$  be another copy of  $r$ . Then we write

$$\begin{aligned} \omega &= (\alpha \otimes \Delta)(\Delta(x)) \\ &= (\alpha \otimes \Delta)([x, r_1] \otimes \alpha(r_2) + (-1)^{|x||r_1|} \alpha(r_1) \otimes [x, r_2]) \\ &= \alpha([x, r_1]) \otimes [\alpha(r_2), r'_1] \otimes \alpha(r'_2) + (-1)^{|r_2||r'_1|} \alpha([x, r_1]) \otimes \alpha(r'_1) \otimes [\alpha(r_2), r'_2] \\ &\quad + (-1)^{|x||r_1|} \alpha^2(r_1) \otimes [[x, r_2], r'_1] \otimes \alpha(r'_2) + (-1)^{|x||r_1|} (-1)^{|x||r'_1|} (-1)^{|r_2||r'_1|} \alpha^2(r_1) \otimes \alpha(r'_1) \otimes [[x, r_2], r'_2]. \end{aligned}$$

Note

$$\begin{aligned} A_1 &= \alpha([x, r_1]) \otimes [\alpha(r_2), r'_1] \otimes \alpha(r'_2), \\ B_1 &= (-1)^{|r_2||r'_1|} \alpha([x, r_1]) \otimes \alpha(r'_1) \otimes [\alpha(r_2), r'_2], \\ C_1 &= (-1)^{|x||r_1|} \alpha^2(r_1) \otimes [[x, r_2], r'_1] \otimes \alpha(r'_2), \\ D_1 &= (-1)^{|x||r_1|} (-1)^{|x||r'_1|} (-1)^{|r_2||r'_1|} \alpha^2(r_1) \otimes \alpha(r'_1) \otimes [[x, r_2], r'_2]. \end{aligned}$$

we get  $\omega = A_1 + B_1 + C_1 + D_1$ .

With these notations, the Hom-super-coJacobi identity of  $\Delta = ad(r)$  (applied to  $x$ ) becomes

$$(1 \otimes 1 \otimes 1 + \xi + \xi^2) \circ (\alpha \otimes \Delta) \circ \Delta(x) = \sum_{i=1}^3 (A_i + B_i + C_i + D_i) = 0. \quad (3.8)$$

Therefore, to prove the equivalence between the Hom-super-coJacobi identity of  $\Delta$  and (3.7), it suffices to show

$$\alpha^{\otimes 3}(ad_x([[r, r]]^\alpha)) = \sum_{i=1}^3 (A_i + B_i + C_i + D_i), \quad (3.9)$$

which we will prove in Lemma 3.13 below.

The proof of Theorem 3.12 will be completed once we prove the Lemma below.  $\square$

**Lemma 3.13.** *The condition (3.9) holds.*

*Proof.* It suffices to show the following three equalities:

$$\alpha^{\otimes 3}(ad_x([r_{12}, r_{13}])) = A_3 + B_2 + C_3 + D_2, \quad (3.10)$$

$$\alpha^{\otimes 3}(ad_x([r_{12}, r_{23}])) = A_1 + B_3 + C_1 + D_3, \quad (3.11)$$

$$\alpha^{\otimes 3}(ad_x([r_{13}, r_{23}])) = A_2 + B_1 + C_2 + D_1, \quad (3.12)$$

where the three bracket, which add up to  $[[r, r]]^\alpha$ , are defined in (3.5). The proofs for the three equalities are very similar, so we will only give the proof of (3.10).

Since  $r = r'$  and  $r_{12} = -r$ , we have

$$\begin{aligned} A_3 &= (-1)^{|x||r_2|}(-1)^{|r'_1||r_1|}[\alpha(r_2), r'_1] \otimes \alpha(r'_2) \otimes \alpha([x, r_1]) \\ &= (-1)^{|x||r_2|}(-1)^{|r_1||r'_1|}[\alpha(r'_2), r_1] \otimes \alpha(r_2) \otimes \alpha([x, r'_1]) \\ &= -(-1)^{|x||r_2|}[\alpha(r'_1), r_1] \otimes \alpha(r_2) \otimes \alpha([x, r'_2]) \\ &= -(-1)^{|x||r_2|}[\alpha^2(r'_1), \alpha^2(r_1)] \otimes \alpha^3(r_2) \otimes \alpha([x, \alpha(r'_2)]) \\ &= \alpha^{\otimes 3}((-1)^{|x||r_2|}(-1)^{|r'_1||r_2|}\alpha([r_1, r'_1]) \otimes \alpha^2(r_2) \otimes [x, \alpha(r'_2)]). \end{aligned} \quad (3.13)$$

In the fourth equality we used  $\alpha^{\otimes 2}(r) = r$ ,  $\alpha^{\otimes 2}(r') = r'$ . In the equality we used the skew-supersymmetry of  $[-, -]$  and  $\alpha([r_1, r'_1]) = [\alpha(r_1), \alpha(r'_1)]$ , we know  $(-1)^{|x||r_1|}(-1)^{|x||r'_1|} = 1$ , just to check the calculations. Similar computation give

$$\begin{aligned} B_2 &= (-1)^{|r'_1||r_1|}[\alpha(r_2), r'_2] \otimes \alpha([x, r_1]) \otimes \alpha(r'_1) \\ &= \alpha^{\otimes 3}((-1)^{|x||r_1|}(-1)^{|x||r'_1|}(-1)^{|r'_1||r_2|}\alpha([r_1, r'_1]) \otimes [x, \alpha(r_2)] \otimes \alpha^2(r'_2)), \end{aligned}$$

$$C_3 = (-1)^{|r'_1||r_2|}[[x, r_2], r'_1] \otimes \alpha(r'_2) \otimes \alpha^2(r_1) = (-1)^{|x||r'_2|}[[r'_2, x], \alpha(r_2)] \otimes \alpha^2(r_1) \otimes \alpha^2(r'_1),$$

$$D_2 = (-1)^{|r'_1||r_2|}[[x, r_2], r'_2] \otimes \alpha^2(r'_2) \otimes \alpha(r'_1) = (-1)^{|r'_1||r_2|}[[x, r_2], \alpha(r'_2)] \otimes \alpha^2(r_1) \otimes \alpha^2(r'_1).$$

Using, in addition, the skew-supersymmetry (1.1) and the Hom-Jacobi identity (1.2) of  $[-, -]$ , we add  $C_3$  and  $D_2$ :

$$\begin{aligned} C_3 + D_2 &= (-1)^{|r'_1||r_2|}((-1)^{|x||r'_2|}(-1)^{|r_2||r'_2|}[[r'_2, x], \alpha(r_2)] + [[x, r_2], \alpha(r'_2)]) \otimes \alpha^2(r_1) \otimes \alpha^2(r'_1) \\ &= (-1)^{|r'_1||r_2|}[\alpha(x), [r_2, r'_2]] \otimes \alpha^2(r_1) \otimes \alpha^2(r'_1) \\ &= (-1)^{|r'_1||r_2|}[\alpha(x), [r_1, r'_1]] \otimes \alpha^2(r_2) \otimes \alpha^2(r'_2) \\ &= (-1)^{|r'_1||r_2|}[\alpha(x), [\alpha(r_1), \alpha(r'_1)]] \otimes \alpha^3(r_2) \otimes \alpha^3(r'_2) \\ &= \alpha^{\otimes 3}((-1)^{|r'_1||r_2|}[x, [r_1, r'_1]] \otimes \alpha^2(r_2) \otimes \alpha^2(r'_2)). \end{aligned}$$

Using the definition (1.6) of  $ad_x$ , we now conclude that:

$$\begin{aligned} A_3 + B_2 + C_3 + D_2 &= \alpha^{\otimes 3}((-1)^{|x||r_2|}(-1)^{|r'_1||r_2|}\alpha([r_1, r'_1]) \otimes \alpha^2(r_2) \otimes [x, \alpha(r'_2)]) \\ &\quad + \alpha^{\otimes 3}((-1)^{|x||r_1|}(-1)^{|x||r'_1|}(-1)^{|r'_1||r_2|}\alpha([r_1, r'_1]) \otimes [x, \alpha(r_2)] \otimes \alpha^2(r'_2)) \\ &\quad + \alpha^{\otimes 3}((-1)^{|r'_1||r_2|}[x, [r_1, r'_1]] \otimes \alpha^2(r_2) \otimes \alpha^2(r'_2)) \\ &= \alpha^{\otimes 3}((-1)^{|r'_1||r_2|}ad_x([r_1, r'_1]) \otimes \alpha(r_2) \otimes \alpha(r'_2)) \\ &= \alpha^{\otimes 3}(ad_x((-1)^{|r'_1||r_2|}[r_1, r'_1] \otimes \alpha(r_2) \otimes \alpha(r'_2))) \\ &= \alpha^{\otimes 3}(ad_x([r_{12}, r_{13}])). \end{aligned}$$

This proves (3.10).

The equalities (3.11) and (3.12) are proved by very similar computations.

Therefore, the equality (3.9) holds. Together with (3.8) we have shown that the the Hom-super-coJacobi identity of  $\Delta = ad(r)$  is equivalent to  $\alpha^{\otimes 3}(ad_x([[r, r]]^\alpha)) = 0$   $\square$

The following result is an immediate consequence of Theorem (3.12). It gives sufficient conditions under which a Hom-Lie superalgebra becomes a quasi-triangular Hom-Lie superbialgebra.

**Corollary 3.14.** *Let  $(\mathfrak{L}, [\cdot, \cdot], \alpha)$  be a multiplicative Hom-Lie superalgebra and  $r \in \mathfrak{L}^{\otimes 2}$  be an element such that  $\alpha^{\otimes 2}(r) = r$ ,  $r_{21} = -r$ , (1.3) and (3.2), ((3.2) and (1.3), i.e.,  $|r| = \bar{0}$ ), and  $[[r, r]]^\alpha = 0$ . Then  $(\mathfrak{L}, [\cdot, \cdot], ad(r), \alpha, r)$  is a multiplicative quasi-triangular Hom-Lie superbialgebra.*

**Theorem 3.15.** *Let  $(\mathfrak{L}, [\cdot, \cdot], \Delta, \alpha, r)$  be a coboundary Hom-Lie superbialgebra. Then the following statements are equivalent,*

- (1)  $\mathfrak{L}$  is a quasi-triangular Hom-Lie superbialgebra, i.e.,  $[[r, r]]^\alpha = 0$  (3.4).
- (2) The equality  $(\alpha \otimes \Delta)(r) = -[r_{12}, r_{13}]$  holds, where the bracket is defined in (3.5).
- (3) The equality  $(\Delta \otimes \alpha)(r) = [r_{13}, r_{23}]$  holds, where the bracket is defined in (3.5).

*Proof.* The equivalence between three statements clearly follows from the equalities. Let  $r' = \sum r'_1 \otimes r'_2$  be another copy of  $r$ . Since  $\Delta = ad(r)$  (3.1),  $r = r'$  and  $r_1 \otimes r_2 = -(-1)^{|r_1||r_2|} r_2 \otimes r_1$ . By calculation we will find results

$$\begin{aligned} (\alpha \otimes \Delta)(r_1 \otimes r_2) &= \alpha(r_1) \otimes \Delta(r_2) = \alpha(r_1) \otimes [r_2, r'_1] \otimes \alpha(r'_2) + (-1)^{|r'_1||r_2|} \alpha(r_1) \otimes \alpha(r'_1) \otimes [r_2, r'_2] \\ &= [r_{12}, r_{23}] + [r_{13}, r_{23}]. \end{aligned}$$

This shows the equivalence between statements (1) and (2). Likewise, we have

$$\begin{aligned} (\Delta \otimes \alpha)(r_1 \otimes r_2) &= \Delta(r_1) \otimes \alpha(r_2) = [r_1, r'_1] \otimes \alpha(r'_2) \otimes \alpha(r_2) + (-1)^{|r_1||r'_1|} \alpha(r'_1) \otimes [r_1, r'_2] \otimes \alpha(r_2) \\ &= -(-1)^{|r'_1||r_2|} [r_1, r'_1] \otimes \alpha(r_2) \otimes \alpha(r'_2) - \alpha(r_1) \otimes [r_2, r'_1] \otimes \alpha(r'_2) \\ &= -[r_{12}, r_{13}] - [r_{12}, r_{23}]. \end{aligned}$$

This shows the equivalence between statements (1) and (3). □

## 4 Cobracket perturbation in Hom-Lie superbialgebras

The purpose of this section is to study perturbation of cobrackets in Hom-Lie superbialgebras, following Drinfel'd's perturbation theory of quasi-Hopf algebras ([6], [9], [10], [11]).

We address the following question : " If  $(\mathfrak{L}, [\cdot, \cdot], \Delta, \alpha)$  is a Hom-Lie superbialgebra (Definition 1.5) and  $t \in \mathfrak{L}^{\otimes 2}$ , under what conditions does the perturbed cobracket  $\Delta_t = \Delta + ad(t)$  give another Hom-Lie superbialgebra  $(\mathfrak{L}, [\cdot, \cdot], \Delta_t, \alpha)$  ?"

Define the perturbed cobracket  $\Delta_t = \Delta + ad(t)$ . For  $x \in \mathfrak{L}$  and  $t = \sum t_1 \otimes t_2 \in \mathfrak{L}^{\otimes 2}$  and also recall the adjoint map  $ad_x : \mathfrak{L}^{\otimes n} \rightarrow \mathfrak{L}^{\otimes n}$  (1.6) we have :

$$\Delta_t(x) = \Delta(x) + ad_x(t) = \Delta(x) + ad_x(t_1 \otimes t_2) = \Delta(x) + [x, t_1] \otimes \alpha(t_2) + (-1)^{|x||t_1|} \alpha(t_1) \otimes [x, t_2].$$

This is a natural question because  $\Delta$  is a 1-cocycle (Remark 1.8),  $ad(t)$  (1.6) is a 1-coboundary when  $\alpha^{\otimes 2}(t) = t$  (Remark 3.7), and perturbation of cocycles by coboundaries is a natural concept in homological algebra. Of course, we have more to worry about than just the cocycle condition (1.8) because  $(\mathfrak{L}, \Delta_t, \alpha)$  must be a Hom-Lie supercoalgebra (Definition 1.2).

In the following result, we give some sufficient conditions under which the perturbed cobracket  $\Delta_t$  gives another Hom-Lie superbialgebra. This is a generalization of [16], which deals with cobracket perturbation in Lie superbialgebras.

A result about cobracket perturbation in a quasi-triangular Hom-Lie superbialgebra, is given after the following result. We also briefly discuss triangular Hom-Lie superbialgebra, which is the Hom-Type version of Drinfel'd's triangular Lie bialgebra [8].

Let us recall some notations first. For  $t = \sum t_1 \otimes t_2 \in \mathfrak{L}^{\otimes 2}$ , the symbol  $t_{21}$  denotes  $\tau(t) = \sum (-1)^{|t_1||t_2|} t_2 \otimes t_1$ . If  $\varphi(x, y)$  is an expression in the elements  $x$  and  $y$ , we set

$$|\varphi(x, y)| = \varphi(x, y) - (-1)^{|x||y|} \varphi(y, x).$$

For example, the compatibility condition  $\Delta([x, y]) = ad_{\alpha(x)}(\Delta(y)) - (-1)^{|x||y|}ad_{\alpha(y)}(\Delta(x))$  (1.8) is equivalent to

$$\Delta([x, y]) = |ad_{\alpha(x)}(\Delta(y))|.$$

Moreover  $[[x, y], \alpha(z)] = |[\alpha(x), [y, z]]|$ , where  $|[\alpha(x), [y, z]]| = [\alpha(x), [y, z]] - (-1)^{|x||y|}[\alpha(y), [x, z]]$ . By calculation the Hom-super-Jacobi identity (1.2) is equivalent to  $[[x, y], \alpha(z)] = |[\alpha(x), [y, z]]|$ . Note that we have

$$|\varphi(x, y) + \psi(x, y)| = |\varphi(x, y)| + |\psi(x, y)|.$$

It is also noted to simplify writing  $(1 \otimes 1 \otimes 1 + \xi + \xi^2) = \circ$ .

**Theorem 4.1.** *Let  $(\mathfrak{L}, [\cdot, \cdot], \Delta, \alpha)$  be a multiplicative Hom-Lie superbialgebra and  $t \in \mathfrak{L}^{\otimes 2}$  be an element such that  $\alpha^{\otimes 2}(t) = t$ ,  $t_{21} = -t$ , (1.3) and (3.2), ((3.2) and (1.3), i.e.,  $|t| = \bar{0}$ ) and*

$$\alpha^{\otimes 3}(ad_x([t, t])^\alpha + \circ(\alpha \otimes \Delta)(t)) = 0 \quad (4.1)$$

for all  $x \in \mathfrak{L}$ . Then  $\mathfrak{L}_t = (\mathfrak{L}, [\cdot, \cdot], \Delta_t = \Delta + ad(t), \alpha)$  is multiplicative Hom-Lie superbialgebra.

*Proof.* To show that  $\mathfrak{L}_t$  is a multiplicative Hom-Lie superbialgebra, we need to prove the following conditions: It is clear that  $(\mathfrak{L}, [\cdot, \cdot], \alpha)$  is a multiplicative Hom-Lie superalgebra.

It remains to show that  $(\mathfrak{L}, \Delta_t, \alpha)$  co-multiplicative Hom-Lie supercoalgebra, and the compatibility condition (1.8) holds for  $\Delta_t$  and  $[\cdot, \cdot]$ .

Precisely we need to prove four things:

- $\alpha^{\otimes 2} \circ \Delta_t = \Delta_t \circ \alpha$ , equality is true because:

$$\begin{aligned} \Delta_t(\alpha(x)) &= \Delta(\alpha(x)) + ad_{\alpha(x)}(t). \\ \alpha^{\otimes 2}(\Delta_t(x)) &= \alpha^{\otimes 2}(\Delta(x)) + \alpha^{\otimes 2}(ad_x(t)) = \Delta(\alpha(x)) + \alpha^{\otimes 2}(ad_x(t_1 \otimes t_2)). \end{aligned}$$

Using  $\alpha^{\otimes 2}(t) = t$ , we have

$$\alpha^{\otimes 2}(ad_x(t_1 \otimes t_2)) = [\alpha(x), t_1] \otimes \alpha(t_2) + (-1)^{|x||t_1|}\alpha(t_1) \otimes [\alpha(x), t_2] = ad_{\alpha(x)}(t).$$

- $\Delta_t$  is skew-supersymmetric because:

$$\begin{aligned} \Delta_t(x) &= \Delta(x) + ad_x(t), \\ \tau(\Delta_t(x)) &= -\Delta(x) + (-1)^{|t_1||t_2|}([x, t_2] \otimes \alpha(t_1) + (-1)^{|x||t_2|}\alpha(t_2) \otimes [x, t_1]). \end{aligned}$$

Then :  $\Delta_t(x) + \tau(\Delta_t(x)) = ad_x(t + (-1)^{|t_1||t_2|}t_2 \otimes t_1) = ad_x(t + t_{21}) = ad_x(0) = 0$ .

- Now, we need to show the compatibility condition (1.8) holds for  $\Delta_t$  and  $[\cdot, \cdot]$ :

$$\Delta_t([x, y]) = ad_{\alpha(x)}(\Delta_t(y)) - (-1)^{|x||y|}ad_{\alpha(y)}(\Delta_t(x)), \quad (4.2)$$

which is equivalent to

$$\Delta_t([x, y]) = |ad_{\alpha(x)}(\Delta_t(y))|. \quad (4.3)$$

Since  $\Delta_t = \Delta + ad(t)$ , (4.3) is equivalent to

$$\Delta([x, y]) + ad_{[x, y]}(t) = |ad_{\alpha(x)}(\Delta(y)) + ad_{\alpha(x)}(ad_y(t))| = |ad_{\alpha(x)}(\Delta(y))| + |ad_{\alpha(x)}(ad_y(t))|.$$

Moreover, since  $\Delta([x, y]) = ad_{\alpha(x)}(\Delta(y)) - (-1)^{|x||y|}ad_{\alpha(y)}(\Delta(x)) = |ad_{\alpha(x)}(\Delta(y))|$ , because  $\mathfrak{L}$  is a Hom-Lie superbialgebra, (4.3) is equivalent to,  $ad_{[x, y]}(t) = |ad_{\alpha(x)}(ad_y(t))|$ , which holds by Lemma 3.11.

- Finally, we must show the Hom-super-coJacobi identity of  $\Delta_t$ , which states

$$\circ(\alpha \otimes \Delta_t)(\Delta_t(x)) = 0 \quad (4.4)$$

for all  $x \in \mathfrak{L}$ . Using the definition  $\Delta_t = \Delta + ad(t)$ . We can rewrite (4.4) as

$$\circ(\alpha \otimes \Delta)(\Delta(x)) + \circ(\alpha \otimes \Delta)(ad_x(t)) + \circ(\alpha \otimes ad(t))(\Delta(x)) + \circ(\alpha \otimes ad(t))(ad_x(t)) = 0. \quad (4.5)$$

We already know that  $\circ (\alpha \otimes \Delta)(\Delta(x)) = 0$ , which is the Hom-super-coJacobi identity of  $\Delta$ . Moreover, in (3.8) and (3.9) (in the proof of Theorem 3.12 with  $t$  instead of  $r$ ), we already showed that

$$\circ (\alpha \otimes ad(t))(ad_x(t)) = \alpha^{\otimes 3}(ad_x([[t, t]]^\alpha)). \quad (4.6)$$

In view of (4.5) and (4.6), the Hom-super-coJacobi identity of  $\Delta_t$  (4.4) is equivalent to

$$\alpha^{\otimes 3}(ad_x([[t, t]]^\alpha)) + \circ ((\alpha \otimes ad(t))(\Delta(x)) + (\alpha \otimes \Delta)(ad_x(t))) = 0. \quad (4.7)$$

Using the assumption (4.1), the condition (4.7) is equivalent to

$$\circ ((\alpha \otimes ad(t))(\Delta(x)) + (\alpha \otimes \Delta)(ad_x(t))) = \alpha^{\otimes 3}(\circ ad_x((\alpha \otimes \Delta)(t))). \quad (4.8)$$

We will prove (4.8) in Lemma 4.2 below.

The proof of Theorem 4.1 will be complete once we prove Lemma 4.2.  $\square$

**Lemma 4.2.** *The condition (4.8) holds.*

*Proof.* Write  $\Delta(x) = \sum x_1 \otimes x_2$  and  $t = \sum t_1 \otimes t_2 \in \mathfrak{L}^{\otimes 2}$ . Then the left-hand side of (4.8) is:

$$\begin{aligned} & \circ ((\alpha \otimes ad(t))(\Delta(x)) + (\alpha \otimes \Delta)(ad_x(t))) \\ &= \circ (\alpha(x_1) \otimes ad_{x_2}(t_1 \otimes t_2) + (\alpha \otimes \Delta)([x, t_1] \otimes \alpha(t_2) + (-1)^{|x||t_1|} \alpha(t_1) \otimes [x, t_2])) \\ &= \circ (\alpha(x_1) \otimes [x_2, t_1] \otimes \alpha(t_2) + (-1)^{|x_2||t_1|} \alpha(x_1) \otimes \alpha(t_1) \otimes [x_2, t_2]) \\ &+ \circ (\alpha([x, t_1]) \otimes \Delta(\alpha(t_2)) + (-1)^{|x||t_1|} \alpha^2(t_1) \otimes \Delta([x, t_2])). \end{aligned}$$

Write  $\Delta(t_2) = \sum t'_2 \otimes t''_2$ . Recall from (1.8) that:  $\Delta([x, t_2]) = ad_{\alpha(x)}(\Delta(t_2)) - (-1)^{|x||t_2|} ad_{\alpha(t_2)}(\Delta(x))$ , because  $\mathfrak{L}$  is a Hom-Lie superbialgebra. We can continue the above computation as follows :

$$\begin{aligned} &= \circ (\alpha(x_1) \otimes [x_2, t_1] \otimes \alpha(t_2) + (-1)^{|x_2||t_1|} \alpha(x_1) \otimes \alpha(t_1) \otimes [x_2, t_2] + \alpha([x, t_1]) \otimes \alpha^{\otimes 2}(\Delta(t_2))) \\ &+ \circ ((-1)^{|x||t_1|} \alpha^2(t_1) \otimes [\alpha(x), t'_2] \otimes \alpha(t''_2) + (-1)^{|x||t_1|} (-1)^{|x||t'_2|} \alpha^2(t_1) \otimes \alpha(t'_2) \otimes [\alpha(x), t''_2]) \\ &- \circ ((-1)^{|x||t_1|} (-1)^{|x||t_2|} \alpha^2(t_1) \otimes [\alpha(t_2), x_1] \otimes \alpha(x_2) \\ &- (-1)^{|x||t_1|} (-1)^{|x||t_2|} (-1)^{|x_1||t_2|} \alpha^2(t_1) \otimes \alpha(x_1) \otimes [\alpha(t_2), x_2]). \end{aligned}$$

It follows the skew-supersymmetry of  $\Delta$  applied to  $x$  (i.e,  $\sum x_1 \otimes x_2 = -\sum (-1)^{|x_1||x_2|} x_2 \otimes x_1$ ),  $|t| = \bar{0}$ ,  $t_{21} = -t$ , and  $\alpha^{\otimes 2}(t) = t$ , We find that

- $\alpha^2(t_1) \otimes [\alpha(t_2), x_1] \otimes \alpha(x_2) = (-1)^{|x_1||x_2|} \alpha(x_2) \otimes \alpha^2(t_1) \otimes [\alpha(t_2), x_1]$ .
- $(-1)^{|x_1||t_2|} \alpha^2(t_1) \otimes \alpha(x_1) \otimes [\alpha(t_2), x_2] = (-1)^{|t_1||t_2|} (-1)^{|x_2||t_1|} \alpha(x_1) \otimes [\alpha(t_2), x_2] \otimes \alpha^2(t_1)$ . That the first two terms and the last two terms above cancel out. Using the commutation of  $\alpha$  with  $[\cdot, \cdot]$  and  $\Delta$  and  $\alpha^{\otimes 2}(t) = t$ , the above computation continues as follows:

$$\begin{aligned} &= \circ (\alpha([x, t_1]) \otimes \alpha^{\otimes 2}(\Delta(t_2)) + (-1)^{|x||t_1|} \alpha^2(t_1) \otimes [\alpha(x), t'_2] \otimes \alpha(t''_2)) \\ &+ (-1)^{|x|(|t_1|+|t'_2|)} \alpha^2(t_1) \otimes \alpha(t'_2) \otimes [\alpha(x), t''_2]) \\ &= \circ (\alpha([x, \alpha(t_1)]) \otimes \alpha^{\otimes 2}(\Delta(\alpha(t_2))) + (-1)^{|x||t_1|} \alpha^3(t_1) \otimes [\alpha(x), \alpha(t'_2)] \otimes \alpha^2(t''_2)) \\ &+ (-1)^{|x|(|t_1|+|t'_2|)} \alpha^3(t_1) \otimes \alpha^2(t'_2) \otimes [\alpha(x), \alpha(t''_2)]) \\ &= \alpha^{\otimes 3}(\circ ([x, \alpha(t_1)] \otimes \Delta(\alpha(t_2)) + (-1)^{|x||t_1|} \alpha^2(t_1) \otimes [x, t'_2] \otimes \alpha(t''_2) + (-1)^{|x|(|t_1|+|t'_2|)} \alpha^2(t_1) \otimes \alpha(t'_2) \otimes [x, t''_2])) \\ &= \alpha^{\otimes 3}(\circ ad_x(\alpha(t_1) \otimes t'_2 \otimes t''_2)) = \alpha^{\otimes 3}(\circ ad_x((\alpha \otimes \Delta)(t))). \end{aligned}$$

This proves (4.8).  $\square$

The following result is a special case of the previous theorem.

**Corollary 4.3.** *Let  $(\mathfrak{L}, [\cdot, \cdot], \Delta, \alpha)$  be a multiplicative Hom-Lie superbialgebra and  $t \in \mathfrak{L}^{\otimes 2}$  be an element such that  $\alpha^{\otimes 2}(t) = t$ ,  $t_{21} = -t$ , (1.3) and (3.2), ((3.2) and (1.3), i.e.,  $|t| = \bar{0}$ ) and  $[[t, t]]^\alpha + \circ (\alpha \otimes \Delta)(t) = 0$  for all  $x \in \mathfrak{L}$ . Then  $\mathfrak{L}_t = (\mathfrak{L}, [\cdot, \cdot], \Delta_t = \Delta + ad(t), \alpha)$  is multiplicative Hom-Lie superbialgebra.*

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**Addresses.**

*M. Fadous* , University of Sfax, Faculty of Sciences Sfax, BP 1171, 3038 Sfax, Tunisia.

*S. Mabrouk*, Université de Gafsa, Faculté des Sciences, Gafsa Tunisia

*A. Makhlouf*, University of Haute Alsace, 4 rue des frères Lumière, 68093 Mulhouse France.

**Emails.** mohamedfadous201001@gmail.com

mabrouksami00@yahoo.fr

Abdenacer.Makhlouf@uha.fr