



The last chapter of the Disquisitiones of Gauss

Laura Anderson, Jasbir S Chahal, Jaap Top

► To cite this version:

Laura Anderson, Jasbir S Chahal, Jaap Top. The last chapter of the Disquisitiones of Gauss. Hardy-Ramanujan Journal, 2022, Special Commemorative volume in honour of Srinivasa Ramanujan - 2021, Volume 44 - Special Commemorative volume in honour of Srinivasa Ramanujan - 2021, pp.152 – 159. 10.46298/hrj.2022.8925 . hal-03498249

HAL Id: hal-03498249

<https://hal.science/hal-03498249>

Submitted on 5 Jan 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The last chapter of the Disquisitiones of Gauss

Laura Anderson, Jasbir S. Chahal and Jaap Top

Abstract. This exposition reviews what exactly Gauss asserted and what did he prove in the last chapter of *Disquisitiones Arithmeticae* about dividing the circle into a given number of equal parts. In other words, what did Gauss claim and actually prove concerning the roots of unity and the construction of a regular polygon with a given number of sides. Some history of Gauss's solution is briefly recalled, and in particular many relevant classical references are provided which we believe deserve to be better known.

Keywords. regular polygon, cyclotomic polynomial, ruler and compasses construction, Disquisitiones

2010 Mathematics Subject Classification. 11-03, 12-03

1. Introduction

A remark by J-P. Serre made towards one of us concerning what the textbook [Cha06] asserts about Gauss' results on the construction of regular n -gons, provided the initial motivation to write this note. A second source of inspiration came from D. Surya Ramana's expository paper [Ram97] on Carl Friedrich Gauss (1777–1855). Various useful additions as well as interesting historical notes are presented in a slightly later text by B. Sury [Sur99]. In 1801 Gauss published his *Disquisitiones Arithmeticae* [Gau86]. The purpose of the present article is to elaborate on the remark of Serre and the comments by Ramana and Sury concerning the last (seventh) chapter of this celebrated textbook. Its last chapter is devoted to the study of roots of unity, i.e., the (complex) solutions to an equation $x^n = 1$. Gauss applied the results he obtained on roots of unity to the problem of constructing a regular polygon of n sides or, what amounts to the same, divide the circle into n equal parts. Here (and everywhere in the present paper) 'constructing' means constructing by using a straightedge and compass alone.

Ramana writes (p. 64 of [Ram97])

... Gauss proved that a regular polygon with n sides, where n is an odd integer, can be constructed using a ruler and compass alone if and only if n can be expressed as a product of distinct Fermat primes.

Gauss indeed discovered and formulated a general result, namely a necessary and sufficient condition on positive integer n for a regular polygon with n sides to be constructible. He states it in the very last Article 366 of Chapter VII of the *Disquisitiones* as follows (whenever we quote from Gauss's text, the 1986 revised English translation [Gau86] is used) :

Hinc colligitur ge-
neraliter, vt circulus geometricè in N partes di-
uidi possit, N esse debere *vel* 2 aut altiore
potestatem ipsius 2, *vel* numerum primum for-
mae $2^m + 1$, *vel* productum e pluribus huius-
modi numeris primis, *vel* productum ex vno tali
primo aut pluribus in 2 aut potestatem altiore
ipsius 2;

“In general therefore in order to be able to divide the circle geometrically into N parts, N must be 2 or a higher power of 2, *or* a prime number of the form $2^m + 1$, *or* the product of several prime numbers of this form, *or* the product of one or several such primes into 2 or a higher power of 2;”

However, Gauss also writes (Article 365) that “*the limits of the present work exclude ... demonstration here.*” and (Article 335) “... *although we could discuss them* (he refers to roots of unity in \mathbb{C} here) *in all their generality, we reduce them to the simplest case ... , both for the sake of brevity and in order that the new principles of this theory may be more easily understood.*” With this last sentence he apparently restricts himself to p -th roots of unity, for (odd) prime numbers p .

The precise modern formulation of the statement by Gauss quoted above is

Theorem 1.1. *A regular N -gon is constructible if and only if $N > 1$ is of the form*

$$N = 2^r p_1 p_2 \dots p_s$$

with $r, s \in \mathbb{Z}_{\geq 0}$ and p_j distinct Fermat primes, i.e., primes of the form $2^m + 1$.

Article 365 of the *Disquisitiones* contains a proof of the fact that if $2^m + 1$ is prime, then $m = 2^\nu$, as well as the observation that Fermat studied these numbers and that Euler discovered $2^{32} + 1$ to be divisible by 641. Although Fermat believed *all* integers $2^{2^\nu} + 1$ to be prime, it is an old conjecture (for example, see [Pas1910, p. 315]) that they are prime for only finitely many integers ν . Likely, $\nu = 4$ is the largest of these.

Gauss finishes the *Disquisitiones* by listing all integers below 300 satisfying the criterion of Theorem 1.1:

2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30,
32, 34, 40, 48, 51, 60, 64, 68, 80, 85, 96, 102,
120, 128, 136, 160, 170, 192, 204, 240, 255,
256, 257, 272.

Many sources assert that Gauss proved Theorem 1.1, for example Chapter 5 of the history textbook [Cha06] written by the second author of the current paper. After receiving a copy of these history notes, Jean-Pierre Serre pointed out to him (as already alluded to in the first lines of this paper) that the *Disquisitiones* does not contain a complete proof of the result. In the remainder of the present note we review some of the famous history of the problem of constructing regular n -gons, and its current mathematical formulation. Note, incidentally, that Serre’s remark was observed by others long before him: see, for example, the title of Pierpont’s 1895 text [Pie1895], a paper discussing among other things a proof of Theorem 1.1. Also Sury’s text [Sur99, p. 48] points out that this theorem is stated but not proven in the *Disquisitiones*.

1.A. Constructions with straightedge and compass

Constructions rely on two axioms already presented in Euclid’s *Elements* (Book 1):

- 1) Any two distinct points may be joined by one and only one (straight) line.
- 2) In the plane, a circle may be drawn with any point as its center and any finite length segment as its radius.

Clearly, they are equivalent to allowing the use of two basic tools:

- 1) A straightedge to join two distinct points, and
- 2) a compass to draw circles with already determined centers and radii.

A convenient approach to this identifies the plane with the complex plane, i.e., with the field of complex numbers \mathbb{C} . This is explained in the first chapter of Klein's 1895 textbook [Kle1897], and in terms of \mathbb{R}^2 in §37 of Hilbert's textbook [Hil1902] of 1899, and using field theory and \mathbb{C} in §65 of Van der Waerden's 1930 classic [Wae91]. It leads to the general criteria below.

Theorem 1.2. *A complex number z is constructible starting from 0 and 1 if and only if one of the following equivalent statements holds:*

- (i) *Field extensions $\mathbb{Q} = K_0 \subseteq K_1 \subseteq \dots \subseteq K_e \subset \mathbb{C}$ exist such that every $K_j \subseteq K_{j+1}$ is quadratic, and $z \in K_e$.*
- (ii) *The splitting field of $\mathbb{Q}(z)$ over \mathbb{Q} has degree a power of 2.*

The Greeks were interested in knowing what kind of figures are possible to construct. In Book 1 (Proposition 1) of his *Elements*, Euclid shows how to construct an equilateral triangle on a given line segment AB . In Book 1 (Proposition 46) it is shown how to construct a square. Finally, Book 4 (Proposition 11) describes a method to construct a regular pentagon.

A regular n -gon, or simply n -gon, is an n sided convex figure in the Euclidean plane with all sides equal, or equivalently with all the interior angles equal to each other. For example, an equilateral triangle is a 3-gon and a regular pentagon is a 5-gon. Since it is possible to bisect any given angle, one can obtain a $2n$ -gon from an n -gon.

More generally, given two positive integers n, m , constructing the regular n -gon and the regular m -gon is equivalent to constructing the regular $\text{lcm}(n, m)$ -gon. Namely, put $k := \text{lcm}(n, m)$. Constructing a regular k -gon is the same as constructing an angle of size $2\pi/k$. Having this, evidently all integral multiples of this angle can be constructed, so in particular those of size $2\pi/n$ and $2\pi/m$. Vice versa, from these two angles one can also construct angles of size $2\pi x/n + 2\pi y/m$ for arbitrary integers x, y . Taking x, y such that $xm + yn = \text{gcd}(n, m)$ (such integers x, y exist by the extended Euclidean algorithm), one obtains

$$x \frac{2\pi}{n} + y \frac{2\pi}{m} = \frac{2\pi}{nm} (xm + yn) = \frac{2\pi}{\text{lcm}(n, m)}.$$

This is classical, and briefly recalled in Article 336 of the *Disquisitiones*.

The construction of an n -gon is equivalent to the construction of the numbers $e^{2\pi im/n} = \cos(2\pi m/n) + i \sin(2\pi m/n)$ in the complex plane, given the numbers 0 and 1. In other words, one tries to construct from 0 and 1 the complex solutions of $x^n = 1$.

Ramana's exposition [Ram97] amply discusses the breakthrough by Gauss concerning the construction of n -gons. In the words of Gauss himself (see [Arc1920]),

“During the winter of 1796 (my first semester in Göttingen), I had already discovered everything related to the separation of the roots of the equation

$$\frac{x^p - 1}{x - 1} = 0$$

into *two* groups, on which the beautiful theorem... depends... After intensive consideration of the relation of all the roots to one another on arithmetical grounds, I succeeded during a holiday in Braunschweig, on the morning of the day alluded to [March 29, 1796] (before I had got out of bed), in viewing this relation in the clearest way, so that I could immediately make special application to the 17-side and to the numerical verification... I announced this discovery... in May or June 1796.”

In Article 365 of the *Disquisitiones* Gauss observes,

“It is certainly astonishing that although the geometric divisibility of the circle into three and five parts was already known in Euclid’s time, nothing was added to this discovery for 2000 years. And all geometers had asserted that, except for those sections and the ones that derive directly from them (that is, division into 15 , $3 \cdot 2^\mu$, $5 \cdot 2^\mu$, and 2^μ parts), there are no others that can be effected by geometric constructions.”

According to the German mathematician Heinrich Weber (1842–1913) famous for the Kronecker-Weber theorem, Gauss requested that the 17-gon be engraved on his tombstone (footnote in § 97 of the first edition of [Web1903] :

1) Disq. arithmeticae, Sectio septima. Wie von Archimedes berichtet wird, er habe bestimmt, daß sein Grabmal die Kugel mit dem Zylinder zeigen solle, so hat Gauß geäußert, er wünsche auf seinem Grabmal die Figur des Siebzehneckes verewigt. Diese kleine Erzählung zeigt, welchen Wert Gauß selbst auf diese Entdeckung legte. Auf dem Grabstein ist diesem Wunsch nicht entsprochen, wohl aber steht bei dem Denkmal, was Gauß in Braunschweig errichtet ist, die Statue (freilich dem Beschauer kaum sichtbar) auf einem Siebzehneck.

Translation: 1) Disq. arithmeticae, Chapter VII. As is recorded that Archimedes ascertained that his epitaph show a ball with a cylinder, Gauss expressed his wish that his epitaph immortalizes the figure of the 17-gon. This little tale shows the importance Gauss himself put on his discovery. On the epitaph this wish was not fulfilled, yet on the monument erected for Gauss in his native town Braunschweig, the statue is placed on a (barely visible for the visitor) 17-gon.

Both Ramana [Ram97] and Sury [Sur99] present a detailed derivation of Gauss’ formula for $\cos(2\pi/17)$; whereas Ramana’s paper does not contain the resulting formula itself, Sury’s one does. Note that the formula given in Article 365 of the *Disquisitiones* actually contains a typo: it reads

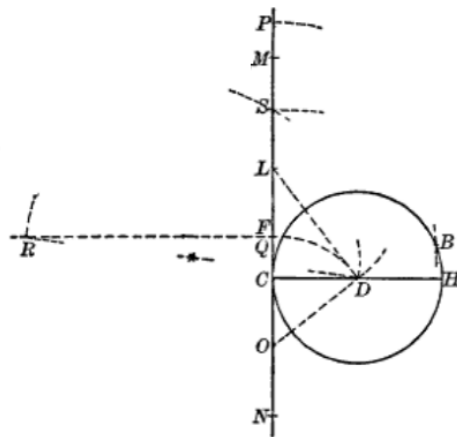
$$-\frac{1}{16} + \frac{1}{16}\sqrt{17} + \frac{1}{16}\sqrt{(34 - 2\sqrt{17})} - \frac{1}{8}\sqrt{(17 + 3\sqrt{17} - \sqrt{(34 - 2\sqrt{17})} - 2\sqrt{(34 + 2\sqrt{17}))}$$

and the rightmost minus sign on the first line should be a +, as confirmed by (e.g.) Sury’s short and clear derivation [Sur99, bottom of p. 46] (using his notation, $\alpha_3/2 = \cos(2\pi/17)$) and the resulting correct formula on [Sur99, p. 47]. From the modern perspective as given in Theorem 1.2 above, Gauss found the explicit tower of quadratic field extensions

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{17}) \subseteq \mathbb{Q}\left(\sqrt{34 - 2\sqrt{17}}\right) \subseteq \mathbb{Q}(\cos(2\pi/17)) \subseteq \mathbb{Q}(e^{2\pi i/17}).$$

Although both the formula and the series of quadratic extensions lead to a construction method of the 17-gon, an actual construction is not discussed in the *Disquisitiones*. According to R.C. Archibald’s historical survey [Arc1920], J.F. Pfaff in 1802 wrote a letter to Gauss in which he quotes the construction from a letter he received from C.F. Pfleiderer. Archibald reproduces the part of Pfaff’s letter containing the construction:

“Diametro CH circuli dati ad extremum eius C normalis ducatur PN recta: a qua abscindantur primum $CF = \frac{1}{4}CH$; tum FM, FN utraque $= FH$. Bifariam in punctis L, O secantur rectae CM, CN : et ad centrum D circuli propositi ductis LD, OD rectis, ab perpendicularo PN abscindantur $LP = LD, OQ = OD$. Ad rectam CQ applicetur rectangulum (CSQ) aequale rectangulo sub DC et SP , et excedens quadrato (quod fit, si recta PN ad Q punctum normalis ducitur $QR = CP$, et super diametro DR circulus describitur secans rectum CP in S). Denique centro C , intervallo CS , describatur circulus, secans propositum circulum in B . Erit HB latus polygoni regularis XVII laterum, circulo dato inscribendi.”



(taken from R.C. Archibald [Arc1920].)

The next Fermat prime after $2^4 + 1 = 17$ is $2^8 + 1 = 257$. In 1831, Friedrich Julius Richelot in his PhD thesis [Ric1832] (supervised by Carl Jacobi) presents a construction of the 257-gon. This was also done in 1887 by E. Pascal [Pas1887] and in 1893 in a short book by Hugo Schwendenwein [Sch1893]. For the final known Fermat prime $2^{16} + 1 = 65537$, Johann Gustav Hermes in 1894 published a construction of the 65537-gon [Her1894]. The practical difficulty in constructing a precise 65537-gon is obvious, for even 257 is so large that on a page of a book, the 257-gon will look like a circle.

2. Proving Gauss's claim

A contemporary proof of Theorem 1.1 goes like this; compare [Pie1895], [Gar86, § 19.3] and many similar texts.

For any integer $n \geq 1$, put

$$\omega = e^{2\pi i/n} \in \mathbb{C}.$$

The complex numbers

$$1, \omega, \omega^2, \dots, \omega^{n-1}$$

are the zeros of the polynomial

$$x^n - 1$$

of degree n .

As remarked earlier, constructing a regular n -gon is equivalent to constructing regular p^m -gons, for every prime power p^m (with $m > 0$) exactly dividing n . Writing $\zeta = e^{2\pi i/p^m}$, by Theorem 1.2 the latter is equivalent to the existence of field extensions

$$\mathbb{Q} = K_0 \subset K_1 \subset \dots \subset K_e = \mathbb{Q}(\zeta)$$

in which every $K_j \subset K_{j+1}$ is quadratic.

The extension $\mathbb{Q}(\zeta)/\mathbb{Q}$ has degree $\phi(p^m) = p^{m-1}(p-1)$ where ϕ is Euler's ϕ -function. By assumption the extension K_e/K_0 has degree 2^e . Hence if the p^m -gon is constructible then either $p = 2$ (and $m > 0$ is arbitrary), or $m = 1$ and p is a Fermat prime. Vice versa, if $\phi(p^m) = 2^e$, then the observation that $\mathbb{Q}(\zeta)/\mathbb{Q}$ is a Galois extension of degree 2^e with abelian Galois group, implies by the fundamental theorem of Galois theory that ζ satisfies the condition for constructibility given in Theorem 1.2. This finishes the proof of Theorem 1.1.

A special instance of Theorem 1.1 is that an 18-gon, and therefore an angle of 20° , cannot be constructed whereas a 6-gon, so an angle of 60° , can. This shows:

Corollary 2.1. *Given only a compass and straightedge, it is impossible to trisect all given angles.*

In 1837, Pierre Laurent Wantzel [1837] published a (different) proof of Corollary 2.1, although it is an immediate consequence of Theorem 1.1. An interesting discourse why Gauss might have emphasized constructibility while somewhat neglecting the impossibility of constructions was recently given by Lützen [Lut09], [Lut10]. These papers moreover discuss earlier proofs of Wantzel’s theorem by R. Descartes (1637), J.É. Montucla (1754), and N.J.A.N. de Condorcet (1775), all preceding the publication of the *Disquisitiones*.

In what follows, we quote what Gauss himself says of the proof of Theorem 1.1 he left out.

Concerning p^λ -th roots of unity with $\lambda > 1$ and p an odd prime, Article 336 of the *Disquisitiones* asserts

“to determine a polygon of p^λ sides we necessarily require the solution of $\lambda - 1$ equations of degree p . Even though the following theory could be extended to this case also, nevertheless we could not avoid so many equations of degree p , and there is no way of reducing their degree if p is prime.”

In today’s terminology, this seems to say that if $\zeta = e^{2\pi i/p}$ and $\omega = e^{2\pi i p^{-\lambda}}$, then the extension $K(\omega)/K(\zeta)$ has degree $p^{\lambda-1}$ and in fact a chain of intermediate fields

$$\mathbb{Q}(\zeta) = L_0 \subset K_1 \subset \dots \subset K_{\lambda-1} = \mathbb{Q}(\omega)$$

exists in which each K_{j+1} has degree p over K_j .

In his discussion of polynomials such that $\zeta = e^{2\pi i/p}$ can be expressed in terms of their zeros (with p an odd prime number not of the form $1 + 2^m$), he writes (*Disquisitiones*, Article 365), the capitalization is by Gauss himself:

“WE CAN SHOW WITH ALL RIGOR THAT THESE HIGHER-DEGREE EQUATIONS CANNOT BE AVOIDED IN ANY WAY NOR CAN THEY BE REDUCED TO LOWER-DEGREE EQUATIONS. The limits of the present work exclude this demonstration here, but we issue this warning lest anyone attempt to achieve geometric constructions for sections other than the ones suggested by our theory (e.g. sections into 7, 11, 13, 19, etc. parts) and so spend his time uselessly.”

In modern terms, this is interpreted as the claim that the degree of $\mathbb{Q}(\zeta)$ over \mathbb{Q} , which equals $p - 1$, is divisible by an odd number > 1 .

For n a Fermat prime, Gauss states (*Disquisitiones*, Article 365; note that he uses P for 2π),

“Whenever... the value of n is 3, 5, 17, 257, 65537, etc. the sectioning of the circle is reduced to quadratic equations only, and the trigonometric functions of the angles P/n , $2P/n$, etc. can be expressed by square roots which are more or less complicated (according to the size of n). Thus in these cases the division of the circle into n parts or the inscription of a regular polygon of n sides can be accomplished by geometric constructions.”

In today’s language, this claims that the field $\mathbb{Q}(\zeta)$ with $\zeta = e^{2\pi i/p}$ and p a Fermat prime, admits a chain of subfields

$$\mathbb{Q} = K_0 \subset K_1 \subset \dots \subset K_e = \mathbb{Q}(\zeta),$$

with each K_{j+1} quadratic over K_j .

Write Φ_n for the minimal monic polynomial with rational coefficients having $\omega = e^{2\pi i/n}$ as a zero. The degree of Φ_n equals the degree of $\mathbb{Q}(e^{2\pi i/n})$ over \mathbb{Q} . Gauss determined in Article 341 of

the *Disquisitiones* that for $n = p$ a prime number, $\Phi_p = x^{p-1} + x^{p-2} + \dots + x + 1$. This may be regarded as a proof of the assertion in the second quote by Gauss mentioned in this section. Note that Entry #136 in Gauss's Diary [Gau84] written in 1808 essentially says that he was able to prove that for every n the degree of Φ_n equals $\phi(n)$. Section IV of Wantzel's 1837 paper [1837] contains the assertion that a slight adaptation of Gauss's argument for determining Φ_p gives Φ_{p^r} for all $r > 0$. Although he refrained from explicitly observing it, Klein's determination (1895) of Φ_{p^2} sketched below in fact extends to the case $\Phi_{p^r} = (x^{p^r} - 1)/(x^{p^{r-1}} - 1)$, confirming Wantzel's assertion. Earlier in 1850 Serret [Ser1850] in a less elementary way computed Φ_n when n is a power of a prime number. Kronecker [Kro1854] in 1854 determined Φ_n for all n , confirming what Gauss asserted in his diary 46 years earlier. A historical survey containing many interesting classical references may be found in [Pas1910, Kapitel IV § 12]. In Section III.8 of [Kle1897] Klein determines Φ_n for $n = p^2$ and p prime, namely, $\Phi_{p^2} = (x^{p^2} - 1)/(x^p - 1)$. In fact, replacing x by $x + 1$ allows one to apply the well known Eisenstein criterion for irreducibility. Klein continues by observing that the degree of this polynomial is a multiple of p . Hence for p odd and n any multiple of p^2 , the n -gon cannot be constructed.

3. Conclusion

Gauss in the *Disquisitiones* stated a correct and complete criterion regarding constructibility of a regular n -gon (Theorem 1.1 above).

Moreover, by determining (stated in modern terms) for any odd prime p the minimal polynomial Φ_p of $e^{2\pi i/p}$, he shows that indeed only the prime 2 and Fermat primes can occur as prime factors of n , in case the n -gon is constructible.

Gauss correctly states the reason why no odd prime p can have the property $p^2|n$ in case the n -gon is constructible. However, he did not provide a proof in this case. Based on a diary entry he wrote 7 years after the publication of the *Disquisitiones*, and on a very simple argument published later by Klein and probably already known to Wantzel, it seems reasonable to assume that indeed Gauss did have a complete proof of this, although maybe not yet in 1801.

Finally by classical arguments, to show that indeed for integers n of the kind described by Gauss the n -gon is constructible, it suffices to show that for any Fermat prime $F_\nu = 2^{2^\nu} + 1$ the F_ν -gon is constructible. Although we have not discussed it here, arguably Gauss knew how to achieve this (see Articles 343-364 and the summary he presents in the first lines of Article 365). The example $\nu = 2$, so $F_\nu = 17$ presented by him illustrates this method.

Acknowledgement. It is our pleasure to thank Sudhir Ghorpade for helpful advise concerning publication of the present note. We are grateful for the comments of a referee, in particular informing us of the paper [Sur99] and of the typo in the original publication of the *Disquisitiones* which we had not noticed.

References

- [Arc1920] Raymond C. Archibald, *Gauss and the Regular Polygon of Seventeen Sides*, The American Mathematical Monthly, 1920.
- [Cha06] Jasbir S. Chahal, *A Historical Perspective of Mathematics*. Heber City, Utah: Kendrick Press, 2006.
- [Euc56] Euclid, *The Elements*, vol. I, II, and III. Dover, 1956.
- [Gar86] D.J.H. Garling, *A Course in Galois Theory*. Cambridge University Press, 1986.
- [Gau86] Carl Friedrich Gauss, *Disquisitiones Arithmeticae*. Leipzig: Gerh. Fleischer, 1801. (First English translation by Arthur A. Clarke, Yale University Press, 1966; Revised edition by William C. Waterhouse, J. Brinkhuis, Cornelius Greither, and A. W. Grootendorst, Springer-Verlag, 1986).
- [Gau84] Carl Friedrich Gauss, *Mathematisches Tagebuch, 1796—1814*. (Translation from Latin into German by E. Schuhmann, Geest & Portig K.-G., 1976. English translation J.J. Gray, *A commentary on Gauss's mathematical diary, 1796–1814*, Expos. Math. **2** (1984), 97–130.

- [Her1894] J. Hermes, *Ueber die Teilung des Kreises in 65537 gleiche Teile*, Nachrichten Ges. der Wissensch. zu Göttingen, Mathematisch-Physikalische Klasse, Nr. 3 (1894), 170–186.
- [Hic06] Angela Hicks, *Translation of portion of letter to C. F. Gauss from J. F. Pfaff*, personal email to the author, August 2006, Portion of original letter in Latin printed in [Arc1920] above.
- [Hil1902] D. Hilbert, *Grundlagen der Geometrie*. Leipzig: B.G. Teubner, 1899. (English translation by E.J. Townsend ‘The Foundations of Geometry’ La Salle, Illinois: The Open Court Publishing Company, 1902.)
- [Kle1897] Felix Klein, *Vorträge über ausgewählte Fragen der Elementargeometrie, ausgearbeitet von F. Tägert*, Leipzig: B.G. Teubner, 1895. (English translation “Famous Problems of Elementary Geometry” by W.W. Beman and D.E. Smith, Boston, etc.: Ginn and Company, 1897.)
- [Kro1854] L. Kronecker, *Mémoire sur les facteurs irréductibles de l’expression $x^n - 1$* . Journ. de math. pures et appl. 1^{re} série **19** (1854), 177–192.
- [Lut09] J. Lützen, *Why was Wantzel overlooked for a century? The changing importance of an impossibility result*, Historia Math. **36** (2009), 374–394.
- [Lut10] J. Lützen, *The Algebra of Geometric Impossibility: Descartes and Montucla on the Impossibility of the Duplication of the Cube and the Trisection of the Angle*, Centaurus **52** (2010), 4–37.
- [Pas1887] E. Pascal, *Sulla costruzione del poligono regolare di 257 lati*. Rend. della R. Acc. delle Scienze Fisiche e Mat. di Napoli, **1** (1887) p. 33.
- [Pas1910] E. Pascal, *Repertorium der höheren Mathematik I: Analysis. Erste Hälfte, Algebra, Differential- und Integralrechnung*. Leipzig und Berlin: Teubner, 1910.
- [Pie1895] James Pierpont, *On an undemonstrated theorem of the Disquisitiones Arithmeticae*, Bull. Amer. Math. Soc. **2** Number 3 (1895), 77–83.
- [Ram97] D. Surya Ramana, *Carl Friedrich Gauss*, Resonance **2** Issue 6 (1997), 60–67.
- [Ric1832] F.J. Richelot, *De resolutione algebraica aequationis $x^{257} = 1$, sive de divisione circuli per bisectionem anguli septies repetitam in partes 257 inter se aequales commentatio coronata*, J. reine u. angewandte Math. **9** (1832), pp. 1–26, 146–161, 209–230, 337–358.
- [Sch1893] H. Schwendenwein, *Das regelmässige Zweihundertsiebenundfünfzigeck*, Teschen Publications, 1893 (58 pages).
- [Ser1850] J.-A. Serret, *Sur une question de théorie des nombres*. Journ. de math. pures et appl. 1^{re} série **15** (1850), p. 296.
- [Sur99] B. Sury, *Cyclotomy and Cyclotomic Polynomials*, Resonance **4** Issue 12 (1999), 41–53.
- [Wae91] B.L. van der Waerden, *Moderne Algebra. Unter Benutzung der Vorlesungen von E. Artin und E. Noether*, 1. Auflage, Die Grundlehren der Mathematischen Wissenschaften Vol. 34, Berlin: Verlag Julius Springer, 1930, 1931. (English edition: Algebra, Springer-Verlag, 1991.)
- [1837] P.L. Wantzel, *Recherches sur les moyens de reconnaître si un Problème de Géométrie peut se résoudre avec la règle et le compas*. J. Math. Pure Appl. **1**, 1837, 366–372.
- [Web1903] H. Weber, *Encyklopädie der elementaren Algebra und Analysis*. Volume 1 of “Encyklopädie der Elementar-Mathematik. Ein Handbuch für Lehrer und Studierende” (Heinrich Weber und Josef Wellstein), Leipzig, B.G. Teubner, 1903.

Laura Anderson

1065 West 10210 South
 South Jordan, UT 84095, USA
e-mail: laurabcannon@gmail.com

Jasbir S. Chahal

Department of Mathematics
 Brigham Young University
 Provo, UT 84602, USA
e-mail: jasbir@math.byu.edu

Jaap Top

Bernoulli Institute
 University of Groningen
 P.O. Box 407, 9700 AK Groningen
 The Netherlands
e-mail: j.top@rug.nl