

² Supplementary Information for

Sediment load determines the shape of rivers

- 4 Predrag Popović, Olivier Devauchelle, Anaïs Abramian, and Eric Lajeunesse
- **5** Corresponding author Predrag Popović.
- 6 E-mail: arpedjo@gmail.com

7 This PDF file includes:

- ⁸ Figs. S1 to S13 (not allowed for Brief Reports)
- ⁹ Tables S1 to S2 (not allowed for Brief Reports)
- 10 SI References

1

11 S1. Experimental parameters and uncertainties

¹² In this section we summarize the parameters used in the experiments of Abramian et al. (1), show the results for each of the ¹³ experimental runs, and describe how we estimate the river properties and their uncertainties.

S.1.1. Details of the experimental runs. In Table S1, we show the experimental parameters and their uncertainties (uncertainties are estimated in Abramian et al. (1)). In Fig. S1, we show the depth and sediment flux profiles for each of the experiments of Abramian et al. (1) and compare them to our model predictions. In Table S2 and Fig. S2, we show the properties of these rivers.

We can see that our model predicts rivers that are slightly wider than the experiments (by about 20%), while the depth, D_{max} , is not biased in an obvious way. The width of the sediment flux profiles is captured quite accurately, but the experimental profiles seem to be somewhat higher (by about 30% for the maximum sediment flux).

In our model, the bed slope at the river bank approximately equals the friction coefficient, $D'|_{D=0} \approx \mu_t$, since the stress, 21 τ , vanishes when D = 0 and the sediment flux, $q_{\mu} \exp[-\xi/\lambda]$, is negligible (Eq. 13 of the main text). However, some of the 22 experiments show a high bank angle, unusual for granular material with irregular grains. This could be due to capillary forces 23 acting near the bank, to rivers being not fully in equilibrium, to surface armoring by removing the loose grains (2), or to 24 the fact that the fluid stress does not necessarily vanish near the bank. In fact, using the value $\mu_t = 1.2$, combined with 25 a length-scale, L_s , smaller by about 20% (which is within the experimental uncertainty), makes our model agree with the 26 experiment well in all metrics. However, to avoid treating μ_t as a tuning parameter, we decided to use $\mu_t = 0.9$, which is the 27 largest value estimated in an independent experiment (3). 28

S.1.2. Estimating the river properties and their uncertainties. We estimated all of the river properties based on the cross-sections 29 30 of Fig. S1. In particular, we estimated the total sediment discharge, Q_s , as the integral of the sediment flux profile found by grain tracking. We identify the uncertainty in the sediment discharge from the fluctuations of Q_s about the mean, once the 31 equilibrium is reached (Fig. 2 in Abramian et al. (1)). Next, we estimate the width, W, as the distance from one bank to 32 the other, where we identify the banks by the sudden change of slope. In doing so, we introduce an error that is of the order 33 of the grain size. However, we can also estimate the natural variability of the river width along its path from the overhead 34 images of the experiment. We find it to be about 5% of the mean — an error larger than that introduced by the cross-section 35 measurement. For this reason, we identify the uncertainty for the width with this natural variability. Next, we estimate the 36 37 depth, $D_{\rm max}$, by taking the minimum of a parabola we fit to the the river bottom, in order to minimize the error due to bed roughness. Abramian et al. (1) estimate the measurement error of depth due to laser inaccuracy to be about 0.5 mm. We do 38 not have access to the natural variability of the depth, although we can say that it is at least of the order of the grain size. 39 Again, this natural variability is greater than the measurement error estimated by Abramian et al. (1), so we take the error of 40 $D_{\rm max}$ to be $d_s = 0.83$ mm for all experiments. We follow a similar procedure to find the maximum sediment flux, $q_{s,\rm max}$ 41 we fit a parabola around the center of the sediment flux profile, and take its maximum to be $q_{s,max}$. Again, we do not have 42 access to the natural variability of $q_{s,\max}$ along the river's path, so we take the error of $q_{s,\max}$ to be the difference between the 43 observed maximum sediment flux profile and the height of the fitted parabola. 44

As we noted in section Dependence on Water and Sediment Discharge of the main text, the downstream slope cannot be measured directly. However, we can estimate it indirectly. The sediment flux at the river center, $q_{s,\max}$, is driven only by the fluid stress, which is about $\tau \approx \rho_f g D_{\max} S$ in the shallow-water approximation. Therefore, dropping the contributions of momentum diffusion and gravity from Eq. 13 of the main text, we find $D_{\max}S/L_s \approx \mu_t + q_{s,\max}/q_{\mu}$, and we can estimate the slope in the experiment as

50

$$S \approx \frac{L_s}{D_{\max}} \left(\mu_t + \frac{q_{s,\max}}{q_{\mu}} \right) .$$
 [S1]

This is the estimate we show in Fig. S2. This estimate is not independent from our model. We can, nevertheless, use it to check the consistency between the parameters in our model and the experiments. We estimate the error for the slope calculated in this way as a combination of the errors for the quantities that enter Eq. S1.

The quantities we described above are all point-measurements and, therefore, are sensitive to the roughness of the bed and 54 measurement error. On the other hand, integral quantities such as the mean sediment flux, $\langle q_s \rangle$, and transport width, W_T , are 55 robust against such errors. In this case, we expect that the uncertainty of $\langle q_s \rangle$ and W_T is mostly due to their natural variability 56 in time and space, rather than by any measurement error. We, however, do not have access to measurements that would allow 57 us to estimate the variability, so we assume that the natural variability of $\langle q_s \rangle$ is of the same order as the variability of the 58 sediment flux profile across the channel. Therefore, we take the uncertainty of $\langle q_s \rangle$ to be equal to the uncertainty of $q_{s,\max}$ — a 59 conservative estimate. Finally, since $W_T = Q_s/\langle q_s \rangle$, we estimate its uncertainty as a combination of the uncertainty of the 60 sediment discharge and that of the mean sediment flux. 61

Definition	Notation	Value	Unit
Grain diameter	d_s	0.83 ± 0.2	mm
Fluid viscosity	ν	10^{-5}	m^2s^{-1}
Fluid density	ρ_f	1160 ± 5	kg m ^{−3}
Sediment density	$ ho_s$	1490	kg m ^{−3}
Fluid discharge	Q_w	0.97 ± 0.05	$I min^{-1}$
Sediment discharge	Q_s	(0, 60)	grains s^{-1}
Threshold Shields parameter	θ_t	0.167 ± 0.003	None
Friction coefficient	μ_t	0.9 ± 0.2	None
Sediment diffusion length	λ	0.10 ± 0.03	mm
Sediment flux scale	q_{μ}	107 ± 30	grains $cm^{-1}s^{-1}$

Table S1. Table of parameters used in the experiments and the model.



Fig. S1. Depth and sediment profiles for all experiments of Abramian et al. (1). The left column are the measured (brown lines) and modeled (blue dashed lines) depth profiles. The right column are the measured (red lines) and modeled (blue lines) sediment flux profiles.

Experiment label	Sediment discharge	Unit
1	0	
2	12.6 ± 1.2	
3	24 ± 2.4	grains s $^{-1}$
4	44.9 ± 4.5	
5	59.7 ± 6.0	

Quantity	Experiment label	Value	Unit
Width, W	1	3.20 ± 0.16	
	2	3.36 ± 0.17	
	3	3.53 ± 0.18	cm
	4	4.83 ± 0.24	
	5	4.56 ± 0.23	
	1	0.744 ± 0.083	
Depth, D _{max}	2	0.748 ± 0.083	
	3	0.530 ± 0.083	cm
	4	0.474 ± 0.083	
	5	0.539 ± 0.083	
Maximum sediment flux, $q_{s,\max}$	1	0	
	2	15.96 ± 0.09	
	3	24.65 ± 1.44	grains s $^{-1}$ cm $^{-1}$
	4	26.97 ± 3.07	
	5	27.42 ± 2.44	
Transport width, W_T	1	n/a	
	2	1.01 ± 0.11	
	3	1.24 ± 0.22	cm
	4	2.16 ± 0.53	
	5	2.46 ± 0.49	
Downstream slope, S	1	0.005 ± 0.002	
	2	0.006 ± 0.002	
	3	0.009 ± 0.004	None
	4	0.011 ± 0.005	
	5	0.009 ± 0.004	



Fig. S2. Comparison of various river properties in dimensional units between our model and experiments of Abramian et al. (1). Slope in panel (c) is estimated using Eq. S1. We only show the weak transport regime for the properties related to the sediment flux.

62 S2. Fluid induced stress, au

In this section we will first derive Eq. 3 of the main text for the fluid stress, τ . Then, we will show that this stress can be written as a series expansion that assumes the aspect ratio of the channel is large. We will show that our model, Eq. 13 of the main text, is the first term in this expansion. Finally, we will test our approximation on several tractable examples.

S.2.1. Deriving the fluid stress equation. In this section, following Devauchelle et al. (4), we derive equation 3 of the main text for the fluid stress, τ . To that end, we first integrate the Stokes' flow equation, Eq. 2 of the main text, over the vertical to get

$$\eta_v \int_{-D}^0 \frac{\partial^2 u}{\partial y^2} dz - \tau_z = -\rho_f g S D , \qquad [S2]$$

where the z-component of the stress is $\tau_z \equiv -\eta_v \left. \frac{\partial u}{\partial z} \right|_{z=-D}$, and we have used the boundary condition that the stress vanishes at the surface, $\tau_z(z=0) = 0$. To get Eq. 3 of the main text, we need to pull the y-derivatives in Eq. S2 outside the integral. Doing this, we get

$$\int_{-D}^{0} \frac{\partial^2 u}{\partial y^2} dz = \frac{\partial^2}{\partial y^2} \int_{-D}^{0} u dz + \frac{dD}{dy} \left. \frac{\partial u}{\partial y} \right|_{z=-D} , \qquad [S3]$$

where we have used the boundary condition that the velocity vanishes at the boundary, u(z = -D) = 0. We can use this same condition to relate the term $\frac{\partial u}{\partial y}\Big|_{z=-D}$ to the shear stress, τ_z . Namely, differentiating the boundary condition, u(z = -D) = 0, vields

$$\left. \frac{\partial u}{\partial y} \right|_{z=-D} = \left. \frac{\mathrm{d}D}{\mathrm{d}y} \left. \frac{\partial u}{\partial z} \right|_{z=-D} \right.$$
 [S4]

⁷⁴ Substituting relations Eqs. S3 and S4 into Eq. S2, we find

$$\eta_v \frac{\mathrm{d}^2}{\mathrm{d}y^2} (D\bar{u}) - \tau_z \left(1 + \left(\frac{\mathrm{d}D}{\mathrm{d}y}\right)^2 \right) = -\rho_f g S D , \qquad [S5]$$

where we have introduced $\bar{u}(y) \equiv \frac{1}{D} \int_{-D}^{0} u dz$. Equation S4 relates the y and z components of the stress as $\tau_y = D' \tau_z$. This means that the total stress, τ , is related to τ_z as

$$\tau = \left(\tau_z^2 + \tau_y^2\right)^{1/2} = \tau_z \left(1 + D^{\prime 2}\right)^{1/2},$$
[S6]

⁷⁷ where prime stands for d/dy. Substituting Eq. S6 into Eq. S5, we can express τ as

$$\tau = \frac{\eta_v (D\bar{u})'' + \rho_f gSD}{(1+D'^2)^{1/2}} , \qquad [S7]$$

⁷⁸ Identifying $(1 + D'^2)^{-1/2}$ with $\cos \phi$, we finally retrieve Eq. 3 of the main text

$$\tau = \left(\eta_v (D\bar{u})'' + \rho_f g S D\right) \cos\phi , \qquad [S8]$$

⁷⁹ So far, we made no approximation to get here from the original Stokes equation (Eq. 2 of the main text).

S.2.2. The stress approximation as a series expansion. In the main text, we used the shallow-water velocity to approximate the fluid stress (Eqs. 12 and 13 of the main text). In this section, we will show that this approximation is the first term in a series expansion that assumes the aspect ratio of the river is large. In this way, this approximation may be systematically improved, assuming that the series converges.

⁸⁴ We begin with the Stokes equation

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{-gS}{\nu}$$
[S9]

If the channel depth, D(y), varies smoothly, then the second y-derivative of velocity scales as U/W^2 while the second z-derivative scales as U/D_{max}^2 , where U is the velocity scale. Therefore, if the aspect ratio, W/D_{max} , is large, the z-derivative is much

⁸⁷ larger than the *y*-derivative. We can make this obvious by rescaling the variables in the Stokes equation as

$$\hat{y} = \frac{y}{W}$$
, $\hat{z} = \frac{z}{D_{\text{max}}}$, $\hat{u} \equiv u \frac{\nu}{gSD_{\text{max}}^2}$ [S10]

In these new coordinates, the channel has unit width and depth, and the fluid is driven with a unit forcing. The Stokes equation then becomes

$$\frac{D_{\max}^2}{W^2}\frac{\partial^2 \hat{u}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{z}^2} = -1$$
[S11]

We can then expand the velocity as

$$\hat{u} = \hat{u}^{(0)} + \hat{u}^{(1)} + \hat{u}^{(2)} + \dots,$$
 [S12]

Predrag Popović, Olivier Devauchelle, Anaïs Abramian, and Eric Lajeunesse

7 of 28

- where $\hat{u}^{(0)}$ is the term of the order unity, $\hat{u}^{(1)}$ is the term of the order D_{\max}^2/W^2 , $\hat{u}^{(2)}$ is the term of the order D_{\max}^4/W^4 , etc.
- ⁹² Equating orders in the Stokes equation, we get (back in the physical coordinates)

$$\frac{\partial^2 u^{(0)}}{\partial z^2} = \frac{-gS}{\nu} , \qquad [S13]$$

$$\frac{\partial^2 u^{(0)}}{\partial y^2} + \frac{\partial^2 u^{(1)}}{\partial z^2} = 0 \ , \ \dots$$
 [S14]

⁹³ The boundary conditions for each term are the same — velocity at each order must vanish on the channel bed and its

z-derivative must vanish on the fluid surface. From Eq. S13, we can see that the zeroth order term is in fact equal to the shallow-water velocity, $u^{(0)} \equiv u_{sw}$. All of the terms in the expansion are polynomial in z, and we can easily find each term of order (n + 1) based on the previous term of order (n) by performing integration over z. In this way, we find for the first two

97 terms:

$$u^{(0)} = \frac{gS}{2\nu} \left(D^2 - z^2 \right) , \qquad [S15]$$

$$u^{(1)} = \frac{gS}{4\nu} \left(D^2 - z^2 \right) \left(D^2 \right)'' \,.$$
 [S16]

The term $u^{(1)}$ is inversely proportional to the square of the aspect ratio since the correcting factor, $(D^2)''$, is of order D_{\max}^2/W^2 . From here, we can find the vertically averaged velocity as an expansion:

$$\bar{u} = \bar{u}^{(0)} + \bar{u}^{(1)} + \dots$$
 [S17]

$$\bar{u}^{(0)} = \frac{gSD^2}{3\nu}$$
, [S18]

$$\bar{u}^{(1)} = \frac{gSD^2}{6\nu} (D^2)^{\prime\prime} .$$
[S19]

Equation S18 for the zeroth order velocity is the same as Eq. 12 of the main text for the shallow-water velocity. With this, we can find the fluid stress. Recall the exact equation for the stress, Eq. 3 of the main text,

$$\tau = \left(\rho_f g S D + \rho_f \nu (\bar{u} D)''\right) \cos \phi .$$
[S20]

¹⁰² We can also write the stress as an expansion

$$\tau = \tau^{(0)} + \tau^{(1)} + \dots , \qquad [S21]$$

where $\tau^{(0)}$ is the leading order term, and $\tau^{(1)}$ is proportional to D_{max}^2/W^2 . Using the expansion for \bar{u} , we see that

$$\tau^{(0)} = \rho_f g S D \cos \phi \tag{S22}$$

$$\tau^{(1)} = \rho_f \nu(\bar{u}^{(0)}D)'' \cos\phi = \left[\rho_f g S \frac{1}{3} \left(D^3\right)''\right] \cos\phi .$$
[S23]

Since $(D^3)'' \sim D_{\max}^3/W^2$, the first-order term is about D_{\max}^2/W^2 times smaller than the leading order term: $\tau^{(1)} \sim \tau^{(0)} D_{\max}^2/W^2$. The cosine term in the above equations is $\cos \phi = (1 + D'^2)^{-1/2}$, and, since $D'^2 \sim D_{\max}^2/W^2$, this term could also be expanded in a series but, for convenience, we keep the entire term here.

¹⁰⁷ The zeroth order term, $\tau^{(0)}$, with $\cos \phi \approx 1$, is the shallow-water stress that was used previously to estimate the shape of ¹⁰⁸ inert rivers (5). Our approximation (Eq. 13 of the main text) amounts to truncating the τ -series at first order, $\tau \approx \tau^{(0)} + \tau^{(1)}$. ¹⁰⁹ In the main text, we showed that the river is formed by the combined action of fluid stress and gravity. The gravity term in ¹¹⁰ the force ratio, $(D')^2$, is of the order of D_{\max}^2/W^2 , and is, therefore, of the same order of magnitude as $\tau^{(1)}$. For this reason, ¹¹¹ keeping only the shallow-water term in the force ratio, μ , is inconsistent, since it neglects a stress term that is comparable to ¹¹² gravity.

Although the expansion above may be corrected to arbitrary order, it is always local — the stress is always a function of the depth and its derivatives. To get non-local effects, whereby the stress depends on the entire bed shape, we would have to add infinitely many terms in the expansion.

S.2.3. Testing the approximation. In the previous section we showed that the fluid velocity and stress can be found using a series expansion when the flow is shallow enough. In this section, we will explicitly test this expansion on several examples for which exact solutions exist.

As a first example, we look at the flow over an inclined surface with an opening angle ϕ , that we assume to be small (Fig. S3a). In this case, the depth profile is given by

$$D(y) = y \tan \phi .$$
 [S24]

¹²¹ This problem is not well-posed since we do not specify the boundary condition on the open edge of the wedge. Nevertheless, if ¹²² the conditions far away do not matter, the solution of Stokes equation and the corresponding stress should be

the conditions far away do not matter, the solution of Stokes equation and the corresponding stress should be $a = p^2 - 2$

$$u = \frac{g_S}{2\nu} \frac{D - z}{1 - (\tan \phi)^2} , \qquad [S25]$$

$$\tau_z = \frac{gS\rho_f D}{1 - (\tan\phi)^2} , \qquad [S26]$$

where τ_z is the vertical component of the stress. We can see that the exact solution has the same form as the shallow-water approximation with the correction factor $1 - (\tan \phi)^2$. Our approximation to first order yields

$$u = \frac{gS}{2\nu} \left(D^2 - z^2 \right) \left(1 + (\tan \phi)^2 \right) , \qquad [S27]$$

$$\tau_z = gS\rho_f D\left(1 + (\tan\phi)^2\right) , \qquad [S28]$$

where we have expanded the cosine terms in Eq. S22 and S23 to first order in ϕ . Equations S27 and S28 show that our approximation is correct to first order in ϕ . We compare the stress in our approximation to the exact solution, and to the shallow-water stress in Fig. S3d. For $\tan \phi \ge 1$, our approximation breaks down. In fact, the exact solution given by Eqs. S25 and S28 also breaks down at $\tan \phi = 1$. For large angles, the exact solution we presented above is not viable because, in that case, the boundary condition at the far end cannot be ignored. This is an example of a non-local effect we mentioned in the previous section — for $\tan \phi \ge 1$, our approximation breaks down because it can only relate the flow to the local bed shape.

As our second example, we consider the flow in an elliptic channel with a ratio of semi-major to semi-minor axes equal to $R = W/(2D_{\text{max}})$, that we assume to be large (Fig. S3b). The depth profile is given by

$$D(y) = \frac{1}{R} \sqrt{\left(\frac{W}{2}\right)^2 - y^2} , \qquad [S29]$$

133 The exact solution of the Stokes equation in such a channel is

$$u = \frac{gS}{2\nu} \frac{\left(D^2 - z^2\right)R^2}{(1+R^2)}$$
[S30]

$$\tau_z = \frac{gS\rho_f DR^2}{1+R^2} , \qquad [S31]$$

134 Our approximation to first order gives

$$u = \frac{gS}{2\nu} \left(D^2 - z^2 \right) \left(1 - \frac{1}{R^2} \right)$$
[S32]

$$\tau_z = gS\rho_f D\left(1 - \frac{1}{R^2}\right) \tag{S33}$$

¹³⁵ Our approximation provides the correct first order when R is large, but fails when $R \ge 1$. The reason is the same as before — ¹³⁶ in a narrow channel, rather than being controlled by the local depth configuration, the fluid velocity becomes dominated by the ¹³⁷ side walls. We compare our approximation with the exact solution and the shallow-water approximation in Fig. S3e.

Finally, we consider the flow the flow over a sinusoidally perturbed bed (Fig. S3c). The depth profile in this case is given by

$$D(y) = D_0 + \delta \sin(ky) , \qquad [S34]$$

¹³⁹ where δ is the amplitude and k is the wavenumber of the perturbation. If δ is small, the full Stokes equation can be linearized to find the stress. Abramian et al. (6) derived the expression for the stress in this case:

$$\tau_z = gS\rho_f D_0 + gS\rho_f \delta \left(1 - kD_0 \tanh(kD_0)\right) \sin(ky) .$$
[S35]

¹⁴¹ We expect our approximation to work when the wavelength of the perturbation is large compared with the flow depth (i.e. ¹⁴² when kD_0 is small), but should not be limited by δ . Therefore, we can compare our approximation to the result of Abramian ¹⁴³ et al. (6) when both δ/D_0 and kD_0 are small. In this case, our approximation yields:

$$\tau_z = gS\rho_f D_0 + gS\rho_f \delta \left(1 - (kD_0)^2\right) \sin(ky)$$
[S36]

Again, this is the correct first order expansion of Eq. S35 for small kD_0 . In the classical shallow-water theory, the stress, $\tau_z = gS\rho_f D$, is always in phase with the depth perturbation. Conversely, in the linearization of Abramian et al. (6), Eq. S35, the phase of the stress can reverse when the wavelength of the perturbation is small enough, so that the stress maximum is where the flow is shallowest. This feature of the full linearized equation is reproduced in our approximation. This shows that our approximation can capture qualitative effects of the cross-stream diffusion of momentum. We compare our first order approximation, linearization of Abramian et al. (6) (Eq. S35), and the shallow-water approximation for a sinusoidal perturbation in Fig. S3f.

It is clear that our approximation cannot always work. For example, in a rectangular channel, our approximation predicts uniform stress and velocity above the entire flat bottom, to all orders of the approximation. This is clearly not the case in reality, since the fluid must slow down near the vertical walls. The effect of the walls is, again, an example of a non-locality the velocity of the fluid is not only determined by the local depth and its derivatives, but is also affected by the far away walls. Therefore, in all cases we considered above, the failure of our approximation was related to the non-locality of the velocity field.



Fig. S3. Comparison of stress between our first order approximation (blue lines), shallow-water approximation (black dashed lines), and a analytical reference solution (red lines) in examples for which an analytical solution is available. The upper row shows the channel shape, along with the velocity field of the analytical solution (lighter blue stands for faster moving fluid). The lower row shows the normalized stress profile, $\tau(y/L)/\rho_f gSL$, where *L* is a reference length scale that is different for each example. The spatial coordinates, *y* and *z*, in upper and lower panels are also normalized by *L*. (a) and (d) Flow over an inclined surface of an opening angle $\phi = 25^{\circ}$ (Eqs. S24 to S28). The reference length scale in this case can be arbitrarily chosen. (b) and (e) Flow over a surface with a ratio of the semi-major to semi-minor axis R = 1.8 (Eqs. S29 to S33). The reference length scale is the channel width, L = W. (c) and (f) Flow over a surface with a small-amplitude, large-wavelength perturbation, $\delta/D_0 = 0.2$ and $kD_0 = 0.9$ (Eqs. S34 to S36). The reference length scale is the upperturbed channel depth, $L = D_0$.

156 S3. River profile as a function of water and sediment discharges

As we mentioned in the main text, our model is completely determined by setting five parameters — μ_t , λ , L_s , S, and ξ . Of those, μ_t , λ , and L_s are directly measurable in the experiments, while S and ξ depend implicitly on discharges Q_w and Q_s . In this section we describe how we numerically find this relation to get the river profiles as a function of the discharges of water and sediment.

S.3.1. Solving the boundary value problem. To get the river as a function of the discharges, we first need to be able to solve 161 our model, Eq. 13 of the main text, for given values of the parameters μ_t , λ , L_s , S, and ξ . This presents a slight challenge. We 162 described this model as a second order boundary value problem with boundary conditions D = 0 when y = 0, and D' = 0163 when y = -W/2. There are two problems here — first, the second derivative, D", given by Eq. 13 of the main text, diverges 164 when $D \to 0$, and, second, the width of the river, W, is unknown a priori, so the second boundary condition is ill-defined. To 165 deal with the first issue, we expand the solution near D = 0, so that we only consider depths larger than some small value, 166 ϵ . Thus, we change the first boundary condition to $D = \epsilon$ when y = 0 (we can arbitrarily shift the solution along y due to 167 translational invariance). To deal with the second issue, we replace the second boundary condition with a new condition for D'168 that, like the first boundary condition, starts at y = 0. In this way, instead of dealing with a boundary value problem, we can 169 solve a simpler initial value problem. To do this, we need to approximate the slope near the river bank, D', when the depth, 170 $D = \epsilon$, is small. We use the fact that D", which is a known function of D and D', diverges when $D \to 0$. In particular, on the 171 left river bank (when $D = \epsilon$ and D' > 0), $D'' \to \infty$ as $\epsilon \to 0$ for all solutions with D' smaller than that of the river, while 172 $D'' \to -\infty$ for all solutions with D' greater than that of the river. Therefore, for small D, the river approximately lies on 173 the curve D''(D, D') = 0. We can thus invert the relation $D''(\epsilon, D') = 0$ to find D' when y = 0, therefore defining the second 174 initial condition. In summary, we solve the initial value problem, $D = \epsilon$ and $D''(\epsilon, D') = 0$ when y = 0, by forward stepping 175 and we stop when we reach the river center, D' = 0. Once we find the left bank, the right bank follows by symmetry. 176

S.3.2. Interpolating the model. Once we can find a river for given model parameters, we fix μ_t , λ , and L_s to their experimental values and then solve our model (Eq. 13 of the main text) for multiple values of S and ξ to create a grid of solutions for a range of model parameters (Fig. S4a). For each numerical solution in this grid, we can find Q_w and Q_s using Eq. 15 of the main text. Interpolating over this grid, we then find S and ξ (or any other property of the river rivers such as the width or depth) as functions of Q_w and Q_s .

One problem is how to choose the range of S and ξ for the solution grid. Namely, increases in the sediment discharge are 182 controlled by minute differences of $\tilde{\xi}$ (on the order of 10^{-9}) from $\tilde{\xi}_c$, where $\tilde{\xi} \equiv \xi S/L_s$ (see section 5 of the main text). To 183 probe a significant range of sediment discharge, we need to explore ξ in a very narrow range around ξ_c . We do not know the 184 value of $\tilde{\xi}_c$ a priori, and we have to find it numerically. Moreover, the value of $\tilde{\xi}_c$ changes with S, so for each S, we need to 185 independently estimate ξ_c to high precision. Since there are no river-like solutions below ξ_c , we can easily identify whether a 186 solution with given parameters S and ξ has $\tilde{\xi} > \tilde{\xi}_c$ or $\tilde{\xi} < \tilde{\xi}_c$. Therefore, we can find $\tilde{\xi}_c$ by a shooting method. Moreover, in 187 the process of approaching ξ_c by interval halving, we solve our model for multiple values of ξ , so we can immediately use the 188 solutions with $\tilde{\xi} < \tilde{\xi}_c$ in our interpolation grid. This also has the useful property that exponentially approaching $\tilde{\xi}_c$ leads to 189 approximately linearly increasing Q_s . Therefore, by finding ξ_c , we probe the space of sediment discharge more or less uniformly. 190 Once we find ξ_c for a given S, we add the solutions to the interpolation grid, we choose randomly a different S, and we repeat 191 the procedure. 192

To probe the relevant range of S that corresponds roughly to the fluid discharge in the experiments, Q_w , we look at inert rivers ($Q_s = 0$). Using Eq. 15 of the main text, we can write the fluid discharge as

$$Q_w = \frac{gL_s^4}{\nu S^3} \int_{-\tilde{W}/2}^{\tilde{W}/2} \frac{\tilde{D}^3}{3} \mathrm{d}\tilde{y} , \qquad [S37]$$

where tildes stand for quantities made non-dimensional with the length scale, L_s/S , as defined in Eq. 16 of the main text. The term $\tilde{Q}_w = \int \tilde{D}^3/3d\tilde{y}$ is the non-dimensional fluid discharge that depends on the shape of the river, but not its size. As we have discussed in section 5 of the main text, the shape of non-dimensional inert rivers only depends on μ_t . So, for an inert river, $\tilde{Q}_{w,0}(\mu_t)$ is independent of the fluid discharge. Therefore, we find that

$$S_0 = \left(\frac{gL_s^4 \tilde{Q}_{w,0}(\mu_t)}{\nu Q_w}\right)^{1/3} \quad \text{when} \quad Q_s = 0 \;.$$
[S38]

¹⁹⁹ Based on this expression, we estimate the order of magnitude of the experimental slope. Then, taking values between, for ²⁰⁰ example, $S_0/5$ and $5S_0$, ensures that we cover the range S that is relevant for our experiment, even when the sediment discharge ²⁰¹ is finite.

S.3.3. Dependence on water and sediment discharges. In Fig. S5, we show how several river properties depend on the discharges of fluid and sediment in our model. Namely, the aspect ratio increases with the sediment discharge, and only depends on the fluid discharge for large values of the sediment discharge. The shape of an inert river ($Q_s = 0$) is independent of the fluid discharge. The maximum sediment flux, $q_{s,max}$, increases with the sediment discharge and is largely independent of the fluid discharge. In fact, for large fluid discharge, Q_w , the maximum sediment flux saturates at $q_{\mu}(\tilde{D}_{max,0} - \mu_t)$, as predicted by

- 207
- the Parker regime. Figure S5f shows that the downstream slope, S, scales approximately as $Q_w^{-1/3}$ (for an inert river, this scaling is exact, Eq. S38). Therefore, the scale of the river, given by L_s/S roughly increases with fluid discharge as $Q_w^{1/3}$. The 208

sediment discharge affects the slope, S, and the size of the river only slightly (Fig. S5c). 209



Fig. S4. An interpolation grid of solutions used to find model dependence on discharges of fluid and sediment. Each grey point is a solution to our model with different values of S and ξ (and fixed μ_t , λ , L_s). For visual clarity, we only show every fifth point. (a) Interpolation grid in the space of model parameters, ξ and S. The horizontal axis shows the logarithm of the difference between $\tilde{\xi}$ and the limiting value $\tilde{\xi}_c$ which is approximately proportional to the sediment discharge. Solid lines correspond to constant sediment discharge (red lines; numbers denote the value in grains s⁻¹) and fluid discharge (blue lines; numbers denote the value in 1 min^{-1}) found after interpolation. The non-smooth appearance of some of these lines for small and large values of S is due to the sparseness of the grid. (b) Interpolation grid in the space of fluid and sediment discharge, Q_w and Q_s . Each point on this grid has a corresponding point in panel (a). Solid lines correspond to constant aspect ratio (red lines) and width (blue lines; numbers denote the value in cred).



Fig. S5. (a-c) River properties in our model as a function of sediment discharge, Q_s , for various values of fluid discharge, Q_w . (a) Aspect ratio, D_{\max}/W . (b) Maximum sediment flux, $q_{s,\max}/q_{\mu}$. (c) Downstream slope, S. (d-f) River properties in our model as a function of fluid discharge, Q_w , for various values of sediment discharge, Q_s . (d) Aspect ratio, D_{\max}/W . (e) Maximum sediment flux, $q_{s,\max}/q_{\mu}$. (f) Downstream slope, S. The black dashed line corresponds to a curve $S = (Q_w^*/Q_w)^{1/3}$, where $Q_w^* \equiv g L_s^4 \tilde{Q}_{w,0}/\nu \approx 2.6 \times 10^{-7} \, \text{I min}^{-1}$ is the characteristic fluid discharge.

210 S4. Non-dimensional river model

In section *Inert, active, and limiting river* of the main text, we discussed our model in non-dimensional form (Eq. 17 of the main text). In this section, we discuss the mathematical properties of Eq. 17, such as its fixed points and its phase portrait. These properties clarify why an infinite, limiting river exists in our model. In addition, we support claims made in sections *Inert, active, and limiting river* and *The Parker regime* of the main text about inert and infinite rivers.

S.4.1. Fixed points. Since Eq. 17 of the main text is of second order, the second derivative of the depth, \tilde{D}'' , is a function of \tilde{D} and \tilde{D}' . Thus, we can view it as a dynamical system that can be integrated by forward-stepping, starting from a given value of \tilde{D} and \tilde{D}' .

Fixed points of this equation are defined as points in the \tilde{D} - \tilde{D}' space (the phase space), that give constant solutions under integration. This is satisfied when $\tilde{D}' = 0$ and $\tilde{D}'' = 0$. Inserting these conditions into Eq. 17 of the main text yields

$$\tilde{D} = \mu_t + e^{(\tilde{D} - \tilde{\xi})/\tilde{\lambda}} .$$
[S39]

Solving this transcendental equation for \tilde{D} yields the depths of the fixed points. For given values of the parameters μ_t , $\tilde{\lambda}$, and $\tilde{\xi}$, Eq. S39 has zero, one, or two solutions. We find that when $\tilde{\xi} = \tilde{\xi}_{\text{bif}}$, Eq. S39 has only one solution at depth \tilde{D}_{bif} (the subscript "bif" stands for "bifurcation"), where

$$\tilde{\xi}_{\text{bif}} = \mu_t + \tilde{\lambda} (1 - \ln \tilde{\lambda}) , \qquad [S40]$$

$$\tilde{D}_{\rm bif} = \mu_t + \tilde{\lambda} \ . \tag{S41}$$

For $\tilde{\xi} < \tilde{\xi}_{\text{bif}}$, there is no fixed point, while for $\tilde{\xi} > \tilde{\xi}_{\text{bif}}$ there are two (at depths \tilde{D}_1 and \tilde{D}_2 ; brown circle and star in Figs. S6 and S7). These fixed points represent flat solutions that extend to infinity in the \tilde{y} -direction (Fig. S6c).

S.4.2. Phase portrait. We now explore how the shape of the river depends on the model parameters μ_t , $\hat{\lambda}$, and $\hat{\xi}$ by looking at the phase portrait, i.e. the trajectories described by Eq. 17 in the \tilde{D} - \tilde{D}' space (7). In addition, we use the phase portrait to unambiguously show that there exists an infinite, limiting river.

In Fig. S6a, we show an example of a phase portrait for given parameters μ_t , $\tilde{\lambda}$, and $\tilde{\xi}$. Each trajectory in the phase portrait corresponds to a bed profile that satisfies Eq. 17 under different boundary conditions (black lines in Fig. S6a). Depending on the boundary conditions, Eq. 17 has multiple, qualitatively different solutions (Fig. S6d-g). However, for a given set of parameters, μ_t , $\tilde{\lambda}$, and $\tilde{\xi}$, there exists only one channel-like solution of Eq. 17 with two banks where the depth vanishes (blue line in Fig. S6a and b). Such a solution is the only one that can represent a river transporting finite amounts of water and sediment.

In Fig. S7, we show how the phase portrait changes as we change $\tilde{\xi}$, but keep μ_t and $\tilde{\lambda}$ fixed. For $\tilde{\xi} \to \infty$ (Fig. S7a), the river is inert and reaches a maximum depth $\tilde{D}_{\max,0} > \mu_t$, while there exists only one fixed point at $\tilde{D}_1 \to \mu_t$. Decreasing $\tilde{\xi}$ to finite values (Fig. S7b), the second fixed point appears, and the river depth lies between the two points, $\tilde{D}_1 < \tilde{D}_{\max} < \tilde{D}_2$. For a particular value $\tilde{\xi} = \tilde{\xi}_c$ (Fig. S7c), the river solution passes through the second fixed point, $\tilde{D}_{\max,c} = \tilde{D}_2$. Therefore, the river becomes infinite in the $\tilde{y} - \tilde{z}$ space. The value of ξ_c depends on μ_t and $\tilde{\lambda}$, but it exists for any value of these parameters.

Reducing $\tilde{\xi}$ below this critical value, $\tilde{\xi}_c$, changes the phase portrait such that no river solution can exist anymore (Fig. S7d) the river solution does not exist since the two banks (solutions starting at $\tilde{D} = 0$) do not join at the center. Reducing $\tilde{\xi}$ even further, below $\tilde{\xi}_{\text{bif}}$ (Fig. S7e), the two fixed points merge and disappear so that there are neither river nor fixed point solutions. We show the positions of the fixed points, \tilde{D}_1 and \tilde{D}_2 , and the river depth, \tilde{D}_{max} , as a function of $\tilde{\xi}$ in the bifurcation diagram, Fig. S7f. There, we can see that, as $\tilde{\xi}$ decreases from $\tilde{\xi} \to \infty$ to finite values, the river depth increases until it meets with \tilde{D}_2 at $\tilde{\xi} = \tilde{\xi}_c$.

The existence of $\tilde{\xi}_c$ implies a limiting flux $q_{s,c}$ of the river (section *Inert, active, and limiting river* of the main text). By the same reasoning, one would be tempted to conclude that the existence of $\tilde{\xi}_{bif}$ indicates a maximum sediment flux for a flat bed (that could be realized, for example, in a flume experiment). This is, however, not true — when $\tilde{\xi} \to \infty$, the fixed points at $\tilde{D}_1 = \mu_t$ and $\tilde{D}_2 \to \infty$ have sediment fluxes $q_{s,1} = 0$ and $q_{s,2} \to \infty$. Decreasing $\tilde{\xi}$ to finite values, $q_{s,1}$ increases while $q_{s,2}$ decreases until they meet, thereby covering the entire range of possible sediment flux values from 0 to ∞ . A flat bed in a flume can, therefore, carry any sediment flux, at least in principle. In practice, however, some of these solutions may become unstable to perturbations (6).

S.4.3. Inert river. In the Parker and weak transport regimes, the shape of the river is determined by that of an inert river. We cannot find the properties of an inert river in our model analytically; instead, to find its profile, $\tilde{D}_0(\tilde{y})$, we have to solve Eq. 17 of the main text with $\tilde{\xi} \to \infty$:

$$\sqrt{\left(\tilde{D}_0 + \frac{1}{3}(\tilde{D}_0^3)''\right)^2 + \tilde{D}_0'^2 - \mu_t} = 0.$$
[S42]

The only parameter in this equation is μ_t , so the inert river shape only depends on this friction coefficient. In Fig. S8, we show how its properties — depth, $\tilde{D}_{\max,0}$, width, \tilde{W}_0 , non-dimensional fluid discharge, $\tilde{Q}_{w,0}$, and aspect ratio, $\tilde{W}_0/\tilde{D}_{\max,0}$ depend on μ_t (blue lines in Fig. S8).

Predrag Popović, Olivier Devauchelle, Anaïs Abramian, and Eric Lajeunesse

We can understand how these properties generally depend on μ_t by comparing them to the inert, shallow-water river whose shape can be found analytically (black dashed lines in Fig. S8). After neglecting the momentum diffusion, Eq. S42 becomes

$$\sqrt{\tilde{D}_{\rm sw,0}^2 + \tilde{D}_{\rm sw,0}^{\prime 2}} - \mu_t = 0 , \qquad [S43]$$

where $\tilde{D}_{sw,0}$ is the depth of an inert shallow-water river. This equation has a simple solution (5)

$$\tilde{D}_{\rm sw,0} = \mu_t \cos \tilde{y} \;. \tag{S44}$$

From here, we find that the shallow-water inert river has $\tilde{D}_{\max,sw,0} = \mu_t$, $\tilde{W}_{sw,0} = \pi$, $\tilde{Q}_{w,sw,0} = 4\mu_t^3/9$, and an aspect ratio of $\tilde{W}_{sw,0}/\tilde{D}_{\max,sw,0} = \pi/\mu_t$. These values represent bounds for our model — Fig. S8 shows that the inert river in our model has a depth greater than μ_t , a width greater than π , a non-dimensional fluid discharge greater than $4\mu_t^3/9$, and an aspect ratio greater than π/μ_t . For small μ_t , our model approaches the shallow-water inert river, while, as we increase μ_t , the non-dimensional inert river in our model becomes less and less like the shallow-water one.

S.4.4. Limiting river depth, $\tilde{D}_{\max,c}$. In the main text, we noted that the value of the non-dimensional depth of an infinite, flat river, $\tilde{D}_{\max,c}$, depends on parameters μ_t and $\tilde{\lambda}$ (section *Inert, active, and limiting river* of the main text), and we explained that for vanishing $\tilde{\lambda}$, $\tilde{D}_{\max,c}$ equals the inert river depth, $\tilde{D}_{\max,0}(\mu_t)$ (section *The Parker regime* of the main text). In Fig. S9, we show how $\tilde{D}_{\max,c}(\tilde{\lambda}, \mu_t)$ depends on $\tilde{\lambda}$ with μ_t fixed. There, we numerically show that, for small $\tilde{\lambda}$, the depth $\tilde{D}_{\max,c}$ behaves as

$$\tilde{D}_{\max,c}(\tilde{\lambda},\mu_t) \approx \tilde{D}_{\max,0}(\mu_t) + \tilde{\lambda} \quad \text{for} \quad \tilde{\lambda} \ll 1 .$$
 [S45]

For experimental parameters ($\mu_t = 0.9$ and $\tilde{\lambda} = 0.02$), we find that the relative error between $\tilde{D}_{\max,c}$ and Eq. S45 is about

For experimental parameters ($\mu_t = 0.9$ and $\lambda = 0.02$), we find that the relative error between $D_{\text{max},c}$ and Eq. 545 is about 0.06%. Although we do not understand exactly why this relationship holds, we found it to be true for all the values of μ_t we tested.



Fig. S6. (a) A phase portrait of Eq. 17 of the main text for $\mu_t = 0.9$, $\tilde{\lambda} = 0.1$, and $\tilde{\xi} = 1.37$. The horizontal axis is the non-dimensional depth, \tilde{D} , and the vertical axis is its \tilde{y} -derivative, \tilde{D}' . Black dashed curves mark the boundaries of the region of \tilde{D} - \tilde{D}' space on which Eq. 17 is well-defined. Black trajectories represent solutions to Eq. 17 for different initial conditions. The blue line represents the river solution shown in panel (b), while the blue dot and blue squares represent the river center and banks. The river solution is unique and acts as a separatrix between two regions of qualitatively different solutions in the phase space (streamwise streaks and underwater channels). Brown circle and brown star represent the two fixed points at depths \tilde{D}_1 and \tilde{D}_2 . Brown lines are solutions ending in the second fixed point and separate regions of the phase space with qualitatively different solutions. A constant depth solution corresponding to the first fixed point, \tilde{D}_1 , is shown in panel (c). Different colored shadings are regions of the phase space with qualitatively different solutions. An example of a solution from each of these regions is marked with a colored dotted line and shown in panels (d)-(g).



Fig. S7. (a)-(e) Phase portraits of Eq. 17 of the main text for $\mu_t = 0.9$, $\tilde{\lambda} = 0.1$, and varying $\tilde{\xi}$. Notation is the same as in Fig. S6. (a) Inert river $(\tilde{\xi} \to \infty)$. There exists only one fixed point at $\tilde{D}_1 = \mu_t$. (b) Active river $(\tilde{\xi} = 1.37 > \tilde{\xi}_c)$. River solution passes between the two fixed points. (c) Infinite, limiting river $(\tilde{\xi} = 1.3237 \approx \tilde{\xi}_c)$. The river solution reaches a maximum depth at the second fixed point, $\tilde{D}_{\max,c} = \tilde{D}_2$. (d) No river solution, but fixed points still exist ($\tilde{\xi} = 1.3 < \tilde{\xi}_c$). The blue line shows the solutions starting at $\tilde{D} = 0$. These solutions do not represent a river since they do not join at the center. (e) No river or fixed point solutions ($\tilde{\xi} = 1.2 < \xi_{\text{bif}} \approx 1.23$). The two fixed points merge and disappear. (f) The bifurcation diagram for $\mu_t = 0.9$ and $\tilde{\lambda} = 0.1$. The brown lines represent the depths of the fixed points as a function of $\tilde{\xi}$ – lower branch corresponds to the first fixed point, \tilde{D}_1 (brown circle in panels a-e), while the upper branch corresponds to the second fixed point, \tilde{D}_2 (brown star in panels a-e). When $\tilde{\xi} \to \infty$, the first fixed point approaches the friction coefficient ($\tilde{D}_1 \to \mu_t$), while the second tends to infinity ($\tilde{D}_2 \to \infty$). The two fixed points $(\tilde{\xi}, \tilde{D}) = (\tilde{\xi}_{\text{bif}}, \tilde{D}_{\text{bif}})$ (brown dot; $\tilde{\xi}_{\text{bif}} \approx 1.23$, $\tilde{D}_{\text{bif}} = 1.2$, $\tilde{D}_{\text{bif}} = 1.2$, $\tilde{D}_{\text{bif}} = 1.2$, $\tilde{D}_{\text{bif}} = 0$. The second fixed point at $(\tilde{\xi}, \tilde{D}) = (\tilde{\xi}_c, \tilde{D}_{\text{max},c})$ (blue star; $\tilde{\xi}_c \approx 1.3237$, $\tilde{D}_{\text{max},c} \approx 1.21$). At this point the river is flat and infinitely wide.



Fig. S8. Properties of non-dimensional inert rivers as a function of the friction coefficient, μ_t . The blue lines correspond to our model (Eq. S42), while the black dashed lines correspond to the shallow-water inert river (Eq. S44). (a) Difference between non-dimensional depth and the friction coefficient, $\tilde{D}_{max,0} - \mu_t$. Black dashed line corresponds to $\tilde{D}_{max,0} - \mu_t = 0$. (b) Non-dimensional width, \tilde{W}_0 . Black dashed line corresponds to $\tilde{W}_0 = \pi$. (c) Non-dimensional fluid discharge, $\tilde{Q}_{w,0} \equiv \int \tilde{D}^3/3 d\tilde{y}$. Black dashed line corresponds to $\tilde{W}_0/\tilde{D}_{max,0} = \pi/\mu_t$.



Fig. S9. The dependence of limiting river depth, $\tilde{D}_{max,c}$, on $\tilde{\lambda}$. The vertical axis shows the difference between the limiting and inert river depth, $\tilde{D}_{max,c} - \tilde{D}_{max,0}$, while the horizontal axis is the value of $\tilde{\lambda}$ ($\mu_t = 0.9$ is fixed). The blue line results from numerical solution of Eq. 17 of the main text. The black dashed line represents the function $\tilde{D}_{max,c} - \tilde{D}_{max,0} = \tilde{\lambda}$. The red shading represents the estimate and the variability of $\tilde{\lambda}$ observed in the experiments.

274 S5. Parker regime

The Parker regime corresponds to a river with no sediment diffusion that splits its channel into inert banks and a flat, active bottom (section *The Parker regime* of the main text). Such a river has a rectangular sediment flux profile of width $W_T^{(P)}$ and height $q_s^{(P)}$.

In Fig. S10, we show that the Parker regime is, in fact, the limit of our model when $\lambda \to 0$. The sediment flux profile of Eq. 278 13 of the main text approaches the rectangular shape of the Parker regime as λ decreases. This is to be expected — when the 279 sediment diffusion length scale, λ , vanishes, the sediment flux, $q_s = q_\mu \exp[(D-\xi)/\lambda]$, also vanishes for $D < \xi$, and becomes 280 infinite for $D > \xi$. Therefore, to have a meaningful solution when $\lambda \to 0$, the river bottom must be flat with a depth, $D_{\max} = \xi$. 281 Physically, vanishing sediment diffusion means that gravity pulls each moving grain of sediment towards the river bottom from 282 which it cannot escape by random collisions with the bed. When λ is finite, the region over which inert banks transition to the 283 flat bottom always has a finite size of the order λ . When the water and sediment discharges are large, both the bank width, 284 W_0 , and the transport width, W_T , are large too, so we can neglect the transition region: this is the gist of the Parker regime. 285 The depth, sediment flux, transport width, and total width of a river in the Parker regime are (as we explained in section 286

²⁸⁷ The Parker regime of the main text):

$$D_{\max}^{(P)} = \frac{L_s}{S^{(P)}} \tilde{D}_{\max,0} \quad , \quad q_s^{(P)} = q_\mu (\tilde{D}_{\max,0} - \mu_t) \quad , \quad W_T^{(P)} = \frac{Q_s}{q_s^{(P)}} \quad , \quad W^{(P)} = \frac{L_s}{S^{(P)}} \tilde{W}_0 + \frac{Q_s}{q_s^{(P)}} \quad .$$
 [S46]

To get the above quantities, we need to find the downstream slope, $S^{(P)}$. This follows from the fluid discharge constraint. According to Eq. 15 of the main text, the fluid discharge is

$$Q_w = \frac{gS^{(P)}}{\nu} \int_{-W/2}^{W/2} \frac{D^3}{3} dy = Q_{w,0} + \frac{gS^{(P)}}{\nu} \frac{D^{(P)3}_{\max}}{3} W_T^{(P)} , \qquad [S47]$$

where $Q_{w,0} \equiv \int_{-W_0/2}^{W_0/2} \frac{D_0^3}{3} dy$ is the contribution from the inert banks. A dimensional fluid discharge, Q_w , can be related to the non-dimensional one, $\tilde{Q}_w = \int \frac{\tilde{D}^3}{3} d\tilde{y}$, through

$$Q_w = \frac{gL_s^4}{\nu S^3} \tilde{Q}_w , \qquad [S48]$$

²⁹² so that Eq. S47 becomes

$$Q_w = \frac{gL_s^4}{\nu S^{(P)3}} \tilde{Q}_{w,0} + \frac{gS^{(P)}}{\nu} \frac{D_{\max}^{(P)3}}{3} W_T^{(P)} , \qquad [S49]$$

where $\tilde{Q}_{w,0} \equiv \int_{-\tilde{W}_0/2}^{\tilde{W}_0/2} \frac{\tilde{D}_0^3}{3} d\tilde{y}$ depends only on μ_t . Combining Eq. S49 with Eqs. S46, we retrieve Eq. 24 of the main text

$$Q_w = \frac{gL_s^4}{\nu S^{(P)3}} \left(\tilde{Q}_{w,0} + \frac{Q_s S^{(P)} \tilde{D}_{\max,0}^3}{3q_\mu L_s (\tilde{D}_{\max,0} - \mu_t)} \right) .$$
[S50]

Depending on the physical parameters that enter it, Eq. S50 can have multiple solutions for $S^{(P)}$, but there is always only one positive, real solution.



Fig. S10. Sediment flux profile of a river in our model with $Q_s = 30$ grains s⁻¹ and $Q_w = 1 | \min^{-1}$ and the varying sediment diffusion length scale, λ . The blue lines are the numerical sediment flux profiles for different values of λ , while the black dashed line is the Parker regime (Eqs. S46) in the limit $\lambda \to 0$.

296 S6. Estimating sediment discharge based on river geometry

²⁹⁷ Our model predicts a link between the river geometry and its sediment load. Here, we will use our experimental dataset to ²⁹⁸ show how to estimate the sediment discharge of a river from the shape of its cross-section.

Equation 22 of the main text relates the sediment discharge to river depth and width, assuming we know the universal quantities, such as the limiting sediment flux and the inert river aspect ratio $(q_s^{(P)} \text{ and } \tilde{W}_0/\tilde{D}_{\max,0} \text{ in Eq. 22})$. Although these universal quantities follow from the theory, various approximations we made may make them inaccurate. Moreover, for a natural river, for which the theory is not yet available, these quantities are unknown a priori. On an ensemble of rivers with different water and sediment discharges, $q_s^{(P)}$ is the maximum allowed sediment flux while $\tilde{W}_0/\tilde{D}_{\max,0}$ is the minimum allowed aspect ratio. Therefore, instead of using our theoretical predictions, here we estimate $q_s^{(P)}$ and $\tilde{W}_0/\tilde{D}_{\max,0}$ as the maximum sediment flux and minimum aspect ratio from our experimental dataset. Thus, we estimate $q_s^{(P)}|_{\exp} \approx 27.4$ grains cm⁻¹s⁻¹ and $\tilde{W}_0/\tilde{D}_{\max,0}|_{\exp.} \approx 4.3$. Using these values, we show in Fig. S11 that the sediment discharge estimated using Eq. 22 of the main text falls within the uncertainty range of the measurements.



Fig. S11. Comparing the measured sediment discharge (x-axis) to that estimated using Eq. 22 of the main text (y-axis). Black dashed line is a one-to-one line.

308 S7. Effects of momentum diffusion

In section *The Parker regime* of the main text, we claimed that momentum diffusion is essential for the existence of a steady-state channel with a non-vanishing sediment discharge, as first suggested by Parker (8). In this section, we will first illustrate this statement by comparing our model to the shallow-water river without momentum diffusion. Then, we will investigate the effect of momentum diffusion in detail by considering the forces acting on the river bed.

5.7.1. Comparison with shallow-water rivers. The shallow-water model, which neglects the cross-stream diffusion of momentum entirely, is defined by a first order ordinary differential equation:

$$\sqrt{\tilde{D}_{\rm sw}^2 + \tilde{D}_{\rm sw}^{\prime 2}} - \mu_t = e^{(\tilde{D}_{\rm sw} - \tilde{\xi})/\tilde{\lambda}} .$$
[S51]

When $\tilde{\xi} \to \infty$, the right-hand side vanishes and the model reduces to Eq. S43 for an inert shallow-water river, which has been successfully used to predict the shape of experimental inert rivers (5). We compare the inert shallow-water river with the experiment and our model in Figs. S12a, c, and e. There, we can see that, while there exists some difference between the models and the experiment, they are largely comparable to each other.

However, when sediment transport is included ($\xi < \infty$), this model is insufficient to reproduce the experiments. We compare an active experimental river with our model and the shallow-water model in Figs. S12b, d, and f. We can see that, while our model and the experiment seem comparable to each other, the shallow-water model is nothing alike. This result may seem somewhat paradoxical — although we expect that the shallow-water approximation should work better for a wider river, we find that the wider the river, the worse the shallow-water approximation is. We resolve this apparent paradox in the next section.

S.7.2. Momentum diffusion in the Parker regime. In this section, we look closely at the components of the force ratio, μ , to understand the role of momentum diffusion. We explain that momentum diffusion controls the sediment transport through a matching condition at the interface of the river banks and the flat bottom — a condition that does not depend on the width of the river.

Figure S13a shows a river in the Parker regime (in non-dimensional coordinates, for convenience) — it is sharply split into curved, inert banks, and a flat, active bottom. Figure S13b shows how the various components of the force ratio, μ , depend on the position within this river. These components are

$$\mu_{\rm sw} \equiv \tilde{D}$$
, [S52]

$$\mu_{\rm md} \equiv \frac{1}{3} (\tilde{D}^3)'' ,$$
 [S53]

$$\mu_{\rm g} \equiv \tilde{D}'$$
, [S54]

$$\mu^2 = (\mu_{\rm sw} + \mu_{\rm md})^2 + \mu_{\rm g}^2 , \qquad [S55]$$

where μ_{sw} is the shallow-water contribution, μ_{md} is the momentum diffusion contribution, and μ_{g} is the gravity contribution to 332 the total force ratio. The banks are kept at threshold by the combined action of the shallow-water stress, momentum diffusion, 333 and gravity, while the bottom only feels the shallow-water stress. At the point where banks and flat bottom connect the banks 334 have a depth $\tilde{D} = \tilde{D}_{\max}$, a slope $\tilde{D}' = 0$, and a second derivative $\tilde{D}'' = -\tilde{\kappa} < 0$, where $\tilde{\kappa}$ is the curvature of the banks at 335 their deepest point. The components of the force ratio are, correspondingly, $\mu_{\rm sw} = \tilde{D}_{\rm max}$, $\mu_{\rm g} = 0$, $\mu_{\rm md} = -\tilde{D}_{\rm max}^2 \tilde{\kappa} < 0$. The 336 threshold condition for the banks is, thus, $\mu_{sw} + \mu_{md} = \mu_t$, which means that the depth must be greater than the friction 337 coefficient, $\tilde{D}_{\text{max}} = \mu_{\text{sw}} = \mu_t - \mu_{\text{md}} > \mu_t$. The bottom feels only the shallow-water stress, and, therefore, must be above the 338 threshold, $\mu|_{\text{bottom}} = \mu_{\text{sw}} = \ddot{D}_{\text{max}} > \mu_t$. 339

Therefore, the absence of sediment transport on the banks translates into excess stress on the flat bottom. The sediment flux on the bottom, $q_s/q_{\mu} = \mu|_{\text{bottom}} - \mu_t = -\mu_{\text{md}}$, corresponds to the jump in the momentum diffusion contribution due to the sudden disappearance of curvature. Thus, the sediment flux is driven by the boundary condition at the junction of the curved banks with the flat bottom.

Written in terms of depth and its derivatives, the sediment flux is $q_s/q_\mu = \tilde{D}_{\max}^2 \tilde{\kappa}$. The curvature, $\tilde{\kappa} \equiv -\tilde{D}''$, scales roughly as $\tilde{\kappa} \sim \tilde{D}_{\max,0}/\tilde{W}_0^2$, so the sediment flux scales as $q_s/q_\mu \sim \tilde{D}_{\max,0}^3/\tilde{W}_0^2$. Taking the depth, $\tilde{D}_{\max,0}$, to be of order one, the sediment flux is inversely proportional to the square of the aspect ratio of an inert river, $W_0^2/D_{\max,0}^2$. Parker (8) likewise found that the distance to threshold is inversely proportional to the aspect ratio squared — a signature of the cross-stream diffusion of momentum.

Without cross-stream momentum diffusion, there can be no sediment transport in the limit $\lambda \to 0$ in our model. When λ is small but non-vanishing, the maximal sediment flux is $q_{s,c}/q_{\mu} \approx \tilde{D}_{\max,0}(\mu_t) - \mu_t + \tilde{\lambda}$, where we used Eq. S45 for the depth of the infinite, limiting river, $\tilde{D}_{\max,c}$. The first term in this equation, $\tilde{D}_{\max,0}(\mu_t) - \mu_t$, is the contribution to sediment transport from the momentum diffusion, while the second term, $\tilde{\lambda}$, is the contribution from the sediment diffusion. For experimental parameters ($\mu_t = 0.9$ and $\tilde{\lambda} = 0.02$), we thus find that momentum diffusion is responsible for about 90% of the sediment transport.



Our model



Shallow water



Fig. S12. River profiles in the experiment (panels (a) and (b)), our model (panels (c) and (d)), and shallow-water approximation (panels (e) and (f)). Profiles in the left column (panels (a), (c), and (e)) are inert rivers, while profiles in the right column (panels (b), (d), and (f)) are active rivers with $Q_s = 24$ grains s⁻¹. Red lines in panels (b), (d), and (f) represent the sediment flux profile where the zero, $q_s = 0$, is vertically shifted in the plots to coincide with the river bottom. While inert rivers in both our model and the shallow-water model are comparable with experiments, momentum diffusion is necessary to capture the shape of active rivers.

(a) Bank	Bottom
$\tilde{D}^{\prime\prime} \neq 0$	$\tilde{D}^{\prime\prime}=0$
$q_s = 0$	$q_s > 0$



Fig. S13. Explanation of the mechanism of sediment transport by momentum diffusion. (a) Half a river in our model for $\lambda \to 0$ and $\mu_t = 0.9$. The river sharply splits into curved, inert banks and a flat, active bottom. (b) Components of the force acting on the bed as a function of the position along the bed. The blue line represents the total force ratio, $\mu = \sqrt{(\mu_{sw} + \mu_{md})^2 + \mu_g^2}$. The dashed brown line represents the shallow-water contribution to the stress, $\mu_{sw} = \tilde{D}$. The solid brown line represents the momentum diffusion contribution to the stress, $\mu_{md} = (\tilde{D}^3)''/3$. The dotted brown line represents the gravity contribution, $\mu_g = \tilde{D}'$. On the banks, all three contributions are non-vanishing and keep the banks at the threshold, $\mu = \mu_t$. On the bottom, only the shallow-water contribution exists and the force ratio is above threshold, $\mu > \mu_t$. The shallow-water and gravity contributions transition continuously from the banks to the bed, but the momentum diffusion contribution experiences a jump which corresponds to a jump in the force ratio, μ . This jump drives sediment transport.

355 References

- A Abramian, O Devauchelle, and E Lajeunesse. Laboratory rivers adjust their shape to sediment transport. *Physical Review E*, 102(5):053101, 2020.
- François Charru, Hélene Mouilleron, and Olivier Eiff. Erosion and deposition of particles on a bed sheared by a viscous flow. Journal of Fluid Mechanics, 519:55–80, 2004.
- 3. Anaïs Abramian. Self-organization of sediment transport in alluvial rivers. PhD thesis, 2018.
- 4. Olivier Devauchelle, Eric Lajeunesse, and Popović Predrag. Viscous transfer of momentum across a shallow laminar flow. *submitted to Journal of Fluid Mechanics*, 2021.
- G Seizilles, O Devauchelle, E Lajeunesse, and F Métivier. Width of laminar laboratory rivers. *Physical Review E*, 87(5):
 052204, 2013.
- 6. Anaïs Abramian, Olivier Devauchelle, and Eric Lajeunesse. Streamwise streaks induced by bedload diffusion. 2019.
- Steven H Strogatz. Nonlinear dynamics and chaos with student solutions manual: With applications to physics, biology, chemistry, and engineering. CRC press, 2018.
- 8. Gary Parker. Self-formed straight rivers with equilibrium banks and mobile bed. part 2. the gravel river. Journal of Fluid mechanics, 89(1):127-146, 1978.