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Alexis Duchesne, Laurent Limat. Circular hydraulic jumps: where does surface tension matter?. Journal of Fluid Mechanics, 2022, 937 (R2), 11 p. 10.1017/jfm.2022.136 . hal-03495814

HAL Id: hal-03495814

<https://hal.science/hal-03495814>

Submitted on 20 Dec 2021

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Circular hydraulic jumps: where does surface tension matter?

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(Received xx; revised xx; accepted xx)

Recently, an unusual scaling law has been observed in circular hydraulic jumps and has been attributed to a supposed missing term in the local energy balance of the flow [Bhagat *et al.* (2018)]. In this paper, we show that - though the experimental observation is valuable and interesting - this interpretation is presumably not the good one. When transposed to the case of a axial sheet formed by two impinging liquid jets, the assumed principle leads in fact to a velocity distribution in contradiction with the present knowledge for this kind of flows. We show here how to correct this approach by keeping consistency with surface tension thermodynamics: for Savart-Taylor sheets, when adequately corrected, we recover the well known $1/r$ liquid thickness with a constant and uniform velocity dictated by Bernoulli's principle.

In the case of circular hydraulic jumps, we propose here a simple approach based on Watson description of the flow in the central region [Watson (1964)], combined with appropriate boundary conditions on the formed circular front. Depending on the specific condition, we find in turn the new scaling by Bhagat *et al.* (2018) and the more conventional scaling law found long ago by Bohr *et al.* (1993). We clarify here a few situations in which one should hold rather than the other, hoping to reconcile Bhagat *et al.* observations with the present knowledge of circular hydraulic jump modeling. However, the question of a possible critical Froude number imposed at the jump exit and dictating logarithmic corrections to scaling remains an opened and unsolved question.

1. Introduction

Stationary axisymmetrical liquid structures formed by jet impacts, have motivated an enormous amount of literature. Three examples that will be important here are sketched on Fig. 1. First of all, the well-known circular hydraulic jump [Rayleigh (1914); Tani (1949); Watson (1964); Craik *et al.* (1981); Bohr *et al.* (1993); Bush & Aristoff (2003); Duchesne *et al.* (2014); Mohajer & Li (2015); Salah *et al.* (2018); Bhagat *et al.* (2018); Wang & Khayat (2019, 2021)], sketched on Fig.1-a, with a well developed liquid film extending all around. Its equivalent on a "dry" surface, possibly superhydrophobic [Jameson *et al.* (2010); Button *et al.* (2010); Maynes *et al.* (2011)], the "rim atomization" is sketched on Fig.1-b. Finally the

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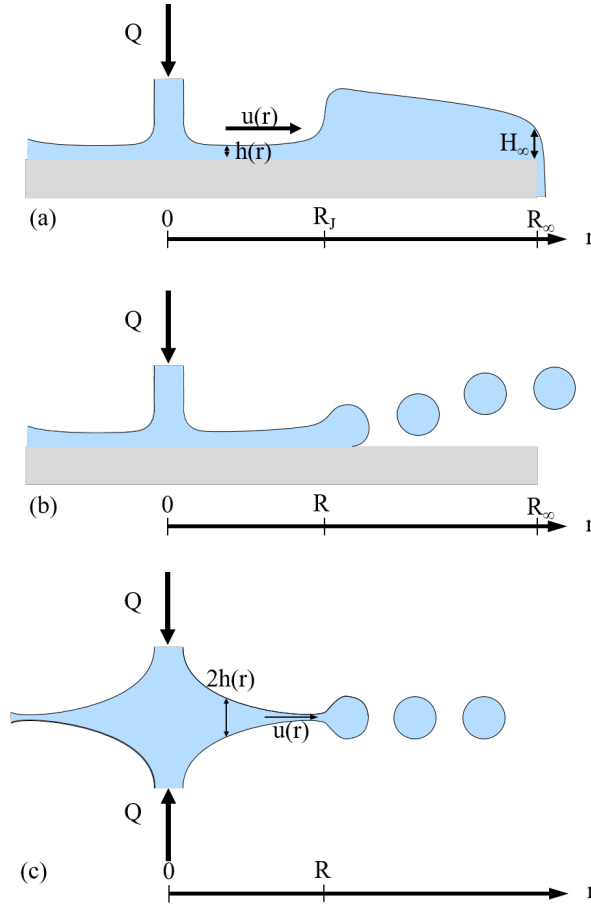


Figure 1: Three axisymmetric film flows are discussed in the present article. (a) The classical circular hydraulic jump formed by a jet impacting a solid disk at its center, (b) atomization ring formed by a jet impacting a dry surface possibly superhydrophobic, (c) liquid sheet formed by impact of two liquid jets of opposite direction.

37 well-known radial liquid sheet [Savart (1833); Huang (1970); Clanet & Villermaux (2002);
 38 Villermaux *et al.* (2013)], formed either by impinging two opposite symmetrical liquid jets,
 39 having the same central axis or by impinging a liquid jet on a solid surface with a diameter
 40 similar to the jet diameter is depicted on Fig.1-c.

41 These three geometries are of course linked together by the same general equation for the
 42 energy balance. In this article, we will therefore show that apparent paradoxes raised by the
 43 modeling of the surface tension on the circular hydraulic jump by Bhagat *et al.* (2018) may
 44 be solved or at least clarified by considering the geometry depicted in Fig.1-c.

45 The selection of jump radius R_J in the circular hydraulic jump case (Fig.1-a) has motivated
 46 many studies. The two most well known approaches are the one from Watson and Bush
 47 [Watson (1964); Bush & Aristoff (2003)], in which the height of the outer film remains a
 48 control parameter, and the one from Bohr *et al.* (1993), rather devised when a liquid film
 49 extends all around at large distance, and inspired from boundary layer theories. As well
 50 known, this second approach leads to a scaling law dependence of R_J upon flow rate Q and
 51 the physical parameters (ν the kinematic viscosity of the fluid, g gravity), that reads:

52
$$R_J \sim Q^{\frac{5}{8}} \nu^{-\frac{3}{8}} g^{-\frac{1}{8}}. \quad (1.1)$$

53 Later, Duchesne *et al.* (2014) emphasized the importance of logarithmic corrections to
 54 scaling, due to viscous dissipation in the outer film, yet observed numerically by Bohr, and
 55 also showed that the prefactor was experimentally linked to the value of the Froude number
 56 at the exit of the jump, that seemed to be locked to a critical value. This phenomenon was
 57 recovered by Mohajer & Li (2015) and by Argentina *et al.* (2017) with a non-linear modeling
 58 of film flow equations including the first finite slope terms.

59 Very recently, an attempt of revision of this picture has been published by Bhagat *et al.*
 60 (2018), who performed new experiments, and reported the observation of a different scaling
 61 in which surface tension γ was involved, but not gravity:

62
$$R_J \sim Q^{\frac{3}{4}} \rho^{\frac{1}{4}} \nu^{-\frac{1}{4}} \gamma^{-\frac{1}{4}}, \quad (1.2)$$

63 in which ρ is the liquid density. To rationalize this finding, these authors claimed that
 64 most available approaches of surface tension influence lead to only small corrections [Bush
 65 & Aristoff (2003)] and that the description of the circular hydraulic jump had thus to be
 66 completely reconsidered. They introduced an energy balance, between two radii r and $r + \delta r$,
 67 that reads:

68
$$\left[\rho \frac{\bar{u}^2}{2} \bar{u} r h \right]_r^{r+\delta r} = [\gamma r \bar{u}]_r^{r+\delta r} - [p \bar{u} h]_r^{r+\delta r} - \left[\rho g \frac{h^2}{2} r \bar{u} \right]_r^{r+\delta r} - r \tau_w \bar{u} \delta r, \quad (1.3)$$

69 with the notation $[A]^{r+\delta r} = A(r + \delta r) - A(r)$, and in which \bar{u} designates the flux-average
 70 radial velocity, r the distance to the axis, $h(r)$ the thickness of the liquid layer, $p(r)$ the
 71 pressure at $z = 0$ and τ_w the wall shear stress. The last term on the right designates the
 72 viscous dissipation by friction on the substrate, while the first one is an additional term
 73 compared to previous approaches, that is presumed to be ‘‘at the origin’’ of the new scaling
 74 (1.2). This conjecture has been contested [Duchesne *et al.* (2019); Bohr & Scheichl (2021)]
 75 (see also Bhagat & Linden (2020) answer), and it is also known that a scaling like (1.2) can
 76 also appear without such an assumption, as shown for instance by Button *et al.* (2010) for
 77 liquid bells formed below a ceiling.

78 We find here useful to have a look on what would happen in the simplified geometry of
 79 Fig. 1-c, when applying this principle. As we shall show in section 2, this modeling leads to
 80 a velocity distribution in complete contradiction with the present knowledge of liquid sheets
 81 (and with Bernoulli’s principle), which suggests that Bhagat *et al.* argument is flawed. In fact
 82 the obtained flow field is not new, and has been proposed in the past by Bouasse (1923)
 83 who attributed the calculation to Hagen (see Hagen (1849)). It will be instructive here to
 84 remind the argument followed by Hagen and Bouasse, in section 3, in a Lagrangian frame,
 85 analyzing a circular expanding piece of film. We will then show, in the same section, how
 86 one can correct the argument to get the more classical and now admitted result deduced
 87 from Bernoulli’s principle of a uniform radial velocity around the impact point, and how,
 88 missing some terms in the balance, one can get the flawed result of Bouasse and Hagen.
 89 Finally, coming back to a Eulerian description, we will explain how these considerations
 90 impact the principle proposed in eq.(1.3). We will show that an extra term exactly equal
 91 and opposite to the capillary contribution should cancel this one, in a way consistent with
 92 classical thermodynamics, leading to the expression usually written from the balance of
 93 momentum.

94 This does not mean, however, that Bhagat *et al.* scaling discovery is of none interest. In
 95 section 4 and 5, we will try to precise to which capillary structures – different from the

96 stationary hydraulic jump observed by Bohr – it could apply, and a possible way to justify its
97 occurrence.

98 2. A look to a simple situation: the axisymmetrical liquid sheet.

99 Let us try to apply the principle suggested in eq.(1.3) to the case suggested on Fig.1-c, i.e. to
100 a axisymmetrical sheet formed by the coaxial impact of two jets in a situation of negligible
101 gravity. The viscous shear on the substrate having disappeared, eq.(1.3) reduces to a very
102 simple balance that reads:

$$103 \quad \left[\rho r h \frac{u^3}{2} \right]_r^{r+\delta r} = [\gamma r u]_r^{r+\delta r}, \quad (2.1)$$

104 where the horizontal velocity u has no dependence upon the transverse direction, and
105 coincides with any of its average values. This implies that the following quantity is constant
106 all over the sheet:

$$107 \quad \rho r h \frac{u^3}{2} - \gamma r u = \text{Cte.} \quad (2.2)$$

108 Combined with the mass balance $Q = 2\pi r h u$, this leads to the following expression for u :

$$109 \quad u = 2\pi \frac{\gamma}{\rho Q} r + \sqrt{u_0^2 - 4\pi \frac{\gamma}{\rho Q} r_0 u_0 + 4\pi^2 \frac{\gamma^2}{\rho^2 Q^2} r^2}, \quad (2.3)$$

110 in which r_0 designates the jet radius at impact and u_0 the asymptotic value for u , reached
111 when $r = r_0$, which satisfies the equality $Q = \pi r_0^2 u_0$ in a quasi-elastic shock approximation
112 [Villermaux *et al.* (2013)]. In the limit of large jet velocity, i.e. $u_0^2 \gg 2\gamma/(\rho r_0)$, this expression
113 reduces to the slowly varying upon r approximate:

$$114 \quad u \approx u_0 + 2\pi \frac{\gamma}{\rho Q} (r - r_0), \quad (2.4)$$

115 which is known to be false, as it has been checked experimentally that the velocity is
116 constant all over the sheet, recovering the Bernoulli's principle (see in particular Fig. 3 in
117 [Villermaux *et al.* (2013)]). It is however amazing to remind that a similar expression is
118 proposed by Bouasse (1923) who attributed this result to Hagen (1849), but with a slight
119 sign change, that is in fact due to a mistake on his own:

$$120 \quad u \approx u_0 - 2\pi \frac{\gamma}{\rho Q} (r - r_0). \quad (2.5)$$

121 Though obtained erroneously, this expression is very seductive and Bouasse used it to
122 calculate the radius of the liquid sheet R_{LS} assuming that the sheet border should stay
123 at the place in which u vanishes which leads to $R_{LS} = (\rho Q u_0)/2\pi\gamma (= \rho r_0^2 u_0^2)/2\gamma$.
124 Surprisingly this result coincides with the right one that is in fact obtained, now, by assuming
125 a constant velocity, dictated by Bernoulli's principle, and the balance of momentum at the
126 sheet perimeter, i.e. $\rho h u^2 = \gamma$ [Villermaux *et al.* (2013)]. But on the other hand, we would
127 like to stress out that the radial velocity is uniform in the sheet of Fig.1-c, which means that
128 the principle proposed in eq.(1.3), and therefore the basis of the theory developed by Bhagat
129 *et al.* (2018) is flawed.

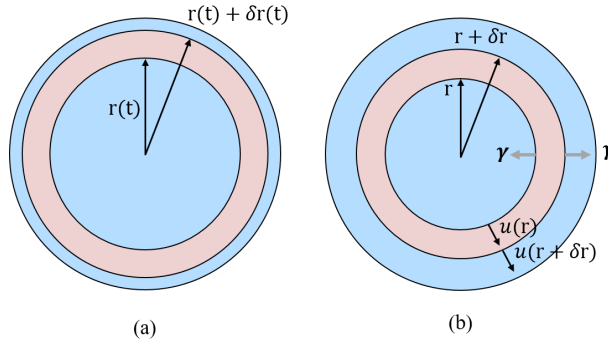


Figure 2: Lagrangien (a) and Eulerian (b) frame for discussing energy balance in a annular portion of a liquid film used in the text.

130 3. Reconsidering Hagen argument, and its implications for hydraulic jump.

131 We try to understand the fault underlying Bouasse and Hagen principle. Their line of thought
 132 is easier to explain considering a Lagrangian frame, and more precisely the balance of energy
 133 on a annular piece of fluid, convected by the radial flow, and it is in fact the method proposed
 134 by Bouasse himself in his treatise of fluid mechanics [Bouasse (1923)].

135 Let us consider a piece of annular piece of film as on Fig.2-a, convected and distorted
 136 by the flow. Mass conservation implies that, at any time $hr\delta r = \text{Cte}$, while the balance of
 137 energy for the whole annulus reads, in the limit of δr small enough to satisfies the condition
 138 $\delta r \frac{\partial u}{\partial r} \ll u$ of a slowly varying velocity field :

$$139 \quad \frac{\partial}{\partial t} \left[2\pi \left(\frac{1}{2} \rho u^2 r h \delta r + \gamma r \delta r \right) \right] \approx 2\pi \gamma \delta r u. \quad (3.1)$$

140 The first term in the left hand side of this equation stands for kinetic energy, and the second
 141 for the surface energy enclosed between r and $r + \delta r$. The right hand term comes from the
 142 work of surface forces, and does not vanish. Indeed, the same surface tension force is pulling
 143 on a different arc length, as the external boundary has a larger perimeter than the other (note
 144 that this is the intuitive argument underlying Bhagat's analysis). Still in the limit of a slowly
 145 varying velocity field at the scale δr , after noting that $\frac{\partial}{\partial t} = u \frac{\partial}{\partial r}$, eq.(3.1) reads:

$$146 \quad r h \delta r u \frac{\partial}{\partial r} \left(\frac{\rho u^2}{2} \right) + \gamma u \delta r \approx \gamma u \delta r. \quad (3.2)$$

147 In fact, the two surface tension terms are canceling each other, which means that the work
 148 provided to the annulus by surface tension of the outer interfaces is completely transformed
 149 into the surface energy stored at the free surface of the annulus, in agreement with simple
 150 thermodynamic considerations. As a result, the fluid velocity is unaffected by surface tension
 151 balance and remains constant as one would deduce from a more classical argument in terms
 152 of Bernoulli's principle, i.e. $u(r)$ is in fact independent of r :

$$153 \quad u(r) = u_0 = \text{Cte}. \quad (3.3)$$

154 Note here that skipping from eq.(3.1) to eq.(3.2) is not completely trivial as there is an
 155 extra γ term remaining, but this one vanishes for the constant and uniform u_0 solution.

156 To reconnect with Bouasse, instead of this, if one would forget the internal surface energy

157 contribution in the left hand member of eq.(3.2), one would get the following equation for u ,
 158 that reduces to:

$$159 \quad rh\delta r \frac{\partial}{\partial r} \left(\frac{\rho u^2}{2} \right) \approx \gamma \delta r. \quad (3.4)$$

160 After simplifying δr , and using the fact that $Q = 2\pi r u h$, this equation leads to:

$$161 \quad \frac{\partial u}{\partial r} \approx 2\pi \frac{\gamma}{\rho Q}, \quad (3.5)$$

162 which leads finally to eq.(2.4). Alternatively, eq.(2.5) is obtained when one forgets the work
 163 provided to the annulus by the outer parts of the liquid sheet, i.e. by neglecting the right hand
 164 member of eq.(3.2), following the intuitive but erroneous idea of Hagen (1849) that surface
 165 tension could slow down the flow. Historically Bouasse followed the first argument, but
 166 committed a sign mistake, obtaining eq.(2.5), that was physically more natural, considering
 167 presumably what Hagen said long ago.

168 To summarize, a correct treatment of the expansion of liquid annula in the flow leads to
 169 the classical result of a uniform velocity, while the approximates defended by Hagen and
 170 Bouasse would follow from neglecting a part of capillary terms. We believe that a similar
 171 problem is involved in eq.(1.3). If we consider now a Eulerian description of the flow, as
 172 suggested on Fig.2-b, the balance of energy should rather read:

$$173 \quad \left[\rho \frac{\bar{u}^2}{2} \bar{u} r h + \gamma r \bar{u} \right]_r^{r+\delta r} = [\gamma r \bar{u}]_r^{r+\delta r} - [p \bar{u} h]_r^{r+\delta r} - \left[\rho g \frac{h^2}{2} r \bar{u} \right]_r^{r+\delta r} - r \tau_w \bar{u} \delta r, \quad (3.6)$$

174 in which we have added in the left hand side the surface energy convected by the film. It
 175 is true that one can consider a capillary force, as in Bhagat *et al*, in the right hand member,
 176 but in this case, one should not miss the flux of surface crossing the two circles displayed on
 177 Fig.2-b in the left hand side of the equation. And just as what happens in a Lagrangian frame,
 178 the physics being the same in both frame, the capillary effects should exactly compensate
 179 each other in this equation, that should then reduce to the more conventional form:

$$180 \quad \left[\rho \frac{\bar{u}^2}{2} \bar{u} r h \right]_r^{r+\delta r} = - [p \bar{u} h]_r^{r+\delta r} - \left[\rho g \frac{h^2}{2} r \bar{u} \right]_r^{r+\delta r} - r \tau_w \bar{u} \delta r, \quad (3.7)$$

181 that apart some coefficients that will depend on the detailed structure of the flow profile is
 182 consistent with what people are used to write starting rather from the balance of momentum
 183 [Bohr *et al.* (1993)]. Therefore, we do not consider, in the interpretation of eq.(1.2), that
 184 one should add a new capillary force distributed all over space as proposed by Bhagat *et al.*
 185 (2018). To our opinion, this would imply to redo the initial mistake of Hagen and Bouasse.
 186 Just as for the calculation of the size of radial liquid sheets, the solution should rather lies
 187 inside the boundary conditions written at the circle which radius is under question. We are
 188 now developing more this idea.

189 **4. Alternative explanation of unusual scaling: the boundary condition at the** 190 **"jump" radius. Comparison with atomization rings.**

191 To interpret the occurrence of Bhagat *et al.* (2018) scaling, we propose an alternative
 192 approach. We just treat the two ideal situations of Fig.1-a and Fig.1-b with the same method,

193 and see what happens. We will then see that the situation obtained in Fig.1-b may be compared
 194 to the one suggested by Bhagat *et al.* (2018).

195 To simplify the analysis, the “internal” flow for $r_0 < r < R_J$ is assimilated to the one
 196 discussed long ago by Watson (1964), in which fluid inertia is progressively dissipated by
 197 viscous friction, i. e. for $r < R_J$:

$$198 \quad u(r, z) = \frac{27c^3}{8\pi^4} \frac{Q^2}{\nu(r^3 + l^3)} f\left(\frac{z}{h}\right), \quad (4.1)$$

199 in which $c \approx 1.402$, $l = 0.567r_0R$ (with R the Reynolds number of the jet) and f is the
 200 function: $f(\eta) = \sqrt{3} + 1 - \frac{2\sqrt{3}}{1+cn(3^{\frac{1}{4}}c(1-\eta))}$. Mass conservation implies that the film thickness
 201 and the flux of momentum are given by:

$$202 \quad \rho h \langle u^2 \rangle = \frac{27\sqrt{3}c^3}{16\pi^6} \frac{\rho Q^3}{R_J \nu (R_J^3 + l^3)}, \quad (4.2)$$

203 where $\langle u^2 \rangle = \int_0^h u^2 dz$. In the case of Fig.1-a, this flow must be matched for $r > R_J$ to a
 204 film flow under the action of gravity, that, according to lubrication [Duchesne *et al.* (2014)],
 205 has a thickness distribution $H(r)$ given by:

$$206 \quad H(r)^4 = H_\infty^4 + \frac{6}{\pi} \frac{\nu Q}{g} \ln\left(\frac{R_\infty}{r}\right), \quad (4.3)$$

207 where R_∞ designates the outer radius of the substrate, where the thickness H reaches a
 208 value called H_∞ that will depend on the specific geometrical conditions of the flow there (see
 209 Fig.1-a for the graphical definition). At $r = R_J$, one has to write some matching condition,
 210 that is consistent with the approximations made on each side of $r = R_J$, and stands for a shock
 211 [Bélanger (1841); Rayleigh (1914)]. If we assume $h \ll H$ and neglect the surface tension
 212 at the shock (i.e. for circular hydraulic jumps large enough such as the ones considered by
 213 Bhagat *et al.* (2018)), this shock condition reads:

$$214 \quad \rho h(r) \langle u(R_J) \rangle^2 \approx \rho g H(R_J)^2, \quad (4.4)$$

215 In the limit of negligible values for H_∞ and r_0 , compared to the other scales, it is easy to
 216 check that these equations lead to the following scaling law for R_J :

$$217 \quad R_J \ln\left(\frac{R_\infty}{R_J}\right)^{\frac{1}{8}} = \frac{(3c)^{\frac{3}{4}}}{2^{\frac{9}{8}} \pi^{\frac{1}{8}} 8} \frac{Q^{\frac{5}{8}}}{\nu^{\frac{3}{8}} g^{\frac{1}{8}}}, \quad (4.5)$$

218 i.e. the scaling obtained by Bohr *et al.* (1993) and modified by logarithmic corrections.

219 We now consider the regime described in Fig.1-b that may be obtained in stationary regime
 220 with particular superhydrophobic treatment [Maynes *et al.* (2011)] or with inverse gravity
 221 [Jameson *et al.* (2010); Button *et al.* (2010)]. In this regime the force opposed to fluid inertia
 222 at the boundaries is dictated only by surface tension and not by gravity, there is no developed
 223 shock, no liquid "wall". In other words, the flux of momentum is only balanced by surface
 224 tension, which means that equations (4.3) and (4.4) are simply replaced now by:

$$225 \quad \rho h(r) \langle u(R_J) \rangle^2 \approx \gamma (1 - \cos \theta), \quad (4.6)$$

226 with θ the static contact angle

227 Using eq.(4.2) in the limit $r = R_J \gg r_0$, this condition yields a new scaling that reads:

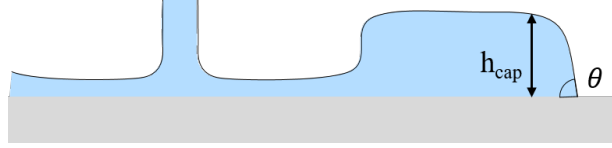


Figure 3: sketch of the intermediate regime for a low viscosity liquid in partial wetting.

$$R_J = \left(\frac{27\sqrt{3}c^3}{16\pi^6} \right)^{\frac{1}{4}} (1 - \cos\theta)^{\frac{1}{4}} Q^{\frac{3}{4}} \nu^{-\frac{1}{4}} \rho^{\frac{1}{4}} \gamma^{-\frac{1}{4}}. \quad (4.7)$$

This scaling is the same than the one suggested by Bhagat *et al.* (2018) and previously by Button *et al.* (2010). It explains why the scaling obtained by Bhagat *et al.* (2018) applies to the experimental data of Jameson *et al.* (2010) even if the theory leading to this scaling is not the right one.

We thus do not believe that there is a “universal” scaling that should hold for any circular “print” formed around an impacting jet. Sometimes one can find Bohr’s scaling and sometimes Bhagat and Button one, it is the analysis of the conditions around the impact that will matter.

5. Another possible occurrence of Bhagat and Button scaling.

We now show that Bhagat’s scaling may also be observed in classical circular hydraulic jumps. In Bhagat *et al.* (2018) paper, the authors consider an intermediate regime where the liquid has not yet reached the edge of the plate (see Fig.3). In their experimental evidence the authors consider partial wetting conditions (they use Perspex, glass and Teflon) and aqueous solutions. Given that the front propagation speed is rather small, we can consider that the liquid front height is approximately given by

$$h_{cap} \approx \left(\frac{\gamma}{\rho g} \right)^{\frac{1}{2}} (1 - \cos\theta)^{\frac{1}{2}}. \quad (5.1)$$

Considering the eq.(4.3) for low viscosity liquid and moderate flow rate one can conclude that:

$$H(r) \approx H_{\infty} \approx h_{cap}. \quad (5.2)$$

Therefore the (simplified) shock condition (4.4) previously obtained leads to:

$$\rho h(r) < u(R_J)^2 > \approx \frac{1}{2} \rho g h_{cap}^2. \quad (5.3)$$

Surprisingly, this argument leads again exactly to the “surface tension dominated” scaling :

$$R_J = \left(\frac{27\sqrt{3}c^3}{16\pi^6} \right)^{\frac{1}{4}} (1 - \cos\theta)^{\frac{1}{4}} Q^{\frac{3}{4}} \nu^{-\frac{1}{4}} \rho^{\frac{1}{4}} \gamma^{-\frac{1}{4}}. \quad (5.4)$$

Following now a remark from Bhagat *et al.* (2018), one can also denote that by defining the Weber number as

$$We \approx \frac{\rho h(r) < u(R_J)^2 >}{\gamma} \approx cste, \quad (5.5)$$

255 i.e., a constant Weber number that replaces the constant Froude number encountered in
 256 fully established hydraulic jump with a complete, flowing outer film.

257 Considering $\theta = \frac{\pi}{2}$, we obtain that:

$$258 \quad \text{We} \approx \frac{1}{2}, \quad (5.6)$$

259 which is the order of magnitude of the Weber number reported in Bhagat *et al.* (2018).

260 6. Conclusion

261 In summary, we have reconsidered the problem of scaling law selection of the "radius of
 262 influence" in the problem of vertical jet impact on a horizontal solid surface. In our opinion,
 263 the ideal law (1.1) proposed by Bohr and coworkers (to which, one should not forget to add
 264 logarithmic corrections as in Duchesne *et al.* (2014)) corresponds to the ideal situation of a
 265 stationary hydraulic jump formed inside a liquid film extending on the whole solid surface.
 266 On the opposite, the scaling (1.2) suggested in ref. [Bhagat *et al.* (2018)] rather holds in
 267 different situations, some of these ones being:

268 - stationary impact of a jet on a dry surface, possibly superhydrophobic, without formation
 269 of the outer film (atomization ring),

270 - stationary impact of a jet on a dry surface in inverse gravity (impact of a jet on a ceiling),

271 - transient regime of circular hydraulic jump formation for low viscosity liquids in partial
 272 wetting.

273 It would be interesting to explore in more details these three situations, and to identify
 274 possible other ones. In our opinion, there is no need to imagine some universal extra capillary
 275 term imposing the scaling (1.2) as imagined in ref [Bhagat *et al.* (2018)]. Though this extra
 276 term really exists, when the control volume contains the free surface of the film instead of
 277 excluding it, it is in practice compensated by another one in a way consistent with classical
 278 thermodynamics. As usual in free surface flows there is no increase or decrease of velocity
 279 that could be due alone to the action of surface tension, except when Marangoni effects are
 280 involved [Marmottant *et al.* (2000)]. Going on in this direction would be just reproducing for
 281 thin film flows on a solid, the initial mistake of Hagen and Bouasse.

282 If we come back to the question of Bohr scaling we have left a bit aside the questions of the
 283 logarithmic corrections and the possible existence of a critical Froude number at the jump
 284 exit, suggested in [Duchesne *et al.* (2014)]. The possible existence of this critical Froude
 285 number leads to a different exponent for the Logarithmic corrections (3/8 instead of 1/8)
 286 and this question is still not solved. As told in the introduction, recent non-linear analytical
 287 treatment of the film flow suggests that such a critical Froude number could exist, but this
 288 remains to be established and convincingly explained.

289 A specific problem of great interest where these considerations should matter is the question
 290 of jet impacts on inclined plates. It is not obvious in this kind of problem that a perfect
 291 hydraulic jump can exist, or not, and the two scaling should compete against each other
 292 in a way that merits to be investigated. The influence of a external fields, here the tangent
 293 component of gravity on a circular shock is a fundamental question of great interest. A
 294 specific effort should be done in this direction [Wilson *et al.* (2012); Duchesne *et al.* (2013)].

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