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Model-based Clustering with Missing Not At Random Data

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Abstract

In recent decades, technological advances have made it possible to collect large data sets. In this context, the model-based clustering is a very popular, flexible and interpretable methodology for data exploration in a well-defined statistical framework. One of the ironies of the increase of large datasets is that missing values are more frequent. However, traditional ways (as discarding observations with missing values or imputation methods) are not designed for the clustering purpose. In addition, they rarely apply to the general case, though frequent in practice, of Missing Not At Random (MNAR) values, i.e. when the missingness depends on the unobserved data values and possibly on the observed data values. The goal of this paper is to propose a novel approach by embedding MNAR data directly within model-based clustering algorithms. We introduce a selection model for the joint distribution of data and missing-data indicator. It corresponds to a mixture model for the data distribution and a general MNAR model for the missing-data mechanism, which may depend on the underlying classes (unknown) and/or the values of the missing variables themselves. A large set of meaningful MNAR sub-models is derived and the identifiability of the parameters is studied for each of the sub-models, which is usually a key issue for any MNAR proposals. The EM and Stochastic EM algorithms are considered for estimation. Finally, we perform empirical evaluations for the proposed sub-models on synthetic data and we illustrate the relevance of our method on a medical register, the TraumaBase[®] dataset.

Keywords. Model-based Clustering, Missing Not At Random (MNAR) Data, Identifiability, EM and Stochastic EM Algorithms, Medical Data.

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1 Introduction

Clustering remains a pivotal tool for readable analysis of large datasets, offering a consistent summary of datasets by grouping individuals. In particular, the model-based paradigm [McLachlan and Basford, 1988, Zhong and Ghosh, 2003, Bouveyron et al., 2019] allows to perform clustering, by providing interpretable models, valuable to understand the connections between the constructed clusters and the features in play. This parametric framework provides a certain plasticity by handling high dimensionality problems [Bouveyron et al., 2007, Bouveyron and Brunet-Saumard, 2014], mixed datasets [Marbac et al., 2017], or even time series and dependent data [Ramoni et al., 2002, Xiong and Yeung, 2004]. The counterpart of performing this multifaceted model-based clustering is the involved modelling work for designing mixture models appropriate to the data structure.

In large scale data analysis, the problem of missing data is ubiquitous, since the more data we have, the more missing values we can expect to have. Classical approaches for dealing with missing data consist of working on a complete dataset [Little and Rubin, 2019], either by using only complete individuals, or by imputing missing values. However, both methods can raise huge problems in the analysis. On the one hand, if we delete the missing values, the remaining observations can form a too small subset or a biased subset of the population, which increases the variance of the estimates, in the worst case. On the other hand, if single imputation is used, the additional variability due to missing values is not taken account into subsequent analysis. Moreover, neither of both strategies are specifically designed for the final clustering task. As an alternative, one can consider likelihood approaches, using for instance Expectation Maximization (EM) type algorithms [Dempster et al., 1977]. We detail such an approach in this paper and develop some clustering methods able to deal with informative missing data in an efficient way.

Notations and typology of the missing values mechanisms To correctly define the missing values mechanisms, some notations must be introduced. The full dataset consists of n individuals $Y = (\mathbf{y}_1 | \dots | \mathbf{y}_n)^T$, where each observation $\mathbf{y}_i = (y_{i1}, \dots, y_{id})^T$ belongs to a space \mathcal{Y} , depending on the kind of data, defined by d features. The pattern of missing data for the full dataset is denoted by $C = (\mathbf{c}_1 | \dots | \mathbf{c}_n)^T \in \{0, 1\}^{n \times d}$, $\mathbf{c}_i = (\mathbf{c}_{i1}, \dots, \mathbf{c}_{id})^T \in \{0, 1\}^d$ being the indicator pattern of missing data for the individual $i \in \{1, \dots, n\}$: $\mathbf{c}_{ij} = 1$ indicates that the value y_{ij} is missing and $\mathbf{c}_{ij} = 0$ otherwise. The observed variables values for individual i will be denoted by $\mathbf{y}_i^{\text{obs}}$. Similarly the missing variables values for individual i is denoted by $\mathbf{y}_i^{\text{mis}}$. In addition, in a clustering context, the target is to estimate an unknown partition of the whole dataset Y into K groups. This partition is denoted by $Z = (\mathbf{z}_1 | \dots | \mathbf{z}_n)^T \in \{0, 1\}^{n \times K}$ with $\mathbf{z}_i = (z_{i1}, \dots, z_{iK})^T \in \{0, 1\}^K$ and where $z_{ik} = 1$ if \mathbf{y}_i belongs to cluster k , $z_{ik} = 0$ otherwise. Consequently, in a clustering context, the missing data are not only the values $\mathbf{y}_i^{\text{mis}}$ but also the partition labels \mathbf{z}_i .

Rubin [1976] distinguishes three missing values mechanisms, namely Missing Completely at Random (MCAR), Missing at Random (MAR) and Missing not at Random (MNAR). The missing data are MCAR when the missingness is independent of all the values, missing or not, and thus can be formalized by $\mathbb{P}(\mathbf{c}_i | \mathbf{y}_i, \mathbf{z}_i; \psi) = \mathbb{P}(\mathbf{c}_i; \psi)$, for all (missing or observed) values $(\mathbf{y}_i, \mathbf{z}_i)$, ψ generically designating a parameter of the multinomial pdf on \mathbf{c}_i . The missing data are MAR when the missingness is independent of the missing values, even if possibly depending on some (or all) observed values, meaning that $\mathbb{P}(\mathbf{c}_i | \mathbf{y}_i, \mathbf{z}_i; \psi) = \mathbb{P}(\mathbf{c}_i | \mathbf{y}_i^{\text{obs}}, \psi)$ for all missing values $(\mathbf{y}_i^{\text{mis}}, \mathbf{z}_i)$. The M(C)AR mechanisms are said ignorable, because estimating the parameter of the data distribution Y does not require the modelisation of $\mathbb{P}(\mathbf{c}_i | \mathbf{y}_i, \mathbf{z}_i; \psi)$, considering that ψ is a nuisance parameter.

Finally, MNAR corresponds to a missing-data mechanism which is not MCAR or MAR. For such missing data, the observed variables are not representative of the population. It is well known that the MNAR mechanism is not-ignorable when the goal is to estimate the parameters of the mixture model [Little and Rubin, 2019]. The MNAR mechanism is actually also not-ignorable when the aim is to recover the partition of the data. Therefore, as the MNAR mechanism is neither ignorable for the density estimation, nor for the clustering, dealing with such data does require the specific modeling effort of $\mathbb{P}(\mathbf{c}_i \mid \mathbf{y}_i, \mathbf{z}_i; \psi)$.

MNAR data In this paper, the data are supposed to be MNAR which is very frequent in practice [Ibrahim et al., 2001, Mohan et al., 2018]. Examples may include surveys where rich people would be less willing to disclose their income or clinical data collected in emergency situations, where doctors may choose to treat patients before measuring heart rate. In both cases, the missingness of income or heart rate depends on the missing values themselves.

The missing-data mechanism must be generally taken into account [Little and Rubin, 2019] by considering the joint distribution of the data and the missing-data pattern. There are mainly two approaches to formulate the joint distribution of the data and the missing-data pattern: (i) the selection model [Heckman, 1979] which factorizes it into the product of the marginal data density and the conditional density of the missing-data pattern given the data i.e. $\mathbb{P}(\mathbf{y}_i, \mathbf{c}_i \mid \mathbf{z}_i) = \mathbb{P}(\mathbf{y}_i \mid \mathbf{z}_i) \mathbb{P}(\mathbf{c}_i \mid \mathbf{y}_i, \mathbf{z}_i)$ (ii) the pattern-mixture model [Little, 1993] which uses the product of the marginal density of the missing-data pattern and the conditional density of the data given the missing-data pattern i.e. $\mathbb{P}(\mathbf{y}_i, \mathbf{c}_i \mid \mathbf{z}_i) = \mathbb{P}(\mathbf{c}_i \mid \mathbf{z}_i) \mathbb{P}(\mathbf{y}_i \mid \mathbf{c}_i, \mathbf{z}_i)$. In this paper, we adopt the selection model strategy, as it is more intuitive to model the distribution of the data (as usually done in parametric clustering approaches) and the cause of the lack according to the data. Although this point of view requires to model the missing-data mechanism, it allows to estimate the parameters of the model-based clustering and the data density and possibly to impute missing values, which are out of reach in pattern-mixture models.

Related works In order to handle missing values in a model-based clustering framework, Hunt and Jorgensen [2003] have implemented the standard EM algorithm [Dempster et al., 1977] based on the observed likelihood. More recently, Serafini et al. [2020] also propose an EM algorithm to estimate Gaussian mixture models in the presence of missing values by performing multiple imputations (with Monte Carlo methods) in the E-step. However, both works only consider M(C)AR data.

In a partition-based framework, Chi et al. [2016] propose an extension of k -means clustering for missing data, called k -Pod, without requiring the missing-data pattern to be modelled, making it suitable for MNAR data. However, like k -means clustering, the k -Pod algorithm cannot identify difficult cluster structures, since it relies on strong assumptions as equal proportions between the clusters. De Chaumaray and Marbac [2020] have proposed to perform clustering via a mixture model using the pattern-mixture approach to formulate the joint distribution, which makes the method not suitable to estimate the density parameters or to impute missing values. For longitudinal data, some authors [Beunckens et al., 2008, Kuha et al., 2018] jointly model the measurements and the dropout process by using an extension of the shared-parameter model, which is another MNAR model assuming that both the data and the dropout process depend on shared latent variables. They introduce for this a latent-class mixture model allowing classification of the subjects into latent groups. However, the MNAR model is restricted to the case where the missingness may

depend on the latent variables but not on the missing variables themselves.

For MNAR data, and specifically in selection models, the main challenge to overcome consists of proving the identifiability of the parameters of both the data and the missing-data pattern distributions. In particular, Molenberghs et al. [2008] prove that the identifiability does not hold when the models are not fixed, i.e. when there is no prior information on the type of the distribution for the missing-data pattern. For fixed models, Miao et al. [2016] provide identifiability results of Gaussian mixture and t-mixture models with MNAR data. However, their identifiability results are restricted to specific missing scenarios in a univariate case (one variable) and no estimation strategy is proposed. In this paper, their identifiability results are extended to more complex missing scenario and to the multivariate case.

Contributions We present and illustrate a relevant inventory of distributions for the MNAR missingness process in the context of unsupervised classification based on mixture models for different types of data (continuous, count, categorical and mixed). We then provide the identifiability of mixture model parameters and missingness process parameters, under certain conditions (including the data type and the link functions governing the missingness mechanism distribution). This is a real issue in the context of MNAR data, as models often lead to unidentifiable parameters. When all variables are continuous or count, all models lead to identifiable parameters. In the categorical and mixed cases, only the models for which the missingness depends on the class membership have identifiable parameters. For each model or sub-model, an EM or Stochastic EM algorithm is proposed, implemented, and made available for reproducibility. We also prove that concerning MNAR models for which the missingness depends on the class membership, the statistical inference can be conducted on the augmented matrix $[\mathbf{Y}, \mathbf{C}]$ considering the MAR mechanism instead; this is a real advantage, especially because the missing-data mechanism does not have to be modelled in this case. This also gives theoretical interpretation of this approach which is often used in practice.

Outline of the paper The rest of the article is organized as follows. We introduce the model-based clustering in presence of missing-data in Section 2 and propose an exhaustive zoology of the possible MNAR specifications in this framework, for which the identifiability issue is addressed. We propose an estimation strategy in Section 3. Section 4 is devoted to numerical experiments on synthetic data in order to assess the performances of our methods. In Section 5, our method is finally illustrated on a public health application, the TraumaBase[®] dataset. Section 6 concludes this paper and gives some perspectives.

2 Missing data in model-based clustering

2.1 Mixture models

Mixture models permit to achieve the clustering aim by modeling the distribution of the observed data $(\mathbf{y}_i^{\text{obs}}, \mathbf{c}_i)$. This distribution can be obtained from the distribution of the couple $(\mathbf{y}_i, \mathbf{c}_i)$ that is supposed to be a mixture model with K components. Thus, using the model selection decomposition for each component, the probability distribution function (pdf) of the couple $(\mathbf{y}_i, \mathbf{c}_i)$ is

$$f(\mathbf{y}_i, \mathbf{c}_i; \theta) = \sum_{k=1}^K \pi_k f_k(\mathbf{y}_i; \lambda_k) f_k(\mathbf{c}_i | \mathbf{y}_i; \psi_k), \tag{1}$$

where $\theta = (\gamma, \psi)$ gathers all the model parameters, $\gamma = (\pi, \lambda)$ groups the parameters related to the marginal distribution of Y_i , $\pi = (\pi_1, \dots, \pi_K)$ is the vector of proportions with $\sum_{k=1}^K \pi_k = 1$ and $\pi_k > 0$ for all $k \in \{1, \dots, K\}$, $\lambda = (\lambda_1, \dots, \lambda_K)$, $f_k(\cdot; \lambda_k)$ is the pdf of the k -th component parameterized by λ_k , $\psi = (\psi_1, \dots, \psi_K)$ groups the parameters of the missingness mechanism and $f_k(\mathbf{c}_i | \mathbf{y}_i; \psi_k)$ is the pdf related to the missingness mechanism under component k . In many cases, the parameter ψ is interpreted as a nuisance parameters. However, we need to consider the whole parameter θ to achieve the clustering since the pdf of the observed data is

$$f(\mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \theta) = \int f(\mathbf{y}_i, \mathbf{c}_i; \theta) d\mathbf{y}_i^{\text{miss}}. \quad (2)$$

Different kinds of pdf $f_k(\cdot; \lambda_k)$ can be considered, depending on the types of features at hand. Thus, if y_i is a vector of continuous variables, the pdf of a d -variate Gaussian distribution [McLachlan and Basford, 1988, Banfield and Raftery, 1993] can be considered for $f_k(\mathbf{y}_i; \lambda_k)$ and thus λ_k groups the mean vector and the covariance matrix. Moreover, if some components of y_i are discrete or categorical, the latent class model (see [Geweke et al., 1994, McParland and Gormley, 2016]) defining $f_k(\mathbf{y}_i; \lambda_k) = \prod_{j=1}^d f_{kj}(y_{ij}; \lambda_{kj})$ can be used. In such case, f_{kj} could be the pdf of a Poisson (multinomial respectively) distribution with parameter λ_{kj} if y_{ij} is an integer (categorical respectively) variable.

It is natural to expect that the individual data \mathbf{Y} convey more information on the partition \mathbf{Z} than the pattern \mathbf{C} of missing data. Thus, it is hazardous to allow the missing data modeling to be more complex than the mixture model itself. Consequently, we assume that the elements of c_i are conditionally independent given $(\mathbf{y}_i, \mathbf{z}_i)$ and that the c_{ij} and $y_{ij'}$ are conditionally independent given (y_{ij}, \mathbf{z}_i) for $j \neq j'$. Moreover, we consider that the conditional distribution of c_{ij} given (y_i, \mathbf{z}_i) is defined by a generalized linear model with link function ρ . Thus, we have

$$f_k(\mathbf{c}_i | \mathbf{y}_i; \psi_k) = \prod_{j=1}^d (\rho(\alpha_{kj} + \beta_{kj}y_{ij}))^{c_{ij}} (1 - \rho(\alpha_{kj} + \beta_{kj}y_{ij}))^{1-c_{ij}}, \quad (3)$$

where $\psi_k = (\alpha_{k1}, \beta_{k1}, \dots, \alpha_{kK}, \beta_{kK})$. Note that the specification for which the coefficients $\alpha_{kj'}$, $j' \neq j$ are not zero could be considered, specifically in the MAR case, but we choose to focus on the MNAR case only. The parameter α_{kj} represents the effect of missingness on the k -th class membership which depends on the variable j (*i.e.*, the effect is not the same for all variables). The parameter β_{kj} represents the direct effect of missingness on the variable j which depends on the class k . We now propose some gradual and parsimonious variants of this core model, while always highlighting their associated interpretation.

2.2 Parsimonious models for the missingness mechanism

The parsimonious models are defined by imposing equality constraints between the parameters of the different missingness mechanisms. First, we introduce two parsimonious models that allow the probability of being missing to depend on both the variable and the class membership. For the MNAR yz^j model, the missingness has a different effect on class membership depending on the variable and it has the same effect on a particular variable regardless the class. This model considers the constraints:

$$\text{MNAR}yz^j: \beta_{1j} = \dots = \beta_{Kj}, \forall j. \quad (4)$$

For the $\text{MNAR}y^kz$ model, we consider that the missingness has the same effect on class membership for all the variables but it has different effect on a particular variable depending on the class,

$$\text{MNAR}y^kz: \alpha_{kj} = \dots = \alpha_{k1}, \forall k. \quad (5)$$

We can consider that the effects on a particular variable and on the class membership are respectively the same for all the classes and for all the variables. It is the purpose of the following $\text{MNAR}yz$ model that considers the constraints:

$$\text{MNAR}yz: \beta_{1j} = \dots = \beta_{Kj}, \forall j \text{ and } \alpha_{kj} = \dots = \alpha_{k1}, \forall k. \quad (6)$$

The probability to be missing can only depend on the variable itself. A special case of missing not at random mechanisms, that is widely used in practice [Mohan, 2018], is the self-masked case, called $\text{MNAR}y$ here, where the only effect of missingness is on the variable j and is the same regardless of the class membership,

$$\text{MNAR}y: \alpha_{11} = \dots = \alpha_{1d} = \alpha_{21} = \dots = \alpha_{Kd} \text{ and } \beta_{1j} = \dots = \beta_{Kj}. \quad (7)$$

A slightly more general case can be considered by allowing the effect of missingness on the variable j to depend on the class, as in the following $\text{MNAR}y^k$ model,

$$\text{MNAR}y^k: \alpha_{11} = \dots = \alpha_{1d} = \alpha_{21} = \dots = \alpha_{Kd}. \quad (8)$$

The probability to be missing can only depend on the class membership. In the $\text{MNAR}z$ model, we consider that the only effect of missingness is on the class membership k which is the same for all variables,

$$\text{MNAR}z: \beta_{kj} = 0, \forall (k, j) \text{ and } \alpha_{kj} = \dots = \alpha_{kd}, \forall k. \quad (9)$$

The $\text{MNAR}z^j$ model is a slightly more general case than the $\text{MNAR}z$ model, because the effect of missingness on the class membership k is not the same for all the variables,

$$\text{MNAR}z^j: \beta_{kj} = 0, \forall (k, j). \quad (10)$$

The last model that we consider is the simplest one, which assumes MCAR values, i.e. each value has the same probability to be missing,

$$\text{MCAR}: \beta_{kj} = 0, \forall (k, j) \text{ and } \alpha_{1j} = \dots = \alpha_{Kj}, \forall j. \quad (11)$$

2.3 About the specificity of some proposed models

The $\text{MNAR}z$ model given in (9) is the simplest of the MNAR models we propose. Roughly speaking, this model assumes that the proportion of missing values can vary among the clusters. However, behind this apparent simplicity, it benefits from interesting properties we underline below.

Although $\text{MNAR}z$ does not directly involve \mathbf{y}_i in its ground definition (9), it does not mean that the pattern \mathbf{c}_i does not depend on \mathbf{y}_i since \mathbf{z}_i depends itself on \mathbf{y}_i . This is illustrated by Figure 1.

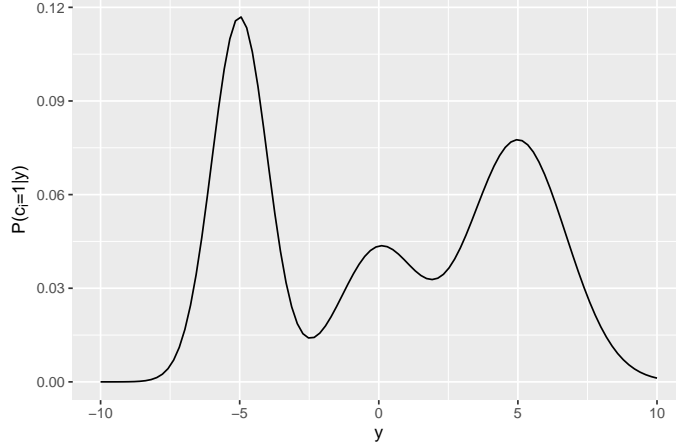


Figure 1: Illustration of dependency between \mathbf{c}_i and \mathbf{y}_i in a MNAR z model by drawing $\mathbb{P}(c_i | \mathbf{y}_i; \pi, \lambda, \psi)$ for a three component univariate Gaussian model with mixing proportions $\pi_1 = \pi_2 = 0.3$ and $\pi_3 = 0.4$, with centers $\mu_1 = \mu_3 = -5$ and $\mu_2 = 0$, and with variances $\sigma_k^2 = k$ ($k \in \{1, 2, 3\}$). The MNAR z parameters are fixed to $\alpha_1 = 2$, $\alpha_2 = 0$ and $\alpha_3 = 1$.

Finally it is important to mention that MNAR z and MNAR z^j can be linked to a MAR-like strategy commonly used in the machine learning community [Josse et al., 2019]. It consists of using a MAR mixture model on the concatenated dataset $\tilde{\mathbf{Y}}^{\text{obs}} = (\mathbf{Y}^{\text{obs}}, \mathbf{C})$ as a way for easily dealing with missing data. In Proposition 1, we prove that the MAR mixture model which is used for this new dataset $\tilde{\mathbf{Y}}^{\text{obs}}$ assuming a MAR missing mechanism is equivalent to the mixture model for \mathbf{Y}^{obs} given in (1) assuming a MNAR z or MNAR z^j model for \mathbf{C} . The proof of this proposition is given in Appendix A.

Proposition 1. *Let us consider the dataset $(\tilde{\mathbf{y}}_1^{\text{obs}}, \dots, \tilde{\mathbf{y}}_n^{\text{obs}})$ such that $\tilde{\mathbf{y}}_i^{\text{obs}} = (\mathbf{y}_i^{\text{obs}}, \mathbf{c}_i)$ for $i \in \{1, \dots, n\}$. Assume that all $\tilde{\mathbf{y}}_i^{\text{obs}}$ arise i.i.d. from the mixture model with a MAR mechanism*

$$\tilde{f}(\tilde{\mathbf{y}}_i^{\text{obs}}; \theta) = \sum_{k=1}^K \pi_k f_k(\mathbf{y}_i^{\text{obs}}; \lambda_k) \prod_{j=1}^d \rho(\alpha_{kj})^{c_{ij}} (1 - \rho(\alpha_{kj}))^{1-c_{ij}}. \quad (12)$$

Then for fixed parameters (π, λ, ψ) , the mixture model for $\tilde{\mathbf{y}}_i^{\text{obs}}$ is the same than the distribution for $\mathbf{y}_i^{\text{obs}}$ with the mixture model (1) under the MNAR z assumption (9) and MNAR z^j assumption (10).

In particular, Proposition 1 implies that the maximum likelihood estimate of (π, λ, ψ) is the same considering $\tilde{\mathbf{y}}_i^{\text{obs}}$ under the MAR assumption and $\mathbf{y}_i^{\text{obs}}$ under the MNAR z assumption (9) or MNAR z^j assumption (10).

2.4 Impact of the missingness mechanism on the clustering

We just have shown that MNAR is not ignorable with the classical definition [Little and Rubin, 2019] but one could check whether it is ignorable for clustering, which is the task of interest. A necessary and sufficient condition to have an ignorable missing process for clustering is that the

distributions of \mathbf{c}_i are equal among the mixture components. Thus, the missingness process is said *ignorable for clustering* if,

$$\forall(\mathbf{y}_i^{\text{obs}}, \mathbf{c}_i), \mathbb{P}(z_{ik} = 1 | \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i = 1; \theta) = \mathbb{P}(z_{ik} = 1 | \mathbf{y}_i^{\text{obs}}; \theta)$$

This is equivalent to have,

$$\forall(\mathbf{y}_i^{\text{obs}}, \mathbf{c}_i), \frac{\pi_k \int f_k(\mathbf{y}_i; \lambda_k) f_k(\mathbf{c}_i | \mathbf{y}_i; \psi_k) d\mathbf{y}_i^{\text{mis}}}{\int \sum_{\ell=1}^K \pi_\ell f_\ell(\mathbf{y}_i; \lambda_\ell) f_\ell(\mathbf{c}_i | \mathbf{y}_i; \psi_\ell) d\mathbf{y}_i^{\text{mis}}} = \frac{\pi_k f_k(\mathbf{y}_i^{\text{obs}}; \lambda_k)}{\sum_{\ell=1}^K \pi_\ell f_\ell(\mathbf{y}_i^{\text{obs}}; \lambda_\ell)}.$$

The MCAR mechanism is thus ignorable for the clustering, because $\mathbb{P}(\mathbf{c}_i | \mathbf{y}_i, z_{ik} = 1; \psi) = \mathbb{P}(\mathbf{c}_i; \psi)$. However, under the MNAR assumption the missingness mechanism is no longer ignorable, even for clustering, and a specific estimation process for the vector parameter (π, θ, ψ) is needed. Obviously, it depends on the MNAR model at hand, i.e. on the missing-pattern distribution $\mathbb{P}(\mathbf{c}_i | \mathbf{y}_i, \mathbf{z}_i; \psi)$.

2.5 Identifiability of the model parameters

This section gives sufficient conditions for the generic identifiability of the parameters for continuous, count, categorical and mixed data (*i.e.*, the space where the parameters are not identifiable has a Lebesgue measure equals to zero; see Allman et al. [2009]).

Proposition 2. Consider $\mathbf{y}_i = (\mathbf{y}_i^{\text{co}}, \mathbf{y}_i^{\text{ca}})$, with \mathbf{y}_i^{co} the set of continuous or count variables of cardinal d_{co} and \mathbf{y}_i^{ca} the set of categorical variables of cardinal $d_{\text{ca}} = d - d_{\text{co}}$.

For the continuous or count components, assume that **A1.** and **A2.** hold and consider the general mechanism defined by (3) (thus all the parsimonious mechanisms are allowed).

For the categorical variables, assume that **A5.** holds and that the mechanism is stated by (9), (10) or (11).

If $d_{\text{ca}} > 0$, assume also **A4.** Under Assumption **A3.**, the parameters of the model in (2) are generically identifiable, up to label swapping.

The whole proof is detailed in Appendix B. The proof for the continuous and count components follows the reasoning used by Teicher [1963, Theorem 2] which proves the identifiability of univariate finite mixture using a total ordering of the mixture densities (**A2.**). Assumption **A1.** requires that the parameters of the marginal mixture are identifiable. For the categorical components, the generic identifiability holds only for the MCAR, MNAR z and MNAR z^j mechanisms. The proof uses Corollary 5 of Allman et al. [2009] which gives the identifiability of finite mixtures of Bernoulli products, under **A5.**, which links minimum number of dimensions and number of classes. By assuming the conditional independence of the features given the group membership (**A4.**; it is a classical assumption for categorical and mixed data, even in the complete case) and that the missing-data mechanism is strictly monotone (**A3.**), the identifiability of mixed data directly follows from the identifiability of continuous and categorical components.

3 Estimation of the proposed MNAR models

As seen in Section 2, MNAR models are not ignorable, thus they require a specific inference procedure for estimating the parameters π , λ and ψ . This section gathers the description of the EM and SEM algorithms for Gaussian, Poisson, multinomial and mixed data with MNAR models for

maximum likelihood estimation. Details of the algorithms are given in Appendix C. These iterative algorithms require to introduce the complete-data log-likelihood is expressed as

$$\ell_{\text{comp}}(\theta; \mathbf{Y}, \mathbf{Z}, \mathbf{C}) = \sum_{i=1}^n \sum_{k=1}^K \log(\pi_k f_k(\mathbf{y}_i; \lambda_k) f_k(\mathbf{c}_i | \mathbf{y}_i; \psi_k)). \quad (13)$$

3.1 The EM algorithm

The EM algorithm [Dempster et al., 1977] is an iterative algorithm that permits to maximize the likelihood function under missingness. Initialized at the point $\theta^{[0]}$, its iteration $[r]$ consists, at the E-step, in computing the expectation of the complete-data log-likelihood $Q(\theta; \theta^{[r-1]}) = \mathbb{E}_{\theta^{[r-1]}} [\ell_{\text{comp}}(\theta; Y, Z, C) | \mathbf{Y}^{\text{obs}}, \mathbf{C}]$, then, at the M-step, updating the parameters by maximizing this function $\theta^{[r]} = \arg \max_{\theta} Q(\theta; \theta^{[r-1]})$. Note that

$$Q(\theta; \theta^{[r-1]}) = \sum_{i=1}^n \sum_{k=1}^K t_{ik}(\theta^{[r-1]}) \left[\log(\pi_k) + \tau_y(\lambda_k; \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i, \theta^{[r-1]}) + \tau_c(\psi_k; \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i, \theta^{[r-1]}) \right],$$

where

$$t_{ik}(\theta^{[r-1]}) = \frac{1}{f(\mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \theta^{[r-1]})} \int \pi_k^{[r-1]} f_k(\mathbf{y}_i; \lambda_k^{[r-1]}) f_k(\mathbf{c}_i | \mathbf{y}_i; \psi_k^{[r-1]}) d\mathbf{y}_i^{\text{miss}},$$

$$\tau_y(\lambda_k; \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i, \theta^{[r-1]}) = \mathbb{E}_{\theta^{[r-1]}} \left[\ln f_k(y_i; \lambda_k) | \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i \right],$$

and

$$\tau_c(\psi_k; \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i, \theta^{[r-1]}) = \mathbb{E}_{\theta^{[r-1]}} \left[\ln f_k(c_i | y_i; \psi_k) | \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i \right].$$

Thus, the iteration $[r]$ of the EM algorithm is defined by

- **E-step:** Computation of

$$t_{ik}(\theta^{[r-1]}), \tau_y(\lambda_k; \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i, \theta^{[r-1]}) \text{ and } \tau_c(\psi_k; \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i, \theta^{[r-1]}).$$

- **M-step:** Updating the parameters

$$\pi_k^{[r]} = \frac{1}{n} \sum_{i=1}^n t_{ik}(\theta^{[r-1]}), \quad \lambda_k^{[r]} = \arg \max_{\lambda_k} \sum_{i=1}^n t_{ik}(\theta^{[r-1]}) \tau_y(\lambda_k; \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i, \theta^{[r-1]}),$$

and

$$\psi_k^{[r]} = \arg \max_{\psi_k} \sum_{i=1}^n t_{ik}(\theta^{[r-1]}) \tau_c(\psi_k; \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i, \theta^{[r-1]}).$$

Note that the difficulty of the computation of quantities defined at the E-step as well as the difficulty of the maximization problem leading to $\lambda_k^{[r]}$ and $\psi_k^{[r]}$, depend on the MNAR model at hand. These steps are straightforward with the MNAR z and MNAR z^j models (see (9) and (10)) but more difficult with all the other MNAR models called in the sequel MNAR y^* (modelling the effect of the missingness depending on y).

Note that computing $\tau_y(\lambda_k; \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i, \theta^{[r-1]})$ requires to integrate over $\mathbf{y}_i^{\text{mis}}$ by considering its conditional distribution given $(\mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i)$ and the parameter $\theta^{[r-1]}$. For MNAR z and MNAR z^j

models, this conditional distribution is equal to the conditional distribution of $\mathbf{y}_i^{\text{mis}}$ by dependence of y given $(\mathbf{y}_i^{\text{obs}}, z_{ik} = 1)$ and the parameter $\theta^{[r-1]}$. This makes $\tau_y(\lambda_k; \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i, \theta^{[r-1]})$ easy to compute in most cases (see Appendix C.1.1 and C.1.2 for both Gaussian and categorical data). The MNAR y^* models consider the effect of the missingness depending on y and lead then to unfeasible computations. The conditional distribution of $\mathbf{y}_i^{\text{mis}}$ given $(\mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i)$ and the parameter $\theta^{[r-1]}$ is explicit in some cases (if the variables are Gaussian, it is a truncated Gaussian as shown in Appendix C.2.1) if the missing-data distribution ρ is probit but it is not a known distribution if ρ is logistic (it would require the use of sampling algorithm, as the Sampling Importance Resampling algorithm Gordon et al. [1993], that are time costly). However, to our knowledge, for both forms of missing-data distributions, $\tau_c(\psi_k; \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i, \theta^{[r-1]})$ and $t_{ik}(\theta^{[r-1]})$ do not have closed forms. In addition, the maximization problem leading to $\psi_k^{[r]}$ is a delicate issue because the function involved is not concave.

3.2 The SEM algorithm for overpassing the EM intractability's

Some distributions lead to computation of untractable integrals at the E-step (*e.g.*, gaussian components with MNAR y^* mecanism defined with logit link). In such case, the SEM algorithm [Celeux and Diebolt, 1985] could avoid this difficulty, by imputing missing values using a Gibbs sampling instead of integrating over them. In addition, it has another possible advantage over the EM algorithm since it is not trapped by the first local maximum encountered of the likelihood function [Celeux and Diebolt, 1985]. The SEM algorithm modifies the E-step of the EM algorithm by considering a stochastic-E step (SE-step) while the M-step is unchanged. Thus, at the iteration $[r]$, the E-step is replaced by the following SE-step:

SE-step: Draw the missing data $(\mathbf{z}_i^{[r]}, \mathbf{y}_i^{\text{mis}[r]})$ according to their conditional distribution given the observed data $(\mathbf{y}_i^{\text{obs}}, \mathbf{c}_i)$ and the current parameter $\theta^{[r-1]}$. Since it is not convenient to simulate this conditional distribution, we simulate instead from the following two easier conditional probabilities using a Gibbs sampling approach:

$$\mathbf{z}_i^{[r]} \sim \mathbf{z}_i \mid \mathbf{y}_i^{[r-1]}, \mathbf{c}_i; \theta^{[r-1]} \quad \text{and} \quad \mathbf{y}_i^{\text{mis}[r]} \sim \mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, \mathbf{z}_i^{[r+1]}, \mathbf{c}_i; \theta^{[r-1]}, \quad (14)$$

where $\mathbf{y}_i^{[r]} = (\mathbf{y}_i^{\text{obs}}, \mathbf{y}_i^{\text{mis}[r]})$.

The sampling of $\mathbf{z}_i^{[r]}$ is performed by a multinomial distribution whose the probabilities of the events are defined by $\pi_k^{[r-1]} f_k(\mathbf{y}_i; \lambda_k^{[r-1]}) f_k(\mathbf{c}_i \mid \mathbf{y}_i; \psi_k^{[r-1]}) / f(\mathbf{y}_i, \mathbf{c}_i; \theta^{[r-1]})$ for $k = 1, \dots, K$.

Note that the conditional distribution of $\mathbf{y}_i^{\text{mis}}$ given $(\mathbf{y}_i^{\text{obs}}, z_{ik}^{[r+1]} = 1, \mathbf{c}_i)$ parametrized by $\theta^{[r-1]}$ is defined by

$$f_k(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \theta^{[r-1]}) = \frac{f_k(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}; \theta^{[r-1]}) f_k(\mathbf{c}_i \mid \mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}; \psi^{[r-1]})}{\int f_k(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}; \theta^{[r-1]}) f_k(\mathbf{c}_i \mid \mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}; \psi^{[r-1]}) d\mathbf{y}_i^{\text{mis}}}.$$

This distribution may not be classical in general. For the MNAR y^* models, the conditional distribution $\mathbf{y}_i^{\text{mis}}$ given $(\mathbf{y}_i^{\text{obs}}, z_{ik}^{[r+1]} = 1, \mathbf{c}_i)$ is not explicit if the components are Gaussian and that the missing-data distribution ρ is logistic, because the product of logistic and Gaussian distributions is not a standard law. Therefore, the SEM algorithm cannot be easily applied. However, if ρ is the probit function, we can make the distribution of interest explicit (it is a truncated Gaussian distribution when the variables are Gaussian). For MNAR z and MNAR z^j models, all the computations remain feasible.

4 Numerical experiments on synthetic data

In this section, for simulated data, we compare the performances of

- the EM algorithm (designed for the MNAR z and MNAR z^j settings, resp. defined in (9) and (10)) which is made explicit in Appendix C.1.1 in the case of Gaussian variables and in Appendix C.1.2 for categorical variables;
- the SEM algorithm (designed for MNAR z and MNAR z^j , MNAR y , MNAR yz settings, resp. defined in (9), (10), (7) and (6)) which is made explicit in Appendix C.2.1 in the case of Gaussian variables and in C.2.2 for categorical variables;
- the classical SEM algorithm considering MCAR data (11);
- several two-step heuristics which consist of first imputing the missing values to get a complete dataset and then applying classical model-based clustering which has been implemented for the case without missing values. We consider two types of imputation strategies:
 - (a) multiple imputations by chained equations [Buuren and Groothuis-Oudshoorn, 2010], implemented in the R package `mice`: it consists of generating M plausible values for each missing value by computing expectation of the missing variables given the observed ones (referred as `Mice`). In such a case, M imputed datasets are computed, the model-based clustering is then performed on each complete dataset, for which the performance is measured. The final performance of this method is computed with the mean.
 - (b) single imputation by chained equations [Buuren and Groothuis-Oudshoorn, 2010], i.e. the same method as in (a) with $M = 1$ (referred as `Ice`).

4.1 Methods training

Measuring the performance To assess the clustering quality, it is possible to use an information criterion such as the Bayesian Information Criterion (BIC) [Schwarz, 1978] or the Integrated Complete-data Likelihood (ICL) [Biernacki et al., 2000]. The BIC criterion is expected to select a relevant mixture model in a density estimation perspective, while ICL is expected to select a relevant mixture model for a clustering purpose. Thus, we consider the later in the following. As the ICL involves an integral which is generally not explicit, we can use an approximate version [Baudry et al., 2015] which we detail when there is missing data. For a model \mathcal{M}_{df} with df parameters, the maximum likelihood estimators are denoted as θ^{MLE} and $\ell(\theta; \mathbf{Y}^{\text{obs}}, \mathbf{C})$ is the observed log-likelihood. One has

$$\text{ICL}(df) = \ell(\theta^{\text{MLE}}; \mathbf{Y}^{\text{obs}}, \mathbf{C}) + \sum_{i=1}^n \sum_{k=1}^K \hat{z}_{ik}^{\text{MAP}}(\theta^{\text{MLE}}) \log(\mathbb{P}(z_{ik} = 1 | \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \theta^{\text{MLE}})), \quad (15)$$

with $\hat{z}_{ik}^{\text{MAP}}(\theta) = \underset{k \in \{1, \dots, K\}}{\text{argmax}} \mathbb{P}(z_{ik} = 1 | \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \theta)$.

In addition, the Adjusted Rand Index (ARI) [Hubert and Arabie, 1985] can be computed between the true partition given by \mathbf{Z} and the estimated one. Obviously other strategies are possible to select a sensible and useful mixture model (see Celeux et al. [2019]).

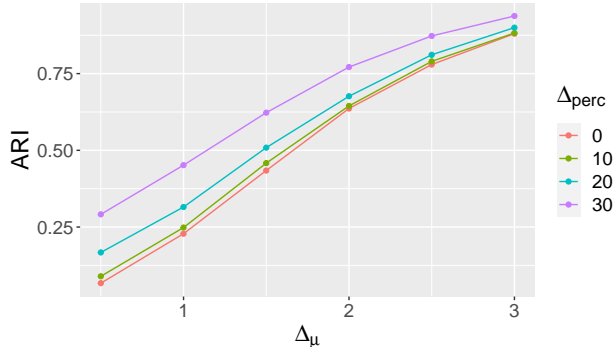


Figure 2: Relative effect of both the mixture component separation strength Δ_μ and the MNAR evidence Δ_{perc} on theoretical ARI. For example, if $\Delta_{\text{perc}} = 10\%$ (green line), it means that the second class has 10% more missing values than the first class.

4.2 Leveraging from MNAR data in clustering illustration

MNAR data is often considered as a real obstacle for statistical processing. Yet, this first numerical experiment illustrates that the MNAR mechanism may help performing the clustering task. Indeed, let us consider a bivariate two-component Gaussian mixture $\mathbf{Y} \sim 0.5\mathcal{N}(\mu_1, I_{2 \times 2}) + 0.5\mathcal{N}(\mu_2, I_{2 \times 2})$. The difference between the centers of both mixture components is taken as $\Delta_\mu = \mu_{21} - \mu_{11} = \mu_{22} - \mu_{12} \in \{0.5, 1, \dots, 3\}$. This cluster overlap controls the mixture separation, which can vary from a low separation ($\Delta_\mu = 0.5$) to a high one ($\Delta_\mu = 3$). By considering the MNAR $_z$ mechanism (9), one can play on the discrepancy between inter-cluster missing proportions $\Delta_{\text{perc}} = |\text{perc}_2 - \text{perc}_1|$, by making it varies in $\{0, 0.1, 0.2, 0.3\}$ ¹. Increasing values of Δ_{perc} corresponds to increase the MNAR evidence: indeed, $\Delta_{\text{perc}} = 0$ corresponds to a MCAR model, whereas a high value of Δ_{perc} corresponds to a high difference of missing pattern proportions between clusters. Finally, 15% of missing values is introduced whatever the MNAR evidence Δ_{perc} and the mixture separation is Δ_μ . Figure 2 gives the theoretical ARI (i.e. we compute the ARI with the theoretical parameters) as a function of the cluster overlap Δ_μ and the MNAR evidence Δ_{perc} . Even though the good classification rate is mostly influenced by the center separation Δ_μ , it also increases with the MNAR evidence Δ_{perc} . This toy example illustrates how clustering can leverage from MNAR missing values, generally considered as a true hindrance for any statistical analysis.

4.3 Toy example: MNAR $_z$ in the Gaussian case

To compare the methods presented above, we now consider a bivariate two-component Gaussian mixture ($n = 2000$, $d = 2$), $\mathbf{Y} \sim \pi_1\mathcal{N}(\mu_1, \Sigma_1) + \pi_2\mathcal{N}(\mu_2, \Sigma_2)$, with $\mu_1 = (0, 0)$, $\Delta_\mu = \mu_2 - \mu_1$, $\pi_1 = 0.3$, $\pi_2 = 0.7$ and $\Sigma_1 = I_{2 \times 2}$, $\Sigma_2 = I_{2 \times 2}$. We make the cluster overlap vary as follows: $\Delta_\mu = (2, 2)$ for a low separation between the two clusters and $\Delta_\mu = (4, 4)$ for a high separation between the two clusters. We introduce missing values with a MNAR $_z$ model (9), leading 3% of missing values in the first class and 35% in the second class (25% in the whole dataset). Thus, the mechanism conveys some information about the different clusters (as already mentioned in Section 4.2). Numerically, we can compute the ARI with true parameter θ : (i) by considering the triplet

¹The value Δ_{perc} means that if the percentage of missing values in the first cluster is perc_1 , the percentage of missing values in the second cluster is $\text{perc}_2 = (\text{perc}_1 + \Delta_{\text{perc}})$.

$(\mathbf{Y}, \mathbf{Z}, \mathbf{C})$ i.e. by computing the estimator of the partition $\tilde{z}_{ik}^{\text{MAP}}$ as in (15) or (ii) by ignoring the missing-data pattern \mathbf{C} and only considering (\mathbf{Y}, \mathbf{Z}) , i.e. computing the estimator of the partition with $\operatorname{argmax}_{k \in \{1, \dots, K\}} \mathbb{P}(z_{ik} = 1 | \mathbf{y}_i^{\text{obs}}; \theta)$.

Cluster separation influence In Figure 3, the algorithms are compared in terms of ARI in the case where the classes are well separated ($\Delta_\mu = (4, 4)$) or not ($\Delta_\mu = (2, 2)$). As expected, all the methods give better performances when the classes are well separated. In both cases, the two-step heuristics (**Ice** and **Mice**) do not seem appropriate for the classification task. Note also that they do not rely on any theoretical guarantee. In the case of clear separation between the clusters ($\Delta_\mu = (4, 4)$), the algorithms designed for MNAR values give similar performances and always outperform the ones assuming MCAR values. Note that the relatively low improvement was expected because the theoretical ARI, taking into account the mechanism, does not increase so much compared to the theoretical ARI without account for it. In addition, note that the MCAR model that we consider is specific since it needs to model \mathbf{C} (see Equation 11), whereas the classical MCAR case ignores the mechanism. It is not a general view of a MCAR situation, that would not include any modeling of \mathbf{C} . In the case where the classes are not well separated ($\Delta_\mu = (2, 2)$), the SEM algorithm assuming MCAR values clearly gives poorer performances than considering MNAR values. When the effect of missingness depending on the class membership is considered (for MNAR_z , MNAR_{z^j} and MNAR_{yz}), the performances are close to the ARI obtained with the true parameters. The algorithm designed for MNAR_y values has poorer performances but still outperforms the one for MCAR values and two-step heuristics. This can be explained by the fact that it only takes into account the effect of the absence of data as a function of variables.

	$\Delta_\mu = (2, 2)$	$\Delta_\mu = (4, 4)$
MNAR z	0.8	1
MNAR y	0	1
MNAR yz	0.3	0.9
MCAR	0	1

Table 1: Proportion of the number of times the correct class number is selected by the ICL, for a bivariate two-component Gaussian mixture, when the true missing scenario is MNAR z .

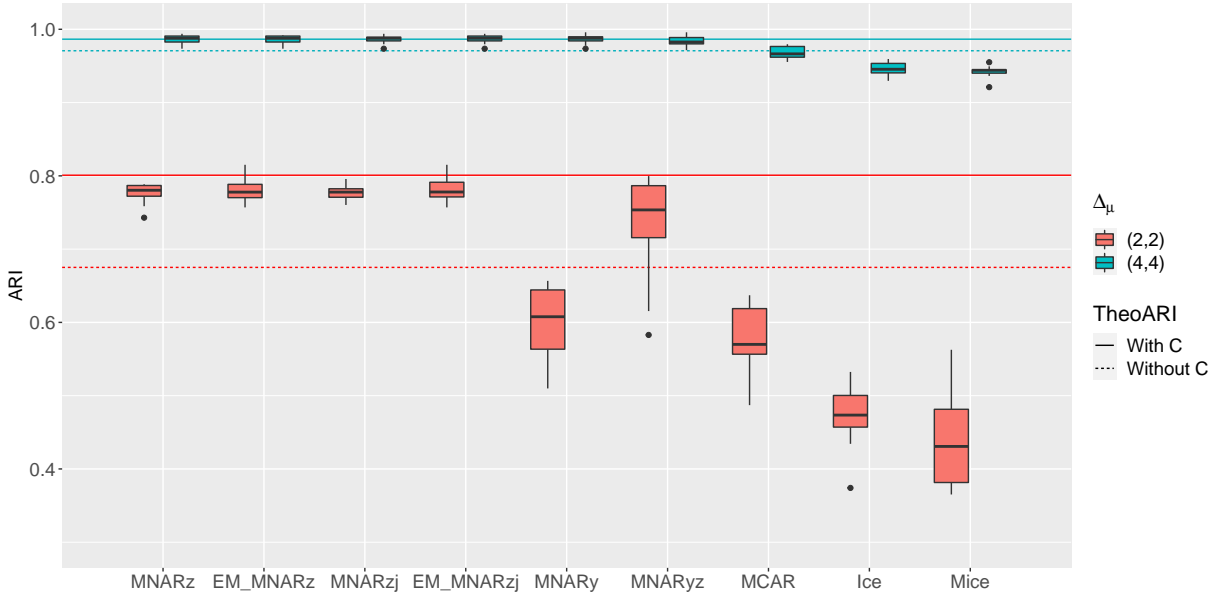


Figure 3: Comparison of the different algorithms in terms of ARI for ivariate two-component Gaussian ($n = 2000$, $d = 2$) when the true missing scenario is MNAR z . The red boxplot corresponds to $\Delta_\mu = (2, 2)$ when the classes are not well separated. The blue boxplot corresponds to $\Delta_\mu = (4, 4)$ when the classes are well separated. The stochasticity comes from the process of drawing 10 times the triplet $(\mathbf{Y}, \mathbf{Z}, \mathbf{C})$. The plain and dashed lines correspond to the mean of the ARI computed with the true parameters by considering $(\mathbf{Y}, \mathbf{Z}, \mathbf{C})$ or (\mathbf{Y}, \mathbf{Z}) respectively.

Automatic choice of the number of clusters In our algorithms, the number of clusters is given as an input, but it can be automatically chosen by using the ICL criterion (15). Indeed, algorithms can be performed with several values of the number $K = 1, 2, 3, 4$ of clusters, and then the clusters number leading to the highest ICL is chosen. In our knowledge, no method propose an automatic choice of the number of clusters in unsupervised classification for the two-step heuristics, which is also a major drawback. The results are summarized in Table 1.

Other numerical experiments can be found in Appendix D. In particular, other oracle missing-data mechanisms than the MNAR z one are considered in Appendix D.1. Besides, the initialization of the algorithms is discussed in D.2.

5 Real medical dataset

In this section, we illustrate our approach on a public health application with the APHP TraumaBase[®] Group (Assistance Publique - Hopitaux de Paris) on the management of traumatized patients. This dataset contains 47 quantitative and categorical variables on 8248 polytraumatized patients who suffer a major trauma (cycle or car accident, injuries) collected from 15 different hospitals. In this dataset, 11% of the data are missing. Figure 4 gives the percentage of missing values per variable and more information on the variables can be found in Appendix E.

The purpose of this real data analysis is twofold: (i) we first want to know if considering the missing values as informative changes the obtained classification, (ii) we compare our method with the classical imputation methods.

5.1 Classifications comparison

After discussion with doctors, some variables can be considered to have informative missing values, such as the variable *HR.ph*, which denotes the heart rate. Indeed, if the heart rate is very low or high, which indicates that the patient’s condition is critical, the heart rate may be not measured by doctors in emergency situations. Therefore, we believe that considering a MNAR mechanism can improve the classification.

In this section, the variables related to the death of the patient and the hand-made classifications made by the doctors (one considers 3 groups, the other 4) are not taken into account for running the algorithms, as they were considered too informative for the classification. A total of 41 variables can thus be used. The SEM algorithm for mixed variables is performed, either by considering MNAR_z mechanism (9) or MCAR mechanism (11).

To compare the classifications, in Table 2, we compute the ARI for the classification assuming MNAR_z (resp. MCAR) mechanism and the home-made classification considered as a ground truth (third column)², the ICL (fourth column), the ARI between the classifications obtained assuming MNAR_z and MCAR mechanisms (fifth column). The ARI with the classification made by the doctors when MNAR_z data are assumed are always higher than when MCAR data are assumed. In terms of ICL, assuming MNAR_z data looks largely beneficial. In both cases of mechanisms, the number of groups chosen is two. Finally, the ARI between the classifications assuming MNAR_z and MCAR data is high for $K = 2$ and $K = 3$ (above 90%, the classifications are similar) but low for $K = 4$ and $K = 5$ (below 90%, the classifications differ). It could be explained by the fact that in more difficult cases, considering MNAR variables brings significant information to the classification.

Note that the difficulty is that we do not have the *true* classification, because the ones made by the doctors (for $K = 3$ and for $K = 4$) are hand-made and they use very few variables (one or two). In addition, the ARI of both their classifications is only 51%, which shows that several classifications can be interesting. The low ARIs (always inferior to 10%) may therefore reflect the fact that we take into account many variables and that our classification is therefore necessarily different from those of physicians who use very few variables.

We assess also the results by using the function `catdes` of the R package FactoMineR [Lê et al., 2008] which allows to see how the cluster of the classification is described by the variables. A cell colored in blue (resp. red) means that the average value of the variable in the given cluster is significantly lower (resp. higher) than in the overall data. In addition, the more transparent the

²We compare to the home-made classification by the doctors which assumes $K = 4$ when $K = 4$ and the one which assumes $K = 3$ for the other cases ($K = 2, 3, 5$).

		ARI	ICL	ARI between the models
K=2	MCAR	9,2%	$-753e^3$	94.6%
	MNAR _z	9,3%	$-623e^3$	
K=3	MCAR	9,3%	$-757e^3$	93.2%
	MNAR _z	9,3%	$-628e^3$	
K=4	MCAR	6.2%	$-761e^3$	44.7%
	MNAR _z	8.4%	$-632e^3$	
K=5	MCAR	7.0%	$-757e^3$	47.3%
	MNAR _z	8.6%	$-632e^3$	

Table 2: Comparison of the MCAR and MNAR_z models for the Traumabase dataset.

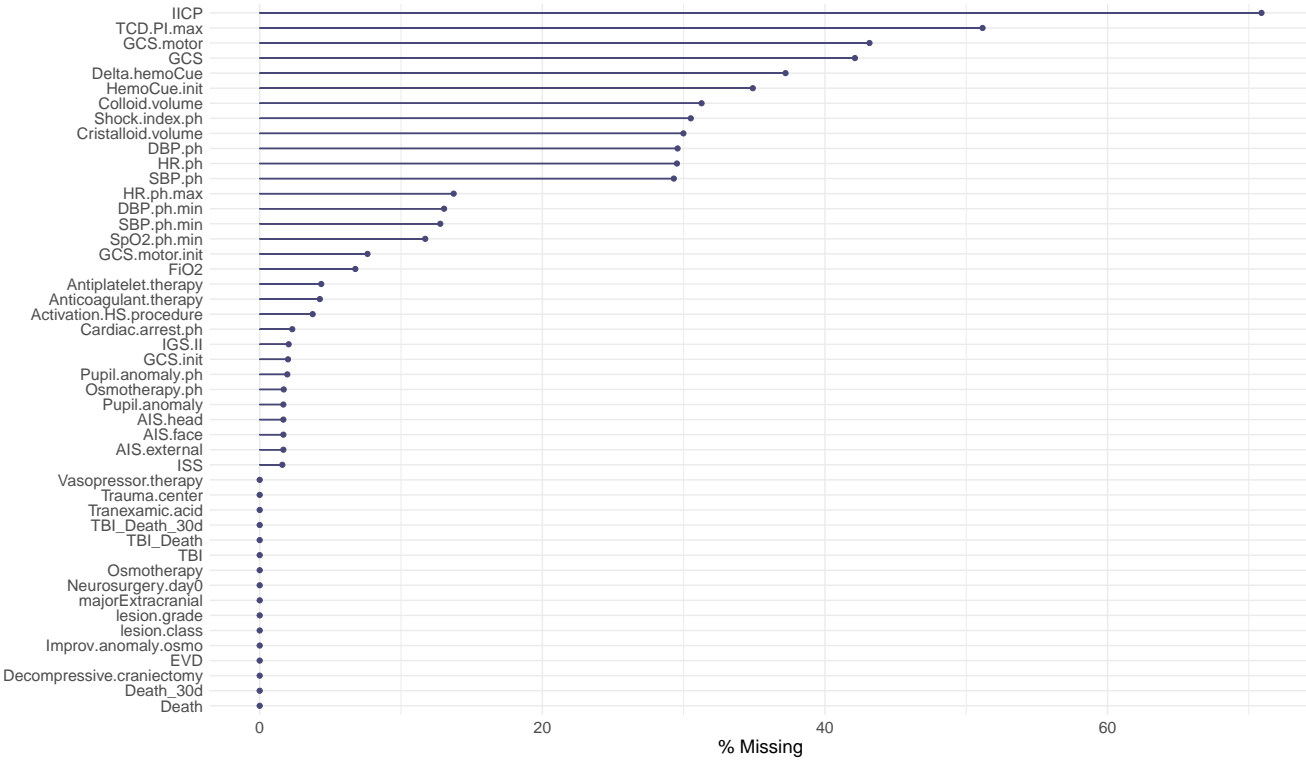


Figure 4: Percentage of missing values per variable for the Traumabase dataset.

cell is, the less significant the difference between the behavior of the variable in the given cluster and in the overall data is. In Figure 5, the results are shown by assuming MNAR z data. The three groups seem to be described by the same characteristics than those given by the doctors: the first group is formed by patients with a higher mortality rate and more severe injuries than the average population and the third group by a lower mortality rate and less serious injuries, whereas the second group may correspond to other cases. Note that this classification was done without using variables related to patient death and that it is quite striking to retrieve the same characteristics. This reinforces the idea that the classification obtained makes sense and provides other information than the one of the doctors.

5.2 Imputation performances

We perform now simulations on the real dataset in order to be able to measure the quality of the imputation of our method compared to the multiple imputation [Buuren and Groothuis-Oudshoorn, 2010] We introduce some additional missing values in three quantitative variables (*TCD.PI.max*, *Shock.index.ph*, *FiO2*) by using the MNAR z mechanism (9). The variables contain initially 51%, 31%, 7% and finally 65%, 48% and 31% missing values. The SEM algorithm for mixed data by assuming MNAR z mechanism is compared with the mean imputation and the multiple imputation in terms of mean squared error (MSE). Denoting $\hat{\mathbf{Y}} \in \mathbb{R}^{n \times d}$ the imputed dataset and $\tilde{\mathbf{C}} \in \mathbb{R}^{n \times d}$ the indicator pattern of newly introduced missing data, the mean squared error is

$$\mathbb{E}[(\hat{\mathbf{Y}} - \mathbf{Y}) \odot \tilde{\mathbf{C}}]_F^2 / \mathbb{E}[\mathbf{Y} \odot \tilde{\mathbf{C}}]_F^2$$

In particular, to impute missing values using the SEM algorithm, we use the assumption for mixed data that the features are independently drawn conditionally to the group membership (see **A4.**). As the continuous variables are assumed to be Gaussian, we have

$$\forall i \in \{1, \dots, n\}, \forall j \in \{1, \dots, d_{\text{cont}}\}, \text{ s.t. } \tilde{c}_{ij} = 1, \hat{y}_{ij} | z_{ik} = 1 \sim \mathcal{N}((\mu_{ik})_j, (\Sigma_{ik})_{jj}).$$

The imputation is performed taking the mean of 10^4 draws. In Figure 6, the SEM algorithm, designed for the MNAR setting, gives smaller error than other methods.

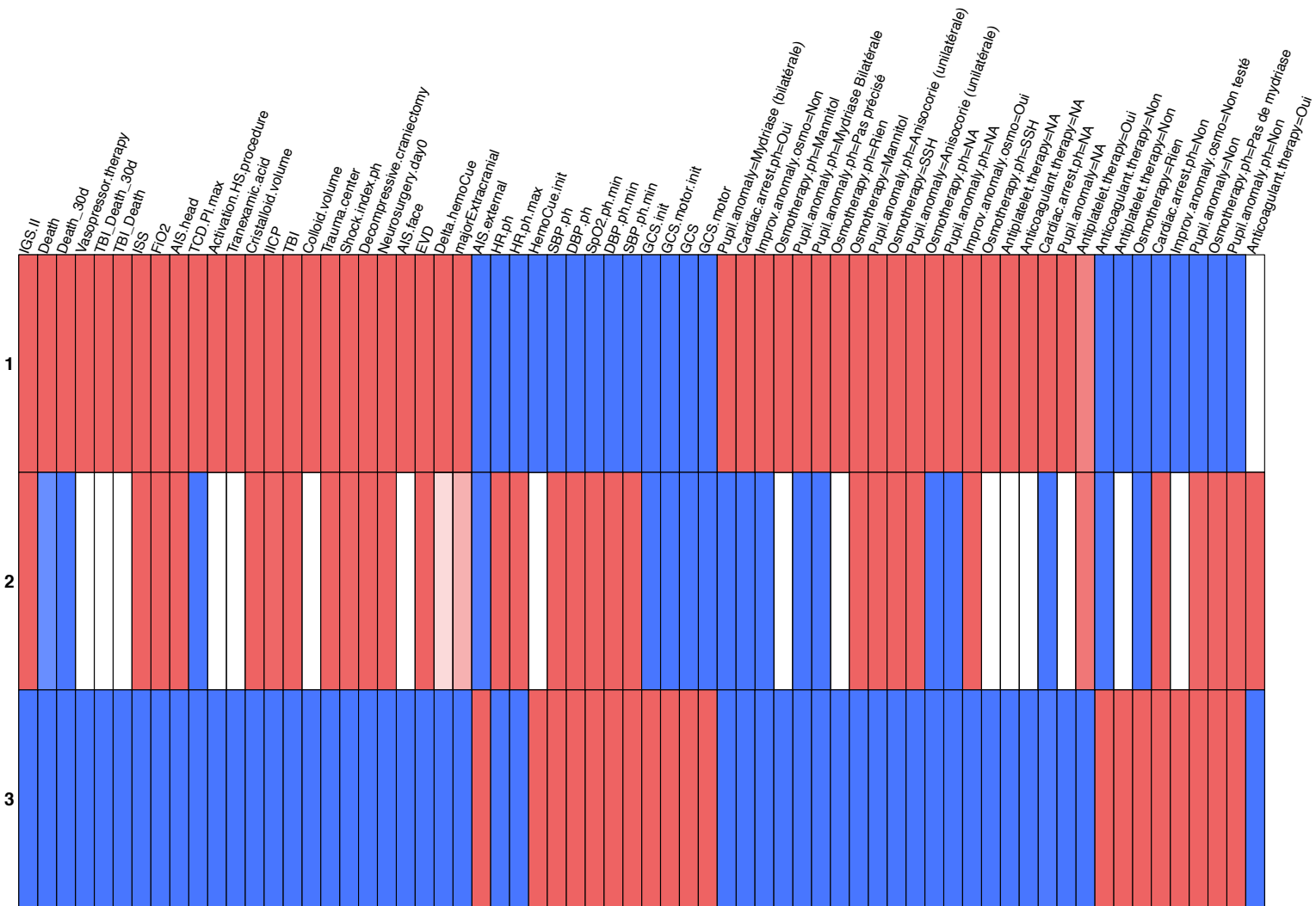


Figure 5: Plot of the catdes output for the SEM algorithm applied to the Traumabase dataset considering MNARz data. We consider $K = 3$ clusters.

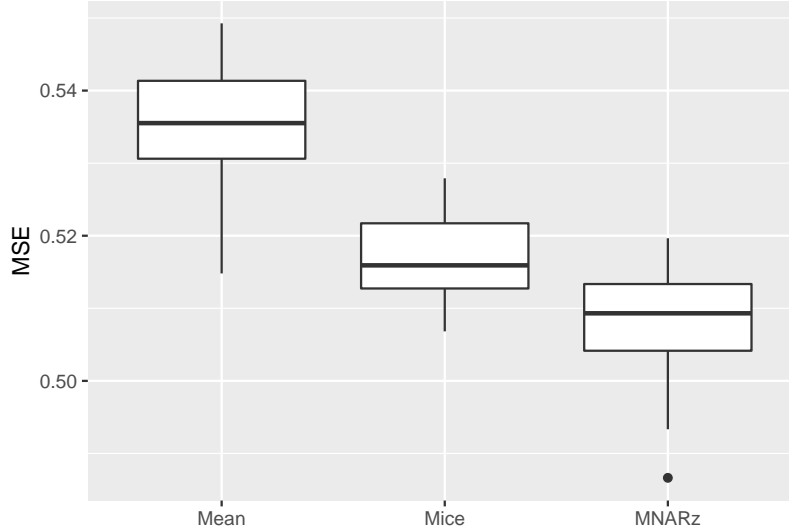


Figure 6: Mean squared error of the imputation task for the Traumabase dataset.

6 Concluding remarks

This paper addresses unsupervised learning when MNAR values occur. The aim was two-fold: (i) to cluster the individuals and (ii) to estimate the parameters of the distributions for each cluster (which also allows to impute missing values). To this end, we have proposed a novel approach which embeds MNAR data directly within model-based clustering algorithms, in particular the EM and SEM algorithms. This work also includes an exhaustive catalog of possible MNAR specifications. The identifiability study showed that the most general models lead to non-identifiable parameters for categorical data. This combined with the numerical experiments leads us to recommend using algorithms considering simple missing-data mechanisms, as the MNAR z mechanism, which models the probability of being missing only depending on the class membership. By its very simplicity, the model-based clustering algorithms considering the MNAR z mechanism are indeed able to deal straightforwardly with any kind of data. Moreover, in Proposition 1, we have proven that the statistical inference may be conducted either on MNAR z data on \mathbf{Y} or on MAR data considering the augmented matrix $[\mathbf{Y}, \mathbf{C}]$, with \mathbf{C} the missing-data pattern. It is worth noticing that this approach was widely used in practice but not theoretically studied. Finally, this mechanism has the advantage of being easily interpretable, which is especially important for real data applications.

The motivation of this work was the application on the Traumabase dataset, which can be extremely interesting, as it is genuinely useful to form groups of similarly-behaving patients for helping doctors in their decisions. To make this work entirely applicable to real datasets, there are still key challenges. First, in most datasets, the variables are not all of the same type (MCAR, MAR and MNAR variables are often coupled). There is probably no theoretical obstacle to consider such a case, but the implementation has not been done yet. This heterogeneity of the mechanisms brings us to a second challenge, which is quite general in missing-data analysis: how to decide if a variable is M(C)AR or MNAR. For the Traumabase dataset, the choice is made after discussions with doctors. However, a less subjective decision could be beneficial. The choice of the MNAR

mechanism is also a challenging question. Finally, note that our methodology can be applied in the case of mixed data (categorical/quantitative) by assuming that the features are independently drawn conditionally to the group membership. This assumption needs to be tested, for example by comparing performance on quantitative variables, assuming it or not.

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A Proof of Proposition 1

Proof of Proposition 1. We denote by $(\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_n)$ the patterns of missing data associated to the observed data $\tilde{\mathbf{y}}^{\text{obs}}$. It is thus the concatenation $\tilde{\mathbf{c}}_i = (\mathbf{c}_i, \mathbf{0}_d)$ of \mathbf{c}_i with the zero vector $\mathbf{0}_d = (0, \dots, 0)$ of length d . Since all c_i values are observed in $\tilde{\mathbf{y}}_i^{\text{obs}}$, it is the reason why the last d values in $\tilde{\mathbf{c}}_i$ are fixed to zero. Then, the MAR assumption indicates that $\mathbb{P}(\tilde{\mathbf{c}}_i | \tilde{\mathbf{y}}_i, \mathbf{z}_i; \zeta) = \mathbb{P}(\tilde{\mathbf{c}}_i | \tilde{\mathbf{y}}_i^{\text{obs}}; \zeta)$, with ζ the related parameter. Consequently, using the MAR assumption and the i.i.d. assumption of all uplets $(\tilde{\mathbf{y}}_i, \mathbf{z}_i, \tilde{\mathbf{c}}_i)$, the whole likelihood can be decomposed into two likelihoods, one has

$$\begin{aligned} L(\theta, \zeta; \tilde{\mathbf{Y}}^{\text{obs}}, \mathbf{C}) &= \int f(\tilde{\mathbf{y}}_i, \tilde{\mathbf{c}}_i; \theta, \zeta) d\tilde{\mathbf{y}}_i^{\text{mis}} \\ &= \int f(\tilde{\mathbf{y}}_i; \pi, \lambda, \psi) f(\tilde{\mathbf{c}}_i | \tilde{\mathbf{y}}_i; \zeta) d\tilde{\mathbf{y}}_i^{\text{mis}} \end{aligned} \quad (16)$$

$$= \prod_{i=1}^n \left[f(\tilde{\mathbf{c}}_i | \tilde{\mathbf{y}}_i^{\text{obs}}; \zeta) \times \int_{\mathcal{Y}_i^{\text{mis}}} f(\tilde{\mathbf{y}}_i; \pi, \lambda, \psi) d\tilde{\mathbf{y}}_i^{\text{mis}} \right] \quad (17)$$

$$= \prod_{i=1}^n L(\zeta; \tilde{\mathbf{c}}_i | \tilde{\mathbf{y}}_i^{\text{obs}}) \times \prod_{i=1}^n L(\pi, \lambda, \psi; \tilde{\mathbf{y}}_i^{\text{obs}}). \quad (18)$$

$$L(\pi, \lambda, \psi, \zeta; \tilde{\mathbf{y}}_i^{\text{obs}}, \tilde{\mathbf{c}}_i) = L(\zeta; \tilde{\mathbf{c}}_i | \tilde{\mathbf{y}}_i^{\text{obs}}) \times L(\pi, \lambda, \psi; \tilde{\mathbf{y}}_i^{\text{obs}}). \quad (19)$$

Providing that (π, λ, ψ) and ζ are functionally independent (ignorability of the MAR mechanism), the maximum likelihood estimate of $\theta = (\pi, \lambda, \psi)$ is obtained by maximizing only $L(\pi, \lambda, \psi; \tilde{\mathbf{y}}_i^{\text{obs}})$, and does not depend on $L(\zeta; \tilde{\mathbf{c}}_i | \tilde{\mathbf{y}}_i^{\text{obs}})$. Finally, by using (12), the observed likelihood $L(\pi, \lambda, \psi; \tilde{\mathbf{y}}_i^{\text{obs}})$ is

$$L(\pi, \lambda, \psi; \tilde{\mathbf{y}}_i^{\text{obs}}) = \sum_{k=1}^K \pi_k f_k(\mathbf{y}_i^{\text{obs}}; \lambda_k) \prod_{j=1}^d \rho(\alpha_{kj})^{c_{ij}} \rho(\alpha_{kj})^{(1-c_{ij})} \quad (20)$$

$$= \sum_{k=1}^K \pi_k f_k(\mathbf{y}_i^{\text{obs}}; \lambda_k) \prod_{j=1}^d \mathbb{P}(c_{ij} | z_{ik} = 1; \psi). \quad (21)$$

As $\mathbb{P}(c_{ij} | z_{ik} = 1; \psi)$ corresponds to the MNAR z definition (9), the observed likelihood $L(\pi, \lambda, \psi; \tilde{\mathbf{y}}_i^{\text{obs}})$ is equal to the full observed likelihood $L(\pi, \lambda, \psi; \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i)$ associated to the MNAR z model,

$$L(\pi, \lambda, \psi; \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i) = \sum_{k=1}^K \pi_k f_k(\mathbf{y}_i^{\text{obs}}; \lambda_k) \prod_{j=1}^d \mathbb{P}(c_{ij} | \mathbf{z}_{ik} = 1; \psi).$$

□

B Identifiability

B.1 Continuous and count data

A1. The parameters (π, λ) of the marginal mixture defined by the density $\sum_{k=1}^K \pi_k f_k(y_i; \lambda_k)$ are identifiable;

A2. There exists a total ordering \preceq of $\mathcal{F}_j \times \mathcal{R}$, for $j \in \{1, \dots, d\}$ fixed, where \mathcal{F}_j is the family of the data densities $\{f_{1j}, \dots, f_{Kj}\}$ and \mathcal{R} is the family of the mechanism densities $\{\rho_1, \dots, \rho_K\} = \{\rho(\cdot; \psi_1), \dots, \rho(\cdot; \psi_K)\}$, where ρ is the cumulative distribution function of any continuous distribution function and $(\psi_k)_{k \in \{1, \dots, K\}}$ its parameter. The total ordering is such that $\forall k < \ell \in \{1, \dots, K\}, \forall j \in \{1, \dots, d\}, F_{kj} \preceq F_{\ell j}$ (denoting $F_{kj} = \rho_k f_{kj}$ and $F_{\ell j} = \rho_\ell f_{\ell j}$) implies $\lim_{u \rightarrow +\infty} \frac{\rho_\ell(u) f_{\ell j}(u)}{\rho_k(u) f_{kj}(u)} = 0$;

A3. The missing-data distribution ρ is assumed to be strictly monotone.

Assumption **A1.** means that the identifiability of the parameters (π, λ, ψ) of the model (2) requires the identifiability of the parameters (π, λ) of the marginal mixture of (\mathbf{Y}, \mathbf{Z}) (i.e. considering the case without missing values). Some authors have already studied the identifiability of the mixture models, when no missing values in \mathbf{Y} occur, especially Teicher [1963] for Gaussian mixtures (continuous variables) and Yakowitz and Spragins [1968] for Poisson mixtures (count variables). Assumption **A2.** is the core ingredient to prove the identifiability of the parameters and we illustrate it by considering concrete examples in the following. Note that under Assumption **A3.** the probit and the logistic functions may be considered, which are the most widely used for MNAR specifications.

Proof of Proposition 2, continuous case. Suppose there exist two sets of parameters $\{\gamma, \psi\}$ and $\{\gamma', \psi'\}$ which have the same observed distribution, i.e. $f(\mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \gamma, \psi) = f(\mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \gamma', \psi')$. More precisely, one has

$$\begin{aligned} \forall \mathbf{y}_i \in \mathbb{R}^d, \forall \mathbf{c}_i \in \{0, 1\}^d, & \sum_{k=1}^K \int_{\mathcal{Y}_i^{\text{mis}}} \pi_k f_k(\mathbf{y}_i; \lambda_k) \prod_{j=1}^d \rho(\alpha_{kj} + \beta_{kj} y_{ij})^{c_{ij}} [1 - \rho(\alpha_{kj} + \beta_{kj} y_{ij})]^{1-c_{ij}} dy \\ & = \sum_{k=1}^{K'} \int_{\mathcal{Y}_i^{\text{mis}}} \pi'_k f_k(\mathbf{y}_i; \lambda'_k) \prod_{j=1}^d \rho((\alpha')_{kj} + (\beta')_{kj} y_{ij})^{c_{ij}} [1 - \rho((\alpha')_{kj} + (\beta')_{kj} y_{ij})]^{1-c_{ij}} dy \end{aligned}$$

Let us consider the case when $c_{ij} = 1$ for all $j = 1, \dots, d$. One has

$$\sum_{k=1}^K \pi_k f_k(\mathbf{y}_i; \lambda_k) \prod_{j=1}^d \rho(\alpha_{kj} + \beta_{kj} y_{ij}) = \sum_{k=1}^{K'} \pi'_k f_k(\mathbf{y}_i; \lambda'_k) \prod_{j=1}^d \rho((\alpha')_{kj} + (\beta')_{kj} y_{ij}).$$

By using the identifiability of the marginal mixture, one obtains $\lambda_k = \lambda'_k$. In addition, integrating out over all the elements but the j -th element, one has for all $y_{ij} \in \mathbb{R}$,

$$\sum_{k=1}^K \pi_k f_{kj}(y_{ij}; \lambda_{kj}) \rho(\alpha_{kj} + \beta_{kj} y_{ij}) = \sum_{k=1}^{K'} \pi'_k f_{kj}(y_{ij}; \lambda_{kj}) \rho((\alpha')_{kj} + (\beta')_{kj} y_{ij}).$$

In the sequel, we use the same reasoning of Theorem 2 in [Teicher, 1963].

Let us denote $F_k(y_{ij}) = f_{kj}(y_{ij}; \lambda_{kj}) \rho(\alpha_{kj} + \beta_{kj} y_{ij})$ and $F'_k(y_{ij}) = f_{kj}(y_{ij}; \lambda_{kj}) \rho((\alpha')_{kj} + (\beta')_{kj} y_{ij})$. Without loss of generality, assume that $F_k \prec F_l$ and $F'_k \prec F'_l$ for $k < l$. If $F_1 \neq F'_1$, we assume also without loss of generality that $F_1 \preceq F'_1$. Then, $F_1 \prec F'_k$ for $1 \leq k \leq K'$. For $u \in T_1$, where $T_1 = S_{F_1} \cap \{u : F_1(u) \neq 0\}$ is the domain of definition of F_1 such that $f_{1j}(u; \lambda_{1j}) \rho(\alpha_{1j} + \beta_{1j} u) \neq 0$, one has

$$\pi_1 + \sum_{k=1}^K \pi_k \frac{F_k(u)}{F_1(u)} = \sum_{k=1}^{K'} \pi'_k \frac{F'_k(u)}{F_1(u)}$$

Letting $u \rightarrow +\infty$, $\pi_1 = 0$ which is in contradiction with the mixture model (where $\pi_k > 0$, $\forall k = 1, \dots, K$). It implies that $F_1 = F'_1$. For any $u \in T_1$, one has

$$\pi_1 + \sum_{k=2}^K \pi_k \frac{F_k(u)}{F_1(u)} = \pi'_1 + \sum_{k=2}^{K'} \pi'_k \frac{F'_k(u)}{F_1(u)}$$

Letting again $u \rightarrow +\infty$, one obtains $\pi_1 = \pi'_1$ and $\sum_{k=2}^K \pi_k \frac{F_k(u)}{F_1(u)} = \sum_{k=2}^{K'} \pi'_k \frac{F'_k(u)}{F_1(u)}$. We repeat this argument to conclude that $F_k = F'_k$ and $\pi_k = \pi'_k$ for $k = 1, \dots, \min\{K, K'\}$. Finally, if $K \neq K'$, say $K > K'$, $\sum_{k=K'+1}^K \pi_k F_k(u) = 0$ implies $\pi_k = 0$ for $K' + 1 \leq k \leq K$ which is in contradiction with the definition of the mixture model. Thus $K = K'$. Note that $F_k = F'_k$ implies that $\rho(\alpha_{kj} + \beta_{kj} y_{ij}) = \rho((\alpha')_{kj} + (\beta')_{kj} y_{ij})$ and thus $\alpha_{kj} = (\alpha')_{kj}$ and $\beta_{kj} = (\beta')_{kj}$, since ρ is an injective function. Indeed, ρ is assumed to be strictly monotone. \square

On identifiability of the Gaussian mixture Finite Gaussian mixtures are identifiable and, for any variable j , there is a total ordering defined by $\sigma_{kj}^2 > \sigma_{(k+1)j}^2$ and $\mu_{kj} > \mu_{(k+1)j}$ if $\sigma_{kj}^2 = \sigma_{(k+1)j}^2$, where μ_{kj} and σ_{kj}^2 are respectively the mean and the variance of variable j under component k . Example 1 shows that the identifiability holds for Gaussian mixtures when there are missing values and that the distribution of the MNAR mechanism is a probit one.

Example 1 (Gaussian + Probit). *Let us consider that ρ is the probit function and f_k (respectively f_{k+1}) the Gaussian density with parameters (μ_k, σ_k) (respectively $(\mu_{k+1}, \sigma_{k+1})$). Suppose without loss of generality that $\beta_k \geq \beta_{k+1}$. One want to prove that*

$$\lim_{u \rightarrow +\infty} E_u := \lim_{u \rightarrow +\infty} \frac{\int_{-\infty}^{\alpha_k + \beta_k u} e^{-t^2/2} dt}{\int_{-\infty}^{\alpha_{k+1} + \beta_{k+1} u} e^{-t^2/2} dt} \frac{\sigma_k \exp\left(-\frac{(u - \mu_{k+1})^2}{2\sigma_{k+1}^2}\right)}{\sigma_{k+1} \exp\left(-\frac{(u - \mu_k)^2}{2\sigma_k^2}\right)} = 0$$

Let us denote $\phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^u e^{-t^2/2} dt$. One has

$$\lim_{u \rightarrow +\infty} \phi(u) = \begin{cases} 1 & \text{if } u > 0 \\ 1/2 & \text{if } u = 0 \\ 0 & \text{if } u < 0 \end{cases} \quad (22)$$

- If $\beta_{k+1} > 0$ (and $\beta_k > 0$):

$$\lim_{u \rightarrow +\infty} E_u = \lim_{u \rightarrow +\infty} \exp - \left(u^2 \left(\frac{1}{2\sigma_{k+1}^2} - \frac{1}{2\sigma_k^2} \right) + u \left(\frac{\mu_k}{\sigma_k} - \frac{\mu_{k+1}}{\sigma_{k+1}} \right) \right) = 0$$

assuming without loss of generality that $\sigma_k^2 > \sigma_{k+1}^2$ or $\mu_k > \mu_{k+1}$ if $\sigma_k^2 = \sigma_{k+1}^2$.

- If $\beta_{k+1} \leq 0$ (and $\beta_k \geq 0$):

$$\lim_{u \rightarrow +\infty} E_u = 0$$

since

$$\lim_{u \rightarrow +\infty} \exp - \left(u^2 \left(\frac{1}{2\sigma_{k+1}^2} - \frac{1}{2\sigma_k^2} \right) + u \left(\frac{\mu_k}{\sigma_k} - \frac{\mu_{k+1}}{\sigma_{k+1}} \right) \right) = 0$$

and

$$\lim_{u \rightarrow +\infty} \frac{\int_{-\infty}^{\alpha_{k+1} + \beta_{k+1}u} e^{-t^2/2} dt}{\int_{-\infty}^{\alpha_k + \beta_k u} e^{-t^2/2} dt} = \begin{cases} 0 & \text{if } \beta_{k+1} < 0 \\ 1/2 & \text{if } \beta_{k+1} = 0 \text{ and } \beta_k > 0 \\ 1 & \text{if } \beta_{k+1} = 0 \text{ and } \beta_k = 0 \end{cases} \quad (23)$$

- If $\beta_{k+1} < 0$ and $\beta_k < 0$: One uses the upper and lower bounds for the probit function.

$$\frac{1}{-t + \sqrt{t^2 + 4}} < \sqrt{\frac{\pi}{2}} \exp \frac{t^2}{2} \phi(t) < \frac{1}{-t + \sqrt{t^2 + 8/\pi}},$$

i.e. $\phi(t) < \sqrt{\frac{2}{\pi}} \frac{1}{-t + \sqrt{t^2 + 8/\pi}} \exp -\frac{t^2}{2}$ and $\frac{1}{\phi(t)} < (-t + \sqrt{t^2 + 4}) \sqrt{\frac{\pi}{2}} \exp \frac{t^2}{2}$. Thus, noting that $\lim_{u \rightarrow +\infty} \phi(\alpha_{k+1} + \beta_{k+1}u) = \lim_{u \rightarrow +\infty} \phi(\beta_{k+1}u)$,

$$\frac{\int_{-\infty}^{\alpha_{k+1} + \beta_{k+1}u} e^{-t^2/2} dt}{\int_{-\infty}^{\alpha_k + \beta_k u} e^{-t^2/2} dt} \underset{u \rightarrow +\infty}{=} \frac{\phi(\beta_{k+1}u)}{\phi(\beta_k u)} \underset{u \rightarrow +\infty}{<} \exp \left(\left(\frac{\beta_k^2}{2} - \frac{\beta_{k+1}^2}{2} \right) u^2 \right) \quad (24)$$

As $\beta_{k+1} \leq \beta_k < 0$, one has $\beta_k^2/2 - \beta_{k+1}^2/2 < 0$ and it implies

$$\lim_{u \rightarrow +\infty} \frac{\int_{-\infty}^{\alpha_{k+1} + \beta_{k+1}u} e^{-t^2/2} dt}{\int_{-\infty}^{\alpha_k + \beta_k u} e^{-t^2/2} dt} = 0.$$

Given that

$$\lim_{u \rightarrow +\infty} \exp - \left(u^2 \left(\frac{1}{2\sigma_{k+1}^2} - \frac{1}{2\sigma_k^2} \right) + u \left(\frac{\mu_k}{\sigma_k} - \frac{\mu_{k+1}}{\sigma_{k+1}} \right) \right) = 0,$$

assuming without loss of generality that $\sigma_k^2 > \sigma_{k+1}^2$ or $\mu_k > \mu_{k+1}$ if $\sigma_k^2 = \sigma_{k+1}^2$, one has

$$\lim_{u \rightarrow +\infty} E_u = 0.$$

This result has been already stated, in the case of univariate distributions, by Miao et al. [2016]. In particular, the identifiability conditions in Miao et al. [2016] (conditions 1 and 2 of their paper) imply the existence of the total ordering defined in Assumption **A2**. However, these conditions excludes the case of Gaussian mixture with a logistic missing-data distribution, which is very used in practice. In Corollary 1, we therefore extend this result to the multivariate case with a logistic missing-data distribution.

Note first that with a logistic distribution, a total ordering cannot be defined. Indeed, for variable j , such an ordering cannot be defined if the two univariate variances are equal (i.e., $\sigma_{kj}^2 = \sigma_{(k+1)j}^2$) and $\mu_{kj} - \beta_{kj} - \mu_{(k+1)j} + \beta_{(k+1)j} = 0$. However, for the specific case of Gaussian mixture where all the univariate variances are different between the components, then conditions of Proposition 2 hold true with a logistic missing-data distribution and so does its identifiability. In addition, for more parsimonious MNAR models for which the effect on the variable j does not depend on the class membership k (i.e. $\beta_{kj} = \beta_{(k+1)j}$), the conditions of Proposition 2 hold true with a logistic missing-data distribution. Finally, as stated by Corollary 1 below, the condition on the covariance matrices (including the case of homoscedastic Gaussian mixture) can be relaxed to obtain the generic identifiability of the model (i.e., all not-identifiable parameter choices lie within a proper submanifold, and thus form a set of Lebesgue zero measure; Allman et al. [2009]).

Corollary 1. Assume that $\sum_{k=1}^K \pi_k f_k(y_i; \lambda_k)$ is a multivariate Gaussian mixture, ρ is the logistic function and that the missingness scenario is defined by (3), (5) or (8), then, the parameters (π, λ, ψ) of the model given by (2) are generically identifiable up to label swapping, i.e. all not-identifiable parameter choices lie within a proper submanifold, and thus form a set of Lebesgue zero measure.

For the other MNAR models given in (4), (6), (7), (9) and (10), the parameters (π, λ, ψ) of the model given by (2) are identifiable up to label swapping.

Proof of Corollary 1. We use Proposition 2. We fix j . By abuse of notation, α_k, β_k, μ_k and σ_k correspond to the parameters $\alpha_{kj}, \beta_{kj}, \mu_{kj}$ and Σ_{kj} of the variable j . Let us first consider the missing scenarios (3), (5) and (8) for which $\beta_k \neq \beta_{k+1}$. To obtain the total ordering, we need to prove that

$$\lim_{u \rightarrow +\infty} E_u = \frac{(1 + e^{-\alpha_k - \beta_k u}) e^{-\frac{(u - \mu_{k+1})^2}{2\sigma_{k+1}^2}} \sigma_k}{(1 + e^{-\alpha_{k+1} - \beta_{k+1} u}) e^{-\frac{(u - \mu_k)^2}{2\sigma_k^2}} \sigma_{k+1}} = 0.$$

- If $\sigma_k^2 > \sigma_{k+1}^2$, $\lim_{u \rightarrow +\infty} E_u = \lim_{u \rightarrow +\infty} \exp -\frac{1}{2} \left(\frac{1}{\sigma_{k+1}^2} - \frac{1}{\sigma_k^2} \right) u^2 = 0$.
- If $\sigma_k^2 = \sigma_{k+1}^2$, one has $\lim_{u \rightarrow +\infty} E_u = \lim_{u \rightarrow +\infty} \exp((\mu_k - \beta_k) - (\mu_{k+1} - \beta_{k+1}))u = 0$ discarding the case where $(\mu_k - \beta_k) - (\mu_{k+1} - \beta_{k+1}) = 0$ and assuming without loss of generality that $(\mu_k - \beta_k) > (\mu_{k+1} - \beta_{k+1})$. The set of nonidentifiable parameters is $\{\mu_k, \beta_k, \mu_{k+1}, \beta_{k+1} \text{ s.t. } (\mu_k - \beta_k) - (\mu_{k+1} - \beta_{k+1}) = 0\}_{k=1, \dots, K}$ and is of Lebesgue zero measure.

Finally, for the missing scenarios (9) and (10), note that $\beta_k = \beta_{k+1} = 0$. For the missing scenarios (4), (6) and (7), one has $\beta_k = \beta_{k+1}$. Following the same reasoning as above, in the case where $\sigma_{k+1}^2 = \sigma_k^2$, one obtains the set of nonidentifiable parameters such that $\mu_k = \mu_{k+1}$, which is empty since $\mu_k \neq \mu_{k+1}$ if $\sigma_k^2 = \sigma_{k+1}^2$. \square

On identifiability of the Poisson mixture Proposition 1 can also be applied for variables with integer value (i.e. count data), as shown below in Examples 2 and 3 for the Poisson mixture with probit or logistic missing-data distributions.

Example 2 (Poisson + Probit). Considering that ρ is the probit function and f_k (respectively f_{k+1}) the Poisson distribution with parameters λ_k (respectively λ_{k+1}). Suppose without loss of generality that $\beta_k > \beta_{k+1}$ and $\lambda_k > \lambda_{k+1}$. One want to prove

$$\lim_{u \rightarrow +\infty} E_u := \lim_{u \rightarrow +\infty} \frac{\int_{-\infty}^{\alpha_{k+1} + \beta_{k+1} u} e^{-t^2/2} dt \lambda_{k+1}^u e^{-\lambda_{k+1}}}{\int_{-\infty}^{\alpha_k + \beta_k u} e^{-t^2/2} dt \lambda_k^u e^{-\lambda_k}} = 0.$$

- If $\beta_{k+1} > 0$ (and $\beta_k > 0$): using (22), one has

$$\lim_{u \rightarrow +\infty} E_u = \lim_{u \rightarrow +\infty} \exp u \ln \frac{\lambda_{k+1}}{\lambda_k} = 0.$$

- If $\beta_{k+1} \leq 0$ (and $\beta_k \geq 0$): one has

$$\lim_{u \rightarrow +\infty} E_u = 0.$$

using

$$\lim_{u \rightarrow +\infty} \exp u \ln \frac{\lambda_{k+1}}{\lambda_k} = 0$$

and (23) for the missing distribution part.

- If $\beta_{k+1} < 0$ and $\beta_k < 0$: using (24), one obtains

$$\lim_{u \rightarrow +\infty} E_u < \lim_{u \rightarrow +\infty} \exp \left(\left(\frac{\beta_k^2}{2} - \frac{\beta_{k+1}^2}{2} \right) u^2 \right) \exp u \ln \frac{\lambda_{k+1}}{\lambda_k} = 0,$$

because $\beta_k^2/2 - \beta_{k+1}^2/2 < 0$.

Example 3 (Poisson + Logistic). Considering that ρ is the logistic function and f_k (respectively f_{k+1}) the Poisson distribution with parameters λ_k (respectively λ_{k+1}). One want to prove that

$$\lim_{u \rightarrow +\infty} E_u = \lim_{u \rightarrow +\infty} \frac{1 + e^{-\alpha_k - \beta_k u}}{1 + e^{-\alpha_{k+1} - \beta_{k+1} u}} \exp u \ln \frac{\lambda_{k+1}}{\lambda_k} = 0.$$

Assume that $\lambda_k > \lambda_{k+1}$ without loss of generality.

- For the missing scenarios (3), (5) and (8) for which $\beta_k \neq \beta_{k+1}$, one obtains the generic identifiability where the set of non-identifiable parameters is $\{\alpha_k, \beta_k, \lambda_k \text{ s.t. } (\ln \lambda_k - \beta_k) - (\ln \lambda_{k+1} - \beta_{k+1}) = 0\}_{k=1, \dots, K}$ and is of Lebesgue zero measure.
- For the missing scenarios (9) and (10), note that $\beta_k = \beta_{k+1} = 0$. For the missing scenarios (4), (6) and (7), one has $\beta_k = \beta_{k+1}$. It implies that identifiability holds since

$$\lim_{u \rightarrow +\infty} E_u = \lim_{u \rightarrow +\infty} \exp u \ln \frac{\lambda_{k+1}}{\lambda_k} = 0.$$

B.2 Categorical data

We assume the following:

- A4.** The feature are independently drawn conditionally to the group membership, i.e.

$$f_k(\cdot; \theta_k) = \prod_{j=1}^d f_{kj}(\cdot; \theta_{kj}); \quad (25)$$

- A5.** The dimension d of the observations is related to the number K of clusters so that

$$d \geq 2 \lceil \log_2 K \rceil + 1,$$

with $\lceil x \rceil$ the least integer greater than or equal to x .

Assumptions **A4.** and **A5.** are classical in the categorical case, even without missing values [Allman et al., 2009]. Proposition 2 states that generic identifiability holds only for the MNAR z and the MNAR z^j missing scenarios and that the other missing scenarios lead to non-identifiable models. The proof uses Corollary 5 of Allman et al. [2009] which gives the identifiability of finite mixtures of Bernoulli products.

Proof of Proposition 2, categorical case. Let us first consider the case where $\beta_{kj} = (0, \dots, 0) \in \mathbb{R}^{\ell_j}, \forall k = 1, \dots, K, \forall j = 1, \dots, d$. Suppose there exists two sets of parameters $\{\gamma, \psi\}$ and $\{\gamma', \psi'\}$ which have the same observed distribution.

$$\begin{aligned} \forall \mathbf{y}_i \in \mathbb{R}^d, \forall \mathbf{c}_i \in \{0, 1\}^d, \sum_{k=1}^K \int_{\mathcal{Y}_i^{\text{mis}}} \pi_k f_k(\mathbf{y}_i; \lambda_k) \prod_{j=1}^d \rho(\alpha_{kj})^{c_{ij}} [1 - \rho(\alpha_{kj})]^{1-c_{ij}} dy \\ = \sum_{k=1}^{K'} \int_{\mathcal{Y}_i^{\text{mis}}} \pi'_k f_k(\mathbf{y}_i; \lambda'_k) \prod_{j=1}^d \rho((\alpha')_{kj})^{c_{ij}} [1 - \rho(\alpha'_{kj})]^{1-c_{ij}} dy \end{aligned}$$

Identifiability of ψ This implies that the marginal distributions of the pattern of missing data for the two sets of parameters ψ and ψ' are equal.

$$\sum_{k=1}^K \pi_k \prod_{j=1}^d \rho(\alpha_{kj})^{c_{ij}} [1 - \rho(\alpha_{kj})]^{1-c_{ij}} = \sum_{k=1}^{K'} \pi'_k \prod_{j=1}^d \rho(\alpha'_{kj})^{c_{ij}} [1 - \rho(\alpha'_{kj})]^{1-c_{ij}}$$

One recognizes the finite mixture of K different Bernoulli products with d components and with parameters $(\rho(\alpha_{k1}), \dots, \rho(\alpha_{kd}))_{k=1, \dots, K}$ and $(\rho(\alpha'_{k1}), \dots, \rho(\alpha'_{kd}))_{k=1, \dots, K}$. The generic identifiability up to a label swapping of these parameters is given by Corollary 5 in Allman et al. [2009]. As the function ρ is strictly monotone, the equality $\rho(\alpha_{kj}) = \rho(\alpha'_{kj})$ implies $\alpha_{kj} = \alpha'_{kj}$.

Identifiability of γ Let us consider the case where all the elements of \mathbf{y}_i are observed, i.e. $c_{ij} = 1, \forall j = 1, \dots, d$. One has

$$\sum_{k=1}^K \pi_k f_k(\mathbf{y}_i; \lambda_k) \prod_{j=1}^d \rho(\alpha_{kj}) = \sum_{k=1}^{K'} \pi'_k f_k(\mathbf{y}_i; \lambda'_k) \prod_{j=1}^d \rho(\alpha'_{kj}),$$

i.e. by independence to the group membership,

$$\begin{aligned} \sum_{k=1}^K \pi_k \prod_{j=1}^d f_{kj}(y_{ij}; \lambda_{kj}) \rho(\alpha_{kj}) &= \sum_{k=1}^{K'} \pi'_k \prod_{j=1}^d f_{kj}(y_{ij}; \lambda'_{kj}) \rho(\alpha'_{kj}), \\ \Leftrightarrow \sum_{k=1}^K \pi_k \prod_{j=1}^d \rho(\alpha_{kj}) \prod_{h=1}^{\ell_j} (\lambda_{kj}^h)^{y_{ij}^h} &= \sum_{k=1}^{K'} \pi'_k \prod_{j=1}^d \rho(\alpha_{kj}) \prod_{h=1}^{\ell_j} ((\lambda'_{kj})^h)^{y_{ij}^h}. \end{aligned}$$

We recognize the finite mixture of K multinomial distributions with d components for $y_{ij} = (y_{ij}^1, \dots, y_{ij}^{\ell_j}), j = 1, \dots, d$ with parameters $(\lambda_{kj}) = (\lambda_{kj}^1, \dots, \lambda_{kj}^{\ell_j}), j = 1, \dots, d$ and proportions $(\pi_k \prod_{j=1}^d \rho(\alpha_{kj}))_{k=1, \dots, K}$. We can thus apply Theorem 4 [Allman et al., 2009] with the model $\mathcal{M}(K; \ell_1, \dots, \ell_d)$ which gives the generic identifiability of the model parameters up to a label swapping, i.e.

$$\begin{aligned} \forall k, \forall j, \lambda_{kj}^h &= (\lambda'_{kj})^h \\ \forall k, \pi_k \prod_{j=1}^d \rho(\alpha_{kj}) &= \pi'_k \prod_{j=1}^d \rho(\alpha'_{kj}) \end{aligned}$$

The second equality implies $\pi_k = \pi'_k$ using the generic identifiability of $\rho(\alpha_{kj}), \forall k, \forall j$ stated above. If $K \neq K'$, say $K > K'$, $\sum_{k=K'+1}^K \pi_k \prod_{j=1}^d \rho(\alpha_{kj}) \prod_{h=1}^{\ell_j} (\lambda_{kj}^h)^{y_{ij}^h} = 0$ implies $\pi_k = 0$ for $K'+1 \leq k \leq K$.

We consider now the missing scenarios for which $\beta_{kj} \neq 0$. The identifiability does not hold. We can present a counter-example. The set of parameters $\psi = \{\alpha = (1, \dots, 1), \beta = (1, \dots, 1)\}$ has the same observed distribution than another set of parameters $\psi' = \{\alpha' = (0, \dots, 0), \beta' = (2, \dots, 2)\}$. Indeed, in the case where $y_{ij} = (1, \dots, 1)$, $\rho(\alpha_{kj} + \beta_{kj}y_{ij}) = \rho(\alpha'_{kj} + \beta'_{kj}y_{ij})$. □

C Detailed algorithms

The algorithms for the different missing scenarios and type of data are given. In particular, for continuous data, we derive the formulae assuming Gaussian data.

C.1 EM algorithm

The EM algorithm consists on two steps iteratively proceeded: the E-step and M-step. For the E-step, one has

$$\begin{aligned} Q(\theta; \theta^{[r-1]}) &= \mathbb{E}[\ell_{\text{comp}}(\theta; \mathbf{y}, \mathbf{z}, \mathbf{c}) | \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \theta^{[r-1]}] \\ &= \sum_{i=1}^n \sum_{k=1}^K \mathbb{E} \left[\log(\pi_k f_k(\mathbf{y}_i; \lambda)) f(\mathbf{c}_i | \mathbf{y}_i, z_{ik} = 1; \psi) \mid \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \pi^{[r-1]}, \lambda^{[r-1]}, \psi^{[r-1]} \right] \\ &= \sum_{i=1}^n \sum_{k=1}^K \int_{\mathcal{Y}_i^{\text{mis}}} \log(\pi_k f_k(\mathbf{y}_i; \lambda)) f(\mathbf{c}_i | \mathbf{y}_i, z_{ik} = 1; \psi) f(\mathbf{y}_i^{\text{mis}}, z_{ik} = 1 | \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \pi^{[r-1]}, \lambda^{[r-1]}, \psi^{[r-1]}) d\mathbf{y}_i^{\text{mis}} \\ &= \sum_{i=1}^n \sum_{k=1}^K t_{ik}(\theta^{[r-1]}) \int_{\mathcal{Y}_i^{\text{mis}}} \log(\pi_k f_k(\mathbf{y}_i; \lambda)) f(\mathbf{c}_i | \mathbf{y}_i, z_{ik} = 1; \psi) f(\mathbf{y}_i^{\text{mis}} | \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i; \pi^{[r-1]}, \lambda^{[r-1]}, \psi^{[r-1]}) d\mathbf{y}_i^{\text{mis}} \end{aligned}$$

using $f(\mathbf{y}_i^{\text{mis}}, z_{ik} = 1 | \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \theta^{[r-1]}) = t_{ik}(\theta^{[r-1]}) f(\mathbf{y}_i^{\text{mis}} | \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i; \theta^{[r-1]})$ with $t_{ik}(\theta^{[r-1]}) = f(z_{ik} = 1 | \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \theta^{[r-1]})$ in the last step.

It leads to the decomposition

$$Q(\theta; \theta^{[r-1]}) = \sum_{i=1}^n \sum_{k=1}^K t_{ik}(\theta^{[r-1]}) \left[\log(\pi_k) + \tau_y(\lambda_k; \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i, \theta^{[r-1]}) + \tau_c(\psi_k; \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i, \theta^{[r-1]}) \right],$$

where the terms involved in this decomposition are now detailed.

- (a) the expectation of the data mixture part over the missing values given the available information (i.e. the observed data and the indicator pattern), the class membership and the current value of the parameters:

$$\tau_y(\lambda_k; \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i, \theta^{[r-1]}) = \mathbb{E}_{\theta^{[r-1]}} \left[\ln f_k(y_i; \lambda_k) \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i \right],$$

- (b) the expectation of the missing mechanism part over the missing values given the available information, the class membership and the current value of the parameters:

$$\tau_c(\psi_k; \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i, \theta^{[r-1]}) = \mathbb{E}_{\theta^{[r-1]}} \left[\ln f_k(c_i | y_i; \psi_k) \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i \right].$$

(c) the conditional probability for an observation i to belong to the class k given the available information and the current value of the parameters:

$$t_{ik}(\theta^{[r-1]}) = f(z_{ik} = 1 \mid \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \theta^{[r-1]}).$$

Terms (a) and (b) require to integrate over the distribution $f(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i; \theta^{[r-1]})$. For Term (a), one has

$$\begin{aligned} f(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i; \theta^{[r-1]}) &= \frac{f(\mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i; \theta^{[r-1]})}{f(\mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i; \theta^{[r-1]})} \\ &= \frac{f(\mathbf{c}_i \mid \mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \psi^{[r-1]}) f(\mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \lambda^{[r-1]})}{\int_{\mathbf{y}_i^{\text{mis}}} f(\mathbf{c}_i \mid \mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \psi^{[r-1]}) f(\mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \lambda^{[r-1]}) d\mathbf{y}_i^{\text{mis}}} \end{aligned} \quad (26)$$

Term (c) corresponds to the conditional probability for an observation i to arise from the k th mixture component with the current values of the model parameter. More particularly, one has

$$\begin{aligned} t_{ik}(\theta^{[r-1]}) &= \frac{f(z_{ik} = 1, \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \theta^{[r-1]})}{f(\mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \theta^{[r-1]})} \\ &= \frac{f(z_{ik} = 1, \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \theta^{[r-1]})}{\sum_{h=1}^K f(z_{ih} = 1, \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \theta^{[r-1]})} \\ &= \frac{\pi_k^{[r-1]} f(\mathbf{y}_i^{\text{obs}} \mid z_{ik} = 1; \lambda_k^{[r-1]}) f(\mathbf{c}_i \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \theta^{[r-1]})}{\sum_{h=1}^K \pi_h^{[r-1]} f(\mathbf{y}_i^{\text{obs}} \mid z_{ih} = 1; \lambda_h^{[r-1]}) f(\mathbf{c}_i \mid \mathbf{y}_i^{\text{obs}}, z_{ih} = 1; \theta^{[r-1]})} \\ &= \frac{\pi_k^{[r-1]} f_k(\mathbf{y}_i^{\text{obs}}; \lambda_k^{[r-1]}) f(\mathbf{c}_i \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \theta^{[r-1]})}{\sum_{h=1}^K \pi_h^{[r-1]} f_h(\mathbf{y}_i^{\text{obs}}; \lambda_h^{[r-1]}) f(\mathbf{c}_i \mid \mathbf{y}_i^{\text{obs}}, z_{ih} = 1; \theta^{[r-1]})} \end{aligned} \quad (27)$$

The quantity that can cause numerical difficulties is the probability $f(\mathbf{c}_i \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \theta^{[r-1]})$.

C.1.1 Gaussian mixture for continuous data

The pdf $f_k(\mathbf{y}_i; \lambda) = \phi(\mathbf{y}_i; \mu_k, \Sigma_k)$ is assumed to be a Gaussian distribution with mean vector μ_k and covariance matrix Σ_k . First, let us detail the terms of the E-step. Term (a) is written as follows:

$$\begin{aligned} \mathbb{E} \left[\log(\phi(\mathbf{y}_i; \mu_k, \Sigma_k)) \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i; \theta^{[r-1]} \right] &= -\frac{1}{2} [n \log(2\pi) + \log(|\Sigma_k|)] \\ &\quad - \frac{1}{2} \mathbb{E} \left[(\mathbf{y}_i - \mu_k)^T (\Sigma_k)^{-1} (\mathbf{y}_i - \mu_k) \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i; \theta^{[r-1]} \right]. \end{aligned}$$

This last term could be expressed using the commutativity and linearity of the trace function:

$$\begin{aligned} \mathbb{E} \left[(\mathbf{y}_i - \mu_k)^T (\Sigma_k)^{-1} (\mathbf{y}_i - \mu_k) \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i; \theta^{[r-1]} \right] \\ = \text{tr} \left(\mathbb{E} \left[(\mathbf{y}_i - \mu_k)(\mathbf{y}_i - \mu_k)^T \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i; \theta^{[r-1]} \right] (\Sigma_k)^{-1} \right). \end{aligned}$$

Finally note that only $\mathbb{E} \left[(\mathbf{y}_i - \mu_k)(\mathbf{y}_i - \mu_k)^T \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i; \theta^{[r-1]} \right]$ has to be calculated.

MNAR_z and MNAR_z^j models For the MNAR_z and MNAR_z^j models, the effect of the missingness is only due to the class membership. Term (a) is the same for both models but (b) and (c) differ. Let us first detail these terms.

- For Term (a), note that

$$f(\mathbf{y}_i^{\text{mis}} | \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i; \theta^{[r-1]}) = f(\mathbf{y}_i^{\text{mis}} | \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \lambda^{[r-1]}),$$

which makes the computation easy. Indeed, using (26),

$$\begin{aligned} f(\mathbf{y}_i^{\text{mis}} | \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i; \theta^{[r-1]}) &= \frac{\prod_{j=1}^d \rho(\alpha_{kj}^{[r-1]})^{c_{ij}} (1 - \rho(\alpha_{kj}^{[r-1]}))^{1-c_{ij}} f(\mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \lambda^{[r-1]})}{\int_{\mathbf{y}_i^{\text{mis}}} \prod_{j=1}^d \rho(\alpha_{kj}^{[r-1]})^{c_{ij}} (1 - \rho(\alpha_{kj}^{[r-1]}))^{1-c_{ij}} f(\mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \lambda^{[r-1]}) d\mathbf{y}_i^{\text{mis}}} \\ &= \frac{f(\mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \lambda^{[r-1]})}{\int_{\mathbf{y}_i^{\text{mis}}} f(\mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \lambda^{[r-1]}) d\mathbf{y}_i^{\text{mis}}} = f(\mathbf{y}_i^{\text{mis}} | \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \lambda^{[r-1]}), \end{aligned}$$

since $\prod_{j=1}^d \rho(\alpha_{kj}^{[r-1]})^{c_{ij}} (1 - \rho(\alpha_{kj}^{[r-1]}))^{1-c_{ij}}$ does not depend on $\mathbf{y}_i^{\text{mis}}$ and is simplified with the numerator. The law of $(\mathbf{y}_i^{\text{mis}} | \mathbf{y}_i^{\text{obs}}, z_{ik} = 1)$ is Gaussian (up to a reorganization of the variables associated to individual i). Noting that

$$\left(\mathbf{y}_i | z_{ik} = 1; \lambda^{[r-1]} \right) = \left(\begin{pmatrix} \mathbf{y}_i^{\text{obs}} \\ \mathbf{y}_i^{\text{mis}} \end{pmatrix} | z_{ik} = 1; \lambda^{[r-1]} \right) \sim \mathcal{N} \left(\begin{pmatrix} (\mu_{ik}^{\text{obs}})^{[r-1]} \\ (\mu_{ik}^{\text{mis}})^{[r-1]} \end{pmatrix}, \begin{pmatrix} (\Sigma_{ik}^{\text{obs,obs}})^{[r-1]} & (\Sigma_{ik}^{\text{obs,mis}})^{[r-1]} \\ (\Sigma_{ik}^{\text{mis,obs}})^{[r-1]} & (\Sigma_{ik}^{\text{mis,mis}})^{[r-1]} \end{pmatrix} \right)$$

one obtains

$$\left(\mathbf{y}_i^{\text{mis}} | \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \lambda^{[r-1]} \right) \sim \mathcal{N} \left((\tilde{\mu}_{ik}^{\text{mis}})^{[r-1]}, (\tilde{\Sigma}_{ik}^{\text{mis}})^{[r-1]} \right). \quad (28)$$

with $(\tilde{\mu}_{ik}^{\text{mis}})^{[r-1]}$ and $(\tilde{\Sigma}_{ik}^{\text{mis}})^{[r-1]}$ the standard expression of the mean vector and covariance matrix of a conditional Gaussian distribution (see for instance Anderson [2003]) detailed as follows

$$(\tilde{\mu}_{ik}^{\text{mis}})^{[r-1]} = (\mu_{ik}^{\text{mis}})^{[r-1]} + (\Sigma_{ik}^{\text{mis,obs}})^{[r-1]} \left((\Sigma_{ik}^{\text{obs,obs}})^{[r-1]} \right)^{-1} \left(\mathbf{y}_i^{\text{obs}} - (\mu_{ik}^{\text{obs}})^{[r-1]} \right), \quad (29)$$

$$(\tilde{\Sigma}_{ik}^{\text{mis}})^{[r-1]} = (\Sigma_{ik}^{\text{mis,mis}})^{[r-1]} - (\Sigma_{ik}^{\text{mis,obs}})^{[r-1]} \left((\Sigma_{ik}^{\text{obs,obs}})^{[r-1]} \right)^{-1} (\Sigma_{ik}^{\text{obs,mis}})^{[r-1]}. \quad (30)$$

Note also that we have

$$(\mathbf{y}_i - \mu_k)(\mathbf{y}_i - \mu_k)^T = \begin{pmatrix} (\mathbf{y}_i^{\text{obs}} - \mu_{ik}^{\text{obs}})^T (\mathbf{y}_i^{\text{obs}} - \mu_{ik}^{\text{obs}}) & (\mathbf{y}_i^{\text{obs}} - \mu_{ik}^{\text{obs}})^T (\mathbf{y}_i^{\text{mis}} - \mu_{ik}^{\text{mis}}) \\ (\mathbf{y}_i^{\text{mis}} - \mu_{ik}^{\text{mis}})^T (\mathbf{y}_i^{\text{obs}} - \mu_{ik}^{\text{obs}}) & (\mathbf{y}_i^{\text{mis}} - \mu_{ik}^{\text{mis}})^T (\mathbf{y}_i^{\text{mis}} - \mu_{ik}^{\text{mis}}) \end{pmatrix}.$$

Therefore, the expected value of each block for the current parameter value is

$$\mathbb{E} \left[(\mathbf{y}_i^{\text{obs}} - \mu_{ik}^{\text{obs}})^T (\mathbf{y}_i^{\text{obs}} - \mu_{ik}^{\text{obs}}) | \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \lambda^{[r-1]} \right] = (\mathbf{y}_i^{\text{obs}} - \mu_{ik}^{\text{obs}})^T (\mathbf{y}_i^{\text{obs}} - \mu_{ik}^{\text{obs}})$$

$$\mathbb{E} \left[(\mathbf{y}_i^{\text{obs}} - \mu_{ik}^{\text{obs}})^T (\mathbf{y}_i^{\text{mis}} - \mu_{ik}^{\text{mis}}) | \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \lambda^{[r-1]} \right] = (\mathbf{y}_i^{\text{obs}} - \mu_{ik}^{\text{obs}})^T ((\tilde{\mu}_{ik}^{\text{mis}})^{[r-1]} - \mu_{ik}^{\text{mis}})$$

$$\mathbb{E} \left[(\mathbf{y}_i^{\text{mis}} - \mu_{ik}^{\text{mis}})^T (\mathbf{y}_i^{\text{mis}} - \mu_{ik}^{\text{mis}}) | \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \lambda^{[r-1]} \right] = ((\tilde{\mu}_{ik}^{\text{mis}})^{[r-1]} - \mu_{ik}^{\text{mis}})^T ((\tilde{\mu}_{ik}^{\text{mis}})^{[r-1]} - \mu_{ik}^{\text{mis}}) + (\tilde{\Sigma}_{ik}^{\text{mis}})^{[r-1]}$$

- For Term (b), $f(\mathbf{c}_i \mid \mathbf{y}_i, z_{ik} = 1; \psi)$ is independent of \mathbf{y}_i , which implies

$$\log(f(\mathbf{c}_i \mid z_{ik} = 1; \psi)) = \begin{cases} \sum_{j=1}^d c_{ij} \log \rho(\alpha_k) + (1 - c_{ij}) \log(1 - \rho(\alpha_k)) & (\text{MNAR}_z) \\ \sum_{j=1}^d c_{ij} \log \rho(\alpha_{kj}) + (1 - c_{ij}) \log(1 - \rho(\alpha_{kj})) & (\text{MNAR}_{z^j}) \end{cases} \quad (31)$$

- For Term (c), one first remark that

$$\mathbb{P}(\mathbf{c}_i \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \theta^{[r-1]}) = \prod_{j=1}^d \mathbb{P}(c_{ij} = 1 \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \theta^{[r-1]})^{c_{ij}} \mathbb{P}(c_{ij} = 0 \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \theta^{[r-1]})^{1-c_{ij}}.$$

In particular, for MNAR_z and MNAR_{z^j} , by independence of \mathbf{y}_i , one has

$$\mathbb{P}(c_{ij} = 1 \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \theta^{[r-1]}) = \mathbb{P}(c_{ij} = 1 \mid z_{ik} = 1; \theta^{[r-1]}) = \begin{cases} \rho(\alpha_k) & (\text{MNAR}_z) \\ \rho(\alpha_{kj}) & (\text{MNAR}_{z^j}) \end{cases}$$

Using (27), one obtains

$$t_{ik}^{[r-1]}(\theta^{[r-1]}) = \begin{cases} \frac{\pi_k^{[r-1]} \phi(\mathbf{y}_i^{\text{obs}}; (\mu_{ik}^{\text{obs}})^{[r-1]}, (\Sigma_{ik}^{\text{obs,obs}})^{[r-1]}) \prod_{j=1}^d \rho(\alpha_k^{[r-1]})^{c_{ij}} (1 - \rho(\alpha_k^{[r-1]}))^{1-c_{ij}}}{\sum_{h=1}^K \pi_h^{[r-1]} \phi(\mathbf{y}_i^{\text{obs}}; (\mu_{ih}^{\text{obs}})^{[r-1]}, (\Sigma_{ih}^{\text{obs,obs}})^{[r-1]}) \prod_{j=1}^d \rho(\alpha_k^{[r-1]})^{c_{ij}} (1 - \rho(\alpha_k^{[r-1]}))^{1-c_{ij}}} & (\text{MNAR}_z) \\ \frac{\pi_k^{[r-1]} \phi(\mathbf{y}_i^{\text{obs}}; (\mu_{ik}^{\text{obs}})^{[r-1]}, (\Sigma_{ik}^{\text{obs,obs}})^{[r-1]}) \prod_{j=1}^d \rho(\alpha_{kj}^{[r-1]})^{c_{ij}} (1 - \rho(\alpha_{kj}^{[r-1]}))^{1-c_{ij}}}{\sum_{h=1}^K \pi_h^{[r-1]} \phi(\mathbf{y}_i^{\text{obs}}; (\mu_{ih}^{\text{obs}})^{[r-1]}, (\Sigma_{ih}^{\text{obs,obs}})^{[r-1]}) \prod_{j=1}^d \rho(\alpha_{kj}^{[r-1]})^{c_{ij}} (1 - \rho(\alpha_{kj}^{[r-1]}))^{1-c_{ij}}} & (\text{MNAR}_{z^j}) \end{cases} \quad (32)$$

If ρ is the logistic distribution, the expression can be written more simply

$$t_{ik}(\theta^{[r-1]}) \propto \pi_k^{[r-1]} \phi(\mathbf{y}_i^{\text{obs}}; \lambda_k^{[r-1]}) \prod_{j=1}^d (1 + \exp(-\delta_{ij} \alpha_{kj}^{[r-1]}))^{-1} \text{ where } \delta_{ij} = \begin{cases} 1 & \text{if } c_{ij} = 1 \\ -1 & \text{otherwise.} \end{cases}$$

Finally, the E-step and the M-step can be sketched as follows in the Gaussian mixture case.

E-step The E-step for Term (a) consists of computing for $k = 1, \dots, K$ and $i = 1, \dots, n$

$$\begin{aligned} (\tilde{\mu}_{ik}^{\text{mis}})^{[r-1]} &= (\mu_{ik}^{\text{mis}})^{[r-1]} + (\Sigma_{ik}^{\text{mis,obs}})^{[r-1]} \left((\Sigma_{ik}^{\text{obs,obs}})^{[r-1]} \right)^{-1} \left(\mathbf{y}_i^{\text{obs}} - (\mu_{ik}^{\text{obs}})^{[r-1]} \right) \\ (\tilde{\Sigma}_{ik}^{\text{mis}})^{[r-1]} &= (\Sigma_{ik}^{\text{mis,mis}})^{[r-1]} - (\Sigma_{ik}^{\text{mis,obs}})^{[r-1]} \left((\Sigma_{ik}^{\text{obs,obs}})^{[r-1]} \right)^{-1} (\Sigma_{ik}^{\text{obs,mis}})^{[r-1]} \\ (\tilde{\mathbf{y}}_{i,k})^{[r-1]} &= (\mathbf{y}_i^{\text{obs}}, (\tilde{\mu}_{ik}^{\text{mis}})^{[r-1]}) \\ \tilde{\Sigma}_{ik}^{[r-1]} &= \begin{pmatrix} 0_i^{\text{obs,obs}} & 0_i^{\text{obs,mis}} \\ 0_i^{\text{mis,obs}} & (\tilde{\Sigma}_{ik}^{\text{mis}})^{[r-1]} \end{pmatrix} \end{aligned}$$

Note that whenever the mixture covariance matrices are supposed diagonal then $(\tilde{\Sigma}_{ik}^{\text{mis}})^{[r-1]}$ is also a diagonal matrix. Term (c) also requires the computation of $t_{ik}(\theta^{[r-1]})$ given in (32) for $k = 1, \dots, K$ and $i = 1, \dots, n$.

M-step The maximization of $Q(\theta; \theta^{[r-1]})$ over (π, λ) leads to, for $k = 1, \dots, K$,

$$\begin{aligned}\pi_k^{[r]} &= \frac{1}{n} \sum_{i=1}^n t_{ik}(\theta^{[r-1]}) \\ \mu_k^{[r]} &= \frac{\sum_{i=1}^n t_{ik}(\theta^{[r-1]})(\tilde{y}_{k,i})^{[r-1]}}{\sum_{i=1}^n t_{ik}(\theta^{[r-1]})} \\ \Sigma_k^{[r]} &= \frac{\sum_{i=1}^n \left[t_{ik}(\theta^{[r-1]}) \left((\tilde{y}_{i,k})^{[r-1]} - \mu_k^{[r]} \right) \left((\tilde{y}_{i,k})^{[r-1]} - \mu_k^{[r]} \right)^T + \tilde{\Sigma}_{ik}^{[r-1]} \right]}{\sum_{i=1}^n t_{ik}(\theta^{[r-1]})}\end{aligned}$$

Then, the maximization of $Q(\theta; \theta^{[r-1]})$ over ψ can be performed using a Newton Raphson algorithm. For $k = 1, \dots, K$, it remains to fit a generalized linear model with the binomial link function for the matrix $(\mathcal{J}_k^{\text{MNAR}z})^{[r]}$ (if the model is MNAR z) or for the matrices $(\mathcal{J}_{kj}^{\text{MNAR}z^j})_{j=1, \dots, d}^{[r]}$ (for the MNAR z model) and by giving $t_{ik}(\theta^{[r-1]})$ as prior weights to fit the process.

$$(\mathcal{J}_k^{\text{MNAR}z})^{[r]} = \begin{matrix} c_{.1} & 1 \\ \vdots & \vdots \\ c_{.d} & 1 \end{matrix} \quad (33)$$

$$(\mathcal{J}_{kj}^{\text{MNAR}z^j})^{[r]} = \begin{matrix} c_{.j} & 1 \end{matrix} \quad (34)$$

The EM algorithm for the MNAR z^j model is described in Algorithm 1 for Gaussian mixture.

MNAR y^* models For missing scenarios which model the effect of the missingness depending on the variable, the computations are more difficult.

- Because of the dependence of y , $f(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i; \theta^{[r-1]}) = f(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \theta^{[r-1]})$ does not hold anymore. Here, one has

$$\begin{aligned}f(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i; \theta^{[r-1]}) &= \frac{\prod_{h=1}^d \rho(\alpha_{kh}^{[r-1]} + \beta_{kh}^{[r-1]} y_{ih}^{\text{mis}})^{c_{ih}} (1 - \rho(\alpha_{kh}^{[r-1]} + \beta_{kh}^{[r-1]} y_{ih}^{\text{obs}}))^{1-c_{ih}} f(\mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \theta^{[r-1]})}{\int_{\mathbf{y}_i^{\text{mis}}} \prod_{h=1}^d \rho(\alpha_{kh}^{[r-1]} + \beta_{kh}^{[r-1]} y_{ih}^{\text{mis}})^{c_{ih}} (1 - \rho(\alpha_{kh}^{[r-1]} + \beta_{kh}^{[r-1]} y_{ih}^{\text{obs}}))^{1-c_{ih}} f(\mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \theta^{[r-1]}) d\mathbf{y}_i^{\text{mis}}} \\ &= \frac{\prod_{h, c_{ih}=1} \rho(\alpha_{kh}^{[r-1]} + \beta_{kh}^{[r-1]} y_{ih}^{\text{mis}}) f(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \theta^{[r-1]})}{\int_{\mathbf{y}_i^{\text{mis}}} \prod_{h, c_{ih}=1} \rho(\alpha_{kh}^{[r-1]} + \beta_{kh}^{[r-1]} y_{ih}^{\text{mis}}) f(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \theta^{[r-1]}) d\mathbf{y}_i^{\text{mis}}}.\end{aligned} \quad (35)$$

which implies that Term (a) requires difficult computations if this distribution is not classical.

- For Term (b), it is the same problem, since $f(\mathbf{c}_i \mid \mathbf{y}_i, z_{ik} = 1; \psi)$ is no longer independent of \mathbf{y} , then it requires a specific numerical integration. Using (35),

$$\begin{aligned}\tau_c(\psi_k; \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i, \theta^{[r-1]}) &= \mathbb{E} \left[\log \left(\prod_{j=1}^d \rho(\alpha_{kj} + \beta_{kj} y_{ij})^{c_{ij}} (1 - \rho(\alpha_{kj} + \beta_{kj} y_{ij}))^{1-c_{ij}} \right) \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i; \theta^{[r-1]} \right] \\ &= \sum_{j=1}^d c_{ij} \int_{\mathbf{y}_i^{\text{mis}}} \log(\rho(\alpha_{kj} + \beta_{kj} y_{ij}^{\text{mis}})) f(y_{ij}^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i; \theta^{[r-1]}) d\mathbf{y}_i^{\text{mis}} \\ &\quad + (1 - c_{ij}) \log(1 - \rho(\alpha_{kj} + \beta_{kj} y_{ij}^{\text{obs}}))\end{aligned}$$

Algorithm 1 EM algorithm for Gaussian mixture and MNAR z^j model

Input: $\mathbf{Y} \in \mathbb{R}^{n \times d}$ (matrix containing missing values), $K \geq 1$, r_{\max} .

Initialize $\pi_k^{[0]}$, $\mu_k^{[0]}$, $\Sigma_k^{[0]}$ and $\psi_k^{[0]}$, for $k \in \{1, \dots, K\}$.

for $r = 0$ **to** r_{\max} **do**

E-step:

for $i = 1$ **to** n , $k = 1$ **to** K **do**

$$(\tilde{\mu}_{ik}^{\text{mis}})^{[r-1]} = (\mu_{ik}^{\text{mis}})^{[r-1]} + (\Sigma_{ik}^{\text{mis,obs}})^{[r-1]} \left((\Sigma_{ik}^{\text{obs,obs}})^{[r-1]} \right)^{-1} (\mathbf{y}_i^{\text{obs}} - (\mu_{ik}^{\text{obs}})^{[r-1]}).$$

$$(\tilde{\Sigma}_{ik}^{\text{mis}})^{[r-1]} = (\Sigma_{ik}^{\text{mis,mis}})^{[r-1]} - (\Sigma_{ik}^{\text{mis,obs}})^{[r-1]} \left((\Sigma_{ik}^{\text{obs,obs}})^{[r-1]} \right)^{-1} (\Sigma_{ik}^{\text{obs,mis}})^{[r-1]}.$$

$$(\tilde{\mathbf{y}}_{i,k})^{[r-1]} = (\mathbf{y}_i^{\text{obs}}, (\tilde{\mu}_{ik}^{\text{mis}})^{[r-1]}).$$

$\tilde{\Sigma}_{ik}^{[r-1]} = \begin{pmatrix} 0_i^{\text{obs,obs}} & 0_i^{\text{obs,mis}} \\ 0_i^{\text{obs,mis}} & (\tilde{\Sigma}_{ik}^{\text{mis}})^{[r-1]} \end{pmatrix}$, where $0_i^{\text{obs,obs}}$ and $0_i^{\text{obs,mis}}$ are the null matrix of size $n_i^{\text{obs}} \times n_i^{\text{obs}}$ and $n_i^{\text{obs}} \times n_i^{\text{mis}}$, with n_i^{obs} (resp. n_i^{mis}) the number of observed (reps. missing) variables for individual i .

$$t_{ik}(\theta^{[r-1]}) \propto \pi_k^{[r-1]} \phi(\mathbf{y}_i^{\text{obs}}; (\mu_{ik}^{\text{obs}})^{[r-1]}, (\Sigma_{ik}^{\text{obs,obs}})^{[r-1]}) \prod_{j=1}^d \rho(\alpha_{kj}^{[r-1]})^{c_{ij}} (1 - \rho(\alpha_{kj}^{[r-1]}))^{1-c_{ij}}$$

end for

M-step:

for $k = 1$ **to** K **do**

$$\pi_k^{[r]} = \frac{1}{n} \sum_{i=1}^n t_{ik}(\theta^{[r-1]}), \quad \mu_k^{[r]} = \frac{\sum_{i=1}^n t_{ik}(\theta^{[r-1]})(\tilde{\mathbf{y}}_{k,i})^{[r-1]}}{\sum_{i=1}^n t_{ik}(\theta^{[r-1]})}$$

$$\Sigma_k^{[r]} = \frac{\sum_{i=1}^n [t_{ik}(\theta^{[r-1]})(\tilde{\mathbf{y}}_{i,k})^{[r-1]} - \mu_k^{[r]}][(\tilde{\mathbf{y}}_{i,k})^{[r-1]} - \mu_k^{[r]}]^T + \tilde{\Sigma}_{ik}^{[r-1]}}{\sum_{i=1}^n t_{ik}(\theta^{[r-1]})}$$

Let $\psi_k^{[r]}$ be the coefficients of a GLM with a binomial link function, by giving prior weights $t_{ik}(\theta^{[r-1]})$. In particular, the optimization problem is, $\forall j \in \{1, \dots, d\}$,

$$\mathcal{M}_{kj} \psi_k^{[r]} = \log \left(\frac{1 - \mathbb{E}[\mathbf{c}_{\cdot j} | \mathcal{M}_{kj}]}{\mathbb{E}[\mathbf{c}_{\cdot j} | \mathcal{M}_{kj}]} \right),$$

for a matrix \mathcal{M}_{kj} depending on the MNAR model (see (33) and (34)) and $\mathbf{c}_{\cdot j}$ the missing data pattern for the variable j .

end for

end for

where

$$\begin{aligned} & f(y_{ij}^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i; \theta^{[r-1]}) \\ &= \frac{\rho(\alpha_{kj}^{[r-1]} + \beta_{kj}^{[r-1]} y_{ij}^{\text{mis}})^{c_{ij}} (1 - \rho(\alpha_{kj}^{[r-1]} + \beta_{kj}^{[r-1]} y_{ij}^{\text{mis}}))^{1-c_{ij}} f(y_{ij}^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \theta^{[r-1]})}{\int_{\mathcal{Y}_{ij}^{\text{mis}}} \rho(\alpha_{kj}^{[r-1]} + \beta_{kj}^{[r-1]} y_{ij}^{\text{mis}})^{c_{ij}} (1 - \rho(\alpha_{kj}^{[r-1]} + \beta_{kj}^{[r-1]} y_{ij}^{\text{mis}}))^{1-c_{ij}} f(y_{ij}^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \theta^{[r-1]}) dy_{ij}^{\text{mis}}}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \tau_c(\psi_k; \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i, \theta^{[r-1]}) \\ &= \sum_{j=1}^d c_{ij} \int_{\mathcal{Y}_{ij}^{\text{mis}}} \log(\rho(\alpha_{kj} + \beta_{kj} y_{ij}^{\text{mis}})) \frac{\rho(\alpha_{kj}^{[r-1]} + \beta_{kj}^{[r-1]} y_{ij}^{\text{mis}})^{c_{ij}} f(y_{ij}^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \theta^{[r-1]})}{\int_{\mathcal{Y}_{ij}^{\text{mis}}} \rho(\alpha_{kj}^{[r-1]} + \beta_{kj}^{[r-1]} x)^{c_{ij}} f(x \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \theta^{[r-1]}) dx} dy_{ij}^{\text{mis}} \\ & \quad + (1 - c_{ij}) \log(1 - \rho(\alpha_{kj} + \beta_{kj} y_{ij}^{\text{obs}})) \end{aligned}$$

- There is no closed-form expression for Term (c).

$$\begin{aligned} & f(c_{ij} \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \theta^{[r-1]}) \\ &= \int_{\mathcal{Y}_{ij}^{\text{mis}}} f(c_{ij} \mid \mathbf{y}_i^{\text{obs}}, y_{ij}^{\text{mis}}, z_{ik} = 1; \psi^{[r-1]}) f(\mathbf{y}_{ij}^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1; \theta^{[r-1]}) dy_{ij}^{\text{mis}} \\ &= c_{ij} \int_{-\infty}^{+\infty} \rho(\alpha_{kj}^{[r-1]} + \beta_{kj}^{[r-1]} y_{ij}^{\text{mis}}) \phi(y_{ij}^{\text{mis}}; (\tilde{\mu}_{ik}^{\text{mis}})_j^{[r-1]}, (\tilde{\Sigma}_{ik}^{\text{mis}})_{jj}^{[r-1]}) dy_{ij}^{\text{mis}} + (1 - c_{ij}) (1 - \rho(\alpha_{kj}^{[r-1]} + \beta_{kj}^{[r-1]} y_{ij}^{\text{obs}})) \end{aligned} \tag{36}$$

Using (27), the probabilities $t_{ik}(\theta^{[r-1]})$ can be deduced from Equation (36).

Let us detail the difficulties for two particular cases, if ρ is logistic or probit.

- **ρ is logistic:** Equation (35) leads to none classical distribution because

$$f(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i; \theta^{[r-1]}) \propto \prod_{h, c_{ih}=1} \frac{1}{\exp(-(\alpha_{kh}^{[r-1]} + \beta_{kh}^{[r-1]} y_{ih}^{\text{mis}}))} \phi(y_i^{\text{mis}}; (\tilde{\mu}_{ik}^{\text{mis}})_j^{[r-1]}, (\tilde{\Sigma}_{ik}^{\text{mis}})_{jj}^{[r-1]})$$

Term (b) is

$$\begin{aligned} & \tau_c(\psi_k; \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i, \theta^{[r-1]}) \\ & \propto \sum_{j=1}^d c_{ij} \int_{\mathcal{Y}_{ij}^{\text{mis}}} -\log(1 + \exp(-(\alpha_{kj} + \beta_{kj} y_{ij}^{\text{mis}}))) \frac{1}{1 + \exp(-(\alpha_{kj}^{[r-1]} + \beta_{kj}^{[r-1]} y_{ij}^{\text{mis}}))} \phi(y_{ij}^{\text{mis}}; (\tilde{\mu}_{ik}^{\text{mis}})_j^{[r-1]}, (\tilde{\Sigma}_{ik}^{\text{mis}})_{jj}^{[r-1]}) dy_{ij}^{\text{mis}} \\ & \quad - (1 - c_{ij}) \log(1 + \exp(\alpha_{kj} + \beta_{kj} y_{ij}^{\text{obs}})), \end{aligned}$$

which amounts to compute the Gaussian moment of $\frac{\log(1 + \exp(-u))}{1 + \exp(-u)}$, but it has no closed form to our knowledge.

Finally, Equation (36) does not have a closed form either because it requires the computation of

$$\int_{-\infty}^{+\infty} \frac{1}{1 + \exp(-(\alpha_{kj}^{[r-1]} + \beta_{kj}^{[r-1]} y_{ij}^{\text{mis}}))} \phi(y_{ij}^{\text{mis}}; (\tilde{\mu}_{ik}^{\text{mis}})_j^{[r-1]}, (\tilde{\Sigma}_{ik}^{\text{mis}})_{jj}^{[r-1]}) dy_{ij}^{\text{mis}},$$

i.e. the computation of the Gaussian moment of $\frac{1}{1 + \exp(-u)}$.

- **ρ is Probit:** One can prove (presented in Appended C.2.1) that the conditional distribution $(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i)$ is a truncated Gaussian, which makes possible the computation of Term (a). Term (b) has no closed form to our knowledge

$$\begin{aligned} & \tau_c(\psi_k; \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i, \theta^{[r-1]}) \\ & \propto \sum_{j=1}^d c_{ij} \int_{\mathbf{y}_{ij}^{\text{mis}}} \log \left(\int_{-\infty}^{\alpha_{kj} + \beta_{kj} y_{ij}^{\text{mis}}} e^{-t^2} dt \right) \frac{1}{1 + \exp(\alpha_{kj}^{[r-1]} + \beta_{kj}^{[r-1]} y_{ij}^{\text{mis}})} \phi(y_{ij}^{\text{mis}}; (\tilde{\mu}_{ik}^{\text{mis}})_j^{[r-1]}, (\tilde{\Sigma}_{ik}^{\text{mis}})_{jj}^{[r-1]}) dy_{ij}^{\text{mis}} \\ & \quad - (1 - c_{ij}) \log \left(1 - \int_{-\infty}^{\alpha_{kj} + \beta_{kj} y_{ij}^{\text{obs}}} e^{-t^2} dt \right), \end{aligned}$$

Equation (36) does not have a closed form either because it requires the computation of

$$\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{\alpha_{kj} + \beta_{kj} y_{ij}^{\text{mis}}} e^{-t^2} dt \right) \phi(y_{ij}^{\text{mis}}; (\tilde{\mu}_{ik}^{\text{mis}})_j^{[r-1]}, (\tilde{\Sigma}_{ik}^{\text{mis}})_{jj}^{[r-1]}) dy_{ij}^{\text{mis}}.$$

C.1.2 Latent class model for categorical data

For categorical data, we have $\phi(\mathbf{y}_i; \lambda_k) = \prod_{j=1}^d \phi(y_{ij}; \lambda_{kj}) = \prod_{j=1}^d \prod_{\ell=1}^{\ell_j} (\lambda_{kj}^{\ell})^{y_{ij}^{\ell}}$.

MNAR z and MNAR z^j models Term (a) is

$$\mathbb{E} \left[\log(\phi(\mathbf{y}_i; p_k)) \mid \mathbf{y}_i^{\text{obs}}, z_{ik} = 1, \mathbf{c}_i; \lambda^{[r-1]} \right] = \sum_{j, c_{ij}=0} \sum_{\ell=1}^{\ell_j} y_{ij}^{\ell} + \sum_{j, c_{ij}=1} \sum_{\ell=1}^{\ell_j} \log(\lambda_{kj}^{\ell}) \quad (37)$$

Term (b) is the same as in the Gaussian case given in (31). Finally, the EM algorithm can be summarized as follows

E step: For $k = 1, \dots, K$ and $i = 1, \dots, n$, compute

$$\begin{aligned} t_{ik}(\theta^{[r-1]}) &= \frac{\pi_k^{[r-1]} \prod_{j, c_{ij}=0} \prod_{\ell=1}^{\ell_j} (\lambda_{kj}^{\ell})^{y_{ij}^{\ell}} \prod_{j=1}^d \rho(\alpha_{kj})}{\sum_{h=1}^K \pi_h^{[r-1]} \prod_{j, c_{ij}=0} \prod_{\ell=1}^{\ell_j} (\lambda_{hj}^{\ell})^{y_{ij}^{\ell}} \prod_{j=1}^d \rho(\alpha_{hj})} \\ (\tilde{y}_{ij,k}^{\ell})^{[r-1]} &= c_{ij} (\theta_{kj}^{\ell})^{[r-1]} + (1 - c_{ij}) y_{ij}^{\ell}, \quad \forall j = 1, \dots, d, \forall \ell = 1, \dots, \ell_j \end{aligned}$$

M step: The maximization of $Q(\theta; \theta^{[r-1]})$ over θ leads to, for $k = 1, \dots, K$,

$$\begin{aligned} \pi_k^{[r]} &= \frac{1}{n} \sum_{i=1}^n t_{ik}(\theta^{[r-1]}) \\ (\theta_{kj}^{\ell})^{[r]} &= \frac{\sum_{i=1}^n t_{ik}(\theta^{[r-1]}) (\tilde{y}_{ij,k}^{\ell})^{[r-1]}}{\sum_{i=1}^n t_{ik}(\theta^{[r-1]})}, \quad \forall j = 1, \dots, d, \forall \ell = 1, \dots, \ell_j \end{aligned}$$

The M-step for ψ consists of performing a GLM with a binomial link and has already been given in detail in Appendix C.1.1 (see (50) and (51)).

C.1.3 Combining Gaussian mixture and latent class model for mixed data

If the data are mixed (continuous and categorical), the formulas can be extended straightforwardly if the continuous and the categorical variables are assumed to be independent knowing the latent clusters.

C.2 SEM algorithm

The SEM algorithm consists on two steps iteratively proceeded as presented in Section 3.2. The key issue is to draw the missing data $(\mathbf{y}_i^{\text{mis}})^{[r]}$ and $\mathbf{z}_i^{[r]}$ according to their current conditional distribution $f(\mathbf{y}_i^{\text{mis}}, \mathbf{z}_i \mid \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \pi^{[r-1]}, \lambda^{[r-1]}, \psi^{[r-1]})$. By convenience, we use a Gibbs sampling and simulate two easier probabilities recalled here

$$\mathbf{z}_i^{[r]} \sim f(\cdot \mid \mathbf{y}_i^{[r-1]}, \mathbf{c}_i; \pi^{[r-1]}, \lambda^{[r-1]}, \psi^{[r-1]}) \quad \text{and} \quad (\mathbf{y}_i^{\text{mis}})^{[r]} \sim f(\cdot \mid \mathbf{y}_i^{\text{obs}}, \mathbf{z}_i^{[r]}, \mathbf{c}_i; \lambda^{[r-1]}, \psi^{[r-1]}),$$

where $\mathbf{y}_i^{[r-1]} = (\mathbf{y}_i^{\text{obs}}, (\mathbf{y}_i^{\text{mis}})^{[r-1]})$. For the first distribution, the membership k of $z_i^{[r]}$ is drawn from the multinomial distribution with probabilities $(\mathbb{P}(z_{ik} = 1 \mid \mathbf{y}_i^{[r-1]}, \mathbf{c}_i; \lambda^{[r-1]}, \psi^{[r-1]}))_{k=1, \dots, K}$ detailed as follows

$$\begin{aligned} \mathbb{P}(z_{ik} = 1 \mid \mathbf{y}_i^{[r-1]}, \mathbf{c}_i; \theta^{[r-1]}) &= \frac{\mathbb{P}(z_{ik} = 1, \mathbf{y}_i^{[r-1]}, \mathbf{c}_i; \pi^{[r-1]}, \lambda^{[r-1]}, \psi^{[r-1]})}{\mathbb{P}(\mathbf{y}_i^{[r-1]}, \mathbf{c}_i; \pi^{[r-1]}, \lambda^{[r-1]}, \psi^{[r-1]})} \\ &= \frac{\mathbb{P}(\mathbf{c}_i \mid \mathbf{y}_i^{[r-1]}, z_{ik} = 1; \psi^{[r-1]}) \mathbb{P}(\mathbf{y}_i^{[r-1]} \mid z_{ik} = 1; \lambda^{[r-1]}) \mathbb{P}(z_{ik} = 1; \pi^{[r-1]})}{\sum_{h=1}^K \mathbb{P}(\mathbf{c}_i \mid \mathbf{y}_i^{[r-1]}, z_{ih} = 1; \psi^{[r-1]}) \mathbb{P}(\mathbf{y}_i^{[r-1]} \mid z_{ih} = 1; \lambda^{[r-1]}) \mathbb{P}(z_{ih} = 1; \pi^{[r-1]})} \\ &= \frac{\mathbb{P}(\mathbf{c}_i \mid \mathbf{y}_i^{[r-1]}, z_{ik} = 1; \psi^{[r-1]}) \mathbb{P}(\mathbf{y}_i^{[r-1]} \mid z_{ik} = 1; \lambda^{[r-1]}) \pi_k^{[r-1]}}{\sum_{h=1}^K \mathbb{P}(\mathbf{c}_i \mid \mathbf{y}_i^{[r-1]}, z_{ih} = 1; \psi^{[r-1]}) \mathbb{P}(\mathbf{y}_i^{[r-1]} \mid z_{ih} = 1; \lambda^{[r-1]}) \pi_h^{[r-1]}} \end{aligned} \quad (38)$$

The conditional distribution of $((\mathbf{y}_i^{\text{mis}}) \mid \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1, \mathbf{c}_i)$ has already been detailed in Equation (35) and recalled here

$$\begin{aligned} f(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1, \mathbf{c}_i; \theta^{[r-1]}) \\ = \frac{\prod_{j, c_{ij}=1} f(c_{ij} = 1 \mid \mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1; \psi^{[r-1]}) f(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1; \lambda^{[r-1]})}{\int_{\mathcal{Y}_i^{\text{mis}}} \prod_{j, c_{ij}=1} f(c_{ij} = 1 \mid \mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1; \psi^{[r-1]}) f(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1; \lambda^{[r-1]}) d\mathbf{y}_i^{\text{mis}}} \end{aligned} \quad (39)$$

C.2.1 Gaussian mixture for continuous data

First note that the probabilities of the multinomial distribution for drawing $\mathbf{z}_i^{[r]}$ given in (38) can be easily computed for all cases.

$$\begin{aligned} \mathbb{P}(z_{ik} = 1 \mid \mathbf{y}_i^{[r-1]}, \mathbf{c}_i; \theta^{[r-1]}) \\ = \frac{\prod_{j=1}^d f(c_{ij} = 1 \mid \mathbf{y}_i^{[r-1]}, z_{ik} = 1; \psi^{[r-1]})^{c_{ij}} f(c_{ij} = 0 \mid \mathbf{y}_i^{[r-1]}, z_{ik} = 1; \psi^{[r-1]})^{1-c_{ij}} \phi(\mathbf{y}_i^{[r-1]}; \lambda_k^{[r-1]}) \pi_k^{[r-1]}}{\sum_{h=1}^K \prod_{j=1}^d f(c_{ij} = 1 \mid \mathbf{y}_i^{[r-1]}, z_{ih} = 1; \psi^{[r-1]})^{c_{ij}} f(c_{ij} = 0 \mid \mathbf{y}_i^{[r-1]}, z_{ih} = 1; \psi^{[r-1]})^{1-c_{ij}} \phi(\mathbf{y}_i^{[r-1]}; \lambda_h^{[r-1]}) \pi_h^{[r-1]}} \end{aligned}$$

where $\phi(\mathbf{y}_i; \lambda_k) = \phi(\mathbf{y}_i; \mu_k, \Sigma_k)$ is assumed to be a Gaussian distribution with mean vector μ_k and covariance matrix Σ_k , and $f(c_{ij} = 1 \mid \mathbf{y}_i^{[r-1]}, z_{ih} = 1; \psi^{[r-1]})$ is specified depending the MNAR model and the distribution ρ . The only difficulty of the SE-step is thus to draw from the distribution $(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1, \mathbf{c}_i)$.

In the sequel, we detail the distribution $(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1, \mathbf{c}_i)$ and the M-step for ψ depending the MNAR model.

MNAR y^* models The conditional distribution $(y_i^{\text{mis}} \mid y_i^{\text{obs}}, z_{ik}^{[r]} = 1, c_i)$ depends on the distribution ρ at hand. For the MNAR y^* models, we will consider two classical distributions for ρ : the logistic function and probit one.

Logistic distribution: For the logistic function, the distribution given in (39) is not classical and drawing y_i^{mis} from it seems complicated. Indeed, one has

$$f(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1, \mathbf{c}_i; \theta^{[r-1]}) \propto \prod_{j=1, c_{ij}=1} \frac{1}{1 + \exp(\alpha_{kj}^{[r-1]} + \beta_{kj}^{[r-1]} y_{ij}^{\text{mis}})} \phi(\mathbf{y}_i^{\text{mis}}; (\tilde{\mu}_{ik}^{\text{mis}})^{[r-1]}, (\tilde{\Sigma}_{ik}^{\text{mis}})^{[r-1]}),$$

where $(\tilde{\mu}_{ik}^{\text{mis}})^{[r-1]}$ and $(\tilde{\Sigma}_{ik}^{\text{mis}})^{[r-1]}$ are given in (29) and (30). We could use the Sampling Importance Resampling (SIR) algorithm which simulates a realization of $(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1, \mathbf{c}_i)$ with a known instrumental distribution (for example: $(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1)$) and includes a re-sampling step. However, this algorithm may be computationnaly costly.

Probit distribution: For the probit function, the distribution in (39) can be made explicit by using a latent variable \mathbf{L}_i .

More particularly, let $\mathbf{L}_i^{[r-1]}$ such that $\mathbf{L}_i = \alpha_k^{[r-1]} + \beta_k^{[r-1]} \mathbf{y}_i + \epsilon_i$, with $\epsilon_i \sim \mathcal{N}(0_d, I_{d \times d})$. Then, \mathbf{c}_i can be viewed as an indicator for whether this latent variable is positive, i.e. for all $j = 1, \dots, d$,

$$c_{ij} = \begin{cases} 1 & \text{if } L_{ij} > 0 \\ 0 & \text{otherwise.} \end{cases} \quad (40)$$

Thus, indeed to draw $(\mathbf{y}_i^{\text{mis}})^{[r]}$ and $\mathbf{z}_i^{[r]}$ according to $f(\mathbf{y}_i^{\text{mis}}, \mathbf{z}_i \mid \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \theta^{[r-1]})$, we draw $\mathbf{L}_i^{[r]}$, $(\mathbf{y}_i^{\text{mis}})^{[r-1]}$ and $\mathbf{z}_i^{[r-1]}$ according to $f(\mathbf{L}_i, \mathbf{y}_i^{\text{mis}}, \mathbf{z}_i \mid \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \theta^{[r-1]})$ by using a Gibbs sampling.

First, we have to draw $\mathbf{L}_i^{[r]}$ according to $f(\cdot \mid \mathbf{y}_i^{[r-1]}, z_{ik}^{[r-1]} = 1, \mathbf{c}_i; \psi^{[r-1]})$. One has

$$\begin{aligned} f(\mathbf{L}_i \mid \mathbf{y}_i^{[r-1]}, z_{ik}^{[r-1]} = 1, \mathbf{c}_i) &\propto f(\mathbf{L}_i, \mathbf{c}_i \mid \mathbf{y}_i^{[r-1]}, z_{ik}^{[r-1]} = 1; \psi^{[r-1]}) \\ &\propto f(\mathbf{c}_i \mid \mathbf{L}_i, \mathbf{y}_i^{[r-1]}, z_{ik}^{[r-1]} = 1; \psi^{[r-1]}) f(\mathbf{L}_i \mid \mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r-1]} = 1; \psi^{[r-1]}) \\ &\stackrel{(i)}{\propto} f(\mathbf{c}_i \mid \mathbf{L}_i; \psi^{[r-1]}) f(\mathbf{L}_i \mid \mathbf{y}_i^{[r-1]}, z_{ik}^{[r-1]} = 1; \psi^{[r-1]}) \\ &\stackrel{(ii)}{=} \mathbb{1}_{\{\mathbf{c}_i=1\} \cap \{\mathbf{L}_i > 0\}} f(\mathbf{L}_i \mid \mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r-1]} = 1; \psi^{[r-1]}) \end{aligned}$$

where we use that \mathbf{L}_i is a function of $\mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik} = 1$ in step (i). Step (ii) is obtained by using (40). By abuse of notation, $\{\mathbf{c}_i = 1\} \cap \{\mathbf{L}_i > 0\}$ means that for all $j = 1, \dots, d$, $\{c_{ij} = 1\} \cap \{L_{ij} > 0\}$. Finally the conditional distribution $(\mathbf{L}_i \mid \mathbf{y}_i^{[r-1]}, z_{ik}^{[r-1]} = 1, \mathbf{c}_i)$ is a multivariate truncated Gaussian distribution denoted as \mathcal{N}_t , as detailed here

$$(\mathbf{L}_i \mid \mathbf{y}_i^{[r-1]}, z_{ik}^{[r-1]} = 1, \mathbf{y}_i) \sim \mathcal{N}_t(\alpha_k^{[r-1]} + \beta_k^{[r-1]} \mathbf{y}_i, I_{d \times d}; \mathbf{a}, \mathbf{b}), \quad (41)$$

with $\mathbf{a} \in \mathbb{R}^d$ and $\mathbf{b} \in \mathbb{R}^d$ the lower and upper bounds such that for all $j = 1, \dots, d$,

$$a_j = \begin{cases} 0 & \text{if } c_{ij} = 1, \\ -\infty & \text{otherwise.} \end{cases}$$

$$b_j = \begin{cases} +\infty & \text{if } c_{ij} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Secondly, we draw the membership k of $\mathbf{z}_i^{[r]}$ from the multinomial distribution with probabilities, for all $k = 1, \dots, K$ detailed as follows

$$\begin{aligned} \mathbb{P}(z_{ik} = 1 \mid \mathbf{L}_i^{[r]}, \mathbf{y}_i^{[r-1]}, \mathbf{c}_i; \theta^{[r-1]}) &= \frac{\mathbb{P}(z_{ik} = 1, \mathbf{L}_i^{[r]}, \mathbf{y}_i^{[r-1]}, \mathbf{c}_i; \theta^{[r-1]})}{\sum_{k=1}^K \mathbb{P}(z_{ik} = 1, \mathbf{L}_i^{[r]}, \mathbf{y}_i^{[r-1]}, \mathbf{c}_i; \theta^{[r-1]})} \\ &= \frac{f(\mathbf{L}_i^{[r]} \mid z_{ik} = 1, \mathbf{y}_i^{[r-1]}, \mathbf{c}_i; \psi^{[r-1]}) f(z_{ik} = 1, \mathbf{y}_i^{[r-1]}, \mathbf{c}_i; \theta^{[r-1]})}{\sum_{k=1}^K f(\mathbf{L}_i^{[r]} \mid z_{ik} = 1, \mathbf{y}_i^{[r-1]}, \mathbf{c}_i; \psi^{[r-1]}) f(z_{ik} = 1, \mathbf{y}_i^{[r-1]}, \mathbf{c}_i; \theta^{[r-1]})} \end{aligned} \quad (42)$$

The part involving $f(z_{ik} = 1, \mathbf{y}_i^{[r-1]}, \mathbf{c}_i; \theta^{[r-1]})$ is given in (38) and $f(\mathbf{L}_i^{[r]} \mid z_{ik} = 1, \mathbf{y}_i^{[r-1]}, \mathbf{c}_i; \psi^{[r-1]})$ is only the density of the multivariate truncated Gaussian distribution described in (41) evaluated in $L_i^{[r]}$.

Finally, $\mathbf{y}_i^{[r]}$ is drawn according to $f(\cdot \mid \mathbf{L}_i^{[r]}, z_{ik}^{[r]} = 1, \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \theta^{[r-1]})$. One has

$$\begin{aligned} &f(\mathbf{y}_i^{\text{mis}} \mid \mathbf{L}_i^{[r]}, z_{ik}^{[r]} = 1, \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i; \theta^{[r-1]}) \\ &\propto f(\mathbf{c}_i, \mathbf{L}_i^{[r]} \mid \mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1; \psi^{[r-1]}) f(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1; \theta^{[r-1]}) \\ &\propto f(\mathbf{c}_i \mid \mathbf{L}_i^{[r]}, \mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1; \psi^{[r-1]}) f(\mathbf{L}_i^{[r]} \mid \mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1; \psi^{[r-1]}) f(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1; \theta^{[r-1]}) \\ &\propto f(\mathbf{c}_i \mid \mathbf{L}_i^{[r]}; \psi^{[r-1]}) f(\mathbf{L}_i^{[r]} \mid \mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1; \psi^{[r-1]}) f(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1; \theta^{[r-1]}) \\ &\propto f(\mathbf{L}_i^{[r]} \mid \mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1; \psi^{[r-1]}) f(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1; \theta^{[r-1]}), \end{aligned}$$

Yet, one has

$$\begin{aligned} f(\mathbf{L}_i^{[r]} \mid \mathbf{y}_i^{\text{mis}}, \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1; \psi^{[r-1]}) &\propto \exp\left(-\frac{1}{2} \left[(\mathbf{L}_i^{[r]} - (\alpha_k^{[r-1]} + \beta_k^{[r-1]} \mathbf{y}_i^{\text{mis}}))^T (\mathbf{L}_i^{[r]} - (\alpha_k^{[r-1]} + \beta_k^{[r-1]} \mathbf{y}_i^{\text{mis}})) \right]\right) \\ f(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1; \theta^{[r-1]}) &\propto \exp\left(-\frac{1}{2} \left[(\mathbf{y}_i^{\text{mis}} - (\tilde{\mu}_{ik}^{\text{mis}})^{[r-1]})^T ((\tilde{\Sigma}_{ik}^{\text{mis}})^{[r-1]})^{-1} (\mathbf{y}_i^{\text{mis}} - (\tilde{\mu}_{ik}^{\text{mis}})^{[r-1]}) \right]\right), \end{aligned}$$

with $(\tilde{\mu}_{ik}^{\text{mis}})^{[r-1]}$ and $(\tilde{\Sigma}_{ik}^{\text{mis}})^{[r-1]}$ given in (28).

Finally combining these two equations one obtains

$$\left(\mathbf{y}_i^{\text{mis}} \mid \mathbf{L}_i^{[r]}, z_{ik}^{[r]} = 1, \mathbf{y}_i^{\text{obs}}, \mathbf{c}_i \right) \sim \mathcal{N}(\mu_{ik}^{\text{SEM}}, \Sigma_{ik}^{\text{SEM}}), \quad (43)$$

where

$$\begin{aligned} \Sigma_{ik}^{\text{SEM}} &= \left(((\tilde{\Sigma}_{ik}^{\text{mis}})^{[r-1]})^{-1} + ((\beta_k^{\text{mis}})^{[r-1]})^T (\beta_k^{\text{mis}})^{[r-1]} \right)^{-1}, \\ \mu_{ik}^{\text{SEM}} &= \Sigma_{ik}^{\text{SEM}} \left[((\tilde{\Sigma}_{ik}^{\text{mis}})^{[r-1]})^{-1} \tilde{\mu}_{ik}^{\text{mis}} + ((\beta_k^{\text{mis}})^{[r-1]})^T (\mathbf{L}_i^{\text{mis}})^{[r]} - ((\beta_k^{\text{mis}})^{[r-1]})^T (\alpha_k^{\text{mis}})^{[r-1]} \right], \end{aligned}$$

with $(\beta_k^{\text{mis}})^{[r-1]}$ (resp. $(\mathbf{L}_i^{\text{mis}})^{[r]}$ and $(\alpha_k^{\text{mis}})^{[r-1]}$) the vector β_k (resp. $(\mathbf{L}_i)^{[r]}$ and $(\alpha_k)^{[r-1]}$) restricted to the coordinates $j \in \mathcal{Y}_i^{\text{mis}}$.

Finally, for fully describing the SEM-algorithm, in the M-step, $\psi^{[r-1]}$ is computed using a GLM with a binomial link function for a matrix depending on the MNAR model. In particular,

- For MNAR y , the coefficient obtained with a GLM for the matrix $(\mathcal{H}_j^{\text{MNAR}y})^{[r]}$ are α_0 and $\beta_1^{[r]}, \dots, \beta_d^{[r]}$, with

$$(\mathcal{H}^{\text{MNAR}y})^{[r]} = \begin{array}{c|cccc} c.1 & 1 & y_{.1}^{[r]} & 0 & \dots & 0 \\ c.2 & 1 & 0 & y_{.2}^{[r]} & \dots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \\ c.d & 1 & 0 & 0 & \dots & y_{.d}^{[r]} \end{array} . \quad (44)$$

- For MNAR y^k , the coefficient obtained with a GLM for the matrix $(\mathcal{H}_{kj}^{\text{MNAR}y^k})^{[r]}$ is α_0 and $\beta_{11}^{[r]}, \dots, \beta_{K1}^{[r]}, \dots, \beta_{Kd}^{[r]}$ with

$$(\mathcal{H}_{kj}^{\text{MNAR}y^k})^{[r]} = \begin{array}{c|cccc} (c_{u1})_{u,z_{u1}^{[r]}=1} & 1 & (y_{u1}^{[r]})_{u,z_{u1}^{[r]}=1} & 0 & \dots & 0 \\ \vdots & \vdots & & & \ddots & \vdots \\ (c_{u1})_{u,z_{uK}^{[r]}=1} & 1 & 0 & (y_{u1}^{[r]})_{u,z_{uK}^{[r]}=1} & & 0 \\ \vdots & \vdots & \vdots & & \ddots & \\ (c_{ud})_{u,z_{uK}^{[r]}=1} & 1 & 0 & 0 & & (y_{ud}^{[r]})_{u,z_{uK}^{[r]}=1} \end{array} . \quad (45)$$

- For MNAR yz , the coefficients obtained with a GLM for the matrix $(\mathcal{H}^{\text{MNAR}yz})^{[r]}$ are $\beta_1^{[r]}, \dots, \beta_d^{[r]}$ and $\alpha_1^{[r]}, \dots, \alpha_K^{[r]}$, with

$$(\mathcal{H}^{\text{MNAR}yz})^{[r]} = \begin{array}{c|ccccccc} c.1 & y_{.1}^{[r]} & 0 & \dots & 0 & z_{.1}^{[r]} & \dots & z_{.K}^{[r]} \\ c.2 & 0 & y_{.2}^{[r]} & \dots & 0 & z_{.1}^{[r]} & \dots & z_{.K}^{[r]} \\ \vdots & & \ddots & \ddots & & \vdots & \vdots & \vdots \\ c.d & 0 & 0 & \dots & y_{.d}^{[r]} & z_{.1}^{[r]} & \dots & z_{.K}^{[r]} \end{array} . \quad (46)$$

- For MNAR yz^j , the coefficients obtained with a GLM for the matrix $(\mathcal{H}_j^{\text{MNAR}yz^j})^{[r]}$ are $\beta_j^{[r]}, \alpha_{1j}^{[r]}, \dots, \alpha_{Kj}^{[r]}$, with

$$(\mathcal{H}_j^{\text{MNAR}yz^j})^{[r]} = c_j \left| \begin{array}{cccc} y_{.j}^{[r]} & z_{.1}^{[r]} & \dots & z_{.K}^{[r]} \end{array} \right. . \quad (47)$$

- For MNAR y^kz , the coefficients obtained with a GLM for the matrix $(\mathcal{H}_k^{\text{MNAR}y^kz})^{[r]}$ are $\beta_{k1}^{[r]}, \dots, \beta_{kd}^{[r]}, \alpha_k^{[r]}$, with

$$(\mathcal{H}_k^{\text{MNAR}y^kz})^{[r]} = \begin{array}{c|cccc} c_{u1} & y_{u1}^{[r]} & 0 & \dots & 0 & 1 \\ c_{u2} & 0 & y_{u2}^{[r]} & \dots & 0 & 1 \\ \vdots & & \ddots & \ddots & & 1 \\ c_{ud} & 0 & 0 & \dots & y_{ud}^{[r]} & 1_{u,z_{uk}^{[r]}=1} \end{array} . \quad (48)$$

- For $\text{MNAR}y^k z^j$, the coefficients obtained with a GLM for the matrix $(\mathcal{H}_{kj}^{\text{MNAR}y^k z^j})^{[r]}$ are β_{kj}, α_{kj} , with

$$(\mathcal{H}_{kj}^{\text{MNAR}y^k z^j})^{[r]} = c_{uj} \mid y_{uj}^{[r]} \quad 1_{u, z_{uk}^{[r]}=1} \quad (49)$$

- For $\text{MNAR}z$, the coefficients obtained with a GLM for the matrix $(\mathcal{H}^{\text{MNAR}z})^{[r]}$ are $\alpha_1, \dots, \alpha_K$, with

$$(\mathcal{H}^{\text{MNAR}z})^{[r]} = \begin{array}{c|ccc} c_{.1} & z_{.1} & \dots & z_{.K} \\ \vdots & \vdots & \vdots & \vdots \\ c_{.d} & z_{.1} & \dots & z_{.K} \end{array} = \begin{array}{c|ccc} c_{11} & z_{11}^{[r]} & \dots & z_{1K}^{[r]} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n1} & z_{n1}^{[r]} & \dots & z_{nK}^{[r]} \\ \vdots & \vdots & \vdots & \vdots \\ c_{1d} & z_{11}^{[r]} & \dots & z_{1K}^{[r]} \\ \vdots & \vdots & \vdots & \vdots \\ c_{nd} & z_{n1}^{[r]} & \dots & z_{nK}^{[r]} \end{array}. \quad (50)$$

- For $\text{MNAR}z^j$, the coefficients obtained with a GLM for the matrix $(\mathcal{H}_j^{\text{MNAR}z^j})^{[r]}$ are $\alpha_{1j}, \dots, \alpha_{Kj}$, with

$$(\mathcal{H}_j^{\text{MNAR}z^j})^{[r]} = c_{.j} \mid z_{.1}^{[r]} \quad \dots \quad z_{.K}^{[r]} \quad (51)$$

When ρ is the probit function, the SEM algorithm can be derived, see Algorithm 2. The initialization and the stopping criterion are discussed in Section 4.

MNAR z and MNAR z^j models For the $\text{MNAR}z$ and $\text{MNAR}z^j$ models, the effect of the missingness is only due to the class membership. We have already proved in Appendix C.1.1 that

$$f(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, \mathbf{z}_i^{[r]}, \mathbf{c}_i; \theta^{[r-1]}) = f(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, \mathbf{z}_i^{[r]}; \lambda^{[r-1]}),$$

and that this conditional distribution is Gaussian given in (28). The M-step for ψ has been specified in the previous paragraph with (50) and (51).

C.2.2 Latent class model for categorical data

For categorical data, we have $\phi(\mathbf{y}_i; \lambda_k) = \prod_{j=1}^d \phi(y_{ij}; \lambda_{kj}) = \prod_{j=1}^d \prod_{\ell=1}^{\ell_j} (\lambda_{kj}^\ell)^{y_{ij}^\ell}$.

MNAR z and MNAR z^j models For drawing from the conditional distribution $(\mathbf{y}_i^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1)$, by independence of the features conditionally to the membership, we can draw for $j = 1, \dots, d$ $y_{ij}^{\text{mis}} = ((y_{ij}^{\text{mis}})^1, \dots, (y_{ij}^{\text{mis}})^{\ell_j})$ from the conditional distribution $(y_{ij}^{\text{mis}} \mid \mathbf{y}_i^{\text{obs}}, z_{ik}^{[r]} = 1)$. This latter is a multinomial distribution with probabilities $(\lambda_{kj}^\ell)_{\ell=1, \dots, \ell_j}$.

D Additional numerical experiments on synthetic data

Algorithm 2 SEM algorithm for Gaussian mixture, MNARy* models, ρ is probit

Input: $\mathbf{Y} \in \mathbb{R}^{n \times d}$ (matrix containing missing values), $K \geq 1$, r_{\max} .

Initialize $\mathbf{Z}^{[0]}$, $\pi_k^{[0]}$, $\mu_k^{[0]}$, $\Sigma_k^{[0]}$ and $\psi_k^{[0]}$, for $k \in \{1, \dots, K\}$.

for $r = 0$ **to** r_{\max} **do**

SE-step:

for $i = 1$ **to** n **do**

Draw $(\mathbf{L}_i)^{[r]}$ from the multivariate truncated Gaussian distribution given in (41).

Draw $\mathbf{z}_i^{[r]}$ from the multinomial distribution with probabilities detailed in (42).

Draw $(\mathbf{y}_i^{\text{mis}})^{[r]}$ from the multivariate Gaussian distribution given in (43).

end for

Let $\mathbf{Y}^{[r]} = (\mathbf{y}_1^{[r]} | \dots | \mathbf{y}_n^{[r]})$ be the imputed matrix.

Let $\mathbf{Z}^{[r]} = (\mathbf{z}_1^{[r]} | \dots | \mathbf{z}_n^{[r]})$ be the partition.

M-step:

for $k = 1$ **to** K **do**

Let $\pi_k^{[r]}$ be the proportion of rows of $\mathbf{Y}^{[r]}$ belonging class k.

Let $\mu_k^{[r]}$, $\Sigma_k^{[r]}$ be the mean and covariance matrix of rows of $\mathbf{Y}^{[r]}$ belonging to class k.

Let $\psi_k^{[r]}$ be the resulted coefficients of a GLM with a binomial link function, i.e. the optimization problem is $\forall j \in \{1, \dots, d\}$,

$$\mathcal{M}_{kj} \psi_k^{[r]} = \log \left(\frac{1 - \mathbb{E}[\mathbf{c}_{.j} | \mathcal{M}_{kj}]}{\mathbb{E}[\mathbf{c}_{.j} | \mathcal{M}_{kj}]} \right),$$

for a matrix \mathcal{M}_{kj} depending on the MNAR model (see (44),(45),(46),(51),(49) and (50)) and $\mathbf{c}_{.j}$ the missing data pattern for the variable j .

end for

end for

	EM			SEM		
	Gaussian		Categorical	Gaussian		Categorical
	Appendix C.1.1		Appendix C.1.2	Appendix C.2.1		Appendix C.1.2
MNAR _z	✓		✓	✓		✓
MNAR _{z^j}	✓		✓	✓		✓
	Probit	Logit		Probit	Logit	
MNARy*	no closed form	no closed form, optim. pb	not identifiable	✓	require algorithms as SIR (costly)	not identifiable

Table 3: Summary of the cases for which the EM and the SEM lead to feasible (or not feasible) computations. The symbol ✓ means that the computations are feasible and that they are derived in Appendix.

D.1 Toy example for other mechanisms

MNAR y , Gaussian case We consider the same case as in Section 4.3 with $\Delta_\mu = (2, 2)$. The missing values are now introduced with a MNAR y model (see (7)). In particular,

$$\forall k \in \{1, 2\}, \mathbb{P}(c_{ij} = 1 | z_{ik} = 1, \mathbf{y}_i) = \Phi(\beta_1 \mathbf{y}_1), \mathbb{P}(c_{ij} = 1 | z_{ik} = 1, \mathbf{y}_i) = \Phi(\beta_2 \mathbf{y}_2),$$

where $\beta = (\beta_1, \beta_2) = (-1, -2)$. It leads to 16% of missing values in the whole dataset: 40% of missing values in the first class and 5% in the second class. Thus, the mechanism still provides some information on the clustering. In Figure 7, the different algorithms are compared in terms of ARI. The SEM algorithms considering MNAR values outperform for all the MNAR scenarios the SEM algorithm designed for MCAR values and the two-step heuristics. Surprisingly, the algorithms assuming MNAR z and MNAR z^j values give better performances than the one specifically designed for MNAR y values, which is however the mechanism considered in the simulation setting. It can be explained by the fact that the parameters lead to a fairly clear separation of proportions of missing data between the two classes, which is a well-modelled case by the SEM algorithms designed for MNAR z and MNAR z^j values. They seem robust to the mechanism misspecification, which is an important fact to keep in mind, because both models has the advantage of being easily implemented (in particular, the EM algorithm can be used) contrary to the algorithm developed MNAR y and MNAR yz values. However, the setting considering here is perfectly adapted to the MNAR z and MNAR z^j scenarios, as the two classes have significantly different percentage of missing values. Therefore, we consider now the case where $\beta = (\beta_1, \beta_2) = (-0.3, -10)$ which leads to 4% of missing values in the first variable and 50% in the second variable. The classes have approximately the same percentage of missing data. In this setting (Figure 8), the SEM algorithm designed MNAR z and MNAR z^j values performs as the one considering MCAR values, which was expected, because the chosen setting does not fit their assumptions. The algorithms assuming MNAR y and MNAR yz values give the best performances.

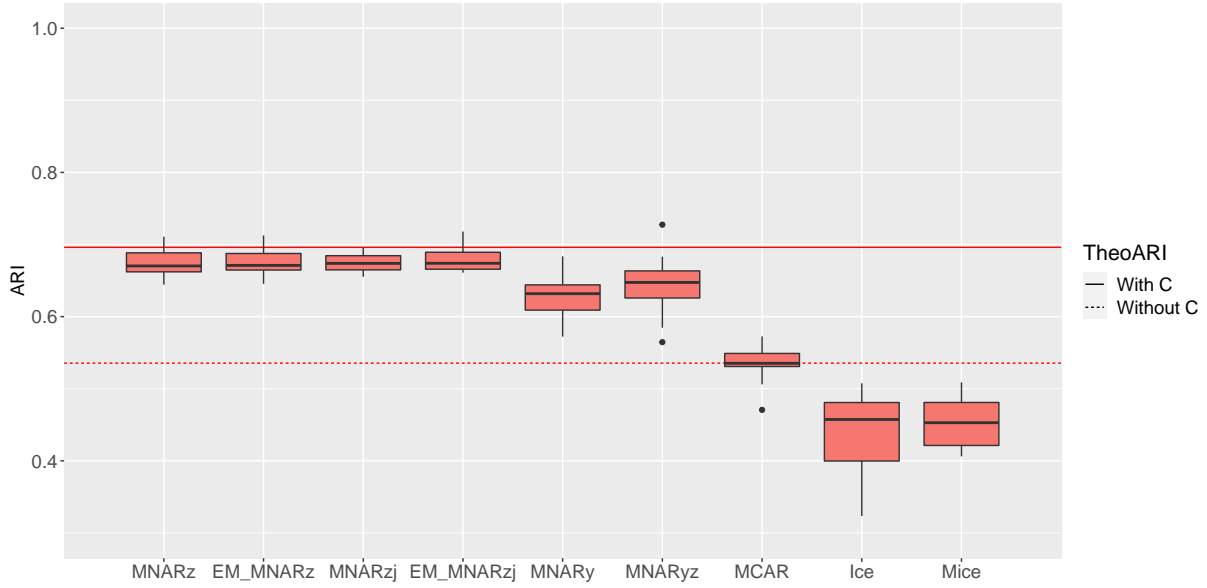


Figure 7: Comparison of the different algorithms in terms of ARI for bivariate Gaussian data ($n = 2000$, $d = 2$) and two classes when the true missing scenario is $MNAR_y$ when $\beta = (-1, -2)$. The stochasticity comes from the process of drawing 10 times the triplet $(\mathbf{Y}, \mathbf{Z}, \mathbf{C})$. The plain and dashed lines correspond to the mean of the ARI computed with the true parameters by considering $(\mathbf{Y}, \mathbf{Z}, \mathbf{C})$ or (\mathbf{Y}, \mathbf{Z}) respectively.

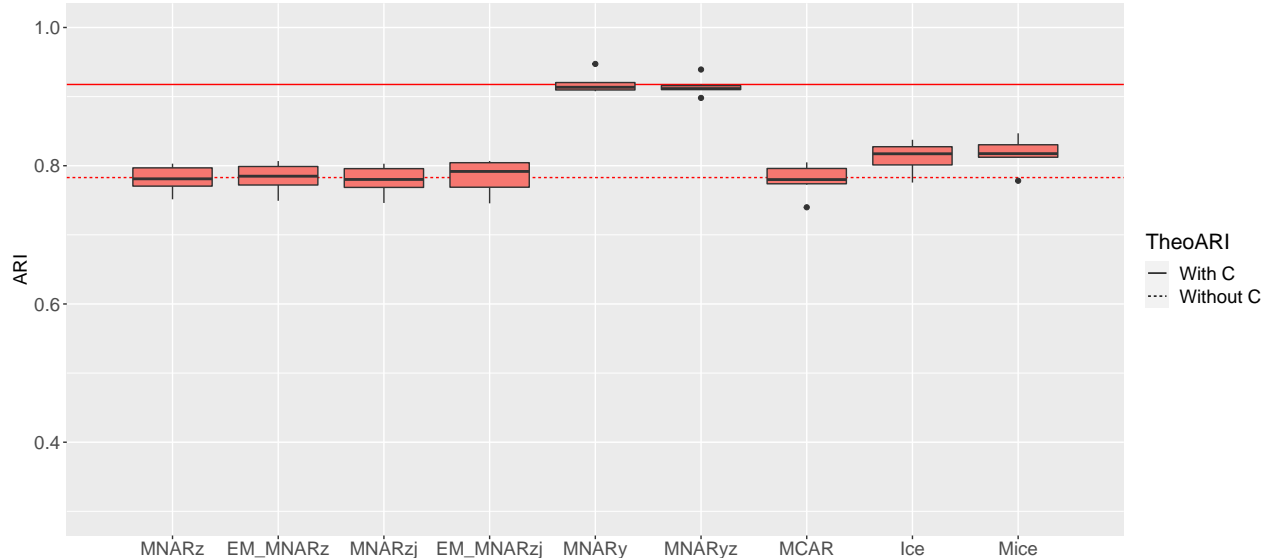


Figure 8: Comparison of the different algorithms in terms of ARI for bivariate Gaussian data ($n = 2000$, $d = 2$) and two classes when the true missing scenario is MNAR_y when $\beta = (-0.3, -10)$. The stochasticity comes from the process of drawing 10 times the triplet $(\mathbf{Y}, \mathbf{Z}, \mathbf{C})$. The plain and dashed lines correspond to the mean of the ARI computed with the true parameters by considering $(\mathbf{Y}, \mathbf{Z}, \mathbf{C})$ or (\mathbf{Y}, \mathbf{Z}) respectively.

MNAR yz , mixed case We consider a model mixture with mixed data, containing 5 categorical variables and 3 continuous variables. There are 3 classes, with probability of belonging $\pi = (0.2, 0.3, 0.5)$. Each categorical variables has 3, 4, 2, 2 or 3 levels, with probabilities fixed randomly. For the continuous variables, they have identity covariance matrix and their mean are $\mu_1 = 0, \mu_2 = 2, \mu_3 = 3$ for the three clusters. Missing values are introduced with a MNAR_{yz} model, leading to 3%, 10% and 16% of missing values in each cluster. In Figure 9, the algorithms considering the MNAR_z scenario for the categorical variables and the MNAR_z , MNAR_{z^j} or MNAR_{yz} scenario (see (6)) for the continuous variables are compared in terms of ARI with the naive model which does not consider MNAR data. This latter has poorer performances than the algorithms considering MNAR values, which all give good results, closed to the one obtained with theoretical parameters. The algorithm assuming MNAR_{yz} values is competitive with the one considering MNAR_{z^j} values, which, however, account for the fact that the effect of the missingness depends directly on the variable (for the continuous data).

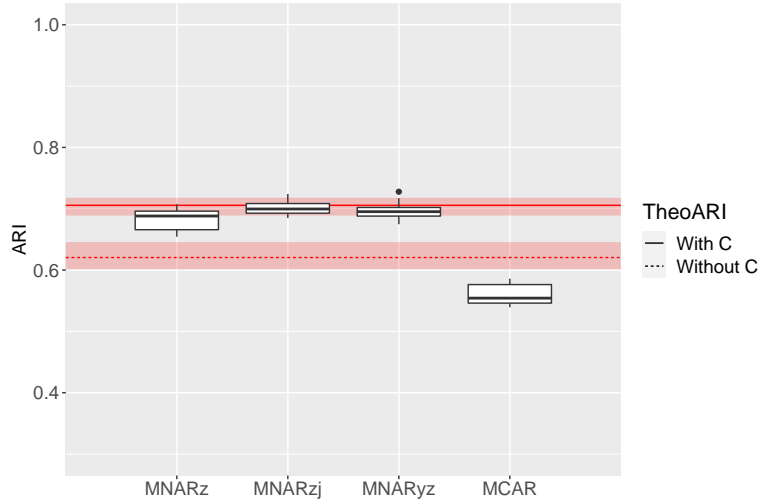


Figure 9: Comparison of the different algorithms in terms of ARI for mixed data ($n = 3000$, $d = 7$) and two classes when the true missing scenario is $MNAR_{yz}$. Here, $MNAR_z$, $MNAR_{z^j}$ and $MNAR_{yz}$ mean that the mechanism is considered as $MNAR_z$ for the categorical variables and $MNAR_z$, $MNAR_{z^j}$ and $MNAR_{yz}$ for the continuous ones. $MCAR$ means that both categorical and continuous variables are considered $MCAR$. The stochasticity comes from the process of drawing 10 times the triplet $(\mathbf{Y}, \mathbf{Z}, \mathbf{C})$. The plain and dashed lines correspond to the mean of the ARI computed with the true parameters by considering $(\mathbf{Y}, \mathbf{Z}, \mathbf{C})$ or (\mathbf{Y}, \mathbf{Z}) respectively.

D.2 Impact of the initialization

In the previous experiments, the SEM algorithm assuming $MNAR$ values is initialized with an SEM algorithm considering $MCAR$ values. In Figure 10, we compare this *MCAR initialization* with a random one in terms of ARI. More precisely, this random initialization is performed 10 times, and we keep the result which maximizes the ICL criterion. Both EM and SEM algorithms developed for $MNAR_z$ and $MNAR_{z^j}$ scenarios behave well for random initialization or for the $MCAR$ initialization. The SEM algorithm assuming $MNAR_{yz}$ values has poorer performances both in terms of ARI and ICL when the initialization is done randomly. On the contrary, the SEM algorithm designed for $MNAR_y$ values has better performance when the initialization is random. Finally, the impact of the initialization seems to be limited and no guideline for the initialization choice emerged from this experiment.

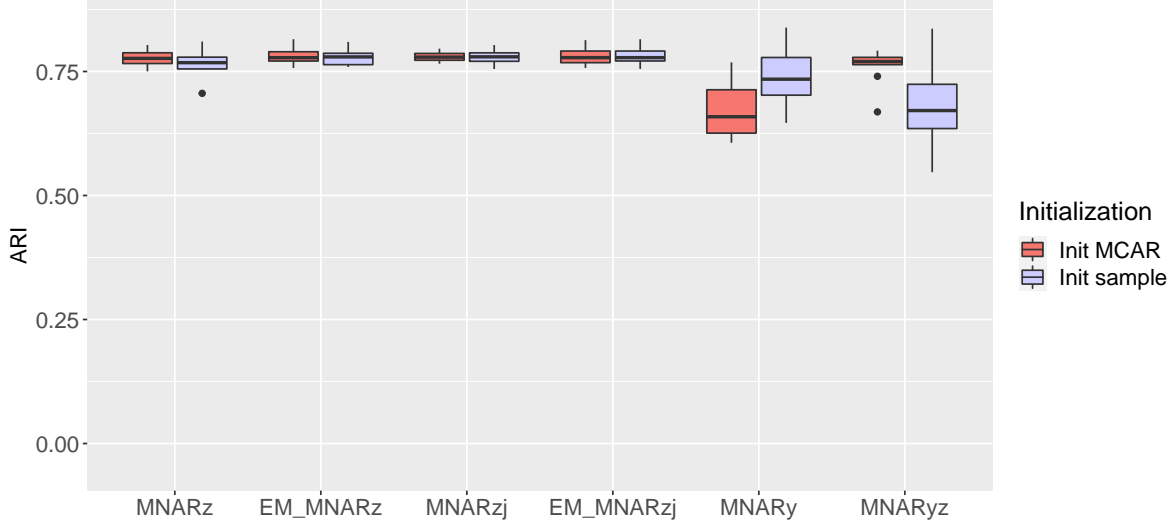


Figure 10: Comparison of the different algorithms in terms of ARI for bivariate Gaussian data ($n = 2000$, $d = 2$) and two classes when the true missing scenario is $MNAR_z$ and $\Delta_\mu = (2, 2)$. The algorithms are initialized either with the MCAR model or with a random initialization. The stochasticity comes from the process of drawing 10 times the triplet $(\mathbf{Y}, \mathbf{Z}, \mathbf{C})$.

E Description of the variables in the Traumabase dataset

A description of the variables which are used in Section 5 is given. The indications given in parentheses ph (pre-hospital) and h (hospital) mean that the measures have been taken before the arrival at the hospital and at the hospital.

- *Trauma.center* (categorical, integers between 1 and 16, no missing values): name of the trauma center (ph & h).
- *Anticoagulant.therapy* (categorical, binary variable, 4.3% NA): oral anticoagulant therapy before the accident (ph).
- *Antiplatelet.therapy* (categorical, binary variable, 4.4% NA): anti-platelet therapy before the accident (ph).
- *GCS.init*, *GCS* (ordinal, integers between 3 and 15, 2% NA & 42% NA): Initial Glasgow Coma Scale (GCS) on arrival on scene of enhanced care team and on arrival at the hospital (GCS = 3: deep coma; GCS = 15: conscious and alert) (ph & h).
- *GCS.motor.init*, *GCS.motor* (ordinal, integers between 1 and 6, 7.6% NA & 43%): Initial Glasgow Coma Scale motor score (GCS.motor = 1: no response; GCS.motor = 6: obeys command/purposeful movement) (ph % h).
- *Pupil.anomaly.ph*, *Pupil.anomaly* (categorical, 3 categories: Non, Anisocore (unilaterale), Mydriase Bilaterale, 2% NA & 1.7%): pupil dilation indicating brain herniation (ph & h).

- *Osmotherapy.ph*, *Osmotherapy* (categorical, 4 categories: Pas de mydriase, SSH, Mannitol, Rien, 1.7% NA and no missing values): administration of osmotherapy to alleviate compression of the brain (either Mannitol or hypertonic saline solution) (ph & h)
- *Improv.anomaly.osmo* (categorical, 3 categories: Non testé, Non, Oui, no missing values): change of pupil anomaly after administration of osmotherapy (ph).
- *Cardiac.arrest.ph* (categorical, binary variable, 2.3% NA): cardiac arrest during pre-hospital phase (ph).
- *SBP.ph*, *DBP.ph*, *HR.ph* (continuous, 29.3% NA & 29.6% NA & 29.5% NA): systolic and diastolic arterial pressure and heart rate during pre-hospital phase (ph).
- *SBP.ph.min*, *DBP.ph.min* (continuous, 12.8% NA & 13% NA): minimal systolic and diastolic arterial pressure during pre-hospital phase (ph).
- *HR.ph.max* (continuous, 13.7 % NA): maximal heart rate during pre-hospital phase (ph).
- *Cristalloid.volume* (continuous, positive values, 30% NA): total amount of prehospital administered cristalloid fluid resuscitation (volume expansion) (ph).
- *Colloid.volume* (continuous, positive values, 31.3% NA): total amount of prehospital administered colloid fluid resuscitation (volume expansion) (ph).
- *HemoCue.init* (continuous, 34.9% NA): prehospital capillary hemoglobin concentration (the lower, the more the patient is probably bleeding and in shock); hemoglobin is an oxygen carrier molecule in the blood (ph).
- *Delta.hemoCue* (continuous, 37.2% NA): difference of hemoglobin level between arrival at the hospital and arrival on the scene (h).
- *Vasopressor.therapy* (continuous, no missing values): treatment with catecholamines in case of physical or emotional stress increasing heart rate, blood pressure, breathing rate, muscle strength and mental alertness (ph).
- *SpO2.min* (continuous, 11.7% NA): peripheral oxygen saturation, measured by pulse oxymetry, to estimate oxygen content in the blood (95 to 100%: considered normal; inferior to 90% critical and associated with considerable trauma, danger and mortality) (ph).
- *TCD.PI.max* (continuous, 51.2% NA): pulsatility index (PI) measured by echodoppler sonographic examen of blood velocity in cerebral arteries (PI > 1.2: indicates altered blood flow maybe due to traumatic brain injury) (h).
- *FiO2* (categorical, in {1, 2, 3, 4, 5}, 6.8% NA): inspired concentration of oxygen on ventilatory support (the higher the more critical; Ventilation = 0: no ventilatory support) (h).
- *Neurosurgery.day0* (categorical, binary variable, no missing values): neurosurgical intervention performed on day of admission (h).
- *IGS.II* (continuous, positive values, 2% NA): Simplified Acute Physiology Score (h).

- *Tranexomic.acid* (categorical, binary variable, no missing values): administration of the tranexomic acid (h).
- *TBI* (categorical, binary variable, no missing values): indicates if the patient suffers from a traumatic brain injury (h).
- *IICP* (categorical, binary variable, 70.9% NA): at least one episode of increased intracranial pressure; mainly in traumatic brain injury; usually associated with worse prognosis (h).
- *EVD* (categorical, binary variable, no missing values): external ventricular drainage (EVD); mean to drain cerebrospinal fluid to reduce intracranial pressure (h).
- *Decompressive.craniectomie* (categorical, binary variable, no missing values): surgical intervention to reduce intracranial hypertension (h).
- *Death* (categorical, binary variable, no missing values): death of the patient (h).
- *AIS.head, AIS.face* (ordinal, discrete, integers between 0 and 6 and 4 1.7% NA & 1.7% NA): Abbreviated Injury Score, describing and quantifying facial and head injuries (AIS = 0: no injury; the higher the more critical) (h).
- *AIS.external* (continuous, discrete, integers between 0 and 5, 1.7% NA): Abbreviated Injury Score for external injuries, here it is assumed to be a proxy of information available/visible during pre-hospital phase (ph/h).
- *ISS* (continuous, discrete, integers between 0 and 75, 1.6% NA): Injury Severity Score, sum of squares of top three AIS scores (h).
- *Activation.HS.procedure* (categorical, binary variable, 3.7% NA): activation of hemorrhagic shock procedure in case of HS suspicion (h).
- *TBI_Death* (categorical, binary variable, no missing values): death of the patients suffering from a traumatic brain injury (h).
- *TBI_Death_30d* (categorical, binary variable, no missing values): death of the patients suffering from a traumatic brain injury in the 30 days (h).
- *TBI_30d* (categorical, binary variable, no missing values): traumatic brain injury in the 30 days (h).
- *Death_30d* (categorical, binary variable, no missing values): death in the 30 days (h).
- *Shock.index.ph* (continuous, positive values, 30.5% NA): ratio of heart rate and systolic arterial pressure during pre-hospital phase (ph).
- *majorExtracranial* (categorical, binary variable, no missing values): major extracranial lesion (h).
- *lesion.class* (no missing values): partition given by the doctors with $K = 4$ classes: axonal, extra, other, intra.
- *lesion.grade* (no missing values): partition given by the doctors with $K = 3$ classes: high, low, other.