An inverse problem: recovering the fragmentation kernel from the short-time behaviour of the fragmentation equation

Marie Doumic, Miguel Escobedo, Magali Tournus

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Abstract

The present paper provides a new representation of the solution to the fragmentation equation as a power series in the Banach space of Radon measures endowed with the total variation norm. This representation is used to justify how the fragmentation kernel, which is one of the two key parameters of the fragmentation equation, can be recovered from short-time experimental measurements of the particle size distributions when the initial condition is a delta function. A new stability result for this equation is also provided using a Wasserstein-type norm. We exploit this stability to prove the robustness of our reconstruction formula with respect to noise and initial data.

1 Introduction

The fragmentation equation is a size-structured PDE describing the evolution of a population of particles and is ubiquitous in modeling physical or biological phenomena (cell division [13], amyloid fibril breakage [21], microtubules dynamics [9]) and technological processes (mineral processing, grinding solids [10], polymer degradation [19] and break-up of liquid droplets or air bubbles). The equation is written as

$$\frac{\partial}{\partial t}u(t,x) = -B(x)u(t,x) + \int_x^\infty \kappa\left(\frac{x}{y}\right) B(y)u(t,y)\frac{dy}{y}, \tag{1}$$

to be understood in a weak sense given below in Definition 1. The two key physical parameters are the fragmentation rate $B(x)$ and the fragmentation kernel κ. They encode fundamental information on the mechanical stability of each particle, and can take quite different forms depending on the particular process considered. To estimate the parameters B and κ using population data (when only particles density can be accessed, not the trajectory of each individual particle) is a challenging problem.

The particular question that lead us to consider such a theoretical question originates from the studies [20, 21], where the authors provide experimental size distribution profiles of different types of amyloid fibrils, in order to estimate their intrinsic division properties (B and κ) and then to relate them to their respective pathogenic properties [1]. It is not possible to follow experimentally each fibril one by one, hence the necessity to draw the characteristic of each particle from the evolution of the whole population.

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Identifying the fragmentation kernel from measurable population data has been a challenging problem these past years. In the seminal paper [10] of Kolmogorov (1940), the large time behaviour of the size distribution is identified for a constant fragmentation rate ($B(x) = B$) and a discrete time model. The self-similar asymptotic behaviour of the fragmentation equation (the shape of the particles size distribution is preserved) has then been established in [8] by Filippov (1961) for the case $B(x) = x^\gamma,\ \gamma > 0$ and the result is now well-known by the scientific community under fairly general balance assumptions on the parameters (see [6] for instance). From the seventies, scientists from physics and chemical departments have been using this similarity concept for the kernel inverse problem. In 1974, a scientist of a department of chemical engineering [17] developed a method to extract information on probabilities of droplet-breakup, and in particular on the daughter-drop-distribution (in modern terms: the fragmentation kernel), as a function of drop sizes data, obtained from an experiment of pure fragmentation in a batch vessel. To do so, he uses the self-similar behaviour of the fragmentation equation, thereby restricted to monomial fragmentation rates, and evaluates the moments of the kernel from the moments of the large time size distribution. To recover the kernel from its moments, a method based on the expansion of the kernel on a specific polynomial basis is suggested. These results are generalized later in the eighties [12] to non-monomial fragmentation rates associated with an adapted definition for the self-similarity of the kernel so as to keep the self-similar asymptotic behaviour of the model. More recently, a reconstruction formula for κ based on the mere knowledge of the asymptotic profile g is proved mathematically in suitable functional spaces [3]. This formula involves the moments of order s of the asymptotic profile g, s being taken along a vertical complex line, i.e. $s = u + iv,\ v \in (-\infty, \infty)$. One of the drawbacks of these methods based on self-similarity is that they require access to high-order moment measurements of the self-similar profile to obtain a fine estimate for κ. Another drawback is the difficulty to obtain a robust inversion formula to deduce the kernel from its Mellin transform. From the late nineties, the large improvements in computer hardware opened the field of numerical investigations of mathematical models, see for instance [11] where the authors provide insights on how the stationary shape of the particle size distribution is impacted by the kernel. Their conclusion is that the inverse problem of assigning a breakage kernel to a known self-similar particular size distribution is ill-posed not only in a mathematical but also in a physical sense since quite different kernels correspond to almost the same particles size distribution. In a recent work [4], we explored the influence of the kernel on the time evolution of the length distribution. We showed that the asymptotic profile is helpful to distinguish whether the fragmentation kernel is of erosion-type kernel (one of the fragments has a size close to that of the parent particle) or produces particles of similar size. We also showed that to obtain a finer estimate on the kernel, there exists a time-window right after the initial time where two kernels result in a maximal difference in length distributions, and where the initial condition that maximizes this difference is a very sharp Gaussian.

In the present paper, we provide a new method to estimate the kernel from population data, using only short-time measurements of the solutions. We assume the fragmentation rate B to be known, and we provide a reconstruction formula for the fragmentation kernel. Unless specific assumptions are stated, we restrict to monomial fragmentation rates

$$B(x) = \alpha x^\gamma, \qquad \gamma > 0, \quad \alpha > 0. \tag{2}$$

This simplification renders our formulae easier to read, but our results can directly be generalized to generic smooth fragmentation rates. The idea is based on the following remark: for Δt small enough, the solution u to the fragmentation equation (14) formally satisfies

$$u(t+\Delta t,x) \approx u(t,x) - \alpha \Delta t x^{\gamma} u(t,x) + \alpha \Delta t \int_x^{\infty} \kappa\left(\frac{x}{y}\right) y^{\gamma-1} u(t,y) dy + o(\Delta t). \tag{3}$$

If we assume that at time t, the size distribution $u(t,x)$ is a Dirac delta function at $x=1$, that is denoted δ_1 or $\delta(x-1)$, then

$$u(t+\Delta t,x) \approx \delta(x-1) - \alpha \Delta t \delta(x-1) + \alpha \Delta t \kappa(x) + o(\Delta t),$$

and thus the kernel κ can be directly expressed from the measurement of the profile u at time $t+\Delta t$ as

$$\kappa(x) \approx \frac{1}{\alpha \Delta t}\left(u(t+\Delta t,x) - (1-\alpha\Delta t)\delta(x-1)\right) + o(1), \qquad \Delta t \ll 1.$$

To make rigorous the above estimate of κ as Δt goes to zero, we expand the solution $u(t,x)$ to the fragmentation equation as a power series about t in the Banach space of Radon measures. Up to our knowledge, such a representation of the solution to the fragmentation equation is novel.

How to use this formula in practice? The cunning to obtain κ directly form the measurement of the distribution profile $u(\Delta t,.)$ is to impose that the initial distribution $u(0,.)$ is a Dirac mass. In other words, at time $t=0$, all particles have the same size. Heuristically, if all particles have the exact same size at $t=0$, after a time t long enough so that almost all particles have broken once, but short enough such that almost no particle has broken twice, it is clear that the kernel κ, sometimes referred to as the "daughter particle distribution" can directly be read on the distribution. No experiment may produce a suspension where all the particles have the same size since it would mean being able to follow each particle one by one. However, we can hope to obtain a suspension where all particles have approximately the same size described by a gaussian distribution. For that reason, the second part of our work focus on the stability of our reconstruction formula with respect to noise and to the error on the initial data. To quantify the stability result first requires to understand what are the types of experimental uncertainties on the initial data coming from the experiments. These are twofold: first, instead of a delta function at $x=x_0$, the initial data is a spread gaussian with variance $\sigma > 0$ (due to the impossibility to obtain a perfectly homogeneous suspension). Second, the gaussian is centered as $x = x_0 + \varepsilon$ for some $\varepsilon > 0$, instead of $x = x_0$ (possible bias on the measurement of the particles size). In order to deal with these uncertainties, the Bounded-Lipshitz (BL) norm is better suited than the total variation norm (TV). For instance, $\forall a \in \mathbb{R}$, $b \in \mathbb{R}$, such that $|b-a| < 2$

$$\|\delta_a - \delta_b\|_{TV} = 2, \qquad \|\delta_a - f_{a,\sigma}\|_{TV} = 2,$$

whereas

$$\|\delta_a - \delta_b\|_{BL} = |b-a|, \qquad \|\delta_1 - f_{1,\sigma}\|_{BL} \leq \frac{2\sqrt{\sigma}}{\sqrt{2\pi}},$$

where $f_{a,\sigma}$ is the density of the gaussian function centered at $x=a$ and with variance σ.

On the other hand, the Mellin transform provides a full range of moments of the fragmentation kernel, in particular mean, variance and skewness, and is then of interest by itself. Our last contribution is a robust reconstruction formula for the Mellin transform of the fragmentation kernel using short-time measurements of the solution to the fragmentation equation endowed with a generic initial condition. Since the equation is autonomous, it means to be able to access two close consecutive measurements of the particles size distribution. The proof of the reconstruction formula requires that the Mellin transform of the solution of the fragmentation equation does not vanish on a vertical strip of the complex plane. An estimate of the error for the variance of the kernel is proved however without such a condition. Recovering the complete kernel from its Mellin transform reveals to be

a hard problem involving noisy deconvolution. The mathematical justification of the robustness of the formula is based on the following property: for a function u regular enough, the asymptotic behaviour of its Mellin transform U on a vertical line of the complex plane (i.e. the behaviour of $U(w+iv)$ for large values of v) does not depend on the vertical line (i.e. does not depend on w). We prove this result on the Mellin transform by using an adaptation of the stationary phase method for oscillatory integrals.

The main novelties brought by this paper are

- a proof of the uniqueness of the solution in the space of measures endowed with total variation norm (Theorem 1),
- a representation of the solution to the fragmentation equation as a power series in the Banach space of measures endowed with total variation (Theorem 2),
- a stability result for the solution to the fragmentation equation for the BL norm, which is a norm adapted to weak convergence of measures (Theorem 4),
- a robust reconstruction formula for the fragmentation kernel involving the short-time solution of the fragmentation equation endowed with a delta function as initial condition. Robustness is to be understood in the sense that if the initial condition is close to a delta function at $x = x_0$ in the BL norm (for instance a rectangular function centered in x_0 or a delta function at $x = x_0+\epsilon$ with ϵ small), then the estimated kernel obtained with the reconstruction formula is close to the real kernel in the BL norm (Theorem 3 and Theorem 5),
- a reconstruction formula for the Mellin transform K of the fragmentation kernel κ involving the short-time solution of the fragmentation equation endowed with any initial condition (Theorem 6).

The outline of the paper is as follows. In the remaining of Section 1, some properties of measures and classical results on measure theory are recalled. Then, uniqueness of solutions to the problem (14) (2) is proved in the space of Radon measures and, their structure is provided for general initial conditions. In Section 2, we provide a new representation of the solution of the fragmentation equation as a power series in the Banach space of Radon measures. In Section 3, the power series representation is used to render explicit the short-time behaviour of the solution to the fragmentation equation with a Dirac delta as initial data. We also explore the robustness of this behaviour, because it is close to impossible to obtain experimentally an exact delta function, the initial data is actually a slightly shifted gaussian distribution. In Section 4, we exhibit a reconstruction formula K^{est} of K, the Mellin transform of the fragmentation κ, only based on short time intervals measurements of the solution to the fragmentation equation, and prove and estimate of the error $K - K^{est}$. An exact expression for K was obtained in [3] from the exact long time self similar asymptotic profile of the solution to the fragmentation equation, but no estimate were obtained of the difference between K and the actual values that may be reached through the experimental data. We end the paper with numerical illustrations of the short-time behaviour of the fragmentation equation, we illustrate the estimation results of Theorems 3 and 5, and we explore how Theorem 6 can be applied to recover the variance of the kernel from the data.

1.1 Short reminder on measure theory

We define $\mathcal{M}(\mathbb{R}^+)$ as the set of Radon measures μ (not necessarily probability measures) such that $\text{supp}\,(\mu) \subset \mathbb{R}^+$. Let us recall that $\mathcal{M}(\mathbb{R}^+)$ is the dual space of the space $(\mathcal{C}(\mathbb{R}^+), \|.\|_\infty)$ of continuous

functions. We denote by (μ^+, μ^-) the Jordan decomposition of μ. We endow $\mathcal{M}(\mathbb{R}^+)$ with two different norms: the total variation norm and the Bounded-Lipschitz norm. As mentioned in the introduction, the final purpose is to obtain stability with respect the Bounded-Lipschitz norm, the TV norm is a technical intermediate tool to reach this purpose. The total variation (TV) of the (signed) measure $\mu \in \mathcal{M}(\mathbb{R}^+)$ is defined as

$$\|\mu\|_{TV} = \sup\{\int_{R^+} \varphi(x)d\mu(x), \quad \varphi \in \mathcal{C}(\mathbb{R}^+) \cap L^1(d|\mu|),\ \|\varphi\|_\infty \leq 1\}. \tag{4}$$

We recall that $(\mathcal{M}(\mathbb{R}^+), \|.\|_{TV})$ is a Banach space. We now introduce the Bounded-Lipschitz norm defined as

$$\|\mu\|_{BL} = \sup\{\int_{R^+} \varphi(x)d\mu(x), \quad \varphi \in \mathcal{C}(\mathbb{R}^+) \cap L^1(d|\mu|),\ \|\varphi\|_\infty \leq 1,\ \|\varphi'\|_\infty \leq 1\}. \tag{5}$$

Comparing (4) and (5), it is clear that

$$\forall\, \mu \in \mathcal{M}(\mathbb{R}^+), \qquad \|\mu\|_{BL} \leq \|\mu\|_{TV}. \tag{6}$$

An optimal transportation point of view is provided in [16, Proposition 23] for the Bounded-Lipschitz norm. It is proven that for any signed Radon measure with finite mass μ we have

$$\|\mu\|_{BL} = \inf\Big\{(\|\mu^+ - \nu\|_{TV} + \|\mu^- - \eta\|_{TV}) + W_1(\nu,\eta), \quad (\nu,\eta) \in \mathcal{M}^+_\mu(\mathbb{R}^+),\Big\} \tag{7}$$

$$\mathcal{M}^+_\mu(\mathbb{R}^+) = \Big\{(\nu,\eta) \in \mathcal{M}^+(\mathbb{R}^+) \times \mathcal{M}^+(\mathbb{R}^+);\ \nu \leq \mu^+, \eta \leq \mu^-, \|\nu\|_{TV} = \|\eta\|_{TV}\Big\} \tag{8}$$

where $\mathcal{M}^+(\mathbb{R}^+)$ is the space of positive Radon measures with support in $\mathbb{R}^+$, and W_1 stands for the classical Wasserstein distance [18] between two positive measures of same mass, namely

$$\begin{gathered} W_1(\nu,\eta) := \inf_{\pi \in \Pi(\nu,\eta)} \textstyle\int_{\mathbb{R}^+} |x-y| d\pi(x,y), \\ \Pi(\nu,\eta) := \big\{\pi \text{ positive measure on } \mathbb{R}^+ \text{ s.t. } \textstyle\int_{\mathbb{R}^+} \pi(x,y)dx = \eta(y), \quad \int_{\mathbb{R}^+} \pi(x,y)dy = \nu(x)\big\}. \end{gathered} \tag{9}$$

Let us recall that for μ, ν two probability measures and for $a > 0$, we have $W_1(a\mu, a\nu) = aW_1(\mu,\nu)$. Formula (7) (8) can be interpreted as follows: the BL norm of the signed measure μ is the BL distance between the two positive measures μ^+ and μ^-. Now take μ^+ and μ^- two positive measures. Consider ν and η two positive measures such that $\nu \leq \mu^+$, $\eta \leq \mu^-$ and $\|\nu\|_{TV} = \|\eta\|_{TV}$. The subpart ν of the measure μ^+ is transported onto the subpart η of the measure μ^-, with a cost $W_1(\nu,\eta)$. The remaining positive measures $(\mu^+ - \nu)$ and $(\mu^- - \eta)$ are both cancelled with a cost $\|\mu^+ - \nu\|_{TV} + \|\mu^- - \eta\|_{TV}$. Among all couples (ν,η) that satisfy $\nu \leq \mu^+$, $\eta \leq \mu^-$ and $\|\nu\|_{TV} = \|\eta\|_{TV}$, we choose one such that the sum $(\|\mu^+ - \nu\|_{TV} + \|\mu^- - \eta\|_{TV}) + W_1(\nu,\eta)$ is minimal (such a couple exists, it is proved in [15] that the infimum is actually a minimum). We provide here three examples.

- Take $\mu = \delta(x-1)$ and $\mu_\varepsilon = \delta(x-(1+\varepsilon))$. Consider $\nu_a = a\mu$ and $\eta_a = a\mu_\varepsilon$ with $0 \leq a \leq 1$. Then $0 \leq \nu_a \leq \mu$, $0 \leq \eta_a \leq \mu_\varepsilon$, and $\|\nu_a\|_{TV} = \|\eta_a\|_{TV} = a$. Using formula (7)(8) we have

$$\begin{aligned} \|\mu - \mu_\varepsilon\|_{BL} &= \inf_{0\leq a\leq 1} \Big\{(\|\mu - \nu_a\|_{TV} + \|\mu_\varepsilon - \eta_a\|_{TV}) + W_1(\nu_a,\eta_a)\Big\} \\ &= \inf_{0\leq a\leq 1} \Big\{2(1-a) + a\varepsilon\Big\} \\ &= \begin{cases} \varepsilon & \text{for } \varepsilon \leq 2, \\ 2 & \text{for } \varepsilon > 2. \end{cases} \end{aligned}$$

- Take $\mu = \delta(x-1)$ and μ_σ is the measure with the rectangular density $\frac{1}{2\sigma\sqrt{3}}\mathbb{1}_{[1-\sigma\sqrt{3},1+\sigma\sqrt{3}]}$ with variance σ^2 for $0 < \sigma < 1$. We take $\nu_a = \mu = \delta_1$ and $\eta_a = \mu_\sigma = f_\sigma dx$, in (7), and obtain

$$\|\mu - \mu_\sigma\|_{BL} \leq W_1(\mu, \mu_\sigma) \leq \int_{1-\sigma\sqrt{3}}^{1+\sigma\sqrt{3}} \frac{|y-1|dy}{2\sqrt{3}} = \frac{\sqrt{3}}{2}\sigma.$$

- Take $\mu = \delta(x-1)$ and μ_σ the Gaussian with mean 1 and variance σ^2. We have

$$\|\mu - \mu_\sigma\|_{BL} \leq W_1(\mu, \mu_\sigma) \leq \int |x| \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} = \frac{2\sigma}{\sqrt{2\pi}}.$$

1.2 Measure-valued solutions to the fragmentation equation

We recall that for $\mu \in \mathcal{M}(\mathbb{R}^+)$ and $T \in \mathcal{C}(\mathbb{R}^+)$, the pushforward η of the measure μ by the function T is defined as the unique measure $\eta = T\#\mu$ such that for all $\varphi \in \mathcal{C}(\mathbb{R}^+)$,

$$\int \varphi(x) d\eta(x) = \int (\varphi \circ T)(x) d\mu(x).$$

For $\ell > 0$, we define the application

$$T_\ell(x) = \ell x, \qquad x \in \mathbb{R}^+. \tag{10}$$

The basis of our analysis in all the remaining of this work are the weak solutions to the Cauchy problem for equation 14 with the initial condition

$$\mu_t(0) = \mu_0, \tag{11}$$

whose precise definition is given below. Throughout the present paper, the following assumptions are used.

Hyp-1 The fragmentation kernel $\kappa \in \mathcal{M}^+(\mathbb{R}^+)$ contains no atom at $x = 0$ and at $x = 1$, and satisfies

$$\text{supp}\,(\kappa) \subset [0,1], \qquad \int_0^1 d\kappa(z) = N < +\infty, \qquad \int_0^1 z d\kappa(z) = 1. \tag{12}$$

Hyp-2 The initial condition $\mu_0 \in \mathcal{M}^+(\mathbb{R}^+)$ is compactly supported

$$\text{supp}\,(\mu_0) \subset [0, M]. \tag{13}$$

Even though κ and μ_t are measures, we sometimes write the fragmentation equation as

$$\frac{\partial}{\partial t}\mu_t(x) = -\alpha x^\gamma \mu_t(x) + \alpha \int_x^\infty \kappa\left(\frac{x}{y}\right) y^{\gamma-1} d\mu_t(y), \qquad \mu_{t=0}(x) = \mu_0(x), \tag{14}$$

or as

$$\frac{\partial}{\partial t}\mu_t(x) = -\alpha x^\gamma \mu_t(x) + \alpha \int_x^\infty \kappa\left(\frac{x}{y}\right) y^{\gamma-1} \mu_t(y) dy, \qquad \mu_{t=0}(x) = \mu_0(x).$$

Definition 1 (Weak solution for (14))**.** *A family $(\mu_t)_{t\geq 0} \subset \mathcal{M}(\mathbb{R}^+)$ is called a measure solution to problem* (1) (2) (11) *with initial data $\mu_0 \in \mathcal{M}(\mathbb{R}^+)$ satisfying* (**Hyp-2**) *if the mapping $t \to \mu_t$ is narrowly continuous and for all $\varphi \in \mathcal{C}(\mathbb{R}^+)$ and all $t \geq 0$,*

$$\int_{\mathbb{R}^+} \varphi(x) d\mu_t(x) = \int_{\mathbb{R}^+} \varphi(x) d\mu_0(x) + \int_0^t ds \int_{\mathbb{R}^+} d\mu_s(x) \alpha x^\gamma \left(-\varphi(x) + \int_0^1 d\kappa(z) \varphi(xz) \right). \tag{15}$$

We recall that μ_n converges narrowly toward μ if for all $\varphi \in C_b(\mathbb{R}^+)$, $\int \varphi d\mu_n \to \int \varphi d\mu$, where $C_b(\mathbb{R}^+)$ denotes the set of continuous and bounded functions defined on $\mathbb{R}^+$. We also recall that T_y is defined in (10).

Theorem 1 (Uniqueness and stability for the fragmentation equation in $(\mathcal{M}(\mathbb{R}^+), \|.\|_{TV})$)**.** *Assume* (**Hyp-1**), (**Hyp-2**), $\gamma \geq 0$, *then there exists at most one measure-valued solution to* (14) *in the sense of Definition 1 in $\mathcal{C}(\mathbb{R}^+, \mathcal{M}(\mathbb{R}^+))$. More precisely, any weak solution satisfies*

$$\text{supp}\,(\mu_t) \subset [0, M], \qquad \|\mu_t\|_{TV} \leq e^{\alpha M^\gamma (1+N) t} \|\mu_0\|_{TV}, \quad \int_{\mathbb{R}^+} x d\mu_t(x) = \int_{\mathbb{R}^+} x d\mu_0(x), \qquad t > 0, \tag{16}$$

where M and N are defined in (**Hyp-2**).

Proof. We start by proving that any weak solution is compactly supported. We define

$$F(t) = \sup\{ \int_{\mathbb{R}^+} \varphi(x) d\mu_t(x), \quad \varphi \in \mathcal{C}(\mathbb{R}^+),\ \|\varphi\|_\infty \leq 1,\ \text{supp}\,(\varphi) \cap [0, M] = \emptyset \},$$

so that $F(0) = 0$ and $F(t) \geq 0$. Take $\varphi \in \mathcal{C}(\mathbb{R}^+)$ such that $\|\varphi\|_\infty \leq 1$ and $\text{supp}\,(\varphi) \cap [0, M] = \emptyset$. Then, using the formulation of Definition 1 combined with (**Hyp-2**), we have

$$\int_{\mathbb{R}^+} \varphi(x) d\mu_t(x) = \int_0^t ds \int_{\mathbb{R}^+} \alpha x^\gamma d\mu_s(x) \left(-\varphi(x) + \int_{z=0}^1 \varphi(xz) d\kappa(z) \right).$$

We set

$$\psi(x) := \alpha x^\gamma \left(-\varphi(x) + \int_{z=0}^1 \varphi(xz) d\kappa(z) \right),$$

then $\Psi \in \mathcal{C}(\mathbb{R}^+)$, $\text{supp}\,(\Psi) \cap [0, M] = \emptyset$ and since $\|\varphi\|_\infty \leq 1$, we have$\|\Psi\|_\infty \leq \alpha M^\gamma (1 + N)$. Thus, with the notation

$$\tilde{\Psi}(x) = \frac{\Psi(x)}{\alpha M^\gamma (1 + N)},$$

we obtain

$$\int_{\mathbb{R}^+} \varphi(x) d\mu_t(x) = \alpha M^\gamma (1 + N) \int_0^t ds \int_{\mathbb{R}^+} \tilde{\Psi}(x) d\mu_s(x) \leq \alpha M^\gamma (1 + N) \int_0^t F(s) ds.$$

Taking the supremum over φ, in the left hand side gives us

$$F(t) \leq \alpha M^\gamma (1 + N) \int_0^t F(s) ds,$$

and the Gronwall lemma implies that for all $t \geq 0$, it holds $F(t) = 0$, hence $\text{supp}\,(\mu_t) \subset [0, M]$.
To prove the BV estimate, we use definition 4, and take now $\varphi \in \mathcal{C}(\mathbb{R}^+)$ such that $\|\varphi\|_\infty \leq 1$. With the same arguments as above, we obtain

$$\int_{\mathbb{R}^+} \varphi(x) d\mu_t(x) \leq \|\mu_0\|_{TV} + \alpha M^\gamma (1+N) \int_0^t \|\mu_s\|_{TV} ds,$$

which implies

$$\|\mu_t\|_{TV} \leq \|\mu_0\|_{TV} + \alpha M^\gamma (1+N) \int_0^t \|\mu_s\|_{TV} ds,$$

and Gronwall Lemma provides us with the result. Finally, mass conservation is obtained by choosing $\varphi(x) = x$ in definition 1. □

1.3 Structure of solutions to the fragmentation equation

Definition 2 (Fundamental solution)**.** *We denote by μ_t^F the fundamental solution of equation* (14)*, i.e. the unique solution to the equation*

$$\frac{\partial}{\partial t}\mu_t(x) = -\alpha x^\gamma \mu_t(x) + \alpha \int_x^\infty \kappa\left(\frac{x}{y}\right) y^{\gamma-1} d\mu_t(y), \qquad \mu_{t=0}(x) = \delta(x-1).$$

Proposition 1 (Fundamental solution rescaled)**.** *Fix $\ell > 0$. Denote by μ_t^F the solution to* (14) *with $\mu_0 = \delta(x-1)$, then the solution to* (14) *with $\mu_0^\ell = \delta(x-\ell)$ is $\mu_t^\ell = T_\ell \# \mu^F_{\ell^\gamma t}$, with $T_\ell(x) = \ell x$.*

Proof. We set $\mu_t^\ell := T_\ell \# \mu^F_{\ell^\gamma t}$. Let us prove that μ_t^ℓ is a solution to (14) with initial condition $\mu_0^\ell = \delta(x-\ell)$ and conclude by uniqueness of the solution. First, $\mu_0^\ell = T_\ell \# \mu_0 = T_\ell \# \delta(x-1) = \delta(x-\ell)$. Then, we obtain that for all $\varphi \in \mathcal{C}_c(\mathbb{R}^+)$

$$\int_{\mathbb{R}^+} \varphi(x) d\mu_t^\ell(x) = \int_{\mathbb{R}^+} \varphi(x) d\left(T_\ell \# \mu^F_{\ell^\gamma t}\right)(x) = \int_{\mathbb{R}^+} (\varphi \circ T_\ell)(x) d\mu^F_{\ell^\gamma t}(x) = \int_{\mathbb{R}^+} \varphi(\ell x) d\mu^F_{\ell^\gamma t}(x).$$

Since μ_t is a weak solution to (14), we have

$$\begin{aligned}\int_{\mathbb{R}^+} \varphi(\ell x) d\mu^F_{\ell^\gamma t}(x) = & \int_{\mathbb{R}^+} \varphi(\ell x) d\mu_0(x) \\ & + \alpha \int_0^{\ell^\gamma t} \int_{\mathbb{R}^+} \left(-x^\gamma \varphi(\ell x) d\mu_s^F(x) + \varphi(\ell x) \int_x^\infty y^{\gamma-1} d\kappa\left(\frac{x}{y}\right) d\mu_s^F(y) \right) ds.\end{aligned}$$

Let us treat each of the three terms of the sum above separately. The first term is

$$\int_{\mathbb{R}^+} \varphi(\ell x) d\mu_0(x) = \int_{\mathbb{R}^+} \varphi(x) d\mu_0^\ell(x).$$

The second term is treated using the change of variables $s = \ell^\gamma u$

$$-\alpha \int_0^{\ell^\gamma t} \int_{\mathbb{R}^+} x^\gamma \varphi(\ell x) d\mu_s^F(x) ds = -\alpha \int_0^t \int_{\mathbb{R}^+} (x\ell)^\gamma \varphi(\ell x) d\mu^F_{\ell^\gamma u}(x) du$$

and then the change of variables $z = T_\ell(x)$ i.e. $d\mu^F_{\ell^\gamma u}(x) = d\left(T_\ell \# \mu^F_{\ell^\gamma u}\right)(z)$

$$\begin{aligned}-\alpha \int_0^t \int_{\mathbb{R}^+} (x\ell)^\gamma \varphi(\ell x) d\mu^F_{\ell^\gamma u}(x) du &= -\alpha \int_0^t \int_{\mathbb{R}^+} z^\gamma \varphi(z) d\left(T_\ell \# \mu^F_{\ell^\gamma u}\right)(z) du \\ &= -\alpha \int_0^t \int_{\mathbb{R}^+} z^\gamma \varphi(z) d\mu_u^\ell(z) du.\end{aligned}$$

For the third term we also use the change of variables $s = \ell^\gamma u$ followed by the change of variables $z = T_\ell(x)$ and to finish the change of variable $w = T_\ell(y)$ i.e. $d\mu^F_{\ell^\gamma u}(y) = d\left(T_\ell \# \mu^F_{\ell^\gamma u}\right)(w) = d\mu^\ell_u(w)$ to get

$$\alpha \int_0^{\ell^\gamma t} \int_{\mathbb{R}^+} \varphi(\ell x) \int_x^\infty y^{\gamma-1} d\kappa\left(\frac{x}{y}\right) d\mu^F_s(y) ds = \alpha \int_0^t \int_{\mathbb{R}^+} \varphi(z) \int_z^\infty w^{\gamma-1} d\kappa\left(\frac{z}{w}\right) d\mu^\ell_u(w) du.$$

To summarize,

$$\int_{\mathbb{R}^+} \varphi(x) d\mu^\ell_t(x) = \int_{\mathbb{R}^+} \varphi(x) d\mu^\ell_0(x) - \alpha \int_0^t \int_{\mathbb{R}^+} \left(z^\gamma \varphi(z) d\mu^\ell_u(z) + \varphi(z) \int_z^\infty w^{\gamma-1} d\kappa\left(\frac{z}{w}\right) d\mu^\ell_u(w) \right) du.$$

Finally, since $t \to \mu^F_t$ is narrowly continuous, then $t \to \mu^\ell_t$ is narrowly continuous as well. This ends the proof of Proposition 1. □

The structure of solutions to (1) is as follows:

Proposition 2 (Superimposition principle)**.** *We denote by μ^ℓ_t the solution to* (14) *with initial condition $\mu^\ell_0 = \delta(x-\ell)$, then, for an initial condition $u_0 \in \mathcal{C}(\mathbb{R}^+)$, the superimposition principle [7] provides the solution u to the linear equation as* (1)

$$u(t,x) = \int_0^\infty u_0(\ell) d\mu^\ell_t(x) d\ell.$$

Proposition 3 (Representation of the solution to (14) when initial condition is a measure)**.** *Consider equation* (14) *endowed with initial condition $\mu_0 \in \mathcal{M}(\mathbb{R}^+)$. Then, the unique solution to* (14) *is the measure μ_t defined for all $t > 0$ as*

$$\int_0^\infty \varphi(x) d\mu_t(x) = \int_0^M \int_0^1 \varphi(\ell x) d\mu^F_{t\ell^\gamma}(x) d\mu_0(\ell), \qquad \varphi \in \mathcal{C}(\mathbb{R}^+),$$

where μ^F_t is the fundamental solution defined in Definition 2.

Proof. The superimposition principle implies

$$\int_0^\infty \varphi(x) d\mu_t(x) = \int_0^\infty \varphi(x) \int_0^\infty d\mu_0(\ell) d\mu^\ell_t(x), \qquad \varphi \in \mathcal{C}(\mathbb{R}^+),$$

where μ^ℓ_t is the solution to (14) with initial condition $\mu_0 = \delta(x-\ell)$ and the conclusion follows by using Proposition 1. □

Let us provide two cases where we have explicit formulations for the fundamental solution to (14) for $\alpha = \gamma = 1$ and $\mu_0 = \delta(x-1)$.

Example 1. *For $\kappa = 2\mathbb{1}_{[0,1]}$, we have [22]*

$$\mu^F_t(x) = e^{-t}\delta(x-1) + (t + (1-x)t^2)e^{-xt},$$

Example 2. *For $\kappa(z) = 2\delta(z-1/2)$ we have [5]*

$$\mu^F_t(x) = e^{-t}\delta(x-1) + \sum_{k=1}^\infty \frac{(4t)^k}{k!} \delta\left(x - \frac{1}{2^k}\right).$$

In both examples, the mass initially located at $x = 1$ decreases exponentially with respect to time and is teleported on $(0,1)$. We end this section with the definition of the Mellin transform.

1.4 Mellin transform

Definition 3. *For a measure $\mu \in \mathcal{M}(\mathbb{R}^+)$, its Mellin transform $M[\mu]$ is defined as*

$$M[\mu](s) = \int_{\mathbb{R}^+} x^{s-1} d\mu(x), \tag{17}$$

for those $s \in \mathbb{C}$ such that (17) is well-defined.

Definition 4 (Multiplicative convolution)**.** *Take μ and ν two compactly supported finite measures on $\mathbb{R}^+$. Their convolutive multiplication is defined as*

$$\forall \varphi \in \mathcal{C}(\mathbb{R}^+), \qquad \langle \mu * \nu, \varphi \rangle = \langle \mu^x \otimes \nu^y, \varphi \circ p \rangle,$$

where $p : (x, y) \to xy$.

If $d\mu(x) = f(x)dx$ and $d\nu(x) = g(x)dx$ for f and g in $L^1(\mathbb{R}^+)$, then $\mu * \nu$ is the measure with density

$$(f * g)(x) = \int_{\mathbb{R}^+} f(y)\, g\left(\frac{x}{y}\right) \frac{dy}{y}.$$

If $d\mu(x) = f(x)dx$ with $f \in \mathcal{C}(\mathbb{R}^+)$ and $\nu = \delta(y - \ell)$, then $\mu * \nu = T_\ell \# \mu$ is the measure with density

$$(f * \nu)(z) = \frac{1}{\ell} f\left(\frac{z}{\ell}\right).$$

Proposition 4 (Mellin transform and multiplicative convolution)**.** *Take μ and ν two compactly supported finite measures on $\mathbb{R}^+$. For the s for which the expression below is defined, we have*

$$M[\mu * \nu](s) = M[\mu](s) M[\nu](s).$$

2 Representation of the solution as a power series

Theorem 2 (Representation of the solution of the fragmentation equation as a power series)**.** *For κ a fragmentation kernel satisfying* (**Hyp-1**)*, for $\gamma \geq 0$ and for μ_0 satisfying* (**Hyp-2**)*, the weak solution to (14) in $\mathcal{C}((0, T), \mathcal{M}(\mathbb{R}^+))$ is*

$$\mu_t = e^{-\alpha x^\gamma t} \mu_0 + \sum_{n=0}^{\infty} (\alpha t)^n \int_x^\infty \ell^{n\gamma} a_n \left(\frac{x}{\ell}\right) \mu_0(\ell) \frac{d\ell}{\ell}, \tag{18}$$

where, for $x \in [0, 1]$,

$$a_0(x) = 0, \qquad a_{n+1}(x) = \frac{1}{n+1} \left(-x^\gamma a_n(x) + \int_x^\infty y^{\gamma - 1} \kappa\left(\frac{x}{y}\right) a_n(y) dy + \kappa(x) \frac{(-1)^n}{n!} \right), \tag{19}$$

Remark 1. *We emphasize that in the formulae (18) and (19), κ, μ_0 and a_n may be measures. In this case, the multiplicative convolution product has to be understood in the sense of measures. For instance, the formula (18) means*

$$\mu_t = e^{-\alpha x^\gamma t} \mu_0 + \sum_{n=0}^{\infty} (\alpha t)^n a_n * b_n$$

where the multiplicative convolution product is defined in Definition 4 and where the measure b_n is the product of the smooth function $\ell \to \ell^{n\gamma}$ and of the measure μ_0. The explicit formula also implies that $Supp(\mu_t) \subset Supp(\mu_0)$ for all $t \geq 0$ (as also shows the superimposition principle).

Proof of Theorem 2 follows from two lemmas.

Lemma 1. *The fundamental solution μ_t^F to (14) (with $\mu_0 = \delta(x-1)$) can be written as*

$$\mu_t^F(x) = e^{-\alpha t}\delta(x-1) + v_t(x), \tag{20}$$

where $v_t(x) \perp \delta(x-1)$ is a positive measure-valued solution to

$$\begin{cases} \partial_t v_t(x) = -\alpha x^\gamma v_t(x) + \alpha \displaystyle\int_x^\infty y^{\gamma-1}\kappa\left(\frac{x}{y}\right) v_t(y)dy + \alpha e^{-\alpha t}\kappa(x), \\ v_0(x) = 0. \end{cases} \tag{21}$$

Proof. We recall that $\mu \perp \nu$ if there exists $E \in \mathcal{B}(\mathbb{R})$ such that $\mu(\mathbb{R}) = \mu(E)$ and $\nu(E) = 0$. The Radon Nikodym decomposition guarantees that μ_t can be decomposed as

$$\mu_t^F = A(t)\delta(x-1) + v_t \tag{22}$$

where $v_t \perp \delta(x-1)$, $A(0) = 1$ and $v_0(x) = 0$. We plug (22) into (14) and get

$$\begin{aligned} A'(t)\delta(x-1) + \partial_t v_t(x) = &-A(t)\alpha x^\gamma \delta(x-1) - v_t(x)\alpha x^\gamma \\ &+ \alpha \int_x^\infty y^{\gamma-1}\kappa\left(\frac{x}{y}\right)\left(A(t)\delta(y-1) + dv_t(y)\right) \end{aligned}$$

which is

$$A'(t)\delta(x-1) + \partial_t v_t(x) = -\alpha A(t)\delta(x-1) - v_t(x)\alpha x^\gamma + \alpha\kappa(x)A(t) + \alpha\int_x^\infty y^{\gamma-1}\kappa\left(\frac{x}{y}\right)dv_t(y).$$

By identification, we get

$$\begin{cases} A'(t) = -\alpha A(t), \quad A(0) = 1, \\ \partial_t v_t(x) = -\alpha x^\gamma v_t(x) + \alpha \displaystyle\int_x^\infty y^{\gamma-1}\kappa\left(\frac{x}{y}\right)dv_t(y) + \alpha\kappa(x)A(t), \quad v_0(x) = 0. \end{cases}$$

The first line gives $A(t) = e^{-\alpha t}$, and Lemma 1 is proved. □

Lemma 2 (Representation of the solution to (21) as a power series)**.** *The power series*

$$v(t,x) = \sum_{n=1}^\infty (\alpha t)^n a_n(x), \qquad \mathcal{C}((0,T), \mathcal{M}(\mathbb{R}^+)), \tag{23}$$

where a_n is the sequence defined in Theorem 2, is the unique weak solution to (21)*.*

Proof. Let us verify that the series (23) converges. Since $(\mathcal{M}(\mathbb{R}^+), \|.\|_{TV})$ is a Banach space, it is enough to prove the normal convergence of the series (23). We first claim that

$$\|a_{n+1}\|_{TV} \le \frac{1}{n+1}\left((N+1)\|a_n\|_{TV} + \frac{N}{n!}\right), \tag{24}$$

This comes directly from the induction formula (19) since $x \in [0,1]$ implies

$$\|x^\gamma a_n\|_{TV} \le \|a_n\|_{TV},$$

and

$$\begin{aligned}\| \int_x^\infty y^{\gamma-1}\kappa\left(\frac{x}{y}\right) a_n(y)dy\|_{TV} &= \int_0^\infty \left| \int_x^\infty y^{\gamma-1}\kappa\left(\frac{x}{y}\right) a_n(y)dy \right| dx \\ &\leq \int_0^\infty \int_x^\infty \left| y^{\gamma-1}\kappa\left(\frac{x}{y}\right) a_n(y) \right| dydx \\ &= \int_0^1 \int_0^\infty |y^\gamma \kappa\left(z\right) a_n(y)|\, dydz \\ &\leq \int_0^1 |\kappa\left(dz\right)| \int_0^\infty |a_n(y)|\, dy \leq N\|a_n\|_{TV},\end{aligned}$$

and finally

$$\|\kappa(x)\frac{(-1)^n}{n!}\|_{TV} = \frac{N}{n!}.$$

We deduce from (24) that for all $n \in \mathbb{N}$,

$$\|a_n\|_{TV} \leq \frac{(N+2)^n}{n!}, \tag{25}$$

hence the normal convergence of the series in $(\mathcal{M}(\mathbb{R}^+), \|.\|_{TV})$. We prove this by induction: (25) is true for $n = 0$ and $n = 1$, and if it is satisfied for $n \geq 1$ we have

$$\|a_{n+1}\|_{TV} \leq \frac{1}{n+1}\left((N+1)\|a_n\|_{TV} + \frac{N}{n!}\right) \leq \frac{1}{(n+1)!}\frac{(N+1)(N+2)^n + N}{n!} \leq \frac{(N+2)^{n+1}}{(n+1)!}.$$

We prove then using the differentiation rule of power series in a Banach space that the power series (23) is a solution to (14). We have

$$\frac{d}{dt}v(t,x) = \frac{d}{dt}\sum_{n=1}^\infty (\alpha t)^n a_n = \alpha \sum_{n=1}^\infty n(\alpha t)^{n-1} a_n = \alpha \sum_{n=0}^\infty (n+1)(\alpha t)^n a_{n+1}$$

and then using the induction hypothesis (19)

$$\begin{aligned}\frac{d}{dt}v(t,x) =& \alpha \sum_{n=0}^\infty (\alpha t)^n \left(-x^\gamma a_n(x) + \int_x^\infty y^{\gamma-1}\kappa\left(\frac{x}{y}\right) a_n(y)dy + \kappa(x)\frac{(-1)^n}{n!}\right) \\ &= -\alpha x^\gamma \sum_{n=0}^\infty (\alpha t)^n a_n + \alpha \int_x^\infty y^{\gamma-1}\kappa\left(\frac{x}{y}\right)\left(\sum_{n=0}^\infty (\alpha t)^n a_n(y)\right) dy + \alpha \sum_{n=0}^\infty (-\alpha t)^n \kappa(x) \\ &= -\alpha x^\gamma v(t,x) + \alpha \int_x^\infty y^{\gamma-1}\kappa\left(\frac{x}{y}\right) v(t,y)dy + \alpha e^{-\alpha t}\kappa(x)\end{aligned}$$

which ends the proof of Lemma 2.

□

Proof of Theorem 2. From the two lemmas above, we have a representation of the fundamental solution as a power series, namely

$$\mu_t^F = e^{-\alpha t}\delta(x-1) + \sum_{n=0}^\infty (\alpha t)^n a_n,$$

where the induction relation between the a_n is expressed in (19). We recall that from Proposition 3 the solution μ_t to the fragmentation equation when initial condition is any measure $\mu_0 \in \mathcal{M}(\mathbb{R}^+)$ can be obtained as

$$\int_0^\infty \varphi(x) d\mu_t(x) = \int_0^\infty \int_0^\infty \varphi(\ell x) d\mu^F_{t\ell^\gamma}(x) d\mu_0(\ell), \qquad \varphi \in \mathcal{C}(\mathbb{R}^+).$$

This ends the proof of Theorem 2.

□

We now use the explicit expression provided by Theorem 2 to obtain error estimates for the fragmentation kernel using the short-time behaviour of the solution.

3 Short-time behaviour

3.1 An estimation for κ using shor time measurements

Let us first investigate the best case for an initial condition, namely when we depart from a Dirac delta function at $x = 1$.

Theorem 3 (An estimate for κ using short-time measurements of the particles size distribution when initial condition is a delta function at $x = 1$.)**.** *Assume κ satisfies* (**Hyp-1**)*. We define*

$$\kappa^{est}(t) = \frac{\mu^F_t - e^{-\alpha t}\delta(x-1)}{\alpha t},$$

where we recall that μ^F_t is the unique fundamental solution to the fragmentation equation (14)*, i.e. endowed with the initial condition $\mu_0 = \delta(x-1)$. Then, for $t \in [0, T]$ and for some $K > 0$ depending on T, we have*

$$\left\| \kappa^{est} - \kappa \right\|_{TV} \leq Kt.$$

Before proving Theorem 3, we point out that another possible formula for the estimated kernel is

$$\kappa^{est}_{bis}(t) = \frac{\mu^F_t - (1-\alpha t)\delta(x-1)}{\alpha t} = 1 + \frac{\mu^F_t - \delta(x-1)}{\alpha t}.$$

Since $e^{-\alpha t} = 1 - \alpha t + o(t)$, we also have

$$\left\| \kappa^{est}_{bis} - \kappa \right\|_{TV} \leq Kt.$$

Proof. We have, using the notations introduced in Lemmas 1 and 2,

$$\frac{\mu^F_t - e^{-\alpha t}\delta(x-1)}{\alpha t} - \kappa = \frac{\sum_{n=1}^\infty (\alpha t)^n a_n}{\alpha t} - \kappa = \sum_{n=1}^\infty (\alpha t)^{n-1} a_n - \kappa = \sum_{n=0}^\infty (\alpha t)^n a_{n+1} - \kappa$$

and since $a_1 = \kappa$, we have

$$\sum_{n=0}^\infty (\alpha t)^n a_{n+1} - \kappa = \sum_{n=1}^\infty (\alpha t)^n a_{n+1} = \alpha t \sum_{n=0}^\infty (\alpha t)^n a_{n+2}.$$

Thus

$$\|\frac{\mu_t^F - e^{-\alpha t}\delta(x-1)}{\alpha t} - \kappa\|_{TV} \leq \alpha t \sum_{n=0}^{\infty} (\alpha t)^n \|a_{n+2}\|_{TV}.$$

The series converges (normal convergence) and thus it is bounded on any compact set, for instance for $t \in [0, T]$. Then Theorem 3 holds for

$$K(t \mapsto T) = \alpha \max_{t\in[0,T]} \sum_{n=0}^{\infty} (\alpha t)^n \|a_{n+2}\|_{TV}.$$

□

From Theorem 3 and Proposition 1, we can deduce an estimate for any Dirac delta function taken as an initial condition.

Corollary 1 (An estimate for κ using short-time measurements of the particles size distribution when initial condition is a delta function at $x = \ell$.)**.** *We denote by κ_ℓ the rescaled fragmentation kernel*

$$\kappa_\ell = T_\ell \# \kappa,$$

where the map T_ℓ is defined in (10)*. We define*

$$\kappa_\ell^{est}(t) = \frac{\mu_t^\ell - e^{-\alpha t \ell^\gamma}\delta(x-\ell)}{\alpha t \ell^\gamma},$$

where μ_t^ℓ is the unique solution to (14) *endowed with the initial condition $\mu_0 = \delta(x-\ell)$. Then we have for $t \in [0, T]$ and for the constant $K > 0$ of Theorem 3 depending on T*

$$\left\|\kappa_\ell^{est}(t) - \kappa_\ell\right\|_{TV} \leq K t \ell^\gamma.$$

Before proving Corollary 1, let us mention that if κ is a function, then

$$\kappa_\ell(z) = \frac{1}{\ell}\kappa\left(\frac{z}{\ell}\right), \qquad 0 \leq z \leq \ell.$$

Proof. We notice that for any continuous map T, we have $\|T\#\mu\|_{TV} \leq \|\mu\|_{TV}$ (with equality if the measure μ is positive, or if T is injective). Let us set $\eta = \mu_{t\ell^\gamma}^F - e^{-\alpha t\ell^\gamma}\delta(x-1)$. We have $T_\ell\#\eta = \mu_t^\ell - e^{-\alpha t\ell^\gamma}\delta(x-\ell)$, hence using Theorem 3

$$\left\|\frac{\mu_t^\ell - e^{-\alpha t\ell^\gamma}\delta(x-\ell)}{\alpha t\ell^\gamma} - \kappa_\ell\right\|_{TV} = \left\|T_\ell\#\Big(\frac{\mu_{t\ell^\gamma}^F - e^{-\alpha t\ell^\gamma}\delta(x-1)}{\alpha t\ell^\gamma} - \kappa\Big)\right\|_{TV} \leq K\alpha\ell^\gamma t,$$

with K the constant of Theorem (3). This ends the proof. □

Experimentally, a Dirac delta function as an initial condition is in most cases out of reach. We thus use the superimposition principle stated in Proposition 3 to obtain an estimate not directly for κ but for a convolution product of κ depending on the initial condition, as stated in the next corollary. Let us first define, for a general initial data u_0

$$\kappa^{est}(u_0; t, x) = \frac{u(t,x) - e^{-\alpha t x^\gamma}u_0(x)}{\alpha t}.$$

Corollary 2 (Generic initial condition)**.** *Assume $\kappa \in C^1([0,1])$. Take $u_0 \in \mathcal{C}(\mathbb{R}^+) \cap L^1(\ell^{2\gamma} d\ell)$. Then the unique solution u to* (14) *endowed with the initial condition u_0 satisfies for $t \in [0,T]$ and for some $K > 0$ depending on T, α, γ and $\|u_0\|_{L^1(\ell^{2\gamma} d\ell)}$*

$$\left\| \kappa^{est}(u_0; t) - w_0 * \kappa(x) \right\|_{TV} \leq Kt.$$

where w_0 denotes the measure with density $\ell \to \ell^\gamma u_0(\ell)$.

Proof. For $\ell > 0$, we multiply the measure

$$X_\ell = \frac{\mu_t^\ell - e^{-\alpha t \ell^\gamma} \delta(x-\ell)}{\alpha t \ell^\gamma} - \kappa_\ell$$

by the smooth function $\ell \to \ell^\gamma$, and apply Corollary 1 to obtain

$$\|Y_\ell\|_{TV} \leq Kt\ell^{2\gamma}.$$

where $Y_\ell = \dfrac{\mu_t^\ell - e^{-\alpha t \ell^\gamma} \delta(x-\ell)}{\alpha t} - \ell^\gamma \kappa_\ell$ We multiply the function $\ell \to Y_\ell$ from $\mathbb{R}^+$ onto $\mathcal{M}(\mathbb{R}^+)$ by $\ell \to u_0(\ell)$and integrate over $\mathbb{R}^+$. Since $(\mathcal{M}(\mathbb{R}^+), \|.\|_{TV})$ is a Banach space, we can use the Bochner integral so that we have

$$\begin{aligned} \left\| \kappa^{est}(u_0; t) - w_0 * \kappa(x) \right\|_{TV} &= \left\| \int_{\mathbb{R}^+} Y_\ell u_0(\ell) d\ell \right\|_{TV} \\ &\leq \int_{\mathbb{R}^+} \|Y_\ell\|_{TV} u_0(\ell) d\ell = Kt \int_{\mathbb{R}^+} u_0(\ell) \ell^{2\gamma} d\ell. \end{aligned}$$

□

Remark 2. *If u_0 satisfies the hypothesese of Corollary 2 it follows,*

$$\|\kappa^{est}(u_0; t) - \kappa\|_{TV} \leq \|\kappa^{est}(u_0; t) - w_0 * \kappa\|_{TV} + \|w_0 * \kappa - \kappa\|_{TV}.$$

In Section 4, we use this corollary to estimate κ for general initial conditions.

3.2 Stability of the κ estimate

Let us now turn to error estimates in more realistic observation cases, where the noise may be twofold: 1/ a model noise, where the initial condition is close to a Dirac delta in the BL distance; and 2/ a measurement noise, where the size distributions μ_0 and μ_t are observed with an error. Before stating our error estimate for κ when taking into account these two sources of noise, we need a stability result for the time-dependent solution with respect to the initial condition. Now that uniqueness of the weak solution to (14) is guaranteed, we then use the explicit expression provided by Theorem 2 to obtain stability with respect to the BL norm.

Theorem 4. *[Stability of the fragmentation equation in $(\mathcal{M}(\mathbb{R}^+), \|.\|_{BL})$] Assume κ satisfies* (**Hyp-1**)*, $\mu_0 \in \mathcal{M}^+(\mathbb{R}^+)$ satisfies* (**Hyp-2**)*, and moreover either $\gamma \geq 1$ or $\operatorname{supp}(\mu_0) \subset [m, M]$ with $m > 0$. Then the unique solution μ_t to the fragmentation equation* (14) *satisfies*

$$\|\mu_t\|_{BL} \leq C(M, N, T)\, \|\mu_0\|_{BL}, \qquad 0 \leq t \leq T.$$

Proof. We use the definition of the BL norm given by (5) and the representation of the solution provided in Theorem 2. Take $\varphi \in \mathcal{C}(\mathbb{R}^+)$ such that $\|\varphi\|_\infty \leq 1$ and $\|\varphi'\|_\infty \leq 1$. Then, using Proposition 3, we have

$$\int_0^{+\infty} \varphi(x) d\mu_t(x) = \int_0^M \int_0^1 \varphi(\ell x) d\mu^F_{t\ell^\gamma}(x) d\mu_0(\ell).$$

We set for $\ell \leq M$,

$$\Psi(\ell) = \int_0^1 \varphi(\ell x) d\mu^F_{t\ell^\gamma}(x).$$

We notice that for any $r \geq 0$, the moment of order r of the absolute value of the fundamental solution μ^F_t is uniformly bounded for $t \in [0,T]$ using the rough estimate based on Theorem 1

$$\int_{\mathbb{R}^+} x^r d|\mu^F_t|(x) \leq M^r \|\mu^F_t\|_{TV} \leq M^r e^{\alpha M^\gamma (N+1)T} \|\delta(x-1)\|_{TV} =: C(M,N,T,r).$$

Then for all $\ell \leq M$,

$$|\Psi(\ell)| \leq \int_0^{+\infty} |\varphi(\ell x)| d|\mu^F_{t\ell^\gamma}|(x) \leq \|\varphi\|_\infty \int_0^{+\infty} d|\mu^F_{TM^\gamma}|(x) \leq C(M,N,T,0),$$

and

$$\left|\Psi'(\ell)\right| \leq \int_0^{+\infty} |\varphi^{'}(\ell x)| x d|\mu^F_{t\ell^\gamma}|(x) + \int_0^{+\infty} |\varphi(\ell x)| t\gamma \ell^{\gamma-1} \left| \frac{\partial}{\partial t} d|\mu^F_{t\ell^\gamma}|(x) \right|$$

where

$$\int_0^{+\infty} |\varphi^{'}(\ell x)| x d|\mu^F_{t\ell^\gamma}|(x) \leq C(M,N,T,1) \|\varphi'\|_\infty,$$

and where

$$\begin{aligned}
&\int_0^\infty |\varphi(\ell x)| t\gamma \ell^{\gamma-1} \left| \frac{\partial}{\partial t} d|\mu^F_{t\ell^\gamma}|(x) \right| \\
&\qquad \leq \|\varphi\|_\infty T \max(M^{\gamma-1}, m^{\gamma-1}) \gamma\alpha \left(\int_0^\infty x^\gamma d|\mu^F_{t\ell^\gamma}|(x) + \int_0^\infty \int_x^\infty \kappa\left(\frac{x}{y}\right) y^{\gamma-1} d|\mu^F_{t\ell^\gamma}|(y) dx \right) \\
&\qquad = \|\varphi\|_\infty T \max(M^{\gamma-1}, m^{\gamma-1}) \gamma\alpha \left(\int_0^\infty x^\gamma d|\mu^F_{t\ell^\gamma}|(x) + \int_0^\infty \int_0^1 \kappa(z)\, dz y^\gamma d|\mu^F_{t\ell^\gamma}|(y) \right) \\
&\qquad \leq T \max(M^{\gamma-1}, m^{\gamma-1}) \gamma\alpha (N+1) C(M,N,T,\gamma).
\end{aligned}$$

We set $C(M,N,T) := C(M,N,T,0) + C(M,N,T,1) + T\max(M^{\gamma-1}, m^{\gamma-1})\gamma\alpha(N+1)C(M,N,T,\gamma)$ and define

$$\tilde{\Psi}(\ell) = \frac{\Psi(\ell)}{C(M,N,T)},$$

then

$$\|\tilde{\Psi}\|_\infty \leq 1, \qquad \|\tilde{\Psi}'\|_\infty \leq 1.$$

We have shown that for any $\varphi \in \mathcal{C}(\mathbb{R}^+)$ satisfying $\|\varphi\|_\infty \leq 1, \|\varphi'\|_\infty \leq 1$, there exists $\tilde{\Psi} \in \mathcal{C}(\mathbb{R}^+)$ such that $\|\tilde{\Psi}\|_\infty \leq 1, \|\tilde{\Psi}'\|_\infty \leq 1$ and

$$\int_0^{+\infty} \varphi(x) d\mu_t(x) \leq C(M,N,T) \int_0^{+\infty} \tilde{\Psi}(x) d\mu_0(x).$$

Thus the conclusion of Theorem 4 holds. □

Remark 3. *For $\gamma < 1$, and for any initial condition μ_0 such that $\mu_0(0) \neq 0$, it can demonstrated that stability with respect to the initial condition is lost.*

We are now ready to state our main stability result.

Theorem 5 (Stability of the κ estimate with respect to the initial condition)**.** *Assume κ satisfies* (**Hyp-1**)*. Take an initial condition μ_0^q satisfying* (**Hyp-2**) *and that is close to a delta function at $x = 1$ in the sense that*

$$\|\mu_0^q - \delta(x-1)\|_{BL} \leq q.$$

Denote by μ_t^q the unique solution to the fragmentation equation (14) *with initial condition μ_0^q. Consider the noisy measurements μ_0^{q,ε_0} and $\mu_t^{q,\varepsilon}$ of the respective measures μ_0^q and μ_t^q such that*

$$\|\mu_0^{q,\varepsilon_0} - \mu_0^q\|_{BL} \leq \varepsilon_0, \qquad \|\mu_t^{q,\varepsilon} - \mu_t^q\|_{BL} \leq \varepsilon.$$

Assume moreover either $\gamma \geq 1$ or $\operatorname{supp}(\mu_0) \subset [m, M]$ with $m > 0$. Then, for all $0 \leq t \leq T$, there are some constants K_1 and K_2 depending on M and T such that

$$\left\| \frac{\mu_t^{q,\varepsilon} - e^{-\alpha t}\mu_0^{q,\varepsilon_0}}{\alpha t} - \kappa \right\|_{BL} \leq K_1 t + \frac{\varepsilon_0 + \varepsilon + K_2 q}{\alpha t}.$$

Proof. We use the triangle inequality to write

$$\begin{aligned}\left\| \frac{\mu_t^{q,\varepsilon} - e^{-\alpha t}\mu_0^{q,\varepsilon_0}}{\alpha t} - \kappa \right\|_{BL} \leq & \frac{\|\mu_t^{q,\varepsilon} - \mu_t^q\|_{BL}}{\alpha t} + \frac{\|\mu_t^q - \mu_t\|_{BL}}{\alpha t} + e^{-\alpha t}\frac{\|\delta(x-1) - \mu_0^q\|_{BL}}{\alpha t} \\ & e^{-\alpha t}\frac{\|\mu_0^q - \mu_0^{q,\varepsilon_0}\|_{BL}}{\alpha t} + \left\| \frac{\mu_t - e^{-\alpha t}\delta(x-1)}{\alpha t} - \kappa \right\|_{BL}.\end{aligned}$$

The first term, the third term and the fourth term are directly controlled using the assumptions of Theorem 5. Theorem 3 combined with (6) guarantee that

$$\left\| \frac{\mu_t - e^{-\alpha t}\delta(x-1)}{\alpha t} - \kappa \right\|_{BL} \leq Kt.$$

For the second term, we use Theorem 4 to obtain

$$\|\mu_t^q - \mu_t\|_{BL} \leq C(M,T)\|\mu_0^q - \mu_0\|_{BL}.$$

Thus with the assumptions of Theorem 5, we obtain

$$\|\mu_t^q - \mu_t\|_{BL} \leq C(M,t)q.$$

This completes the proof of Theorem 5 with $K_1 = K$ and $K_2 = 1 + C(M,T)$. □

4 Reconstruction formula using Mellin coordinates

We have seen in the previous section, how to approximate κ for general initial conditions, and obtain $\kappa^{est}(u_0;.)$. However, it is also interesting to have an approximation of the Mellin of κ. Of course the best method to this end is not to use the Mellin transform of the approximation $\kappa^{est}(u_0;.)$ of κ. We use instead the series representation of μ_t to obtain a series representation of its Mellin transform, and then deduce an approximation. We denote by $U(t,.)$ the spatial Mellin transform of the solution μ_t to (14), and we denote by K the Mellin transform of κ, i.e.

$$U(t,s) = \int_0^{+\infty} x^{s-1} d\mu_t(x), \quad K(s) = \int_0^{+\infty} z^{s-1} d\kappa(z), \quad W(t,h,s) = \int_0^{+\infty} x^{s-1} e^{-\alpha h x^\gamma} d\mu_t(x), \quad t \geq 0.$$

for complex values of s for which the integrals are defined and for $h > 0$.

4.1 A formula for U

Lemma 3 (Representation of U as a power series)**.** *Take κ satisfying* (**Hyp-1**) *and $\mu_0 \in \mathcal{M}(\mathbb{R}^+)$. Then, the Mellin transform U of the solution μ_t to* (14) *satisfies*

$$U(t+\Delta t,s) = W(t,\Delta t,s) + \sum_{n=1}^{\infty} \frac{(\alpha\Delta t)^n}{n!} U(t,s+\gamma n) \sum_{j=0}^{n-1} (-1)^{n-1-j} K(s+j\gamma) \prod_{m=0}^{j-1} (K(s+m\gamma)-1), \tag{26}$$

with the convention

$$\prod_{n\in\emptyset} b_n = 1.$$

Proof. Since the fragmentation equation is autonomous, Theorem 2 implies that for all $t > 0$, $\Delta t > 0$, we have

$$\mu_{t+\Delta t} = e^{-\alpha x^\gamma \Delta t}\mu_t + \sum_{n=0}^{\infty} (\alpha\Delta t)^n \int_0^\infty \ell^{n\gamma} a_n\left(\frac{x}{\ell}\right) \mu_t(\ell)\frac{d\ell}{\ell},$$

We pass the above equality into Mellin coordinates to get, using Proposition 4

$$U(t+\Delta t,s) = W(t,\Delta t,s) + \sum_{n=0}^{\infty} (\alpha\Delta t)^n U(t,s+n\gamma) A_n(s),$$

where we denote by A_n the Mellin transform of the measure a_n. Passing (19) into the Mellin coordinates, the sequence A_n satisfies

$$A_0 = 0, \quad A_{n+1}(s) = \frac{1}{n+1}\left((K(s)-1)A_n(s+\gamma) + \frac{(-1)^n}{n!}K(s)\right).$$

By induction, we deduce

$$A_n(s) = \frac{1}{n!}\left((-1)^{n-1}K(s) + \sum_{j=1}^{n-1}(-1)^{n-1-j}K(s+j\gamma)\prod_{m=0}^{j-1}(K(s+m\gamma)-1)\right),$$

and Lemma 3 is proved. □

Inversion of the formula provided by Lemma 3, i.e. expressing K as a function of μ_t is not easy. However, since κ is supported on $[0,1]$, it follows that $K(s+n\gamma) \to 0$ as $n \to \infty$, and then an approximation formula for K is obtained by truncation at $n = 1$ of the second term at the right hand side of (26). In order to estimate the error of such approximation the following condition is needed

Hyp-3 There exists a compact interval $I \subset \mathbb{R}^+$ such that the Mellin transform U satisfies

$$U(t, w+\gamma+iv) \neq 0, \qquad \forall t \geq 0, \quad \forall w \in I, \quad \forall v \in \mathbb{R}.$$

(the Mellin transform U of the solution μ_t to the fragmentation equation does not vanish on some vertical strip of the complex plane.)

Remark 4. *It was proved in [3] that condition* **Hyp**-*3 is fulfilled by the Mellin transform of the self similar solution of the fragmentation equation* (14)*. If, following the basic assumption that for sufficiently large times t, it may be considered that measuring μ_t provides a measurement of the self similar solution (c.f. [17], [14]), it could be considered that the condition* **Hyp**-*3 is fulfilled for t large enough.*

4.2 A reconstruction for K using short times

Definition 5 (Approximation formula for the Mellin transform of the kernel)**.** *For those s such that $s = w + iv$ with $w \in I$ and where I is defined in* (**Hyp**-3)*, we define an estimate K^{est} for the Mellin transform K.*

$$K^{est}(s,t,\Delta t) = \frac{U(t+\Delta t,s) - W(t,\Delta t,s)}{\alpha \Delta t U(t,s+\gamma)}.$$

Theorem 6 (Reconstruction formula for K)**.**

Let us assume that the fragmentation kernel satisfies (**Hyp-1**) *and that additionally $\kappa \in C^1([0,1])$. We assume that $u_0 \in \mathcal{C}^3([0,M])$ and that either $u_0(M) > 0$, or $u_0(M) = 0$ and $u_0'(M) < 0$. In addition to that, we assume that* (**Hyp**-3) *holds. Let us define R as the difference between K and K^{est}*

$$R(s,t,\Delta t) = K(s) - K^{est}(s,t,\Delta t), \tag{27}$$

then there exists $C > 0$ depending on I, T, and M such that

$$|R(s,t,\Delta t)| \leq \frac{C\alpha\Delta t}{(1+|s|)}, \qquad \Re(s) \in I, \quad \Im(s) \in \mathbb{R}, \quad t \in [0,T].$$

We state one technical lemma that is used in the proof of Theorem 6.

Lemma 4 (Regularity and support of the solution to the fragmentation equation)**.** *Assume the fragmentation kernel κ satisfies* (**Hyp-1**) *and that $\kappa \in \mathcal{C}^1(0,1)$. Take $u_0 \in \mathcal{C}^3([0,M])$ such that* $\mathrm{supp}\,(u_0) = [0,M]$*. Then, if we denote by u the solution to the fragmentation equation* (14)*, it holds*

1. *The function $x \to u(t,x)$ is in $\mathcal{C}^3([0,M])$ for all $t > 0$.*
2. $\mathrm{supp}\,(u(t,.)) = [0,M]$.
3. *If $u_0(M) > 0$, then for all $t > 0$, $u(t,M) = e^{-\alpha M^\gamma t} u_0(M) > 0$.*
   If $u_0(M) = 0$ and $u_0'(M) < 0$, then $u(t,M) = 0$ and $\partial_x u(t,M) = e^{-\alpha M^\gamma t} u_0'(M) < 0$ for all $t > 0$.

The proof of Lemma 4 is postponed at the end of Section 4.

Proof of Thereom 6. Combining (27) with Lemma 3, we have the expression for the rest R

$$R(s,t,\Delta t) = \frac{1}{\alpha\Delta t} \sum_{n=2}^{\infty} \frac{(\alpha\Delta t)^n}{n!} \frac{U(t,s+\gamma n)}{U(t,s+\gamma)} \sum_{j=0}^{n-1} (-1)^{n-1-j} K(s+j\gamma) \prod_{m=0}^{j-1} (K(s+m\gamma) - 1). \tag{28}$$

Step 1. Estimate for K.
We prove here that for some $\tilde{C} > 0$ depending on I it holds

$$|K(s)| \leq \frac{\tilde{C}}{1+|s|}, \qquad \Re(s) \in I, \quad \Im(s) \in \mathbb{R}. \tag{29}$$

We have

$$K(s) = \int_0^1 \kappa(x) x^{s-1} dx = \frac{\kappa(1)}{s} - \frac{1}{s} \int_0^1 \kappa'(x) x^s dx.$$

Then

$$|K(s)| \leq \frac{\kappa(1)}{|s|} + \frac{1}{|s|}\int_0^1 |\kappa'(x)|dx \leq \frac{C}{|s|}$$

for some $C > 0$. This implies (29)

Step 2. Estimate for U. We prove here that for some $C > 0$ it holds

$$\left|\frac{U(t, w + n\gamma + iv)}{U(t, w + \gamma + iv)}\right| \leq Cn(n-1)M^{(n-1)\gamma}, \qquad w \in I, \quad n \geq 2, \quad v \in \mathbb{R}. \tag{30}$$

For $|v|$ large, we follow the calculation of [2, Chapter IV, Section 4] where the stationary phase method is used to study the behaviour of oscillatory integrals. For $w > 0$, we have for $v \neq 0$

$$\begin{aligned} U(t, w + iv) = \int_0^M u(t, x)x^{w-1}x^{iv}dx &= \int_0^M u(t, x)x^{w-1}e^{iv\ln(x)}dx \\ &= \frac{1}{iv}\int_0^M u(t, x)x^w \frac{d}{dx}\left(e^{iv\ln(x)}\right)dx. \end{aligned}$$

since $\dfrac{d}{dx}(e^{iv\ln(x)}) = \dfrac{iv}{x}e^{iv\ln(x)}$. We perform an integration by part and we obtain

$$U(t, w + iv) = \frac{1}{iv}u(t, M)M^w e^{iv\ln(M)} - \frac{1}{iv}\int_0^M e^{iv\ln(x)}\frac{\partial}{\partial x}\left(u(t, x)x^w\right)dx,$$

which we rewrite, using the same trick than above

$$U(t, w + iv) = \frac{1}{iv}u(t, M)M^w e^{iv\ln(M)} - \left(\frac{1}{iv}\right)^2 \int_0^M x\frac{\partial}{\partial x}\left(u(t, x)x^w\right)\frac{d}{dx}\left(e^{iv\ln(x)}\right)dx.$$

We perform another integration by part to obtain

$$\begin{aligned} U(t, w + iv) = \frac{1}{iv}u(t, M)M^w e^{iv\ln(M)} &- \left(\frac{1}{iv}\right)^2 M^w\left(M\frac{\partial}{\partial x}u(t, M) + wu(t, M)\right)e^{iv\ln(M)} \\ &+ \left(\frac{1}{iv}\right)^2 \int_0^M \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\left(u(t, x)x^w\right)\right)e^{iv\ln(x)}dx. \end{aligned}$$

The third term of right hand side above can be expanded using

$$\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\left(u(t, x)x^w\right)\right) = w^2x^{w-1}u(t, x) + x^w(1 + 2w)\frac{\partial}{\partial x}u(t, x) + x^{w+1}\frac{\partial^2}{\partial x^2}u(t, x).$$

Then we have

$$\begin{cases} U(t, w + \gamma + iv) = \dfrac{C(t, w, v)}{iv} + \dfrac{C'(t, w, v)}{(iv)^2}, \\ U(t, w + n\gamma + iv) = nM^{(n-1)\gamma}\left(\dfrac{C(t, w, v)}{niv} + \dfrac{C''(t, w, v, n)}{(iv)^2}\right) \end{cases} \tag{31}$$

for some complex constants $C(t,w,v)$, $C'(t,w,v)$ and $C''(t,w,v,n)$ defined as

$$
\begin{aligned}
&C(t,w,v) = u(t,M)M^{w+\gamma}e^{iv\ln(M)},\\
&C'(t,w,v) = -M^{w+\gamma}\left(M\frac{\partial u}{\partial x}(t,M) + (w+\gamma)u(t,M)\right)e^{iv\ln(M)},\\
&+\int_0^M \left((w+\gamma)^2 x^{w+\gamma-1}u(t,x) + x^{w+\gamma}(1+2(w+\gamma))\frac{\partial}{\partial x}u(t,x) + x^{w+\gamma+1}\frac{\partial^2}{\partial x^2}u(t,x)\right)e^{iv\ln(x)}dx\\
&C''(t,w,v,n) = M^{w+\gamma}\left(M\frac{\partial}{\partial x}u(t,M) + \frac{w+n\gamma}{n}u(t,M)\right)e^{iv\ln(M)}\\
&+\int_0^M \left((w+n\gamma)^2\left(1+x^{w+n\gamma-1}\right)u(t,x) + x^{w+n\gamma}(1+2(w+n\gamma))\frac{\partial u}{\partial x}(t,x) + x^{w+n\gamma+1}\frac{\partial^2 u}{\partial x^2}(t,x)\right)dx.
\end{aligned}
$$

If $u_0(M) > 0$, then Lemma 4 guarantees that $u(t,M) > 0$ as well. Then we have the following estimates on C, C' and C''

$$0 < C_0 \le |C(t,w,v)| \le C_1, \quad |C'(t,w,v)| \le C_2, \qquad w \in [0,a],\ t \in [0,T],\ v \in \mathbb{R}$$

$$|C''(t,w,v,n)| \le C_3, \qquad w \in [0,a],\ t \in [0,T],\ v \in \mathbb{R},\ n \ge 1.$$

Then, using (31), there exists $V > 0$ such that for $|v| \ge V$ and $w \in I$,

$$\left|\frac{U(t,w+n\gamma+iv)}{U(t,w+\gamma+iv)}\right| \le nM^{(n-1)\gamma}\frac{|Civ+C'|}{|Civ+C''|} \le nM^{(n-1)\gamma}\left(1+\frac{|C''-C'|}{|Civ+C'|}\right) \le nM^{(n-1)\gamma}K(V). \tag{32}$$

Now, using Hypothesis (**Hyp**-3), we have by continuity

$$|U(t,w+\gamma+iv)| \ge \varepsilon(V), \qquad w \in I, \quad v \in [-V,V]$$

for some $\varepsilon(V) > 0$. On the other hand, for $w \in I$,

$$
\begin{aligned}
|U(t,w+n\gamma+iv)| = \left|\int_0^M u(t,x)x^{w+n\gamma-1}e^{iv\ln(x)}dx\right| &\le M^{(n-1)\gamma}\left|\int_0^M u(t,x)x^{w+\gamma-1}dx\right|\\
&\le M^{(n-1)\gamma}M^{a+\gamma+1}\|u(t,.)\|_\infty.
\end{aligned}
$$

Then,

$$\left|\frac{U(t,w+n\gamma+iv)}{U(t,w+\gamma+iv)}\right| \le \frac{2}{n\varepsilon(V)}M^{a+\gamma+1}\|u(t,.)\|_\infty M^{(n-1)\gamma}, \qquad w \in I,\ v \in [-V,V]. \tag{33}$$

Combining (32) and (33) proves formula (30).

Now if $u_0(M) = 0$, then Lemma 4 guarantees that $u(t,M) = 0$ as well. Thus

$$U(t,w) = -\left(\frac{1}{iv}\right)^2\int_0^M x\frac{\partial}{\partial x}\left(u(t,x)x^w\right)\frac{d}{dx}\left(e^{iv\ln(x)}\right)dx.$$

In that case, $u_0'(M) < 0$ so that Lemma 4 guarantees that $\partial_x u(t,M) < 0$ as well, and we go one step further in the expansion and write

$$U(t,w) = -\left(\frac{1}{iv}\right)^2 M^{w+1} e^{iv\ln(M)} \frac{\partial}{\partial x}\left(u(t,x)x^w\right)\Big|_{x=M} + \left(\frac{1}{iv}\right)^3 M e^{iv\ln(M)} \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\left(u(t,x)x^w\right)\right)\Big|_{x=M} + \left(\frac{1}{iv}\right)^3 \int_0^M \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\left(u(t,x)x^w\right)\right)\right) e^{iv\ln(x)} dx$$

Using the same types of arguments than above, we end the proof of formula (30).

Step 3. Estimate for R.

Using formula (28) and the triangle inequality, we have

$$|R_(s,y,\Delta t)| \le \alpha\Delta t \sum_{n=2}^{\infty} \frac{(\alpha\Delta t)^{n-2}}{n!} \left|\frac{U(t,s+\gamma n)}{U(t,s+\gamma)}\right| \sum_{j=0}^{n-1} |K(s+j\gamma)| \prod_{m=0}^{j-1} |K(s+m\gamma)-1|.$$

Using now (29) and (30) we obtain for $\Re(s) \in I$ and $\Im(s) \in \mathbb{R}$

$$|R_(s,y,\Delta t)| \le \alpha\Delta t M^\gamma \sum_{n=2}^{\infty} \frac{(\alpha\Delta t)^{n-2}}{(n-2)!} (M^\gamma)^{n-2} \sum_{j=0}^{n-1} \frac{\tilde{C}}{1+|s|} \tilde{C}^j.$$

which implies

$$|R_(s,y,\Delta t)| \le \alpha\Delta t \frac{M^\gamma \tilde{C}}{1+|s|} \sum_{n=2}^{\infty} \frac{(\alpha\Delta t)^{n-2}}{(n-2)!} (M^\gamma)^{n-2} \frac{1-\tilde{C}^n}{1-\tilde{C}},$$

and Theorem 6 is proved for $C = M^\gamma \tilde{C} \sum_{n=2}^{\infty} \frac{(\alpha\Delta t)^{n-2}}{(n-2)!} (M^\gamma)^{n-2} \frac{1-\tilde{C}^n}{1-\tilde{C}} < \infty$. □

Corollary 3 (A better estimate for kernels not allowing erosion)**.** *Assume the fragmentation kernel* $\kappa \in C^2([0,1])$ *satisfies* (**Hyp-1**) *and* $\kappa(0) = 0$. *We assume that* $u_0 \in \mathcal{C}^3([0,M])$ *and that either* $u_0(M) > 0$, *or* $u_0(M) = 0$ *and* $u_0'(M) < 0$. *In addition to that, we assume that* (**Hyp-3**) *holds. Then, there exists* $C > 0$ *depending on* I, T *and* M *such that*

$$|R(s,t,\Delta t)| \le \frac{^C\alpha\Delta t}{(1+|s|^2)}, \qquad \Re(s) \in I, \quad \Im(s) \in \mathbb{R}.$$

Proof. The proof is the same than the proof of Theorem 6, except that the estimate for the Mellin transform in Step 1 becomes

$$K(s) = \int_0^1 \kappa(x) x^{s-1} dx = -\frac{1}{s}\int_0^1 \kappa'(x) x^{s-1} dx = -\frac{1}{s(s+1)}\left(\kappa'(1) - \kappa'(0) - \int_0^1 \kappa''(x) x^{s+1} dx\right).$$

Thus for some $\bar{C} > 0$

$$|K(s)| \le \frac{\bar{C}}{|s|^2+1}.$$

□

Corollary 4 (Estimate of the variance of the kernel)**.** *Let us assume that the fragmentation kernel* $\kappa \in C^1([0,1])$ *satisfies* (**Hyp-1**). *We assume that* $u_0 \in \mathcal{C}^3([0,M])$ *and that either* $u_0(M) > 0$, *or* $u_0(M) = 0$ *and* $u_0'(M) < 0$. *We denote by* $Var[\kappa]$ *the variance of the fragmentation kernel* κ. *Then, there is* $C > 0$ *that depends on* $t > 0$ *such that*

$$\left|Var[\kappa] - Var[\kappa^{est}(s,t,\Delta t)]\right| \le C\alpha\Delta t.$$

Proof. Notice that (**Hyp**-3) is not needed for this corollary, since $U(t,s) > 0$ for $s \in \mathbb{R}^+$, and in particular for $s = 3$. We express the variance of the kernel in terms of its Mellin transform

$$Var\left[\frac{\kappa}{2}\right] = \frac{1}{2}\int_{(0,1)} \left|x - \frac{1}{2}\right|^2 \kappa(x)dx = \frac{1}{2}K(3) - \frac{1}{2}K(2) + \frac{1}{8}K(1) = \frac{1}{2}K(3) - \frac{1}{4}.$$

and we apply Theorem 6. □

4.3 Proof of Lemma 4.

Proof of Lemma 4. The arguments rely on the formula obtained in Theorem 2.

1. We use the formula (18) obtained in Theorem 2. Note that it can be rewritten using the change of variables

$$z = \frac{x}{\ell}, \quad dz = -\frac{z^2}{x}d\ell, \tag{34}$$

as

$$u(t,x) = e^{-\alpha x^\gamma t}u_0(x) + \sum_{n=0}^{\infty}(\alpha t)^n \int_0^1 \frac{x^{n\gamma}}{z^{n\gamma}}a_n(z)u_0\left(\frac{x}{z}\right)\frac{dz}{z}. \tag{35}$$

The first term of the sum is clearly C^1, since u_0 is. To deal with the second term, set

$$I_n(x) = \int_0^1 \frac{x^{n\gamma}}{z^{n\gamma}}a_n(z)u_0\left(\frac{x}{z}\right)\frac{dz}{z}.$$

The fisrt step is to prove by induction that for all $x_0 > 0$, for all $n \geq 0$, $z \to a_n(z) \in \mathcal{C}^1[x_0, 1]$. The function a_0 is clearly $\mathcal{C}^1$, since it is identically zero. Let us assume that for some $n \geq 0$, $z \to a_n(z) \in \mathcal{C}^1[x_0, 1]$. The function a_{n+1} satisfies (19) and is composed with three terms. The first term and third term are clearly $\mathcal{C}^1$ since $a_n(z) \in \mathcal{C}^1[x_0, 1]$ and since $\kappa \in \mathcal{C}^1(0, 1)$. We focus on the second term

$$J_n(x) = \int_x^\infty y^{\gamma-1}\kappa\left(\frac{x}{y}\right)a_n(y)dy.$$

Once again, it can be rewritten using the change of variables (34)

$$J_n(x) = \int_{x_0}^1 \frac{x^\gamma}{z^\gamma}\kappa(z)a_n\left(\frac{x}{z}\right)\frac{dz}{z}.$$

The dominated convergence theorem guarantees that $J_n \in \mathcal{C}^1[x_0, 1]$ and that

$$J_n'(x) = \int_{x_0}^1 \frac{x^\gamma}{z^\gamma}\kappa(z)a_n'\left(\frac{x}{z}\right)\frac{dz}{z^2} + \int_{x_0}^1 \gamma\frac{x^{\gamma-1}}{z^\gamma}\kappa(z)a_n\left(\frac{x}{z}\right)\frac{dz}{z}.$$

Indeed

$$\left|\frac{x^\gamma}{z^\gamma}\kappa(z)a_n'\left(\frac{x}{z}\right)\frac{1}{z^2}\right| \leq \frac{1}{x_0^{2+\gamma}}\|a_n'\|_{\mathcal{C}^0[x_0,1]}, \quad \left|\gamma\frac{x^{\gamma-1}}{z^\gamma}\kappa(z)a_n\left(\frac{x}{z}\right)\frac{1}{z}\right| \leq \frac{\gamma\max\{x_0^{\gamma-1}, 1\}}{x_0^{1+\gamma}}\|a_n\|_{\mathcal{C}^0[x_0,1]}.$$

We have proven that $a_n \in \mathcal{C}^1(0, 1)$ since it is $\mathcal{C}^1(K)$ for all K compact of $(0, 1)$.

The fisrt step is to prove that $I_n \in \mathcal{C}^1([0,M])$. To do so, we use the dominated convergence to prove that for all $x_0 > 0$, we have $I_n \in \mathcal{C}^1([x_0,M])$ and that

$$I_n'(x) = n\gamma x^{n\gamma-1}\int_0^1 \frac{1}{z^{n\gamma}}a_n(z)u_0\left(\frac{x}{z}\right)\frac{dz}{z} + \int_0^1 \frac{x^{n\gamma}}{z^{n\gamma}}a_n(z)u_0'\left(\frac{x}{z}\right)\frac{dz}{z^2}, \quad x \in [x_0,M]. \tag{36}$$

Indeed, the conclusion of the dominated convergence holds: $u_0 \in \mathcal{C}^1([0,M])$, hence the the integrand is in $\mathcal{C}^1([0,M])$ as well. The domination is as follows: since $\text{supp}\,(u_0) \subset [0,M]$, the bounds of the integral I_n are $z \in \left[\frac{x}{M},1\right] \subset \left[\frac{x_0}{M},1\right]$, and thus

$$\left|\frac{x^{n\gamma-1}}{z^{n\gamma}}a_n(z)u_0\left(\frac{x}{z}\right)\frac{1}{z}\right| \leq \frac{\max\{\left(\frac{x_0}{M}\right)^{n\gamma-1}, M^{n\gamma-1}\}}{x_0^2}\|u_0\|_\infty a_n(z),$$
$$\left|\frac{x^{n\gamma}}{z^{n\gamma}}a_n(z)u_0'\left(\frac{x}{z}\right)\frac{dz}{z^2}\right| \leq \frac{\max\{\left(\frac{x_0}{M}\right)^{n\gamma}, M^{n\gamma}\}}{x_0^2}\|u_0'\|_\infty a_n(z),$$

and it was proved in (25) that $\|a_n\|_{TV} \leq \dfrac{(N+2)^n}{n!}$.

Now we claim that the function S defined as

$$S(x) = \sum_{n=1}^{\infty}(\alpha t)^n I_n(x) \tag{37}$$

is of class $\mathcal{C}^1([x_0,M])$ for all $x_0 > 0$. Indeed, we just saw that $I_n \in \mathcal{C}^1([0,M])$, and that I_n' is given by (36). For $x \in [x_0,M]$, we can control each of the two terms of the sum (36) by two sequences that converge. Indeed using again (25), we have

$$n\int_0^1 \left|\frac{x^{n\gamma-1}}{z^{n\gamma}}a_n(z)u_0\left(\frac{x}{z}\right)\frac{1}{z}\right|dz \leq \frac{\max\{\left(\frac{x_0}{M}\right)^{n\gamma-1}, M^{n\gamma-1}\}}{x_0^2}\|u_0\|_\infty n\frac{(N+2)^n}{n!},$$
$$\int_0^1 \left|\frac{x^{n\gamma}}{z^{n\gamma}}a_n(z)u_0'\left(\frac{x}{z}\right)\frac{dz}{z^2}\right|dz \leq \frac{\max\{\left(\frac{x_0}{M}\right)^{n\gamma}, M^{n\gamma}\}}{x_0^2}\|u_0'\|_\infty \frac{(N+2)^n}{n!},$$

and

$$\sum_{n=1}^{\infty}(\alpha t)^n\left(\frac{M^\gamma}{x_0^\gamma}\right)^n\frac{(N+2)^n}{n!} < \infty, \qquad \sum_{n=1}^{\infty}(\alpha t)^n\left(\max\{x_0^\gamma, M^\gamma\}\right)^n n\frac{(N+2)^n}{n!} < \infty,$$

This ends the proof of 1, and we have in addition for $x \in [x_0,M]$ an expression of the spatial derivative of u

$$\begin{aligned}\frac{\partial}{\partial x}u(t,x) =& e^{-\alpha x^\gamma t}u_0'(x) - \alpha\gamma x^{\gamma-1}te^{-\alpha x^\gamma t}u_0(x)\\ &+\sum_{n=1}^{\infty}(\alpha t)^n\left(n\gamma x^{n\gamma-1}\int_0^1\frac{1}{z^{n\gamma}}a_n(z)u_0\left(\frac{x}{z}\right)\frac{dz}{z} + \int_0^1\frac{x^{n\gamma}}{z^{n\gamma}}a_n(z)u_0'\left(\frac{x}{z}\right)\frac{dz}{z^2}\right).\end{aligned} \tag{38}$$

Similar arguments hold to guarantee that $x \to u(t,x) \in C^3([0,M])$.

2. First, we claim that $\text{supp}\,(u(t,.)) \subset [0,M]$. Indeed, this is a consequence of formula (35) and of the fact that $\text{supp}\,(a_n) \subset [0,1]$ for $n \geq 0$. Let us now prove that $\text{supp}\,(u(t,.)) = [0,M]$. Take $y \in [0,M]$ and set $Y(t) = u(t,y)$. The fragmentation equation (14) implies

$$Y'(t) \geq -\alpha y^\gamma Y(t),$$

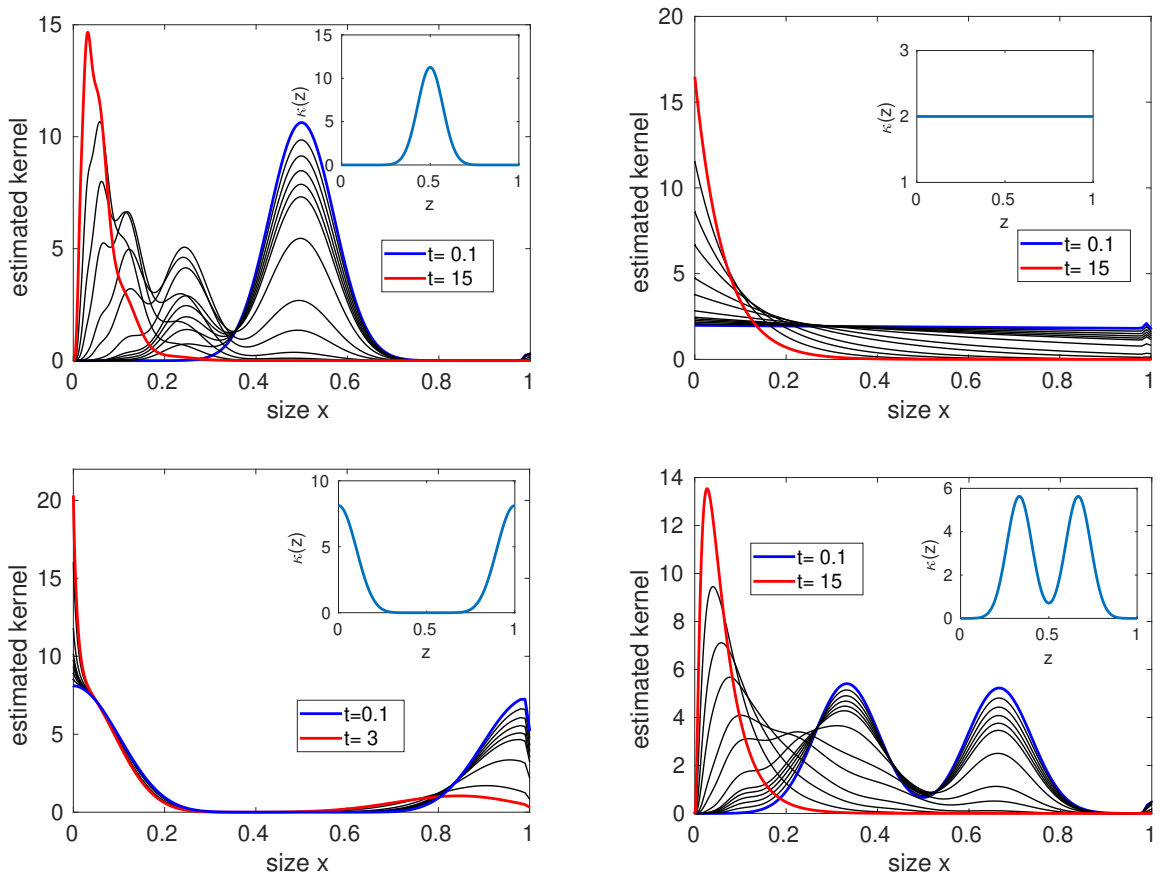

Figure 1: On each plot, we display the estimated kernel $\kappa^{est}(t)$ defined in Theorem 3 for different time points t. All plots are in black, except for the very early time point ($t = 0.1$, blue), and a late time point (red).

i.e.

$$u(t,y) = Y(t) \geq e^{-\alpha y^\gamma t} Y(0) = e^{-\alpha y^\gamma t} u_0(y).$$

If $u_0(y) \neq 0$, then for all $t \geq 0$, $u(t,y) \neq 0$. If $u_0(y) = 0$, since $y \in \text{supp}\,(u_0)$, for all $\varepsilon > 0$, there exists y_ε such that $|y - y_\varepsilon| < \varepsilon$ and $u_0(y_\varepsilon) \neq 0$ and then $u(t, y_\varepsilon) \neq 0$, which implies that $y \in \text{supp}\,(u(t,.))$. Thus $\text{supp}\,(u(t,.)) = [0, M]$.

3. It is clear from formula (18) that $u(t,M) = e^{-\alpha M^\gamma t} u_0(M)$. Then, if $u_0(M) > 0$, we have $u(t,M) > 0$. If $u_0(M) = 0$ and $u_0'(M) < 0$, we have $u(t,M) = 0$, and formula (38) implies

$$\frac{\partial}{\partial x} u(t,M) = e^{-\alpha M^\gamma t} u_0'(M) < 0.$$

□

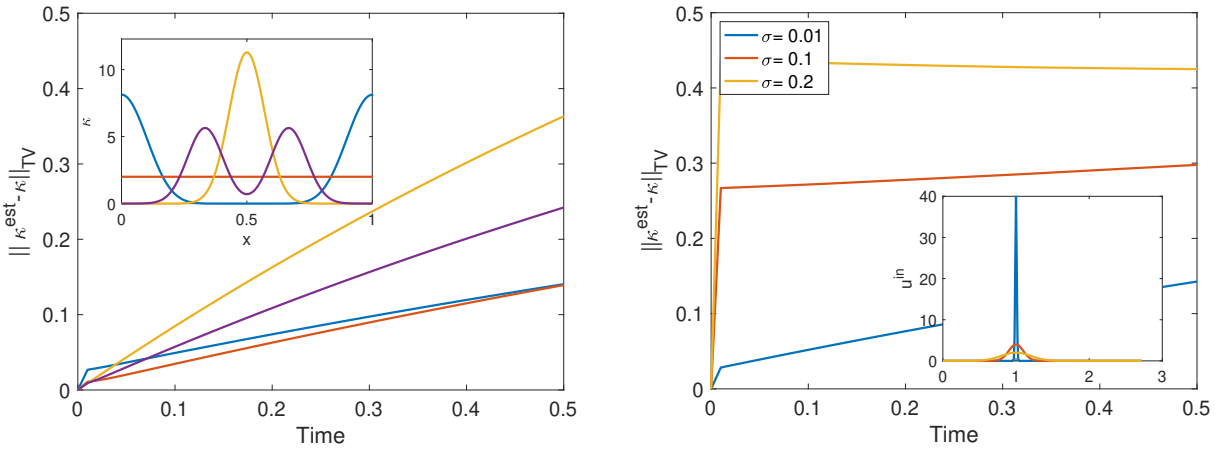

Figure 2: Time evolution of the distance in the Total Variation norm between the fragmentation kernel and its first order estimate given by κ^{est}, departing from $u_0 = \delta_1$ (Left) or departing from a Gaussian curve centred at $x = 1$ with a standard deviation $\sigma = 0.01$, $\sigma = 0.1$ and $\sigma = 0.2$ respectively (Right). Left: the corresponding kernel is displayed on the inset with the same colour as the error curve. Right: the fragmentation kernel is the one in Fig 1 bottom left (in blue on the inset of the left figure). The corresponding initial condition is displayed on the inset with the same colour as the error curve.

5 Numerical simulations

5.1 Illustration of the short-time behaviour

We illustrate on Figure 1 the formula obtained in Theorem 3 for the estimated kernel κ^{est}, that is

$$\kappa^{est}(t) = \frac{u(t,x) - e^{-\alpha t}\delta(x-1)}{\alpha t},$$

where u is the solution to the fragmentation equation endowed with initial condition $\delta(x-1)$. It is observed on the plots that the above formula for the estimated kernel is valid for early time points. As time goes by, the size distribution is driven towards stationary state and the information on the kernel is lost. On Figure 2 Left, we see the linear time evolution of the error $\left\|\kappa^{est}(t) - \kappa\right\|_{TV} \leq Kt$ as in Theorem 3 for the four kernels of Figure 1. In Figure 2 Right, are drawn the curves of the error $\left\|\kappa^{est}(u_0, t) - \kappa\right\|_{TV}$ as in Remark 2, for three initial conditions u_0 given by (truncated) Gaussians of standard deviation $\sigma = 0.01$, $\sigma = 0.1$ and $\sigma = 0.2$. In that case, an extra constant error is added, related to the distance between δ_1 and u_0. For large standard deviations, this error in the total variation norm becomes so large that it is no more meaningful: we see the interest to turn to the Bounded Lipshitz norm.

Figures 3 and 4 illustrate Corollary 2 for the fragmentation kernel of erosion type higher at the edges (Fig 1 bottom left). We see how the estimation is differently impacted around $x = 0$ and around $x = 1$,. This gives interesting hints on how the kernel symmetry could be used to improve the theoretical estimates.

Finally, we tested the assumptions illustrate the results of Theorem 5, computing the W_1 distance instead of the BL-norm (this is valid for our numerical case studies). In Fig **??**, we display the results obtained for the two-peak kernel (bottom right on Fig 5). The time evolution of the W_1- error first

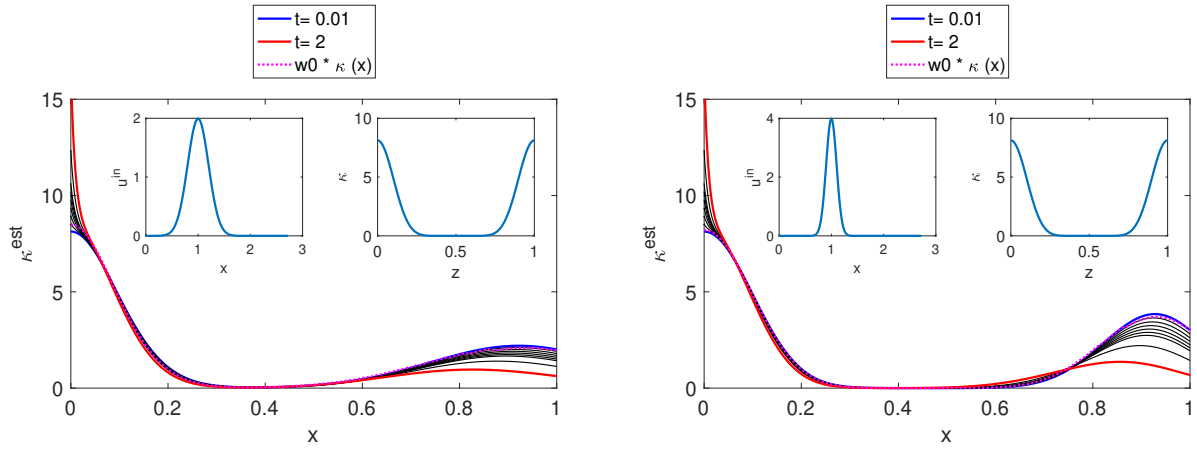

Figure 3: Estimation of the fragmentation kernel κ^{est} at various times (black curves), first time point in blue, latest time point in red. Left: for an initial data with variance $\sigma = 0.2$, Right: with variance $\sigma = 0.1$. In dotted pink is what is truly estimated, namely the convolution $w_0 * \kappa$, see Corollary 2.

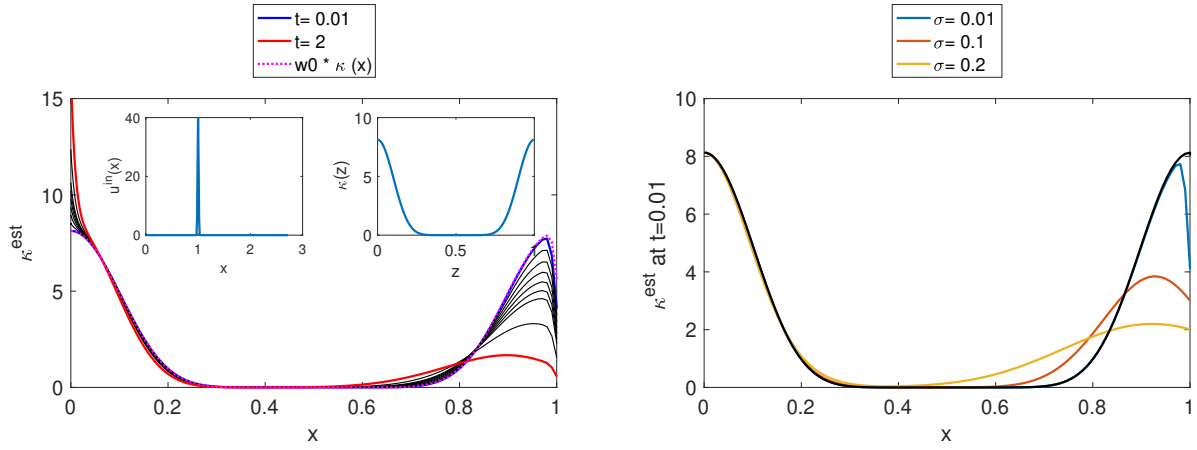

Figure 4: Estimation of the fragmentation kernel at various times (black curves), first time point in blue, latest time point in red. Left: for an initial data with variance $\sigma = 0.01$, Right: estimate at an early time point $t = 0.01$ given by the three initial conditions of respective variance $\sigma = 0.01$, $\sigma = 0.1$ and $\sigma = 0.2$.

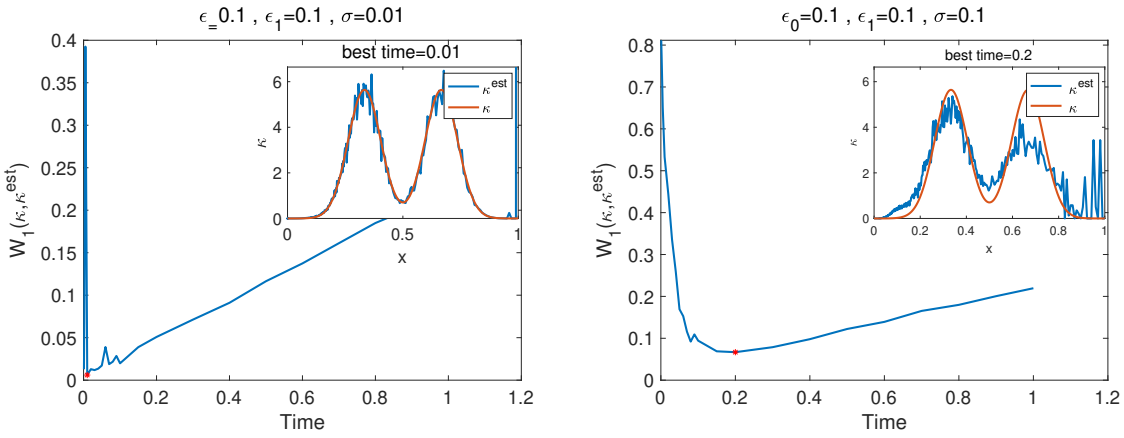

Figure 5: Time evolution of the error estimate in W_1- distance between the fragmentation kernel and its estimate, for $\epsilon_0 = \epsilon_1 = 0.1$ and for $\sigma = 0.01$ (Left), $\sigma = 0.1$ (Right). The insets display the best estimate, obtained at the timepoints where the W_1 distance is minimal.

decreases and then increases, as expected by the estimate of Theorem 5; we also displayed the best estimated κ, taken at the optimal time where the error reaches its minimum.

5.2 Recovering the variance of the fragmentation kernel

In this section we explore how formulae from Theorem 6 enable us to recover the variance from simulated data. We consider 6 different typical fragmentation kernels. In Table 1, we give the value for their variance and standard deviation. We recall that the variance and standard deviation are given by

$$Var = Var\left[\frac{\kappa}{2}\right] = \frac{1}{2}K(3) - \frac{1}{4}, \qquad SD = \sqrt{Var},$$

and we define similarly, provided that $\frac{1}{2}K^{est}(3,t,\Delta t) - \frac{1}{4} > 0$

$$Var^{est} = \frac{1}{2}K^{est}(3,t,\Delta t) - \frac{1}{4},$$

else, $Var^{est} = 0$. Also $SD^{est} = \sqrt{Var^{est}}$, where the formula for K^{est} is given in Definition 5.

In Figure 5.2, we explore the influence of Δt on the estimation SD^{est} for standard deviation SD. The relative error on the standard deviation is defined as

$$\text{Relative Error on the Standard Deviation} = \frac{|SD^{est} - SD|}{SD}.$$

We observe that for $\alpha\Delta t = 0.05$, we are able to have a precise idea of the standard deviation of the kernel. The worst error we have is for the kernel in blue. In Table 1, we see that this corresponds to the kernel whose SD is of 0.1001. A 50% error gives us an estimated SD of 0.15, which is still below the SD of the spread gaussian.

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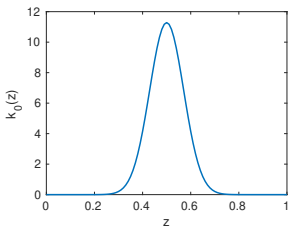
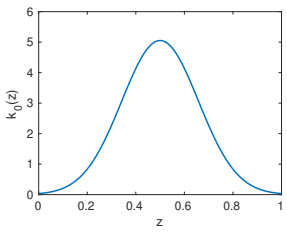
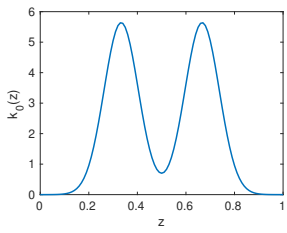
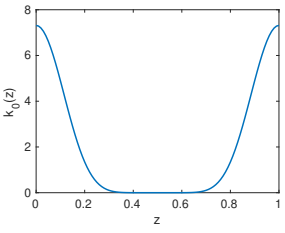
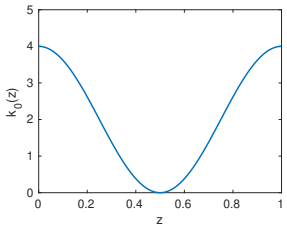
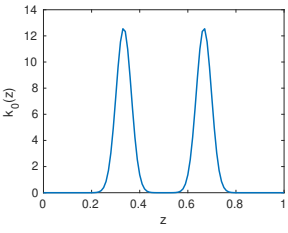
Kernel			
Var	0.0100	0.0295	0.0378
SD	0.1001	0.1718	0.1944
Kernel			
Var	0.1625	0.1290	0.0338
SD	0.4031	0.3591	0.1839

Table 1: Variance and Standard Deviation for 6 typical fragmentation kernels.

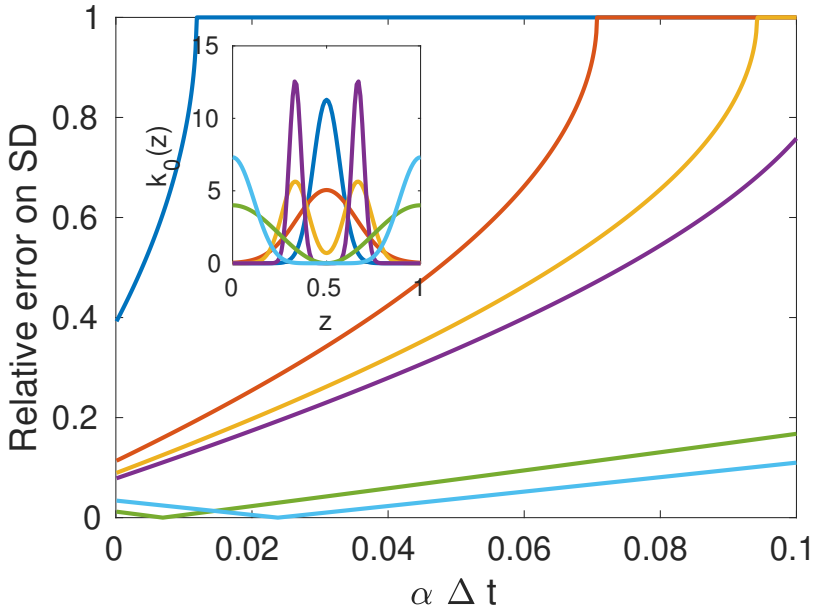


Figure 6: Influence of Δt on the relative error on the standard deviation. The other parameters are $\alpha = \gamma = 1$, the initial condition is a very thin gaussian centered at $x = 2$.

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