

An inverse problem: recovering the fragmentation kernel from the short-time behaviour of the fragmentation equation

Marie Doumic, Miguel Escobedo, Magali Tournus

▶ **To cite this version:**

Marie Doumic, Miguel Escobedo, Magali Tournus. An inverse problem: recovering the fragmentation kernel from the short-time behaviour of the fragmentation equation. 2023. hal-03494439v2

HAL Id: hal-03494439
https://hal.science/hal-03494439v2

Preprint submitted on 11 Feb 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

An inverse problem: recovering the fragmentation kernel from the short-time behaviour of the fragmentation equation

Marie Doumic * Miguel Escobedo † Magali Tournus ‡

Abstract

Given a phenomenon described by a self-similar fragmentation equation, how to infer the fragmentation kernel from experimental measurements of the solution ? To answer this question at the basis of our work, a formal asymptotic expansion suggested us that using short-time observations and initial data close to a Dirac measure should be a well-adapted strategy. As a necessary preliminary step, we study the direct problem, *i.e.* we prove existence, uniqueness and stability with respect to the initial data of non negative measure-valued solutions when the initial data is a compactly supported, bounded, non negative measure. A representation of the solution as a power series in the space of Radon measures is also shown. This representation is used to propose a reconstruction formula for the fragmentation kernel, using short-time experimental measurements when the initial data is close to a Dirac measure. We prove error estimates in Total Variation and Bounded Lipshitz norms; this gives a quantitative meaning to what a "short" time observation is. For general initial data in the space of compactly supported measures, we provide estimates on how the short-time measurements approximate the convolution of the fragmentation kernel with a suitably-scaled version of the initial data. The series representation also yields a reconstruction formula for the Mellin transform of the fragmentation kernel κ and an error estimate for such an approximation. Our analysis is complemented by a numerical investigation.

1 Introduction

The fragmentation equation is a size-structured PDE describing the evolution of a population of particles. It is ubiquitous in modelling physical or biological phenomena (cell division [40], amyloid fibril breakage [49], microtubules dynamics [30]) and technological processes (mineral processing, grinding solids [32], polymer degradation [47] and break-up of liquid droplets or air bubbles). As presented in [36], the equation may be written as follows

$$\frac{\partial}{\partial t}u(t,x) = -\frac{u(t,x)}{x}\int_0^x yF(x,y)dy + \int_x^\infty u(t,y)F(y,x)dy, \tag{1}$$

where $u(t,x)$ represents the concentration of particles at time t of size x, and the fragmentation measure $F(y,x)$ the creation of particles of size x out of fragmenting particles of size y. The mathematical properties of the fragmentation equation have been extensively studied using a great variety of methods (statistical physics; formal asymptotics; real, complex and functional analysis;

*Sorbonne Université, Inria, CNRS, Université de Paris, Laboratoire Jacques-Louis Lions, 4 place Jussieu, 75005 Paris, France. Email: marie.doumic@inria.fr

†Departamento de Matemáticas Universidad del País Vasco (UPV/EHU) Apartado 644, Bilbao 48080 Spain. Email: miguel.escobedo@ehu.es

‡Aix Marseille Univ, CNRS, I2M, Marseille, Centrale Marseille, France. Email: magali.tournus@math.cnrs.fr

linear semigroup theory; probability methods). Only a few references are given here among the vast existing mathematical literature as: on particular solutions [47, 51], on the existence and uniqueness of solutions for the Cauchy problem [45, 36, 4, 21, 35, 5], on detailed properties of the solutions [41, 3, 14, 11, 26]. For a rather complete list of references the interested reader may consult [5, 6, 12].

Due to its importance in the modelling (see for example [31, 44]), and its very rich mathematical properties, a class of fragmentation measures F have proved to be particularly fruitful: the measures composed with a fragmentation rate $B(x)$, that depends on the particle size, and a fragmentation kernel $\kappa(y/x)$, that describes the probability that a particle of size y is created by fragmentation of a particle of size x:

$$F(x,y) = \frac{B(x)}{x}\kappa\left(\frac{y}{x}\right). \tag{2}$$

The fact that the probability to obtain a particle of size y out of a particle of size x only depends on the ratio y/x is a classical assumption often referred to as a 'self-similarity property' [11, 13, 14]. In order to be coherent with the modelling, the fragmentation kernel must be a finite measure compactly supported on $(0,1)$ OR [0,1]? and such that $zd\kappa(z)$ is a probability measure. With such a fragmentation measure, equation (1) reads then

$$\frac{\partial}{\partial t}u(t,x) = -B(x)u(t,x) + \int_x^\infty \kappa\left(\frac{x}{y}\right) B(y)u(t,y)\frac{dy}{y}. \tag{3}$$

The two key physical parameters B and κ encode fundamental information on the mechanical stability of each particle, and can take different forms depending on the particular process considered. To estimate the parameters B and κ using population data (when only the particles density $u(t,x)$ can be accessed, not the trajectory of each individual particle) is a challenging mathematical problem, important for the applications. The specific application that led us to its study originates from the works [48, 49], where the authors provide experimental size distribution profiles of different types of amyloid fibrils, in order to estimate their intrinsic division properties (B and κ) and then to relate them to their respective pathogenic properties [9]. It is not possible to follow experimentally each fibril one by one, hence the necessity to draw the characteristic features of each particle from the evolution of the whole population.

1.1 Review on existing results to estimate the fragmentation kernel

Identifying the fragmentation kernel κ from observed population data has been a challenging problem for some time. As detailed below, in most of the cases up to now, the analysis of this problem has been based on the idea of self-similar long-time asymptotic behaviour of the solutions to (3), see [3, 11, 13, 16, 21].

In the seminal paper [32] of A. N. Kolmogorov (1940) on general random processes of particle grinding, the self-similar large time behaviour of the size distribution is identified in a slightly different but closely related equation, discretised in time and with a constant fragmentation rate B. The self-similar asymptotic behaviour of the fragmentation equation written for the cumulative distribution function was established in [24] by Filippov (1961) for the case $B(x) = x^\gamma$, $\gamma > 0$ and the result is now well-known by the scientific community under fairly general balance assumptions on the parameters (see for instance [41, 23, 3, 14]).

From the seventies, scientists from physics and chemical departments have been using this similarity concept for the kernel inverse problem. In 1974, a scientist of a department of chemical engineering [44] developed a method to extract information on probabilities of droplet-breakup, and in particular on the daughter-drop-distribution (in modern terms: the fragmentation kernel), as a function of drop sizes data, obtained from an experiment of pure fragmentation in a batch vessel.

To do so, the self-similar behaviour of the solutions of the fragmentation equation, written here too for the cumulative distribution function, is assumed, thereby restricted to power law fragmentation rates (*i.e.* $B(x) = \alpha x^\gamma$ with $\alpha, \gamma > 0$), and the moments of the kernel are estimated from the moments of the large time size distribution. To recover the kernel from its moments, a method based on the expansion of the kernel on a specific polynomial basis is suggested. These results are generalised later in 1980 [38] to non-power law fragmentation rates associated with an adapted definition for the self-similarity of the kernel so as to keep the self-similar asymptotic behaviour of the model.

From the late nineties, the large improvements in computer hardware opened the field of numerical investigations of mathematical models. In [33] the authors provide insights on how the stationary shape of the particle size distribution is impacted by the kernel. Their conclusion is that the inverse problem of assigning a breakage kernel to a known self-similar particular size distribution is ill-posed not only in a mathematical but also in a physical sense since quite different kernels correspond to almost the same particles size distribution. This conclusion has been confirmed by the theoretical results of [3, 18]: in these articles, key properties of the fragmentation kernel have been proved to be linked to unobservable quantities of the asymptotic profile, namely its behaviour for very small or very large sizes. In [18], we proposed a reconstruction formula for κ based on the mere knowledge of the long-time asymptotic profile g of the solutions of (3) in suitable functional spaces [18]. This formula involves the moments of order s of the asymptotic profile g, s being taken along a vertical complex line, *i.e.* $s = u + iv,\ v \in (-\infty, \infty)$. However, due to its severely ill-posedness on the one hand, and to the impossibility of observing the asymptotic profile for very small or very large sizes on the other hand, this reconstruction formula revealed of little practical use. Of note, a similar estimate in the case of the growth-fragmentation equation with constant growth and division rates has been carried out in a statistical setting in [29], together with a consistency result and a numerical study.

We thus explored further the influence of the kernel on the time evolution of the length distribution [19]. We showed that despite the previously seen limitations, the asymptotic profile remains helpful to distinguish whether the fragmentation kernel is an erosion-type kernel (one of the fragments has a size close to that of the parent particle) or produces particles of similar sizes. By statistical testing, we also showed that departing from the same initial condition, there exists a time-window right after the initial time where two different kernels give rise to a maximal difference of their corresponding size distribution solutions, and that the initial condition that maximizes this difference is a very sharp Gaussian. This last remark led us to explore further the short-time behaviour of the solution, which is the basis of our present study.

Inverse problems for fragmentation equations related with our "short-time" approach appeared in 2002 [1] and in 2013 [2]. In the first article the authors consider the reconstruction of a source term in a coagulation-fragmentation equation. The equation is linearized assuming that for short times the solution $c(t, x)$ of the equation may be approximated by the initial data c_0, and keeping only linear terms in the perturbation. The inverse problem for the linear equation is then solved using optimal control methods, the solvability theory of operator equations, and iteration algorithms. In the second article the authors solve the linearization of the inverse problem for (1), obtained assuming $F = 1 + f$ with $|f|$ small, $u = c_0 + g$ where c_0 is the solution of (1) with $F = 1$, assuming that $|g|$ is also small, and keeping in the equation only principal terms.

1.2 Outline of our main results

In the present article we revisit the question of estimating κ from measurements on the population density $u(t, x)$, and we introduce two main novelties. First, a new method, that only uses

short-time measurements of the solutions. As pointed out in the above review, this is a very different idea from those generally used up to now since these are based on the long time self-similar behaviour of the solutions. Second, a reconstruction formula for the Mellin transform of κ and an estimate of the error of the approximation. More precisely, we assume the fragmentation rate B to be known, and provide a reconstruction formula for the sole fragmentation kernel.

Unless specific assumptions are stated, we restrict the study to power law fragmentation rates

$$B(x) = \alpha x^\gamma, \qquad \gamma > 0, \quad \alpha > 0. \tag{4}$$

The guiding idea of our study is based on the following remark: for Δt small enough, the solution μ to the fragmentation equation (3) with B defined by (4) formally satisfies

$$\mu(t+\Delta t, x) \approx \mu(t,x) - \alpha \Delta t x^\gamma \mu(t,x) + \alpha \Delta t \int_x^\infty \kappa\left(\frac{x}{y}\right) y^{\gamma-1} \mu(t,y) dy + o(\Delta t). \tag{5}$$

If we assume that at time t, the size distribution $\mu(t,x)$ is a Dirac delta function at $x=1$, that is denoted δ_1 or $\delta(x-1)$, then

$$\mu(t+\Delta t, x) \approx \delta(x-1) - \alpha \Delta t \delta(x-1) + \alpha \Delta t \kappa(x) + o(\Delta t),$$

and thus the kernel κ can be directly estimated from the measurement of the profile μ at time $t+\Delta t$ as

$$\kappa(x) \approx \frac{1}{\alpha \Delta t} \left(\mu(t+\Delta t, x) - (1-\alpha \Delta t)\delta(x-1)\right) + o(1), \qquad \Delta t \ll 1.$$

To make the above estimate of κ rigorous, we first prove the uniqueness of a non negative solution μ to the Cauchy problem for the equation (3) when κ and the initial data μ_0 are non negative measures satisfying some suitable conditions (see Theorem 1 below). Then we expand the solution $\mu(t,x)$ as a power series about t in the Banach space of Radon measures. Up to our knowledge, such representation of the measure-valued solution of the fragmentation equation with a fragmentation kernel measure κ is new, though some explicit solutions of the fragmentation equation in form of series are given in [51, 50] for particular continuous fragmentation functions and particular initial data μ_0.

To estimate κ from the measurement of the distribution profile $\mu(\Delta t, .)$ for small values of Δt, the cunning observation is to impose that the initial distribution μ_0 is a Dirac mass. In other words, at time $t=0$, all particles should have the same size. Heuristically, if all particles have the exact same size at $t=0$, after a time t long enough so that a non-negligible quantity of particles have broken once, but short enough so that a negligible quantity of particles has broken twice, it is clear that the kernel κ, sometimes referred to as the "daughter particle distribution" can directly be read on the distribution of particles strictly smaller than initially.

Of course no experiment may produce a suspension where all the particles have the same size since it would mean being able to follow each particle one by one. However, we can hope to obtain a suspension where all particles have approximately the same size, described for instance by a gaussian distribution that would be not too far from a Dirac delta function. For that reason the stability of our estimates of κ with respect to measurement noise and to the error on the initial data μ_0 must be estimated . It is then necessary to consider measure-valued solutions. The existence and uniqueness of such solutions to coagulation or fragmentation equations has been already studied in the literature of mathematics, for example in [39], [10] for the coagulation equation, [13], [7] for a growth-fragmentation equation, [16] for a fragmentation equation but where only the case $\gamma = 0$ would satisfy the hypothesis.

Quantifying the stability result first requires to understand what are the types of experimental uncertainties on the initial data coming from the experiments. These are twofold: first, instead of a delta function at $x = x_0$, the initial data is a spread distribution with variance $\sigma > 0$ (due to the impossibility to obtain a perfectly homogeneous suspension). Second, this distribution is centered at $x = x_0 + \varepsilon$ for some $\varepsilon > 0$, instead of $x = x_0$ (possible bias on the measurement of the particles' sizes). In order to deal with these uncertainties, the Bounded-Lipshitz (BL) norm is better suited than the total variation norm (TV). For instance, $\forall a \in \mathbb{R}$, $b \in \mathbb{R}$, such that $|b - a| < 2$

$$\|\delta_a - \delta_b\|_{TV} = 2, \qquad \|\delta_a - f_{a,\sigma}\|_{TV} = 2,$$

whereas

$$\|\delta_a - \delta_b\|_{BL} = |b - a|, \qquad \|\delta_1 - f_{1,\sigma}\|_{BL} \leq \frac{2\sqrt{\sigma}}{\sqrt{2\pi}},$$

where $f_{a,\sigma}$ is the density of the gaussian function centered at $x = a$ with variance σ.

However, in the case of a generic initial data not necessarily close to a delta function, a reconstruction formula may still be obtained through the use of the moments of the solution. From the very beginning of the study of inverse problems for the fragmentation equation, the moments of the solution μ ([44, 38]), and then its Mellin transform, have been extensively used. Of note, the Mellin transform of κ (denoted by K from now on), is of interest by itself since it provides a range of moments of the fragmentation kernel, in particular variance and skewness. An exact expression for K was obtained in [18] from the long-time self similar asymptotic profile of the solution μ in terms of an (in general) oscillatory integral, but no way to approximate this integral and estimate the error was given. The exact series representation of the solution μ to (3) obtained in the present paper may be used in order to deduce an approximation of K and estimate the error of such an approximation.

Our last contribution is thus a robust reconstruction formula of K. To this end, we use short-time measurements of the solution μ to equation (3) for generic initial data μ_0, not necessary close to a Dirac measure, and the initial data itself. This dependence on the initial data μ_0 contrasts with the result in [18] where the reconstruction formula (see Theorem 2 in [18]) only involves the long-time asymptotic profile of the solution. Since the equation is autonomous, this means to be able to access two close consecutive measurements of the particles size distribution - an experimental setting much more realistic than to depart from a mono-disperse suspension.

To sum-up, the main novelties brought by this paper are

- a proof of the uniqueness and stability of the solution in the space of non negative measures endowed with the total variation norm (Theorem 1),
- a representation of a solution to the fragmentation equation (endowed with any non-negative measure as initial data) as a power series in the Banach space of measures endowed with the total variation norm (Theorem 2), implying in particular existence of measure-valued solutions to (3),
- a proof of the non-negativity of the power series solution (Theorem 3),
- as a consequence of the three previous items and summarized in Corollary 1, a statement of existence of a unique non-negative measure-valued solution to (3), accompanied with a power series representation of this unique solution,
- a stability result for the solution to the fragmentation equation for the BL norm, which is a norm adapted to weak convergence of measures (Theorem 4),

- a robust reconstruction formula for the fragmentation kernel involving the short-time solution of the fragmentation equation endowed with a delta function as initial condition. Robustness is to be understood in the sense that if the initial condition is close to a delta function at $x = x_0$ in the BL norm (for instance a rectangular function centered in x_0 or a delta function at $x = x_0+\epsilon$ with ϵ small), then the estimated kernel obtained with the reconstruction formula is close to the real kernel in the BL norm (Theorem 5 and Theorem 6),
- a reconstruction formula for the Mellin transform K of the fragmentation kernel κ involving the short-time solution of the fragmentation equation endowed with any initial condition (Theorem 7).

The outline of the paper is as follows. In the remaining of Section 1, some properties of measures and classical results on measure theory are recalled, as well as the definition of Mellin transform and Mellin convolution. Section 2 is devoted to the proof of the existence, uniqueness, non negativity and series representation of solutions to the problem (3) (with B defined by (4)) in the space of Radon measures, and their stability with respect to the initial data in the TV norm. In Section 3, estimates of the fragmentation kernel and bounds for the error of such estimates are obtained using, for small values of the time variable, the expression as a series of the solution μ provided by Theorem 2. The stability of these estimates with respect to the initial data and noise measurements is also considered in BL norm. In Section 4, we study the Mellin transform K of κ. Under some regularity assumption on κ and on the initial data μ_0, a reconstruction formula K^{est} of K is obtained, only based on short time-intervals measurements of the solution to the fragmentation equation and an estimate of the error $K - K^{est}$ is obtained. An estimate of the variance of κ is then deduced, and under a stronger regularity assumption on κ, a pointwise estimate of the difference of κ and the inverse Mellin transform of K^{est} is proved. We end the paper with a numerical investigation of the short-time behaviour of the fragmentation equation, we illustrate the estimation results of Theorems 5 and 6, and we explore how Theorem 7 can be applied to recover the variance of the kernel from the data. For every theorem, the constants arising in estimates and depending continuously on parameters $p_1, p_2, \dots$ are denoted by $C(p_1, p_2, \dots)$.

1.3 Short reminder on measure theory

We define $\mathcal{M}(\mathbb{R}^+)$ as the set of Radon measures μ (not necessarily probability measures) such that $\text{supp}\,(\mu) \subset \mathbb{R}^+$. Let us recall that $\mathcal{M}(\mathbb{R}^+)$ is the dual space of the space $(\mathcal{C}(\mathbb{R}^+), \|.\|_\infty)$ of continuous functions. We denote by (μ^+, μ^-) the Jordan decomposition of μ. We endow $\mathcal{M}(\mathbb{R}^+)$ with two different norms: the total variation norm and the Bounded-Lipschitz norm. As mentioned in the introduction, the final purpose is to obtain stability with respect to the BL norm, the TV norm being a technical intermediate tool to reach this purpose. The TV norm of the (signed) measure $\mu \in \mathcal{M}(\mathbb{R}^+)$ is defined as

$$\|\mu\|_{TV} = \sup\{\int_{R^+} \varphi(x) d\mu(x), \quad \varphi \in \mathcal{C}(\mathbb{R}^+) \cap L^1(d|\mu|),\ \|\varphi\|_\infty \leq 1\}. \tag{6}$$

We recall that $(\mathcal{M}(\mathbb{R}^+), \|.\|_{TV})$ is a Banach space. We now define the BL norm as

$$\|\mu\|_{BL} = \sup\{\int_{R^+} \varphi(x) d\mu(x), \quad \varphi \in \mathcal{C}(\mathbb{R}^+) \cap L^1(d|\mu|),\ \|\varphi\|_\infty \leq 1,\ \|\varphi'\|_\infty \leq 1\}. \tag{7}$$

Comparing (6) and (7), it is clear that

$$\forall\, \mu \in \mathcal{M}(\mathbb{R}^+), \qquad \|\mu\|_{BL} \leq \|\mu\|_{TV}. \tag{8}$$

An optimal transportation point of view is provided in [43, Proposition 23] for the BL norm. It is proven that for any signed Radon measure with finite mass μ we have

$$\|\mu\|_{BL} = \inf\Big\{(\|\mu^+ - \nu\|_{TV} + \|\mu^- - \eta\|_{TV}) + W_1(\nu,\eta), \quad (\nu,\eta) \in \mathcal{M}^+_\mu(\mathbb{R}^+),\Big\} \tag{9}$$

$$\mathcal{M}^+_\mu(\mathbb{R}^+) = \Big\{(\nu,\eta) \in \mathcal{M}^+(\mathbb{R}^+) \times \mathcal{M}^+(\mathbb{R}^+);\, \nu \le \mu^+, \eta \le \mu^-, \|\nu\|_{TV} = \|\eta\|_{TV}\Big\} \tag{10}$$

where $\mathcal{M}^+(\mathbb{R}^+)$ is the space of positive Radon measures with support in $\mathbb{R}^+$, and W_1 stands for the classical Wasserstein distance [46] between two positive measures of same mass, namely

$$\begin{gathered} W_1(\nu,\eta) := \inf_{\pi\in\Pi(\nu,\eta)} \textstyle\int_{\mathbb{R}^+} |x-y| d\pi(x,y), \\ \Pi(\nu,\eta) := \big\{\pi \text{ positive measure on } \mathbb{R}^+ \text{ s.t. } \textstyle\int_{\mathbb{R}^+} \pi(x,y)dx = \eta(y), \quad \int_{\mathbb{R}^+} \pi(x,y)dy = \nu(x)\big\}. \end{gathered} \tag{11}$$

Let us recall that for μ,ν two probability measures and for $a>0$, we have $W_1(a\mu,a\nu) = aW_1(\mu,\nu)$. Formula (9) (10) can be interpreted as follows: the BL norm of the signed measure μ is the BL distance between the two positive measures μ^+ and μ^-. Now take μ^+ and μ^- two positive measures. Consider ν and η two positive measures such that $\nu \le \mu^+$, $\eta \le \mu^-$ and $\|\nu\|_{TV} = \|\eta\|_{TV}$. The subpart ν of the measure μ^+ is transported onto the subpart η of the measure μ^-, with a cost $W_1(\nu,\eta)$. The remaining positive measures $(\mu^+ - \nu)$ and $(\mu^- - \eta)$ are both cancelled with a cost $\|\mu^+ - \nu\|_{TV} + \|\mu^- - \eta\|_{TV}$. Among all couples (ν,η) that satisfy $\nu \le \mu^+$, $\eta \le \mu^-$ and $\|\nu\|_{TV} = \|\eta\|_{TV}$, we choose one such that the sum $(\|\mu^+ - \nu\|_{TV} + \|\mu^- - \eta\|_{TV}) + W_1(\nu,\eta)$ is minimal (such a couple exists, it is proved in [42] that the infimum is actually a minimum). Let us give three examples.

- Take $\mu = \delta(x-1)$ and $\mu_\varepsilon = \delta(x-(1+\varepsilon))$. Consider $\nu_a = a\mu$ and $\eta_a = a\mu_\varepsilon$ with $0 \le a \le 1$. Then $0 \le \nu_a \le \mu$, $0 \le \eta_a \le \mu_\varepsilon$, and $\|\nu_a\|_{TV} = \|\eta_a\|_{TV} = a$. Using formula (9)(10) we have

$$\begin{aligned} \|\mu - \mu_\varepsilon\|_{BL} &= \inf_{0\le a\le 1} \Big\{(\|\mu - \nu_a\|_{TV} + \|\mu_\varepsilon - \eta_a\|_{TV}) + W_1(\nu_a,\eta_a)\Big\} \\ &= \inf_{0\le a\le 1} \Big\{2(1-a) + a\varepsilon\Big\} \\ &= \begin{cases} \varepsilon & \text{for } \varepsilon \le 2, \\ 2 & \text{for } \varepsilon > 2. \end{cases} \end{aligned}$$

- Take $\mu = \delta(x-1)$ and μ_σ is the measure with the rectangular density $\frac{1}{2\sigma\sqrt{3}}\mathbb{1}_{[1-\sigma\sqrt{3},1+\sigma\sqrt{3}]}$ with variance σ^2 for $0<\sigma<1$. We take $\nu_a = \mu = \delta_1$ and $\eta_a = \mu_\sigma = f_\sigma dx$, in (9), and obtain

$$\|\mu - \mu_\sigma\|_{BL} \le W_1(\mu,\mu_\sigma) \le \int_{1-\sigma\sqrt{3}}^{1+\sigma\sqrt{3}} \frac{|y-1|dy}{2\sqrt{3}} = \frac{\sqrt{3}}{2}\sigma.$$

- Take $\mu = \delta(x-1)$ and μ_σ the Gaussian with mean 1 and variance σ^2. We have

$$\|\mu - \mu_\sigma\|_{BL} \le W_1(\mu,\mu_\sigma) \le \int |x| \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} = \frac{2\sigma}{\sqrt{2\pi}}.$$

We recall that for $\mu \in \mathcal{M}(\mathbb{R}^+)$ and $T \in \mathcal{C}(\mathbb{R}^+)$, the pushforward η of the measure μ by the function T is defined as the unique measure

$$\eta = T\#\mu$$

such that for all $\varphi \in \mathcal{C}(\mathbb{R}^+)$,

$$\int \varphi(x) d\eta(x) = \int (\varphi \circ T)(x) d\mu(x).$$

For $\ell > 0$, we define the application

$$T_\ell(x) = \ell x, \qquad x \in \mathbb{R}^+. \tag{12}$$

1.4 Mellin transform

Definition 1. *For a measure $\mu \in \mathcal{M}(\mathbb{R}^+)$, its Mellin transform $M[\mu]$ is defined as*

$$M[\mu](s) = \int_{\mathbb{R}^+} x^{s-1} d\mu(x), \tag{13}$$

for $s \in \mathbb{C}$ such that (13) is well-defined.

Definition 2 (Mellin convolution (cf. [37])**.** *Take μ and ν two compactly supported finite measures on $\mathbb{R}^+$. Their Mellin convolution (sometimes referred to as multiplicative convolution) is defined as*

$$\forall \varphi \in \mathcal{C}(\mathbb{R}^+), \qquad \langle \mu * \nu, \varphi \rangle = \langle \mu^x \otimes \nu^y, \varphi \circ p \rangle,$$

where $p : (x, y) \to xy$.

If $d\mu(x) = f(x)dx$ and $d\nu(x) = g(x)dx$ for f and g in $L^1(\mathbb{R}^+)$, then $\mu * \nu$ is the measure with density

$$(f * g)(x) = \int_{\mathbb{R}^+} f(y)\, g\left(\frac{x}{y}\right) \frac{dy}{y}.$$

If $d\mu(x) = f(x)dx$ with $f \in \mathcal{C}(\mathbb{R}^+)$ and $\nu = \delta(y - \ell)$, then $\mu * \nu = T_\ell \# \mu$ is the measure with density

$$(f * \nu)(z) = \frac{1}{\ell} f\left(\frac{z}{\ell}\right).$$

Proposition 1 (Mellin transform and Mellin convolution)**.** *Take μ and ν two compactly supported finite measures on $\mathbb{R}^+$. For the s for which the expression below is defined, we have*

$$M[\mu * \nu](s) = M[\mu](s) M[\nu](s).$$

2 Measure-valued solutions to the fragmentation equation: existence, uniqueness, stability and series representation.

The basis of our analysis in all the remaining of this work are the weak solutions to the Cauchy problem for equation 17 with the initial condition

$$\mu_t(t = 0) = \mu_0, \tag{14}$$

whose precise definition is given below. Throughout the present paper, the following assumptions are used.

Hyp-1 The fragmentation kernel $\kappa \in \mathcal{M}^+(\mathbb{R}^+)$ contains no atom at $x = 0$ and at $x = 1$, and satisfies

$$\text{supp}\,(\kappa) \subset [0, 1], \qquad \int_0^1 d\kappa(z) = N < +\infty, \qquad \int_0^1 z d\kappa(z) = 1. \tag{15}$$

Hyp-2 The initial condition $\mu_0 \in \mathcal{M}^+(\mathbb{R}^+)$ is compactly supported

$$\text{supp}\,(\mu_0) \subset [0, L]. \tag{16}$$

Even though κ and μ_t are measures, we sometimes write the fragmentation equation as

$$\frac{\partial}{\partial t}\mu_t(x) = -\alpha x^\gamma \mu_t(x) + \alpha \int_x^\infty \kappa\left(\frac{x}{y}\right) y^{\gamma-1} d\mu_t(y), \qquad \mu_{t=0}(x) = \mu_0(x), \tag{17}$$

or as

$$\frac{\partial}{\partial t}\mu_t(x) = -\alpha x^\gamma \mu_t(x) + \alpha \int_x^\infty \kappa\left(\frac{x}{y}\right) y^{\gamma-1} \mu_t(y) dy, \qquad \mu_{t=0}(x) = \mu_0(x).$$

Definition 3 (Weak solution for (17))**.** *A family $(\mu_t)_{t\geq 0} \subset \mathcal{M}(\mathbb{R}^+)$ is called a measure-valued solution to problem* (3) (4) (14) *with initial data $\mu_0 \in \mathcal{M}(\mathbb{R}^+)$ satisfying* (**Hyp-2**) *if the mapping $t \to \mu_t$ is narrowly continuous and for all $\varphi \in \mathcal{C}(\mathbb{R}^+)$ such that $x \mapsto \varphi(x)/(1+x)$ is bounded on $[0, \infty)$, and all $t \geq 0$,*

$$\int_{\mathbb{R}^+} \varphi(x) d\mu_t(x) = \int_{\mathbb{R}^+} \varphi(x) d\mu_0(x) + \int_0^t ds \int_{\mathbb{R}^+} d\mu_s(x) \alpha x^\gamma \left(-\varphi(x) + \int_0^1 d\kappa(z)\varphi(xz)\right). \tag{18}$$

We recall that μ_n converges narrowly toward μ if for all $\varphi \in C_b(\mathbb{R}^+)$, $\int \varphi d\mu_n \to \int \varphi d\mu$, where $C_b(\mathbb{R}^+)$ denotes the set of continuous and bounded functions defined on $\mathbb{R}^+$.

Although several results may be found in the references given in the introduction about the existence and uniqueness of solutions to fragmentation equations, none of them covers exactly the hypotheses that we have in mind for κ and the initial data μ_0. For the sake of completeness, our first result is then an existence and uniqueness of compactly supported and non negative measure-valued solutions to (17) under assumptions (**Hyp-1**), (**Hyp-2**). We begin with a uniqueness and stability result.

Theorem 1 (Uniqueness and TV-stability for the fragmentation equation in $(\mathcal{M}_+(\mathbb{R}^+), \|.\|_{TV})$)**.** *Assume* (**Hyp-1**), (**Hyp-2**) *and $\gamma \geq 0$. Suppose that $\mu_t \in \mathcal{C}(\mathbb{R}^+, \mathcal{M}(\mathbb{R}^+))$, is a non negative measure-valued solution to* (17)*, in the sense of Definition 3. Then, for all $t > 0$,*

$$\text{supp}\,(\mu_t) \subset [0, L] \tag{19}$$

$$\|\mu_t\|_{TV} \leq \|\mu_0\|_{TV} e^{\alpha(2L)^\gamma(1+N)t} \tag{20}$$

$$\int_{\mathbb{R}^+} x d\mu_t(x) = \int_{\mathbb{R}^+} x d\mu_0(x) \tag{21}$$

where N is defined in (**Hyp-1**) *and L is defined in* (**Hyp-2**)*. In particular such a solution is unique.*

Proof. Consider $\mu_t \in \mathcal{C}(\mathbb{R}^+, \mathcal{M}(\mathbb{R}^+))$ a non negative measure-valued solution to (17) in the sense of Definition (3). We start proving property (19). To this end we first notice that

$$\alpha x^\gamma \mu_t(x) = \alpha \mu_t(x) \int_0^x \frac{y}{x} x^{\gamma-1} \kappa\left(\frac{y}{x}\right) dy.$$

Then in the right-hand side of (18) we write

$$\int_{\mathbb{R}^+} d\mu_s(x)\alpha x^\gamma \left(-\varphi(x) + \int_0^1 d\kappa(z)\varphi(xz)\right) = \\ = \int_0^\infty \int_0^x b(x,y)\, y \left(\frac{\varphi(y)}{y} - \frac{\varphi(x)}{x}\right) dy d\mu_s(x).$$

where

$$b(x,y) = \alpha x^{\gamma-1}\kappa\left(\frac{y}{x}\right). \tag{22}$$

Consider then the test function

$$\varphi(x) = \begin{cases} 0 & \forall x \in [0,L] \\ x(x-L) & \forall x \in [L, L+1] \\ x & \forall x \geq L+1. \end{cases}$$

Since $x \to \varphi(x)/(1+x)$ is bounded and non decreasing on $[0,\infty)$, by (18)

$$\int_{\mathbb{R}^+} \varphi(x) d\mu_t(x) = \int_{\mathbb{R}^+} \varphi(x) d\mu_0(x) + \int_0^t \int_0^\infty \int_0^x b(x,y)\, y \left(\frac{\varphi(y)}{y} - \frac{\varphi(x)}{x}\right) dy d\mu_s(x) ds \\ \leq \int_{\mathbb{R}^+} \varphi(x) d\mu_0(x) = 0,$$

where the last inequality is justified since $\mu_t \geq 0$ for $t \geq 0$ and since $x \to \varphi(x)/x$ is non decreasing as well. Since $\varphi \geq 0$ and $\mu_t \geq 0$ for all $t \geq 0$ it follows that for $\varphi(x)d\mu_t(x) = 0$ for all $t > 0$ and almost every $x > 0$. Since by construction $\varphi(x) > 0$ for all $x > L$ we deduce that for every $t > 0$, $\text{supp}\,(\mu_t) \subset [0, L]$.

To prove the BV estimate (20), we use definition (6), and take $\varphi \in \mathcal{C}(\mathbb{R}^+)$ such that $\|\varphi\|_\infty \leq 1$. By (18),

$$\int_{\mathbb{R}^+} \varphi(x) d\mu_t(x) = \int_{\mathbb{R}^+} \varphi(x) d\mu_0(x) + \int_0^t ds \int_{\mathbb{R}^+} \alpha x^\gamma d\mu_s(x) \left(-\varphi(x) + \int_{z=0}^1 \varphi(xz) d\kappa(z)\right).$$

Let $\chi \in C([0,\infty))$ such that $||\chi||_\infty = 1$, $\chi(x) = 1$ for all $x \in [0, L]$ and $\chi(x) = 0$ for $x > 2L$ and consider the function defined as

$$\psi(x) := \alpha x^\gamma \chi(x) \left(-\varphi(x) + \int_{z=0}^1 \varphi(xz) d\kappa(z)\right),$$

It satisfies $\psi \in \mathcal{C}(\mathbb{R}^+)$ and, since $\|\varphi\|_\infty \leq 1$, and $\text{supp}\,(\chi) \subset [0, 2L]$,

$$\sup_{0 \leq x \leq L} |\psi(x)| \leq \alpha(2L)^\gamma (1+N).$$

Therefore,

$$\int_{\mathbb{R}^+} \varphi(x) d\mu_t(x) \leq \|\mu_0\|_{TV} + \alpha(2L)^\gamma (1+N) \int_0^t \|\mu_s\|_{TV} ds,$$

which implies

$$\|\mu_t\|_{TV} \leq \|\mu_0\|_{TV} + \alpha(2L)^\gamma (1+N) \int_0^t \|\mu_s\|_{TV} ds,$$

and Gronwall Lemma yields (20). Finally, mass conservation property (21) is obtained by choosing $\varphi(x) = x$ in definition 3 and using the last statement of (**Hyp-1**). □

The following proposition provides us with a solution to the fragmentation equation when initial condition is a delta mass localized at $x = \ell$ in terms of a fundamental solution.

Proposition 2 (Fundamental solution rescaled)**.** *Fix $\ell > 0$. Assume that μ_t^F is a fundamental solution to* (17)*, i.e. a solution to* (17) *when the initial data is $\mu_0 = \delta(x-1)$. Then, $\mu_t^\ell = T_\ell \# \mu^F_{\ell^\gamma t}$, with $T_\ell(x) = \ell x$, is a solution to* (17) *with $\mu_0^\ell = \delta(x-\ell)$.*

Proof. We set $\mu_t^\ell := T_\ell \# \mu^F_{\ell^\gamma t}$. Let us prove that μ_t^ℓ is a solution to (17) with initial condition $\mu_0^\ell = \delta(x-\ell)$ and conclude by uniqueness of the solution. First, $\mu_0^\ell = T_\ell \# \mu_0 = T_\ell \# \delta(x-1) = \delta(x-\ell)$. Then, we obtain that for all $\varphi \in \mathcal{C}_c(\mathbb{R}^+)$

$$\int_{\mathbb{R}^+} \varphi(x) d\mu_t^\ell(x) = \int_{\mathbb{R}^+} \varphi(x) d\left(T_\ell \# \mu^F_{\ell^\gamma t}\right)(x) = \int_{\mathbb{R}^+} (\varphi \circ T_\ell)(x) d\mu^F_{\ell^\gamma t}(x) = \int_{\mathbb{R}^+} \varphi(\ell x) d\mu^F_{\ell^\gamma t}(x).$$

Since μ_t is a weak solution to (17), we have

$$\begin{aligned}\int_{\mathbb{R}^+} \varphi(\ell x) d\mu^F_{\ell^\gamma t}(x) = \ & \int_{\mathbb{R}^+} \varphi(\ell x) d\mu_0(x) \\ & + \alpha \int_0^{\ell^\gamma t} \int_{\mathbb{R}^+} \left(-x^\gamma \varphi(\ell x) d\mu_s^F(x) + \varphi(\ell x) \int_x^\infty y^{\gamma-1} d\kappa\left(\frac{x}{y}\right) d\mu_s^F(y) \right) ds.\end{aligned}$$

Let us treat each of the three terms of the sum above separately. The first term is

$$\int_{\mathbb{R}^+} \varphi(\ell x) d\mu_0(x) = \int_{\mathbb{R}^+} \varphi(x) d\mu_0^\ell(x).$$

The second term is treated using the change of variables $s = \ell^\gamma u$

$$-\alpha \int_0^{\ell^\gamma t} \int_{\mathbb{R}^+} x^\gamma \varphi(\ell x) d\mu_s^F(x) ds = -\alpha \int_0^t \int_{\mathbb{R}^+} (x\ell)^\gamma \varphi(\ell x) d\mu^F_{\ell^\gamma u}(x) du,$$

and then the change of variables $z = T_\ell(x)$ *i.e.* $d\mu^F_{\ell^\gamma u}(x) = d\left(T_\ell \# \mu^F_{\ell^\gamma u}\right)(z)$

$$\begin{aligned}-\alpha \int_0^t \int_{\mathbb{R}^+} (x\ell)^\gamma \varphi(\ell x) d\mu^F_{\ell^\gamma u}(x) du &= -\alpha \int_0^t \int_{\mathbb{R}^+} z^\gamma \varphi(z) d\left(T_\ell \# \mu^F_{\ell^\gamma u}\right)(z) du \\ &= -\alpha \int_0^t \int_{\mathbb{R}^+} z^\gamma \varphi(z) d\mu_u^\ell(z) du.\end{aligned}$$

For the third term we also use the change of variables $s = \ell^\gamma u$ followed by the change of variables $z = T_\ell(x)$ and to finish the change of variable $w = T_\ell(y)$ *i.e.* $d\mu^F_{\ell^\gamma u}(y) = d\left(T_\ell \# \mu^F_{\ell^\gamma u}\right)(w) = d\mu_u^\ell(w)$ to get

$$\alpha \int_0^{\ell^\gamma t} \int_{\mathbb{R}^+} \varphi(\ell x) \int_x^\infty y^{\gamma-1} d\kappa\left(\frac{x}{y}\right) d\mu_s^F(y) ds = \alpha \int_0^t \int_{\mathbb{R}^+} \varphi(z) \int_z^\infty w^{\gamma-1} d\kappa\left(\frac{z}{w}\right) d\mu_u^\ell(w) du.$$

To summarize,

$$\int_{\mathbb{R}^+} \varphi(x) d\mu_t^\ell(x) = \int_{\mathbb{R}^+} \varphi(x) d\mu_0^\ell(x) - \alpha \int_0^t \int_{\mathbb{R}^+} \left(z^\gamma \varphi(z) d\mu_u^\ell(z) + \varphi(z) \int_z^\infty w^{\gamma-1} d\kappa\left(\frac{z}{w}\right) d\mu_u^\ell(w) \right) du.$$

Finally, since $t \to \mu_t^F$ is narrowly continuous, then $t \to \mu_t^\ell$ is narrowly continuous as well. This ends the proof of Proposition 2. □

Theorem 2 (Existence of a solution to (17) represented as a power series)**.** *For any fragmentation kernel κ satisfying* (**Hyp-1**)*, $\gamma \geq 0$ and μ_0 satisfying* (**Hyp-2**)*, there exists a weak solution $\mu_t \in \mathcal{C}(\mathbb{R}^+, \mathcal{M}(\mathbb{R}^+))$ to* (17) *in the sense of Definition 3. This solution is given by the following everywhere convergent series*

$$\mu_t = e^{-\alpha x^\gamma t}\mu_0 + \sum_{n=0}^{\infty}(\alpha t)^n \int_0^\infty \ell^{n\gamma} a_n\left(\frac{x}{\ell}\right)\mu_0(\ell)\frac{d\ell}{\ell}, \tag{23}$$

where the sequence a_n is defined as follows for $x \in [0,1]$,

$$a_0(x) = 0, \qquad a_{n+1}(x) = \frac{1}{n+1}\left(-x^\gamma a_n(x) + \int_x^\infty y^{\gamma-1}\kappa\left(\frac{x}{y}\right)a_n(y)dy + \kappa(x)\frac{(-1)^n}{n!}\right). \tag{24}$$

In particular,

$$\int_0^\infty \varphi(x)d\mu_t(x) = \int_0^L\int_0^1 \varphi(\ell x)d\mu^F_{t\ell^\gamma}(x)d\mu_0(\ell), \qquad \forall \varphi \in \mathcal{C}(\mathbb{R}^+), \tag{25}$$

and

$$\operatorname{supp}(\mu_t) \subset \operatorname{supp}(\mu_0), \qquad \text{for all } t > 0. \tag{26}$$

Remark 1. *We emphasize that in* (24) *and* (23)*, κ, μ_0 and a_n may be measures. In this case, the Mellin convolution product has to be understood in the sense of measures. For instance, the formula* (23) *means*

$$\mu_t = e^{-\alpha x^\gamma t}\mu_0 + \sum_{n=0}^{\infty}(\alpha t)^n a_n * b_n \tag{27}$$

$$db_n(y) = y^{n\gamma}d\mu_0(y), \tag{28}$$

where the Mellin convolution product $*$ is defined in Definition 2.

Proof of Theorem 2. **Step 1: A power series representation for the fundamental solution.** In this step, we prove that

$$\mu^F_t(x) = e^{-\alpha t}\delta(x-1) + v_t, \tag{29}$$

where

$$v_t = \sum_{n=1}^{\infty}(\alpha t)^n a_n \in \mathcal{C}((0,T), \mathcal{M}(\mathbb{R}^+)), \qquad v_t(0) = 0, \qquad v_t \perp \delta(x-1), \tag{30}$$

is a fundamental solution to (17), and we prove that for all $t > 0$, this solution satisfies

$$\operatorname{supp}(\mu^F_t) \subset [0,1]. \tag{31}$$

Fix $T > 0$. Assume that μ_t^F is a fundamental solution to (17) on $[0,T] \times \mathbb{R}^+$. We recall that $\mu \perp \nu$ if there exists $E \in \mathcal{B}(\mathbb{R})$ such that $\mu(\mathbb{R}) = \mu(E)$ and $\nu(E) = 0$. The Radon-Nikodym decomposition guarantees that for all $t > 0$, μ_t^F can be decomposed as

$$\mu_t^F = A(t)\delta(x-1) + v_t, \tag{32}$$

where $v_t \perp \delta(x-1)$, $A(0) = 1$ and $v_0(x) = 0$. We plug (32) into (17) and get

$$\begin{aligned} A'(t)\delta(x-1) + \partial_t v_t(x) = &-A(t)\alpha x^\gamma \delta(x-1) - v_t(x)\alpha x^\gamma + \\ &+ \alpha \int_x^\infty y^{\gamma-1}\kappa\left(\frac{x}{y}\right)\left(A(t)\delta(y-1) + dv_t(y)\right) \end{aligned}$$

which is

$$A'(t)\delta(x-1) + \partial_t v_t(x) = -\alpha A(t)\delta(x-1) - v_t(x)\alpha x^\gamma + \alpha\kappa(x)A(t) + \alpha \int_x^\infty y^{\gamma-1}\kappa\left(\frac{x}{y}\right) dv_t(y).$$

By identification, we get that necessarily

$$\begin{cases} A'(t) = -\alpha A(t), \quad A(0) = 1, \\ \partial_t v_t(x) = -\alpha x^\gamma v_t(x) + \alpha \displaystyle\int_x^\infty y^{\gamma-1}\kappa\left(\frac{x}{y}\right) dv_t(y) + \alpha\kappa(x)A(t), \quad v_0(x) = 0. \end{cases} \tag{33}$$

The first line gives $A(t) = e^{-\alpha t}$. This proves that μ_t^F is necessarily equal to $e^{-\alpha t} + v_t$, where v_t satisfies the second line of (33). Now let us verify that the series (30) converges in $\mathcal{C}((0,T), \mathcal{M}(\mathbb{R}^+))$. Since $(\mathcal{M}(\mathbb{R}^+), \|.\|_{TV})$ is a Banach space, it is enough to prove the normal convergence of the series (30). We first claim that

$$\|a_{n+1}\|_{TV} \leq \frac{1}{n+1}\left((N+1)\|a_n\|_{TV} + \frac{N}{n!}\right), \tag{34}$$

This comes directly from the induction formula (24) since $x \in [0,1]$ implies

$$\|x^\gamma a_n\|_{TV} \leq \|a_n\|_{TV},$$

and

$$\begin{aligned} \left\| \int_x^\infty y^{\gamma-1}\kappa\left(\frac{x}{y}\right) a_n(y)dy \right\|_{TV} &= \int_0^\infty \left| \int_x^\infty y^{\gamma-1}\kappa\left(\frac{x}{y}\right) a_n(y)dy \right| dX \\ &\leq \int_0^\infty \int_x^\infty \left| y^{\gamma-1}\kappa\left(\frac{x}{y}\right) a_n(y) \right| dydx \\ &= \int_0^1 \int_0^\infty \left| y^\gamma \kappa(z)\, a_n(y) \right| dydz \\ &\leq \int_0^1 |\kappa(dz)| \int_0^\infty y^\gamma \left| a_n(y) \right| dy \leq N\|a_n\|_{TV}, \end{aligned}$$

and finally

$$\left\|\kappa(x)\frac{(-1)^n}{n!}\right\|_{TV} = \frac{N}{n!}.$$

We deduce from (34) that for all $n \in \mathbb{N}$,

$$\|a_n\|_{TV} \leq \frac{(N+2)^n}{n!}, \tag{35}$$

hence the normal convergence of the series (30) in $(\mathcal{M}(\mathbb{R}^+), \|.\|_{TV})$ for all $t > 0$. We prove (35) by induction: (35) is true for $n = 0$ and $n = 1$, and if it is satisfied for $n \geq 1$ we have

$$\|a_{n+1}\|_{TV} \leq \frac{1}{n+1}\left((N+1)\|a_n\|_{TV} + \frac{N}{n!}\right) \leq \frac{1}{(n+1)!}\frac{(N+1)(N+2)^n + N}{n!} \leq \frac{(N+2)^{n+1}}{(n+1)!}.$$

Then the series v_t defined in (30) converges in the Banach space $\mathcal{C}((0,T), \mathcal{M}(\mathbb{R}^+))$. Since from the induction rule (24) $\text{supp}\,(a_n) \subset [0,1]$ for all $n \geq 0$, it follows that $\text{supp}\,(v_t) \subset [0,1]$. We prove then, using the differentiation rule of power series in a Banach space, that the power series (30) is a solution to the second line of (33). We have

$$\frac{d}{dt}v(t,x) = \frac{d}{dt}\sum_{n=1}^{\infty}(\alpha t)^n a_n = \alpha\sum_{n=1}^{\infty} n(\alpha t)^{n-1}a_n = \alpha\sum_{n=0}^{\infty}(n+1)(\alpha t)^n a_{n+1}.$$

Using the induction hypothesis (24) we get

$$\begin{aligned}
\frac{d}{dt}v(t,x) =& \alpha\sum_{n=0}^{\infty}(\alpha t)^n\left(-x^\gamma a_n(x) + \int_x^\infty y^{\gamma-1}\kappa\left(\frac{x}{y}\right)a_n(y)dy + \kappa(x)\frac{(-1)^n}{n!}\right)\\
=& -\alpha x^\gamma\sum_{n=0}^{\infty}(\alpha t)^n a_n + \alpha\int_x^\infty y^{\gamma-1}\kappa\left(\frac{x}{y}\right)\left(\sum_{n=0}^{\infty}(\alpha t)^n a_n(y)\right)dy + \alpha\sum_{n=0}^{\infty}(-\alpha t)^n\kappa(x)\\
=& -\alpha x^\gamma v(t,x) + \alpha\int_x^\infty y^{\gamma-1}\kappa\left(\frac{x}{y}\right)v(t,y)dy + \alpha e^{-\alpha t}\kappa(x),
\end{aligned}$$

which is (33). The property of the support $\text{supp}\,(\mu_t^F) \subset [0,1]$ follows from the hypothesis on the support of κ and by inspection of formulas (30) and (24).

Step 2: A power series representation of a solution with a generic initial condition. By the classical superposition principle, if

$$\mu_t(x) = \int_0^\infty \mu_0(\ell)\mu_t^\ell(x)\,d\ell \tag{36}$$

converges in $\mathcal{C}((0,T), \mathcal{M}(\mathbb{R}^+))$ (where μ_t^ℓ is the scaled fundamental solution obtained in Proposition 2 from μ_t^F the fundamental solution obtained in Step 1), μ_t will be a solution to the fragmentation equation with initial condition $\mu_0 \in \mathcal{M}(\mathbb{R}^+)$. Notice that we have the following equality for the integral (36):

$$\int_0^\infty \mu_0(\ell)\mu_t^\ell(x)d\ell = e^{-\alpha x^\gamma t}\mu_0 + \int_0^\infty\sum_{n=0}^{+\infty}(\alpha t)^n\ell^{n\gamma}a_n\left(\frac{x}{\ell}\right)\mu_0(\ell)\frac{d\ell}{\ell}. \tag{37}$$

Since for every n, by (35)

$$\begin{aligned}
\sum_{n=0}^{m}\left|\left|(\alpha t)^n\int_0^\infty \ell^{n\gamma}a_n\left(\frac{x}{\ell}\right)\mu_0(\ell)\frac{d\ell}{\ell}\right|\right|_{TV} &\leq ||\mu_0||_{TV}\sum_{n=0}^{m}(\alpha t)^n||a_n||_{TV}L^{\gamma n}\\
&\leq ||\mu_0||_{TV}\sum_{n=0}^{m}(\alpha t)^n\frac{(N+2)^n}{n!}L^{\gamma n},
\end{aligned}$$

the series in the right-hand side of (37) converges absolutely in the Banach space $\mathcal{C}((0,T),\mathcal{M}(\mathbb{R}^+))$ for all $T>0$. The integral (36) is then absolutely convergent and defines a solution to the fragmentation equation with initial condition $\mu_0 \in \mathcal{M}(\mathbb{R}^+)$ and for $t \in [0,T]$. Property (25) follows then from the definition of $\mu^F_{t\ell^\gamma}$. Since $\mathrm{supp}\,(a_n) \subset [0,1]$ for every $n \geq 0$, it follows from (23) that $\mathrm{supp}\,(\mu_t) \subset [0,L]$ for all $t>0$. Using a classical diagonal argument, and since the property on the support of the solution does not depend on T, the power series defines a solution in $\mathcal{C}(\mathbb{R}^+,\mathcal{M}(\mathbb{R}^+))$ This ends the proof of Theorem 2]. □

Theorem 3 (Non negativity of the power series solution)**.** *Assume the fragmentation kernel κ satisfies* (**Hyp-1**) *with $\gamma \geq 0$ and take μ_0 that satisfies* (**Hyp-2**)*. Then, the power series solution* (23) *to the fragmentation equation* (17) *is non-negative.*

Remark 2. *Up to our knowledge, no proof of positivity for the fragmentation equation is available in the literature in the case where either the initial condition μ_0 is a measure or the fragmentation kernel κ is a measure. In our case, both are measures.*

Proof. We first prove the non negativity of the fundamental solution μ^F_t defined by (29) (30) using an approximation argument. Consider to this end the function χ such that:

$$\chi \in \mathcal{C}(0,\infty),\ \chi \geq 0,\ ||\chi||_\infty = 1, \quad \chi(x) = 1\ \forall x \in [0,2L]; \quad \chi(x) = 0\ \forall x > 3L,$$

and a mollifier $\theta \in \mathcal{C}(0,\infty)$, such that $\theta \geq 0$, $\mathrm{supp}\,\theta \subset [0,1]$ and the sequence $\theta_m(x) = m\theta(m(x-1))$. Let us then denote by κ_m a regularization of the fragmentation kernel

$$\kappa_m = \kappa * \theta_m,$$

by a a regularization of the fragmentation rate

$$a(x) = x^\gamma \chi(x),$$

and by θ_k a regularization of the initial condition $\delta(x-1)$. By construction $||a||_\infty \leq (3L)^\gamma$ and for each $m \geq 1$, κ_m is a regular function satisfying

$$\mathrm{supp}\,[\kappa_m] \subset [0,2], \quad \lim_{m\to\infty} ||\kappa_m - \kappa||_{BL} = 0, \quad ||\kappa_m||_{TV} \leq ||\kappa||_{TV}||\theta_m||_{TV} \leq N. \tag{38}$$

Consider for every $m \geq 1$ the sequence of functions $\{a_{n,m}\}_{n\in\mathbb{N},m\in\mathbb{N}}$ defined as follows,

$$\begin{aligned} &a_{0,m}(x) = 0,\\ &a_{n+1,m}(x) = \frac{1}{n+1}\left(-a(x)a_{n,m}(x) + \int_0^\infty a(y)\kappa_m\left(\frac{x}{y}\right)a_{n,m}(y)\frac{dy}{y} + \kappa_m(x)\frac{(-1)^n}{n!}\right). \end{aligned} \tag{39}$$

It immediately follows, for all $n \geq 1, m \geq 1$,

$$a_{n,m} \in \mathcal{C}([0,\infty)),\ \mathrm{supp}\,[a_{n,m}] \subset [0,2]$$

and then, the sequence $\{a_{n,m}\}_{n\in\mathbb{N},m\in\mathbb{N}}$ satisfies also,

$$a_{n+1,m}(x) = \frac{1}{n+1}\left(-x^\gamma a_{n,m}(x) + \int_0^\infty y^\gamma \kappa_m\left(\frac{x}{y}\right)a_{n,m}(y)\frac{dy}{y} + \kappa_m(x)\frac{(-1)^n}{n!}\right). \tag{40}$$

Since supp $[a_{n,m}] \subset [0,2]$ it follows that

$$||x \to x^\gamma a_{n,m}(x)||_{TV} \leq 2^\gamma ||a_{n,m}||_{TV}.$$

. Then, for every $m \geq 1$ fixed, as for the proof of (35) it follows now that

$$\|a_{n,m}\|_{TV} \leq \frac{2^{\gamma n}(N+2)^n}{n!}, \ \forall n \geq 1, \forall m \geq 1. \tag{41}$$

Step 1: The solution u_{km} to the regularized problem is non negative. Consider the regularized problem ($m < \infty$ and $k < \infty$)

$$\begin{cases} \dfrac{\partial}{\partial t} u_{k,m}(t,x) = -\alpha a(x) u_{k,m}(t,x) + \alpha \displaystyle\int_x^\infty \kappa_m \left(\frac{x}{y}\right) a(y) u_{k,m}(t,y) \frac{dy}{y}, \\ u_{k,m}(0,x) = \theta_k(x). \end{cases} \tag{42}$$

We define for $k \geq 1$ and $m \geq 1$ the sequence of functions

$$u_{k,m}(t,x) = e^{-\alpha x^\gamma t}\theta_k + \sum_{n=0}^\infty (\alpha t)^n \int_0^\infty \ell^{n\gamma} a_{n,m}\left(\frac{x}{\ell}\right) \theta_k(\ell) \frac{d\ell}{\ell}, \tag{43}$$

For $k \geq 1$ and $m \geq 1$, the series is absolutely convergent since Moreover, by construction supp $[u_{k,m}] \subset [0,2]$. Then, $u_{k,m}$ satisfies (42). By Lemma 3 of [36], of which the equation (42) and the initial data θ_k satisfy the hypothesis, the Cauchy problem for (42) with initial data θ_k possesses a global solution bounded, continuous, non negative, analytic in t for each $x > 0$ and integrable in x for every $t > 0$. Moreover, by construction, for all $T > 0$ and $t \in (0,T)$,

$$\begin{aligned} \int |u_{km}(t,x)|dx &= \int_0^{infty} e^{-\alpha x^\gamma t}\theta_k(x)dx + \int_0^\infty \sum_{n=0}^\infty (\alpha t)^n \int_0^\infty \ell^{n\gamma} \Big| a_{n,m}\left(\frac{x}{\ell}\right) \Big| \theta_k(\ell) \frac{d\ell}{\ell} dx \\ &\leq 1 + \sum_{n=0}^\infty (\alpha t)^n 2^{n\gamma} \frac{(N+2)^n}{n!} < \infty. \end{aligned}$$

Therefore, by Lemma 4 in [36], the function $u_{k,m}$ is the unique solution of (42)that satisfies (**??**). Moreover, this solution is non negative.

Step 2: limit $k \to \infty$. Consider the problem ($m < \infty$)

$$\begin{aligned} &\frac{\partial}{\partial t} u_m(t,x) = -\alpha a(x) u_m(t,x) + \alpha \int_x^\infty \kappa_m \left(\frac{x}{y}\right) a(y) u_m(t,y) \frac{dy}{y}, \\ &u_m(0,x) = \delta(x-1). \end{aligned} \tag{44}$$

We define the sequence of measures u_m as

$$u_m(t,x) = e^{-\alpha t}\delta(x-1) + \sum_{n=0}^\infty (\alpha t)^n a_{n,m} \tag{45}$$

The series in (45) is absolutely convergent in TV norm for every $m \geq 1$ and it defines a measure $u_m \in \mathcal{M}(\mathbb{R}_+)$ such that

$$||u_m(t)||_{TV} \leq \exp(\alpha t 2^\gamma (N+2)) + e^{-\alpha t}. \tag{46}$$

The measure $u_m(t,.)$ satisfies (44) but since $\operatorname{supp}[u_m(t)] \subset [0,2]$ it also satisfies,

$$\frac{\partial}{\partial t}u_m(t,x) = -\alpha x^\gamma u_m(t,x) + \alpha \int_x^\infty \kappa_m\left(\frac{x}{y}\right) y^{\gamma-1} u_m(t,y)dy.$$

We claim now that $(u_{km})_{k\geq 0}$ converges weakly towards u_m. Indeed, on one hand, for all $\varphi \in \mathcal{C}_C(\mathbb{R}^+)$,

$$\lim_{k\to\infty} \int_0^\infty e^{-\alpha x^\gamma t}\theta_k(x)\varphi(x)dx = e^{-\alpha t}\varphi(1), \tag{47}$$

and on the other hand, let us notice that for each $n \geq 1$ and $m \geq 1$ fixed,

$$\lim_{k\to\infty} \int_0^\infty \ell^{n\gamma} a_{n,m}\left(\frac{x}{\ell}\right)\theta_k(\ell)\frac{d\ell}{\ell} = a_{n,m}(x), \quad x \in [0,2].$$

Moreover, for all $k \geq 1$, $n \geq 1$ and $m \geq 1$

$$\int_0^\infty \ell^{n\gamma} a_{n,m}\left(\frac{x}{\ell}\right)\theta_k(\ell)\frac{d\ell}{\ell} = \int_0^\infty \left(\frac{x}{y}\right)^{n\gamma-1} a_{n,m}(y) k\,\theta\left(k\frac{x}{y}\right)dy.$$

Therefore, if

$$\psi_k(z) = z^{n\gamma-1}k\theta(kz)$$

then $\psi_k \in \mathcal{C}([0,\infty)$ and $\psi_k \geq 0$. Since $\operatorname{supp}\theta \subset [0,1]$ and $||\theta||_\infty \leq 1$, for all n such that $n\gamma > 2$,

$$\sup_{z>0}\psi_k(z) = \sup_{z>0} z^{n\gamma-1}k\theta(kz) = \sup_{z\in[0,k^{-1}]} z^{n\gamma-2}(kz)\theta(kz) \leq 1$$

It follows that for all $m \geq 1$, $n \geq 1$, $k \geq 1$ and $x > 0$,

$$\left|\int_0^\infty \ell^{n\gamma} a_{n,m}\left(\frac{x}{\ell}\right)\theta_k(\ell)\frac{d\ell}{\ell}\right| \leq ||a_{n,m}||_{TV}$$

It follows by the Lebesgue's convergence that

$$\lim_{k\to\infty}\sum_{n=0}^\infty (\alpha t)^n \int_0^\infty \ell^{n\gamma} a_{n,m}\left(\frac{x}{\ell}\right)\theta_k(\ell)\frac{d\ell}{\ell} = \sum_{n=0}^\infty (\alpha t)^n a_{n,m}(x), \quad x \in [0,2], \tag{48}$$

and then, combining (47) with (48) gives us

$$\lim_{k\to\infty}\int_0^\infty \varphi(x)u_{k,m}(t,x)dx = \int_0^\infty \varphi(x)u_m(t,x)dx, \quad \varphi \in \mathcal{C}_C(\mathbb{R}^+),$$

and then

$$u_m \geq 0. \tag{49}$$

Step 3: limit $m \to \infty$. We prove here that u_m converges weakly towards μ_t^F. To do so, we prove by induction that

$$\|a_{n,m} - a_n\|_{BL} \underset{m\to\infty}{\to} 0. \tag{50}$$

For $n=1$, $a_{1,m}=\kappa_m$, $a_1=\kappa$, and by construction $||\kappa_m-\kappa||_{BL} \underset{m\to\infty}{\to}$. Assume then $||a_{n,m}-a_n||_{BL}\to 0$. In order to prove that the same property holds for the sequence $\{a_{n+1,m}\}_{m\in\mathbb{N}}$ it is sufficient to prove

$$\lim_{m\to\infty}\left|\left|\int_x^\infty y^{\gamma-1}\kappa_m\left(\frac{x}{y}\right)a_{n,m}(y)dy-\int_x^\infty y^{\gamma-1}\kappa\left(\frac{x}{y}\right)a_n(y)dy\right|\right|_{BL}=0. \tag{51}$$

If, for the sake of notation we define the functions $\tilde{a}_{n,m}$ and $\tilde{a}_n$ as

$$\tilde{a}_{n,m}(\ell)=\ell^\gamma a_{n,m}(\ell),\qquad \tilde{a}_n(\ell)=\ell^\gamma a_n(\ell),$$

then property (51) reads,

$$\lim_{m\to\infty}||\tilde{a}_{n,m}*\kappa_m-\tilde{a}_n*\kappa||_{BL}=0. \tag{52}$$

Notice indeed that, for all test function φ such that $||\varphi||_\infty\le 1$ and $||\varphi'||_\infty\le 1$,

$$\begin{aligned}\int_0^\infty\varphi(x)\left(\tilde{a}_{n,m}*\kappa_m(x)-\tilde{a}_n*\kappa(x)\right)dx&=\int_0^\infty\varphi(x)\left(\tilde{a}_{n,m}-\tilde{a}_n\right)*\kappa_m(x)dx+\\&+\int_0^\infty\varphi(x)\left(\kappa_m-\kappa\right)*\tilde{a}_n(x)dx\end{aligned} \tag{53}$$

The two terms in the right-hand side of (53) may be bounded with the same arguments. Consider for example the first.

$$\begin{aligned}\left|\int_0^\infty\varphi(x)\left(\tilde{a}_{n,m}-\tilde{a}_n\right)*\kappa_m(x)dx\right|&=\left|\int_0^\infty\varphi(x)\int_0^\infty\left(\tilde{a}_{n,m}-\tilde{a}_n\right)\left(\frac{x}{y}\right)\kappa_m(y)\frac{dy}{y}dx\right|\\&=\left|\int_0^2\kappa_m(y)\int_0^\infty\left(\tilde{a}_{n,m}-\tilde{a}_n\right)\left(\frac{x}{y}\right)\varphi(x)dx\frac{dy}{y}\right|\\&\le||\kappa_m||_{TV}\sup_{y\in[0,2]}\left|\int_0^\infty\left(\tilde{a}_{n,m}-\tilde{a}_n\right)(z)\,\varphi(zy)dz\right|\end{aligned}$$

For each $y\in[0,2]$,

$$\left|\int_0^\infty\left(\tilde{a}_{n,m}-\tilde{a}_n\right)(z)\,\varphi(zy)dz\right|\le||\tilde{a}_{n,m}-\tilde{a}_n||_{BL}\left(||\varphi||_\infty+2||\varphi'||_\infty\right)$$

from where, by (38)

$$||(\tilde{a}_{n,m}-\tilde{a}_n)*\kappa_m||_{BL}\le 3N||\tilde{a}_{n,m}-\tilde{a}_n||_{BL}.$$

A similar arguments shows, using (35),

$$||(\kappa_m-\kappa)*\tilde{a}_n||_{BL}\le 3||k_m-k||_{BL}||\tilde{a}_n||_{TV}\le 3\frac{(N+2)^n}{n!}||k_m-k||_{BL}$$

and then, (51) holds true.
Now, for any $\varphi\in\mathcal{C}^1([0,\infty))$, such that $||\varphi||_\infty+||\varphi'||_\infty<\infty$, by definition of the measure u_m

$$\begin{aligned}\int_0^\infty u_m(t,x)\varphi(x)dx&=e^{-\alpha t}\varphi(1)+\int_0^\infty\left(\sum_{n=0}^\infty(\alpha t)^n a_{n,m}(x)\right)\varphi(x)dx\\&=e^{-\alpha t}\varphi(1)+\sum_{n=0}^\infty(\alpha t)^n\int_0^\infty a_{n,m}(x)\varphi(x)dx.\end{aligned}$$

Since for every $n \geq 1$,

$$\lim_{m\to\infty} \int_0^\infty a_{n,m}(x)\varphi(x)dx = \int_0^\infty a_n(x)\varphi(x)dx$$

and

$$\begin{aligned}(\alpha t)^n \int_0^\infty |a_{n,m}(x)\varphi(x)|dx &\leq (\alpha t)^n ||\varphi||_\infty ||a_{n,m}||_{TV} \\ &\leq (\alpha t)^n ||\varphi||_\infty \frac{2^{\gamma n}(N+2)^n}{n!}\end{aligned}$$

one has,

$$\lim_{m\to\infty} \int_0^\infty \varphi(x)u_m(t,x)dx = e^{-\alpha t}\varphi(1) + \sum_{n=0}^\infty (\alpha t)^n \int_0^\infty a_n(x)\varphi(x)dx = \int_0^\infty \varphi(x)d\mu_t^F$$

and it follows

$$\mu_t^F \geq 0.$$

This ends the proof of Theorem 3. □

The following corollary follows now easily from Theorem 1, Theorem (2), Theorem 3 and Proposition (2).

Corollary 1 (Well-posedness of the fragmentation equation)**.** *Assume the fragmentation kernel κ satisfies* (**Hyp-1**) *with $\gamma \geq 0$ and take μ_0 that satisfies* (**Hyp-2**)*. Then, there exists a unique non-negative solution μ_t to the fragmentation equation in $\mathcal{C}(\mathbb{R}^+, \mathcal{M}(\mathbb{R}^+))$ that has a finite TV norm. Moreover, this solution satisfies*

$$\text{supp}\,(\mu_t) \subset \text{supp}\,(\mu_0)$$

and can be represented as the power series (23)*.*

Let us provide two cases where we have explicit formulations for the fundamental solution to (17) for $\mu_0 = \delta(x-1)$.

Example 1. *For $\alpha = \gamma = 1$ and $\kappa = 2\mathbb{1}_{[0,1]}$, we have [51, formula 11]*

$$\mu_t^F(x) = e^{-t}\delta(x-1) + (2t + (1-x)t^2)e^{-xt},$$

Example 2. *For $\alpha = 1$, $\gamma = 0$, and $\kappa(z) = 2\delta(z-1/2)$ we have [20, Proposition 1]*

$$\mu_t^F(x) = e^{-t}\delta(x-1) + e^{-t}\sum_{k=1}^\infty \frac{(4t)^k}{k!}\delta\left(x - \frac{1}{2^k}\right).$$

In both examples, the mass initially located at $x = 1$ decreases exponentially with respect to time and is teleported on $(0, 1)$.

The stability of the solution with respect to the TV norm has been proved in Theorem 1. The stability in the BL norm is deduced now from the explicit expression provided by Theorem 2.

Theorem 4 (Stability of the fragmentation equation in $(\mathcal{M}(\mathbb{R}^+), \|.\|_{BL})$)**.** *Assume κ satisfies* (**Hyp-1**)*, $\mu_0 \in \mathcal{M}^+(\mathbb{R}^+)$ satisfies* (**Hyp-2**)*, and moreover either $\gamma \geq 1$ or $\text{supp}\,(\mu_0) \subset [m, M]$ with $m > 0$. Then the unique solution μ_t to the fragmentation equation* (17) *satisfies*

$$\|\mu_t\|_{BL} \leq C(L, N, T, \alpha, \gamma, m^{\gamma-1})\ \|\mu_0\|_{BL}, \qquad 0 \leq t \leq T,$$

Proof. We use the definition of the BL norm given by (7) and the representation of the solution provided in Theorem 2. Take $\varphi \in \mathcal{C}(\mathbb{R}^+)$ such that $\|\varphi\|_\infty \leq 1$ and $\|\varphi'\|_\infty \leq 1$. Then by (25) in Theorem 2,

$$\int_0^{+\infty} \varphi(x) d\mu_t(x) = \int_0 \int_0^1 \varphi(\ell x) d\mu^F_{t\ell^\gamma}(x) d\mu_0(\ell).$$

For $\ell \leq M$, we set

$$\Psi(\ell) = \int_0^1 \varphi(\ell x) d\mu^F_{t\ell^\gamma}(x).$$

We notice that for any $r \geq 0$, the moment of order r of the absolute value of the fundamental solution μ^F_t is uniformly bounded for $t \in [0,T]$ using the rough estimate based on Theorem 1

$$\int_{\mathbb{R}^+} x^r d|\mu^F_t|(x) \leq L^r \|\mu^F_t\|_{TV} \leq L^r e^{\alpha L^\gamma (N+1)T} \|\delta(x-1)\|_{TV} =: \tilde{C}(L,N,T,r,\alpha,\gamma).$$

Then for all $\ell \leq L$,

$$|\Psi(\ell)| \leq \int_0^{+\infty} |\varphi(\ell x)| d|\mu^F_{t\ell^\gamma}|(x) \leq \|\varphi\|_\infty \int_0^{+\infty} d|\mu^F_{TL^\gamma}|(x) \leq \tilde{C}(L,N,T,0,\alpha,\gamma),$$

and

$$\left|\Psi'(\ell)\right| \leq \int_0^{+\infty} |\varphi^{'}(\ell x)| x d|\mu^F_{t\ell^\gamma}|(x) + \int_0^{+\infty} |\varphi(\ell x)| t\gamma \ell^{\gamma-1} \left| \frac{\partial}{\partial t} d|\mu^F_{t\ell^\gamma}|(x) \right|$$

where

$$\int_0^{+\infty} |\varphi^{'}(\ell x)| x d|\mu^F_{t\ell^\gamma}|(x) \leq \tilde{C}(L,N,T,1,\alpha,\gamma) \|\varphi'\|_\infty,$$

and where

$$\begin{aligned}
&\int_0^\infty |\varphi(\ell x)| t\gamma \ell^{\gamma-1} \left| \frac{\partial}{\partial t} d|\mu^F_{t\ell^\gamma}|(x) \right| \\
&\qquad \leq \|\varphi\|_\infty T \max(L^{\gamma-1}, m^{\gamma-1}) \gamma\alpha \left(\int_0^\infty x^\gamma d|\mu^F_{t\ell^\gamma}|(x) + \int_0^\infty \int_x^\infty \kappa\left(\frac{x}{y}\right) y^{\gamma-1} d|\mu^F_{t\ell^\gamma}|(y) dx \right) \\
&\qquad = \|\varphi\|_\infty T \max(L^{\gamma-1}, m^{\gamma-1}) \gamma\alpha \left(\int_0^\infty x^\gamma d|\mu^F_{t\ell^\gamma}|(x) + \int_0^\infty \int_0^1 \kappa(z)\, dz y^\gamma d|\mu^F_{t\ell^\gamma}|(y) \right) \\
&\qquad \leq T \max(L^{\gamma-1}, m^{\gamma-1}) \gamma\alpha (N+1) \tilde{C}(L,N,T,\gamma,\alpha,\gamma).
\end{aligned}$$

We set $C(L,N,T,\alpha,\gamma,m^{\gamma-1}) := \tilde{C}(L,N,T,0,\alpha,\gamma) + \tilde{C}(L,N,T,1,\alpha,\gamma) + T\max(L^{\gamma-1}, m^{\gamma-1})\gamma\alpha(N+1)\tilde{C}(L,N,T,\gamma,\alpha,\gamma)$ and define

$$\tilde{\Psi}(\ell) = \frac{\Psi(\ell)}{C(L,N,T,\alpha,\gamma,m^{\gamma-1})},$$

then

$$\|\tilde{\Psi}\|_\infty \leq 1, \qquad \|\tilde{\Psi}'\|_\infty \leq 1.$$

We have shown that for any $\varphi \in \mathcal{C}(\mathbb{R}^+)$ satisfying $\|\varphi\|_\infty \leq 1, \|\varphi'\|_\infty \leq 1$, there exists $\tilde{\Psi} \in \mathcal{C}(\mathbb{R}^+)$ such that $\|\tilde{\Psi}\|_\infty \leq 1, \|\tilde{\Psi}'\|_\infty \leq 1$ and

$$\int_0^{+\infty} \varphi(x) d\mu_t(x) \leq C(L,N,T,\alpha,\gamma,m^{\gamma-1}) \int_0^{+\infty} \tilde{\Psi}(x) d\mu_0(x).$$

Thus the conclusion of Theorem 4 holds. □

Remark 3. *For $\gamma < 1$, and for any initial condition μ_0 such that $\mu_0(0) \neq 0$, $m = 0$ and thus Theorem* (4) *does not provide any estimate on $\|\mu_t\|_{BL}$. Stability with respect to the initial condition is lost.*

3 Inverse problem for the fragmentation kernel

In this section, estimates of the fragmentation kernel and bounds of the error of such estimates are obtained using the series expression of the solution μ of (17) provided by Theorem 2 for short values of the time variable.

3.1 An estimation for κ using short-time measurements

Let us first investigate the best possible case, when the initial data μ_0 is a Dirac delta function at $x = 1$.

Theorem 5 (An estimate for κ using short-time measurements of the particles size distribution when initial condition is a delta function at $x = 1$.)**.** *Assume κ satisfies* (**Hyp-1**) *and define*

$$\kappa^{est}(t) = \frac{\mu_t^F - e^{-\alpha t}\delta(x-1)}{\alpha t}, \tag{54}$$

where μ_t^F is the unique fundamental solution to the fragmentation equation (17) *with the initial data $\mu_0 = \delta(x-1)$. Then we have*

$$\left\|\kappa^{est} - \kappa\right\|_{TV} \leq C(N,T,\alpha)\, t,\ \forall t \in (0,T], \tag{55}$$

for

$$C = \alpha \max_{t\in[0,T]} \sum_{n=0}^{\infty} (\alpha t)^n \|a_{n+2}\|_{TV} = \alpha \sum_{n=0}^{\infty} (\alpha T)^n \|a_{n+2}\|_{TV}. \tag{56}$$

Before proving Theorem 5, we point out that another possible formula for the estimated kernel is

$$\kappa_{bis}^{est}(t) = \frac{\mu_t^F - (1-\alpha t)\delta(x-1)}{\alpha t} = 1 + \frac{\mu_t^F - \delta(x-1)}{\alpha t}.$$

Since $e^{-\alpha t} = 1 - \alpha t + o(t)$, we also have

$$\left\|\kappa_{bis}^{est} - \kappa\right\|_{TV} \leq Ct.$$

Proof. We have, using the notations introduced in Lemma **??** and Proposition2,

$$\frac{\mu_t^F - e^{-\alpha t}\delta(x-1)}{\alpha t} - \kappa = \frac{\sum_{n=1}^{\infty}(\alpha t)^n a_n}{\alpha t} - \kappa = \sum_{n=1}^{\infty}(\alpha t)^{n-1} a_n - \kappa = \sum_{n=0}^{\infty}(\alpha t)^n a_{n+1} - \kappa$$

and since $a_1 = \kappa$, we have

$$\sum_{n=0}^{\infty}(\alpha t)^n a_{n+1} - \kappa = \sum_{n=1}^{\infty}(\alpha t)^n a_{n+1} = \alpha t \sum_{n=0}^{\infty}(\alpha t)^n a_{n+2}.$$

Thus

$$\|\frac{\mu_t^F - e^{-\alpha t}\delta(x-1)}{\alpha t} - \kappa\|_{TV} \leq \alpha t \sum_{n=0}^{\infty} (\alpha t)^n \|a_{n+2}\|_{TV}.$$

The series converges (normal convergence) and thus it is bounded on any compact set, for instance for $t \in [0, T]$. This ends the proof of Theorem 5. □

When the initial data is a Dirac delta at $x = \ell > 0$ Theorem 5 and Proposition 2 give an estimate of the following rescaled fragmentation kernel,

$$\kappa_\ell = T_\ell \# \kappa,$$

where the map T_ℓ is defined in (12). Recall that if κ is a function, then

$$\kappa_\ell(z) = \frac{1}{\ell}\kappa\left(\frac{z}{\ell}\right), \qquad 0 \leq z \leq \ell.$$

Corollary 2 (An estimate for κ using short-time measurements of the particles size distribution when initial condition is a delta function at $x = \ell$)**.** *We define*

$$\kappa_\ell^{est}(t) = \frac{\mu_t^\ell - e^{-\alpha t \ell^\gamma}\delta(x-\ell)}{\alpha t \ell^\gamma},$$

where μ_t^ℓ is the unique solution to (17) *with the initial condition $\mu_0 = \delta(x-\ell)$. Then, for all $T > 0$,*

$$\left\|\kappa_\ell^{est}(t) - \kappa_\ell\right\|_{TV} \leq Ct\ell^\gamma, \ \forall t \in (0, T].$$

where C is the constant given in (56)

Proof. We notice that for any continuous map T, we have $\|T\#\mu\|_{TV} \leq \|\mu\|_{TV}$ (with equality if the measure μ is positive, or if T is an injection). Let us set $\eta = \mu_{t\ell^\gamma}^F - e^{-\alpha t\ell^\gamma}\delta(x-1)$. We have $T_\ell\#\eta = \mu_t^\ell - e^{-\alpha t\ell^\gamma}\delta(x-\ell)$, hence using Theorem 5

$$\left\|\frac{\mu_t^\ell - e^{-\alpha t\ell^\gamma}\delta(x-\ell)}{\alpha t\ell^\gamma} - \kappa_\ell\right\|_{TV} = \left\|T_\ell\#\Big(\frac{\mu_{t\ell^\gamma}^F - e^{-\alpha t\ell^\gamma}\delta(x-1)}{\alpha t\ell^\gamma} - \kappa\Big)\right\|_{TV} \leq C\alpha\ell^\gamma t,$$

where C is the constant in (56). This ends the proof. □

In most of the cases, a Dirac delta as an initial condition is experimentally out of reach. However, as proved in the next corollary, for all initial data μ_0 satisfying (**Hyp-2**), it is possible to estimate not the kernel κ itself but the convolution $\kappa * w_0$ where $dw_0(\ell) = \ell^\gamma d\mu_0(\ell)$. Moreover, if the initial data μ_0 becomes closer and closer, in some suitable sense, to $\delta(x-1)$, so does $\kappa * w_0$ and the estimate of $\kappa * w_0$ gives an estimate of κ itself.

If μ_0 satisfies (**Hyp-2**) and μ is the unique solution given by Theorem 2 of the equation (17) with initial data μ_0, define

$$\kappa^{est}(\mu_0; t, x) = \frac{\mu(t,x) - e^{-\alpha t x^\gamma}\mu_0(x)}{\alpha t}. \tag{57}$$

We have the following corollary.

Corollary 3 (Generic initial condition)**.** *Assume κ satisfies* (**Hyp-1**) *and μ_0 satisfies* (**Hyp-2**)*. Then, for all $T > 0$,*

$$\left\|\kappa^{est}(\mu_0;t) - w_0 * \kappa(x)\right\|_{TV} \leq CL^{2\gamma}\|\mu_0\|_{TV}\, t,\ \forall t \in (0,T], \tag{58}$$

where w_0 denotes the measure with density $\ell \to \ell^\gamma \mu_0(\ell)$, C is given in (56) and $\kappa^{est}(\mu_0;t)$ is defined by (57)*.*

If $\{\mu_{0,n}\}_{n\in\mathbb{N}} \subset \mathcal{M}^+(\mathbb{R}^+)$ is a sequence such that

$$\begin{cases} \lim\limits_{n\to\infty} ||\mu_{0,n} - \delta(x-1)||_{BL} = 0 \\ \sup\limits_{n\in\mathbb{N}} ||\mu_{0,n}||_{TV} < \infty \end{cases} \tag{59}$$

or if

$$\lim_{n\to\infty} ||\mu_{0,n} - \delta(x-1)||_{TV} = 0, \tag{60}$$

then for all $\varepsilon > 0$ there exists n_0 such that for all $T > 0$,

$$\left\|\kappa^{est}(\mu_{0,n};t) - \kappa(x)\right\|_{BL} \leq CL^{2\gamma} \sup_{n\in\mathbb{N}} ||\mu_{0,n}||_{TV}\, t + \varepsilon, \quad \forall t \in (0,T], \quad \forall n > n_0. \tag{61}$$

Proof. For $\ell > 0$, we multiply the measure

$$X_\ell = \frac{\mu_t^\ell - e^{-\alpha t \ell^\gamma}\delta(x-\ell)}{\alpha t \ell^\gamma} - \kappa_\ell$$

by the smooth function $\ell \to \ell^\gamma$, and apply Corollary 2 to obtain

$$\|Y_\ell\|_{TV} \leq Ct\ell^{2\gamma},$$

with C the constant given in (56) and $Y_\ell = \dfrac{\mu_t^\ell - e^{-\alpha t\ell^\gamma}\delta(x-\ell)}{\alpha t} - \ell^\gamma \kappa_\ell$. We multiply the function $\ell \to Y_\ell$ from $\mathbb{R}^+$ onto $\mathcal{M}(\mathbb{R}^+)$ by $\ell \to \mu_0(\ell)$ and integrate over $\mathbb{R}^+$. Since $(\mathcal{M}(\mathbb{R}^+), \|.\|_{TV})$ is a Banach space, we can use the Bochner integral so that we have

$$\begin{aligned} \left\|\kappa^{est}(\mu_0;t) - w_0 * \kappa(x)\right\|_{TV} &= \left\| \int_{\mathbb{R}^+} Y_\ell d\mu_0(\ell)\right\|_{TV} \\ &\leq \int_{\mathbb{R}^+} \|Y_\ell\|_{TV} d\mu_0(\ell) \leq t \int_{\mathbb{R}^+} \ell^{2\gamma} d\mu_0(\ell), \end{aligned}$$

and (58) follows.

If we suppose now that $\{\mu_{0,n}\}_{n\in\mathbb{N}}$ satisfies (59), then so does $\{w_{0,n}\}_{n\in\mathbb{N}}$ and therefore $||w_{0,N} * \kappa - \kappa||_{BL} \underset{n\to\infty}{\longrightarrow} 0$. We deduce by (8) and (58),

$$\begin{aligned} ||\kappa^{est}(\mu_{0,n};t) - \kappa||_{BL} &\leq ||\kappa^{est}(\mu_{0,n};t) - w_{0,n} * \kappa||_{TV} + ||\kappa - \kappa * w_{0,n}||_{BL} \\ &\leq CL^{2\gamma}||\mu_{0,n}||_{TV} t + ||\kappa - \kappa * w_{0,n}||_{BL} \end{aligned}$$

from where (61) follows.

Now if $\{\mu_{0,n}\}_{n\in\mathbb{N}}$ satisfies ((60), $\{w_{0,n}\}_{n\in\mathbb{N}}$ does too and therefore

$$\lim_{n\to\infty} ||\kappa - \kappa * w_{0,n}||_{TV} = 0.$$

We deduce,

$$||\kappa^{est}(\mu_{0,n};t) - \kappa||_{TV} \leq CL^{2\gamma}||\mu_{0,n}||_{TV}t + ||\kappa - \kappa * w_{0,n}||_{TV}$$

and (61) follows. □

Remark 4. *If $\{\mu_{0,n}\}_{n\in\mathbb{N}}$ is such that*

$$\begin{aligned}
&\mu_{0,n} \underset{n\to\infty}{\rightharpoonup} \delta(x-1), \text{ in the weak sense of measures,}\\
&\sup_{n\in\mathbb{N}} ||\mu_{0,n}||_{TV} < \infty,\\
&\exists\ Q \subset [0,\infty),\ \textit{compact}\,;\ \operatorname{supp}\mu_{0,n} \subset Q,\ \forall n \in \mathbb{N},
\end{aligned}$$

then, by Proposition 4 in [27], $||w_{0,n} - \delta(x-1)||_{BL} \underset{n\to\infty}{\longrightarrow} 0$. It follows that Property (59) is satisfied and (61) holds. Notice however that property (59) is not satisfied for any weakly-converging sequence $\mu_{0,n}$.

3.2 Stability of the κ estimate colorvert with respect to model and measurement noises.

Let us now turn to error estimates in more realistic observation cases, where the noise may be twofold: 1/ a model noise, where the initial condition is close to a Dirac delta in the BL distance; and 2/ a measurement noise, where the size distributions μ_0 and μ_t are observed with an error. A stability result for the time-dependent solution with respect to the initial condition μ_0 has already been proved in Theorem 4.

Theorem 6 (Stability of the κ estimate with respect to noises on the initial condition and the measurements)**.** *Assume κ satisfies* (**Hyp-1**)*. Take an initial condition μ_0^q satisfying* (**Hyp-2**) *and that is close to a delta function at $x = 1$ in the sense that*

$$\|\mu_0^q - \delta(x-1)\|_{BL} \leq q.$$

We denote by μ_t^q the unique solution to the fragmentation equation (17) *with initial condition μ_0^q. Consider the noisy measurements μ_0^{q,ε_0} and $\mu_t^{q,\varepsilon}$ of the respective measures μ_0^q and μ_t^q such that*

$$\|\mu_0^{q,\varepsilon_0} - \mu_0^q\|_{BL} \leq \varepsilon_0, \qquad \|\mu_t^{q,\varepsilon} - \mu_t^q\|_{BL} \leq \varepsilon.$$

Assume moreover either $\gamma \geq 1$ or $\operatorname{supp}(\mu_0) \subset [m, L]$ with $m > 0$. Then, for all $0 \leq t \leq T$, there are some constants $C_1(N, T, \alpha)$ and $C_2(L, N, T, \alpha, \gamma, m^{\gamma-1})$ such that

$$\left\| \frac{\mu_t^{q,\varepsilon} - e^{-\alpha t}\mu_0^{q,\varepsilon_0}}{\alpha t} - \kappa \right\|_{BL} \leq C_1 t + \frac{\varepsilon_0 + \varepsilon + C_2 q}{\alpha t}. \tag{62}$$

Proof. We use the triangle inequality to write

$$\begin{aligned}\left\|\frac{\mu_t^{q,\varepsilon}-e^{-\alpha t}\mu_0^{q,\varepsilon_0}}{\alpha t}-\kappa\right\|_{BL} \leq & \frac{\|\mu_t^{q,\varepsilon}-\mu_t^q\|_{BL}}{\alpha t}+\frac{\|\mu_t^q-\mu_t^F\|_{BL}}{\alpha t}+e^{-\alpha t}\frac{\|\delta(x-1)-\mu_0^q\|_{BL}}{\alpha t} \\ & e^{-\alpha t}\frac{\|\mu_0^q-\mu_0^{q,\varepsilon_0}\|_{BL}}{\alpha t}+\left\|\frac{\mu_t-e^{-\alpha t}\delta(x-1)}{\alpha t}-\kappa\right\|_{BL}.\end{aligned} \tag{63}$$

The first, third and fourth terms in the right-hand side of (63) are directly controlled using the assumptions of Theorem 6.In the last term at the right-hand side of (63), Theorem 5 combined with (8) guarantee that

$$\left\|\frac{\mu_t-e^{-\alpha t}\delta(x-1)}{\alpha t}-\kappa\right\|_{BL}\leq C(N,T,\alpha)\,t.$$

For the second term, we use Theorem 4 to obtain

$$\|\mu_t^q-\mu_t\|_{BL}\leq C(L,N,T,\alpha,\gamma,m^{\gamma-1})\|\mu_0^q-\delta(x-1)\|_{BL}$$

Thus with the assumptions of Theorem 6, we obtain

$$\|\mu_t^q-\mu_t\|_{BL}\leq C(L,N,T,\alpha,\gamma,m^{\gamma-1})q.$$

This completes the proof of Theorem 6 with $C_1=C(N,T,\alpha)$ and $C_2=1+C(L,N,T,\alpha,\gamma,m^{\gamma-1})$. □

Remark 5. *We notice that* (62) *presents a balance between two terms, which is classically encountered in the field of inverse problems [22] and which is also reminiscent of the classical bias-variance tradeoff in nonparametric statistics [25]. The time interval t plays the same role as a regularisation parameter: if too small, the noise is not smoothed and the right-hand side of* (62) *tends to infinity; if too large, the estimate loses its accuracy, the right-hand side being not small. There is a time* t^* *such that the estimate provided by Theorem* (6) *is optimal, namely*

$$t^*=\sqrt{\frac{\varepsilon_0+\varepsilon+C_2q}{\alpha C_1}}. \tag{64}$$

For this value, the error estimate is in the order of $\sqrt{\varepsilon_0+\varepsilon+q}$*, vanishing when the noise levels vanish, though at a lower speed than the noises themselves - the rate of convergence in the order of* $\sqrt{\varepsilon}$ *being reminiscent of mildly ill-posed problems.*

Remark 6. *Using short-time measurements to estimate parameters of a given time-dependent equation is an idea that has appeared for other types of equations. Recently, a very similar approach has been used for estimating the tumbling kernel of a mesoscopic equation for chemotaxis [28]; in their approach, convergence of their estimate is obtained, but no quantitative error estimate as* (62)*. Further away from our equation, it has been used to estimate the exponent of a time-fractional diffusion equation [34], or yet the diffusion parameter in the heat equation [15]. However, up to our knowledge, no systematic approach which would analyse the "short-time method" in a general framework, and which would justify our analogy of the time window of the observation with a regularisation parameter, has yet been developed.*

4 Reconstruction formula in Mellin variables

We have seen in the previous section how to approximate κ when the initial condition is not too far from a Dirac measure, and how to approximate $w_0 * \kappa$ by $\kappa^{est}(\mu_0;.)$ for generic initial condition. This section is devoted to the deduction of a reconstruction formula for the Mellin transform of the fragmentation kernel κ in the case of generic initial condition, and to estimate the error of such an approximation, in terms of short-time measurements of the population data and the initial data. The best method to this end is not to use the Mellin transform of the approximation $\kappa^{est}(\mu_0;.)$ of κ obtained in Corollary 3. Instead, the series representation of μ_t is used to deduce a series representation of its Mellin transform U, and then an approximation of the Mellin transform directly.

Suppose that κ satisfies (**Hyp-1**), μ_0 satisfies (**Hyp-2**) and let μ be the solution to (17) with initial condition μ_0 given by Theorem 2 . We denote by $U(t,.)$ the x-Mellin transform of μ_t to (17), and we denote by K the Mellin transform of κ, *i.e.*

$$U(t,s) = \int_0^{+\infty} x^{s-1} d\mu_t(x), \quad K(s) = \int_0^{+\infty} z^{s-1} d\kappa(z).$$

We also define

$$W(t,h,s) = \int_0^{+\infty} x^{s-1} e^{-\alpha h x^\gamma} d\mu_t(x), \ \forall t \geq 0, \ \forall h > 0.$$

It follows from (**Hyp-1**) and Theorem 2 that K is analytic in $s \in S_1^+ = \{s \in \mathbb{C};\ \Re e(s) > 1\}$ and so are $U(t)$ and $W(t,h)$ for all $t > 0$ and $h > 0$.

4.1 A formula for U

Lemma 1 (Representation of U as a power series)**.** *Take κ satisfying* (**Hyp-1**) *and $\mu_0 \in \mathcal{M}(\mathbb{R}^+)$ that satisfies* (**Hyp-2**)*. Then, the Mellin transform U of the solution μ_t to* (17) *satisfies*

$$\begin{aligned} U(t+\Delta\tau, s) =& W(t,\Delta\tau,s) + \sum_{n=1}^{\infty} \frac{(\alpha\Delta\tau)^n}{n!} U(t, s+\gamma n) \times \\ & \times \sum_{j=0}^{n-1} (-1)^{n-1-j} K(s+j\gamma) \prod_{m=0}^{j-1} (K(s+m\gamma) - 1), \end{aligned} \tag{65}$$

for $t > 0$ and $\Delta\tau > 0$, with the convention

$$\prod_{n \in \emptyset} b_n = 1.$$

Proof. Since the fragmentation equation is autonomous, Theorem 2 implies that for all $t > 0$, $\Delta\tau > 0$, we have

$$\mu_{t+\Delta\tau} = e^{-\alpha x^\gamma \Delta\tau} \mu_t + \sum_{n=0}^{\infty} (\alpha\Delta\tau)^n \int_0^{\infty} \ell^{n\gamma} a_n\left(\frac{x}{\ell}\right) \mu_t(\ell) \frac{d\ell}{\ell},$$

We apply the Mellin transform to both sides of the above equality and use Proposition 1: it follows

$$U(t+\Delta\tau, s) = W(t,\Delta\tau,s) + \sum_{n=0}^{\infty} (\alpha\Delta\tau)^n U(t, s+n\gamma) A_n(s),$$

where we denote by A_n the Mellin transform of the measure a_n. Passing (24) into the Mellin coordinates, the sequence A_n satisfies

$$A_0 = 0, \quad A_{n+1}(s) = \frac{1}{n+1}\left((K(s)-1)A_n(s+\gamma) + \frac{(-1)^n}{n!}K(s)\right).$$

By induction, we deduce

$$A_n(s) = \frac{1}{n!}\left((-1)^{n-1}K(s) + \sum_{j=1}^{n-1}(-1)^{n-1-j}K(s+j\gamma)\prod_{m=0}^{j-1}(K(s+m\gamma)-1)\right),$$

and Lemma 1 is proved. □

4.2 A reconstruction for K using short times

Since κ is supported on $[0,1]$, it follows that $K(s+n\gamma) \to 0$ as $n \to \infty$, and then an approximation formula for K may be obtained by truncation at $n = 1$ of the second term at the right-hand side of (65). To this end, let us give the following definitions.

Definition 4 (Approximation formula for the Mellin transform of the kernel)**.** *For $s \in \mathbb{C}$, we denote*

$$K^{est}(s,t,\Delta\tau) = \frac{U(t+\Delta\tau,s) - W(t,\Delta\tau,s)}{\alpha\Delta\tau U(t,s+\gamma)}$$
$$R(s,t,\Delta\tau) = K(s) - K^{est}(s,t,\Delta\tau). \tag{66}$$

The error term $R(s,t,\Delta\tau)$ may be estimated uniformly for s on some vertical strip of the complex plane such that $|\Im m(s)| > V$ for $V > 0$ large enough. This requires some further regularity on the kernel κ, the initial data μ_0 and the solution μ that ensure that U and K decay fast enough at infinity.

Hyp-3 There exists an interval $I \subset (0,\infty)$ such that κ and the function $x \mapsto s^{-1}x^s\kappa(x)$ are absolutely continuous functions on $x \in [0,1]$, for all $s \in S_I$, where

$$S_I = \{s \in \mathbb{C};\ \Re e(s) \in I\}.$$

Hyp-4 Let $\mu_0 \equiv u_0 \in \mathcal{C}^3([0,L])$ and either $u_0(L) > 0$, or $u_0(L) = 0$ and $u_0'(L) < 0$.

The decay at infinity of the Mellin transform U follows from the condition (**Hyp**-4) on u_0 thanks to the following lemma whose proof is postponed until the end of Section 4.

Lemma 2 (Regularity and support of the solution to the fragmentation equation)**.** *Assume the fragmentation kernel κ satisfies* (**Hyp-1**)*. Take $u_0 \in \mathcal{C}^3([0,L])$ such that* $\mathrm{supp}\,(u_0) = [0,L]$*. Then, if we denote $\mu = u$ the solution to the fragmentation equation* (17) *with $\mu_0 = u_0$, it holds*

1. *The function $x \to u(t,x)$ is in $\mathcal{C}^3([0,L])$ for all $t > 0$.*
2. $\mathrm{supp}\,(u(t,.)) = [0,L]$.
3. *If $u_0(L) > 0$, then for all $t > 0$, $u(t,L) = e^{-\alpha L^\gamma t}u_0(L) > 0$.*
   If $u_0(L) = 0$ and $u_0'(L) < 0$, then $u(t,L) = 0$ and $\partial_x u(t,L) = e^{-\alpha L^\gamma t}u_0'(L) < 0$ for all $t > 0$.

For any interval $I \subset (0,\infty)$ and $V > 0$ let us define the domain

$$D_{I,V} = \{s \in \mathbb{C};\, \Re es \in I,\, |\Im ms| > V\}$$

We have then the following

Theorem 7 (Reconstruction formula for K)**.** *Suppose that the fragmentation kernel κ satisfies* (**Hyp-1**) *and* (**Hyp**-3) *and u_0 satisfies* (**Hyp-2**) *and* (**Hyp**-4)*. Then, the following holds.*
(i) For all $T > 0, \tau_0 > 0$ and $V > 0$ sufficiently large, there exists a constant $C > 0$ depending on $\alpha, \gamma, V, I, T, \tau_0$, and L, such that for all $t \in (0,T)$ and all $\Delta\tau \in (0,\tau_0)$

$$|R(s,t,\Delta\tau)| \leq \frac{C\alpha\Delta\tau}{|s|}, \quad \forall s \in D_{I,V}. \tag{67}$$

(ii) For all $T > 0$, all $\tau_0 > 0$ and $s \in \mathbb{R}$ such that $s > 1$ there exists a constant $C = C(t,s,\tau_0) > 0$ such that

$$|R(s,t,\Delta\tau)| \leq C\alpha\Delta\tau,\ \forall t \in (0,T),\ \forall \Delta\tau \in (0,\tau_0) \tag{68}$$

Proof of Thereom 7. We first prove (i). Combining (66) with Lemma 1, we have the expression for the rest R

$$R(s,t,\Delta\tau) = \frac{1}{\alpha\Delta\tau}\sum_{n=2}^{\infty}\frac{(\alpha\Delta\tau)^n}{n!}\frac{U(t,s+\gamma n)}{U(t,s+\gamma)}\sum_{j=0}^{n-1}(-1)^{n-1-j}K(s+j\gamma)\prod_{m=0}^{j-1}(K(s+m\gamma)-1). \tag{69}$$

Step 1. Estimate for K. We prove here that for some $\tilde{C} > 0$ depending on I it holds

$$|K(s)| \leq \frac{\tilde{C}(I)}{1+|s|}, \quad \forall s \in S_I. \tag{70}$$

By (**Hyp-1**), $K(s)$ is well defined and analytic for $\Re es > 0$. Take $s \in I$. Since by (**Hyp**-3), $\kappa \in C([0,1])$ and $\Re e(s) > 0$ it follows that $x^s\kappa(x) \underset{x\to 0}{\longrightarrow} 0$. And since κ and $x \to s^{-1}x^s\kappa(s)$ are absolutely continuous on $[0,1]$

$$K(s) = \int_0^1 \kappa(x)x^{s-1}dx = \frac{\kappa(1)}{s} - \frac{1}{s}\int_0^1 \kappa'(x)x^s dx,\ \forall s \in S_I.$$

Because κ is absolutely continuous on $[0,1]$ there exists two non-decreasing functions on $[0,1]$, κ_1 and κ_2, such that $\kappa = \kappa_1 - \kappa_2$ on $[0,1]$, κ_i' are measurable and non negative on $[0,1]$ for $i = 1,2$ and

$$\begin{aligned}
&\int_0^1 \kappa_i'(x) \leq \kappa_i(1) - \kappa_i(0),\ i = 1,2\\
&\int_0^1 |\kappa'(x)|dx \leq \int_0^1 (\kappa_1'(x) + \kappa_2'(x))dx\\
&\qquad\qquad \leq \kappa_1(1) + \kappa_2(1) - \kappa_1(0) - \kappa_2(0).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|K(s)| &\leq \frac{\kappa(1)}{|s|} + \frac{1}{|s|}\int_0^1 |\kappa'(x)|dx,\ \forall s \in S_I\\
&\leq \frac{\kappa(1)}{|s|} + \frac{1}{|s|}(\kappa_1(1) + \kappa_2(1) - \kappa_1(0) - \kappa_2(0))
\end{aligned}$$

from where (70) follows. **Step 2. Estimate for** U. We prove here that for every $T>0$ and for V large enough, there exists a constant $C=C(L,T,V,\alpha,\gamma)>0$ such that for all $t\in(0,T)$ and $n\geq 2$,

$$\left|\frac{U(t,w+n\gamma+iv)}{U(t,w+\gamma+iv)}\right|\leq C(L,T,\alpha,\gamma)n(n-1)L^{(n-1)\gamma},\ \forall s\in D_{I,V}. \tag{71}$$

We follow, for $|v|$ large, the calculation of [17, Chapter IV, Section 4] where the stationary phase method is used to study the behaviour of oscillatory integrals. For $w>0$, we have for $v\neq 0$

$$\begin{aligned}U(t,w+iv)=\int_0^L u(t,x)x^{w-1}x^{iv}dx&=\int_0^L u(t,x)x^{w-1}e^{iv\ln(x)}dx\\&=\frac{1}{iv}\int_0^L u(t,x)x^w\frac{d}{dx}\left(e^{iv\ln(x)}\right)dx.\end{aligned}$$

since $\frac{d}{dx}(e^{iv\ln(x)})=\frac{iv}{x}e^{iv\ln(x)}$. We perform an integration by part and we obtain

$$U(t,w+iv)=\frac{1}{iv}u(t,L)L^we^{iv\ln(L)}-\frac{1}{iv}\int_0^L e^{iv\ln(x)}\frac{\partial}{\partial x}\left(u(t,x)x^w\right)dx,$$

which we rewrite, using the same trick than above

$$U(t,w+iv)=\frac{1}{iv}u(t,L)M^we^{iv\ln(L)}-\left(\frac{1}{iv}\right)^2\int_0^L x\frac{\partial}{\partial x}\left(u(t,x)x^w\right)\frac{d}{dx}\left(e^{iv\ln(x)}\right)dx.$$

We perform another integration by part to obtain

$$\begin{aligned}U(t,w+iv)=\frac{1}{iv}u(t,L)^we^{iv\ln(L)}&-\left(\frac{1}{iv}\right)^2L^w\left(L\frac{\partial}{\partial x}u(t,L)+wu(t,M)\right)e^{iv\ln(L)}\\&+\left(\frac{1}{iv}\right)^2\int_0^L\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\left(u(t,x)x^w\right)\right)e^{iv\ln(x)}dx.\end{aligned}$$

The third term of the right-hand side above can be expanded using

$$\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\left(u(t,x)x^w\right)\right)=w^2x^{w-1}u(t,x)+x^w(1+2w)\frac{\partial}{\partial x}u(t,x)+x^{w+1}\frac{\partial^2}{\partial x^2}u(t,x).$$

Then we have

$$\begin{cases}U(t,w+\gamma+iv)=\dfrac{C(L,t,\alpha,\gamma,w,v)}{iv}+\dfrac{C'(L,t,\alpha,\gamma,w,v)}{(iv)^2},\\[2ex] U(t,w+n\gamma+iv)=nL^{(n-1)\gamma}\left(\dfrac{C(L,t,\alpha,\gamma,w,v)}{niv}+\dfrac{C''(L,t,\alpha,\gamma,w,v,n)}{(iv)^2}\right)\end{cases} \tag{72}$$

for some complex constants $C(L,t,\alpha,\gamma,w,v)$, $C'(L,t,\alpha,\gamma,w,v)$ and $C''(L,t,\alpha,\gamma,w,v,n)$ defined

as

$$C(L,t,\alpha,\gamma,w,v) = u(t,L)L^{w+\gamma}e^{iv\ln(L)},$$
$$C'(L,t,\alpha,\gamma,w,v) = -L^{w+\gamma}\left(L\frac{\partial u}{\partial x}(t,L) + (w+\gamma)u(t,L)\right)e^{iv\ln(L)},$$
$$+\int_0^L \left((w+\gamma)^2 x^{w+\gamma-1}u(t,x) + x^{w+\gamma}(1+2(w+\gamma))\frac{\partial}{\partial x}u(t,x) + x^{w+\gamma+1}\frac{\partial^2}{\partial x^2}u(t,x)\right)e^{iv\ln(x)}dx$$
$$C''(L,t,\alpha,\gamma,w,v,n) = L^{w+\gamma}\left(L\frac{\partial}{\partial x}u(t,L) + \frac{w+n\gamma}{n}u(t,L)\right)e^{iv\ln(L)}$$
$$+\int_0^L \left((w+n\gamma)^2\left(1+x^{w+n\gamma-1}\right)u(t,x) + x^{w+n\gamma}(1+2(w+n\gamma))\frac{\partial u}{\partial x}(t,x) + x^{w+n\gamma+1}\frac{\partial^2 u}{\partial x^2}(t,x)\right)dx.$$

If $u_0(L) > 0$, then Lemma 2 guarantees that $u(t,L) > 0$ as well. Then we have the following estimates on C, C' and C''

$$0 < C_0 \le |C(L,t,\alpha,\gamma,w,v)| \le C_1, \quad |C'(L,t,\alpha,\gamma,w,v)| \le C_2, \quad w \in I,\ t \in [0,T],\ v \in \mathbb{R},$$
$$\text{and} \quad |C''(L,t,\alpha,\gamma,w,v,n)| \le C_3, \quad w \in I,\ t \in [0,T],\ v \in \mathbb{R},\ n \ge 1.$$

Then, using (72), there exists $V > 0$ such that for $|v| \ge V$ and $w \in I$,

$$\begin{aligned}\left|\frac{U(t,w+n\gamma+iv)}{U(t,w+\gamma+iv)}\right| &\le nL^{(n-1)\gamma}\frac{|Civ+C'|}{|Civ+C''|}\\ &\le nL^{(n-1)\gamma}\left(1+\frac{|C''-C'|}{|Civ+C'|}\right) \le nL^{(n-1)\gamma}C(V). \end{aligned} \tag{73}$$

for some constant $C(V) > 0$ that depends on V, and formula (71) follows.

Now if $u_0(L) = 0$, then Lemma 2 guarantees that $u(t,L) = 0$ as well. Thus

$$U(t,w) = -\left(\frac{1}{iv}\right)^2 \int_0^L x\frac{\partial}{\partial x}\left(u(t,x)x^w\right)\frac{d}{dx}\left(e^{iv\ln(x)}\right)dx.$$

In that case, $u_0'(L) < 0$ so that Lemma 2 guarantees that $\partial_x u(t,L) < 0$ as well, and we go one step further in the expansion and write

$$\begin{aligned}U(t,w) = &-\left(\frac{1}{iv}\right)^2 L^{w+1}e^{iv\ln(L)}\frac{\partial}{\partial x}\left(u(t,x)x^w\right)\Big|_{x=L}\\ &+\left(\frac{1}{iv}\right)^3 Le^{iv\ln(L)}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\left(u(t,x)x^w\right)\right)\Big|_{x=L}\\ &+\left(\frac{1}{iv}\right)^3\int_0^L \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\left(u(t,x)x^w\right)\right)\right)e^{iv\ln(x)}dx\end{aligned}$$

Using the same types of arguments than above, formula (71) holds again.

Step 3. Estimate for R.

Using formula (69) and the triangle inequality, we have

$$|R(s,y,\Delta\tau)| \le \alpha\Delta\tau\sum_{n=2}^{\infty}\frac{(\alpha\Delta\tau)^{n-2}}{n!}\left|\frac{U(t,s+\gamma n)}{U(t,s+\gamma)}\right|\sum_{j=0}^{n-1}|K(s+j\gamma)|\prod_{m=0}^{j-1}|K(s+m\gamma)-1|.$$

Using now (70) and (71) we obtain for $\Re(s) \in I$ and $\Im(s) \in \mathbb{R}$

$$|R(s,y,\Delta\tau)| \leq \frac{\alpha\Delta\tau L^\gamma}{1+|s|} \sum_{n=2}^{\infty} \frac{(\alpha\Delta\tau)^{n-2}}{(n-2)!} (L^\gamma)^{n-2} \sum_{j=0}^{n-1} \tilde{C}^{j+1}. \tag{74}$$

which implies

$$|R(s,y,\Delta\tau)| \leq \alpha\tau_0 \frac{L^\gamma \tilde{C}}{1+|s|} \sum_{n=2}^{\infty} \frac{(\alpha\Delta\tau)^{n-2}}{(n-2)!} (L^\gamma)^{n-2} \frac{1-\tilde{C}^n}{1-\tilde{C}},$$

and Theorem 7 is proved for $C = L^\gamma \tilde{C} \sum_{n=2}^{\infty} \frac{(\alpha\tau_0)^{n-2}}{(n-2)!} (L^\gamma)^{n-2} \frac{1-\tilde{C}^n}{1-\tilde{C}} < \infty$.

In order to prove now (ii), suppose that $s \in \mathbb{R}$ is fixed such that $s > 1$, estimate (70) still holds. Moreover, since $\mu(t) \geq 0$ for all $t > 0$ it follows that $U(t,s) > 0$. Since $U(\cdot, s) \in C(0,\infty)$ it follows that for all $T > 0$ there exists a constant $C = C(T,s) > 0$ such that $U(t, s+\gamma) > C^{-1}$ for all $t \in (0,t)$. It then follows, arguing as for (74),

$$\begin{aligned}
\left| \frac{U(t,s+n\gamma)}{U(t,s+\gamma)} \right| &\leq C L^{\gamma n - 1} ||\mu(t)||_{TV} \\
|R(s,t,\Delta\tau)| &\leq C||\mu(t)||_{TV} \alpha\Delta\tau \sum_{n=2}^{\infty} \frac{(\alpha\tau_0)^{n-2}}{n!} \sum_{j=0}^{n-1} \tilde{C}^{j+1} \\
&\leq \left(C||\mu(t)||_{TV} \sum_{n=2}^{\infty} \frac{(\alpha\tau_0)^{n-2}}{n!} \frac{1-\tilde{C}^n}{1-\tilde{C}} \right) \alpha\Delta\tau
\end{aligned}$$

where $\tilde{C}$ comes from Step 1. □

The estimate in Theorem 7 may be improved under stronger assumptions on κ. For example,

Corollary 4 (A better estimate for kernels not allowing erosion)**.** *Suppose that the hypothesis of Theorem 7 hold. Suppose moreover that κ' and $x \mapsto \kappa'(x)x^{s+1}$ are absolutely continuous on $x \in [0,1]$, and $\kappa(1) = 0$ for $s \in S_I$. Then, for all $T > 0$ and $\tau_0 > 0$ there exists two constants $V > 0$ and $C(L,T,\alpha,\gamma,I,V,\tau_0) > 0$ such that for all $t \in (0,T)$, $\Delta\tau \in (0,\tau_0)$*

$$|R(s,t,\Delta\tau)| \leq \frac{C(L,T,\alpha,\gamma,I,V,\tau_0)\alpha\Delta\tau}{|s|^2}, \forall s \in D_{I,V}. \tag{75}$$

Proof. The proof is the same than the proof of Theorem 7, except that the estimate for the Mellin transform in Step 1 becomes,

$$\begin{aligned}
K(s) &= \int_0^1 \kappa(x) x^{s-1} dx = -\frac{1}{s} \int_0^1 \kappa'(x) x^s dx \\
&= -\frac{1}{s(s+1)} \left(\kappa'(1) - \kappa'(0) - \int_0^1 \kappa''(x) x^{s+1} dx \right), \forall s \in S_I
\end{aligned}$$

and thus for some $\bar{C} > 0$

$$|K(s)| \leq \frac{\bar{C}}{|s|^2+1}, \quad \forall s \in S_I.$$

□

A natural question arising from Theorem 7 and Corollary 4 is if, and in what sense the inverse Mellin transform of $K^{est}(t,\Delta\tau)$, $\mathcal{M}^{-1}\left(K^{est}(t,\Delta\tau)\right)$, is an approximation of the kernel κ itself. By (**Hyp**-3), for all $s \in I$,

$$\sup_{x\in[0,1]} x^s|\kappa(x)| < \infty, \tag{76}$$

and then, for any $s_0 \in I$, (76) holds for all $s > s_0$. But (76) is not known to be true for $s < \inf\{\sigma; \sigma \in I\}$.

Theorem 8. *Suppose that the hypothesis of Corollary 4 are satisfied and denote $I = (a,b)$ for some $b > a \geq 0$. Then, for every $t > 0$, every $\tau_0 > 0$ and $\delta > 0$ sufficiently small, there exists a positive constant C that depends on t, τ_0 and δ such that, for all $s \in (a,b)$,*

$$\sup_{x\in[0,1]} x^s \left|\kappa(x) - \mathcal{M}^{-1}\left(K^{est}(t,\Delta\tau)\right)(x)\right| \leq C\Delta\tau, \quad \forall\, \Delta\tau \in (0,\tau_0). \tag{77}$$

Proof. By hypothesis, for all $T > 0$ and $\tau_0 > 0$ there exist two constants $V > 0$ and $C > 0$ such that (75) holds for $t \in [0,T]$, $\Delta\tau \in (0,\tau_0)$, $s \in S_I$ and $|\Im m(s)| > V$.
Moreover, for each $t \in (0,T)$ the function $U(t, s+\gamma)$ is analytic on the domain $\{s \in \mathbb{C}; \Re es > 1-\gamma\}$ and $\gamma \geq 1$. Then, for any $\delta > 0$ sufficiently small to have $(a+\delta/3, a+2\delta) \subset$ the function $U(t, s+\gamma)$ is analytic on

$$Q_{\delta,V} = \{s \in S_I; \Re es \in (a+\delta/3, a+2\delta, |\Im ms| < 2V\}$$

and it may then have only a finite number of zeros in $Q_{\delta,V}$. Therefore there exists a closed subinterval $J \subset (a+\delta/2, a+\delta)$ such that $U(t,s+\gamma) \neq 0$ for $s \in S_J \cap Q_{\delta,V}$. It follows by continuity that for some constant $C(t,J,V,\delta) > 0$ that may depend on t, J, V, δ,

$$|U(t,s+\gamma)| \geq C(t,J,V,\delta) > 0, \ \ \forall s \in S_J, \ |\Im s| < V. \tag{78}$$

On the other hand, for $w \in J$,

$$\begin{aligned}|U(t,w+n\gamma+iv)| = \left|\int_0^L u(t,x)x^{w+n\gamma-1}e^{iv\ln(x)}dx\right| &\leq L^{(n-1)\gamma}\left|\int_0^L u(t,x)x^{w+\gamma-1}dx\right| \\ &\leq L^{(n-1)\gamma}L^{a+\gamma+1}\|u(t,.)\|_\infty.\end{aligned}$$

Then,

$$\left|\frac{U(t,w+n\gamma+iv)}{U(t,w+\gamma+iv)}\right| \leq \frac{2}{nC(t,J,V,\delta)}L^{a+\gamma+1}\|u(t,.)\|_\infty L^{(n-1)\gamma}, \qquad w \in J, \ v \in [-V,V]. \tag{79}$$

Therefore, by (73) and (79), for every $t \in (0,T)$, $\tau_0 > 0$ and $\delta > 0$ small enough, there exists a closed interval $J \subset I$ and a constant $C > 0$ depending on t, J, τ_0, δ such that

$$|R(s,t,\Delta\tau)| \leq \frac{C}{1+|s|^2} \ \forall s \in S_J, \ \forall \Delta\tau \in (0,\tau_0). \tag{80}$$

The inverse Mellin transform of $R(s,t,\Delta\tau)$ is then a well-defined function for all $x > 0$ and $\Delta\tau \in (0,\tau_0)$, given by

$$\mathcal{M}^{-1}(R(t,\Delta\tau))(x) = \frac{1}{2i\pi}\int_{r-i\infty}^{r+i\infty} R(s,t,\Delta\tau)x^{-s}ds, \ \ r \in J.$$

and is such that

$$\mathcal{M}^{-1}(R(t,\Delta\tau)) \in E'_J$$

where E'_J is the space of distributions whose Mellin transform is analytic on S_J (cf. [37]). Since it also holds for $x > 0$,

$$\kappa(x) \equiv \mathcal{M}^{-1}(K)(x) = \frac{1}{2i\pi}\int_{r-i\infty}^{r+i\infty} K(x)x^{-s}ds,\ r \in J.$$

it follows from (66) that the inverse Mellin transform of $K^{est}(t,\Delta\tau)$ is also well defined for $\Delta\tau \in (0,\tau_0)$ and $x > 0$, and given by a similar integral expression. It is then possible to apply the inverse Mellin transform to both sides of (66) to obtain for $t > 0$, $\Delta\tau \in (0,\tau_0)$, $x > 0$ and some constant $C > 0$ depending on t, τ_0 and δ,

$$\begin{aligned}\left|\kappa(x) - \mathcal{M}^{-1}(K^{est}(t,\Delta\tau)(x)\right| &= \left|\mathcal{M}^{-1}(R(t,\Delta\tau))(x)\right| \\ &\leq C\Delta\tau\, x^{-r}\int_{\mathbb{R}}(1+|v|^2)^{-1}dv,\ \forall x \in (0,1),\ \forall r \in J.\end{aligned}$$

Since $r \in J \subset (a+\delta/2, a+\delta)$,

$$\left|\kappa(x) - \mathcal{M}^{-1}\left(K^{est}(t,\Delta\tau)\right)(x)\right| \leq C_\delta x^{-a-\delta}\Delta\tau \quad \forall x \in (0,1),\ \forall \Delta\tau \in (0,\tau_0).$$

For every $s \in (a,b)$ there exists $\delta > 0$ such that $s > a+\delta$ and then, for all $x \in [0,1]$, $\Delta\tau \in (0,\tau_0)$,

$$x^s\left|\kappa(x) - \mathcal{M}^{-1}\left(K^{est}(t,\Delta\tau)\right)(x)\right| \leq C_\delta x^{-a-\delta+s}\Delta\tau \leq C_\delta\Delta\tau,$$

and estimate (77) follows. □

A different reconstruction formula of κ was already obtained in Theorem 2, (iii) of [18]. We notice that there are similarities in the formulae: both of them are the inverse Mellin transform of a ratio between two Mellin transforms of linear functionals of the solution, the numerator taken in s and the denominator taken in $s+\gamma$. This reveals a serious drawback when noise is considered: both formulae then fall in the scope of so-called *severely ill-posed* inverse problems, exactly as for deconvolution problems, see e.g. [8], ch. 4 for an introduction. However, despite the fact that this new formula is an approximation whereas the previous one was exact, its advantages are many.

- Experimentally, it is possible to use the measurement of the solutions at several pairs of close time points, thus making the most of experimental data, see [9]. On the contrary, with the formula in [18], only the large-time asymptotic profile can be used.
- In [18] there are specific difficulties linked to the measurement of the asymptotic profile: first, as time passes, the distribution is closer and closer to zero-size particles, making the measurement all the more noisy ; second, one needs to assess the validity of considering that the asymptotic behaviour is reached - the distance to the true asymptotics being a second source of noise ; third, it has been proved by numerical simulation that different fragmentation kernels may give rise to very close asymptotic size-distribution of particles [19].
- Experimentally, it should be possible to depart from several very different initial conditions, and then use the superimposition principle to combine them in such a way that we get the most information. This is a direction for future research.

Remark 7. *If only the hypothesis of Theorem 7 are assumed, the same argument as above still shows (78) for some interval $J \subset I$ and some constant C depending on t, J and V. But instead of (80) only the following holds for some constant $C = C(t, J, \tau_0)$,*

$$|R(s,t,\Delta\tau)| \leq \frac{C}{1+|s|} \ \forall s \in S_J, \ \forall \Delta\tau \in (0,\tau_0). \tag{81}$$

The inverse Mellin transform of $R(t,\Delta\tau)$ is then still well defined, but its expression is now

$$\mathcal{M}^{-1}(R(t,\Delta\tau)) = -\left(x\frac{\partial}{\partial x}\right)(h(t,\Delta\tau))$$
$$\forall x > 0: \ h(t,\Delta\tau,x) = \frac{1}{2i\pi}\int_{r-i\infty}^{r+i\infty} R(s,t,\Delta\tau)s^{-1}x^{-s}ds, \ r \in J,$$

where $h(t,\Delta\tau,x)$ is a function, defined for all $x > 0$ and all $\Delta\tau \in (0,\tau_0)$, such that $h(t,\Delta\tau_0) \in E'_J$. As above, the inverse Mellin transform of $K^{est}(t,\Delta\tau)$ is then well defined too, but no point wise estimate like (77) holds.

Remark 8. *At first sight, regularity assumptions such as* (**Hyp**-3), (**Hyp**-4) *and the non-erosion of Corollary 4 may be surprising. It is however classical in the field of inverse problems to assume regularity on the object we want to estimate, and to gain a better convergence rate when the regularity increases, see for instance [22]. Here however it is not only on κ that regularity is required - moreover* (**Hyp**-3) *is satisfied for a large class of measures and for all the classical fragmentation kernels - but also on the initial condition, which is less expected. We thus have been able to reconstruct the fragmentation kernel in two extreme cases: either very singular initial condition, given by a Dirac delta function, or very regular ones, at least $\mathcal{C}^3$.*

Point (ii) of Theorem 7 may also be used to estimate the statistical parameters of the kernel κ like mean, variance, skewness, kurtosis, since all of them may be expressed in terms of $K(s)$ for integer values of s. Consider for example the variance given by

$$Var\left[\frac{\kappa}{2}\right] = \frac{1}{2}\int_{(0,1)}\left|x-\frac{1}{2}\right|^2\kappa(x)dx = \frac{1}{2}K(3) - \frac{1}{2}K(2) + \frac{1}{8}K(1) = \frac{1}{2}K(3) - \frac{1}{4}$$

It is then possible to estimate $Var[\kappa]$ using 65 and defining:

$$\left(Var\left[\frac{\kappa}{2}\right]\right)^{est}(t,\Delta\tau) = \frac{1}{2}K^{est}(3,t,\Delta\tau) - \frac{1}{4}.$$

The following Corollary immediately follows from Point (ii) of Theorem 7 for $s = 3$.

Corollary 5 (Estimate of the variance of the kernel)**.** *Suppose that the assumptions of Theorem 7 are satisfied. Then for all $T > 0$ and $\tau_0 > 0$ there is $C(L,T,\tau_0,\alpha,\gamma)$ such that*

$$\left|Var[\kappa] - (Var[\kappa])^{est}(t,\Delta\tau)\right| \leq C(L,T,\alpha,\gamma)\alpha\Delta\tau, \ \forall t \in (0,T), \forall \Delta\tau \in (0,\tau_0).$$

4.3 Proof of Lemma 2.

Proof of Lemma 2. The arguments rely on the formula obtained in Theorem 2.

1. We use the formula (23) obtained in Theorem 2. Note that it can be rewritten using the change of variables

$$z = \frac{x}{\ell}, \quad dz = -\frac{z^2}{x} d\ell, \tag{82}$$

as

$$u(t,x) = e^{-\alpha x^\gamma t} u_0(x) + \sum_{n=0}^{\infty} (\alpha t)^n \int_0^1 \frac{x^{n\gamma}}{z^{n\gamma}} a_n(z) u_0\left(\frac{x}{z}\right) \frac{dz}{z}. \tag{83}$$

The first term of the sum is clearly C^1, since u_0 is. To deal with the second term, set

$$I_n(x) = \int_0^1 \frac{x^{n\gamma}}{z^{n\gamma}} a_n(z) u_0\left(\frac{x}{z}\right) \frac{dz}{z}.$$

The fisrt step is to prove by induction that for all $x_0 > 0$, for all $n \geq 0$, $z \to a_n(z) \in \mathcal{C}^1[x_0, 1]$. The function a_0 is clearly $\mathcal{C}^1$, since it is identically zero. Let us assume that for some $n \geq 0$, $z \to a_n(z) \in \mathcal{C}^1[x_0, 1]$. The function a_{n+1} satisfies (24) and is composed with three terms. The first term and third term are clearly $\mathcal{C}^1$ since $a_n(z) \in \mathcal{C}^1[x_0, 1]$. We focus on the second term

$$J_n(x) = \int_x^\infty y^{\gamma-1} \kappa\left(\frac{x}{y}\right) a_n(y) dy.$$

Once again, it can be rewritten using the change of variables (82)

$$J_n(x) = \int_{x_0}^1 \frac{x^\gamma}{z^\gamma} \kappa(z) a_n\left(\frac{x}{z}\right) \frac{dz}{z}.$$

The dominated convergence theorem guarantees that $J_n \in \mathcal{C}^1[x_0, 1]$ and that

$$J_n'(x) = \int_{x_0}^1 \frac{x^\gamma}{z^\gamma} \kappa(z) a_n'\left(\frac{x}{z}\right) \frac{dz}{z^2} + \int_{x_0}^1 \gamma \frac{x^{\gamma-1}}{z^\gamma} \kappa(z) a_n\left(\frac{x}{z}\right) \frac{dz}{z}.$$

Indeed

$$\left| \frac{x^\gamma}{z^\gamma} \kappa(z) a_n'\left(\frac{x}{z}\right) \frac{1}{z^2} \right| \leq \frac{1}{x_0^{2+\gamma}} \|a_n'\|_{\mathcal{C}^0[x_0,1]}, \quad \left| \gamma \frac{x^{\gamma-1}}{z^\gamma} \kappa(z) a_n\left(\frac{x}{z}\right) \frac{1}{z} \right| \leq \frac{\gamma \max\{x_0^{\gamma-1}, 1\}}{x_0^{1+\gamma}} \|a_n\|_{\mathcal{C}^0[x_0,1]}.$$

We have proven that $a_n \in \mathcal{C}^1(0,1)$ since it is $\mathcal{C}^1(K)$ for all K compact of $(0,1)$.

The fisrt step is to prove that $I_n \in \mathcal{C}^1([0, L])$. To do so, we use the dominated convergence to prove that for all $x_0 > 0$, we have $I_n \in \mathcal{C}^1([x_0, L])$ and that

$$I_n'(x) = n\gamma x^{n\gamma-1} \int_0^1 \frac{1}{z^{n\gamma}} a_n(z) u_0\left(\frac{x}{z}\right) \frac{dz}{z} + \int_0^1 \frac{x^{n\gamma}}{z^{n\gamma}} a_n(z) u_0'\left(\frac{x}{z}\right) \frac{dz}{z^2}, \quad x \in [x_0, L]. \tag{84}$$

Indeed, the conclusion of the dominated convergence holds: $u_0 \in \mathcal{C}^1([0, L])$, hence the the integrand is in $\mathcal{C}^1([0, L])$ as well. The domination is as follows: since $\operatorname{supp}(u_0) \subset [0, L]$, the bounds of the integral I_n are $z \in \left[\frac{x}{L}, 1\right] \subset \left[\frac{x_0}{L}, 1\right]$, and thus

$$\left| \frac{x^{n\gamma-1}}{z^{n\gamma}} a_n(z) u_0\left(\frac{x}{z}\right) \frac{1}{z} \right| \leq \frac{\max\{\left(\frac{x_0}{L}\right)^{n\gamma-1}, L^{n\gamma-1}\}}{x_0^2} \|u_0\|_\infty a_n(z),$$

$$\left| \frac{x^{n\gamma}}{z^{n\gamma}} a_n(z) u_0'\left(\frac{x}{z}\right) \frac{dz}{z^2} \right| \leq \frac{\max\{\left(\frac{x_0}{L}\right)^{n\gamma}, L^{n\gamma}\}}{x_0^2} \|u_0'\|_\infty a_n(z),$$

and it was proved in (35) that $\|a_n\|_{TV} \leq \dfrac{(N+2)^n}{n!}$.

Now we claim that the function S defined as

$$S(x) = \sum_{n=1}^{\infty} (\alpha t)^n I_n(x) \tag{85}$$

is of class $\mathcal{C}^1([x_0, L])$ for all $x_0 > 0$. Indeed, we just saw that $I_n \in \mathcal{C}^1([0, L])$, and that I_n' is given by (84). For $x \in [x_0, L]$, we can control each of the two terms of the sum (84) by two sequences that converge. Indeed using again (35), we have

$$\begin{aligned} n \int_0^1 \left| \frac{x^{n\gamma-1}}{z^{n\gamma}} a_n(z) u_0\left(\frac{x}{z}\right) \frac{1}{z} \right| dz &\leq \frac{\max\{\left(\frac{x_0}{L}\right)^{n\gamma-1}, L^{n\gamma-1}\}}{x_0^2} \|u_0\|_\infty n \frac{(N+2)^n}{n!}, \\ \int_0^1 \left| \frac{x^{n\gamma}}{z^{n\gamma}} a_n(z) u_0'\left(\frac{x}{z}\right) \frac{dz}{z^2} \right| dz &\leq \frac{\max\{\left(\frac{x_0}{L}\right)^{n\gamma}, L^{n\gamma}\}}{x_0^2} \|u_0'\|_\infty \frac{(N+2)^n}{n!}, \end{aligned}$$

and

$$\sum_{n=1}^{\infty} (\alpha t)^n \left(\frac{L^\gamma}{x_0^\gamma}\right)^n \frac{(N+2)^n}{n!} < \infty, \qquad \sum_{n=1}^{\infty} (\alpha t)^n \left(\max\{x_0^\gamma, L^\gamma\}\right)^n n \frac{(N+2)^n}{n!} < \infty,$$

This ends the proof of 1, and we have in addition for $x \in [x_0, L]$ an expression of the spatial derivative of u

$$\begin{aligned} \frac{\partial}{\partial x} u(t,x) =& e^{-\alpha x^\gamma t} u_0'(x) - \alpha\gamma x^{\gamma-1} t e^{-\alpha x^\gamma t} u_0(x) \\ &+ \sum_{n=1}^{\infty} (\alpha t)^n \left(n\gamma x^{n\gamma-1} \int_0^1 \frac{1}{z^{n\gamma}} a_n(z) u_0\left(\frac{x}{z}\right) \frac{dz}{z} + \int_0^1 \frac{x^{n\gamma}}{z^{n\gamma}} a_n(z) u_0'\left(\frac{x}{z}\right) \frac{dz}{z^2} \right). \end{aligned} \tag{86}$$

Similar arguments hold to guarantee that $x \to u(t,x) \in C^3([0, L])$.

2. First, we claim that $\text{supp}\,(u(t,.)) \subset [0, L]$. Indeed, this is a consequence of formula (83) and of the fact that $\text{supp}\,(a_n) \subset [0, 1]$ for $n \geq 0$. Let us now prove that $\text{supp}\,(u(t,.)) = [0, L]$. Take $y \in [0, L]$ and set $Y(t) = u(t, y)$. The fragmentation equation (17) implies

$$Y'(t) \geq -\alpha y^\gamma Y(t),$$

i.e.

$$u(t,y) = Y(t) \geq e^{-\alpha y^\gamma t} Y(0) = e^{-\alpha y^\gamma t} u_0(y).$$

If $u_0(y) \neq 0$, then for all $t \geq 0$, $u(t, y) \neq 0$. If $u_0(y) = 0$, since $y \in \text{supp}\,(u_0)$, for all $\varepsilon > 0$, there exists y_ε such that $|y - y_\varepsilon| < \varepsilon$ and $u_0(y_\varepsilon) \neq 0$ and then $u(t, y_\varepsilon) \neq 0$, which implies that $y \in \text{supp}\,(u(t,.))$. Thus $\text{supp}\,(u(t,.)) = [0, L]$.

3. It is clear from formula (23) that $u(t, L) = e^{-\alpha L^\gamma t} u_0(L)$. Then, if $u_0(L) > 0$, we have $u(t, L) > 0$. If $u_0(L) = 0$ and $u_0'(L) < 0$, we have $u(t, L) = 0$, and formula (86) implies

$$\frac{\partial}{\partial x} u(t, L) = e^{-\alpha L^\gamma t} u_0'(L) < 0.$$

□

5 Numerical simulations

In this section, we illustrate the different theoretical results and investigate the convergence errors of the reconstruction formulae.

- **Illustration of Theorem 5:** We show on Figure 1 the profile of the estimated kernel $\kappa^{est}(t)$ defined in formula (54), for $\gamma = \alpha = 1$, for four different kernels κ and for different times t. For each plot, the kernel κ is displayed in an inset on the upper right. The initial condition μ_0 is a highly peaked gaussian centered at $x = 1$ and the numerical solution μ_t used to build $\kappa^{est}(t)$ is obtained using a numerical scheme with a time step $\Delta t = 0.01$. We observe that the estimate $\kappa^{est}(t)$ is valid for early time points: indeed, at the naked eye, $\kappa^{est}(t)$ and κ look alike. As time goes by, the size distribution is driven towards the stationary state and the information on the kernel is lost.
  On Figure 2 Left, we illustrate the estimate (55) and show that the time evolution of the error

$$e_{TV}(t) = \left\| \kappa^{est}(t) - \kappa \right\|_{TV}$$

  increases linearly with time t for the same four kernels κ considered in Figure 1 and for an initial condition μ_0 very close to $\delta(x = 1)$. We observe that the slope of $t \to e_{TV}(t)$ is small for kernels of erosion type ($\kappa(0) \neq 0$), and large for kernels producing daughter particles of similar sizes. This may be linked to a larger constant in (55) for more peaked kernels; this provides us with an interesting direction for future work.

- **Illustration of Corollary 3** In Figure 2 Right, we draw the curves of the error $e_{TV}(t)$ for three initial conditions μ_0 given by (truncated) gaussians of standard deviation $\sigma = 0.01$, $\sigma = 0.1$ and $\sigma = 0.2$ and for the kernel κ in black on the left figure. As seen on the formula (61), the increase of $e_{TV}(t)$ is linear with respect to time t, but an extra constant error ε is added, related to the distance between δ_1 and μ_0. We notice that a small error term ε was already observed in Figure 2, Left, due to the distance between $\delta(x = 1)$ and its numerical approximation on a discrete grid. For large standard deviations (e.g. $\sigma = 0.2$), the error ε becomes so large that the estimate in Total Variation norm is no more meaningful: we see the interest to turn to the Bounded Lipshitz norm.
  In Figure 3, we display the shape of the estimated kernel $\kappa^{est}(t)$ for a small value of t, for a kernel κ of erosion type and with three initial conditions being gaussians with various spreading. It can be observed at the naked eye that the thiner the gaussian is, the better the approximation $\kappa^{est}(t)$ is as well. We observe how the estimated kernel κ^{est} is differently impacted around $x = 0$ and around $x = 1$: this gives interesting hints on how the kernel symmetry could be used to improve the theoretical estimates.

- **Illustration of Theorem 6**: In Fig 4, we display the error

$$e_{BL}(t) = \|\kappa^{est}(t) - \kappa\|_{BL}$$

  as a function of time for a two-peaked gaussian kernel κ, for a gaussian initial condition μ_0 of variance σ^2 and for a noise ε_0 on the measurement of the initial data μ_0 and a noise ε_1 on the measurement of the solution μ_t used in the calculation of $\kappa^{est}(t)$. The standard deviation σ thus plays the role of q in Theorem 6. To simulate the noise on the solution observed, we add a multiplicative uniform noise on $[-0.5\varepsilon_i, +0.5\varepsilon_i]$ to the simulation. Numerically, we approximate the BL norm by the Wasserstein distance W_1, since 1/it is easier to compute using

the monotone rearrangement theorem, 2/the BL norm is close to the Wasserstein distance between two measures of approximately same mass and whose supports are not too far. The error $e_{BL}(t)$ first decreases and then increases, as expected by Remark 5. In the inset, we superimposed the kernel κ (in red) with the best estimated κ, namely $\kappa^{est}(t^*)$ (in blue) taken at the optimal time where the error reaches its minimum.

- **Illustration of Remark 5** In Figure 5, we investigated how the minimal error and the optimal time, *i.e.* the time displaying the minimal error (drawn on Figs. 4 as the red asterisk), evolve with respect to the noise level. To do so, we take an equal level of noise for the three noise sources ε_0, ε_1 and q (with $q = \sigma$ the standard deviation of the gaussian taken for the initial size distribution). We ran fifty simulations - to take into account the fact that the noise we simulate is random - and we draw the optimal time (blue asterisks in Fig. 5) giving the optimal error (green asterisks in Fig. 5). We then compare the mean curves over these fifty simulations, and compare it with the curve $x \mapsto \sqrt{x}$. We observe a good qualitative agreement with the expected rate of convergence.

- **Illustration of Corollary 5** We illustrate how we recover the variances of the 6 different typical fragmentation kernels described in Table 1. We recall that the variance and standard deviation are given by

$$Var = Var\left[\frac{\kappa}{2}\right] = \frac{1}{2}K(3) - \frac{1}{4}, \qquad SD = \sqrt{Var},$$

and we define the estimated variance of the kernel as,

$$\begin{cases} Var^{est}(t, \Delta t) = \dfrac{1}{2}K^{est}(3, t, \Delta t) - \dfrac{1}{4}, & \text{if } \dfrac{1}{2}K^{est}(3, t, \Delta t) - \dfrac{1}{4} > 0 \\ Var^{est}(t, \Delta t) = 0, & \text{else.} \end{cases}$$

where the formula for K^{est} is given in Definition 4. Let us recall that the estimation of the variance Var^{est} is not a priori the variance of the estimated kernel κ^{est}. We also define $SD^{est}(t, \Delta t) = \sqrt{Var^{est}(t, \Delta t)}$. In Figure 2, we assume $\gamma = \alpha = 1$, we consider the six kernels κ described in Table 1 and the initial condition is a peaked gaussian centered at $x = 2$. We plot the relative error on the standard deviation defined as

$$\text{Relative Error on the Standard Deviation} = \frac{|SD^{est}(0, \Delta t) - SD|}{SD}$$

as a function of Δt. We observe that for large values of Δt the relative error is saturated and equal to 1 for the kernels in blue, red and yellow, corresponding to kernels with small variances. For these kernels, the estimated variance becomes negative from a certain value for Δt, so that Var^{est} is then SD^{est} are taken to be zero. The worst estimation of the relative standard deviation we have is for the kernel in blue, *i.e.* for the kernel with a very small standard deviation (SD=0.1001): the estimation of the standard deviation SD^{est} is zero, and then the relative error is equal to 1. For $\alpha\Delta t = 0.1$, we are able to have a good idea of the ordering of standard deviations of the six kernels.

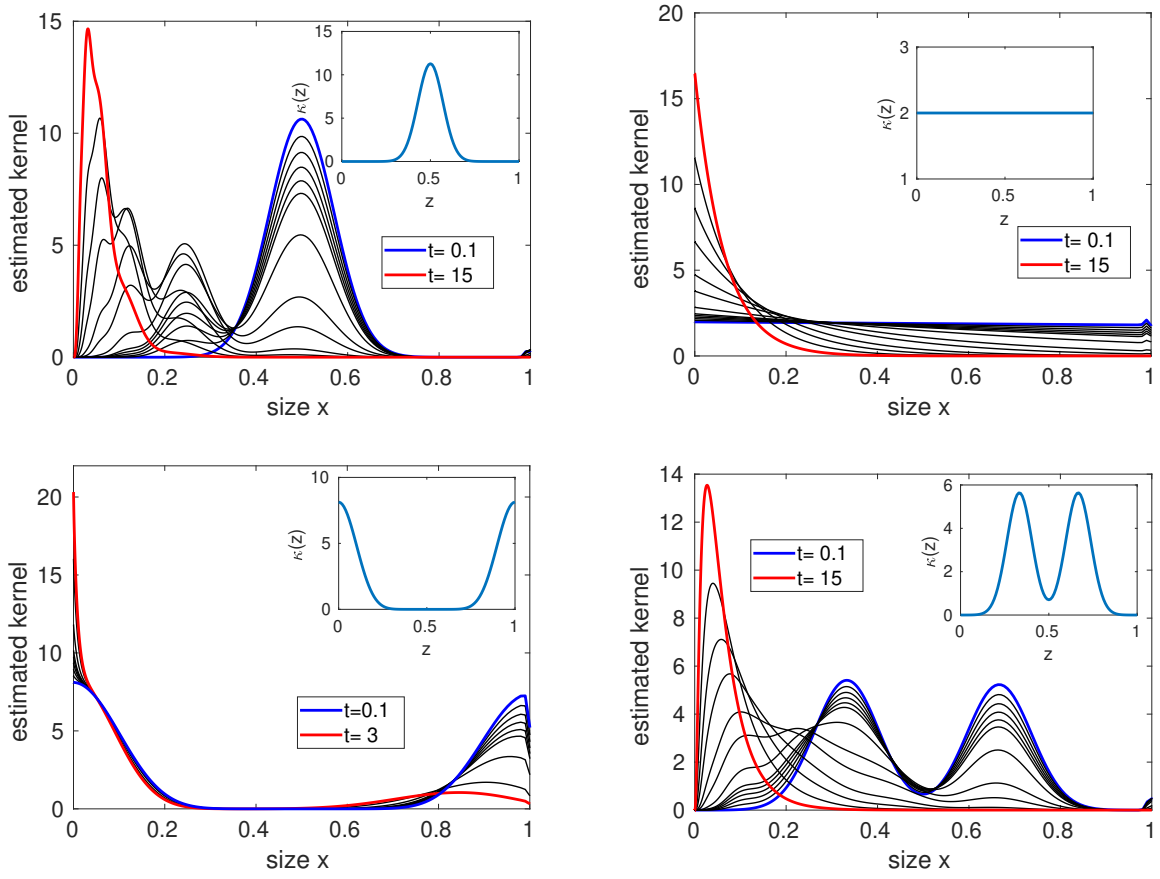


Figure 1: **Profile of the estimated kernel** $\kappa^{est}(t)$ for $\gamma = \alpha = 1$. Upper-right inset, light blue: the kernel κ. Blue: $\kappa^{est}(t)$ for $t = 0.1$. Red: $\kappa^{est}(t)$ for a large value of t. Black: $\kappa^{est}(t)$ for intermediate times t.

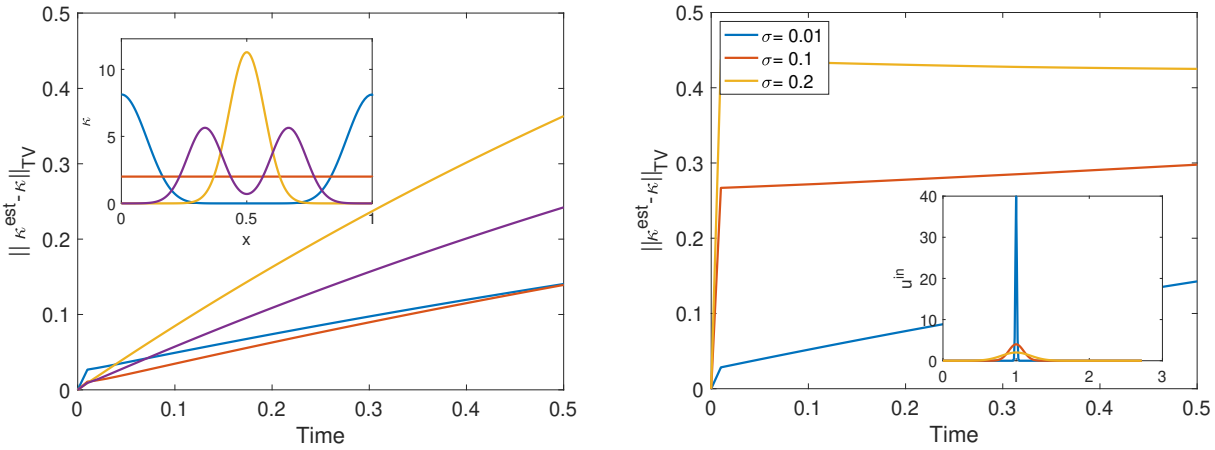


Figure 2: **Time evolution of the distance in the Total Variation norm between the fragmentation kernel and its first order estimate** given by κ^{est}, departing from $u_0 = \delta_1$ (Left) or departing from a Gaussian curve centered at $x = 1$ with a standard deviation $\sigma = 0.01$, $\sigma = 0.1$ and $\sigma = 0.2$ respectively (Right). Left: the corresponding kernel is displayed on the inset with the same colour as the error curve. Right: the fragmentation kernel is the one in Fig 1 bottom left (in blue on the inset of the left figure). The corresponding initial condition is displayed on the inset with the same colour as the error curve.

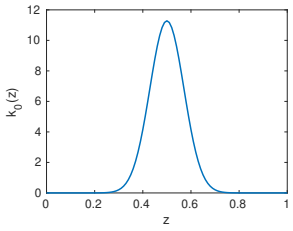
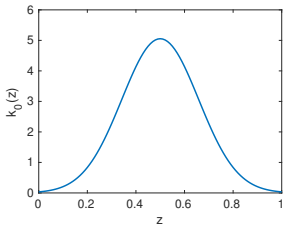
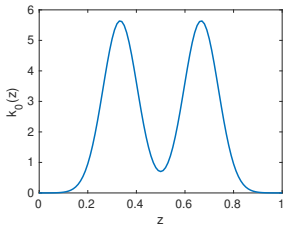
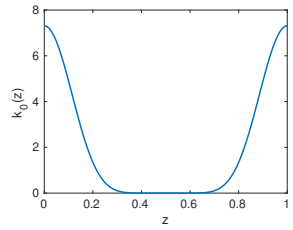
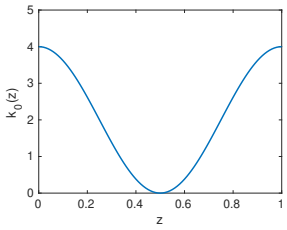
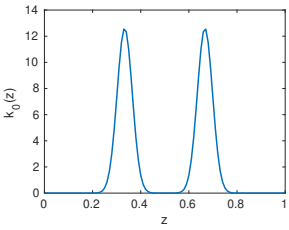
Kernel			
Var	0.0100	0.0295	0.0378
SD	0.1001	0.1718	0.1944
Kernel			
Var	0.1625	0.1290	0.0338
SD	0.4031	0.3591	0.1839

Table 1: Variance and Standard Deviation for 6 typical fragmentation kernels.

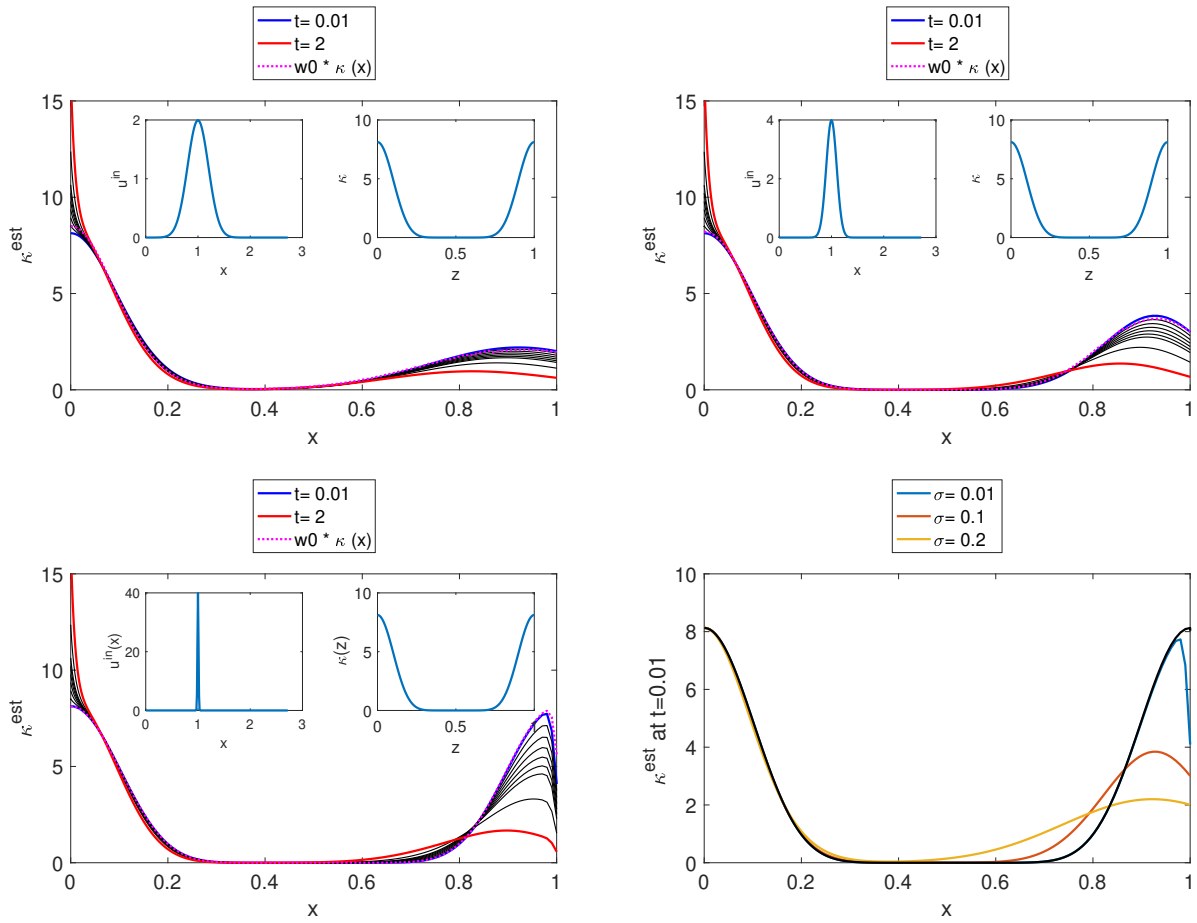


Figure 3: **Estimation of the fragmentation kernel** $\kappa^{est}(t)$ at various times t (black curves), first time point in blue, latest time point in red. Upper left: for an initial data μ_0 with variance $\sigma = 0.2$, Upper Right: with variance $\sigma = 0.1$, Bottom Left: with variance $\sigma = 0.01$. In dotted pink is what is truly estimated, namely the convolution $w_0 * \kappa$ (see notations of Corollary 3). Bottom Right: Superimposition of the three estimates $\kappa^{est}(t)$ at an early time point $t = 0.01$ $t = 0.01$ corresponding to the three initial conditions of respective variance $\sigma = 0.01$, $\sigma = 0.1$ and $\sigma = 0.2$.

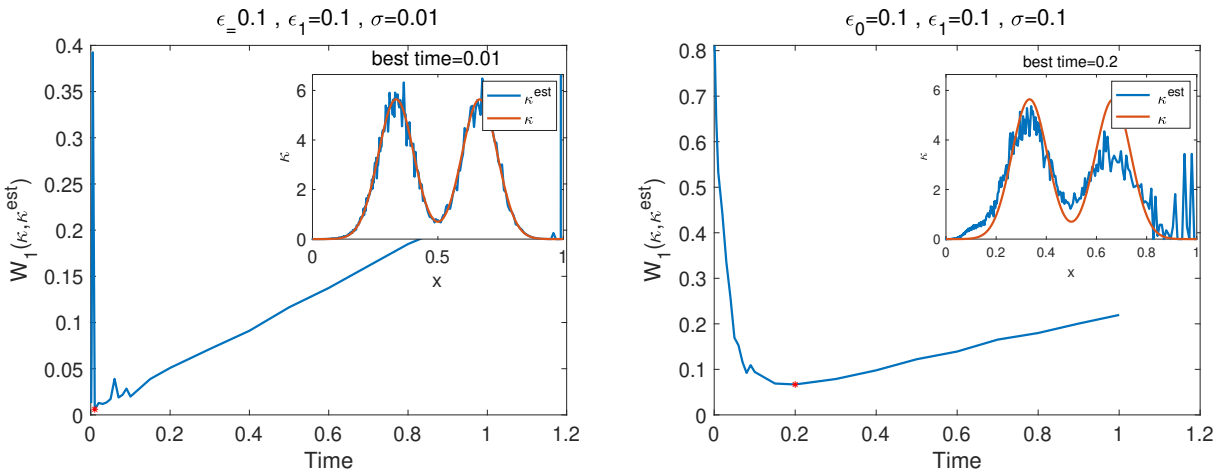


Figure 4: **Time evolution of the error estimate** in BL norm (approximated here by the W_1- distance) between the fragmentation kernel κ and its estimate $\kappa^{est}(t)$, for $\epsilon_0 = \epsilon_1 = 0.1$ and for $q = 0.01$ (Left), $q = 0.1$ (Right). The insets display the best estimate $\kappa^{est}(t_0)$, obtained at the timepoint t_0 where the W_1 distance is minimal.

References

[1] V. I. Agoshkov and P. B. Dubovski. Solution of the reconstruction problem of a source function in the coagulation-fragmentation equation. *Russian Journal of Numerical Analysis and Mathematical Modelling*, 17(4):319–330, 2002.

[2] O. Alomari and P.B. Dubovski. Recovery of the integral kernel in the kinetic fragmentation equation. *Inverse Problems in Science and Engineering*, 21(1):171–181, 2013.

[3] D. Balagué, J. Cañizo, and P. Gabriel. Fine asymptotics of profiles and relaxation to equilibrium for growth-fragmentation equations with variable drift rates. *Kinetic and related models*, 6(2):219–243, 2013.

[4] J. M. Ball and J. Carr. The discrete coagulation-fragmentation equations: Existence, uniqueness, and density conservation. *Journal of Statistical Physics*, 61(1-2):203–234, October 1990.

[5] J. Banasiak and L. Arlotti. *Perturbations of Positive Semigroups with Applications*. Springer Monographs in Mathematics. Springer London, 2006.

[6] J. Banasiak, W. Lamb, and P. Laurencot. *Analytic Methods for Coagulation-Fragmentation Models, I & II*. Chapman et Hall, 2019.

[7] V. Bansaye, B. Cloez, P. Gabriel, and A. Marguet. A non-conservative harris ergodic theorem. *Journal of the London Mathematical Society*, 106(3):2459–2510, 2022.

[8] J. Baumeister and A. Leitão. *Topics in inverse problems*. Publicações Matemáticas do IMPA. [IMPA Mathematical Publications]. Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2005. 25o Colóquio Brasileiro de Matemática. [25th Brazilian Mathematics Colloquium].

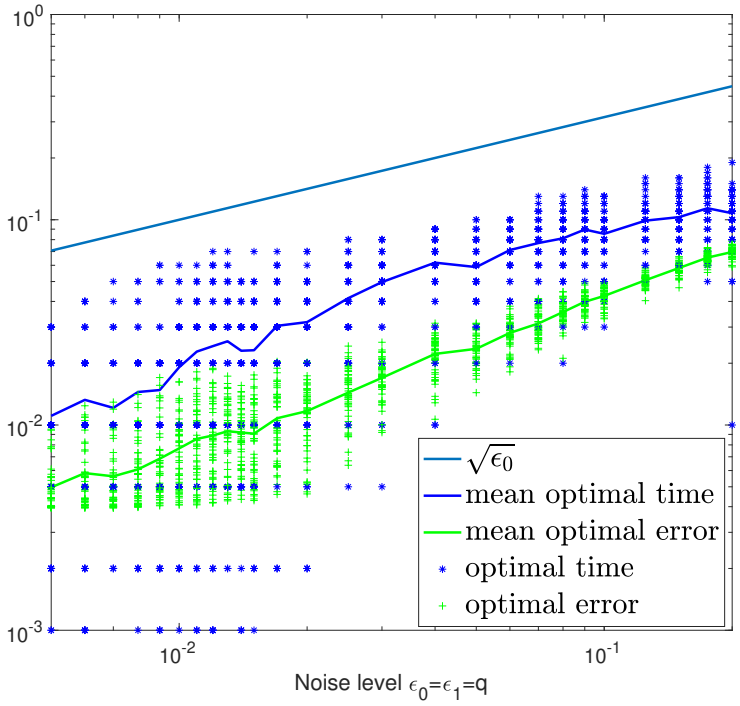
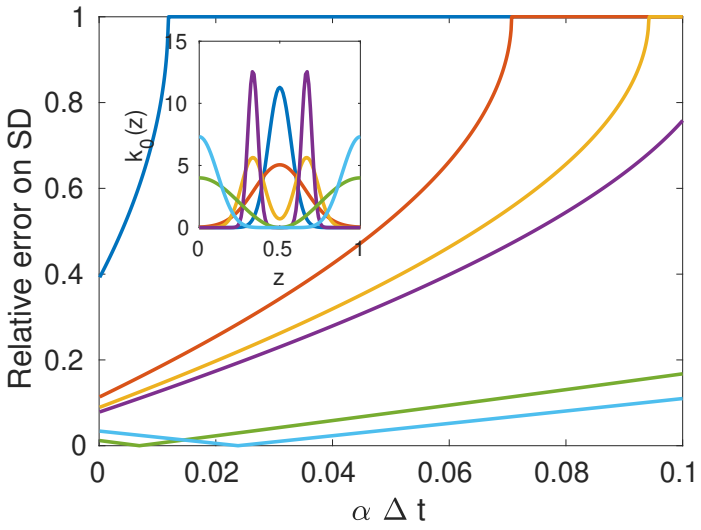


Figure 5: **Optimal error and optimal time with respect to the noise level.** We choose an equal level for the three noise sources ε_0, ε_1 and q (i.e. $q = \sigma$ the standard deviation of the initial gaussian), and for 50 simulations we draw the optimal time (blue asterisks) giving the optimal error (green asterisks). In plain lines we draw the mean over the 50 simulations, to be compared with the curve $\varepsilon \mapsto \sqrt{\varepsilon}$ (lighter blue plain line). We observe a good agreement with the expected rate of convergence.



$\alpha\Delta t = 0.02$		$\alpha\Delta t = 0.1$	
SD	SD^{est}	SD	SD^{est}
0.1001	0	0.1001	0
0.1718	0.1296	0.1718	0
0.1944	0.1574	0.1944	0
0.2065	0.1718	0.2065	0.0537
0.3591	0.3515	0.3591	0.2997
0.4031	0.4060	0.4031	0.3595

Table 2: Parameters: $\alpha = \gamma = 1$. Initial condition: a thin gaussian centered at $x = 2$. For the 6 kernels described in Table 1 and displayed in the inset, we plot the relative error on the standard deviation estimate as a function of $\alpha\Delta t$. We detail two cases: $\alpha\Delta t = 0.02$ and $\alpha\Delta t = 0.1$ on the table (Right)

[9] D. M. Beal, M. Tournus, R. Marchante, T. Purton, D. P. Smith, M. F. Tuite, M. Doumic, and W-F. Xue. The division of amyloid fibrils. *iScience*, 23(9), 2020.

[10] J. Bertoin. Eternal solutions to Smoluchowski's coagulation equation with additive kernel and their probabilistic interpretations. *The Annals of Applied Probability*, 12(2):547 – 564, 2002.

[11] J. Bertoin. The asymptotic behavior of fragmentation processes. *Journal of the European Mathematical Society*, 005:395–416, 2003.

[12] J. Bertoin. *Random Fragmentation and Coagulation Processes*, volume 102 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, August 2006.

[13] J. Bertoin and A. R. Watson. Probabilistic aspects of critical growth-fragmentation equations. *Advances in Applied Probability*, 48(A):37–61, 2016.

[14] M. J. Cáceres, J. A. Cañizo, and S. Mischler. Rate of convergence to the remarkable state for the self-similar fragmentation and growth-fragmentation equations. *Journal de Mathematiques Pures et Appliquees*, 96(4):334–362, 2011.

[15] H. Cao and S. V. Pereverzev. Natural linearization for the identification of a diffusion coefficient in a quasi-linear parabolic system from short-time observations. *Inverse Problems*, 22(6):2311, 2006.

[16] J.A. Carrillo, R.M. Colombo, P. Gwiazda, and A. Ulikowska. Structured populations, cell growth and measure valued balance laws. *Journal of Differential Equations*, 252(4):3245–3277, 2012.

[17] J. Dieudonné. *Calcul infinitésimal*. Hermann, Paris, 1968.

[18] M. Doumic, M. Escobedo, and M. Tournus. Estimating the division rate and kernel in the fragmentation equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 35(7):1847–1884, 2018.

[19] M. Doumic, M. Escobedo, M. Tournus, and W-F. Xue. Insights into the dynamic trajectories of protein filament division revealed by numerical investigation into the mathematical model of pure fragmentation. *Plos Computational Biology*, 2021.

[20] M. Doumic and B. van Brunt. Explicit solution and fine asymptotics for a critical growth-fragmentation equation. In *CIMPA School on Mathematical Models in Biology and Medicine*, volume 62 of *ESAIM Proc. Surveys*, pages 30–42. EDP Sci., Les Ulis, 2018.

[21] P. B. Dubovski and I. W. Stewart. Existence, uniqueness and mass conservation for the coagulation-fragmentation equation. *Mathematical Methods in The Applied Sciences*, 19:571–591, 1996.

[22] H.W. Engl, M. Hanke, and A. Neubauer. *Regularization of inverse problems*, volume 375 of *Mathematics and its Applications*. Springer, 1996.

[23] M. Escobedo, S. Mischler, and M. R. Ricard. On self-similarity and stationary problem for fragmentation and coagulation models. *Annales de l'institut Henri Poincaré (C) Analyse non linéaire*, 22(1):99–125, 2005.

[24] A. F. Filippov. On the distribution of the sizes of particles which undergo splitting. *Theory of Probability & Its Applications*, 6(3):275–294, 1961.

[25] E. Giné and R. Nickl. *Mathematical foundations of infinite-dimensional statistical models*. Cambridge Series in Statistical and Probabilistic Mathematics, [40]. Cambridge University Press, New York, 2016.

[26] B. Haas. Asymptotic behavior of solutions of the fragmentation equation with shattering: An approach via self-similar markov processes. *The Annals of Applied Probability*, page 382–429, 2010.

[27] L. G Hanin. An extension of the Kantorovich norm. *Monge Ampère equation: applications to geometry and optimization (Deerfield Beach, FL, 1997) Contemp. Math., 226, Amer. Math. Soc., Providence, RI, 1999*, pages 113–130.

[28] Kathrin Hellmuth, Christian Klingenberg, Qin Li, and Min Tang. Kinetic chemotaxis tumbling kernel determined from macroscopic quantities. *arXiv preprint arXiv:2206.01629*, 2022.

[29] V. H. Hoang, T.M. Pham Ngoc, V. Rivoirard, and V.C. Tran. Nonparametric estimation of the fragmentation kernel based on a PDE stationary distribution approximation. March 2019. Preprint.

[30] S. Honoré, F. Hubert, M. Tournus, and D. White. A growth-fragmentation approach for modeling microtubule dynamic instability. *Bull. Math. Biol.*, 81(3):722–758, 2019.

[31] O.Bilous K.J. Valentas and N.R. Amundson. Analysis of breakage in dispersed phase systems. *Ind. Eng. Chem. Fundamen*, 5:271–279, 1966.

[32] A. N. Kolmogorov. On the logarithmic normal distribution of particle sizes under grinding. *Dokl. Akad. Nauk. SSSR 31*, pages 99–101, 1941.

[33] M. Kostoglou and A.J. Karabelas. On the self-similar solution of fragmentation equation: Numerical evaluation with implications for the inverse problem. *Journal of Colloid and Interface Science*, 284:571–581, 2005.

[34] Zhiyuan Li, Xing Cheng, and Gongsheng Li. An inverse problem in time-fractional diffusion equations with nonlinear boundary condition. *Journal of Mathematical Physics*, 60(9):091502, 2019.

[35] D. J. McLaughlin, W. Lamb, and A. C. McBride. An existence and uniqueness result for a coagulation and multiple-fragmentation equation. *SIAM Journal on Mathematical Analysis*, 28(5):1173–1190, 1997.

[36] Z. A. Melzak. A scalar transport equation. *Trans. Amer. Math. Soc.*, 85:547–560, 1957.

[37] O. P. Misra and J. L. Lavoine. *Transform analysis of generalized functions*. North-Holland, 1986.

[38] G. Narsimhan, D. Ramkrishna, and J. P. Gupta. Analysis of drop size distributions in lean liquid-liquid dispersions. *AIChE Journal*, 26(6):991–1000, 1980.

[39] J. R. Norris. Smoluchowski's coagulation equation: uniqueness, nonuniqueness and a hydrodynamic limit for the stochastic coalescent. *The Annals of Applied Probability*, 9(1):78 – 109, 1999.

[40] B Perthame. *Transport equations in biology*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2007.

[41] Benoît Perthame and Lenya Ryzhik. Exponential decay for the fragmentation or cell-division equation. *Journal of Differential Equations*, 210(1):155–177, 2005.

[42] B. Piccoli and F. Rossi. Generalized Wasserstein distance and its application to transport equations with source. *Arch. Ration. Mech. Anal.*, 211(1):335–358, 2014.

[43] B Piccoli, F. Rossi, and M. Tournus. A wasserstein norm for signed measures, with application to non local transport equation with source term. *Preprint*, 2017.

[44] D. Ramkrishna. Drop-breakage in agitated liquid-liquid dispersions. *Chemical Engineering Science*, 29:987–992, 1974.

[45] I.W. Stewart. On the coagulation-fragmentation equation. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, 41(6):917–924, 1990.

[46] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[47] E. W. Montroll and R. Simha. Theory of depolymerization of long chain molecules. *The Journal of Chemical Physics*, 8:721–726, 09 1940.

[48] W-F Xue, S. W. Homans, and S. E. Radford. Amyloid fibril length distribution quantified by atomic force microscopy single-particle image analysis. *Protein engineering design selection PEDS*, 22(8):489–496, 2009.

[49] W-F Xue and S. E. Radford. An imaging and systems modeling approach to fibril breakage enables prediction of amyloid behavior. *Biophys. Journal*, 105:2811–2819, 2013.

[50] R. M. Ziff and E. D. McGrady. Kinetics of polymer degradation. *Macromolecules*, 19(10):2513–2519, 1986.

[51] R.M. Ziff and E. D. McGrady. The kinetics of cluster fragmentation and depolymerisation. *J. Phys. A: Math. Gen*, 18:3027–3037, 1985.