A general model of binary opinions updating

Alexis Poindron

A general model of binary opinions updating

Alexis Poindron*

October 6, 2020

Abstract

We generalise Grabisch and Rusinowska (2013) to non-conformist societies. Agents in a network are iteratively picking a yes/no opinion, where updating stems from mutual influence. We introduce a notion of groups based on the signs of influence. We examine a few canonical societies, namely, conformist, communitarian, with leaders, with anti-conformist agents. We investigate stability issues. Any kind of opinion updating model can be hosted by our formalism, provided that: (i) alternatives are binary; (ii) opinion adoption is reversible and independent among agents; (iii) the process is Markovian and stationary; (iv) the number of agents is finite; (v) time is discrete.

Key words: influence graphs, opinion dynamics, groups, stability, synchronism, asynchronism

JEL classification: C7, D7, D85

1 Introduction

We consider a society N of agents invited to express a yes/no opinion. The state of the world $S \subseteq N$ is defined as the set of agents saying yes. Agents update their opinions simultaneously (or synchronously) at each period of time, as follows: (i) each agent aggregates the opinions S of the society on the basis of the influences that are exerted on him, which influences can be positive, negative or null; (ii) the aggregate $p_i(S)$ is his probability to say yes at the next period. Dynamics of opinions are described by the transition graph $\mathbf{B}$. The aim of this paper is to relate the properties of $\mathbf{p} = (p_i)_{i \in N}$ with the dynamics obtained.

Prior to votes, where profiles are aggregated to produce a social preference (see, e.g., Arrow (1963), Suzumura (1983)), agents interact with each other during debates, discussions, advertisements, etc. This process, called opinion formation, is the one studied in the present paper. It is embedded into the literature on opinion formation, diffusion and dynamics in social networks, and into the literature of Ising-like models. This literature is particularly fertile in physics, where agents are particles and where, for this reason, the framework is typically anonymous and the approach statistic. In economics, however, the framework is often non-anonymous, as well as in biology, where Boolean networks are used to model neuronal activity or interactions between

*Université Paris 1 Panthéon Sorbonne. 106-112, boulevard de l'Hôpital, 75013 Paris, France. alexis.poindron@laposte.net. Declarations of interest: none.

genes and proteins. Grabisch and Rusinowska (2013) proposed a non-anonymous model of influence where agents are conformist, meaning that all influences are positive. However, in order to generate rich and various opinion dynamics, and therefore, to better match the dynamics of the real world, it is essential to allow for negative influence. In Grabisch et al. (2019a), a population is composed of a mixture of conformist and anti-conformist agents, in an anonymous framework though. As compared with Förster et al. (2013), an anonymous model of influence with conformist agents only, the presence of anti-conformist agents enormously enriches the landscape of dynamics. We now lack a model which would be both non-anonymous and with negative influence, not only anti-conformism but also as heterogeneous influences as desired: agents would be positively influenced by some neighbours, and negatively by some others.

The present paper fills this gap. Our contribution is mostly to clarify the mechanisms of a vast literature and propose a tractable and unifying formalism to serve as a basis for further investigations. Consistently with this intention, we evidence bridges between some canonical societies that have been studied separately in the literature, namely, conformist, with leaders, communitarian (anti-coordination) and mixed (with anti-conformist agents). In the vein of Grabisch and Rusinowska (2013), Förster et al. (2013) and Grabisch et al. (2019a), we investigate the reachable dynamics and evidence several channels of stability. Not surprisingly, coordination plays in favour of stability, but also anonymity and autonomy (independence from other agents opinions). We evidence similar dynamics for the conformist and the communitarian societies. The mixed society and societies with leaders have the same dynamics too. Besides these clarifying contributions, which pursue the concerns of the above-mentioned papers, our paper examines the utility foundations of the model and the updating scheme. In particular, we briefly discuss its asynchronous version, suggesting that the asynchronous scheme produces more intuitive dynamics.

The paper is structured as follows. Section 2 exposes the model. A summary is proposed in Section 2.9 to ease the reading. Section 3 discusses some particular restrictions on $\mathbf{p}$ that one might want to impose or not. Section 4 introduces a notion of groups based on the influence graph and examines a few canonical societies. Section 5 investigate stability issues. Related literature is relegated to Section 6. Section 7 concludes. Appendix A provides the proofs of the propositions. Appendix B relaxes the assumption of synchronous updating by investigating asynchronous and correlated updating. Appendix C discusses the utility foundations of the model, in particular, the myopic behaviour of agents. Appendix D provides algebraic considerations of an important tool of our model, along with further connections with Grabisch and Rusinowska (2013). A technical discussion concerning the connections between influences and opinion dynamics can be found in Appendix E.

2 Description of the model

We consider a set $N := \{1, \ldots, n\}$ of agents iteratively picking an opinion in a binary alternative set (yes or no, to adopt or not to adopt, being active or inactive, choose the first or the second candidate, accept or reject the null hypothesis, etc.). Each agent starts from an initial opinion and updates it due to mutual influence. Opinions are picked simultaneously[1] by agents on the basis of the current *state of the world* S, simply called *state*, being the set of agents whose opinion is yes. We use the following convention: 'Yes' is coded by 1 and 'No' by 0. There is no 'Maybe' opinion. When the set S must be explicited, we index agents by numbers. A common

[1]We relax this assumption in Appendix B.

convention ignores the curly brackets and commas. For example, $\{1,2,3\}$ can be written 123. The notations $S \in 2^N$ and $S \subseteq N$ are equivalent.

2.1 A few notations relative to sets

For two sets $A \subseteq B$ in 2^N, we write $[A,B] := \{S \mid A \subseteq S \subseteq B\}$; in particular, $[A,B]$ is empty if $A \nsubseteq B$. A collection of sets of this form is called an *interval*. The cardinality of a set is denoted by the corresponding lower case, e.g., $s = |S|$. The symmetric difference Δ on sets S_1 and S_2 is defined by $S_1 \Delta S_2 := (S_1 \cup S_2) \setminus (S_1 \cap S_2) = (S_1 \setminus S_2) \cup (S_2 \setminus S_1)$. The complement set of S is written S^c. If $\mathcal{P} := \{S_1, \ldots, S_k\}$ is a collection of sets, we write $\mathcal{P}^{(c)} := \{S_1^c, \ldots, S_k^c\}$ and $\mathcal{P}^c := 2^N \setminus \mathcal{P}$. We also write $\mathcal{P}\Delta Z := \{S_1 \Delta Z, \ldots S_k \Delta Z\}$.

2.2 The transition graph

Opinion formation is entirely described by the probability vector $\mathbf{p} := (p_i)_{i \in N}$ defined as follows. When the state of the society is S, the probability for agent i to say yes at the next period, that is, the probability that agent i belongs to the next state, is $p_i(S)$. The next state is therefore given by the realisation of n Bernoulli random variables of parameters $p_i(S)$, like in Asavathiratham (2000) and Grabisch and Rusinowska (2013).[2]

Given the state S of the society, the probability that the next state is T is written by $b_{S,T}$. Assuming that the process iterates, we obtain a stochastic process, called an *influence process*. The process is assumed to be: (i) Markovian, i.e, $b_{S,T}$ only depends on S and T, and not on the whole history of the process; (ii) stationary, i.e., time does not appear in the computation of $b_{S,T}$; in particular, $\mathbf{p}$ is fixed. If the probabilities $p_i(S)$ are independent among agents, as it is assumed in this paper, and since agents update at the same time, we have:

$$b_{S,T} = \prod_{i \in T} p_i(S) \prod_{i \notin T} (1 - p_i(S)). \tag{2.1}$$

We propose a qualitative description of the process described by the $2^n \times 2^n$ row-stochastic transition matrix $\mathbf{B} := [b_{S,T}]_{S,T \subseteq N}$; more precisely, we are interested in the absorbing classes and their type (for example, whether they are periodic or not). For this reason, we define the *reduced matrix* $\tilde{\mathbf{B}}$, where:

$$\tilde{b}_{S,T} = \begin{cases} 1 \text{ if } b_{S,T} > 0 \\ 0 \text{ if } b_{S,T} = 0 \end{cases} \tag{2.2}$$

When $b_{S,T} > 0$, we also write $S \to T$. If $b_{S,T} = 1$, we write $S \xrightarrow{1} T$. The reduced matrix can be equivalently represented by a *transition graph* $\tilde{\Gamma} = (2^N, E)$, where E is the set of directed arcs (S,T), and an arc exists if and only if $\tilde{b}_{S,T} = 1$. It is obtained from the *weighted transition graph* $\Gamma = (2^N, E, \mathbf{B})$ by ignoring the weights $b_{S,T}$ on the arcs.

The shape of transitions is given by Property 1 (see Appendix B for a more general exposure):

Property 1. All transitions are of the form $S \xrightarrow{1} [T, T \cup K]$.

[2]This approach allows to bypass a utility-based approach, by working directly with best responses. Let us make a few technical remarks. Firstly, we may think that S must be common knowledge, but actually we need not such a strong requirement: there is no impediment to encompassing the misobservations of agents into p_i. Note the difference of this approach with DeGroot (1974): the aggregate p_i gives a probability distribution of agent i opinions at the next period, whereas in the DeGroot model, the aggregate itself is the new opinion of the agent. Moreover, in the DeGroot model, updating is synchronous, while this assumption can be relaxed in our approach.

In (2.1), $\mathbf{B}$ is obtained from $\mathbf{p}$. Conversely, we can recover $\mathbf{p}$ from $\mathbf{B}$, as stated by the following property:

Property 2. $\mathbf{p}$ and $\mathbf{B}$ are isomorphic.

We will refer to the p_i as *aggregation functions*, that is, functions which aggregates a vector into a single number:

Definition 1. An *aggregation function* is a mapping from $\{0,1\}^n$ to $[0,1]$. An aggregation function taking values in $\{0,1\}$ is called *Boolean*.

Definition 1 is not standard: an aggregation function is usually assumed to be increasing with respect to the inclusion set order, e.g, in the voter model or in Grabisch and Rusinowska (2013). However, our model aims precisely at dealing also with negative influence; hence our more general definition. When needed, we will mention the vector of aggregation functions in superscript, e.g., $\Gamma^{\mathbf{p}}$. When all the p_i are Boolean, all the transitions are with probability 1; in this case, the transition graph and the society are said to be *deterministic*[3].

Remark 1. If $p_i(S) = 1 - p_i(S^c)$ for all S, agent i treats in the same way the 'yes' and 'no' opinions, i.e., it is *unbiased* toward the word 'yes' or the word 'no' (otherwise, it is called 'biased'). Assuming that aggregation functions are unbiased can be relevant to model elections for a candidate, but not to model adoption of a new technology, in which case there is an investment cost. Aggregation functions representing majority influence, for example, are unbiased.

2.3 Absorbing classes and dynamics

We first recall some basic notions of Markov chains. Let $\mathcal{C}$ be a non-empty collection of states. $\mathcal{C}$ is *strongly connected* if either it is of the form $\mathcal{C} = \{S\}$, or for every distinct $S, T \in \mathcal{C}$, there is a path in $\mathcal{C}$ from S to T. Moreover, $\mathcal{C}$ is a *class* if it is strongly connected and maximal for this property, i.e., if no super-collection of $\mathcal{C}$ is strongly connected. It is *absorbing* if there is no arc from some $S \in \mathcal{C}$ to some $T \notin \mathcal{C}$. An *absorbing state* S is one such that $S \xrightarrow{1} S$. An absorbing class is *periodic* if for some $k \geq 2$ there exists a partition $\{P_1, \ldots, P_k\}$ of $\mathcal{C}$ such that when the process is at a state belonging to P_i at time t, then it will be in a state of P_{i+1} at time $t+1$, with $P_{k+1} := P_1$. When each P_i reduces to a single state, we refer to the periodic class as a *cycle*.[4]

Theorem 1 below gives the shape of absorbing classes. This theorem does not involve any assumption on aggregation functions.

Theorem 1. (Grabisch and Rusinowska, 2013)
Consider an influence process based on aggregation functions $\mathbf{p}$. Absorbing classes are:

(i) either singletons (absorbing states) $\{S\}$, $S \in 2^N$,

(ii) or cycles: $S_1 \xrightarrow{1} S_2 \xrightarrow{1} \ldots \xrightarrow{1} S_k \xrightarrow{1} S_1$,

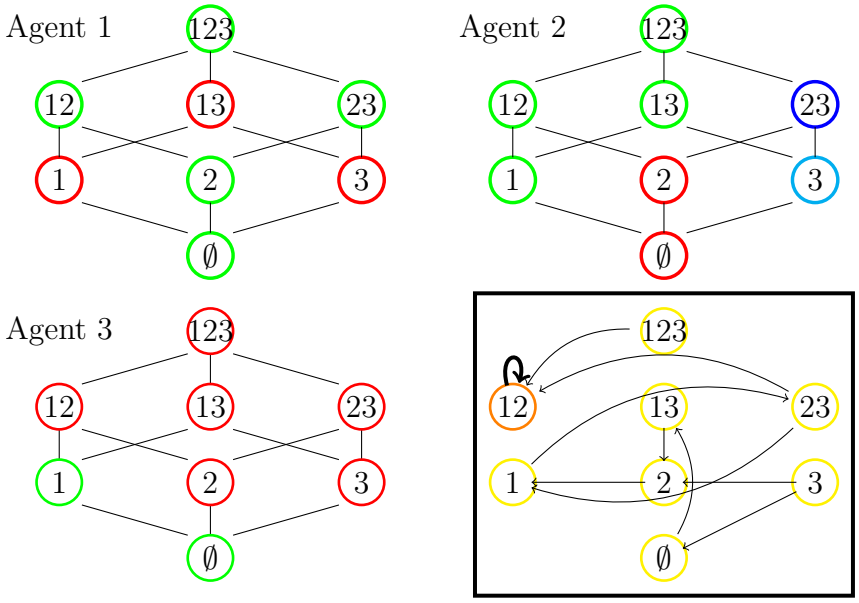
[3]In this case we recover the literature on Boolean networks. The next state T is obtained by operating a global transition function on the current state S: $T = f(S)$, as in Richard (2018). However, our Property 7 in Appendix B shows that a similar formalism can be produced even in non-deterministic frameworks.

[4]Some authors like Remy et al. (2008) refer to cycles as sequences of transitions, not necessarily sure ones, which we rather call "loops" (this notion of loops is used only in Appendix B). Our notion of cycles (always attractive, by definition) corresponds to their notion of attractive cycle. In order to make the distinction between cycles of states and cycles in the influence graph (Definition 5), some authors rather refer to latter ones as "circuits", but we are not investigating this notion in the current paper.

(iii) or collections $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_p$, where each collection $\mathcal{C}_j$ is an interval $[S_j, S_j \cup K_j]$ with at least one non-empty K_j.

When the society is deterministic, absorbing classes are states or cycles. Collections $\mathcal{C}^i$ in absorbing classes of the third kind are not necessarily pairwise disjoint, but they must be so if the transitions are sure from one $\mathcal{C}^i$ to another (periodic classes). In the sequel, we will refer to the *dynamics of a society* as being the kind of absorbing classes reached: (i), (ii) or (iii). The last category is actually extremely various; see, e.g., Grabisch et al. (2019a) where a finer classification is undertaken. Appendix B adapts Theorem 1 to the asynchronous framework.

Figures 1 and 2 show two examples of transition graphs $\tilde{\Gamma}$ (in yellow, at the bottom right) producing different kinds of dynamics in societies with three agents. Their p_i are represented with coloured lattices. In the transition graph, we represent in orange the states belonging to an absorbing class.



$p_i(S) = 1$ *at green states.* $p_i(S) = \frac{2}{3}$ *at blue states.*
$p_i(S) = \frac{1}{3}$ *at cyan states.* $p_i(S) = 0$ *at red states.*

Figure 1: From the aggregation functions to the transition graph (boxed). 12 is an absorbing state ($12 \xrightarrow{1} 12$). The probabilities of transitions can be computed from (2.1). For example: $b_{23,12} = 1 \cdot \frac{2}{3} \cdot (1 - 0) = \frac{2}{3}$. Monotonicity collections (Definition 3) are $\mathcal{B}_1 = \{2\}$, $\mathcal{B}_2 = \{123\}$ and $\mathcal{B}_3 = [\emptyset, 1]$, that is, Agent 2 is conformist and Agent 3 is anti-conformist (see Example 1). Agent 1 is neither conformist nor anti-conformist. p_1 and p_3 are Boolean.

At this stage, our concern is to know where the probabilities p_i come from. This concern is the subject of Section 2.5; but we need before to introduce a tool which is specific to our paper, concerned with encompassing negative influence. This being done, we will be in a position to wisely restrict the p_i, which in turn will allow us to pursue our investigations on the transition graph.

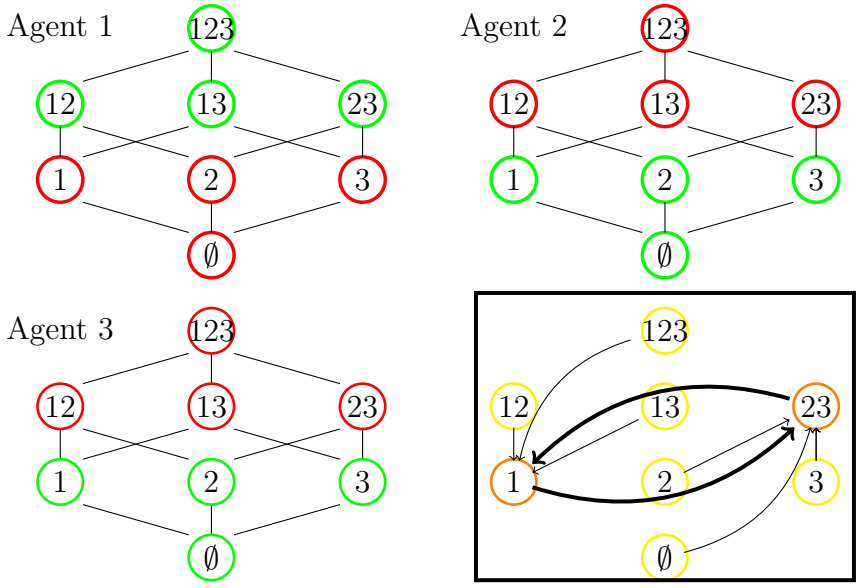


Figure 2: From the aggregation functions to the transition graph (boxed). $1 \xrightarrow{1} 23 \xrightarrow{1} 1$ is a cycle. All the p_i are Boolean; hence, all the transitions are deterministic. In this example, $\mathcal{B}_2 = \mathcal{B}_3 = \{\emptyset\}$ and $\mathcal{B}_1 = \{123\}$. Agent 1 is conformist and Agents 2 and 3 are anti-conformist. It is therefore a mixed society (see Section 4.2: $N_1 = 1$ and $N_2 = 23$) with distance-based aggregation functions (Section 3.1). In this example, the mixed society produces a cycle which is made of blocks (Section 4).

2.4 The partial order $\preccurlyeq_B$ and related notions

As compared with Grabisch and Rusinowska (2013), the main difficulty of our generalisation is not the negative influence per se, but rather the heterogeneity of influences received from one agent to another. Each agent aggregates differently the opinions of the society. In order to model such heterogeneity, we first need to introduce a collection of partial orders, in Definition 2.

Definition 2. (Partial order $\preccurlyeq_B$)
Let $S_1, S_2, B \in 2^N$. We define the $\preccurlyeq_B$ partial order (or simply the $\preccurlyeq_B$ order) on sets by:

$$S_1 \preccurlyeq_B S_2 \Leftrightarrow \begin{cases} S_1 \cap B \subseteq S_2 \cap B \\ S_1 \cap B^c \supseteq S_2 \cap B^c \end{cases} \tag{2.3}$$

and $S_1 \prec_B S_2$ if at least one of these two inclusions is strict.

That is, $S_1 \preccurlyeq_B S_2$ if S_1 is a subset of S_2 within B, and a superset of S_2 outside B (Figure 3). $\preccurlyeq_N$ is the usual set order $\subseteq$, while $\preccurlyeq_\emptyset$ is the reverse set order $\supseteq$. More details about the algebraic structures of $\preccurlyeq_B$ can be found in Appendix D.

2.5 Monotonicity collections

At this stage, we did not impose any constraint on $\mathbf{p}$. Our concern is now to impose meaningful restrictions to our very general framework. In order to lighten the exposure of the model, the

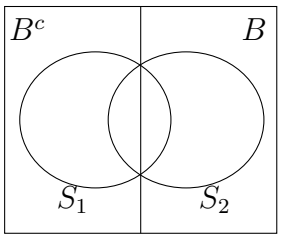


Figure 3: Illustration of $S_1 \preccurlyeq_B S_2$: S_2 is more engaged into B than S_1.

utility foundations of $\mathbf{p}$ have been relegated in Appendix C, which explains in particular why our model is myopic and how it does relate with the restrictions that we impose on $\mathbf{p}$ in this section.

Definition 3. (Monotonicity collection) The collection $\mathcal{B}_i := \{B \in 2^N \text{ such that } S_1 \preccurlyeq_B S_2 \Rightarrow p_i(S_1) \leq p_i(S_2)\}$ is called the *monotonicity collection of agent i*.

An agent i such that $B \in \mathcal{B}_i$ says yes with a higher probability when more agents in B say yes, and less agents outside B say yes. Let us see two important particular cases.

Example 1. (Conformism, Anti-conformism).

- (Conformism). An agent i is said to be *conformist* if $N \in \mathcal{B}_i$, that is, if for all S_1, S_2 with $S_1 \preccurlyeq_N S_2$ (i.e., $S_1 \subseteq S_2$), it holds that $p_i(S_1) \leq p_i(S_2)$ (i.e., p_i is non decreasing). Conformist agents are those who say "yes" when more agents say "yes" ("more" in the inclusion sense).
- (Anti-conformism). An agent i is said to be *anti-conformist* if $\emptyset \in \mathcal{B}_i$, that is, if for all S_1, S_2 with $S_1 \preccurlyeq_\emptyset S_2$ (i.e., $S_1 \supseteq S_2$), it holds that $p_i(S_1) \leq p_i(S_2)$ (i.e., p_i is non increasing). Anti-conformist agents are those who say "yes" when more agents say "no" ("more" in the inclusion sense).

Conformism is also called agoraphily and anti-conformism is also called "agoraphoby" or "counterconformism"[5]. Conformism can be justified, e.g., by benefits from coordination, like in Rohlfs (1974) and Katz and Shapiro (1985). An interesting example of anti-conformism is an attempt to beat the market, called 'contrarian investing' (Dreman, 2007). In Figure 2, Agents 2 and 3 are anti-conformist, while Agent 1 is conformist.

Example 2. (Monotonicity collections) In Figure 1, $\mathcal{B}_1 = \{2\}$ (though $p_1(123) = 1$, Agent 1 is not conformist!), $\mathcal{B}_2 = \{123\}$ (conformist agent) and $\mathcal{B}_3 = [\emptyset, 1]$ (anti-conformist agent). In Figure 2, $\mathcal{B}_2 = \mathcal{B}_3 = \{\emptyset\}$ and $\mathcal{B}_1 = \{123\}$. In Figure 4, $\mathcal{B}_i = [3, 123]$.

Definition 4. The *polarisation of* $\mathbf{p}$ (or the 'polarisation of the society') is the collection $(\mathcal{B}_i)_{i \in N}$.

A *conformist polarisation* is such that all agents in the society are conformists. This is the framework of Grabisch and Rusinowska (2013), to which we refer as being the "conformist model". A mixture of conformist and anti-conformist agents is call a *mixed polarisation*. As

[5]We sometimes meet the term "nonconformism", which is slightly more general since it may also refer to independence.

shown in Property 3, when $\mathcal{B}_i \neq \emptyset$, agent i can split all agents into two categories: his friends and his enemies[6], *independently on the state of the world* (no congestion, no satiation). The more of its friends say yes, the more inclined is agent i to say yes. Example 3 exposes some examples where $\mathcal{B}_i = \emptyset$, which situations will be dismissed by the subsequent Fundamental Assumption. Note that $\mathcal{B}_i = \emptyset$ (Example 3) is different from $\mathcal{B}_i = \{\emptyset\}$ (which represents a particular case of anti-conformism).

Property 3. The following statements are equivalent:

(i) $\mathcal{B}_i \neq \emptyset$.

(ii) For any $j \in N$, either $p_i(S \cup j) \geq p_i(S)$ for all $S \in 2^N$, or $p_i(S \cup j) \leq p_i(S)$ for all $S \in 2^N$.

Example 3. ($\mathcal{B}_i = \emptyset$: congestion, saturation)
Let $p_i(S) = s(n-s)$ for all S. Then for any j, $p_i(S \cup j) \geq p_i(S)$ for $s \leq \frac{n}{2} - 1$ and $p_i(S \cup j) \leq p_i(S)$ for $s \geq \frac{n}{2}$. This corresponds to situations like the frequentation of a restaurant. When there are not too many people in the restaurant, the dominant effect is the herd behaviour, since agents extract information on the quality of the restaurant from the number of clients. But when there are too many people, the congestion effect dominates. Another example is the following: agent i is such that $\mathcal{B}_i = \emptyset$ is an agent having a low self-esteem. It is positively influenced by himelf when a lot of people share his opinion and negatively influenced by himself when few people share his opinion. One can think also of diplomatic situations, where peers can be friends in some states of the world, and enemies in other states of the world. For example, mafias or nations which compete over small pieces of territory can unite against a common bigger enemy. Let us mention also the "mixed aggregation rule" (Grabisch et al., 2019a) where the agent randomises to pick an aggregation function before aggregating. Finally, the swarming rule describes the incentive for agents to cluster, though they must observe physical distances to avoid collisions.

In the light of the previous example, where situations exposed are quite subtle, it seems reasonable to impose the following Fundamental Assumption. Relaxing it would ask for a totally different mathematical treatment. The utility foundations of this assumption can be found in Appendix C.

Fundamental Assumption. $\mathcal{B}_i \neq \emptyset$ for all $i \in N$.

2.6 The influence index

Having imposed the non-emptiness of the $\mathcal{B}_i$ in the previous subsection, point (ii) of Property 3 becomes a property of our model. As a consequence, we are incentivised to introduce the influence index in Definition 5. Since all the terms $(p_i(S \cup j) - p_i(S))$ of the sum are of the same sign (in the large sense) and since only the sign of the sum is of interest for us, we do not need to put any coefficients in front of the terms.

[6]To fix ideas, we refer to interactions in terms of friendship and negative influence in terms of enmity. This terminology is informal and only for the purpose of easing the exposure. There are no moral connotations in these terms. For example, an anti-coordination games is one where agents exert negative influence on each other, though it can simply be for management purposes. Go shopping or not? Sharing the tasks is optimal: in a couple, if one of the two goes shopping, then the other one should not. One can think also of excludable goods. Also, in gene networks, inhibition is described by negative influence (Paulevé et al., 2020).

Definition 5. We define the *influence index* of j on i as follows:

$$\phi(j \to i) := \sum_{S \subseteq N \setminus j} (p_i(S \cup j) - p_i(S)). \tag{2.4}$$

If $\phi(j \to i) > 0$ (respectively $<$, $=$), we say that j has a *positive* (respectively *negative*, *null*) *influence* on i. The *influence graph* is $\mathcal{G} := (N, \mathcal{E})$ where $\mathcal{E}$ is the disjoint union of $\mathcal{E}^+$ and $\mathcal{E}^-$, respectively the set of positive influence arcs and the set of negative influence arcs, where there is a positive (resp. negative) influence arc from j to i if $\phi(j \to i) > 0$ (resp. < 0). An agent exerting a null influence on i is also called *irrelevant* for i.

An influence (positive or negative) is obtained when at least one $S \subseteq N \setminus j$ is such that $p_i(S \cup j) \neq p_i(S)$, but not necessarily all of them[7]. Figure 6 shows an example of influence graph.

2.7 Shape of the monotonicity collections

Theorem 2 below is used throughout the paper. It states that the collection $\mathcal{B}_i$ is an interval which can be expressed with the influence index. Agents in $\underline{B}_i$ exert a strict positive influence on i. Agents in $(\overline{B}_i)^c$ exert a strict negative influence on i. Agents in $\overline{B}_i \setminus \underline{B}_i$ are irrelevant for i. In other words, the baseline set of $\mathcal{B}_i$ is $\underline{B}_i$ and we can add some irrelevant agents to $\underline{B}_i$.

Theorem 2. $\mathcal{B}_i = [\underline{B}_i, \overline{B}_i]$, where:

$$\underline{B}_i := \bigcap_{B \in \mathcal{B}_i} B = \{j \in N \mid \phi(j \to i) > 0\} \tag{2.5}$$

$$\overline{B}_i := \bigcup_{B \in \mathcal{B}_i} B = \{j \in N \mid \phi(j \to i) \geq 0\} \tag{2.6}$$

Theorem 2 implies this important remark: the influence graph is entirely characterised by $\mathcal{B}_i$ and vice versa. This means that the structure of interactions is entirely contained in $\mathcal{B}_i$. Since $\mathcal{B}_i$ is very handy, we prefer in the sequel to work with $\mathcal{B}_i$. To illustrate this theorem, we can characterise conformism and anti-conformism (see Example 1) as follows:

$$i \text{ is a conformist agent} \Leftrightarrow N \in \mathcal{B}_i \Leftrightarrow \overline{B}_i = N \tag{2.7}$$

$$i \text{ is an anti-conformist agent} \Leftrightarrow \emptyset \in \mathcal{B}_i \Leftrightarrow \underline{B}_i = \emptyset \tag{2.8}$$

That is, an agent is conformist (resp. anti-conformist) if and only if no agent has a strictly negative (resp. strictly positive) influence on him.

The presence of irrelevant agents, that is, agents that we do not see, is the reason why our model is embedded into the literature on social networks. Since these agents play no role in the opinion formation of agent i, some flexibility is allowed in the choice of $B \in \mathcal{B}_i$. This flexibility, illustrated by Figure 4, is the key of our notion of groups introduced in Section 4.

Example 4. (Bipartite graphs) If the influence graph is bipartite, meaning that N is the disjoint union of some sets N_1 and N_2 such that $\phi(j \to i) \neq 0 \Rightarrow i \in N_1$ and $j \in N_2$ or $i \in N_2$ and $j \in N_1$, then: $\underline{B}_i \cup (\overline{B}_i)^c \subseteq N_2$ for all $i \in N_1$ and $\underline{B}_i \cup (\overline{B}_i)^c \subseteq N_1$ for all $i \in N_2$[8]. In particular if all agents are conformist: $\mathcal{B}_i \supseteq [N_2, N]$ for all $i \in N_1$ and $\mathcal{B}_i \supseteq [N_1, N]$ for all $i \in N_2$.

[7] This fact is at the root of the wall between the influence graph and the opinion dynamics, as discussed in Appendix E. This is why the literature on Boolean networks (with variations in the terminology) resorts to the notion of influence at a given state: see for instance Chevalier et al. (2019). We use a similar notion in Appendix B, namely the set of dissatisfied agents at a given set.

[8] Note that N_1 and N_2 are not necessarily groups in the sense of Section 4.

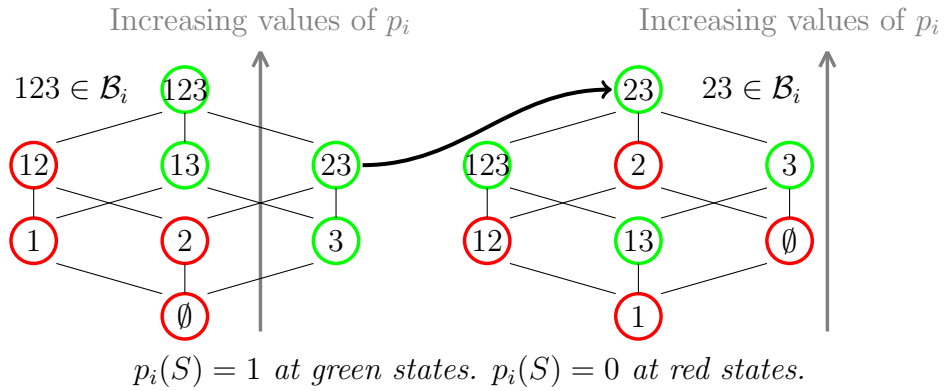


Figure 4: Theorem 2: from left to right, the lattice is simply rearranged. Any $B \in \mathcal{B}_i = [3,123]$ induces a partial order $\preccurlyeq_B$ for p_i. Agents in $\{1,2,3\} \setminus \{3\} = \{1,2\}$ are irrelevant on i; by removing or adding any irrelevant agent, p_i is left unchanged. Graphically, any path from the bottom to the top is going "from the red to the green".

2.8 Non-anonymous thresholds: the $\mathcal{P}^1$ and $\mathcal{P}^0$ collections

We end the general presentation of the model with the key tools to study the process qualitatively.

Definition 6. Let $i \in N$.

$$\mathcal{P}_i^1 := \{S \mid p_i(S) = 1\}. \tag{2.9}$$

$$\mathcal{P}_i^0 := \{S \mid p_i(S) = 0\}. \tag{2.10}$$

$$\mathcal{P}_i^* := 2^N \setminus (\mathcal{P}_i^0 \cup \mathcal{P}_i^1) = \{S \mid 0 < p_i(S) < 1\}. \tag{2.11}$$

Assumption BC ('boundary conditions') naturally completes the assumption on the non-emptiness of the $\mathcal{B}$: it equivalently says that the $\mathcal{P}^1$ and $\mathcal{P}^0$ collections are non-empty. It is not assumed by default in this paper; in particular, it was not required in Theorem 2.

Assumption BC. For any i and $B \in \mathcal{B}_i$:

- $p_i(B) = 1$,
- $p_i(B^c) = 0$.

The $\mathcal{P}^1$ and $\mathcal{P}^0$ collections, which are such that the agent i says yes or no for sure can be regarded as some kinds of 'non-anonymous thresholds'. We can check that p_i is Boolean if and only if $\mathcal{P}_i^* = \emptyset$ and that Boolean aggregation functions are entirely defined with $\mathcal{P}_i^1$ (or $\mathcal{P}_i^0$). Under Assumption BC, we have in particular $\mathcal{B}_i \subseteq \mathcal{P}_i^1$ and $\mathcal{B}_i^{(c)} \subseteq \mathcal{P}_i^0$. Moreover, in the case where aggregation functions are Boolean, equalities hold if and only if $\mathcal{P}_i^1$ is an interval (see Section 3.2 on boss sets for more details). One important remark to bear in mind is that $(\mathcal{P}_i^0, \mathcal{P}_i^1)_{i \in N}$ (not the polarisation $(\mathcal{B}_i)_{i \in N}$ of the society) is key to describing the opinion dynamics. This is the main insight of the following straightforward property.

Property 4. $\tilde{\mathbf{B}}$ and $(\mathcal{P}_i^0, \mathcal{P}_i^1)_{i \in N}$ are isomorphic.

It stems from Property 4 that $(\mathcal{P}_i^0, \mathcal{P}_i^1)_{i\in N}$ contain as much information as the transition graph. In particular, it suffices to compute the absorbing classes. The converse, however, is wrong. Figure 5 at the end of the summary subsection below helps visualising the articulations between these objects.

At this stage of the paper, let us summarise the notions introduced. The reader how would like to read more about the algebra behind the model is referred to Appendix D; he will find also more connections with Grabisch and Rusinowska (2013), along with more details about the transition and the influence graphs.

2.9 Summary of the notions introduced

The following summary can be read keeping an eye on Figure 5.

(0) The primitive of the model is $\mathbf{p}$. Utility foundations of $\mathbf{p}$ ($(u_i)_{i\in N}$ on Figure 5) are exposed in Appendix C.

(i) From $\mathbf{p}$, we extract the transition graph $\mathbf{B} = (b_{S,T})_{S,T\in 2^N}$:

$$b_{S,T} = \prod_{i\in T} p_i(S) \prod_{i\notin T} (1 - p_i(S)).$$

$\mathbf{p}$ and $\mathbf{B}$ are isomorphic.

(ii) Absorbing classes are obtained from $\tilde{\mathbf{B}}$.

(iii) We define the $\preccurlyeq_B$ order and the influence index ϕ:

$$\phi(j \to i) = \sum_{S\subseteq N\setminus j} [p_i(S \cup j) - p_i(S)].$$

(iv) The monotonicity collections $\mathcal{B}_i$ are extracted from the aggregation functions p_i:

$$B \in \mathcal{B}_i \text{ if and only if for any } S_1, S_2 \in 2^N,\ S_1 \preccurlyeq_B S_2 \Rightarrow p_i(S_1) \leq p_i(S_2).$$

We assume that $\mathcal{B}_i \neq \emptyset$ for all i (hence, all the terms in the expression of ϕ are of the same sign).

(v) We relate the interval $\mathcal{B}_i$ with the influence indexes: $\mathcal{B}_i = [\underline{B}_i, \overline{B}_i]$, where:

$$\underline{B}_i := \bigcap_{B\in\mathcal{B}_i} B = \{j \in N \mid \phi(j \to i) > 0\} \text{ and } \overline{B}_i := \underline{B}_i \cup K_i,$$

with K_i the set of irrelevant agents on i.

(vi) $(\mathcal{P}_i^0, \mathcal{P}_i^1)_{i\in N}$ are obtained from p_i:

$$\mathcal{P}_i^1 := \{S \mid p_i(S) = 1\}$$

and

$$\mathcal{P}_i^0 := \{S \mid p_i(S) = 0\}.$$

$\tilde{\mathbf{B}}$ and $(\mathcal{P}_i^0, \mathcal{P}_i^1)_{i\in N}$ are isomorphic.

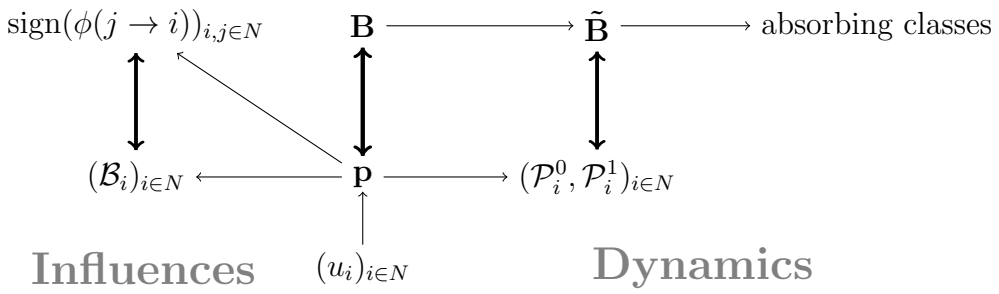


Figure 5: Summary and relations between the introduced notions.

It is essential to note that Figure 5 is composed of two hermetic sides. On the right, the dynamical side is described by absorbing classes and related notions; on the left, the influence side, where signs of influence and monotonicity collections appear; they are related by Theorem 2. In this paper, there will be no bridge between the two sides and indeed, surprisingly, we cannot characterise the dynamics of opinions with the influence graph (and vice versa). Only a statistical approach can break the wall, which will be done in a future paper; the reader is referred to Appendix E for more details on this issue. If the analysis is restricted to a purely algebraic approach then, besides $(\mathcal{P}_i^0, \mathcal{P}_i^1)_{i\in N}$, the appropriate notion to characterise the opinion dynamics is the one of influential coalitions, exposed in Appendix D.

Remark 2. That updating scheme (synchronous or asynchronous) does not affect the notions introduced. The only exception is (i). Indeed, the identity $b_{S,T} = \prod_{i\in T} p_i(S) \prod_{i\notin T}(1 - p_i(S))$ enlists all agents to update simultaneously. See Appendix B for more general updating schemes.

3 Some particular p

3.1 Distance-based aggregation functions and Generalised Weighted Means

One important class of aggregation functions is the anonymous ones. We say that p_i is *anonymous* when $p_i(S)$ can be written only with the cardinality of S, that is, when there exists a function q from $\mathbb{R}$ to $\mathbb{R}$ such that $p_i(S) = q(s)$ for all $S \in 2^N$ (in this case we rather write $p_i(S) \equiv p_i(s)$ by an abuse of notation). The distance-based aggregation functions generalise the notion of anonymous influence.

Definition 7. p_i is said to be *distance-based* if there exists a state T such that: $|S_1 \Delta T| = |S_2 \Delta T| \Rightarrow p_i(S_1) = p_i(S_2)$.

A distance-based aggregation function is anonymous if and only if the agent is conformist or anti-conformist; in both cases we can either take $T = N$ or $T = \emptyset$. The aggregation functions of Figure 2 are anonymous, and therefore distance-based. Let us introduce the *neighborhood* of T (of radius r) as follows:

$$\mathcal{V}^r(T) := \{S : |S\Delta T| \leq r\}. \tag{3.1}$$

Distance-based aggregation functions have $\mathcal{P}_i^0$ and $\mathcal{P}_i^1$ of the form $\mathcal{P}_i^1 = \mathcal{V}^{r_1}(T)$ and $\mathcal{P}_i^0 = \mathcal{V}^{r_0}(T^c)$. We generalise below the weighted means introduced in Grabisch and Rusinowska

(2013)[9] and we relate them with distance-based aggregation functions in Proposition 1.(3).

Definition 8. The family of *Generalised Weighted Means* (GWM) is defined by aggregation functions $p_i(S) = \frac{1}{2}\left(1 + f^{-1}\left(\sum_{j=1}^{n} w_i^j f(x_j)\right)\right)$ where f is a continuous automorphism on $[-1,1]$, $x_j = -1$ if $j \notin S$, $x_j = 1$ if $j \in S$ and $w_i^1, \ldots, w_i^n \in [-1,1]$, called the *weights*, are such that $\sum_{i=1}^{n} |w_i^j| = 1$. Such aggregation functions are called *GWM aggregation functions*, and a society where all agents aggregate according to a GWM function is called a *GWM society*.

The GWM aggregation function of an agent i is unbiased (see Remark 1) if and only if f is an odd function. Proposition 1 gives some precisions about the GWM aggregation functions.

Proposition 1. Assume that i is an agent who aggregates opinions according to a GWM aggregation function with weights $(w_i^j)_{j=1,\cdots,n}$. Then:

(1) $\mathcal{B}_i \neq \emptyset$ for all i and Assumption BC holds.

(2) $\phi(j \to i) > 0$ (resp. $<, = 0$) if and only if $w_i^j > 0$ (resp. $<, = 0$).

(3) p_i is a distance-based aggregation if and only if $|w_i^j| = \frac{1}{n}$ for all j.

3.2 Boss sets

Boss sets (Hu and Shapley, 2003a,b) are key to describing aggregation functions of agents exerting a strong discrimination in their opinion formation. They capture extremely non-anonymous influences; as such, they can be seen as the opposite of distance-based aggregation functions.

3.2.1 Definition

Definition 9. (Boss sets) Assume that p_i is Boolean. We say that K is a *boss set for* i if and only if i, K verify one of the following situations:

(i) $\mathcal{P}_i^1 = [K, N] = \uparrow K$.
i says yes for sure if and only if all agents of K say yes. In other words, $K \subseteq S \Leftrightarrow p_i(S) = 1$.

(ii) $\mathcal{P}_i^1 = [\emptyset, K] = \downarrow K$.
i says yes for sure if and only if no agent outside K says yes. In other words, $S \subseteq K \Leftrightarrow p_i(S) = 1$.

(iii) $\mathcal{P}_i^0 = [K, N] = \uparrow K$.
i says no for sure if and only if all agents of K say yes. In other words, $S \subseteq K \Leftrightarrow p_i(S) = 0$.

(iv) $\mathcal{P}_i^0 = [\emptyset, K] = \downarrow K$.
i says no for sure if and only if no agent outside K says yes. In other words, $K \subseteq S \Leftrightarrow p_i(S) = 0$.

If $K = \{j\}$, we say that j is a *boss for* i.

Example 5. (Star influence graph) Assume that the p_i are Boolean and non-constant, and that the influence graph is a star, meaning that there exists $i \in N$ such that $\phi(j \to k) \neq 0 \Rightarrow i \in \{j, k\}$. Then i is a boss for all agents $j \in N \setminus i$.

[9] Aracena et al. (2006) used a similar function, called the Heaviside step function H, to model opinion updating (gene activation/inhibition), with the difference that H is a 0-1 valued threshold function, which are of a special interest to model genes or nervous activity, while the GWM functions are built with an automorphism f.

3.2.2 Application of boss sets to time-varying influence graphs

Boss sets can be be used to model time-varying influence graphs. Let us assume to simplify that aggregation functions are Boolean and that all agents are conformist. We want to formalise the idea that when i gets linked to j, i.e., when $\phi(i \to j)$ gets strictly positive, then j becomes a bit closer to taking i as a boss.

Let $p_1, \ldots, p_m$ be m aggregation functions. We define the convex hull of $\{p_1, \ldots, p_m\}$, denoted by $Conv(p_1, \ldots, p_m)$, as the collection of aggregation functions p verifying the following condition: there exists no $S \in 2^N$ such that $p_1(S) = \ldots = p_m(S) \neq p(S)$. Now, let p_j be the aggregation function of agent j. If j gets connected to agent i (receives influence from him), then it should be that its new aggregation function p'_j is such that $p'_j \in Conv(p_j, p_{(i)})$, where $p_{(i)}$ is such that $\mathcal{P}^1 =\uparrow i$. The loss of a link from agent i would be formalised by $p'_j \in Conv(p_j, p_{(i)^c})$, where $p_{(i)^c}$ is such that $\mathcal{P}^1 =\downarrow (N \setminus i)$.

This is not the only way of modeling a new or a lost link. Instead of adopting the *opinion* of agent i, i.e., being closer to adopting i as a boss, one could imagine that agent j adopts the *behaviour* of agent i, that is, agent j gets closer to adopting the aggregation function of agent i: $p'_j \in Conv(p_j, p_i)$. The mere comparison of the two approaches would deserve a whole paper. More generally, mimicry is not only adopting an opinion, but also the behaviour that produces this opinion. One might argue that there is no reason why we should treat the two aspects in a different manner and, in some situations, it not even clear whether we can actually distinguish them. This is the case for example when the opinion is choosing an influence graph of friendships: not only agents choose their opinion, but also they choose who will influence them: in this case (and in this case only) sharing the same behaviour is some kind of guarantee to share the same beliefs. Therefore, the aggregation function should be endogenous in the following sense: that the choice of p_i and the choice of the opinion should obey to the same rules. This suggests a model of influence where we would restrict our analysis to a collection of exogenous aggregation functions, as many as opinions. In our binary opinions models, at each step, not only agents would revise their opinion, but also they would revise their aggregation function, picking it from the alternative (p_A, p_B), using the *same* aggregation function at the current period.

3.3 Discussion on the self-influence

A natural idea is that agents positively weigh their own opinion, i.e., they "agree with themselves". Assumption SI formalises this idea, but is not assumed by default in this paper.

Assumption SI: $i \in \underline{B}_i$ for all i.

Assumption SI means that all agents exert a strict positive influence on themselves. Equivalently (by Property 3), for all i, there exists $S \subseteq N \setminus i$ such that $p_i(S) < p_i(S \cup i)$[10]. Though Assumption SI sounds very natural, it might be relevant, in some contexts, to assume that $\phi(i \to i) < 0$. Here are a few examples.

- Like in SIR models, 'saying yes' can be seen as 'being infected'. Stating $\phi(i \to i) < 0$ allows to model a recovery from the disease. More generally, it can model an excitation which calls for a subsequent inhibition in a biologic cycle.

[10]One might prefer, as a weaker assumption, to state Assumption SI as $i \in \overline{B}_i$ for all i (agents do not have a strict negative influence on themselves), but our formulation is easier to work with in Proposition 3.

- 'Saying yes' can be 'passing on a piece of information'. If there is a cost for passing the information, then the agent will convey the information for one or a few periods, until enough people received it. Then, he will stop.
- From a psychological point of view, a vote can be the acquisition of an object, and failure of Assumption SI would model, either a state-varying passive utility function[11], or an active utility function[12] expressing the idea that utility for a given object decreases once we acquire it, that is, that desire is exacerbated by privation. For the anthropological foundations of this idea, see Girard (1966), Girard (1977).
- From a managerial point of view, a vote can be a check of the quality of a product in the manufacturing process. An agent probing the product at time t is less inclined to probe it again at time $t+1$ since he gains from spreading his checks over time rather than concentrating them on a contained period.
- The mixed and the leadership model in Section 4.2 are other examples where Assumption SI does not hold.

We expect Assumption SI to be involved in stability issues, which are of particular interest for us. This is the case indeed; see Proposition 3 in Section 5.

4 Behaviour groups: a non-topological approach to distance in networks

We tend to think of distances in networks in terms of topology: distance between nodes, cliques, conventions (Jackson and Storms, 2018), etc. But distance can also be regarded on a behavioural point of view: agents are close when they similarly react to their environment. In large communitarian groups, e.g., cultural, political, religious, etc. agents have similar aggregation functions though they might not know each others and might not even share the same friends. We expose in this section a non-topological notion of groups that brings together agents who are 'on the same wavelength', that is, whose opinions tend to move comonotonically.

4.1 Definition, characterisation, particular cases

Definition 10. Let $(N_1, \dots, N_g)$ be a partition of N into g blocks of agents. These blocks are called *behaviour groups*, or simply *groups*, if $\bigcap_{i \in N_k} \mathcal{B}_i \neq \emptyset$ for all $k \in \{1, \dots, g\}$. We write $N_{(i)}$ for the group that agent i belongs to.

Example 6. $\mathcal{B}_1 = [13, 123]$; $\mathcal{B}_2 = [12, 123]$. Then, $\mathcal{B}_1 \cap \mathcal{B}_2 = \{123\}$, which is non-empty: hence, agents 1 and 2 can belong to the same group.

Members of a group need not influence each other but are similar in respect to who influences them. Agents belong to the same group if their aggregation functions can be polarised in the same direction, meaning that there exists a common $\preccurlyeq_B$ for all agents of the group. A partition into groups always exists since $\{\{1\}, \dots, \{n\}\}$ is a possible one, but it might be not unique, as Example 7 shows:

[11] See Appendix C. The vector $(u_i(S))_{S \in 2^N}$ *itself* depends on the current state of the world (no stationarity anymore).
[12] See Appendix C.

Example 7. (Non-uniqueness of the partition) Assume that $N = \{1, 2, 3\}$. Let $\mathcal{B}_1 := [\emptyset, 1]$, $\mathcal{B}_2 := [1, 123]$ and $\mathcal{B}_3 := [23, 123]$. We can partition N into $\{12, 3\}$ or into $\{1, 23\}$. Irrelevant agents are the key of the non-uniqueness of the partition. This flexibility is arguably a strength of our definition, since it allows to incorporate additional criteria, e.g., topological ones, to choose among several partitions.

It is important to note that agents of the same group do not necessarily exert a positive influence on each other (nor any kind of influence). Indeed, in Example 7, $\phi(2 \to 1) < 0$. However, $\{12, 3\}$ is a possible partition into groups. Figure 6 helps visualising the notion of groups: agents of the same group receive the same influences in the large sense: the only forbidden situation for two agents a, b of the same group is $\phi(c \to a) > 0$ and $\phi(c \to b) < 0$ for some $c \in N$, as stated by Property 5:

Property 5. (Characterisation of groups) A partition $(N_1, \dots, N_g)$ is a possible partition into groups if and only if for any a, b belonging to any same block N_k and for any $c \in N$, $\phi(c \to a) > 0 \Leftrightarrow \neg(\phi(c \to b) < 0)$.

Property 5 can be seen as a "no-veto" characterisation of groups: a friend of an agent of the group cannot be the enemy of another. This being imposed, a flexible space remains in the notion of groups, thanks to the irrelevant agents. Indeed, the non-emptiness of $\bigcap_{i \in N_k} \mathcal{B}_i$ in Definition 10 does not impose that agents in the same group N_k should be *strictly* influenced by the same people, as the next example shows.

Example 8. Assume that $B = \{a, b, c, d\} \in \mathcal{B}_1 \cap \mathcal{B}_2$. The nonemptiness of $\mathcal{B}_1 \cap \mathcal{B}_2$ implies that agents 1 and 2 can form a group together. Agents a, b, c, d (and only them) are exerting a positive or null influence on agents 1 and 2. It could be that agents b, c and d are irrelevant on 1 (only a exerting a strict positive influence on him) and that agent a is irrelevant on 2 (only b, c and d exerting a strict positive influence on him). In this case, Agents 1 and 2 would belong to the same group while not sharing the same friends.

The following constraint, not assumed by default in this paper, requires that no coarser partition exists. It is very natural, but in some circumstances, it is convenient not to impose it.

Assumption C: For all $j, k \in \{1, \dots, g\}$, $\bigcap_{i \in N_j \cup N_k} \mathcal{B}_i = \emptyset$.

Definition 11. Let $N = N_1 \cup \ldots \cup N_g$ be a partition into groups of N. We say that a state S is *made of blocks* if there exists $J \subseteq [1, g]$, called the *indices of* S, such that S is a union of groups: $S = \bigcup_{j \in J} N_j$. We say that a class $\mathcal{C}$ is *made of blocks* if $\mathcal{C}$ is a singleton made of blocks, or a cycle where each state is made of blocks, or a union of intervals where each S_j, K_j is made of blocks (with the notations of Theorem 1.(iii)).

Recall that Theorem 2 states that $\mathcal{B}_i$ is an interval, i.e., of the form $[\underline{B}_i, \overline{B}_i]$. Assumption BA ("blocks assumption") states that the set of agents influencing a given agent is made of blocks. This means that all agents have perceptions of groups as a whole (it can be interpreted as a society where everyone exerts a gender, racial, etc. discrimination). It is not assumed by default in this paper. See Figure 6 for an influence graph where Assumption BA does not hold.

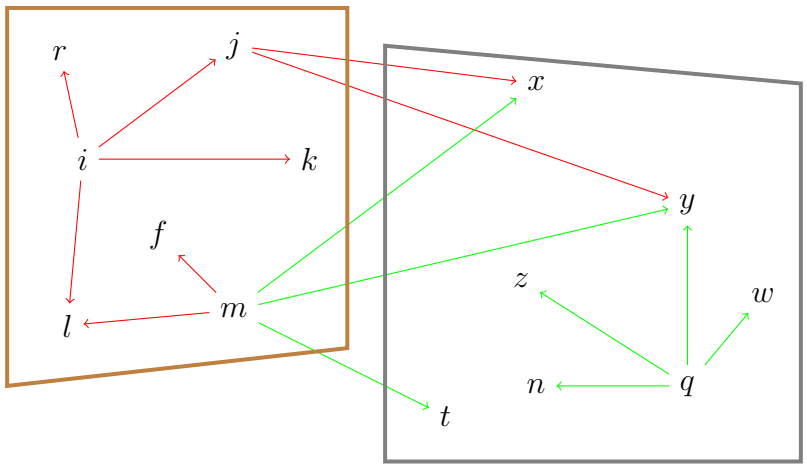


Figure 6: Influence graph. One group boxed in brown, the other group in grey. Positive (rep. negative) arrows of the influence graph are represented in green (rep. red) . Assumption BA does not hold, since x and y, which belong to the same group, receive positive influence from some agents of the brown group (m) and negative influence from some others (j). Agents can a priori exert negative influence on agents of their own group (red arrows within the brown group). The topology of the influence graph plays no role in the definition of groups. For example, neither t nor x is linked with any agent of his group, while they receive influences agents of the other group.

Assumption BA: There exists a partition into groups such that for all $i = 1, \dots, n$, $\underline{B}_i$ and $\overline{B}_i$ are made of blocks.

It can be shown that under Assumptions SI and BA, agents of the same group exert non-negative influences on each other[13]. A particular case is the one of communitarian groups, i.e., communities of agents who not only have a positive perception of agents from their group, but also a negative perception from agents outside. Example 9 introduces this notion.

Example 9. A *communitarian group* is a group N' such that $N' \in \mathcal{B}_i$ for all $i \in N'$. When all groups are communitarian, we say that the society is *communitarian*. The agent says yes when the state of the society is 'not too far' from his group. The particular case of two communitarian groups is referred to in the literature as an 'anti-coordination model'[14].

As we mentioned in introduction, and as we can see from the definition of groups, the topology of the influence graph plays no role. Appendix B provides a notion of correlated updating which hits two birds with one stone: (i) it extends our model to correlated updating (which relaxes the synchronous updating assumption); (ii) it restitutes a role to the topology by arguing that agents who closely interact with each other should update their opinion at the same time.

[13]Imposing these assumptions gives a more natural notion of groups, of agents as 'friends'.

[14]Bramoullé et al. (2004) present the first anti-coordination model embedding this game in a network formation game. Links are costly and the cost is the same across all players. On the contrary, in López-Pintado (2009), the cost of forming links is shared among individuals. Finally, Bramoullé (2007) compares the network effects on coordination games and anti-coordination games.

4.2 Analysis of the case $g = 2$

This subsection is devoted to the analysis of the case of two groups N_1 and N_2. It is convenient not to impose Assumption C here. However we impose Assumption BA. By a slight abuse of notation, we denote:

$$\overline{B}_1 := \bigcap_{i \in N_1} \overline{B}_i \tag{4.1}$$

$$\overline{B}_2 := \bigcap_{i \in N_2} \overline{B}_i. \tag{4.2}$$

Up to relabelling, four among the sixteen possible types of polarisations call for particular interest: the conformist, mixed, leadership and communitarian polarisations, which we define in Table 1. Under Assumption BC, some important transitions are given in Table 2.

Conformist polarisation C $N \in \mathcal{B}_i$ for all $i \in N$	Mixed polarisation M_1 $N \in \mathcal{B}_i$ for $i \in N_1$ $\emptyset \in \mathcal{B}_i$ for $i \in N_2$
Leadership polarisation L_1 $N_1 \in \mathcal{B}_i$ for all $i \in N$	Communitarian polarisation R_1 $N_1 \in \mathcal{B}_i$ for $i \in N_1$ $N_2 \in \mathcal{B}_i$ for $i \in N_2$

Table 1: The C, M_1, L_1 and R_1 polarisations.

Conformist polarisation C $\emptyset \xrightarrow{1} \emptyset$ $N \xrightarrow{1} N$	Mixed polarisation M_1 $\emptyset \xrightarrow{1} N_2$ $N \xrightarrow{1} N_1$
Leadership polarisation L_1 $N_2 \xrightarrow{1} \emptyset$ $N_1 \xrightarrow{1} N$	Communitarian polarisation R_1 $N_1 \xrightarrow{1} N_1$ $N_2 \xrightarrow{1} N_2$

Table 2: Some important transitions (Assumption BC must hold)

(C) *Conformist polarisations* have already been defined in the sequel of Example 1. It is such that $N \in \mathcal{B}_i$ for all $i \in N$.

(R_1) *Communitarian polarisations* have been defined in Example 9. The case of two groups translates into $N_1 \subseteq \overline{B}_1$, $N_2 \subseteq \overline{B}_2$, $N_2 \subseteq \overline{B}_1^c$ and $N_1 \subseteq \overline{B}_2^c$. Influences are reciprocal in the large sense: $\phi(i \to j) \geq 0 \Leftrightarrow \phi(j \to i) \geq 0$ and $\phi(i \to j) \leq 0 \Leftrightarrow \phi(j \to i) \leq 0$. We note that $\mathcal{E} \cap N_1 \subseteq \mathcal{E}^+$, $\mathcal{E} \cap N_2 \subseteq \mathcal{E}^+$ and that $\phi(i \to j) < 0$ implies that i, j belong to different groups. In particular, the number of negative arcs in a cycle of the influence graph must be even; hence the influence graph is a balanced signed graph in the sense of Harary (1953). The two groups play an anti-coordination game. In accordance with Altafini (2012, 2013), showing that balanced influence graphs result in opinion polarisation, we evidence that this configuration is involved in conditions of stability (Section 5).

(M_1) *Mixed polarisations.*

It holds that $N_1 \subseteq \overline{B}_1$, $N_2 \subseteq \overline{B}_1$, $N_1 \subseteq \overline{B}_2^c$ and $N_2 \subseteq \overline{B}_2^c$. A mixed polarisation is one where agents in N_1 are conformists ($\overline{B}_1 = N$) and agents in N_2 are anti-conformists ($\underline{B}_2 = \emptyset$). Grabisch et al. (2019a) investigates the dynamics of mixed polarisations.

(L_1) *Leadership polarisations.*

It holds that $N_1 \subseteq \overline{B}_1$, $N_2 \subseteq \overline{B}_1^c$, $N_1 \subseteq \overline{B}_2$ and $N_2 \subseteq \overline{B}_2^c$. Hence, $\overline{B}_1 = \overline{B}_2 = N_1$. Thus, $\bigcap_{i \in N} \mathcal{B}_i \neq \emptyset$: a partition into a single group is possible. Taking $B \in \bigcap_{i \in N} \mathcal{B}_i$, the society can be split into B and B^c: those who are unanimously followed (B), those who are unanimously disesteemed, including by themselves (B^c). Writing $\bigcap_{i \in N} \mathcal{B}_i = [\underline{B}, \overline{B}]$, where $\underline{B} := \bigcup_{i=1}^{n} \underline{B}_i$ and $\overline{B} := \bigcap_{i=1}^{n} \overline{B}_i$, we have $\underline{B} \cap \bigcup_{i=1}^{n} \overline{B}_i^c = \emptyset$ and $\overline{B}^c \cap \bigcup_{i=1}^{n} \underline{B}_i = \emptyset$, from which we infer that agents in $\underline{B}$ are weighted strictly negatively by no agents, and agents outside $\overline{B}$ are weighted strictly positively by no agents.

Polarisations M_2, L_2 and R_2 are defined by exchanging the roles of N_1 and N_2. For M_2 and L_2, this is a mere relabelling, but R_2 is of a new kind of society, which we may call 'anti-communitarianism': agents weight positively agents outside their group, and negatively agents of their group.

We now investigate isomorphisms between the four important classes of polarisations, C, M, R and L. To do this, we introduce the following transformations on the weighted transition graphs:

Definition 12. Consider the transition $S \xrightarrow{p} T$ and $Z \subseteq N$.

- The *Z-right transformation* of the transition $S \xrightarrow{p} T$ is defined by $S \xrightarrow{p} T\Delta Z$.
- The *Z-left transformation* of the transition $S \xrightarrow{p} T$ is defined by $S\Delta Z \xrightarrow{p} T$.
- The *Z-left-right transformation* of the transition $S \xrightarrow{p} T$ is defined by $S\Delta Z \xrightarrow{p} T\Delta Z$.

A vector of aggregation functions **p'** is said to be obtained from **p** by a right/left/left-right transformation when a right/left/left-right transformation is operated on all transitions of $\Gamma^{\mathbf{p}}$ to produce $\Gamma^{\mathbf{p'}}$. The Z-left-right transformation is a graph homomorphism (but the Z-left and the Z-right are not). A consequence of this fact is stated in Proposition 2.

We now expose three related propositions in a row. Theorem 3 expresses the monotonicity collections obtained from the Δ transformations. Corollary 1 applies this theorem to the four particular polarisations we are particularly interested in. Finally, Proposition 2 focuses on the absorbing classes obtained from a left-right transformation; in particular, switching from the absorbing classes of C to the ones of R, or from the ones of M to the ones of L (and vice versa), is extremely simple.

Theorem 3. (Transformations on transition graphs and $\mathcal{B}$)

Assume that **p'** is obtained from **p** by :

(i) a Z-left transformation.

Then, $\mathcal{B}_i^{\mathbf{p'}} = \mathcal{B}_i^{\mathbf{p}}\Delta Z$ for all $i \in N$.

(ii) a Z-right transformation.
Then: $\mathcal{B}_i^{\mathbf{p}'} = \mathcal{B}_i^{\mathbf{p}}$ for all $i \notin Z$, and $\mathcal{B}_i^{\mathbf{p}'} = (\mathcal{B}_i^{\mathbf{p}})^{(c)}$ for all $i \in Z$.

(iii) a Z-left-right transformation.
Then: $\mathcal{B}_i^{\mathbf{p}'} = \mathcal{B}_i^{\mathbf{p}} \Delta Z$ for all $i \notin Z$, and $\mathcal{B}_i^{\mathbf{p}'} = \mathcal{B}_i^{\mathbf{p}} \Delta Z^c$ for all $i \in Z$.

Corollary 1. (Relations between the polarisations C, M, L and R) Assume that the society is split into the groups N_1 and N_2.

(i) Let **p'** be the aggregation function obtained from the aggregation function **p** with the N_2-right transformation. Then:
   (a) The polarisation of **p** is conformist if and only if the polarisation of **p'** is mixed.
   (b) The polarisation of **p** is leadership if and only if the polarisation **p'** is communitarian.

(ii) Let **p'** be the aggregation function obtained from the aggregation function **p** with the N_2-left-right transformation. Then:
   (a) The polarisation of **p** is conformist if and only if the polarisation of **p'** is communitarian.
   (b) The polarisation of **p** is leadership if and only if the polarisation of **p'** is mixed.

(iii) Let **p'** be the aggregation function obtained from the aggregation function **p** with the N_2-left transformation. Then:
   (a) The polarisation of **p** is conformist if and only if the polarisation of **p'** is leadership.
   (b) The polarisation of **p** is mixed if and only if the polarisation **p'** is communitarian.

Proposition 2. (Left-right transformation) If **p'** is the aggregation function obtained from **p** by a $Z-$left-right transformation of its weighted transition graph, then $\mathcal{C}$ is an absorbing class of **p** if and only if $\mathcal{C}\Delta Z$ is an absorbing class of **p'**. In particular, if the polarisation of **p** is conformist or communitarian, then there exists no periodic class of length $k > \binom{n}{[n/2]}$.

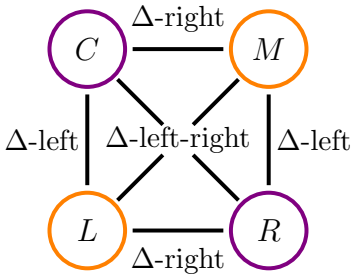


Figure 7: Corollary 1 and Proposition 2: polarisations C and R on the one hand, and M and L on the other hand, have the same dynamics.

It follows from Corollary 1.(ii) and the first part of Proposition 2 that the dynamics of communitarian and the conformist polarisations are the same[15] in other words opinions in a communautarian society tend to separate like in a decantation of oil from vinegar, and that the

[15] As noticed also by Altafini (2013) using the notion of gauge transformation.

dynamics of leadership and the mixed polarisations are the same[16]. It is essential to note that in general, even though weighted transition graphs are isomorphic under a left-transformation, or under a right-transformation, absorbing classes are entirely reshuffled[17].

5 Stability

It is not surprising that a high proportion of anti-conformist agents results in cycling opinions (see Figure 2, where two agents are anti-conformist); but it is more surprising that conformist societies can produce cycles as well, as evidenced in Grabisch and Rusinowska (2013). The fact that all agents would like to coordinate does not ensure convergence towards an absorbing state! This phenomenon has two main sources:

- **A behavioural origin for instability.** The influence graph is a complex system of hierarchies where interactions fail to coordinate. This happens particularly in "very non-anonymous" cases, i.e., when agents have boss sets (Section 3.2). We can show also (not exposed in this article) using an index of entropy that the notion of natural dynamic can be rigorously defined, giving a justification to the fact that conformist societies tend to end in an absorbing state, even though we can design tricky conformist societies with cycling opinions.
- **A synchronism origin for instability.** The surprising fact that conformist agents may sometimes fail to coordinate is inherent to the synchronous framework. We show in Appendix B that conformist societies updating asynchronously always end up in an absorbing state: asynchronism produces more intuitive dynamics.

In this section, we focus on the behavioural determinants of instability. We refer to opinions (or the society itself) as being *stable* when an absorbing state is reached[18]. In biology, contradictory impulsions resulting in cycles are not necessarily an undesirable phenomenon: we can think of hormonal cycles. In ethology, competing drives are standard, e.g., agression or escape, or on the occasion of courtship displays, resulting in behavioral cycles (Lorenz, 1969). On the contrary, in social choice theory, unstable societies are such that the outcome of the vote depends on the voting day, resulting in difficult and unreliable predictions. This remark motivates investigating influence processes which does not converge towards an absorbing state, i.e., by Theorem 1, which reach a cycle or an interval collection. In this section, we focus on cycles and we leave aside the absorbing classes of the third kind, whose analysis is more complex. Other results related to stability issues are given in Appendix B, by the investigating synchronous singularities, and in Appendix D where we characterise absorbing states and intervals using the

[16]That leadership societies are prone to opinion fluctuations might seem more surprising but it has already been observed by Acemoglu et al. (2010), in a different context though, since it considers two sources of leadership.

[17]To figure it out, we can consider Proposition 2, which states that lengths of cycles in polarisations C and R do not exceed $\binom{n}{[n/2]}$; however we can show that there exists leadership and mixed societies with cycle lengths exceeding $\binom{n}{[n/2]}$. The reader is referred to Aracena et al. (2004) for the connection between the length of cycles with the topology of the influence graph in an influence model with $m \geq 2$ states.

[18]Admittedly, this definition of stability is strong. We could also define quantitative notions of stability, e.g., we could study the invariants $I(\mathcal{C}) := \left(\bigcap_{S\in\mathcal{C}} S\right) \cup \left(\bigcup_{S\in\mathcal{C}} S\right)^c$ of the absorbing class $\mathcal{C}$, which are the agents who never change their opinion in $\mathcal{C}$. Another approach to stability could be to compute the average cardinality of states in $\mathcal{C}$, and its variance. Another argument why absorbing states deserve special interest is that they are robust to passing from a synchronous to an asynchronous scheme, as shown in Appendix B by the equivalence of the points (i) and (ii) of Proposition 4.

notion influential of coalitions[19].

At this level of generality, delivering necessary and sufficient conditions to forbid cycles seems to be out of reach. We only propose a few sufficient conditions.

Proposition 3. (Stability of opinions) If one of the following statements holds, then there is no cycle.

(i) The society is GWM and Assumption SI holds.

(ii) There exists $i \in N$ such that $\phi(i \to i) \geq 0$ and such that $\mathcal{P}_i^0$ and $\mathcal{P}_i^1$ are singletons.

(iii) (Autonomous opinions)
All aggregation functions are Boolean, agents of the same group have the same aggregation function, $\mathcal{P}_i^1 = \mathcal{V}^{r_i}(N_{(i)})$ for all i and there exists i such that $\min(r_i, n-r_i) \leq \min_{n_j \neq n_l} |n_j - n_l|$.

(iv) All agents have distance-based aggregation functions, the polarisation is conformist or made of two communitarian groups.

(v) Each $\mathcal{P}_i^*$ can be partitioned into $\mathcal{P}_i^* = \mathcal{P}_i \cup \mathcal{P}_i'$ such that the vector of Boolean aggregation functions **p'** verifies one of the previous statements, where p_i' is determined by $\mathcal{P}_i'^1 := \mathcal{P}_i^1 \cup \mathcal{P}_i$ and $\mathcal{P}_i'^0 := \mathcal{P}_i^0 \cup \mathcal{P}_i'$.

We can group the 'ingredients of stability' of Proposition 3 into four main groups. More or less, they all seem obvious; but the interesting point is that combinations of several ingredients are needed to get a sufficient receipt to dodge out cycling opinions. Typically, none of them is sufficient on its own. Besides the take-home message aspect of the following list, its interest is to evidence that qualities which a priori look different, actually all play in favour of stability when combined with each others.

(1) **Enough self-confidence**. Assumption SI, its weaker version ($\phi(i \to i) \geq 0$ for all i, or for at least one agent i), are involved in all points (i) to (iv), either explicitly or implicitly. Though it is not a necessary condition for reaching an absorbing state, it is hard to bypass it when designing sufficient conditions. Another aspect of self confidence is having small thresholds (small $\mathcal{P}^1/\mathcal{P}^0$ collections; this is also the insight of Theorems 7 and 8 in Appendix D), like in (ii): when agents are little inclined to adopt sure strategies (say yes or no for sure) then obviously this is an impediment to cycles.

(2) **Enough anonymity**. Anonymity plays a role in favour of stability in points (i), (iii) and (iv). Indeed, GWM (see Section 3.1, Definition 8) is some kind of 'semi-anonymous' aggregation function: all agents are equally considered, but maybe with different signs. Distance-based aggregation functions, in points (iii) and (iv), form a particular case of GWM aggregation functions (Proposition 1.3). Essentially, cycles are incapacities of agents to coordinate themselves. A boss structure of interactions, as opposed to anonymity, typically causes coordination failure, resulting in cycles. Anonymity, as opposed to discrimination, can be seen as tolerance, or apathie, lack of responsiveness, viscosity, inertia, etc.

[19]The connection between influences the dynamics is quite active in the network literature applied to biology, as they can model interaction and regulation among genes or their products. See, e.g., Thomas (1981); Aracena et al. (2006); Paulevé et al. (2020), whose approach are close to ours. A concise review of the different connections, in the deterministic framework, between the circuits in the the influence graph and the existence and/or unicity of absorbing states, called the Thomas' conjectures (Thomas, 1981), can be found in Richard (2011).

since agents do not harmonise their opinion on the one of some specific agents (leadership tends to be unstable, as discussed in Section 4.2) but rather peg it on the opinion of the mass. Lack of discrimination results in stable opinions.

(3) **Enough autonomy.** This point interestingly balances the previous idea that anonymity results in stable opinions. Indeed, an opinion is very autonomous when the agent has a large biais for yes or for no: since he strongly stays on its position, many agents are required to weight in the opposite direction to finally convince him to switch his opinion. This inertia plays against cycles. Even though following the mass can stabilise opinions, on the contrary, not listening to the mass can make the same job! Opinions pegged on the mass can be unstable; this phenomenon, well-known in politics and history, occurs when information circulates freely, opinion updating is fast and agents are unbiased.

(4) **Enough coordination**. Let us put it with different words: having a common objective plays in favour of stability. By "common objective", we do not mean that agents should have the willingness to share the same opinion. For example, two enemies have the same willingness to anti-coordinate: in this case, where they form a balanced graph, it is quite intuitive that their interaction should result in more stable opinions than in unbalanced ones. This is what (iv) confirms since conformist and communitarian configurations are balanced.

Since deterministic societies end up either in absorbing states or cycles, the absence of cycles ensures the stability of opinions; this is what Corollary 2 states.

Corollary 2. If all aggregation functions are Boolean and if one of the statements of Proposition 3 holds, then the society converges towards an absorbing state.

6 Related literature

The literature on binary opinions updating is active is various fields. In biology the deterministic framework, where the influence graphs are called Boolean networks, receives a particular consideration (Remy et al., 2008; Paulevé and Richard, 2012; Comet et al., 2013; Richard, 2018): they are used to model neuronal activity or interactions between genes and proteins (McCulloch and Pitts, 1943; Aracena et al., 2006). This literature is also particularly active on the border of physics and economics, where the concerns of the two disciplines are actually so close that this hybrid field, which belongs to the literature of agent-based models whose focus is to explain macroscopic features from the microscopic ones, deserved a name: Galam (2008) coined the word 'sociophysics'. Nyczka and Sznajd-Weron (2013) label the agents 'spinson', a portmanteau combining 'spin' and 'person'. Historically, the first agent-based model is actually a spin-based model, namely the Ising model, introduced by Lenz and Ising (1925). In the Ising model, spins can be either 'up' or 'down'. Interactions take place between the nearest neighbours; updating is reversible, symmetric, local and asynchronous. The Ising model can be formulated with two kinds of interactions: ferromagnetic (conformism) and anti-ferromagnetic (anti-conformism). The cellular automata, originally synchronous, have been introduced by Ulam (1960) and von Neumann (1966): each cell updates its state depending on its current state and the states of its neighbours, whose Conway's game of life is the most famous one (Gardner, 1970). The voter model (Liggett, 1985, 1999) has been introduced by Clifford and Sudbury (1973) to model the competition of species over a spatial territory: agents may swap their positions or unify their opinions when they meet (invasion process). Many variants have been investigated, like voter models with inflexible agents, with memory, and voter models in

networks Berenbrink et al. (2016). Among the other notable variants, the q-voter model is such that agents adopt the opinion of q neighbours chosen randomly (Castellano et al., 2009). Nyczka and Sznajd-Weron (2013) and Nyczka et al. (2012) study the q-voter model with anti-conformist and inflexible/independent agents. They show that the two kinds of agents can be qualitatively indistinguishable at the macroscopic level in some circumstances. This finding is confirmed in Grabisch et al. (2019a), in which model some agents are not chosen conformist or anti-conformist once for all, but they randomly draw their type. The latter agents, called the mixed agents, can be regarded as some kind of independent agents. In the list of twenty possible dynamics, characterised with the cardinalities in conformist, anti-conformist and mixed agents, we see that in some cases there is one degree of freedom: agents of one type can be converted into another type without altering the global dynamics. In Galam (2004), contrarian (anti-conformist) agents are incorporated to the original Galam model (Galam, 1986), a multi-layer voting system where the population is refined by successive votes until the president is elected. In Galam and Jacobs (2007), inflexible agents are incorporated and their impact on the outcome is found to be close to the one of the anti-conformist agents. The effect of static (baseline) opinions are also investigated in Taylor (1968); Acemoglu et al. (2010). Instead of assuming that agents are influenced by their neighbours, Sznajd-Weron and Sznajd (2001) proposed a model where agents are influenced by their agreeing neighbours. The CODA model of continuous opinions with discrete actions is studied in Martins (2007, 2008). The workhorse model of continuous opinions in network updating is French (1956) and DeGroot (1974), where agents simply average the opinions of their neighbours in a linear way, simultaneously. Many linear models have been developed in the sequel, among which Abelson (1964), Taylor (1968), Friedkin and Johnsen (1990) and Buechel et al. (2014). Granovetter (1978) introduces a threshold model of riots, where opinions are irreversible. Cascades are revisited by Watts (2002) in networks. Morris (2000) studies a local game investigating contagion in a network where agents have an interest to coordinate. Models with anti-conformist agents are particularly interesting in a dynamical framework, since they tend to cause fluctuations of opinions. For this reason, opinions are typically assumed to be reversible. Jull and Porter (2019), however, is an exception: in the vein of Granovetter (1978) and Watts (2002), this article presents a threshold model that accounts for cascades of adoptions from seeds of innovators. Adoption is irreversible and agents can be either conformist or anti-conformist, with some probability. Similarly, Touboul mixes conformist and anti-conformist agents (hipsters), but time in this model flows continuously and there are information delays. An adaptation of the Granovetter model including anti-conformist agents is proposed by Grabisch and Li (2019). In the previous models, anti-conformism is choosing the opposite of the prevailing opinion; in Buechel et al. (2015), however, conformism and anti-conformism receive a different interpretation: agents have true opinions which differ from the opinions they claim to have. The stated opinion is distorted from the true opinion by the level of conformity of the agent: disagreeing with people is regarded as discomfortable by conformist agents. If the agent is anti-conformist, he exaggerates his true opinion, and if the agent is honest, he reports his true opinion truthfully.

7 Concluding remarks

Grabisch and Rusinowska (2013) approach to iterated updating with aggregation functions is a convenient approach to bypass utility functions, by directly resorting to best responses. This approach is naturally non-linear and non-anonymous, which meets the needs of economists. The contribution of our paper, as compared with Grabisch and Rusinowska (2013) is to encompass

negative and heterogeneous influences. We imposed the following assumptions: (i) alternatives are binary; (ii) opinion adoption is reversible and independent among agents; (iii) the process is Markovian and stationary; (iv) the number of agents is finite; (v) time is discrete. The previous constraints are not all on an equal footing. The stationarity constraint could be relaxed in a strategic version of the model[20]. The point (iv) is relaxed in Grabisch et al. (2019b). A model of influence with iterated updating with m variables has been proposed by Aracena et al. (2004). However, continuous opinions would require a different approach. The point (ii) also is clearly essential, along with the Markovian framework and the discrete time. Putting them into question would require a different mathematical treatment.

Our handy formalism allows to treat positive and negative influence in the same manner, using appropriate partial orders on the states of the world. We evidenced the similar structure of the conformist societies with the communitarian societies with two groups (anti-coordination), along with the similar structure of the mixed societies (with anti-conformist agents) with the society with leaders, meaning that they result is similar patterns of opinion dynamics. We listed a few conditions ensuring stability of opinions. In particular, we evidenced that coordination plays in favour of stability. But also, anonymity and autonomy, though being contradicting features, both tend to play in favour of stability, which suggests that more investigation is needed to clarify how these two aspects interact with each others. We discussed the utility foundations of the model along with the updating scheme. Paradoxical situations, like the possible existence of cycles in the conformist model, disappear in the asynchronous scheme, which suggests that paradoxical situations are mostly synchronous singularities. We distinguish between two notions of groups, based on two different ways of considering the distance between agents: distance can be a topological notion, or it can refer to the behaviour of agents in terms of signs of influence. Our notion of behaviour groups (or simply groups) is based on the latter, while our notion of updating groups is based on the former.

Besides the obvious motivation for a non-binary and/or multivariate version of the model, issues related to prediction and information extraction call for a specific treatment. The predictive potential of the model is promising and applications are expected, for example, in opinions forecasting. At this stage of the research, there is a wall between dynamics and influences: a statistical approach is needed to navigate between the two worlds. A more technical discussion of this question is exposed in Appendix E.

Acknowledgements

Article written within my Ph.D thesis at the Centre d'Economie de la Sorbonne, ED 465, under the supervision of Michel Grabisch and Agnieszka Rusinowska, to whom I address all my thanks for their constant help. I would like to thank the CES laboratory and the network seminar group. Fen Li, Antoine Mandel, Akylai Taalaibekova and Xavier Venel are specially thanked for our insightful discussions. The author is grateful to Berno Buechel for his thorough reading and for challenging the foundations of the model, to Serge Galam for our talks about anti-conformism and inflexibility, to Andrea Galeotti for his remarks on congestion and saturation, to Katarzyna Sznajd-Weron and Dunia López-Pintado for their insights about synchronous singularities. The author would like to thank also Nizar Allouch, Grazia Cecere, Margherita Comola, Stéphane Gonzalez and Laura Hernandez, along with an anonymous referee for valuable improvements.

[20]Section 3.2 gave a flavour of the treatment that could be imposed on aggregation functions, but the robustness of dynamics with respect to changes in aggregation functions has not been investigated in our paper. For a non-stationary model, see also Proskurnikov et al. (2016).

The Agence Nationale de la Recherche is thanked for financial support under contracts ANR-13-BSHS1-0010 (DynaMITE) and ANR-10-LABX-93-01 (Labex OSE).

A Proofs

Proof. (Property 1) For any $S \in 2^N$, $\mathbf{p}(S)$ is of the form $\mathbf{p}(S) = (\mathbf{1}_T, \mathbf{x}_K, \mathbf{0}_{(T\cup K)^c})$ with $\mathbf{x}_K \in (0,1)^K$. (This means that agents in T will say yes for sure, that agents in K might say yes or no, while agents in $(T \cup K)^c$ will say no for sure.) We obtain from (2.1) that the states Q such that $b_{S,Q} > 0$ are exactly the ones of the collection $[T, T \cup K]$, that is, $S \xrightarrow{1} [T, T \cup K]$. □

Proof. (Property 2) $p_i(S)$ is the probability that the next state contains i. Hence, using Property 1: $p_i(S) = \sum_{\substack{T' \ni i \\ T' \in [T, T\cup K]}} b_{S,T'}$.

(This property can be found in Grabisch and Rusinowska (2010).) □

Proof. (Property 3) ($\Rightarrow$) Let $B \in \mathcal{B}_i$. Let $j \in N$. (i) Either $j \notin B$, that is, $j \in B^c$, in which case we have $S \prec_{B^c} S \cup j$, which can also be written $S \cup j \prec_B S$ (that is, $\preccurlyeq_{B^c}$ is the dual order of $\preccurlyeq_B$). As a consequence, $p_i(S \cup j) \leq p_i(S)$ for all $S \in 2^N$; (ii) or $j \in B$ and then $p_i(S) \leq p_i(S \cup j)$ for all $S \in 2^N$. ($\Leftarrow$) Assume that there exists $j \in N$ such that (a) there exists S such that $p_i(S \cup j) \geq p_i(S)$, and that: (b) there exists S' such that $p_i(S' \cup j) < p_i(S')$. If there exists $B \in \mathcal{B}_i$, then either $j \in B$, which contradicts (b), or $j \notin B$, which contradicts (a). □

Proof. (Theorem 1) This is Theorem 2 in the conformist context of Grabisch and Rusinowska (2013). It suffices to note that the proof involves the conformist monotonicity conditions only for the precisions given on the maximal length of cycles and the intersection of the interval collections with $\emptyset$ and N (which precisions, however, cannot be exported to the general case), but not for to establish the list of three dynamics, which remains correct in our general framework. □

Proof. (Theorem 2) For any sets $A, B \in 2^N$, we write $[[A, B]] := [A \cap B, A \cup B]$.

Step 1. *Assume that* $B_1, B_2 \in \mathcal{B}_i$. *Then,* $[[B_1, B_2]] \subseteq \mathcal{B}_i$.

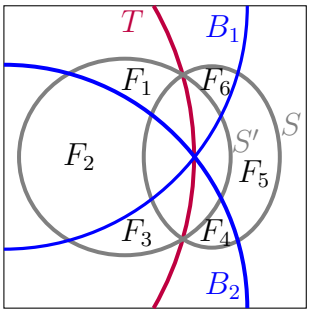


Figure A.1: $T \in [[B_1, B_2]]$

Let $T \in [[B_1, B_2]]$. Assume that $S \preccurlyeq_T S'$ and let:

- $S_1 := S \cup F_1$ with $F_1 = (S \Delta S') \cap (T \setminus B_2)$
- $S_2 := S_1 \cup F_2$ with $F_2 = (S \Delta S') \cap (B_1 \cap B_2)$
- $S_3 := S_2 \cup F_3$ with $F_3 = (S \Delta S') \cap (T \setminus B_1)$
- $S_4 := S_3 \setminus F_4$ with $F_4 = (S \Delta S') \cap (B_2 \setminus T)$
- $S_5 := S_4 \setminus F_5$ with $F_5 = (S \Delta S') \cap (B_1 \cup B_2)^c$
- $S' = S_5 \setminus F_6$ with $F_6 = (S \Delta S') \cap (B_1 \setminus T)$

Noticing respectively that: $F_1 \in B_1$; $F_2 \in B_1$; $F_3 \in B_2$; $F_4 \in B_1^c$; $F_5 \in B_1^c$ and $F_6 \in B_2^c$, we deduce the following inequalities: $p_i(S) \leq p_i(S_1) \leq p_i(S_2) \leq p_i(S_3) \leq p_i(S_4) \leq p_i(S_5) \leq p_i(S')$.

Step 2. $\mathcal{B}_i = [\underline{B}_i, \overline{B}_i]$.
The previous step shows that $\mathcal{B}_i$ is an interval. From the definition of $\underline{B}_i$ and $\overline{B}_i$, we actually have $\mathcal{B}_i = [\underline{B}_i, \overline{B}_i]$.

Step 3. $\phi(j \to i) = 0 \;\Leftrightarrow\; [B \setminus j \in \mathcal{B}_i \Leftrightarrow B \cup j \in \mathcal{B}_i]$.
If $\phi(j \to i) = 0$, then by Property 3: for all $S \in 2^N$: $p_i(S \setminus j) = p_i(S \cup j)$. In particular: $B \setminus j \in \mathcal{B}_i \Leftrightarrow B \cup j \in \mathcal{B}_i$. Converse: Assume that $\phi(j \to i) \neq 0$ (without loss of generality, $\phi(j \to i) > 0$). We want to show that the equivalence $[B \setminus j \in \mathcal{B}_i \Leftrightarrow B \cup j \in \mathcal{B}_i]$ does not hold. By Property 3, $p_i(S \setminus j) \leq p_i(S \cup j)$ for all S, and exists S^* such that $p_i(S^* \setminus j) < p_i(S^* \cup j)$. Let $B \in \mathcal{B}_i$; let us show that necessarily $j \in B$ (and therefore, that there is no $B \in \mathcal{B}_i$ such that $B \setminus j \in \mathcal{B}_i$). Assume that $j \in B^c$. Then $p_i(S \cup j) \leq p_i(S \setminus j)$ for all S. In particular for $S = S^*$, then $p_i(S^* \cup j) \leq p_i(S^* \setminus j)$, a contradiction.

Step 4. *Assume that $B_1, B_2 \in \mathcal{B}_i$. For all $j \in B_1 \Delta B_2$: $\phi(j \to i) = 0$. Conversely, if $\phi(j \to i) = 0$, there exists $B_1, B_2 \in \mathcal{B}_i$: $j \in B_1 \Delta B_2$.*
($\Leftarrow$) Let $B_1, B_2 \in \mathcal{B}_i$. By the Step 1: $[[B_1, B_2]] \subseteq \mathcal{B}_i$. Let $j \in (B_1 \cup B_i) \setminus (B_1 \cap B_2)$: then for any $B \in [[B_1, B_2]]$: $[B \setminus j \in \mathcal{B}_i \Leftrightarrow B \cup j \in \mathcal{B}_i]$. Result follows from the converse direction of the previous step. ($\Rightarrow$) If $\phi(j \to i) = 0$, then for any $B \in \mathcal{B}_i$, by Step 3: either $B \ni j$, and therefore: $B \setminus j \in \mathcal{B}_i$, or $j \notin B$, and therefore $B \cup j \in \mathcal{B}_i$. In both cases we have found $B_1, B_2 \in \mathcal{B}_i$ such that $j \in B_1 \Delta B_2$.

Step 5. $\phi(j \to i) = 0$ *if and only if* $j \in \overline{B}_i \setminus \underline{B}_i$.
From $\overline{B}_i \setminus \underline{B}_i = \overline{B}_i \Delta \underline{B}_i = \{j \mid \exists B_1, B_2 \in \mathcal{B}_i : j \in B_1 \Delta B_2\}$ and the previous step.

Step 6. $\overline{B}_i = \{j : \phi(j \to i) \geq 0\}$ *and* $\underline{B}_i = \{j : \phi(j \to i) > 0\}$.
If $\phi(j \to i) < 0$ and $j \in B \in \mathcal{B}_i$, then there exists S^* such that $p_i(S^* \cup j) < p_i(S^*)$. By $j \in B \in \mathcal{B}_i$, it holds that $S \prec_B S \cup j$ for and all S. In particular $S^* \prec_B S^* \cup j$, which implies $p_i(S^*) \leq p_i(S^* \cup j)$, a contradiction. If $\phi(j \to i) > 0$, then by the same argument as in Step 3, $j \in \bigcap_{B \in \mathcal{B}_i} B = \underline{B}_i$. If $\phi(j \to i) = 0$, then $j \in \overline{B}_i \setminus \underline{B}_i$.

□

Proof. (Proposition 1) The first two points are obvious. For the third, note that if there exists j, k with $|w_i^j| \neq |w_i^k|$, then $p_i(T \Delta j) \neq p_i(T \Delta k)$. □

Proof. (Property 5) We recall that for any interval collections $[A, B], [C, D]$, it holds that $[A, B] \cap [C, D] = [A \cup C, B \cap D]$. Property 5 is an application of Theorem 2.($\Rightarrow$) Assume that $\phi(c \to a) < 0$ and $\phi(c \to b) > 0$. Then, $c \in \overline{B}_a^c \cap \underline{B}_b$. If it was also that a, b belong to the same group, i.e., that $\underline{B}_a \cup \underline{B}_b \subseteq \overline{B}_a \cap \overline{B}_b$, then we would have in particular $\underline{B}_b \subseteq \overline{B}_a$. But then, $\overline{B}_a^c \cap \underline{B}_b$ would be empty; hence the contradiction. ($\Leftarrow$) For any block N_k and any $c \in N$, either $\phi(c \to i) \geq 0$ for any $i \in N_k$, or $\phi(c \to i) \leq 0$ for any $i \in N_k$. This means that either

$c \in \bigcap_{i \in N^k} \overline{B}_i$, or $c \in \bigcap_{i \in N_k} \underline{B}_i^c = (\bigcup_{i \in N_k} \underline{B}_i)^c$, that is, either c exerts a positive influence on all agents of N_k, or it exerts a negative influence on all agents of N_k. Writing $K^+ := \bigcap_{i \in N_k} \overline{B}_i$ and $K^- := (\bigcup_{i \in N_k} \underline{B}_i)^c$, we thus have $K^- \cup K^+ = N$, which implies that $(K^-)^c \subseteq K^+$. This means that $\bigcup_{i \in N_k} \underline{B}_i \subseteq \bigcap_{i \in N_k} \overline{B}_i$, which means precisely that $\bigcap_{i \in N_k} \mathcal{B}_i \neq \emptyset$. □

Proof. (Theorem 3) We will use the following straightforward property:

Property 6. For any $S_1, S_2, B, Z \in 2^N$, $S_1 \preccurlyeq_B S_2 \Leftrightarrow S_1 \Delta Z \preccurlyeq_{B \Delta Z} S_2 \Delta Z$.

(i) Let S_1, S_2 be any two states in the weighted transition graph characterised by $\mathbf{p}$. Let $B \in \mathcal{B}_i^{\mathbf{p}}$. Then, in the weighted transition graph characterised by **p'** obtained from $\mathbf{p}$ by a left transformation: $S_1 \Delta Z \preccurlyeq_B S_2 \Delta Z \Rightarrow p_i'(S_1) \leq p_i'(S_2)$. By Property 6, $S_1 \preccurlyeq_{B \Delta Z} S_2 \Rightarrow p_i'(S_1) \leq p_i'(S_2)$. Therefore, $B \Delta Z \in \mathcal{B}_i^{\mathbf{p'}}$. We have established $\mathcal{B}_i^{\mathbf{p'}} \supseteq \mathcal{B}_i^{\mathbf{p}} \Delta Z$; the left-transformation being a symmetry, the same reasoning establishes the reverse inclusion.

(ii) Recall that for any sets $A, B \in 2^N$, we denote $[[A, B]] := [A \cap B, A \cup B]$.

For any $S \in 2^N$, there exists T, K such that $S \xrightarrow[\mathbf{p}]{1} [T, T \cup K]$. By Property 2:

$$p_i^{\mathbf{p}}(S) = \sum_{\substack{T' \ni i \\ T' \in [T, T \cup K]}} b_{S,T'}^{\mathbf{p}} \tag{A.1}$$

Let **p'** obtained by a Z-right transformation. By definition, for all $S, T \in 2^N$: $b_{S,T}^{\mathbf{p}} = b_{S,T \Delta Z}^{\mathbf{p'}}$. That is:

$$S \xrightarrow[\mathbf{p'}]{1} [[T \Delta Z, (T \cup K) \Delta Z]] \tag{A.2}$$

with:

$$p_i^{\mathbf{p'}}(S) = \sum_{\substack{T' \ni i \\ T' \in [[T \Delta Z, (T \cup K) \Delta Z]]}} b_{S,T'}^{\mathbf{p'}}. \tag{A.3}$$

We write $\mathcal{D}^{\mathbf{p}}(S, i)$ the collection of sets $T' \in [T, T \cup K]$ containing i, for a given transition $S \xrightarrow[\mathbf{p}]{1} [T, T \cup K]$. We rewrite:

$$p_i^{\mathbf{p}}(S) = \sum_{T' \in \mathcal{D}^{\mathbf{p}}(S,i)} b_{S,T'}^{\mathbf{p}} \tag{A.4}$$

Noticing that:

$$\mathcal{D}^{\mathbf{p'}}(S, i) = \begin{cases} (\mathcal{D}^{\mathbf{p}}(S, i))\, \Delta Z & \text{if } i \notin Z \\ ([T, T \cup K] \setminus \mathcal{D}^{\mathbf{p}}(S, i)) \Delta Z & \text{if } i \in Z \end{cases}, \tag{A.5}$$

we rewrite:

$$p_i^{\mathbf{p'}}(S) = \begin{cases} \displaystyle\sum_{T' \in (\mathcal{D}^{\mathbf{p}}(S,i)) \Delta Z} b_{S,T'}^{\mathbf{p'}} & \text{if } i \notin Z \\ \displaystyle\sum_{T' \in ([T, T \cup K] \setminus \mathcal{D}^{\mathbf{p}}(S,i)) \Delta Z} b_{S,T'}^{\mathbf{p'}} & \text{if } i \in Z \end{cases} \tag{A.6}$$

If $i \notin Z$, then $p_i^{\mathbf{p'}} = p_i^{\mathbf{p}}$. If $i \in Z$, then $p_i^{\mathbf{p'}} = 1 - p_i^{\mathbf{p}}$.

We get that finally for all i and $B \in \mathcal{B}_i^{\mathbf{p}}$:

$$S \preccurlyeq_B S' \Rightarrow \begin{cases} p_i^{\mathbf{p'}}(S) \leq p_i^{\mathbf{p'}}(S') & \text{if } i \notin Z \\ p_i^{\mathbf{p'}}(S) \geq p_i^{\mathbf{p'}}(S') & \text{if } i \in Z \end{cases} \tag{A.7}$$

Therefore, $B \in \mathcal{B}_i^{\mathbf{p'}}$ for $i \notin Z$ and $B^c \in \mathcal{B}_i^{\mathbf{p'}}$ for $i \in Z$. That is, $\mathcal{B}_i^{\mathbf{p'}} \supseteq \mathcal{B}_i^{\mathbf{p}}$ for all $i \notin Z$, and $\mathcal{B}_i^{\mathbf{p'}} \supseteq (\mathcal{B}_i^{\mathbf{p}})^{(c)}$ for all $i \in Z$. As in the previous point, the reverse inclusions hold too.

(iii) A left-right transformation is a (commutative) composition of a left and a right-transformation.

□

Proof. (Corollary 1)
It is a direct application of Theorem 3. For example, let us prove that the transformation of a polarisation M_1 by a N_2-right transformation is a polarisation R_1. A polarisation M_1 is such that $N \in \mathcal{B}_i$ for $i \in N_1$ and $\emptyset \in \mathcal{B}_i$ for $i \in N_2$. Apply a N_2 right transformation on the weighted transition graph determined by $\mathbf{p}$ (Property 2): from Theorem 3, for $i \in N_1$: $N \Delta N_2 = N_1 \in \mathcal{B}_i^{\mathbf{p'}}$ and for $i \in N_2$: $\emptyset \Delta N_2 = N_2 \in \mathcal{B}_i^{\mathbf{p'}}$, i.e., the polarisation obtained is a communitarian one.

□

Proof. (Proposition 2)
The left-right transformation is a graph homomorphism. Therefore, $\mathcal{C}$ in an absorbing class in $\Gamma^{\mathbf{p}}$ if and only if $\mathcal{C}\Delta Z$ in an absorbing class in $\Gamma^{\mathbf{p'}}$. Let us now establish the precision concerning the conformist and communitarian polarisations. Assume that $\mathbf{p}$ is the aggregation vector of communitarian polarisation where the two groups are N_1 and N_2. Let the periodic class: $[T_1, T_1 \cup H_1] \xrightarrow{1} [T_2, T_2 \cup H_2] \xrightarrow{1} \ldots \xrightarrow{1} [T_k, T_k \cup H_k] \xrightarrow{1} [T_1, T_1 \cup H_1]$, where some H_p are possibly empty. Let $\mathbf{p}$' obtained from $\mathbf{p}$ with a N_2-left-right transformation. Then the polarisation of $\mathbf{p'}$ is conformist and by the first part of Corollary 1, we may consider without loss of generality the following periodic class in a conformist model: $[S_1, S_1 \cup K_1] \xrightarrow{1} [S_2, S_2 \cup K_2] \xrightarrow{1} \ldots \xrightarrow{1} [S_k, S_k \cup K_k] \xrightarrow{1} [S_1, S_1 \cup K_1]$. Sperner's theorem states that the upper bound of the length of an antichain with the set order is $\binom{n}{[n/2]}$. Therefore if we show that $S_1, S_2, \ldots, S_k$ must be incomparable with the order $\subset$, then we will have proved that $\binom{n}{[n/2]}$ is an upper-bound on the length of the periodic classes. Assume that $S_p \subset S_q$ for some p and q such that $p \equiv q[k]$ does not hold. Then $S_p \xrightarrow{1} [S_{p+1}, S_{p+1} \cup K_{p+1}]$ and $S_q \xrightarrow{1} [S_{q+1}, S_{q+1} \cup K_{q+1}]$ with $S_{p+1} \subseteq S_{q+1}$ and K_{p+1}, K_{q+1} possibly empty (where actually $S_{p+1} \subset S_{q+1}$ because cycles are taken minimal). Iterating $q - p$ times: $S_q \subset S_{2q-p}$. In the ring of integers modulo k, $\mathbb{Z}/k\mathbb{Z} = \{\overline{0}, \overline{1}, \ldots, \overline{k-1}\}$, for any $m \in \mathbb{Z}$, we identify $\overline{m}$ with S_m. Hence our previous argument proved that for any p, m such that $m \equiv p[q-p]$, we have $S_p \subset S_m$. Calling d be the period of $\overline{q-p}$, i.e., the smallest positive integer such that $\overline{d(q-p)} = \overline{0}$ (Lang, 2002), we get $S_p \subset S_{p+d(q-p)} = S_p$, which is absurd. □

Proof. (Proposition 3)
In all these points, we assume that $S_1 \xrightarrow{1} S_2 \xrightarrow{1} \ldots \xrightarrow{1} S_k \xrightarrow{1} S_1$ with $k \geq 2$.

(i) Let $W_j := \{i \mid w_j^i = 0\}$, so that $\phi(i \to j) = 0 \Leftrightarrow i \in W_j$ (Proposition 1.(2)). Let the transition $S_p \xrightarrow{1} S_{p+1}$. In a GWM model (from Proposition 1.(1)+(2)), we have for all $j \in S_{p+1}$: $S_p \setminus W_j = \bigcup_{w_j^q > 0} q$ and for all $j \notin S_{p+1}$: $S_p \setminus W_j = \bigcup_{w_j^q < 0} q$. By assumption SI,

$w_j^j > 0$ for all $j \in N$. Therefore $S_{p+1} \subseteq S_p \setminus W_j$ for all $j \in S_{p+1}$, which implies that $S_{p+1} \subseteq S_p$, where the inclusion is actually strict, since the transition is embedded into a cycle. Finally, $S_1 \subset S_2 \subset \ldots \subset S_k \subset S_1$, which is impossible.

(This is an adaptation of Proposition 1.(iii) in Grabisch and Rusinowska (2013)).

(ii) Under these assumptions, $\mathcal{B}_i$ necessarily contains a single set B_i. Therefore, $p_i(S) \in \{0,1\}$ if and only if $S \in \{B_i, B_i^c\}$. There is only one candidate cycle to examine: $B_i \xrightarrow{1} B_i^c \xrightarrow{1} B_i$. Since $\phi(i \to i) \geq 0$, $i \notin B_i^c$. Since $B_i \xrightarrow{1} B_i^c$, we have: $p_i(B_i) = 0$. Since $\mathcal{B}_i = \{B_i\}$, this implies $p_i(S) = 0$ for all $S \in 2^N$, contradicting the non-emptiness of $\mathcal{P}_i^1$.

(This is an adaptation of Proposition 1.(ii) in Grabisch and Rusinowska (2013).)

(iii) Under these assumptions, agents of the same group have the same aggregation function ($p_i = p_j$ when i and j belong to the same group). Therefore the following cycle is made of blocks: $S_1 \xrightarrow{1} S_2 \xrightarrow{1} \ldots \xrightarrow{1} S_p \xrightarrow{1} S_1$, and we call $J_1, \ldots, J_p$ the indices of the $(S_q)_{q=1..p}$. This succession of p transitions imposes the following conditions:

$$\begin{cases} S_q \in \mathcal{V}^{r_i}(N_{(i)}) & \text{for all } i \in S_{q+1} \\ S_q \notin \mathcal{V}^{r_i}(N_{(i)}) & \text{for all } i \notin S_{q+1} \end{cases} \tag{A.8}$$

Since $i \in S_q$ if and only if the index of the group that i belong to belongs to J_q (which we write $(i) \in J_q$), the previous conditions translate into:

$$\begin{cases} \sum\limits_{j \in J_q} n_j - n_{(i)} \leq r_i & \text{for all } i \text{ such that } (i) \in J_{q+1} \\ \sum\limits_{j \in J_q} n_j + n_{(i)} > r_i & \text{for all } i \text{ such that } (i) \notin J_{q+1} \end{cases} \tag{A.9}$$

Hence $S_1 \xrightarrow{1} S_2 \xrightarrow{1} \ldots \xrightarrow{1} S_k \xrightarrow{1} S_1$ is a cycle if and only if for all $i \in N$:

$$r_i \in \left[\max_{q \in [1,k]} \left[\sum_{j \in J_q} n_j - \min_{j \in J_{q+1}} n_j \right] \; ; \; \min_{q \in [1,k]} \left[\sum_{j \in J_q} n_j + \min_{j \notin J_{q+1}} n_j \right] - 1 \right]. \tag{A.10}$$

(where the interval $[a,b]$ is empty if $b < a$).
If there exists some r_i falling outside the interval, then cycles are impossible. By noticing that we have:

$$r_i \geq \min_{q \in [1,k]} \left[\sum_{j \in J_q} n_j + \min_{j \notin J_{q+1}} n_j \right]$$

if and only if we have:

$$n - r_i \leq \max_{q \in [1,k]} \left[n - (\sum_{j \in J_q} n_j + \min_{j \notin J_{q+1}} n_j) \right] = \max_{q \in [1,k]} \left[\sum_{j \notin J_q} n_j - \min_{j \notin J_{q+1}} n_j \right],$$

we get that there is no cycle if there exists i such that either:

$$r_i \leq \max_{q \in [1,k]} \left[\sum_{j \in J_q} n_j - \min_{j \in J_{q+1}} n_j \right]$$

or:

$$n - r_i \leq \max_{q \in [1,k]} \left[\sum_{j \notin J_q} n_j - \min_{j \notin J_{q+1}} n_j \right].$$

Noticing that there exists at least two consecutive states with decreasing cardinalities in the large sense, and at least two consecutive states with increasing cardinalities in the large sense, we respectively get that $\min_{n_j \neq n_l} |n_j - n_l| \leq \max_{q \in [1,k]} \left[\sum_{j \in J_q} n_j - \min_{j \in J_{q+1}} n_j \right]$ and $\min_{n_j \neq n_l} |n_j - n_l| \leq \max_{q \in [1,k]} \left[\sum_{j \notin J_q} n_j - \min_{j \notin J_{q+1}} n_j \right]$ the condition stated in (iii) ensures that at least one i is such that (A.10) is not verified.

(iv) Let us show it for conformist polarisations; by Proposition 2, this holds in communitarian polarisations too.

In a society where agents have distance-based aggregation functions, $|S| = |S'|$ and $S \xrightarrow{1} T$ implies $S' \xrightarrow{1} T$. Therefore states must be of different cardinality within a cycle; hence there exists S_i, S_j such that $|S_i| < |S_j|$. But this implies $|S_{i+1}| \leq |S_{j+1}|$, because all agents are conformist. Iterating k times and using a similar argument as in the proof of Proposition 2, we get $|S_i| < |S_i|$, which is absurd.

(This result was established in Förster et al. (2013), Proposition 3).

(v) If there is no cycle in a society where the set of sure transitions is larger, then there is no cycle in the primitive society.

□

Proof. (Corollary 2) When aggregation functions are Boolean, absorbing classes are either cycles or absorbing states. Therefore, if there are no cycles, there are only absorbing states. □

B Correlated updating: a topological ingredient of the model

In physics and in biology, most models are asynchronous. Besides the need to relate our approach to the natural sciences literature, the main reason why we should tell a word on asynchronous updating is the serious issue of the so-called "synchronous singularities", i.e., dynamics which are specific to the synchronous updating. Robustness issues are well-studied in the literature, e.g., D. Cornforth and Newth (2005). Fatès and Morvan (2004) discusses small perturbations to the synchronous updating in cellular automata. The synchronous and asynchronous updating dynamics are compared by Blok and Bergersen (1999) in the game of life and in Paulevé and Richard (2012) in Boolean networks[21]. A notion correlated updating (also called intermediate level of asynchronism) using a synchrony rate, has been proposed by Bouré et al. (2013) in the context of lattice-gas models and by Grilo and Correia (2011) in 2-player evolutionary games, showing that, in general, asynchronism plays in favour of cooperation.

While there is only one way of being synchronous, there are many ways of being asynchronous; our approach to asynchronous updating is that agents spontaneously update their

[21] In Paulevé and Richard (2012), the notion of "generalised iteration graphs" is also investigated: it corresponds in our model to the union of all correlated updating graphs.

status. However, we want also to allow for correlated updating: agents influencing each others a lot should be inclined to update their opinion at the same time. This appendix hits two birds with one stone. Not only it does relax the assumption that agents all update at the same time, but also it is the opportunity to make a use of an importance source of information which we neglected so far, namely the topology of the influence graph. In network theory, distance issues receive a lot of consideration. But in fact, there are (at least) two ways of defining the distance between two individuals. By "close agents", we may: (a) stick to the very topological meaning of distance, meaning that agents know each others, for example because they belong to a same clique, or at least they are separated by few nodes; or: (b) we rely on what the common language means by "being close", that is, agents who share similar beliefs and/or behaviour, no matter how far they can be from each other in the network. Point (b) corresponds to the notion of groups introduced in Section 4, which was based on the behaviour. Point (a) introduces another notion of groups: it is based on a classical notion of distance: agents reshaping their opinions at the same time should be closely interacting agents, no matter how "friends" or "enemies" they can be; they will form a so-called "updating group". Disentangling the two notions of distance comes down to consider separately the arcs of the influence graph from their signs, where the arcs determine the updating scheme and their signs the (behaviour) groups. In this appendix, our point is not to discuss how the updating groups are obtained: we can import any existing result or model from the literature on the topology of networks to determine a partition $\mathcal{R}$ of groups of closely interacting agents. Of course our approach to correlated opinion updating using a fixed partition partition $\mathcal{R}$ is over-simplifying. It would be more realistic to allow strongly connected component of the influence graph to update at the same time, in which case the updating scheme is not driven by a partition $\mathcal{R}$ anymore, but by a cover. The insight, however, would be unchanged: the network should determine the updating scheme that structures the process. In this view time can be seen, not as a canevas, but as a variable emerging from the activity of the network, in some sense, its temperature, the quantity of information exchanged, the tightness of interactions.

The society N is partitioned as follows: $\mathcal{R} := \{N^1, \ldots, N^p\}$, where a block N^k, $k \in [1, p]$, is called an *updating group*. $\mathcal{R}$ is called the *updating partition*; it expresses the updating scheme of the process. Agents of the same updating group update at the same time when invited to update. When $p = 1$, i.e., when there is only one updating group, $\mathcal{R} = \{N\} =: \mathcal{R}_\blacklozenge$: this is the synchronous scheme studied in this paper. When $p = n$, i.e., when $\mathcal{R} = \{\{1\}, \ldots, \{n\}\} =: \mathcal{R}_\diamondsuit$, the model is called *asynchronous*[22]. We distinguish a primitive time, which is continuous and flows continuously. At each point of this primitive time, an updating group can be invited to update. This invitation occurs according to an instantaneous probability: there exists for each updating group a strictly positive parameter λ_i, called the rate, similar to decay constant in radioactivity[23]. The rates of invitation to update are assumed to be independent across updating groups; as a consequence, the probability that any two updating groups are invited to update their opinion at the same time is zero. In particular, in the asynchronous model, no two agents can update at the same time, which implies that any two consecutive states can differ of at most one agent.

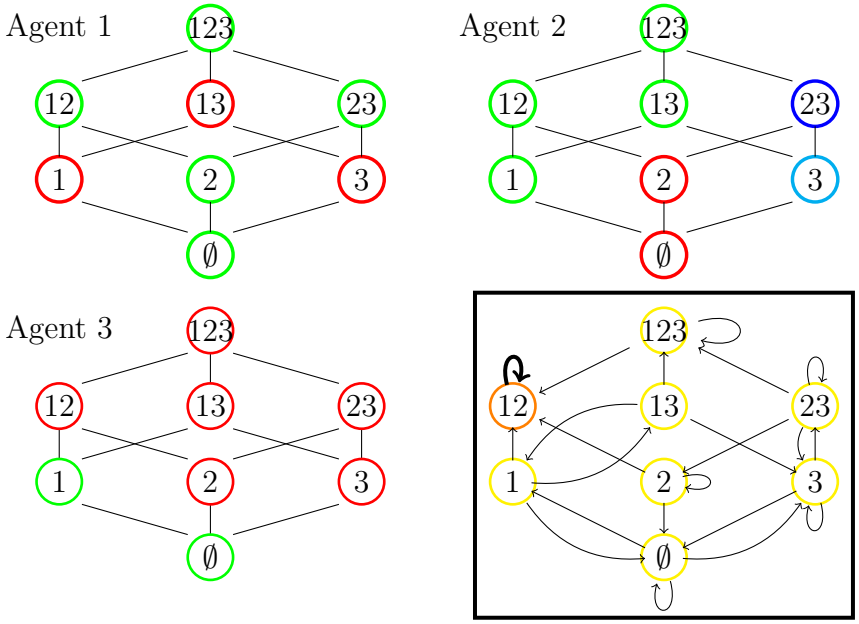
We first notice that most tools introduced in this paper are unchanged. The monotonicity collection, the influence index, the influential coalitions, etc. and all notions derived from

[22]The set of all partitions can be ordered with a coarsening order relation (Grabisch, 2016), defining a lattice whose $\mathcal{R}_\diamondsuit$ and $\mathcal{R}_\blacklozenge$ are respectively the bottom and the top.

[23]In a qualitative model, the value of the λ_i plays no role.

a single p_i, which are not implicated into the updating process. Transitions however are different, and therefore, the dynamics also. Indeed, they not described by the identity $b_{S,T} = \prod_{i\in T} p_i(S) \prod_{i\notin T}(1-p_i(S))$ anymore, since not all agents are invited to update at the same time. Nevertheless, notions of absorbing states and cycles remain unchanged. While the notion of loop played no role in the synchronous framework, we are now lead to define a *loop* as a sequence of k transitions $S_1 \to S_2 \to \ldots, \to S_1$. (A cycle is a loop with sure transitions.) For example in Figure B.1, $1 \to 13 \to 3 \to \emptyset \to 1$ is a loop.

Example 10. (Asynchronous updating: $\mathcal{R} = \mathcal{R}_\Diamond$) Figures B.1 and B.2 are the asynchronous versions ($\mathcal{R} = \mathcal{R}_\Diamond$) of Figures 1 and 2 (where $\mathcal{R} = \mathcal{R}_\blacklozenge$). There exists a link from $S \subseteq N \setminus i$ to $S \cup i$ if and only if $S \in \mathcal{P}_i^1 \cup \mathcal{P}_i^*$ and there exists a transition from $S \ni i$ to $S \setminus i$ if and only if $S \in \mathcal{P}_i^0 \cup \mathcal{P}_i^*$. There exists a self-loop at S if and only if there exists $i \in S$ such that $p_i(S) > 0$ or $i \notin S$ such that $p_i(S) < 1$. We recover the notion of asynchronism exposed in Remy et al. (2008), Comet et al. (2013) and Paulevé et al. (2020).



$p_i(S) = 1$ *at green states.* $p_i(S) = \frac{2}{3}$ *at blue states.*
$p_i(S) = \frac{1}{3}$ *at cyan states.* $p_i(S) = 0$ *at red states.*

Figure B.1: Asynchronous version of the society represented in Figure 1 in the synchronous case. 12 remains an absorbing state, in accordance with Proposition 4.(i)+(ii).

Let us now investigate the dynamics. First of all, we need to examine the shape of transitions. We are lead to introduce the notion of "dissatisfied agent", namely, agents who are inclined to reverse their opinion if invited to do so. They are the agents currently saying yes (resp. no), but who have a probability $p_i(S) < 1$ (resp. > 0) to say yes if invited to update. We define Π to be the following correspondance:

$$\Pi : \left\{ \begin{array}{l} 2^N \times 2^N \to 2^N \\ (S,K) \mapsto S\Delta[\underline{\Phi}(S,K), \overline{\Phi}(S,K)] \end{array} \right. , \tag{B.1}$$

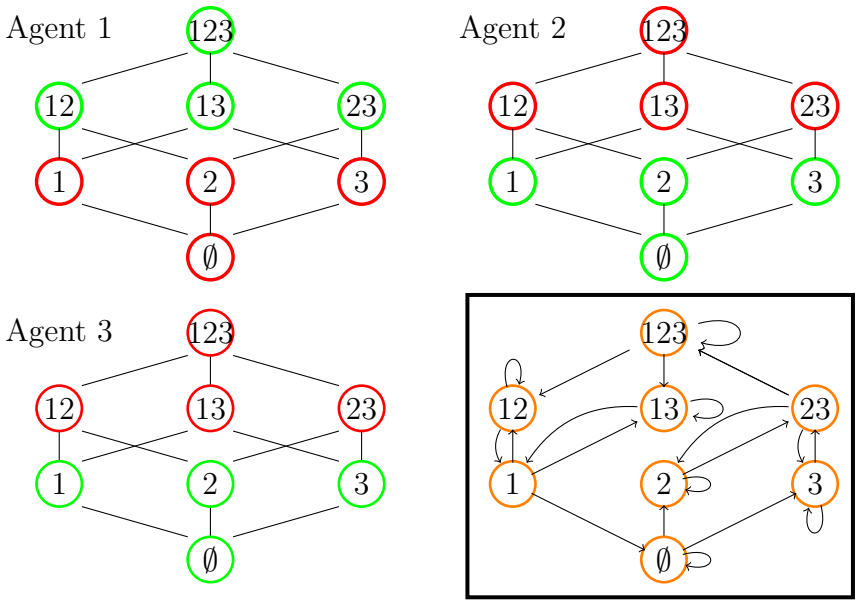


Figure B.2: Asynchronous version of the society represented in Figure 2 in the synchronous case. As compared with Figure 2, the cycle $1 \xrightarrow{1} 23 \xrightarrow{1} 1$ has disappeared. Instead, 2^N is absorbing.

where:

$$\underline{\Phi}(S,K) = K \cap \{i : i \in S \in \mathcal{P}_i^0 \text{ or } i \notin S \in \mathcal{P}_i^1\},$$
$$\overline{\Phi}(S,K) = K \cap \{i : i \in S \in \mathcal{P}_i^0 \cup \mathcal{P}_i^* \text{ or } i \notin S \in \mathcal{P}_i^1 \cup \mathcal{P}_i^*\}.$$

$\overline{\Phi}(S,N)$ is called the *set of dissatisfied agents at* S[24]. We note that for any $\Phi \in [\underline{\Phi}, \overline{\Phi}]$, $\Phi(S,K) = \Phi(S,\Phi(S,K))$; therefore, Φ is a projection with respect to its second entry. By interpreting $\Pi(S,K)$ as the output of a composition law, we introduce $\odot$ to be such that $\Pi(S,K) \equiv S \odot K$. The shape of transitions is given by Property 7. Recall that $S \odot K$ is a collection of sets and that the union $\bigcup_{N^k \in \mathcal{R}} S \odot N^k$ is to be heard at the collection level.

Property 7. Transitions are of the form $S \xrightarrow{1} S \odot \mathcal{R}$.
(That is, of the form $S \xrightarrow{1} \bigcup_{N^k \in \mathcal{R}} S \odot N^k$.)

Example 11. When $\mathcal{R} = \mathcal{R}_\blacklozenge$, all transitions are of the form $S \xrightarrow{1} [T, T \cup V]$, where $\underline{\Phi}(S,N) = T = \{i : p_i(S) = 1\}$ and $\overline{\Phi}(S,N) = T \cup V = \{i : p_i(S) > 0\}$, that is, we recover Property 1 of the synchronous scheme. When $\mathcal{R} = \mathcal{R}_\Diamond$, transitions are of the form $S \xrightarrow{1} S \odot K$, where K is the set of agents i such that either $i \notin S$ and $p_i(S) > 0$ or $i \in S$ and $p_i(S) < 1$. Property 7 allows to write transitions in a functional form, despite that the process is non-deterministic; indeed, writing $\mathbf{F}(S) = S \odot \mathcal{R}$, we have $S \xrightarrow{1} \mathbf{F}(S)$, which notation is familiar in the literature on Boolean networks.

[24]The set of dissatisfied agents is analogous to the derivative in analysis, since it expresses to inclination of the current state to evolve. Knowing the initial state and the dissatisfied agents at each point of the process suffices to compute the transitions, just like in the theory of ordinary differential equations where, under some conditions, the initial condition and the derivative suffice to recover the entire function.

It is straightforward to characterise an absorbing state with the notions introduced. Not surprisingly, a state is absorbing when no agent is dissatisfied.

Proposition 4. (Absorbing states) The following properties are equivalent:

(i) S is absorbing ($S \xrightarrow{1} S$) for some $\mathcal{R}$.

(ii) S is absorbing ($S \xrightarrow{1} S$) for all $\mathcal{R}$.

(iii) $\overline{\Phi}(S, N) = \emptyset$ (no agent is dissatisfied at S).

(iv) $S \odot \mathcal{R} = S \odot \emptyset$ ($= S$).

Proof. Only the implication (iii) $\Rightarrow$ (iv) is not obvious. It comes from the observation that if $\mathcal{R}'$ coincides with $\mathcal{R}$ on the set $\overline{\Phi}(S, N)$ of dissatisfied agents, then $S \odot \mathcal{R} = S \odot \mathcal{R}'$. □

Theorem 4 adapts Theorem 1 to any $\mathcal{R}$:

Theorem 4. (Absorbing classes with any updating scheme) Consider an influence process based on aggregation functions $\mathbf{p}$ and any updating scheme $\mathcal{R}$. Absorbing classes are:

(i) either singletons (absorbing states) $\{S\}$, $S \in 2^N$,

(ii) or collections $\mathcal{C} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_p$, where each collection $\mathcal{L}_j$ is a loop.

Proof. Absorbing classes can be singletons: for example, $\emptyset$ and N are absorbing in a conformist model. Now, let $\mathcal{C}$ be an absorbing class which is not a singleton, and let $S_1 \in \mathcal{C}$. Since $\mathcal{C}$ is absorbing, there exists a sequence $S_1 \to S_2 \to \ldots \to S_1$. We call $\mathcal{L}_1$ this sequence and, by assumption, $\mathcal{L}_1 \in \mathcal{C}$. If there exists $S_1' \in \mathcal{L}_1$ such that $S_1' \to S_2' \notin \mathcal{L}_1$, then there exists a sequence $S_1' \to S_2' \to \ldots \to S_1'$. We call $\mathcal{L}_2$ this sequence. Pursuing the process with the states of $\mathcal{L}_1 \cup \mathcal{L}_2$, and so on, we get a union of loops $\bigcup_{k=1\cdots p} \mathcal{L}_k$ such that, finally (because of the finite number of states), there exists no $S \in \bigcup_{k=1\cdots p} \mathcal{L}_k$ and $T \notin \bigcup_{k=1\cdots p} \mathcal{L}_k$ with $S \to T$. This means that $\bigcup_{k=1\cdots p} \mathcal{L}_k$ is an absorbing class. (If $p = 1$, $\mathcal{L}_1$ is a cycle.)

□

Proposition 4 stated that the fact that S is an absorbing state does not depend on $\mathcal{R}$. This convenient result does not hold in general for other dynamics. Since cycles can induce jumps in the cardinality of states, they are typically destroyed when switching from the synchronous to the asynchronous framework. As for the third kind of dynamics (Theorem 1), namely unions of intervals, which is more various, nothing can be said in general. However intervals (not *unions* of intervals) roughly speaking are robust in the asynchronous model, as stated by Proposition 5. This is not surprising since absorbing intervals are some kind of blurry absorbing states

Proposition 5. If $[S, S \cup K]$ is an absorbing class in $\mathbf{p}^{\mathcal{R}}$ for some $\mathcal{R}$, then for any $\mathcal{R}'$ there exists a non-empty sub-collection of $[S, S \cup K]$ which is absorbing in $\mathbf{p}^{\mathcal{R}'}$.

Proof. It suffices to show that in $(N, \mathbf{p}, \mathcal{R})$, no arc of the transition graph goes from some state of $[S, S \cup K]$ outside $[S, S \cup K]$. Since $[S, S \cup K]$ is absorbing in $\mathcal{R}$, $p_i(T) = 1$ for all $i \in S$ and $T \in [S, S \cup K]$, $p_i(T) = 0$ for all $i \notin S \cup K$ and $T \in [S, S \cup K]$. These necessary conditions are still valid in $\mathcal{R}'$. As a consequence, a process starting from $S_0 \in [S, S \cup K]$ ends up in an absorbing class which is a sub-collection of $[S, S \cup K]$.

□

Example 12. In general, absorbing intervals shrink when switching to any other $\mathcal{R}$. Assume that:

$$\mathcal{P}_i^* = \begin{cases} \{S\} & \text{for all } i \in K \\ \emptyset & \text{for } i \in K^c \end{cases} \quad \text{and } \mathcal{P}_i^1 = \begin{cases} [S, S\cup K] & \text{for all } i \in S \\ [S, S\cup K]^c & \text{for all } i \in S^c \end{cases}.$$

In the synchronous framework $[S, S\cup K]$ is absorbing (more precisely, $S \overset{1}{\to} [S, S\cup K] \overset{1}{\to} S$), while in the asynchronous framework, $\{S, S\cup i_1, \ldots, S\cup i_k\}$ is an absorbing class, where $K = \{i_1, \ldots, i_k\}$.

The right and left transformations do not make sense anymore in the non-synchronous framework: only the left-right one remains. Theorem 5, stated in any updating partition $\mathcal{R}$, is proved similarly as Theorem 3, except that we must treat the left and the right together, whereas in Theorem 3 we decomposed the left-right as the composition of the left and the right. It also generalises Proposition 2.

Theorem 5. Assume that **p'** is obtained from **p** by a Z-left-right transformation. Then:

- $\mathcal{C}$ is an absorbing class of **p** if and only if $\mathcal{C}\Delta Z$ is an absorbing class of **p'**.
- $\mathcal{B}_i^{\mathbf{p'}} = \mathcal{B}_i^{\mathbf{p}}\Delta Z$ for all $i \notin Z$, and $\mathcal{B}_i^{\mathbf{p'}} = \mathcal{B}_i^{\mathbf{p}}\Delta Z^c$ for all $i \in Z$.

Proof. The Z-transformation is a graph homomorphism (preservation of the structure); hence the first statement. Let us establish the second one. For any $S \in 2^N$, there exists T, K such that $S \overset{1}{\underset{\mathbf{p}}{\to}} S \odot \mathcal{R}$. The probabilities for agents to say yes can be recovered from the transition graph as follows:

$$p_i^{\mathbf{p}}(S) = \sum_{\substack{T' \ni i \\ T' \in S\odot\mathcal{R}}} b_{S,T'}^{\mathbf{p}} \tag{B.2}$$

Let **p'** obtained by a Z-left-right transformation. By definition, for all $S, T \in 2^N$: $b_{S,T}^{\mathbf{p}} = b_{S\Delta Z, T\Delta Z}^{\mathbf{p'}}$. That is:

$$S\Delta Z \overset{1}{\underset{\mathbf{p'}}{\to}} (S\Delta Z) \odot \mathcal{R} \tag{B.3}$$

with:

$$p_i^{\mathbf{p'}}(S\Delta Z) = \sum_{\substack{T' \ni i \\ T' \in (S\Delta Z)\odot\mathcal{R}}} b_{S\Delta Z, T'}^{\mathbf{p'}}. \tag{B.4}$$

By the same argument as in the proof of Theorem 3, $\mathcal{D}^{\mathbf{p}}(S,i)$ being the collection of sets $T' \in S \odot \mathcal{R}$ containing i, for the transition $S \overset{1}{\underset{\mathbf{p}}{\to}} S \odot \mathcal{R}$:

$$p_i^{\mathbf{p'}}(S\Delta Z) = \begin{cases} \displaystyle\sum_{T' \in Z\Delta\mathcal{D}^{\mathbf{p}}(S\Delta Z, i)} b_{S\Delta Z, T'}^{\mathbf{p'}} & \text{if } i \notin Z \\ \displaystyle\sum_{T' \in Z\Delta[(S\odot\mathcal{R})\setminus\mathcal{D}^{\mathbf{p}}(S\Delta Z, i)]} b_{S\Delta Z, T'}^{\mathbf{p'}} & \text{if } i \in Z \end{cases} \tag{B.5}$$

If $i \notin Z$, then $p_i^{\mathbf{p'}} = p_i^{\mathbf{p}}$. If $i \in Z$, then $p_i^{\mathbf{p'}} = 1 - p_i^{\mathbf{p}}$.
We get that finally for all i and $B \in \mathcal{B}_i^{\mathbf{p}}$:

$$S \preccurlyeq_B S' \Leftrightarrow S\Delta Z \preccurlyeq_{B\Delta Z} S'\Delta Z \Rightarrow \begin{cases} p_i^{\mathbf{p'}}(S\Delta Z) \le p_i^{\mathbf{p'}}(S'\Delta Z) & \text{if } i \notin Z \\ p_i^{\mathbf{p'}}(S\Delta Z) \ge p_i^{\mathbf{p'}}(S'\Delta Z) & \text{if } i \in Z \end{cases} \tag{B.6}$$

Therefore, $B\Delta Z \in \mathcal{B}_i^{\mathbf{p}'}$ for $i \notin Z$ and $(B\Delta Z)^c \in \mathcal{B}_i^{\mathbf{p}'}$ for $i \in Z$ (then, note that $(B\Delta Z)^c = B\Delta Z^c$). The reverse inclusions work the same way.

□

Proposition 6 states that in the asynchronous scheme, the process always reaches an absorbing state when all agents are conformist. This was wrong in the synchronous framework; hence, an asynchronous framework plays in favour of intuition.

Proposition 6. (Conformist and asynchronous model) Assume that $\mathcal{R} = \mathcal{R}_\Diamond$, that all agents are conformist and that Assumption BC holds. Then the process ends up in an absorbing state.

Proof. By Assumption BC, and since all agents are conformist, $\emptyset \xrightarrow{1} \emptyset$ and $N \xrightarrow{1} N$. Assume that there exists an absorbing class of the second type: $\mathcal{C} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_k$, where each collection $\mathcal{L}_j$ is a loop. Let S be a set of maximal cardinality in $\mathcal{C}$. Since $\emptyset \xrightarrow{1} \emptyset$, $S \neq \emptyset$. Since $\mathcal{R} = \mathcal{R}_\Diamond$, transitions between two distinct sets differ by only one agent. We distinguish between two cases:

- *Case (i)*: $S \nrightarrow S$. This implies that $p_i(S) = 0$ for all $i \in S$, otherwise, an agent $i \in S$ such that $p_i(S) > 0$, if invited to update, could maintain its opinion, which would imply that $S \to S$. Since S is of maximal cardinality in $\mathcal{C}$, we also have $p_i(S) = 0$ for all $i \notin S$. But then, if $S \neq N$, the existence of an agent $j \in S^c$ who would say no if invited to update implies that $S \to S$, a contradiction. This implies that $S = N$. Since, $N \xrightarrow{1} N$, we get a contradiction with $S \nrightarrow S$. Hence this case can be eliminated.
- *Case (ii)*: $S \to S$. Since S is of maximal cardinality in $\mathcal{C}$, $p_i(S) = 0$ for all $i \notin S$. Since by assumption S is not absorbing, there exists $i \in S$ such that $p_i(S) = 0$. Since i is conformist, $p_i(T) = 0$ for all $T \in\!\downarrow S$. Therefore there exists $T_1 \not\ni i$ and a loop $\mathcal{L} \in \mathcal{C}$ containing the transition $T_1 \to T_1 \cup i \neq S$, and necessarily $T_1 \nsubseteq S$. Consider a walk $\mathcal{W}$ from T_1 to S: $T_1 \to \ldots \to S \setminus k \to S$. We call T_2 the last state of the walk, from T_1, which does not belong to $\downarrow S$, so that $T_2 \to T_3 \to \ldots \to T_k \to S$ with $T_3 \to \ldots \to S$ a walk of states in $\downarrow S$. Since $T_2 \notin\!\downarrow T_3$ (otherwise it would belong to $\downarrow S$), we can write $T_2 = T_3 \cup j$ for some j. But then, since all agents are conformist, $T_2 = T_3 \cup j \to \ldots \to T_k \cup j \to S \cup j$, contradicting that S is of maximal cardinality in $\mathcal{C}$. This case can therefore be eliminated too.

We proved that unions of loops cannot be absorbing. But then, Theorem 4 ensures that all absorbing classes are actually states.

□

The specificities of the asynchronous framework, as compared with the synchronous one, evidence the crucial interest of investigating the spectrums of dynamics indexed by the updating schemes $\mathcal{R}$ from $\mathcal{R}_\blacklozenge$ to $\mathcal{R}_\Diamond$.

C Utility foundations of p

The aim of this appendix is to provide a utility justification to the fundamental assumption of the present paper, namely, $\mathcal{B}_i \neq \emptyset$ for all $i \in N$, which can be reformulated as follows:

$$\text{For any } i \in N, \text{ there exists } B \text{ such that } S_1 \preccurlyeq_B S_2 \Rightarrow p_i(S_1) \leq p_i(S_2). \tag{C.1}$$

We are going to redefine the monotonicity collection with utilities, denoting this collection $\mathcal{B}^u$, which will happen to be a subset of $\mathcal{B}$. There are no technicalities in this appendix: it

simply aims at evidencing the articulations between each assumption. In particular, we point out where exactly one can insert psychological and cognitive considerations.

C.1. Active/Passive utility functions

We distinguish between two kinds of utility functions. The passive utility function only attributes to the agent a level of satisfaction at each state of the world. The active one, as compared with the passive one, evaluates the utility of an opinion adoption at a given state of the world. Since no confusion is possible, we use the same letter to designate the two.

Definition 13. (Active and passive utility functions)

- A *passive utility function* u_i is a function: $2^N \to \mathbb{R}$ which attributes to agent i a utility $u_i(S)$ at a state of the world $S \in 2^N$.
- An *active utility function* is a function u_i: $\{0,1\} \times 2^N \to \mathbb{R}$ which attributes to agent i a utility $u_i(x,S)$ for an opinion $x \in \{0,1\}$ at a state of the world $S \in 2^N$.

The articulation between the active to the passive utility function is the flexible space where we can impose psychological, intellectual or informational assumptions, as discussed in Examples 13 and 14. To remain general, our model is built with active utility functions.

Example 13. The identification:

$$u_i(x,S) \equiv \begin{cases} u_i(S \cup i) \text{ if } x = 1 \\ u_i(S \setminus i) \text{ if } x = 0 \end{cases}$$

comes down to assuming that agent i only takes into account its deviation into its new passive utility function, either because the impact of his adoption on the rest of the population is neglectable, or because he cannot compute the effect of his adoption on the adoption of his neighbours. It is a specific kind of myopic behaviour. It can be shown that this agent cannot exert negative influence on itself, an option which in some circumstances can be defensible, as explained in Section 3.3.

Example 14. (Consistent utility) If for any $S \in 2^N$ it holds that $u(0, S \setminus i) = u(0,S)$ and $u(1, S \cup i) = u(1,S)$, the agent i is said to be *consistent*; this notion captures some idea of stability in decision making, or in the utility conferred to the possession on object (Girard, 1966, 1977). If the identification of Example 13 holds, then the agent is consistent.

C.2. Opinion adoption under a myopic behaviour

The model which we now expose the foundations is embedded into a myopic framework. This is be heard in two aspects:

(1) **Agents are more inclined to adopt the opinion of their friends.** At first sight, it might be hard to see why it is a myopic behaviour. This is because this assumption suppresses any kind of strategic aspect and, in particular, any kind of manipulation. In this sense, our model is close to the concerns of physicists, who study particles rather than clever agents. This assumption is essential, e.g, in mean-field games Guéant et al. (2010), or in financial networks (Yang, 2013), where agents cannot observe the entire structure of interactions. Sometimes also, our neighbours are simply agents that we *see*, be on an aggregate or statistical scale only. For example, reading the newspapers with opinion

surveys on presidential elections connects the whole country to us; hence the number of "neighbours" is typically huge. In some circumstances, the network is complete, like in Grabisch et al. (2019a).

(2) **With regard to the fact that utility functions are stochastic**. Not only agents cannot compute the effect of their opinion adoption on the adoption of their neighbours, in particular do not know what the next state will be, but even if they knew it, their utility at this state would remain stochastic. Strzalecki (2017) lists a few reasons why utility may be stochastic, among which: fluctuating tastes, noisy signals (in our model, imperfect observation of the state of the world), trembling hands, experimentation. The idea of noisy signals or imperfect observation traces back to the experiments by Fechner (1860). For more details on stochastic utility functions, the reader is referred, e.g., to Camerer (1989) or Blavatskyy (2007).

We denote $\tilde{u} \leq \tilde{v}$ when $\tilde{v}$ first-order stochastically dominates $\tilde{u}$.

Definition 14. The *utility monotonicity collection* $\mathcal{B}_i^v$ is the collection of sets B such that:

$$S_1 \preccurlyeq_B S_2 \;\Rightarrow\; \begin{cases} \tilde{u}_i(1, S_1) \leq \tilde{u}_i(1, S_2) \\ \tilde{u}_i(0, S_1) \geq \tilde{u}_i(0, S_2) \end{cases}.$$

Proposition 7 formalises the idea that if we prefer to choose a given opinion *even* when a friend does not choose it, then *a fortiori* we still prefer to choose this opinion when this friend chooses it, ceteris paribus.

Proposition 7. The following assertions are equivalent:

(1) $\mathcal{B}_i^v \neq \emptyset$.

(2) For any $j \in N$:

- either $\tilde{u}_i(1, S \cup j) \geq \tilde{u}_i(1, S \setminus j)$ and $\tilde{u}_i(0, S \cup j) \leq \tilde{u}_i(0, S \setminus j)$ for all S (in which case j is said to have a *positive utility-influence* on i, and a *strictly positive utility-influence* on i if at least one inequality at some state S is strict),
- or $\tilde{u}_i(0, S \cup j) \geq \tilde{u}_i(0, S \setminus j)$ and $\tilde{u}_i(1, S \cup j) \leq \tilde{u}_i(1, S \setminus j)$ for all S (in which case j is said to have a *negative utility-influence* on i, and a *strictly negative utility-influence* on i if at least one inequality at some state S is strict).

A positive and negative utility-influence is called *null*. An agent exerting a null utility-influence on a given agent is also said to be *irrelevant* for him.

Proof. (Proposition 7)

($\Rightarrow$) Since $\mathcal{B}_i^v \neq \emptyset$, there exists $B \in \mathcal{B}_i^v$. Let $j \in N$. Either $j \in B$, and then $\tilde{u}_i(1, S \setminus j) \leq \tilde{u}_i(1, S \cup j)$ for all $S \in 2^N$ and $\tilde{u}_i(0, S \setminus j) \geq \tilde{u}_i(0, S \cup j)$ for all $S \in 2^N$; or $j \notin B$, and then $\tilde{u}_i(1, S \setminus j) \geq \tilde{u}_i(1, S \cup j)$ for all $S \in 2^N$ and $\tilde{u}_i(0, S \setminus j) \leq \tilde{u}_i(0, S \cup j)$ for all $S \in 2^N$.

($\Leftarrow$) Assume that there exists $j \in N$ such that the following two assertions hold:

(a) there exists S such that:

$$\tilde{u}_i(1, S \cup j) < \tilde{u}_i(1, S \setminus j) \text{ or } \tilde{u}_i(0, S \cup j) > \tilde{u}_i(0, S \setminus j)$$

(b) there exists S' such that:

$$\tilde{u}_i(0, S' \cup j) < \tilde{u}_i(0, S' \setminus j) \text{ or } \tilde{u}_i(1, S' \cup j) > \tilde{u}_i(1, S' \setminus j).$$

If there exists $B \in \mathcal{B}_i^v$, then either $j \in B$, which contradicts (a), or $j \notin B$, which contradicts (b).

□

The point (2) states that any given agent is either a friend or an enemy, *independently on the state of the world*. This sounds reasonable in many circumstances. As a consequence, we are in a position to admit the following Fundamental Assumption I:

Fundamental Assumption I. $\mathcal{B}_i^v \neq \emptyset$ for all $i \in N$.

Example 15. Let 1_S be the indicator function of S, i.e., $1_S(i) = 1$ if $i \in S$ and 0 otherwise. The following utility function for agent i:

$$u_i(S) := |\{j \in N, 1_S(i) = 1_S(j)\}| - |\{j \in N, 1_S(i) \neq 1_S(j)\}|,$$

is increasing in the number of agents sharing his opinion. In particular, $u_i(N) = u_i(\emptyset) \geq u_i(S)$ for any $S \subseteq N$. This aggregation function is the one a conformist agent, i.e., an agent who like sharing the opinion of as many as possible agents[25].

By a *strategy* $\tilde{x}$, we mean a random variable which takes values in $\{0, 1\}$. Let $p_i(S)$ be the probability for agent i to say yes at the next period when the current state is S. :

$$p_i(S) := \mathbb{P}(\tilde{x} = 1 \text{ at state } S). \tag{C.2}$$

We now fill the gap between $\tilde{u}_i$ and p_i. The following second fondamental assumption is a praxeological axiom.

Fundamental Assumption II. $\begin{cases} \tilde{u}_i(1, S_1) \leq \tilde{u}_i(1, S_2) \\ \tilde{u}_i(0, S_1) \geq \tilde{u}_i(0, S_2) \end{cases} \Rightarrow p_i(S_1) \leq p_i(S_2).$

Since first-order stochastic dominance implies expected utility domination, the Fundamental Assumption II is weaker that the maximum expected utility axiom. This means that we are not imposing a priori that agents should choose their opinion according to expected utilities (a challenging and strong axiom; consider, e.g., the St. Petersburg paradox), but only that they should abide by a minimal consistency in their decision making.

[25]In our model, there is neither any strategic aspect nor distinction between opinion and communication, since agents opinions are well-observable. However it would be possible to frame the model in a signalling framework. There is no impediment to encompass imperfect observation, either of the true state of the world that agents would try to extract, or of the decision of agents, which latter case comes down to considering votes and communication separately. In the synchronous updating framework with perfect observation, like in Example 15, the only way an agent could possibly manipulate its vote would be by creating a cascade of opinion in its favour. Grabisch et al. (2019a) evidences that such cascades can be triggered by the presence of anti-conformist agents, in a non-strategic setting though. The literature on herd behaviour, fads and fashion was initiated by Banerjee (1992) and Bikhchandani et al. (1992), in which papers the updating is sequential (a special kind of asynchronism in one shot). Other models of herds and cascades are gathered in Chamley (2004). There exists an important literature on deliberation and voting encompassing strategic behaviour, e.g., Austen-Smith and Banks (1996) which challenges the role of the sincerity assumption in the Condorcet Jury theorem, and models of jury decision making (Feddersen and Pesendorfer, 1998; Coughlan, 2000; Gerardi and Yariv, 2007). In the literature of information extraction from signals, Nitzan and Paroush (1982) and Shapley and Grofman (1984) are concerned with the optimal weighted voting rule aggregating the choices of agents receiving signals about the true state of the world; based on which optimal weights Buechel and Mechtenberg (2019) proposes a one round setting with common interest of agents, where the influence graph induces correlated information among voters.

Remark 3. If the utility function of the agent is deterministic, then its aggregation function is Boolean, but that the reverse might not hold.

From Assumption II, we get the following property:

Property 8. $\mathcal{B}_i^v \subseteq \mathcal{B}_i$.
(That is, for any $B \in \mathcal{B}_i^v$, we have the implication $S_1 \preccurlyeq_B S_2 \Rightarrow p_i(S_1) \leq p_i(S_2)$.)

Since the Fundamental Assumption I states that $\mathcal{B}_i^v \neq \emptyset$, the non-emptiness of $\mathcal{B}_i$ is implied by Property 8. As a consequence, we have established (C.1). Note that the inclusion can actually be strict. In general, for a given $\mathbf{u} := (u_i)_{i \in N}$, there exists many $(p_i)_{i \in N}$ which do not violate Assumptions I and II. The $\mathbf{p}$ chosen can result in a "loss of information" in the following sense:

Corollary 3. If j exerts a **strictly positive** utility-influence on i, then it exerts a **positive or null** influence on i (in the sense of Definition 5). If j exerts a **strictly negative** utility-influence on i, then it exerts a **negative or null** influence on i.

Proof. $\mathcal{B}^u = [\underline{B}^u, \overline{B}^u] \subseteq \mathcal{B} = [\underline{B}, \overline{B}]$ if and only if $\underline{B} \subseteq \underline{B}^u \subseteq \overline{B}^u \subseteq \overline{B}$. In particular, $\underline{B}^u \subseteq \overline{B}$ and $(\overline{B}^u)^c \subseteq \underline{B}^c$. □

In general, these inclusions can be strict. In other words, an agent j exerting a strict positive utility-influence on i, can very well be an irrelevant agent in p_i[26]. The following example enlightens the behavioural assumptions which one might enclose in the articulation from $\mathbf{u}$ to $\mathbf{p}$.

Example 16. An agent i such that $p_i(S) = 1$ (resp. 0) if and only if $\tilde{u}_i(1, S) \geq \tilde{u}_i(0, S)$ (resp. $\leq$) is an agent which chooses an opinion for sure, yes or no, when and only when one of the two opinion first-order stochastically dominates the other. First-order stochastic dominance is a rather strong requirement but not over-demanding. We call this agent *reasonable*; this notion is involved in Proposition 8. Another assumption that we could impose is the following: $p_i(S) > 0 \Leftrightarrow \mathbb{P}[\tilde{u}_i(1, S) > \tilde{u}_i(0, S)] \neq 0$ and $p_i(S) < 1 \Leftrightarrow \mathbb{P}[\tilde{u}_i(1, S) < \tilde{u}_i(0, S)] \neq 0$. Under this assumption, the agent never dodges entirely an opinion as long as it could possibly grant him with higher utility ex post. This captures some kind of extreme prudence; hence such an agent is call *prudent*. In Example 17, the agent is assumed to be prudent.

Of course, cognitive, psychological, etc. behaviours, by shaping the u_i which in turn determine the p_i, have a central role in the process. But since the link from the u_i to the p_i induces a loss of information, we can imagine some behaviours to be undistinguishable from one another or to disappear at the aggregation function scale. More generally, some behaviours could be absorbed into the successive losses of information: from the u_i to the p_i, from the p_i to $\tilde{\mathbf{B}}$, from $\tilde{\mathbf{B}}$ to the absorbing classes, thus, it is reasonable to argue that some behaviours would not perturb the dynamics and that various behaviours could produce the same opinion dynamics.

Example 17. In this example, we make the following assumptions on agent i:

(i) he is conformist;

(ii) he aggregates anonymously (only the cardinality of the states enters into its utility function so that $\tilde{u}(x, S) \equiv \tilde{u}(x, s)$ and $p_i(S) \equiv p_i(s)$);

[26] On a practical point of view, this remark can be borne in mind as a criterium to discriminate among several possible partitions into groups.

(iii) he is prudent (see Example 16);

The first two assumptions translate into:

$$s_1 \leq s_2 \Rightarrow \tilde{u}_i(1, s_1) \leq \tilde{u}_i(1, s_2) \text{ and } \tilde{u}_i(0, s_1) \geq \tilde{u}_i(1, s_2). \tag{C.3}$$

Let:

$$\underline{u}_i(x, s) := \inf\{u_i(x, s) \mid f_{\tilde{u}_i(x,s)} \neq 0\}. \tag{C.4}$$

$$\overline{u}_i(x, s) := \sup\{u_i(x, s) \mid f_{\tilde{u}_i(x,s)} \neq 0\}, \tag{C.5}$$

where $f_{\tilde{u}_i(x,s)}$ is the density probability function of $\tilde{u}_i(x, s)$. In other words, the intervals $[\underline{u}_i(0, s), \overline{u}_i(0, s)]$ and $[\underline{u}_i(1, s), \overline{u}_i(1, s)]$ are the smallest intervals containing the support of density of $\tilde{u}(0, s)$ and $\tilde{u}(1, s)$. Roughly speaking, it is a stochastic version of Example 15. The functions $\underline{u}_i(0, s)$ and $\overline{u}_i(0, s)$ are represented in red. The functions $\underline{u}_i(1, s)$ and $\overline{u}_i(1, s)$ are represented in green. Because of (C.3), $\underline{u}_i(1, s)$ and $\overline{u}_i(1, s)$ are increasing in s and $\underline{u}_i(0, s)$ and $\overline{u}_i(0, s)$ are decreasing in s. The prudence assumption operates when drawing p_i: $p_i(s) = 1 \Leftrightarrow \underline{u}_i(1, s) \geq \overline{u}_i(0, s)$ and $p_i(s) = 0 \Leftrightarrow \underline{u}_i(0, s) \geq \overline{u}_i(1, s)$. A similar example can be found in Grabisch et al. (2019a). Incidentally, this example evidences the connection of our model with fuzzy hypothesis testing; see, e.g., Chukhrova and Johannssen (2020). To fix ideas, we have represented the intermediate part of $p_i(s)$ as a linear function, but it only has to be increasing. The sigmoid function capturing phase-transitions by allowing mixed decisions/compositions is familiar to natural sciences, e.g., the Doppler effect when the vehicle passes nearby the observer, or the titration curves in chemistry.

C.3. Connection between absorbing states and Nash equilibria

The absorbing states correspond to stable opinions: agents in S say yes forever, agents outside S say no forever. The connection with Nash equilibria in clarified in Proposition 8. The notion of reasonability is introduced in Example 16.

Definition 15. We say that S is a *Nash equilibrium* if $\tilde{u}_i(1, S) \geq \tilde{u}_i(0, S)$ for all $i \in S$ and $\tilde{u}_i(1, S) \leq \tilde{u}_i(0, S)$ for all $i \notin S$.

Proposition 8. Assume that all agents are reasonable. Then, S is an absorbing state if and only if it is a Nash equilibrium.

D Algebraic considerations on $\preccurlyeq_B$ and the analysis of coalitions

Let $f_B : \begin{cases} 2^N \to 2^N \\ S \mapsto S\Delta B^c. \end{cases}$ It is a bijection from 2^N onto itself. Note that $f_N = \mathrm{id}$ and that $f_\emptyset$ transforms each set into its complement. Importantly, 2^N endowed with the partial order $\preccurlyeq_B$ is a lattice $\mathcal{L}_B := (2^N, \preccurlyeq_B)$, with top element and bottom elements B and B^c, and with infimum and supremum $\wedge_B$, $\vee_B$ given for any $S, T \in 2^N$ by:

$$S \wedge_B T := [(S \cap T) \cap B] \cup [(S \cup T) \cap B^c] \tag{D.1}$$

$$S \vee_B T := [(S \cup T) \cap B] \cup [(S \cap T) \cap B^c]. \tag{D.2}$$

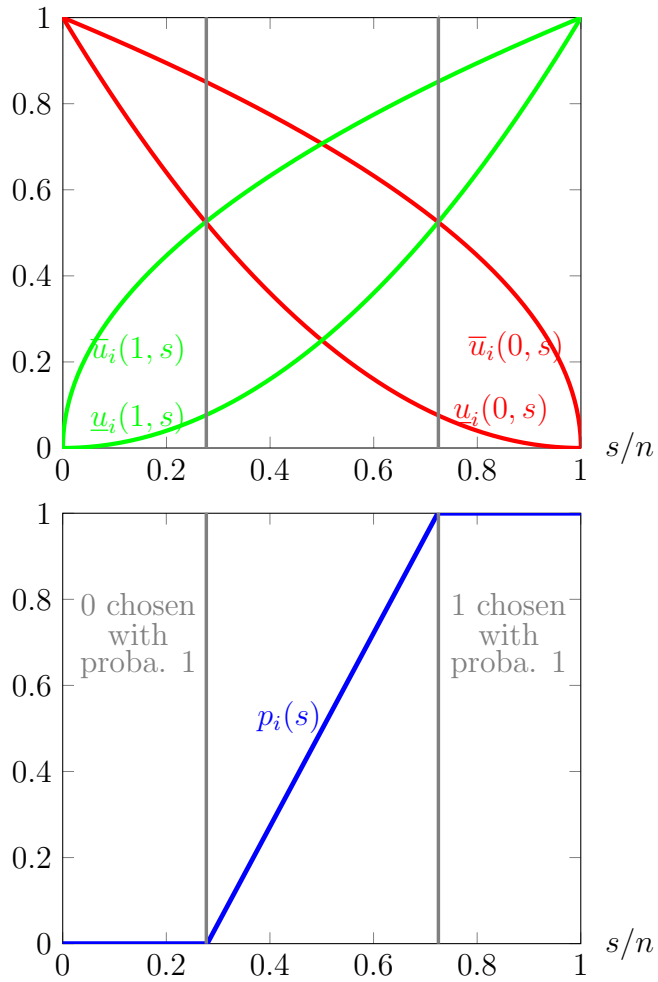


Figure C.1: Opinion adoption for a prudent agent

For $\wedge_B$, we take the intersection within B and the union outside B. For $\vee_B$, we take the union within B and the intersection outside B (see Figure D.1). $\wedge_B$ and $\vee_B$ coincide with $\cap$ and $\cup$ when $B = N$. We can also define the *upset* and *downset* of a given set S under the partial order $\preccurlyeq_B$, by $\uparrow_B S := \{S' \mid S \preccurlyeq_B S'\}$ and $\downarrow_B := \{S' \mid S' \preccurlyeq_B S\}$. If $B = N$, we simply denote them by $\uparrow S$ $(= [S, N])$ and $\downarrow S$ $(= [\emptyset, S])$.

f_B is a lattice isomorphism from $\mathcal{L}_N = (2^N, \subseteq)$ to $\mathcal{L}_B = (2^N, \preccurlyeq_B)$. Indeed, for any $S, S' \in 2^N$:

$$f_B(S \cap S') = f_B(S) \wedge_B f_B(S') \tag{D.3}$$

and

$$f_B(S \cup S') = f_B(S) \vee_B f_B(S'). \tag{D.4}$$

Note that: (i) $f_B(\emptyset) = B^c$; (ii) $f_B(N) = B$ (bounded-lattice properties) and that we can write

$$S_1 \preccurlyeq_B S_2 \Leftrightarrow f_B^{-1}(S_1) \subseteq f_B^{-1}(S_2) \Leftrightarrow f_B(S_1) \subseteq f_B(S_2) \tag{D.5}$$

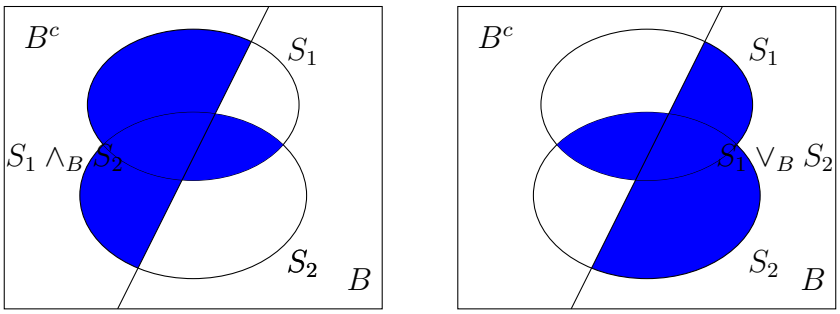


Figure D.1: Infimum and supremum

where the first equivalence comes from the fact that f_B is a lattice isomorphism and the second equivalence comes from the fact that f_B is an involution: $f_B \circ f_B = \mathrm{id}$. In particular, we recover Property 6 exposed in the proof of Theorem 3. Note also that $f_{B^c}(S) = f_B(S^c)$.

We now define the notion of yes and no influential coalitions, introduced in Grabisch and Rusinowska (2013) in the conformist society, which can be generalised in our heterogeneous setting using f_B:

Definition 16. (Yes/No influential coalitions) Let $i \in N$ with its monotonicity collection $\mathcal{B}_i = [\underline{B}_i, \overline{B}_i]$:

A set S is a yes-influential coalition on $i \in N$ if:

- $p_i(f_{\overline{B}_i}(S)) > 0$,
- $p_i(f_{\overline{B}_i}(S')) = 0$ for all $S' \subset S$.

A set S is a no-influential coalition on $i \in N$ if:

- $p_i(f_{\overline{B}_i^c}(S)) < 1$,
- $p_i(f_{\overline{B}_i^c}(S')) = 1$ for all $S' \subset S$.

The collections of yes and no-influential coalitions on i are denoted by $\mathcal{Y}_i$ and $\mathcal{N}_i$. Note that $\mathcal{Y}_i$ and $\mathcal{N}_i$ are left unchanged by the operation f_B on states (relabelling).

Example 18. If $\overline{B}_i = N$ (conformist agent), then $f_{\overline{B}_i}(S) = S\Delta\emptyset = S$. Hence, S is a yes-influential coalition on $i \in N$ if $p_i(S) > 0$ and $p_i(S') = 0$ for all $S' \subset S$. Hence, our definition of $\mathcal{Y}_i$ indeed embraces Grabisch and Rusinowska (2013) (same reasoning for $\mathcal{N}_i$). See Figure D.2.

Stating the definition with $\overline{B}_i$ rather than with any other $B \in \mathcal{B}_i$ is a matter of taste since the yes and no-influential coalitions contain no irrelevant agents. Indeed, we can easily show that:

$$\bigcup \mathcal{Y}_i \cup \bigcup \mathcal{N}_i \subseteq \{j \mid \phi(j \to i) \neq 0\} \quad (= \underline{B}_i \cup \overline{B}_i^c). \tag{D.6}$$

In general this inclusion is strict but we can prove that equality holds when p_i is Boolean.

Theorem 6 generalises the first theorem of Grabisch and Rusinowska (2013) on the transition graph.

Theorem 6. Assume that Assumption BC holds.

(1) (From $\tilde{\mathbf{B}}$ to $(\mathcal{Y}_i, \mathcal{N}_i)_{i \in N}$)
For all $S, T \in 2^N$, $\tilde{b}_{S,T} = 1$ if and only if:

– For each $i \notin T$, there exists a non-empty $M \subseteq f_{B_i}(S^c)$ (where $B_i \in \mathcal{B}_i$) such that M is no-influential on i.
– For each $i \in T$, there exists a non-empty $Y \subseteq f_{B_i}(S)$ (where $B_i \in \mathcal{B}_i$) such that Y is yes-influential on i and

(2) (From $\mathbf{B}$ to $(\mathcal{Y}_i, \mathcal{N}_i)_{i \in N}$)

Suppose that $S \xrightarrow{1} T$. Then, there cannot be a yes-influential coalition $Y \subseteq f_{B_i}(S)$ on i (where $B_i \in \mathcal{B}_i$) if $i \notin T$, or a no-influential coalition $M \subseteq f_{B_i^c}(S)$ on i (where $B_i \in \mathcal{B}_i$) if $i \in T$.

Proof. (1) $S \to T$ if and only if $p_i(S) > 0$ for all $i \in T$ and $p_i(S) < 1$ for all $i \notin T$. For $B_i \in \mathcal{B}_i$, from the monotonicity of p_i with respect to the $\preccurlyeq_{B_i}$ order and (from Assumption BC) the fact that $p_i(B_i^c) = 0$, we deduce that $p_i(S) > 0$ if and only if there exists a yes-influential coalition Y on i with $Y \subseteq S \Delta B_i^c = f_{B_i}(S)$. On the other hand, since p_i is monotonous with respect to the $\preccurlyeq_{B_i}$ order and (from Assumption BC) $p_i(B_i) = 1$; therefore, $p_i(S) < 1$ if and only if there exists a no-influential coalition M on i with $M \subseteq S \Delta B_i = f_{B_i^c}(S) = f_{B_i}(S^c)$.

(2) $S \xrightarrow{1} T$ is equivalent to $p_i(S) = 1$ if $i \in T$ and $p_i(S) = 0$ if $i \notin T$. If $Y \subseteq S \Delta B_i^c$ (by (D.5), this is equivalent to $Y \Delta B_i^c \preccurlyeq_{B_i} S$) is yes-influential on $i \notin T$, $0 < p_i(Y \Delta B_i^c) \leq p_i(S) = 0$, a contradiction. By the same token, if $M \subseteq S \Delta B_i$ (by (D.5), this is equivalent to $M \Delta B_i \preccurlyeq_{B_i^c} S$, i.e., $M \Delta B_i \succcurlyeq_{B_i} S$) is no-influential on $i \in T$, $1 = p_i(S) \leq p_i(M \Delta B_i) < 1$, a contradiction.

□

We can use Theorem 6 to establish the useful Property 9, namely that the yes and no-influential coalitions contain all the information on $\tilde{\mathbf{B}}$ (i.e., on $(\mathcal{P}_i^0, \mathcal{P}_i^1)_{i \in N}$), and vice versa. In particular, knowing them suffices to compute the absorbing classes. Figure D.3 completes Figure 5 by adding $(\mathcal{Y}_i, \mathcal{N}_i)_{i \in N}$.

Property 9. $\mathcal{Y}_i$ (respectively $\mathcal{N}_i$) and $\mathcal{P}_i^0$ (respectively $\mathcal{P}_i^1$) correspond bijectively.

Proof. It is useful to keep an eye on Figure D.2. Assume that the agent is conformist; since the yes and no-influential coalitions are invariant under a f_B transformation, this is without loss of generality. By definition, $\mathcal{P}_i^0$ (resp. $\mathcal{P}_i^1$) uniquely determines $\mathcal{Y}_i$ (resp. $\mathcal{N}_i$). We must show, conversely, that $\mathcal{Y}_i$ (resp. $\mathcal{N}_i$) uniquely determines $\mathcal{P}_i^0$ (resp. $\mathcal{P}_i^1$). Let us show that: $\mathcal{P}_i^* \cup \mathcal{P}_i^1 = \bigcup_{Y \in \mathcal{Y}_i} \uparrow Y$ and $\mathcal{P}_i^* \cup \mathcal{P}_i^0 = \bigcup_{M \in \mathcal{N}_i} \downarrow (N \setminus M)$. We have $\mathcal{P}_i^* \cup \mathcal{P}_i^1 \supseteq \bigcup_{Y \in \mathcal{Y}_i} \uparrow Y$, otherwise the monotonicity conditions would be violated. Conversely, $\mathcal{P}_i^* \cup \mathcal{P}_i^1 \subseteq \bigcup_{Y \in \mathcal{Y}_i} \uparrow Y$; indeed, if there existed $S \notin \bigcup_{Y \in \mathcal{Y}_i} \uparrow Y$ such that $p_i(S) > 0$, by Theorem 6.(1) there would exists $Y \in \downarrow S$ such that $Y \in \mathcal{Y}_i$, i.e., $S \in \uparrow Y$, which is absurd. (Similar argument for no-influential coalitions.) □

To see why the influence coalitions, and not the topology of the influence graph, are the appropriate tool to characterise opinion dynamics, let us have a look on Figure D.3. The influence graph, as captured with the signs of ϕ, is clearly separated from the absorbing classes: neither its topology nor the signs of its arcs can be exploited. We further discuss this problem in Appendix E. However the same figure clearly shows that the influential coalitions $(\mathcal{Y}_i, \mathcal{N}_i)_{i \in N}$ are the proper tool. This can be seen also in (D.6), where we see that agents involved in influential coalitions on i form a subset of agents exerting influence on him; in short, this means that the influence graph is too rough. Theorems 7 and 8 below generalise Theorem 3 and 4 of Grabisch and Rusinowska (2013) by characterising absorbing states and intervals using influential coalitions.

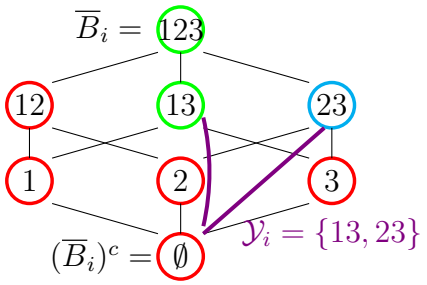


Figure D.2: Property 9: $\mathcal{Y}_i$ (violet) and $\mathcal{P}_i^0$ (red) correspond bijectively.

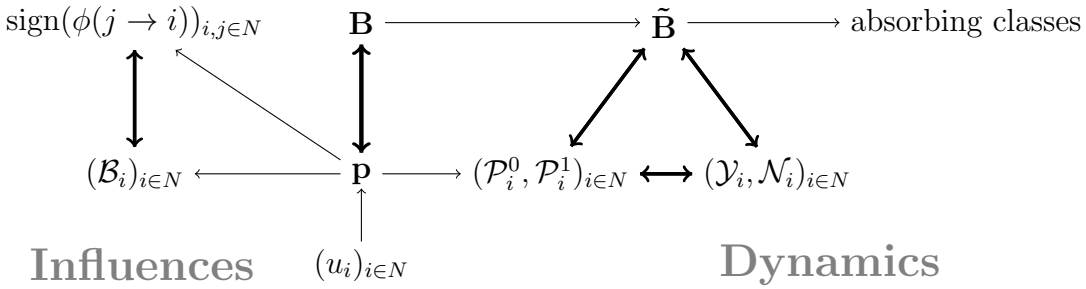


Figure D.3: Figure 5 completed with $(\mathcal{Y}_i, \mathcal{N}_i)_{i \in N}$

Theorem 7. Consider an influence process $\mathbf{B}$ based on aggregation functions $\mathbf{p}$. A subset $S \subseteq N$ is an absorbing state if and only if the following two conditions hold:

(i) For any $i \in S$ and $B_i \in \mathcal{B}_i$, there exists no $M \in \mathcal{N}_i$ such that $M \subseteq f_{B_i}(S^c)$.

(ii) For any $i \notin S$ and $B_i \in \mathcal{B}_i$, there exists no $Y \in \mathcal{Y}_i$ such that $Y \subseteq f_{B_i}(S)$.

Proof. S is absorbing if and only if $S \in \mathcal{P}_i^0$ for all $i \notin S$ and $S \in \mathcal{P}_i^1$ for all $i \in S$. We note that $M \subseteq f_{\overline{B}_i^c}(S) \Leftrightarrow S \preccurlyeq_{B_i} M \Delta B_i \Leftrightarrow S \notin \mathcal{P}_i^1$ and $Y \in \mathcal{Y}_i$ such that $Y \subseteq f_{\overline{B}_i}(S) \Leftrightarrow Y \Delta B_i^c \preccurlyeq_{B_i} S \Leftrightarrow S \notin \mathcal{P}_i^0$.

□

Example 19. Assume that that the society is conformist. By Theorem 7, S is an absorbing state if and only if the following two conditions hold:

(i) For any $i \in S$, there exists no $M \in \mathcal{N}_i$ such that $M \subseteq S^c$.

(ii) For any $i \notin S$, there exists no $Y \in \mathcal{Y}_i$ such that $Y \subseteq S$.

Let us moreover assume that all influential coalitions are actually singletons (in which case we refer to the influential coalitions as "influential agents"). Let the *graph of yes-influence* to be the directed graph $G^{yes} := (N, \mathcal{E})$ whose set of nodes is N and such that there is an arc (j, i) from j to i if j is yes-influential on i; we define the *graph of no-influence* similarly. We write $(G)^*$ the graph G with all arcs inverted. Corollary 2.(iii) in Grabisch and Rusinowska (2013) states that if there is an absorbing state $S \notin \{\emptyset, N\}$ if and only if $G := (G^{yes})^* \cup G^{no}$ is not strongly connected.

Theorem 8. Consider an influence process $\mathbf{B}$ based on aggregation functions $\mathbf{p}$ and assume that $[S, S \cup K]$ is strongly connected. This interval is an absorbing interval if and only if the following two conditions hold:

(i) There exists no $i \in S$ and $M \in \mathcal{N}_i$ such that $M \subseteq f_{B_i}\left(\bigvee_{B_i}[(S \cup K)^c, S^c]\right)$, for some $B_i \in \mathcal{B}_i$.

(ii) There exists no $i \in (S \cup K)^c$ and $Y \in \mathcal{Y}_i$ such that $Y \subseteq f_{B_i}\left(\bigvee_{B_i}[S, S \cup K]\right)$, for some $B_i \in \mathcal{B}_i$.

Proof. Since it is assumed to be connected, $[S, S \cup K]$ is an absorbing interval if and only if the following two conditions hold:

(i) There exists no $i \in S$ and $T \in [S, S \cup K]$ such that $T \in \mathcal{P}_i^0 \cup \mathcal{P}_i^*$.

(ii) There exists no $i \in (S \cup K)^c$ and $T \in [S, S \cup K]$ such that $T \in \mathcal{P}_i^1 \cup \mathcal{P}_i^*$.

By Property 9, it is equivalent to say that $[S, S \cup K]$ is an absorbing interval if and only if the following two conditions hold:

(i) There exists no $T \in [S, S \cup K]$ and $i \in S$ with some $B_i \in \mathcal{B}_i$ and $M \in \mathcal{N}_i$ such that $T \preccurlyeq_{B_i} M \Delta B_i$. (That is, $M \subseteq T \Delta B_i = f_{B_i^c}(T)$ by Property 6.)

(ii) There exists no $T \in [S, S \cup K]$ and $i \in (S \cup K)^c$ with some $B_i \in \mathcal{B}_i$ and $Y \in \mathcal{Y}_i$ such that such that $Y \Delta B_i^c \preccurlyeq_{B_i} T$. (That is, $Y \subseteq T \Delta B_i^c = f_{B_i}(T)$ by Property 6.)

By noticing that $\{f_{B_i}(T) \mid T \in [S, S \cup K]\}$ is an interval, we obtain to write the conditions as follows:

(i) There exists no $i \in S$ and $M \in \mathcal{N}_i$ such that $M \subseteq \bigcup_{T \in [S, S \cup K]} f_{B_i^c}(T)$, for some $B_i \in \mathcal{B}_i$.

(ii) there exists no $i \in (S \cup K)^c$ and $Y \in \mathcal{Y}_i$ such that such that $Y \subseteq \bigcup_{T \in [S, S \cup K]} f_{B_i}(T)$, for some $B_i \in \mathcal{B}_i$.

Since f_{B_i} is an involution, we obtain from (D.4), $S \leftarrow f_{B_i}(S)$ and $S' \leftarrow f_{B_i}(S')$ that $f_{B_i}(S) \cup f_{B_i}(S') = f_i(S \vee_{B_i} S')$. Moreover, $f_{B_i^c}\left(\bigvee_{\substack{B_i^c \\ T \in [S, S \cup K]}} T\right) = f_{B_i}\left(\bigvee_{\substack{B_i \\ T \in [S, S \cup K]}} T^c\right)$ and the conditions can be written at the collection level as stated in the theorem. (In these conditions, we can take any $B_i \in \mathcal{B}_i$ since the yes and no influential coalitions contain no irrelevant agents.) □

When $K = \emptyset$, the absorbing interval $[S, S \cup K]$ degenerates into an absorbing state and we recover the conditions of Theorem 7. Interpreting the intervals $[S, S \cup K]$ as blurry absorbing states, we can say that stability of opinions is reached when yes and no-influential coalitions are large enough, which means that sizeable coalitions are necessary to displace agent's opinions. Since the influential coalitions can be regarded as the boundaries of the $\mathcal{P}^0$ and $\mathcal{P}^1$ collections, sizeable influential coalitions are tantamount to large thresholds in anonymous models, which is indeed some guarantee of stability (Grabisch et al., 2019a).

E Influence and dynamics: two worlds apart?

In this appendix, we provide a technical discussion concerning the relations between influences and dynamics. By influences, we may refer, either to the influence graph (Definition 5) based on

the influence index (2.4), or to the influential coalitions (Definition 16). Appendix D explains why opinion dynamics are better characterised by influential coalitions than by the influence graph.

Nevertheless, would it be possible to impose restrictions on p_i so that all terms in (2.4) are strictly positive, or all strictly negative, or all null ? The only way to obtain this convenient property would be to build the model by letting $W_i := \{j : p_i(S \setminus j) = p_i(S \cup j)\ \forall S \in 2^N\}$ the set of irrelevant agents on i and then to define $\mathcal{B}_i^w := \{B \in 2^N$ such that $S_1 \setminus W_i \prec_B S_2 \setminus W_i \Rightarrow p_i(S_1) < p_i(S_2)\}$. Then we would impose $\mathcal{B}_i^w \neq \emptyset$ for all i. In this case, the yes and no influential coalitions would be $\mathcal{Y}_i = \mathcal{N}_i = \{\{j\} \mid j \in \underline{B}_i \cup (\overline{B}_i)^c\}$, where $\underline{B}_i$ and $\overline{B}_i$ are the ones of our paper. This means that the p_i are decomposable and symmetric in the sense of Definition 7 in Grabisch and Rusinowska (2013) and we can show that their Theorem 7 can be generalised to our setting: such p_i are qualitatively equivalent to the GWM functions presented in Section 3.1. In particular we would get that the $\mathcal{P}^1$ and $\mathcal{P}^0$ collections are intervals, and actually quite special ones: $\mathcal{P}_i^1 = \mathcal{B}_i$ and $\mathcal{P}_i^0 = (\mathcal{B}_i)^c$; in particular they would capture the influence graph. This means that in our summary figure, we could draw a double arrow between $(\mathcal{P}_i^0, \mathcal{P}_i^1)$ and $(\mathcal{B}_i)_{i \in N}$ to express a bijective relation: the wall between influences and dynamics would be broken.

Back to our unrestricted model, we leave as an open problem the relation between the influence graph and the dynamics at a statistical level, using, e.g., a maximal likelihood estimation. We would like to be able to answer the following questions: is there some kind of "natural absorbing class" associated with a given influence graph ? Conversely, can we recover a reasonable influence graph when observing an absorbing class ? Let us remember also that in this paper we made no use of the absolute value of the influences $\phi(j \to i))_{i,j \in N}$. Allied with the influential coalitions, this neglected source of information seems to indicate that a connection with cooperative game theory remains to be uncovered.

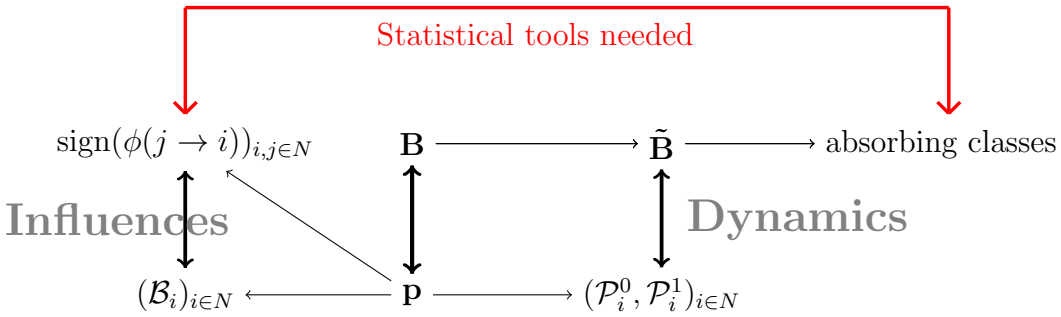


Figure E.1: A wall between influences and dynamics

Bibliography

R. P. Abelson. Mathematical models of the distribution of attitudes under controversy. pages 142–160. 1964.

D. Acemoglu, G. Como, F. Fagnani, and A. Ozdaglar. Opinion fluctuations and disagreement in social network. *Mathematics of Operations Research*, 38(1):1–27, 2010.

C. Altafini. Dynamics of opinion forming in structurally balanced social networks. *PLoS ONE*, 7(6), 2012.

C. Altafini. Consensus problems on networks with antagonistic interactions. *IEEE Transactions on Automatic Control*, 58(4):935–946, 2013.

J. Aracena, J. Demongeot, and E. Goles. On limit cycles of monotone functions with symmetric connection graph. *Theoretical Computer Science*, 2004.

J. Aracena, M. González, A. Zu niga, M. Mendez, and V. Cambiazo. Regulatory network for cell shape changes during the drosophila ventral furrow formation. *Journal of Theoretical Biology*, 239:49–62, 2006.

K. J. Arrow. *Social Choice and Individual Values*. New York: Wiley, 1963.

C. Asavathiratham. *Influence model: a tractable representation of networked Markov chains*. PhD thesis, Massachusetts Institute of Technology, Cambridge, MA, 2000.

D. Austen-Smith and J. S. Banks. Aggregation, rationality, and the condorcet jury theorem. *The American Political Science Review*, 90(1):34–35, 1996.

A. V. Banerjee. A simple model of herd behavior. *Quarterly Journal of Economics*, 107(3): 797–817, 1992.

P. Berenbrink, G. Giakkoupis, A. Kermarrec, and F. Mallmann-Trenn. Bounds on the voter model in dynamic networks. 2016.

S. Bikhchandani, D. Hirshleifer, and I. Welch. A theory of fads, fashion, custom, and cultural change as informational cascades. *Journal of Political Economy*, 100:992–1026, 1992.

P. Blavatskyy. Stochastic expected utility theory. *Journal of Risk and Uncertainty*, 34:259–286, 2007.

H. Blok and B. Bergersen. Synchronous versus asynchronous updating in the "game of life". *Physical Review E*, 54(4), April 1999.

O. Bouré, N. Fatès, and V. Chevrier. First steps on asynchronous lattice-gas models with an application to a swarming rule. *Natural Computing*, 12:551–560, 2013.

Y. Bramoullé. Anti-coordination and social interactions. *Games and Economic Behavior*, 58: 30–49, 2007.

Y. Bramoullé, D. López-Pintado, S. Goyal, and F. Vega-Redondo. Network formation and anti-coordination games. *International Journal of Game Theory*, 33(1):1–19, 2004.

B. Buechel and L. Mechtenberg. The swing voter's curse in social networks. *Games and Economic Behavior*, 118:241–268, 2019.

B. Buechel, T. Hellmann, and M. Pichler. The dynamics of continuous cultural traits in social networks. *Journal of Economic Theory*, 154:274–309, 11 2014.

B. Buechel, T. Hellmann, and S. Klößner. Opinion dynamics and wisdom under conformity. *Journal of Economic Dynamics and Control*, 52:240–257, 2015.

C. F. Camerer. An experimental test of several generalized utility theories. *Journal of Risk and Uncertainty*, 2:61–104, 1989.

C. Castellano, M. Mu noz, and R. Pastor-Satorras. Nonlinear q-voter model. *Physical Review. E, Statistical, Nonlinear, and Soft Matter Physics*, 80:041129, 2009.

C. P. Chamley. *Rational Herds: Economic Models of Social Learning.* Cambridge University Press, 2004.

S. Chevalier, C. Froidevaux, L. Paulevé, and A. Zinovyev. Synthesis of boolean networks from biological dynamical constraints using answer-set programming. In *2019 IEEE 31st International Conference on Tools with Artificial Intelligence (ICTAI)*, ICTAI, Portland, Oregon, United States, 2019. IEEE.

N. Chukhrova and A. Johannssen. Fuzzy hypothesis testing for a population proportion based on set-valued information. *Fuzzy Sets and Systems*, 387:127–157, 2020.

P. Clifford and A. Sudbury. A model of spatial conflict. *Biometrika*, 60(3):581, 1973.

J-P. Comet, M. Noual, A. Richard, J. Aracena, L. Calzone, J. Demongeot, M. Kaufman, A. Naldi, E. H. Snoussi, and D. Thieffry. On circuit functionality in boolean networks. *Bulletin of Mathematical Biology*, 75(6), 2013.

P. Coughlan. In defense of unanimous jury verdicts: Mistrials, communication, and strategic voting. *The American Political Science Review*, 94:375–393, 2000.

D. Green D. Cornforth and D. Newth. Ordered asynchronous processes in multi-agent systems. *Physica D*, 204(1-2):70–82, May 2005.

M. H. DeGroot. Reaching a consensus. *Journal of the American Statistical Association*, 69(345): 118–121, 1974.

D. Dreman. Contrarian investment strategies. *Simon and Schuster Ltd*, 2007.

N. Fatès and M. Morvan. An experimental study of robustness to asynchronism for elementary cellular automata. *Complex Systems*, 16, 2004.

G. Fechner. *Elements of Psychophysics.* 1860.

T. Feddersen and W. Pesendorfer. Convicting the innocent: The inferiority of unanimous jury verdicts under strategic voting. *The American Political Science Review*, 92(1):23–35, 1998.

M. Förster, M. Grabisch, and A. Rusinowska. Anonymous social influence. *Games and Economic Behavior*, 82:621–635, 2013.

J. French. A formal theory of social power. *Psychological Review*, 63(3):181–194, 1956.

N. Friedkin and E. Johnsen. Social influences and opinion. *Journal of Mathematical Sociology*, 15:193–205, 1990.

S. Galam. Majority rule, hierarchical structures, and democratic totalitarianism: A statistical approach. *Journal of Mathematical Psychology*, 30(4):426–434, 1986.

S. Galam. Contrarian deterministic effects on opinion dynamics: the hung elections scenario. *Physica A*, 333:453–460, 2004.

S. Galam. Sociophysics: A review of galam models. *International Journal of Modern Physics C*, 19, 2008.

S. Galam and F. Jacobs. The role of inflexible minorities in the breaking of democratic opinion dynamics. *Physica A: Statistical Mechanics and its Applications*, 381:366–376, 2007.

M. Gardner. Mathematical games - the fantastic combinations of john conway's new solitaire game 'life'. *Scientific American*, 223:120–123, 1970.

D. Gerardi and L. Yariv. Deliberative voting. *Journal of Economic Theory*, 134(1):317 – 338, 2007.

R. Girard. *Deceit, Desire and the Novel: Self and Other in Literary Structure*. Baltimore: Johns Hopkins University, 1966.

R. Girard. *Violence and the Sacred*. Baltimore: Johns Hopkins University, 1977.

M. Grabisch. Set functions, games and capacities in decision making. *Springer International Publishing AG*, 2016.

M. Grabisch and F. Li. Anti-conformism in the threshold model of collective behavior. *Dynamic Games and Applications, Springer Verlag*, 2019.

M. Grabisch and A. Rusinowska. Iterating influence between players in a social network. *Documents de travail du Centre d'Economie de la Sorbonne*, 2010.

M. Grabisch and A. Rusinowska. A model of influence based on aggregation functions. *Mathematical Social Sciences*, 66:216–330, 2013.

M. Grabisch, A. Poindron, and A. Rusinowska. A model of anonymous influence with anti-conformist agents. *Journal of Economic Dynamics and Control*, 109, 2019a.

M. Grabisch, A. Rusinowska, and X. Venel. Diffusion in countably infinite networks. *Documents de Travail du Centre d'Economie de la Sorbonne*, 2019b.

M. Granovetter. Threshold models of collective behavior. *American Journal of Sociology,*, 83: 1420–1443, 1978.

C. Grilo and L. Correia. Effects of asynchronism on evolutionary games. *Journal of Theoretical Biology*, 2011.

O. Guéant, J-M Lasry, and P-L. Lions. *Mean Field Games and Applications*. 2010.

F. Harary. On the notion of balance of a signed graph. *The Michigan Mathematical Journal*, 2(2):143–146, 1953.

X. Hu and S. Shapley. On authority distributions in organizations: Equilibrium. *Games and Economic Behavior*, 45:132–152, 2003a.

X. Hu and S. Shapley. On authority distributions in organizations: Controls. *Games and Economic Behavior*, 45:153–170, 2003b.

E. Ising. *Contribution to the Theory of Ferromagnetism.* PhD thesis, 1925.

M. O. Jackson and E. C. Storms. Behavioral communities and the atomic structure of networks. 2018.

J. S. Jull and M. A. Porter. Hipsters on networks: How a minority group of individuals can lead to an antiestablishment majority. *Physical Review*, E 99, 022313, 2019.

M. L. Katz and C. Shapiro. Network externalities, competition, and compatibility. *The American Economic Review*, 75(3):424–440, 1985.

S. Lang. Algebra, third edition. *Springer*, 2002.

T. M. Liggett. *Interacting Particle Systems.* Springer, 1985.

T. M. Liggett. *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes.* Springer, Heidelberg, 1999.

D. López-Pintado. Network formation, cost sharing and anti-coordination. *International Game Theory Review*, 11(1):53–76, 2009.

K. Lorenz. *L'agression, chapter 6: Le grand parlement des instincts.* Flammarion, 1969.

A. Martins. Continuous opinions and discrete actions in opinion dynamics problems. *International Journal of Modern Physics C*, 19, 2007.

A. Martins. Mobility and social network effects on extremist opinions. *Physical Review. E, Statistical, Nonlinear, and Soft Matter Physics*, 78:036104, 2008.

W. S. McCulloch and W. Pitts. A logical calculus of the ideas immanent in nervous activity. *Bulletin of Mathematical Biophysics*, 5:115–133, 1943.

S. Morris. Contagion. *Review of Economic Studies*, 67(1):57–78, 2000.

S. Nitzan and J. Paroush. Optimal decision rules in uncertain dichotomous choice situations. *International Economic Review*, 23(2):289–297, 1982.

P. Nyczka and K. Sznajd-Weron. Anticonformity or independence? Insights from statistical physics. *Journal of Statistical Physics*, 151:174–202, 2013.

P. Nyczka, K. Sznajd-Weron, and J. Cislo. Phase transitions in the q-voter model with two types of stochastic driving. *Physical Review*, 86, 2012.

L. Paulevé and A. Richard. Static analysis of boolean networks based on interaction graphs: A survey. *Electronic Notes in Theoretical Computer Science*, 284:93 – 104, 2012.

L. Paulevé, T. Chatain J. Kolčá, and S. Haar. Reconciling qualitative, abstract, and scalable modelling of biological networks. *Nature Communications*, 11, 2020.

A. Proskurnikov, A. Matveev, and M. Cao. Opinion dynamics in social networks with hostile camps: Consensus vs. polarization. *IEEE Transactions on Automatic Control*, 61 (6):1524–1536, 2016.

E. Remy, P. Ruet, and D. Thieffry. Graphic requirements for multistability and attractive cycles in a boolean dynamical framework. *Advances in Applied Mathematics*, 41:335–350, 2008.

A. Richard. Local negative circuits and fixed points in non-expansive boolean networks. *Discrete Applied Mathematics*, 159(11):1085 – 1093, 2011.

A. Richard. Fixed points and connections between positive and negative cycles in boolean networks. *Discrete Applied Mathematics*, 2018.

J. Rohlfs. A theory of interdependent demand for a communications service. *The Bell Journal of Economics and Management Science*, 5(1):16–37, 1974.

L. Shapley and B. Grofman. Optimizing group judgmental accuracy in the presence of interdependencies. *Public Choice*, 43(3):329–343, 1984.

T. Strzalecki. Stochastic choice, hotelling lectures in economic theory, econometric society european meeting, lisbon, 2017.

K. Suzumura. *Rational choice, collective decisions, and social welfare.* Cambridge Cambridgeshire New York: Cambridge University Press, 1983.

K. Sznajd-Weron and J. Sznajd. Opinion evolution in closed community. *International Journal of Modern Physics*, 11(6), 2001.

M. Taylor. Towards a mathematical theory of influence and attitude change. *Human Relations*, 21:121–139, 1968.

R. Thomas. On the relation between the logical structure of systems and their ability to generate multiple steady states or sustained oscillations. *Numerical Methods in the Study of Critical Phenomena*, pages 180–193, 1981.

J. Touboul. The hipster effect: When anticonformists all look the same. *Discrete & Continuous Dynamical Systems - B.*

S. M. Ulam. A collection of mathematical problems. 1960.

J. von Neumann. *Theory of Self-Reproducing Automata.* University of Illinois Press, Champaign, IL, USA, 1966.

D. J. Watts. A simple model of global cascades on random networks. *Proceedings of the National Academy of Sciences of the United States of America*, 99(9):5766–5771, 2002.

B. Yang. Communication, excess comovement and factor structures. *Journal of Business Finance and Accounting*, 40(9) and (10):1304–1325, November/December 2013.