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## HYDRODYNAMIC DESIGN OPTIMIZATION USING NON STATIONARY POROUS MEDIA MODEL

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**ABSTRACT.** In this paper, we focus on the penalty finite element method for the non stationary porous media model. We begin by showing the existence and uniqueness of the solution for the initial problem. Error estimates for the velocity and the pressure are obtained via the energy method. We introduce a time discretization by the use of a backward Euler scheme combined with fully discrete finite element method to approximate the penalized problem and establish an error estimate for the velocity and the pressure which will be used to show the convergence of the approximate solution to the solution of the initial problem. The shape optimization problem is to find the shape which is optimal in that it minimizes a cost functional related to a comfort fish populations. We derive the adjoint system associated to the penalized problem. We compute the gradient in terms of state and adjoint variables. The optimization procedure is implemented using the continuous adjoint method and the finite element method. Numerical simulations are presented to show the efficiency and the robustness of the proposed method.

**Key Words** unsteady flow, finite element method, optimal shape design, fishways.

**Mathematics Subject Classifications** 35B45. 35Q93. 65N30. 76D55

### 1. INTRODUCTION

In this work, we are interested in studying the non-stationary Navier-Stokes Forchheimer (NSF) equations

$$\left\{ \begin{array}{ll} \mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + a |\mathbf{u}|^\alpha \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \times [0, T], \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \times [0, T], \\ \mathbf{u} = 0 & \text{on } \partial\Omega \times [0, T], \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \text{on } \Omega, \end{array} \right. \quad (1.1)$$

where  $\Omega$  is an open bounded set in  $\mathbb{R}^d$  ( $d = 2$  or  $d = 3$ ) with a sufficiently smooth boundary  $\partial\Omega$ . The unknown functions here are  $\mathbf{u}(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$  and  $p = p(x, t)$ , which stand for the velocity field and the pressure of the flow, respectively. In damping term,  $1 \leq \alpha \leq 2$  and  $a > 0$  are two constants. The given function  $\mathbf{u}_0 = \mathbf{u}_0(x)$  is the initial velocity and the constant  $\nu > 0$  represents the viscosity coefficient of the flow.

In the literature, there are several papers devoted to the mathematical study of NSF equations without the nonlinear convection terms. References [2], [14, 17] reported results concerning the structural stability for the coefficients in flows. We refer to the work of [21] for the asymptotic behavior of solutions. In [12], the analysis of the pressure stabilization method for the unsteady incompressible Brinkman-Forchheimer equations is proposed. Error estimates for the velocity and the pressure are established. The existence of regular dissipative solutions with the nonlinearity of an arbitrary polynomial growth rate is investigated in [10].

Among the applications of these water flow model are the problems associated to the optimization of fishways structure. In fact, some types of fish accomplish their daily or annual migrations and for long or short distances to complete their life cycles. These migratory fish species are threatened by urbanization, agriculture, or dams.

Dams are built through centuries by men for agriculture, navigation, electricity or economic reasons. They represent impassable barriers for migratory species for their reproduction or feeding areas during the rise of the rivers. To meet this challenge, man thought to built fishways for permitting fish to pass with comfortable conditions. Fishways (fish ladder, or fish passage) are hydraulic structures placed near from dams or weirs to allow migratory fishes to cross to upstream river. Several types of fish passage are known nowadays. The best general reference Clay [5] for the pool and weir type, the standard work of Katopodis et al. [11] for the Denil fishways and the classical work for the vertical slot type Rajaratnam et al. [18]. We are involved in the third type of fishways: vertical slot fishway (VSF). VSF has an area of low velocity for fish to rest during upstream passage, a good energy dissipation. This makes them suitable to sites where fish passage is needed at different times of the year and for a maximum number of small species.

In actual engineering, in order to reduce the flow velocity in the fishway pool, sand gravels are usually laid at the bottom which can increase the roughness of the fishway and bring the flow information in the fishway much closer to a natural river and induce a familiar flow patterns for fish. The flow behavior can only modeled accurately if the structure of porous media is properly described. Transition from the Darcy flow regime to the Forchheimer regime can be observed by the investigations of flow in pore space. Many researchers have attempted to clarify the physical meaning of the linearity in the Forchheimer equation by recovering the Forchheimer equation from the Navier-Stokes equation by using the basic principles underlying the theory of hydrodynamics. Despite the diverse opinions on the origin of the nonlinearity, it is now generally agreed that the quadratic term involves the separation of boundary layers and wake formation behind solid obstacles. Detailed discussions can be found in [3, 9, 13].

Our purpose is to investigate these structures using the non stationary porous media model and to assess the possibility to compute the flow pattern in vertical slot fish ladder and deduce an optimal structure allowing fish to cross the obstacle in a convenient conditions.

The resolution of the problem (1.1) could be difficult numerically, caused by the incompressibility constraint  $\nabla \cdot \mathbf{u} = 0$ . A popular strategy to overcome this difficulty is to relax the incompressibility term with the penalty approach. Our purpose is to use a penalty method, introduced by Courant [6] and used by Temam [20] for the approximation of the Navier Stokes equations, to treat the Navier Stokes Forchheimer system.

The penalty method is to establish an approximation of the couple  $(\mathbf{u}, p)$  solution of (1.1) by  $(\mathbf{u}^\varepsilon, p^\varepsilon)$  solution of the following penalized problem

$$\left\{ \begin{array}{ll} \mathbf{u}_t^\varepsilon - \nu \Delta \mathbf{u}^\varepsilon + (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{u}^\varepsilon + \frac{1}{2} (\nabla \cdot \mathbf{u}^\varepsilon) \mathbf{u}^\varepsilon + a |\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon + \nabla p^\varepsilon = \mathbf{f} & \text{in } \Omega \times [0, T], \\ \nabla \cdot \mathbf{u}^\varepsilon + \varepsilon p^\varepsilon = 0 & \text{in } \Omega \times [0, T], \\ \mathbf{u}^\varepsilon = 0 & \text{on } \partial\Omega \times [0, T], \\ \mathbf{u}^\varepsilon|_{t=0} = \mathbf{u}_0 & \text{in } \Omega. \end{array} \right. \quad (1.2)$$

We prove the existence and uniqueness of the initial problem solution. We analyze the error behavior for the linearized NSF model. Then we treat the penalty method for the fully nonlinear NSF system.

The penalized model is considered to study an optimization problem related to fish pass structure, which consist to find an optimal shape of fishway by minimizing a cost function connected to fish comfort. We derive an adjoint equation and compute the shape gradient of the objective function.

We propose and analyze a fully discrete finite element method for the penalized problem. The numerical approach combines a finite element method to solve the non-stationary porous media equations and a gradient type algorithm called spectral projected gradient to solve the optimization problem.

The remainder of this paper is organized as follows. In the next section, we introduce notations and recall some preliminary results. We use also the classical Faedo–Galerkin method and the Fourier transform technique to prove the existence of a weak solution. The Section 3 deals with the shape design. We define a shape optimization problem. We then derive the adjoint system and compute the shape gradient of the cost function. The Section 4 treats a time discretization by the use of a backward Euler scheme combined with a finite element method to approximate the penalized problem and obtain an error estimate for the velocity and the pressure. Finally, the last section is devoted to the gradient method to solve the shape optimization problem and the presentation of some numerical results to prove the efficiency of the suggested technique.

## 2. MATHEMATICAL MODEL

**2.1. Preliminaries.** The Navier Stokes Forchheimer equations governing the motion of an incompressible viscous fluid inside  $\Omega$  read:

$$\mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + a |\mathbf{u}|^\alpha \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \times [0, T] \quad (2.1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times [0, T], \quad \mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0 \quad (2.2)$$

We aim to describe some of the notation and results which will be frequently used in this paper. For the mathematical setting of the Navier Stokes Forchheimer system, we introduce the Hilbert spaces<sup>1</sup>

$$\mathbf{H} = \{ \mathbf{v} \in \mathbf{L}^2(\Omega), : \nabla \cdot \mathbf{v} = 0, \quad \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0 \}$$

$$\mathbf{V} = \{ \mathbf{v} \in H_0^1(\Omega) : \nabla \cdot \mathbf{v} = 0 \}$$

<sup>1</sup> The vector functions and vector spaces will be indicated by boldface type.

$\mathbf{X} = (H_0^1(\Omega))^d$  ( $d = 2, 3$ ),  $Y = L^2(\Omega)$ ,  $\hat{Y} = \{q \in H^1(\Omega)\}$ ,  $M = \left\{q \in L^2(\Omega); \int_{\Omega} q dx = 0\right\}$

$V_0(\Omega) = \{\mathbf{u} \in (H^2(\Omega))^d | u = 0 \text{ on } \Gamma_1\}$ , where  $\partial\Omega = \Gamma_1 \cup \Gamma_2$

$\mathbb{X} = \{\mathbf{v} \in (H_0^1(\Omega))^d | (q, \nabla \cdot \mathbf{v}) = 0, \forall q \in L^2(\Omega)\}$

$H^{-2}(\Omega)$ : the dual space of  $H^2(\Omega) \cap H_0^1(\Omega)$

$P_{\mathbf{H}}$ : the orthogonal projector in  $\mathbf{L}^2(\Omega)$  onto  $\mathbf{H}$

$A = -P_{\mathbf{H}}\Delta = \tilde{\Delta}$ : the Stokes operator, which is an unbounded positive self-adjoint closed operator in  $\mathbf{H}$  with domain  $D(A) = \mathbf{H}^2(\Omega) \cap \mathbf{V}$ . We denote also by  $\|\cdot\|_k$  the norm of  $\mathbf{H}^k(\Omega)$ , In particular, we will use  $\|\cdot\|$  to denote the norm in  $\mathbf{L}^2(\Omega)$  and  $(\cdot, \cdot)$  to denote the scalar product in  $\mathbf{L}^2(\Omega)$ .

We now recall some classical properties associated with the nonlinear terms in Navier Stokes equations.

$$B(\mathbf{u}, \mathbf{v}) = (\mathbf{u} \cdot \nabla) \mathbf{v}, \quad \tilde{B}(\mathbf{u}, \mathbf{v}) = (\mathbf{u} \cdot \nabla) \mathbf{v} + \frac{1}{2}(\nabla \cdot \mathbf{u}) \mathbf{v},$$

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (B(\mathbf{u}, \mathbf{v}), \mathbf{w}), \quad \tilde{b}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\tilde{B}(\mathbf{u}, \mathbf{v}), \mathbf{w}).$$

We observe that

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0 \quad \forall \mathbf{u} \in \mathbf{H}, \mathbf{v} \in \mathbf{H}_0^1(\Omega), \quad (2.3)$$

and

$$\tilde{b}(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0 \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}_0^1(\Omega). \quad (2.4)$$

We have also the two following inequalities which will be used repeatedly in the upcoming sections.

$$\tilde{b}(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq \|\mathbf{u}\|_1 \|\mathbf{v}\|_1^{\frac{1}{2}} \|\mathbf{v}\|_2^{\frac{1}{2}} \|\mathbf{w}\| \quad \forall \mathbf{v} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega), \quad \forall \mathbf{u}, \mathbf{w} \in \mathbf{H}_0^1(\Omega). \quad (2.5)$$

$$\tilde{b}(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq \begin{cases} \|\mathbf{u}\|_1 \|\mathbf{v}\|_1 \|\mathbf{w}\|_1 & \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1(\Omega), \\ \|\mathbf{u}\|_2 \|\mathbf{v}\| \|\mathbf{w}\|_1 & \forall \mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega), \quad \forall \mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1(\Omega), \\ \|\mathbf{u}\|_2 \|\mathbf{v}\|_1 \|\mathbf{w}\| & \forall \mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega), \quad \forall \mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1(\Omega), \\ \|\mathbf{u}\|_1 \|\mathbf{v}\|_2 \|\mathbf{w}\| & \forall \mathbf{v} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega), \quad \forall \mathbf{u}, \mathbf{w} \in \mathbf{H}_0^1(\Omega). \end{cases} \quad (2.6)$$

Let introduce the perturbed  $A_{\varepsilon} u = -\nu \Delta u - \frac{1}{\varepsilon} \nabla(\nabla \cdot \mathbf{u})$ , which is the operator related to the penalty method. It is a positive self-adjoint operator from  $\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$  onto  $\mathbf{L}^2(\Omega)$  and the powers  $A^{\beta}$  of  $A$  ( $\beta \in \mathbb{R}$ ) are well defined and we have

$$(A^{1/2} \mathbf{u}, A^{1/2} \mathbf{v}) = (\nabla \mathbf{u}, \nabla \mathbf{v}), \quad (A_{\varepsilon}^{1/2} \mathbf{u}, A_{\varepsilon}^{1/2} \mathbf{v}) = (A^{1/2} \mathbf{u}, A^{1/2} \mathbf{v}) + \frac{1}{\varepsilon} (\operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v})$$

for all  $\mathbf{u}, \mathbf{v} \in X$ .

For  $\varepsilon$  sufficiently small,

$$\begin{aligned} \|\Delta \mathbf{u}\| &\leq C \|A_{\varepsilon} \mathbf{u}\| \quad \forall \mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{X}, \\ \|\nabla \mathbf{u}\| &\leq C \|A_{\varepsilon}^{\frac{1}{2}} \mathbf{u}\| \quad \forall \mathbf{u} \in \mathbf{X}, \\ \|A_{\varepsilon}^{-1} \mathbf{u}\| &\leq C \|\mathbf{u}\|_{-2} \quad \forall \mathbf{u} \in \mathbf{H}^{-2}(\Omega). \end{aligned} \quad (2.7)$$

$C$  is the generic constant that can take different values in different places.

If the boundary  $\partial\Omega$  is smooth enough, for any  $v \in \mathbf{V} \cap \mathbf{H}^2(\Omega)$

$$\|\mathbf{v}\|_2 \leq \|\tilde{\Delta} \mathbf{v}\|. \quad (2.8)$$

It is well known that the following Gagliardo-Nirenberg inequalities hold:

**Lemma 2.1.**

$$\|\mathbf{v}\|_{L^4} \leq c \|\mathbf{v}\|^{1/2} \|A^{1/2} \mathbf{v}\|^{1/2}, \quad \|\mathbf{v}\| \leq c \|A^{1/2} \mathbf{v}\|, \quad \forall v \in \mathbf{X} \quad (2.9)$$

$$\|\nabla \mathbf{v}\|_{L^4} \leq c \|A^{1/2} \mathbf{v}\|^{1/2} \|A \mathbf{v}\|^{1/2}, \quad \|A^{1/2} \mathbf{v}\| \leq c \|A \mathbf{v}\|, \quad \forall v \in D(A) \quad (2.10)$$

$$\|\mathbf{v}\|_{L^\infty} \leq c\|\mathbf{v}\|^{1/2}\|\mathbf{A}\mathbf{v}\|^{1/2}, \quad \forall \mathbf{v} \in D(A). \quad (2.11)$$

Furthermore, we have the following properties for the Stokes operator.

**Lemma 2.2.** *There exists a constant  $c_0 > 0$  depending on  $\Omega$  and such that if  $c\varepsilon \leq 1$ ,*

$$\|\mathbf{A}\mathbf{v}\| \leq c_0\|\mathbf{A}_\varepsilon\mathbf{v}\|, \quad \|\mathbf{A}^{1/2}\mathbf{v}\| \leq c_0\|\mathbf{A}_\varepsilon^{1/2}\mathbf{v}\| \quad (2.12)$$

Also, we remind two Gronwall's lemmas type which will be often used.

**Lemma 2.3.** *(Gronwall lemma) Let  $y(t)$ ,  $h(t)$ ,  $g(t)$  and  $f(t)$  be non negative functions verifying*

$$y(t) + \int_0^t h(s)ds \leq y(0) + \int_0^t (g(s)y(s) + f(s))ds \quad \forall 0 \leq t \leq T, \quad \text{with} \quad \int_0^T g(t)dt \leq M,$$

then

$$y(t) + \int_0^t h(s)ds \leq \exp(M)(y(0) + \int_0^t f(s)ds) \quad \forall 0 \leq t \leq T.$$

**Lemma 2.4.** *(discrete Gronwall lemma). Let  $y^n$ ,  $h^n$ ,  $g^n$ ,  $f^n$  be nonnegative series verifying*

$$y^m + k \sum_{n=0}^m h^n \leq k \sum_{n=0}^m (g^n y^n + f^n), \quad \text{with} \quad k \sum_{n=0}^{T/k} g^n \leq M \quad \forall 0 \leq m \leq T/k.$$

Suppose  $kg^n < 1 \quad \forall n$  and let  $\sigma = \max_{0 \leq n \leq T/R} (1 - kg^n)^{-1}$ , then

$$y^m + k \sum_{n=1}^m h^n \leq \exp(\sigma M)(B + k \sum_{n=0}^m f^n), \quad \forall m \leq T/k$$

It is worth pointing out that the mapping  $F: \mathbf{x} \mapsto |\mathbf{x}|^\alpha \mathbf{x}$  is monotone, then:

$$\forall \mathbf{u}, \mathbf{v} \in \mathbf{X}, \quad (|\mathbf{u}|^\alpha \mathbf{u} - |\mathbf{v}|^\alpha \mathbf{v}, \mathbf{u} - \mathbf{v}) \geq 0. \quad (2.13)$$

The following Sobolev inequality is useful to deal with the nonlinear form F.

$$\|\cdot\|_p \leq \|\nabla \mathbf{v}\|, \quad 1 \leq p \leq 6 \quad (2.14)$$

It is convenient to consider the functional  $G$  defined by  $G(\mathbf{u}) = PF(\mathbf{u}) \quad \forall \mathbf{u} \in \mathbf{X}$ .

In the next lemma, we provide the estimate for the velocity's second derivative, the velocity's derivative with respect to time and the pressure.

**Lemma 2.5.** *We assume that the given data  $\mathbf{u}_0$  and  $\mathbf{f}$  satisfy the following regularity*

$$\mathbf{u}_0 \in \mathbf{V} \cap \mathbf{H}^2(\Omega), \quad \mathbf{f}, \mathbf{f}_t \in \mathbf{L}^2(\Omega) \quad (S1)$$

Then there exists  $T_1 \leq T$  such that the solution of the system (2.1)-(2.2) satisfies:

$$\|\mathbf{u}(t)\|_2 + \|\mathbf{u}_t(t)\| + \|p(t)\|_1 \leq C, \quad \forall t \in [0, T_1]. \quad (2.15)$$

*Proof.* Take the inner product of the equation (2.1) with  $\tilde{\Delta} \mathbf{u}$  and integrate all to get

$$\frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{u}\|^2 + \gamma \|\tilde{\Delta} \mathbf{u}\|^2 = -b(\mathbf{u}, \mathbf{u}, \tilde{\Delta} \mathbf{u}) - a(|\mathbf{u}|^\alpha \mathbf{u}, \tilde{\Delta} \mathbf{u}) + (\mathbf{f}, \tilde{\Delta} \mathbf{u}). \quad (2.16)$$

Combining the inequalities (2.7)-(2.11) with Young inequality, we have

$$b(\mathbf{u}, \mathbf{u}, \tilde{\Delta} \mathbf{u}) \leq \frac{\gamma}{6} \|\tilde{\Delta} \mathbf{u}\|^2 + C' \|\nabla \mathbf{u}\|^6, \quad (2.17)$$

$$a(|\mathbf{u}|^\alpha \mathbf{u}, \tilde{\Delta} \mathbf{u}) \leq \frac{\gamma}{6} \|\tilde{\Delta} \mathbf{u}\|^2 + C \|\nabla \mathbf{u}\|^{2(\alpha+1)}. \quad (2.18)$$

Substituting (2.17) and (2.18) into (2.16) and using the fact that

$$(f, \tilde{\Delta} \mathbf{u}) \leq \frac{\gamma}{6} \|\tilde{\Delta} \mathbf{u}\|^2 + \frac{3}{2\gamma} \|f\|^2,$$

we obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{u}\|^2 + \frac{\gamma}{2} \|\tilde{\Delta} \mathbf{u}\|^2 \leq C' \|\nabla \mathbf{u}\|^6 + \frac{3}{2\gamma} \|f\|^2. \quad (2.19)$$

In particular, it is equivalent to solve the following equation (by assuming  $C' \geq 1$ ):

$$\frac{d}{dt} y(t) \leq C' y(t)^3, \quad \text{with } y(t) = \|\nabla \mathbf{u}(t)\|^2 + C_1, \quad y(0) = \|\nabla \mathbf{u}(0)\|^2 + C_1$$

and  $C_1 = \frac{3}{2\gamma} \sup_{0 \leq t \leq T} \|f\|^2$ . The equation above has a solution in  $[0, T_1]$  where  $T_1 \leq \frac{1}{2C'y(0)^2} = \frac{1}{C_0}$  and

$$y(t) \leq \sqrt{2}y(0), \quad t \in [0, T_1].$$

So, for all  $t \in [0, T_1]$  with  $T_1 = \min\{T_1, \frac{1}{C_0}\}$ , we have

$$\|\nabla \mathbf{u}(t)\|^2 \leq \sqrt{2}(\|\nabla \mathbf{u}(0)\|^2 + C_1) := C_3, \quad (2.20)$$

Consequently, using (2.19) and (2.20), to infer that

$$\sup_{0 \leq t \leq T_1} \|\nabla \mathbf{u}(t)\|^2 + \gamma \int_0^{T_1} \|\tilde{\Delta} \mathbf{u}(t)\|^2 dt \leq C, \quad \forall t \in [0, T_1]. \quad (2.21)$$

Deriving one time the equation (2.1) with respect to  $t$ , we obtain

$$\mathbf{u}_{tt} - \gamma \Delta \mathbf{u}_t + aF'(\mathbf{u})\mathbf{u}_t + B(\mathbf{u}_t, \mathbf{u}) + B(\mathbf{u}, \mathbf{u}_t) + \nabla p_t = \mathbf{f}_t \quad (2.22)$$

Multiplying the relation (2.22) with  $\mathbf{u}_t$ , we get

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}_t\|^2 + \gamma \|\nabla \mathbf{u}_t\|^2 = -a(F'(\mathbf{u})\mathbf{u}_t, \mathbf{u}_t) - b(\mathbf{u}_t, \mathbf{u}, \mathbf{u}_t) + (\mathbf{f}_t, \mathbf{u}_t).$$

Since  $(F'(\mathbf{u})\mathbf{u}_t) \cdot \mathbf{u}_t$  is positive definite, using the inequalities (2.7), Hölder's and Sobolev's inequalities yields to

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{u}_t\|^2 + \gamma \|\nabla \mathbf{u}_t\|^2 &\leq c \|\mathbf{u}_t\| \|\mathbf{u}\|_1^{\frac{1}{2}} \|\mathbf{u}\|_2^{\frac{1}{2}} \|\nabla \mathbf{u}_t\| + \frac{1}{2} \|\mathbf{f}_t\|^2, \\ &\leq \frac{\gamma}{2} \|\nabla \mathbf{u}_t\|^2 + C(\|\mathbf{u}\|_1 \|\mathbf{u}\|_2) \|\mathbf{u}_t\|^2 + \frac{1}{2} \|\mathbf{f}_t\|^2. \end{aligned}$$

Combining Cauchy-Schwarz inequality, estimates (2.8) and (2.21), to have:

$$\int_0^{T_1} \|\mathbf{u}(s)\|_1 \|\mathbf{u}(s)\|_2 ds \leq \left( \int_0^{T_1} \|\mathbf{u}(s)\|_1^2 ds \right)^{\frac{1}{2}} \left( \int_0^{T_1} \|\mathbf{u}(s)\|_2^2 ds \right)^{\frac{1}{2}} \leq C. \quad (2.23)$$

Applying Gronwall Lemma to (2.23), we obtain:

$$\|\mathbf{u}_t\|^2 + \gamma \int_0^{T_1} \|\nabla \mathbf{u}_t(s)\|^2 ds \leq C, \quad \forall t \in [0, T_1]. \quad (2.24)$$

Multiplying the equations (2.1) by  $\tilde{\Delta} \mathbf{u}$ , we get

$$\begin{aligned} \gamma \|\tilde{\Delta} \mathbf{u}\|^2 &= -(\mathbf{u}_t, \tilde{\Delta} \mathbf{u}) - a(|\mathbf{u}|^\alpha \mathbf{u}, \tilde{\Delta} \mathbf{u}) - b(\mathbf{u}, \mathbf{u}, \tilde{\Delta} \mathbf{u}) - (\mathbf{f}, \tilde{\Delta} \mathbf{u}) \\ &\leq \frac{\gamma}{8} \|\tilde{\Delta} \mathbf{u}\|^2 + 2\gamma \|\mathbf{u}_t\|^2 + a(|\mathbf{u}|^\alpha \mathbf{u}, \tilde{\Delta} \mathbf{u}) + |b(\mathbf{u}, \mathbf{u}, \tilde{\Delta} \mathbf{u})| + \frac{\gamma}{8} \|\tilde{\Delta} \mathbf{u}\|^2 + 2\gamma \|f\|^2. \end{aligned}$$

Similarly as in (2.18) and (2.17) we control the nonlinear terms. Using (2.20), (2.24) and (2.8), we infer that

$$\|\mathbf{u}\|_2 \leq C, \quad \forall t \in [0, T_1]. \quad (2.25)$$

For the pressure estimate, we use the equation (2.1) to have

$$\|\nabla p\| \leq (\|\mathbf{u}_t\| + \gamma\|\nabla \mathbf{u}\| + a\|\mathbf{u}\|^{\alpha+1}\| + \|B(\mathbf{u}, \mathbf{u})\|). \quad (2.26)$$

Using the Hölder's and the inequalities (2.7), the estimates (2.20) and (2.25), since  $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^4(\Omega)$ , we observe that:

$$\begin{aligned} \|B(\mathbf{u}, \mathbf{u})\|^2 &\leq C\|\mathbf{u}\|_{\alpha+2}\|\nabla \mathbf{u}\|^{\frac{2(\alpha+2)}{\alpha}} \\ &\leq C\|\mathbf{u}\|_{\alpha+2}\|\Delta \mathbf{u}\|_2^{\frac{(\frac{10-\alpha}{\alpha+8})}{\alpha}}\|\mathbf{u}\|_{\alpha+2}^{2(\frac{\alpha-1}{\alpha+8})} \\ &\leq C\|\Delta \mathbf{u}\|_2^{2(\frac{10-\alpha}{\alpha+8})}\|\mathbf{u}\|_{\alpha+2}^{3(\frac{\alpha+2}{\alpha+8})} \leq C. \end{aligned}$$

Substituting the inequalities (2.20), (2.24) and (2.25) in (2.26), we obtain the pressure estimate.  $\square$

**Lemma 2.6.** *Under the same assumptions of the lemma 2.5, we suppose in addition that*

$$\mathbf{f}_t, \mathbf{f}_{tt} \in C([0, T], \underline{L}^2(\Omega)) \quad (\text{S3})$$

Then, the solution of the system (2.1) satisfies

$$\|\mathbf{u}_t(t)\|_2^2 + \|\nabla p_t\| + \int_{t_0}^t (\|\mathbf{u}_{tt}(s)\|_2^2 + \|p_{tt}(s)\|_1^2) ds \leq C, \quad \forall t \in [t_0, T_1]. \quad (2.27)$$

*Proof.* Deriving one time the equation (2.1) with respect to  $t$ , to have

$$\mathbf{u}_{tt} - \gamma\tilde{\Delta}\mathbf{u}_t + aG'(\mathbf{u})\mathbf{u}_t + B(\mathbf{u}_t, \mathbf{u}) + B(\mathbf{u}, \mathbf{u}_t) = \mathbf{f}_t \quad (2.28)$$

Multiplying the equation (2.28) by  $\tilde{\Delta}\mathbf{u}_t$ , we obtain:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}_t\|^2 + \gamma\|\tilde{\Delta}\mathbf{u}_t\|^2 &= -(\mathbf{f}_t, \tilde{\Delta}\mathbf{u}_t) + b(\mathbf{u}_t, \mathbf{u}, \tilde{\Delta}\mathbf{u}_t) + b(\mathbf{u}, \mathbf{u}_t, \tilde{\Delta}\mathbf{u}_t) \\ &\quad + a(F'(\mathbf{u})\mathbf{u}_t, \tilde{\Delta}\mathbf{u}_t) \\ &\leq \frac{\gamma}{8}\|\tilde{\Delta}\mathbf{u}_t\|^2 + 2\|\mathbf{f}_t\|^2 + |b(\mathbf{u}_t, \mathbf{u}, \tilde{\Delta}\mathbf{u}_t)| + |b(\mathbf{u}, \mathbf{u}_t, \tilde{\Delta}\mathbf{u}_t)| \\ &\quad + a|(F'(\mathbf{u})\mathbf{u}_t, \tilde{\Delta}\mathbf{u}_t)| \end{aligned}$$

With a similar manner as in lemma 2.5 we write

$$|b(\mathbf{u}_t, \mathbf{u}, \tilde{\Delta}\mathbf{u}_t)| \leq \frac{\gamma}{8}\|\tilde{\Delta}\mathbf{u}_t\|^2 + C\|\mathbf{u}\|_2^2\|\nabla \mathbf{u}_t\|^2 \quad (2.29)$$

$$|b(\mathbf{u}, \mathbf{u}_t, \tilde{\Delta}\mathbf{u}_t)| \leq \frac{\gamma}{8}\|\tilde{\Delta}\mathbf{u}_t\|^2 + C\|\mathbf{u}\|_2^2\|\nabla \mathbf{u}_t\|^2. \quad (2.30)$$

Next,

$$a|(F'(\mathbf{u})\mathbf{u}_t, \tilde{\Delta}\mathbf{u}_t)| \leq C(\alpha+1)\|\mathbf{u}\|_\infty^\alpha\|\mathbf{u}_t\|\|\tilde{\Delta}\mathbf{u}_t\| \leq \frac{\gamma}{8}\|\tilde{\Delta}\mathbf{u}_t\|^2 + C\|\mathbf{u}\|_\infty^{2\alpha}\|\nabla \mathbf{u}_t\|^2 \quad (2.31)$$

Combining the previous estimates together and using the inequality (2.14), (2.20) and (2.25), to obtain

$$\frac{d}{dt}\|\nabla \mathbf{u}_t\|^2 + \gamma\|\tilde{\Delta}\mathbf{u}_t\|^2 \leq 4\|\mathbf{f}_t\|^2 + C\|\nabla \mathbf{u}_t\|^2. \quad (2.32)$$

Thanks to Gronwall's inequality, we infer that

$$\sup_{t_0 \leq t \leq T_1} \|\nabla \mathbf{u}_t(t)\|^2 + \gamma \int_{t_0}^t \|\tilde{\Delta}\mathbf{u}_t(s)\|^2 ds \leq C, \quad \forall t \in [t_0, T_1]. \quad (2.33)$$

Now, differentiating the equation (2.22) with respect to  $t$ , we have

$$\mathbf{u}_{ttt} - \gamma\Delta \mathbf{u}_{tt} + aF''(\mathbf{u})|\mathbf{u}_t|^2 + aF'(\mathbf{u})\mathbf{u}_{tt} + 2B(\mathbf{u}_t, \mathbf{u}_t) + B(\mathbf{u}_{tt}, \mathbf{u}) + B(\mathbf{u}, \mathbf{u}_{tt}) + \nabla p_{tt} = \mathbf{f}_{tt}. \quad (2.34)$$

Taking the inner product of the relation (2.34) with  $\mathbf{u}_{tt}$ , to get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathbf{u}_{tt}\|^2 + \gamma\|\nabla \mathbf{u}_{tt}\|^2 &= -a(F''(\mathbf{u})|\mathbf{u}_t|^2, \mathbf{u}_{tt}) - a(F'(\mathbf{u})\mathbf{u}_{tt}, \mathbf{u}_{tt}) \\ &\quad - 2b(\mathbf{u}_t, \mathbf{u}_t, \mathbf{u}_{tt}) - b(\mathbf{u}_{tt}, \mathbf{u}, \mathbf{u}_{tt}) + (\mathbf{f}_{tt}, \mathbf{u}_{tt}). \end{aligned}$$

Since  $(F'(\mathbf{u})\mathbf{u}_{tt}) \cdot \mathbf{u}_{tt}$  is positive definite, using Young's inequality, we have

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}_{tt}\|^2 + \gamma \|\nabla \mathbf{u}_{tt}\|^2 \leq \frac{1}{2} \|\mathbf{f}_{tt}\|^2 + |\beta(F''(\mathbf{u})|\mathbf{u}_t|^2, \mathbf{u}_{tt})| + 2|b(\mathbf{u}_t, \mathbf{u}_t, \mathbf{u}_{tt})| + |b(\mathbf{u}_{tt}, \mathbf{u}, \mathbf{u}_{tt})|.$$

Thanks to (2.14), (2.20) and (2.33), since  $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^4(\Omega)$ ,

$$\begin{aligned} |(F''(\mathbf{u})|\mathbf{u}_t|^2, \mathbf{u}_{tt})| &\leq \alpha(\alpha+1)(|\mathbf{u}|^{\alpha-1}|\mathbf{u}_t|^2, |\mathbf{u}_{tt}|) \\ &\leq \alpha(\alpha+1)\|\mathbf{u}\|_\infty^{\alpha-1}\|\mathbf{u}_t\|_4^2\|\mathbf{u}_{tt}\| \\ &\leq C\|\mathbf{u}_{tt}\|^2 + C\|\mathbf{u}\|_\infty^{2(\alpha-1)}\|\nabla \mathbf{u}_t\|^4 \leq C\|\mathbf{u}_{tt}\|^2 + C. \end{aligned} \quad (2.35)$$

Using Young, Hölder inequalities, the inequality (2.7)- (2.11) and estimate (2.15), we have

$$|b(\mathbf{u}_{tt}, \mathbf{u}, \mathbf{u}_{tt})| \leq \|\mathbf{u}_{tt}\|_1 \|\mathbf{u}\|_2 \|\mathbf{u}_{tt}\| \leq \frac{\gamma}{2} \|\nabla \mathbf{u}_{tt}\|^2 + C\|\mathbf{u}_{tt}\|^2. \quad (2.36)$$

$$|b(\mathbf{u}_t, \mathbf{u}_t, \mathbf{u}_{tt})| \leq \|\mathbf{u}_t\|_2 \|\mathbf{u}_t\|_1 \|\mathbf{u}_{tt}\| \leq C\|\nabla \mathbf{u}_t\|^2 + \|\mathbf{u}_t\|_2^2 \|\mathbf{u}_{tt}\|^2. \quad (2.37)$$

Replacing (2.35)-(2.37) in (2.35), using (2.33) and applying Gronwall Lemma yield

$$\|\mathbf{u}_{tt}\|^2 + \gamma \int_{t_0}^{T_1} \|\nabla \mathbf{u}_{tt}(s)\|^2 ds \leq C. \quad (2.38)$$

Multiplying the equations (2.1) with  $\tilde{\Delta} \mathbf{u}_t$ , we obtain:

$$\begin{aligned} \gamma \|\tilde{\Delta} \mathbf{u}_t\|^2 &= (\mathbf{u}_{tt}, \tilde{\Delta} \mathbf{u}_t) - (\mathbf{f}_t, \tilde{\Delta} \mathbf{u}_t) - a(F'(\mathbf{u})\mathbf{u}_t, \tilde{\Delta} \mathbf{u}_t) - b(\mathbf{u}_t, \mathbf{u}, \tilde{\Delta} \mathbf{u}_t) \\ &\quad - b(\mathbf{u}, \mathbf{u}_t, \tilde{\Delta} \mathbf{u}_t) \\ &\leq \frac{5\gamma}{2} \|\mathbf{u}_{tt}\|^2 + \frac{\gamma}{5} \|\tilde{\Delta} \mathbf{u}_t\|^2 + \frac{5\gamma}{2} \|\mathbf{f}_t\|^2 + a|(F'(\mathbf{u})\mathbf{u}_t, \tilde{\Delta} \mathbf{u}_t)| \\ &\quad + |b(\mathbf{u}_t, \mathbf{u}, \tilde{\Delta} \mathbf{u}_t)| + |b(\mathbf{u}, \mathbf{u}_t, \tilde{\Delta} \mathbf{u}_t)| \end{aligned}$$

For bounding the nonlinear terms we apply a similar estimations as in (2.29)-(2.31), thanks to (2.8), (2.38) and (2.33), we deduct

$$\|\mathbf{u}_t\|_2 \leq C, \quad \forall t \in [t_0, T_1]. \quad (2.39)$$

We can obtain the remaining estimates in (2.27) by following same arguments of Lemma 2.5. Notice that (2.33) is an analogue of  $\int_{t_0}^t \|\tilde{\Delta} \mathbf{u}_{tt}(s)\|^2 ds$  for  $\mathbf{u}_{tt}$  instead of  $\mathbf{u}_t$ .  $\square$

Moreover, if we assume that

$$t\mathbf{f}_t \in L^2(0, T_1; \mathbf{H}^1(\Omega)), \quad (S2)$$

We obtain the pressure estimate

$$tp_t \in L^2(0, T_1; \mathbf{H}^1(\Omega)) \quad (2.40)$$

**Lemma 2.7.** *Assume that  $1 \leq \alpha \leq 2$  and  $u_0 \in \mathbf{H}$ , then the solution  $u$  of the problem (2.1)-(2.2) satisfies*

$$\sup_{t \in [0, T]} \|\mathbf{u}\| \leq C \quad (2.41)$$

*Proof.* We multiply the equation (2.1) by  $\mathbf{u}_t + \mathbf{u}$  in  $L^2(\Omega)$

$$\begin{aligned} 2\|\mathbf{u}_t(t)\|^2 + \frac{d}{dt} \left[ \nu \|\nabla \mathbf{u}(t)\|^2 + \frac{2a}{\alpha+2} \int_{\Omega} |\mathbf{u}(x, t)|^{\alpha+2} dx \right] + 2\nu \|\nabla \mathbf{u}(t)\|^2 + 2a \int_{\Omega} |\mathbf{u}(x, t)|^{\alpha+2} dx \\ + b(\mathbf{u}, \mathbf{u}, \mathbf{u}_t) = (\mathbf{f}, \mathbf{u}_t) + (\mathbf{f}, \mathbf{u}) \leq c\|\mathbf{f}\|^2 + \frac{1}{2}\|\mathbf{u}_t\|^2 + \frac{\nu}{2}\|\mathbf{u}\|^2 \end{aligned} \quad (2.42)$$

The equation (2.42) implies

$$\|\mathbf{u}_t(t)\|^2 + \frac{d}{dt} \left[ \nu \|\nabla \mathbf{u}(t)\|^2 + \frac{2a}{\alpha+2} \int_{\Omega} |\mathbf{u}(x, t)|^{\alpha+2} dx \right] + \nu \|\nabla \mathbf{u}(t)\|^2 + 2a \int_{\Omega} |\mathbf{u}(x, t)|^{\alpha+2} dx \leq c\|\mathbf{f}\|^2 \quad (2.43)$$

We define the function

$$\psi(t) = \nu \|\nabla \mathbf{u}(t)\|^2 + \frac{2a}{\alpha + 2} \int_{\Omega} |\mathbf{u}(x, t)|^{\alpha+2} dx$$

$\psi$  satisfies the inequality  $\frac{d}{dt}\psi(t) + \psi(t) \leq c\|\mathbf{f}\|^2$ . we have the estimate

$$\nu \|\nabla \mathbf{u}(t)\|^2 + \frac{2a}{\alpha + 2} \int_{\Omega} |\mathbf{u}(x, t)|^{\alpha+2} dx \leq Ke^{-t} \quad (2.44)$$

where  $K = \nu \|\nabla \mathbf{u}_0\|^2 + \frac{2a}{\alpha + 2} \int_{\Omega} |\mathbf{u}_0(x)|^{\alpha+2} dx$  □

The next result gives the control of the mapping  $F$ .

**Lemma 2.8.** *Consider  $\mathbf{u}, \mathbf{v} \in X$  such that (2.41) holds for  $u$  and  $v$ , then*

$$\|F(\mathbf{u}) - F(\mathbf{v})\| \leq C\|\mathbf{u} - \mathbf{v}\| \quad (2.45)$$

Our next point is the existence and uniqueness of weak solutions to system(2.1)-(2.2).

**Theorem 2.1.** *Under the assumption (S1) and  $\mathbf{u}_0 \in \mathbf{H}$ , the system(2.1)-(2.2) has at least one solution  $\mathbf{u} \in L^2(0, T; H_0^1(\Omega)) \cap L^{\alpha+2}(0, T; L^{\alpha+2}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ .*

*Proof.* We consider an orthonormal basis  $(w_k)$  in  $X$  and define the sequence  $(\mathbf{u}_m) \in X$  satisfying

$$\left(\frac{\partial \mathbf{u}_m(t)}{\partial t}, w_k\right) + \nu(\nabla \mathbf{u}_m(t), \nabla w_k) + b(\mathbf{u}_m(t), \mathbf{u}_m(t), w_k) + a(|\mathbf{u}_m(t)|^\alpha \mathbf{u}_m(t), w_k) = (\mathbf{f}(t), w_k), \quad \text{for } k = 1, \dots, m \quad (2.46)$$

with initial conditions

$$\mathbf{u}_m(0) = \mathbf{u}_{0m} \quad (2.47)$$

where for each  $m \in \mathbb{N}$

$$\mathbf{u}_m(t) = \sum_{k=0}^m \xi_{km}(t) w_k \quad (2.48)$$

and  $\mathbf{u}_{0m}$  is the projection of  $\mathbf{u}_0$  from the space  $X$  to the space  $\text{span}\{w_1, \dots, w_m\}$ .

The Cauchy problem (2.46)-(2.47) is a first order nonlinear ordinary differential equation in  $t$  for the functions  $\xi_{1m}(t), \dots, \xi_{mm}(t)$  with the initial conditions

$$\xi_{km}(0) = \text{the } k\text{th component of } \mathbf{u}_{0m}, \quad k = 1, \dots, m$$

Applying the properties of an ordinary differential equations, there exist a solution of the Cauchy problem  $\mathbf{u}_m$  in the time interval  $[0, t_n)$ ,  $0 < t_n \leq T$  and we have if  $t_n < T$ , then  $\lim_{t \rightarrow t_n^-} \|\mathbf{u}_m\| = +\infty$ . Otherwise  $t_n = T$ .

We multiply the equation (2.46) by  $\xi_{km}$  and sum with respect to  $k$  from 1 to  $m$ , to obtain

$$\left(\frac{\partial \mathbf{u}_m}{\partial t}, \mathbf{u}_m\right) + \nu \|\nabla \mathbf{u}_m\|^2 + a \|\mathbf{u}_m\|_{L^{\alpha+2}}^{\alpha+2} = (\mathbf{f}, \mathbf{u}_m) \quad (2.49)$$

Using Poincaré inequality and Hölder inequality yields to

$$\begin{aligned} & \frac{d}{dt} \|\mathbf{u}_m\|^2 + 2\nu \|\nabla \mathbf{u}_m\|^2 + 2a \|\mathbf{u}_m\|_{L^{\alpha+2}}^{\alpha+2} \\ & \leq 2(\mathbf{f}, \mathbf{u}_m) \leq 2\|\mathbf{f}\| \|\mathbf{u}_m\| \leq \frac{1}{\nu\lambda} \|\mathbf{f}\|^2 + \nu\lambda \|\mathbf{u}_m\|^2 \\ & \leq \frac{1}{\nu\lambda} \|\mathbf{f}\|^2 + \nu \|\nabla \mathbf{u}_m\|^2 \end{aligned} \quad (2.50)$$

Therefore

$$\frac{d}{dt} \|\mathbf{u}_m\|^2 + \nu \|\nabla \mathbf{u}_m\|^2 + 2a \|\mathbf{u}_m\|_{L^{\alpha+2}}^{\alpha+2} \leq \frac{1}{\nu\lambda} \|\mathbf{f}\|^2 \quad (2.51)$$

We integrate the inequality (2.51) with respect to  $t$  from 0 to  $s$  ( $\leq t_n$ ) to have

$$\begin{aligned}\|\mathbf{u}_m\|^2 &\leq \|\mathbf{u}_{0m}\|^2 + \frac{1}{\nu\lambda} \int_0^s \|\mathbf{f}(t)\|^2 dt \\ &\leq \|\mathbf{u}_0\|^2 + \frac{1}{\nu\lambda} \int_0^T \|\mathbf{f}(t)\|^2 dt < +\infty\end{aligned}\quad (2.52)$$

When  $t_n = T$  we obtain

$$\sup_{t \in [0, T]} \|\mathbf{u}_m\|^2 \leq \|\mathbf{u}_0\|^2 + \frac{1}{\nu\lambda} \int_0^T \|\mathbf{f}(t)\|^2 dt \quad (2.53)$$

We integrate (2.51) with respect to  $t$  from 0 to  $T$ , to get

$$\int_0^T \|\nabla \mathbf{u}_m\|^2 dt \leq \frac{1}{\nu} \|\mathbf{u}_0\|^2 + \frac{1}{\nu^2 \lambda} \int_0^T \|\mathbf{f}(t)\|^2 dt \quad (2.54)$$

$$\int_0^T \|\mathbf{u}_m\|_{L^{\alpha+2}}^{\alpha+2} dt \leq \frac{1}{2a} \|\mathbf{u}_0\|^2 + \frac{1}{2a\nu\lambda} \int_0^T \|\mathbf{f}(t)\|^2 dt \quad (2.55)$$

In order to estimate the derivative term with respect to  $t$  of  $\mathbf{u}_m(t)$  for passing to the limit in the nonlinear term in (2.46), we define

$$\mathbf{v}_m(t) = \mathbf{f}(t) - \nu \Delta \mathbf{u}_m(t) - (\mathbf{u}_m(t) \cdot \nabla) \mathbf{u}_m(t) - a |\mathbf{u}_m(t)|^\alpha \mathbf{u}_m(t)$$

We rewrite the equation (2.46) as

$$\left( \frac{\partial \mathbf{u}_m(t)}{\partial t}, w_k \right) = (\mathbf{v}_m(t), w_k), \quad \text{for } k = 1, 2, \dots, m \quad (2.56)$$

We define

$$\tilde{\mathbf{u}}_m(t) = \begin{cases} \mathbf{u}_m(t), & t \in [0, T] \\ 0, & t \in \mathbb{R} \setminus [0, T] \end{cases}$$

$\tilde{\mathbf{u}}_m(t)$  is discontinuous at  $t = 0$  and  $t = T$ . For all the set  $\mathbb{R}$ , we have

$$\frac{d}{dt} (\tilde{\mathbf{u}}_m(t), w_k) = (\tilde{\mathbf{v}}_m(t), w_k) + (\mathbf{u}_{0m}, w_k) \delta_{(0)} - (\mathbf{u}_m(T), w_k) \delta_{(T)}, \quad k = 1, 2, \dots, m \quad (2.57)$$

with  $\delta_{(0)}$  and  $\delta_{(T)}$  are the Dirac distributions at 0 and  $T$ , and

$$\tilde{\mathbf{v}}_m(t) = \begin{cases} \mathbf{v}_m(t), & t \in [0, T] \\ 0, & t \in \mathbb{R} \setminus [0, T] \end{cases}$$

Applying the Fourier transform on the both sides in equation (2.56) yields to

$$2i\pi\tau (\hat{\mathbf{u}}_m(\tau), w_k) = (\hat{\mathbf{v}}_m(\tau), w_k) + (\mathbf{u}_{0m}, w_k) - (\mathbf{u}_m(T), w_k) \exp(-2i\pi\tau T), \quad k = 1, \dots, m \quad (2.58)$$

With a similar manner, we define  $\tilde{\xi}_{km}$  and we denote by  $\hat{\xi}_{km}$  the Fourier transform of  $\tilde{\xi}_{km}$ .

We multiply the equation (2.56) by  $\hat{\xi}_{km}$  and sum with respect to  $k$ , to obtain

$$2i\pi\tau \|\hat{\mathbf{u}}_m(\tau)\|^2 = (\hat{\mathbf{v}}_m(\tau), \hat{\mathbf{u}}_m(\tau)) + (\mathbf{u}_{0m}, \hat{\mathbf{u}}_m(\tau)) - (\mathbf{u}_m(T), \hat{\mathbf{u}}_m(\tau)) \exp(-2i\pi\tau T) \quad (2.59)$$

Using the inequalities (2.52)-(2.55), yields to

$$\begin{aligned}2\pi\tau \|\hat{\mathbf{u}}_m(\tau)\|^2 &\leq |(\hat{\mathbf{v}}_m(\tau), \hat{\mathbf{u}}_m(\tau))| + \|\mathbf{u}_{0m}\| \|\hat{\mathbf{u}}_m(\tau)\| + \|\mathbf{u}_m(T)\| \|\hat{\mathbf{u}}_m(\tau)\| \\ &\leq |(\hat{\mathbf{v}}_m(\tau), \hat{\mathbf{u}}_m(\tau))| + 2\sqrt{K} \|\hat{\mathbf{u}}_m(\tau)\|\end{aligned}\quad (2.60)$$

where  $K = \|\mathbf{u}_0\|^2 + \frac{1}{\nu\lambda} \int_0^T \|\mathbf{f}(t)\|^2 dt$ .

The term  $|(\hat{\mathbf{v}}_m(\tau), \hat{\mathbf{u}}_m(\tau))|$  can be raised as

$$\begin{aligned}|(\hat{\mathbf{v}}_m(\tau), \hat{\mathbf{u}}_m(\tau))| &\leq |(\hat{\mathbf{f}}(\tau), \hat{\mathbf{u}}_m(\tau))| + \nu \|\nabla \hat{\mathbf{u}}_m(\tau)\|^2 + b(\hat{\mathbf{u}}_m(\tau), \hat{\mathbf{u}}_m(\tau), \hat{\mathbf{u}}_m(\tau)) + a \|\hat{\mathbf{u}}_m(\tau)\|_{L^{\alpha+2}}^{\alpha+2} \\ &\leq \|\hat{\mathbf{f}}(\tau)\|^2 + \|\hat{\mathbf{u}}_m(\tau)\|^2 + \nu \|\hat{\mathbf{u}}_m(\tau)\|^2 + a \|\hat{\mathbf{u}}_m(\tau)\|_{L^{\alpha+2}}^{\alpha+2}\end{aligned}\quad (2.61)$$

Combining the inequalities (2.60)-(2.61) yields to

$$2\pi\tau\|\hat{\mathbf{u}}_m(\tau)\|^2 \leq \|\hat{\mathbf{f}}(\tau)\|^2 + \|\hat{\mathbf{u}}_m(\tau)\|^2 + \nu\|\hat{\mathbf{u}}_m(\tau)\|^2 + a\|\hat{\mathbf{u}}_m(\tau)\|_{L^{\alpha+2}}^{\alpha+2} + 2\sqrt{K}\|\hat{\mathbf{u}}_m(\tau)\| \quad (2.62)$$

Consider  $\gamma \in [0, 1/4]$ , we have the inequality:  $|\tau|^{2\gamma} \leq (2\gamma + 1)\frac{1 + |\tau|}{1 + |\tau|^{1-2\gamma}}, \forall \tau \in \mathbb{R}$ . Therefore

$$\begin{aligned} \int_{-\infty}^{+\infty} |\tau|^{2\gamma}\|\hat{\mathbf{u}}_m(\tau)\|^2 d\tau &\leq (2\gamma + 1) \int_{-\infty}^{+\infty} \frac{\|\hat{\mathbf{u}}_m(\tau)\|^2}{1 + |\tau|^{1-2\gamma}} d\tau + (2\gamma + 1) \int_{-\infty}^{+\infty} \frac{|\tau|\|\hat{\mathbf{u}}_m(\tau)\|^2}{1 + |\tau|^{1-2\gamma}} d\tau \\ &\leq (2\gamma + 1) \int_{-\infty}^{+\infty} \|\hat{\mathbf{u}}_m(\tau)\|^2 d\tau + (2\gamma + 1) \int_{-\infty}^{+\infty} \frac{|\tau|\|\hat{\mathbf{u}}_m(\tau)\|^2}{1 + |\tau|^{1-2\gamma}} d\tau \end{aligned} \quad (2.63)$$

For raising the first term on the right hand side in (2.63), we use the Perceval equality, the Poincaré inequality and the inequality (2.54) to have

$$\begin{aligned} (2\gamma + 1) \int_{-\infty}^{+\infty} \|\hat{\mathbf{u}}_m(\tau)\|^2 d\tau &= (2\gamma + 1) \int_{-\infty}^{+\infty} \|\hat{\mathbf{u}}_m(t)\|^2 dt \\ &= (2\gamma + 1) \int_{-\infty}^{+\infty} \|\mathbf{u}_m(t)\|^2 dt \leq c(\gamma, d) \end{aligned} \quad (2.64)$$

It remains the second term on the right hand side of the inequality (2.63). We use the inequalities (2.53)-(2.55) and (2.62) to get

$$\begin{aligned} (2\gamma + 1) \int_{-\infty}^{+\infty} \frac{|\tau|\|\hat{\mathbf{u}}_m(\tau)\|^2}{1 + |\tau|^{1-2\gamma}} d\tau &\leq \frac{(2\gamma + 1)}{2\pi} \int_{-\infty}^{+\infty} \|\hat{\mathbf{f}}\|^2 d\tau + \frac{(2\gamma + 1)a}{\pi} \int_{-\infty}^{+\infty} \|\hat{\mathbf{u}}_m(\tau)\|_{L^\alpha}^\alpha d\tau \\ &\quad + \frac{(2\gamma + 1)\nu}{2\pi} \int_{-\infty}^{+\infty} \|\nabla\hat{\mathbf{u}}_m(\tau)\|^2 d\tau + \frac{(2\gamma + 1)\sqrt{K}}{2\pi} \int_{-\infty}^{+\infty} \frac{\|\nabla\hat{\mathbf{u}}_m(\tau)\|}{1 + |\tau|^{1-2\gamma}} d\tau \\ &\leq c \int_{-\infty}^{+\infty} \left( \|\hat{\mathbf{f}}\|^2 + \|\hat{\mathbf{u}}_m(t)\|_{L^\alpha}^\alpha + \|\nabla\hat{\mathbf{u}}_m(t)\|^2 \right) dt \\ &\quad + c \left( \int_{-\infty}^{+\infty} \frac{d\tau}{1 + |\tau|^{1-2\gamma}} \right)^{1/2} \left( \int_{-\infty}^{+\infty} \|\hat{\mathbf{u}}_m(t)\|^2 dt \right)^{1/2} \\ &\leq c \int_0^T \left( \|\hat{\mathbf{f}}\|^2 + \|\hat{\mathbf{u}}_m(t)\|_{L^\alpha}^\alpha + \|\nabla\hat{\mathbf{u}}_m(t)\|^2 \right) dt \\ &\quad + c \left( \int_{-\infty}^{+\infty} \frac{d\tau}{1 + |\tau|^{1-2\gamma}} \right)^{1/2} \left( \int_0^T \|\hat{\mathbf{u}}_m(t)\|^2 dt \right)^{1/2} \leq c(\pi, \gamma, a, \nu, d, T) \end{aligned} \quad (2.65)$$

Using the inequalities (2.63)-(2.65), there exists a subsequence denoted without lose of generality by  $\mathbf{u}_m$  and  $\mathbf{u}$  such that

$$\mathbf{u}_m \overset{*}{\rightharpoonup} \mathbf{u} \quad \text{in } L^\infty(0, T; L^2(\Omega)) \quad (2.66)$$

$$\mathbf{u}_m \rightharpoonup \mathbf{u} \quad \text{in } L^2(0, T; H_0^1(\Omega)) \quad (2.67)$$

$$\mathbf{u}_m \longrightarrow \mathbf{u} \quad \text{in } L^2(0, T; L^2(\Omega)) \quad (2.68)$$

$$\mathbf{u}_m \rightharpoonup \mathbf{u} \quad \text{in } L^{\alpha+2}(0, T; L^{\alpha+2}(\Omega)) \quad (2.69)$$

Consider  $\phi(t) \in C_0^\infty(0, T)$ . Multiplying the equation (2.46) by  $\phi(t)$ , integrating from 0 to T and integrating the first term of the result by parts to obtain

$$\begin{aligned} - \int_0^T (\mathbf{u}_m(t), w_k) \phi'(t) dt + \nu \int_0^T (\nabla\mathbf{u}_m(t), \phi(t)\nabla w_k) dt + \int_0^T b(\mathbf{u}_m(t), \mathbf{u}_m(t), w_k \phi(t)) dt \\ a \int_0^T (|\mathbf{u}_m(t)|^\alpha \mathbf{u}_m(t), w_k \phi(t)) dt = \int_0^T (\mathbf{f}, w_k \phi(t)) dt, \quad k = 1, 2, \dots, m \end{aligned} \quad (2.70)$$

Now, we study the convergence of the nonlinear terms in (2.70). We begin by the trilinear form

$$\begin{aligned}
\left| \int_0^T b(\mathbf{u}_m, \mathbf{u}_m, w_k \phi(t)) dt - \int_0^T b(\mathbf{u}, \mathbf{u}, w_k \phi(t)) dt \right| &\leq \frac{1}{2} \left| \int_0^T b(\mathbf{u}_m, \mathbf{u}_m, w_k \phi(t)) dt - \int_0^T b(\mathbf{u}, \mathbf{u}, w_k \phi(t)) dt \right| \\
&+ \frac{1}{2} \left| \int_0^T b(\mathbf{u}_m, w_k \phi, \mathbf{u}_m) dt - \int_0^T b(\mathbf{u}, w_k \phi, \mathbf{u}) dt \right| \\
&\leq \frac{1}{2} \left| \int_0^T b(\mathbf{u}_m - \mathbf{u}, \mathbf{u}_m, w_k \phi(t)) dt \right| + \frac{1}{2} \left| \int_0^T b(\mathbf{u}, \mathbf{u}_m - \mathbf{u}, w_k \phi(t)) dt \right| \\
&+ \frac{1}{2} \left| \int_0^T b(\mathbf{u}_m - \mathbf{u}, w_k \phi(t), \mathbf{u}_m) dt \right| + \frac{1}{2} \left| \int_0^T b(\mathbf{u}, w_k \phi(t), \mathbf{u}_m - \mathbf{u}) dt \right| \\
&= b_1 + b_2 + b_3 + b_4
\end{aligned} \tag{2.71}$$

where

$$\begin{aligned}
b_1 = \frac{1}{2} \left| \int_0^T b(\mathbf{u}_m - \mathbf{u}, \mathbf{u}_m, w_k \phi(t)) dt \right| &\leq \frac{1}{2} \sup_{t \in [0, T]} |\phi(t)| \sup_{x \in \Omega} |w_k| \int_0^T \|\mathbf{u}_m - \mathbf{u}\| \|\nabla \mathbf{u}\| dt \\
&\leq c \|\mathbf{u}_m - \mathbf{u}\|_{L^2(0, T; L^2(\Omega))} \|\mathbf{u}_m\|_{L^2(0, T; H_0^1(\Omega))} \\
&\rightarrow 0 \text{ as } m \rightarrow \infty
\end{aligned} \tag{2.72}$$

$$\begin{aligned}
b_2 = \frac{1}{2} \left| \int_0^T b(\mathbf{u}, \mathbf{u}_m - \mathbf{u}, w_k \phi(t)) dt \right| &= \left| \frac{1}{2} \int_0^T \sum_{i, j=1}^3 \int_{\Omega} (\mathbf{u})_i \frac{\partial(\mathbf{u}_m - \mathbf{u})}{\partial x_j} (\phi(t) w_k)_j dx dt \right| \\
&\leq \frac{1}{2} \int_0^T \sum_{i, j=1}^3 \left| \int_{\Omega} (\mathbf{u}_m - \mathbf{u})_j \frac{\partial(\mathbf{u})_i}{\partial x_j} (\phi(t) w_k)_j dx dt \right| \\
&+ \frac{1}{2} \int_0^T \sum_{i, j=1}^3 \left| \int_{\Omega} (\mathbf{u}_m - \mathbf{u})_j (\mathbf{u})_i \phi(t) \frac{\partial(w_k)_j}{\partial w_j} dx dt \right| \\
&\leq \frac{1}{2} \sup_{t \in [0, T]} |\phi(t)| \sup_{x \in \Omega} |w_k| \int_0^T \|\mathbf{u}_m - \mathbf{u}\| \|\nabla \mathbf{u}\| dt \\
&+ \frac{1}{2} \sup_{t \in [0, T]} |\phi(t)| \sup_{x \in \Omega} |\nabla w_k| \int_0^T \|\mathbf{u}_m - \mathbf{u}\| \|\mathbf{u}\| dt \\
&\leq c \|\mathbf{u}_m - \mathbf{u}\|_{L^2(0, T; L^2(\Omega))} (\|\mathbf{u}\|_{L^2(0, T; H_0^1(\Omega))} + \|\mathbf{u}\|_{L^2(0, T; L^2(\Omega))}) \\
&\rightarrow 0 \text{ as } m \rightarrow \infty
\end{aligned} \tag{2.73}$$

$$\begin{aligned}
b_3 = \frac{1}{2} \left| \int_0^T b(\mathbf{u}_m - \mathbf{u}, w_k \phi(t), \mathbf{u}_m) dt \right| &\leq \sup_{t \in [0, T]} |\phi(t)| \sup_{x \in \Omega} |\nabla w_k| \int_0^T \|\mathbf{u}_m - \mathbf{u}\| \|\nabla \mathbf{u}_m\| dt \\
&\leq c \|\mathbf{u}_m - \mathbf{u}\|_{L^2(0, T; L^2(\Omega))} \|\mathbf{u}_m\|_{L^2(0, T; H_0^1(\Omega))} \\
&\rightarrow 0 \text{ as } m \rightarrow \infty
\end{aligned} \tag{2.74}$$

$$\begin{aligned}
b_4 = \frac{1}{2} \left| \int_0^T b(\mathbf{u}, w_k \phi(t), \mathbf{u}_m - \mathbf{u}) dt \right| &\leq \sup_{t \in [0, T]} |\phi(t)| \sup_{x \in \Omega} |\nabla w_k| \int_0^T \|\mathbf{u}_m - \mathbf{u}\| \|\mathbf{u}\| dt \\
&\leq c \|\mathbf{u}_m - \mathbf{u}\|_{L^2(0, T; L^2(\Omega))} \|\mathbf{u}\|_{L^2(0, T; L^2(\Omega))} \\
&\rightarrow 0 \text{ as } m \rightarrow \infty
\end{aligned} \tag{2.75}$$

Using (2.72)-(2.75) to deduce that

$$\int_0^T b(\mathbf{u}_m, \mathbf{u}_m, w_k \phi(t)) dt \longrightarrow \int_0^T b(\mathbf{u}, \mathbf{u}, w_k \phi(t)) dt \quad (2.76)$$

It remains the Forchheimer term. We write

$$\begin{aligned} \left| \int_0^T (|\mathbf{u}_m|^\alpha \mathbf{u}_m, w_k \phi(t)) dt - \int_0^T (|\mathbf{u}|^\alpha \mathbf{u}, w_k \phi(t)) dt \right| &\leq \int_0^T \|\mathbf{u}_m|^\alpha \mathbf{u}_m - |\mathbf{u}|^\alpha \mathbf{u}\| |w_k| |\phi(t)| dt \\ &\leq c \int_0^T \|\nabla(\mathbf{u}_m - \mathbf{u})\| dt \\ &\leq c \|\mathbf{u}_m - \mathbf{u}\|_{L^2(0,T,H_0^1(\Omega))} \longrightarrow 0 \text{ as } m \rightarrow \infty \end{aligned} \quad (2.77)$$

Equation (2.77) implies

$$a \int_0^T (|\mathbf{u}_m|^\alpha \mathbf{u}_m, w_k \phi(t)) dt \longrightarrow a \int_0^T (|\mathbf{u}|^\alpha \mathbf{u}, w_k \phi(t)) dt \quad (2.78)$$

Now, for fixed  $k = 1, 2, \dots, m$ , we pass to the limit in (2.70) and using (2.66)-(2.69), (2.76) and (2.78) to infer that

$$\begin{aligned} & - \int_0^T (\mathbf{u}, w_k) \phi'(t) dt + \nu \int_0^T (\nabla \mathbf{u}, \phi(t) \nabla w_k) dt + \int_0^T b(\mathbf{u}, \mathbf{u}, w_k \phi(t)) dt \\ & + a \int_0^T (|\mathbf{u}|^\alpha \mathbf{u}, w_k \phi(t)) dt = \int_0^T (\mathbf{f}(t), w_k \phi(t)) dt, \quad k = 1, 2, \dots, m \end{aligned} \quad (2.79)$$

As it is known the nonlinear terms in (2.79). By continuity properties we have  $\forall w \in \mathbf{H}$

$$\begin{aligned} & - \int_0^T (\mathbf{u}, w) \phi'(t) dt + \nu \int_0^T (\nabla \mathbf{u}, \phi(t) \nabla w) dt + \int_0^T b(\mathbf{u}, \mathbf{u}, w \phi(t)) dt \\ & + a \int_0^T (|\mathbf{u}|^\alpha \mathbf{u}, w \phi(t)) dt = \int_0^T (\mathbf{f}(t), w \phi(t)) dt \end{aligned} \quad (2.80)$$

Equation (2.80) implies in the distribution sense

$$\frac{d}{dt} (\mathbf{u}, w) + \nu (\nabla \mathbf{u}, \nabla w) + b(\mathbf{u}, \mathbf{u}, w) + a (|\mathbf{u}|^\alpha \mathbf{u}, w) = (\mathbf{f}, w), \quad \forall w \in \mathbf{H} \quad (2.81)$$

At least, we verify if the velocity  $\mathbf{u}$  satisfies the initial condition  $\mathbf{u}(0) = \mathbf{u}_0$ . We have  $\mathbf{u} \in L^2(0, T; H_0^1(\Omega)) \cap L^{\alpha+2}(0, T; L^{\alpha+2}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ , and we use the fact that

$$a \int_0^T \| |\mathbf{u}(t)|^\alpha \mathbf{u}(t) \|_{L^{(\alpha+2)/3}}^{(\alpha+2)/3} dt \leq c \int_0^T \|\mathbf{u}\|_{L^{\alpha+2}}^{\alpha+2} dt \quad (2.82)$$

$a|\mathbf{u}|^\alpha \mathbf{u}$  belongs in  $L^{(\alpha+2)/3}(0, T; L^{(\alpha+2)/3}(\Omega)) \subset L^{4/3}(0, T; H^{-1}(\Omega))$

We use the fact that

$$\begin{aligned} \int_0^T \|B(u(t), u(t))\|_{H^{-1}}^{4/3} dt &\leq c \int_0^T \|\mathbf{u}(t)\|^{2/3} \|\mathbf{u}(t)\|_{H_0^1}^2 dt \\ &\leq c \|\mathbf{u}(t)\|_{L^\infty(0,T;L^2(\Omega))}^{2/3} \int_0^T \|\mathbf{u}(t)\|_1^2 dt \end{aligned} \quad (2.83)$$

$B(\mathbf{u}, \mathbf{u}) \in L^{4/3}(0, T, H^{-1}(\Omega))$ . Using (2.81)

$$\frac{\partial \mathbf{u}}{\partial t} \in L^{4/3}(0, T, H^{-1}(\Omega)) \quad (2.84)$$

By the embedding theorem  $\mathbf{u} \in C(0, T; L^2(\Omega))$ .

Consider  $\phi \in C^1([0, T])$  such that  $\phi(0) \neq 0$  and  $\phi(T) = 0$ , then we have

$$\begin{aligned} & - \int_0^T (\mathbf{u}, w) \phi'(t) dt + \nu \int_0^T (\nabla \mathbf{u}, \phi(t) \nabla w) dt + \int_0^T b(\mathbf{u}, \mathbf{u}, w \phi(t)) dt \\ & + a \int_0^T (|\mathbf{u}|^\alpha \mathbf{u}, w \phi(t)) dt = (\mathbf{u}_0, w) \phi(0) + \int_0^T (\mathbf{f}(t), w \phi(t)) dt \quad \forall w \in \mathbf{H} \end{aligned} \quad (2.85)$$

Multiplying the equation (2.81) by  $\phi(t)$  and integrating yields to

$$\begin{aligned} & - \int_0^T (\mathbf{u}, w) \phi'(t) dt + \nu \int_0^T (\nabla \mathbf{u}, \phi(t) \nabla w) dt + \int_0^T b(\mathbf{u}, \mathbf{u}, w \phi(t)) dt \\ & + a \int_0^T (|\mathbf{u}|^\alpha \mathbf{u}, w \phi(t)) dt = (\mathbf{u}(0), w) \phi(0) + \int_0^T (\mathbf{f}(t), w \phi(t)) dt \quad \forall w \in \mathbf{H} \end{aligned} \quad (2.86)$$

We look at the analogy between the equations (2.85) and (2.86) to infer that

$$(\mathbf{u}_0 - \mathbf{u}(0), w) = 0 \quad \forall w \in \mathbf{H} \quad (2.87)$$

Therefore  $\mathbf{u}(0) = \mathbf{u}_0$ .

The existence of the pressure follows from the existence of the velocity and for each  $t \in [0, T]$ ,  $p(t) \in L^2(\Omega)$ .

Moreover  $\nabla p \in C(0, T; H^{-1}(\Omega))$  and we have

$$p \in C(0, T; L^2(\Omega)) \quad (2.88)$$

and satisfies in the distribution sense that

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + a |\mathbf{u}|^\alpha \mathbf{u} + \nabla p = \mathbf{f} \quad (2.89)$$

which complete the proof.  $\square$

**Lemma 2.9.** *Under the assumption (S1), there exists  $T_2 < T$  and a positive constant  $C$  independent of  $\varepsilon$  such that*

$$\|\mathbf{u}^\varepsilon\|^2 + \frac{\nu}{2} \int_0^t \|\mathbf{u}^\varepsilon\|_1^2 ds + a \int_0^t \|\mathbf{u}^\varepsilon\|_{L^{\alpha+2}}^{\alpha+2} ds + \varepsilon \int_0^t \|p^\varepsilon\|^2 ds \leq C, \quad \forall t \in [0, T_2]. \quad (2.90)$$

*Proof.* We use the inner product of (3.1) with  $\mathbf{u}^\varepsilon$  and (3.2) with  $p^\varepsilon$  to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{u}^\varepsilon\|^2 + \nu \|\nabla \mathbf{u}^\varepsilon\|^2 + a \|\mathbf{u}^\varepsilon\|_{L^{\alpha+2}}^{\alpha+2} + \varepsilon \|p^\varepsilon\|^2 &= (\mathbf{f}, \mathbf{u}^\varepsilon) \\ &\leq \frac{1}{2\nu} \|\mathbf{f}\|^2 + \frac{\nu}{2} \|\nabla \mathbf{u}^\varepsilon\|^2 \end{aligned}$$

By integrating with respect to  $t$ , we get the inequality (2.90).  $\square$

### 3. STUDY OF THE PERTURBED PROBLEM

In this section we focus on the existence, the uniqueness and the regularity for the penalized problem associated to porous media model (2.1)-(2.2).

The associated penalized problem to the equations (2.1)-(2.2) writes

$$\mathbf{u}_t^\varepsilon - \nu \Delta \mathbf{u}^\varepsilon + \tilde{B}(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon) + a |\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon + \nabla p^\varepsilon = \mathbf{f}, \quad \text{in } \Omega \times [0, T], \quad (3.1)$$

$$\nabla \cdot \mathbf{u}^\varepsilon + \varepsilon p^\varepsilon = 0, \quad \text{in } \Omega \times [0, T], \quad \mathbf{u}^\varepsilon|_{\partial\Omega} = 0, \quad \mathbf{u}^\varepsilon|_{t=0} = \mathbf{u}_0, \quad (3.2)$$

where  $\tilde{B}(\mathbf{u}, \mathbf{v}) = (\mathbf{u} \cdot \nabla) \mathbf{v} + \frac{1}{2} (\nabla \cdot \mathbf{u}) \mathbf{v}$  is the modified bilinear term.

Using the operator  $A_\varepsilon$ , we can reformulate the perturbed system (2.1)-(2.2) as

$$\mathbf{u}_t^\varepsilon + A_\varepsilon \mathbf{u}^\varepsilon + \tilde{B}(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon) + a |\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon = \mathbf{f}, \quad (3.3)$$

$$\mathbf{u}^\varepsilon|_{\partial\Omega} = 0, \quad \mathbf{u}^\varepsilon|_{t=0} = \mathbf{u}_0 \quad (3.4)$$

**Proposition 3.1.** *Under the assumption (S1), there exists  $T_2 < T$  and a positive constant  $C_0$  and  $C_1$  such that the solution  $\mathbf{u}$  the system (3.1)-(3.2) has one solution  $(u^\varepsilon, p^\varepsilon)$  which belongs in  $L^2(0; T_0; H^2(\Omega)) \cap L^\infty(0; T_0; H^1(\Omega)) \times L^\infty(0; T_0; H_1(\Omega))$  and we have*

$$\|\mathbf{u}^\varepsilon\|^2 + \frac{\nu}{2} \int_0^t \|\mathbf{u}^\varepsilon\|_1^2 ds + a \int_0^t \|\mathbf{u}^\varepsilon\|_{L^{\alpha+2}}^{\alpha+2} ds + \varepsilon \|\nabla p^\varepsilon\|^2 \leq C_0, \quad \forall t \in [0, T_2]. \quad (3.5)$$

and

$$\sup_{0 \leq t \leq T_0} \|\nabla \mathbf{u}^\varepsilon\|^2 + \frac{\nu}{2} \int_0^t \|\Delta \mathbf{u}^\varepsilon\|^2 ds \leq C_1, \quad \forall t \in [0, T_2]. \quad (3.6)$$

*Proof.* With a similar manner as done for the system (2.1)-(2.2), we establish the existence of solution via a Faedo-Galerkin method. To derive the regularity of the velocity  $u^\varepsilon$  and the pressure  $p^\varepsilon$ ; First, we use the inner product of (3.1) with  $\mathbf{u}^\varepsilon$  and the equation (3.2) by  $p^\varepsilon$  to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{u}^\varepsilon\|^2 + \nu \|\nabla \mathbf{u}^\varepsilon\|^2 + a \|\mathbf{u}^\varepsilon\|_{L^{\alpha+2}}^{\alpha+2} + \varepsilon \|\nabla p^\varepsilon\|^2 &= (\mathbf{f}, \mathbf{u}^\varepsilon) \\ &\leq \frac{1}{2\nu} \|\mathbf{f}\|^2 + \frac{\nu}{2} \|\nabla \mathbf{u}^\varepsilon\|^2 \end{aligned} \quad (3.7)$$

By integrating with respect to  $t$ , we get the inequality (3.5) with  $C_0 = \frac{1}{2\nu} \|\mathbf{f}\|^2 + \|u_0^\varepsilon\|^2$ .

Next, to obtain more regularity for the velocity  $\mathbf{u}$ , we multiply the system (2.1) by  $-\Delta \mathbf{u}$  to infer

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{u}^\varepsilon\|^2 + \nu \|\Delta \mathbf{u}^\varepsilon\|^2 &= a(|\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon, \Delta \mathbf{u}^\varepsilon) + \tilde{b}(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon, \Delta \mathbf{u}^\varepsilon) - (\mathbf{f} - \nabla p^\varepsilon, \Delta \mathbf{u}^\varepsilon) \\ &\leq \frac{\nu}{6} \|\Delta \mathbf{u}^\varepsilon\|^2 + \frac{3}{2\nu} (\|\mathbf{f}\|^2 + \|\nabla p^\varepsilon\|^2) \end{aligned} \quad (3.8)$$

By applying Cauchy-Schwartz inequality and Young's inequality to obtain

$$\begin{aligned} a(|\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon, \Delta \mathbf{u}^\varepsilon) &\leq a \|\mathbf{u}^\varepsilon\|^{\alpha+1} \|\Delta \mathbf{u}^\varepsilon\| \\ &\leq \frac{\nu}{6} \|\Delta \mathbf{u}^\varepsilon\|^2 + a^3 \frac{3}{2\nu} \|\nabla \mathbf{u}^\varepsilon\|^{2(\alpha+1)} \end{aligned}$$

Taking  $1 \leq \alpha \leq 2$  to deduce

$$a(|\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon, \Delta \mathbf{u}^\varepsilon) \leq \frac{\nu}{6} \|\Delta \mathbf{u}^\varepsilon\|^2 + C \|\nabla \mathbf{u}^\varepsilon\|^6 \quad (3.9)$$

Using Agmon's inequality to bound  $\tilde{b}(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon, \Delta \mathbf{u}^\varepsilon)$  as follow

$$\begin{aligned} \tilde{b}(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon, \Delta \mathbf{u}^\varepsilon) &\leq C \|\mathbf{u}^\varepsilon\|_\infty \|\mathbf{u}^\varepsilon\|_1 \|\mathbf{u}^\varepsilon\|_2 \\ &\leq \frac{\nu}{6} \|\Delta \mathbf{u}^\varepsilon\|^2 + \frac{C}{\nu} \|\nabla \mathbf{u}^\varepsilon\|^2 \end{aligned} \quad (3.10)$$

Substituting (3.9) and (3.10) in (3.7) to have

$$\frac{d}{dt} \|\nabla \mathbf{u}^\varepsilon\|^2 + \nu \|\Delta \mathbf{u}^\varepsilon\|^2 \leq C_2 + C_3 \|\nabla \mathbf{u}^\varepsilon\|^6 \quad (3.11)$$

Let  $y(t) = \|\nabla \mathbf{u}^\varepsilon\|^2 + K_1$ . Note that for  $K_2 \geq 1$

$$\frac{d}{dt} \leq K_2 y(t)^3, \quad \text{where } y(t) = \|\nabla \mathbf{u}^\varepsilon\|^2 + K_1 \text{ and } y(0) = \|\nabla \mathbf{u}^\varepsilon\|^2 + K_1 \quad (3.12)$$

From (3.10) we derive a solution defined on  $t$  ( $t \geq 0$ ) satisfies

$$t \leq \frac{1}{2K_2 y(0)^2} = \frac{1}{K_2} \quad (3.13)$$

This leads to

$$y(t) \leq \sqrt{2} y(0) \quad (3.14)$$

Set  $T_2 = \min(T, \frac{1}{K_2})$ , for  $t \in [0, T_2]$ , we deduce

$$\|\nabla \mathbf{u}^\varepsilon(t)\|^2 \leq \sqrt{2}\|\nabla \mathbf{u}^\varepsilon(0)\|^2 + K_1 := K_3 \quad (3.15)$$

To summarize, we combine the equations (3.7) and (3.10) to conclude

$$\sup_{0 \leq t \leq T_0} \|\nabla \mathbf{u}^\varepsilon\|^2 + \frac{\nu}{2} \int_0^t \|\Delta \mathbf{u}^\varepsilon\|^2 ds \leq C_1, \quad \forall t \in [0, T_2]. \quad (3.16)$$

□

Let us consider  $T_0 = \min(T_1, T_2)$ . For simplicity we denote by  $\|\mathbf{v}\|_{L^p(X)}$  the norm  $(\int_0^{T_0} \|\mathbf{v}\|_X^p dt)^{1/p}$  in  $L^p(0, T_0; X)$ .

#### 4. ERROR ANALYSIS

The main result of this section is stated in Theorem 4.2. The proof is split into two steps: the error of the linear case is given in the first step and the second one is related to the error behavior for the fully nonlinear problem.

**4.1. Error estimates for a linearly perturbed problem.** Primarily, we will focus on the analysis of the error for the linear case.

We consider the linearized Navier Stokes equations at  $\mathbf{u} = 0$  :

$$\begin{aligned} \mathbf{u}_t - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega \times [0, T] \\ \nabla \cdot \mathbf{u} &= 0, \quad \text{in } \Omega \times [0, T] \\ \mathbf{u}|_{\partial\Omega} &= 0, \quad \mathbf{u}(0) = \mathbf{u}_0 \end{aligned} \quad (4.1)$$

The associated perturbed problem to the system (4.1) takes the form

$$\begin{aligned} \mathbf{u}_t^\varepsilon - \nu \Delta \mathbf{u}^\varepsilon + \nabla p^\varepsilon &= \mathbf{f} \quad \text{in } \Omega \times [0, T] \\ \nabla \cdot \mathbf{u}^\varepsilon + \varepsilon p^\varepsilon &= 0, \quad \text{in } \Omega \times [0, T] \\ \mathbf{u}^\varepsilon|_{\partial\Omega} &= 0, \quad \mathbf{u}^\varepsilon(0) = \mathbf{u}_0 \end{aligned} \quad (4.2)$$

Let  $\mathbf{e} = \mathbf{u} - \mathbf{u}^\varepsilon$  and  $q = p - p^\varepsilon$ . Subtracting (4.2) from (4.1) to get

$$\mathbf{e}_t - \nu \Delta \mathbf{e} + \nabla q = 0 \quad (4.3)$$

$$\nabla \cdot \mathbf{e} + \varepsilon q = \varepsilon p, \quad \mathbf{e}(0) = 0 \quad (4.4)$$

The following result summarizes the error relating to the linearized problem

**Lemma 4.1.** *Let assume (S1), we have*

$$\|\mathbf{e}\|_{L^\infty(\mathbf{L}^2)} + \sqrt{\nu}\|\mathbf{e}\|_{L^2(\mathbf{H}^1)} + \sqrt{\varepsilon}\|q\|_{L^2(\mathbf{L}^2)} \leq C\sqrt{\varepsilon} \quad (4.5)$$

$$\|\mathbf{e}\|_{L^2(\mathbf{L}^2)} \leq C\varepsilon \quad (4.6)$$

*Proof.* Multiplying (4.3) by  $\mathbf{e}$  and (4.4) by  $q$  and summing up, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{e}\|^2 + \nu \|\nabla \mathbf{e}\|^2 + \varepsilon \|q\|^2 = \varepsilon (p, q) \leq \frac{\varepsilon}{2} \|q\|^2 + \frac{\varepsilon}{2} \|p\|^2$$

We integrate the previous inequality from 0 to  $t \leq T_0$ , and we use the fact that  $\mathbf{e}(0) = 0$  and lemma 2.5 to have

$$\|\mathbf{e}\|_{L^\infty(\mathbf{L}^2)}^2 + \nu \|\mathbf{e}\|_{L^2(\mathbf{H}^1)}^2 + \varepsilon \|q\|_{L^2(\mathbf{L}^2)}^2 \leq C\varepsilon$$

which is equivalent to (4.5).

To deduce the assertion (4.6), we need to introduce the auxiliary problem: for  $0 < t \leq T_0$  we define  $(\mathbf{w}, \phi)$

$$\begin{aligned} \mathbf{w}_s + \nu \Delta \mathbf{w} + \nabla \phi &= \mathbf{e}(s) \quad \forall 0 < s \leq t, \\ \nabla \cdot \mathbf{w} &= 0, \quad \mathbf{w}(t) = 0. \end{aligned} \quad (4.7)$$

Firstly, we derive the following inequality

$$\nu \|\mathbf{w}\|_{L^2(\mathbf{H}^2)} + \|\nabla \phi\|_{L^2(\mathbf{L}^2)} \leq C \|\mathbf{e}\|_{L^2(\mathbf{L}^2)} \quad (4.8)$$

For this aim, we use the inner product of (4.7) with  $A\mathbf{w}$ , and we integrate from 0 to  $t$ , to obtain

$$\nu \|\mathbf{w}\|_{L^2(\mathbf{H}^2)} + \|\nabla \mathbf{w}(0)\|_{L^2(\mathbf{L}^2)} \leq C \|\mathbf{e}\|_{L^2(\mathbf{L}^2)}$$

Using the projection operator  $P_{\mathbf{H}}$  on (4.7) to get

$$\|\mathbf{w}_s\|_{L^2(\mathbf{L}^2)} \leq C \|\mathbf{e}\|_{L^2(\mathbf{L}^2)}$$

We use the equation (4.7) again to get

$$\|\nabla \phi\|_{L^2(\mathbf{L}^2)} \leq C \|\mathbf{e}\|_{L^2(\mathbf{L}^2)}$$

This latter inequality completes the demonstration of (4.8).

We now multiply the equation (4.7) by  $\mathbf{e}(s)$ , using (4.3) and fact  $\nabla \cdot \mathbf{w} = 0$  we obtain

$$\begin{aligned} \|\mathbf{e}\|^2 &= (\mathbf{w}_s, \mathbf{e}) + \nu(\Delta \mathbf{w}, \mathbf{e}) + (\nabla \phi, \mathbf{e}) \\ &= \frac{d}{ds}(\mathbf{w}, \mathbf{e}) - (\mathbf{e}_s, \mathbf{w}) - \nu(\nabla \mathbf{e}, \nabla \mathbf{w}) + (\nabla \phi, \mathbf{e}) \\ &= \frac{d}{ds}(\mathbf{w}, \mathbf{e}) + (\nabla q, \mathbf{w}) - (\phi, \nabla \cdot \mathbf{e}) = \frac{d}{ds}(\mathbf{w}, \mathbf{e}) - \varepsilon(\phi, p^\varepsilon) \end{aligned}$$

We integrate from 0 to  $t$ , and the fact  $\mathbf{w}(t) = \mathbf{e}(0) = 0$  gives

$$\int_0^t \|\mathbf{e}\|^2 ds = - \int_0^t \varepsilon(\phi, p^\varepsilon) ds \leq \delta \|\phi\|_{L^2(\mathbf{L}^2)}^2 + C_\delta \varepsilon^2 \|p^\varepsilon\|_{L^2(\mathbf{L}^2)}^2$$

with  $C_\delta$  is a positive constant depending only of  $\delta$ . Applying (4.8) and choosing  $\delta$  small enough to have

$$\int_0^t \|\mathbf{e}\|^2 ds \leq C \varepsilon^2 \|p^\varepsilon\|_{L^2(\mathbf{L}^2)}^2 \leq C \varepsilon^2 \quad \forall t \in [0, T_0].$$

□

The following result is necessary for the subsequent investigations.

**Lemma 4.2.** *Provided (S1) and (S2) are satisfied, we have*

$$\int_0^t s^2 \|p_t^\varepsilon\|^2 ds \leq C \quad \forall t \in [0, T_0].$$

*Proof.* We derive one time with respect to  $t$  on (4.3) and (4.4), we get

$$\mathbf{e}_{tt} - \nu \Delta \mathbf{e}_t + \nabla q_t = 0 \quad (4.9)$$

$$\nabla \cdot \mathbf{e}_t + \varepsilon q_t = \varepsilon p_t, \quad \mathbf{e}(0) = 0 \quad (4.10)$$

Multiplying (4.3) by  $t\mathbf{e}_t$  and (4.10) by  $tq$  to get

$$\begin{aligned} t\|\mathbf{e}\|^2 + \frac{\nu}{2} \frac{d}{dt} t \|\nabla \mathbf{e}\|^2 + \frac{\varepsilon}{2} \frac{d}{dt} t \|q\|^2 &= \frac{\nu}{2} \|\nabla \mathbf{e}\|^2 + \frac{\varepsilon}{2} \|q\|^2 + \varepsilon t(p_t, q) \\ &\leq \frac{\nu}{2} \|\nabla \mathbf{e}\|^2 + \varepsilon \|q\|^2 + \frac{\varepsilon t^2}{2} \|p_t\|^2 \end{aligned}$$

Integrating from 0 to  $t$ , using Lemma 4.6, (2.40) and Gronwall lemma, we derive

$$\int_0^t s \|\mathbf{e}_t\|^2 ds + t \|\mathbf{e}\|_1^2 + \varepsilon t \|q\|^2 \leq C\varepsilon \quad (4.11)$$

We Multiply (4.9) by  $t^2 \mathbf{e}_t$  and (4.10) by  $t^2 q_t$  and we sum up we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} t^2 \|\mathbf{e}_t\|^2 + \nu t^2 \|\nabla \mathbf{e}_t\|^2 + \varepsilon t^2 \|q_t\|^2 &= t \|\mathbf{e}_t\|^2 + \varepsilon t^2 (p_t, q_t) \\ &\leq t \|\mathbf{e}_t\|^2 + \frac{\varepsilon t^2}{2} \|q_t\|^2 + \frac{\varepsilon t^2}{2} \|p_t\|^2 \end{aligned}$$

Integrating over  $[0, t]$ , using (4.11) and (2.40) we get

$$\varepsilon \int_0^t s^2 \|q_t\|^2 ds \leq C \int_0^t s \|\mathbf{e}_t\|^2 ds + C\varepsilon \int_0^t s^2 \|p_t\|^2 ds \leq C\varepsilon$$

The results follows from the previous inequality and (2.40).  $\square$

**Lemma 4.3.** *We suppose that (S1) and (S2) are satisfied then*

$$t \|\mathbf{e}(t)\|^2 + \nu \int_0^t s \|\nabla \mathbf{e}(s)\|^2 ds + \varepsilon \int_0^t s \|q(s)\|^2 ds \leq C\varepsilon^2 \quad \forall t \in [0, T_0], \quad (4.12)$$

$$t^2 \|\nabla \mathbf{e}(t)\|^2 + \int_0^t s^2 \|q(s)\|^2 ds \leq C\varepsilon^2 \quad \forall t \in [0, T_0]. \quad (4.13)$$

*Proof.* We consider the decomposition

$$\mathbf{H}_0^1(\Omega) = V \oplus V^\perp, \text{ where } V^\perp = \{(-\Delta)^{-1} \nabla q, q \in L^2(\Omega)\}$$

and  $\mathbf{v} = (-\Delta)^{-1} \nabla q$  if and only if  $-\Delta \mathbf{v} = \nabla q$  and  $\mathbf{v}|_{\partial\Omega} = 0$ . We note that for  $p(t) \in L^2(\Omega)/\mathbb{R}$  there exists a unique  $\varphi(t) \in V^\perp$  such that  $\nabla \cdot \varphi(t) = p(t)$  with

$$\|\varphi(t)\|_1 \leq C \|p(t)\| \quad \forall t \in [0, T_0]. \quad (4.14)$$

Moreover, if  $p_t(t) \in L^2(\Omega)/\mathbb{R}$ , we then have  $\nabla \cdot \varphi_t(t) = p_t(t)$  such that

$$\|\varphi_t(t)\|_1 \leq C \|p_t(t)\| \quad \forall t \in [0, T_0]. \quad (4.15)$$

Multiplying (4.3) by  $t\mathbf{e}$  and (4.4) by  $tq$ , we sum up and use (4.3), we infer

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} t \|\mathbf{e}\|^2 + t\nu \|\nabla \mathbf{e}\|^2 + \varepsilon t \|q\|^2 &= \frac{1}{2} \|\mathbf{e}\|^2 + \varepsilon t (p, q) = \frac{1}{2} \|\mathbf{e}\|^2 + \varepsilon t (\nabla \cdot \varphi, q) \\ &= \frac{1}{2} \|\mathbf{e}\|^2 - \varepsilon t (\nabla q, \varphi) = \frac{1}{2} \|\mathbf{e}\|^2 + \varepsilon t (\mathbf{e}_t, \varphi) + \varepsilon \nu t (\nabla \mathbf{e}, \nabla \varphi) \\ &= \frac{1}{2} \|\mathbf{e}\|^2 + \varepsilon \frac{d}{dt} t (\mathbf{e}, \varphi) - \varepsilon (\mathbf{e}, \varphi) - \varepsilon t (\mathbf{e}, \varphi_t) + \varepsilon \nu t (\nabla \mathbf{e}, \nabla \varphi) \end{aligned} \quad (4.16)$$

By the assumption (S1), we have  $\sqrt{t}p \in L^\infty(0, T_0; H^1(\Omega)/\mathbb{R})$ . Using (4.14) we obtain

$$\varepsilon t (\mathbf{e}(t), \varphi(t)) \leq \frac{t}{4} \|\mathbf{e}(t)\|^2 + \varepsilon^2 t \|\varphi\|^2 \leq \frac{t}{4} \|\mathbf{e}(t)\|^2 + C\varepsilon^2$$

By integrating (4.16) from 0 to  $t$ , using the previous inequality, the Cauchy-Schwartz inequality, Lemma 4.6, (2.40) and (4.15) we deduce

$$\begin{aligned} t \|\mathbf{e}(t)\|^2 + \int_0^t (\nu s \|\nabla \mathbf{e}(s)\|^2 + \varepsilon s \|q(s)\|^2) ds \\ \leq C\varepsilon^2 + C \int_0^t \|\mathbf{e}(s)\|^2 ds + c\varepsilon^2 \int_0^t \|\varphi(s)\|^2 ds + C\varepsilon^2 \int_0^t s^2 \|\varphi_t(s)\|^2 ds \\ \leq C\varepsilon^2 + C\varepsilon^2 \int_0^t (\|p\|^2 + s^2 \|p_t\|^2) ds \leq C\varepsilon^2 \end{aligned}$$

Taking the inner product of (4.3) with  $t^2\mathbf{e}_t$ , (4.10) with  $t^2q$  we deduct

$$t^2\|\mathbf{e}_t\|^2 + \frac{1}{2}\frac{d}{dt}t^2\|\nabla e\|^2 + \frac{\varepsilon}{2}\frac{d}{dt}t^2\|q\|^2 = t\|e\|_1^2 + \varepsilon t\|q\|^2 + \varepsilon t^2(p_t, q) \quad (4.17)$$

Using equation (4.3) and the inequality (4.15) to obtain

$$\begin{aligned} \varepsilon t^2(p_t, q) &= \varepsilon t^2(\nabla \cdot \varphi_t, q) = -\varepsilon t^2(\varphi_t, \nabla q) = \varepsilon t^2(\mathbf{e}_t, \varphi_t) + \varepsilon t^2(\nabla e, \nabla \varphi_t) \\ &\leq \frac{t^2}{2}\|\mathbf{e}_t\|^2 + c\varepsilon^2 t^2\|\varphi_t\|^2 + t^2\|\nabla e\|^2 + \varepsilon^2 t^2\|\nabla \varphi_t\|^2 \\ &\leq \frac{t^2}{2}\|\mathbf{e}_t\|^2 + t^2\|\nabla e\|^2 + C\varepsilon^2 t^2\|p_t\|^2 \end{aligned}$$

We integrate (4.17) and using the Gronwall lemma to have

$$\int_0^t s^2\|\mathbf{e}_t(s)\|^2 ds + t^2\|\nabla e\|^2 + \varepsilon t^2\|q(t)\|^2 \leq C\varepsilon^2 \quad \forall t \in [0, T_0]$$

Again from the equation (4.3)

$$\|q\|^2 \leq C\|\nabla q\|_{-1}^2 \leq C(\|\Delta e\|_{-1}^2 + \|e_t\|_{-1}^2) \leq C(\|\mathbf{e}\|_1^2 + \|\mathbf{e}_t\|^2)$$

Consequently

$$\int_0^{T_0} s^2\|q\|^2 ds \leq C \int_0^{T_0} s^2(\|\mathbf{e}\|_1^2 + \|\mathbf{e}_t\|^2) ds \leq C\varepsilon^2$$

□

The result below describes the behavior of the linear error part.

**Theorem 4.1.** *Assume the assumption (S1). Then, there exists a constant  $C$  depending on the given data such that*

$$\|\mathbf{e}(t)\| + \left( \int_0^t \|\mathbf{e}(s)\|_1^2 ds \right)^{\frac{1}{2}} \leq C\sqrt{\varepsilon} \quad \forall t \in [0, T_0].$$

If we assume moreover the hypothesis (S2), then

$$\sqrt{t}\|\mathbf{e}(t)\| + t\|\mathbf{e}(t)\|_1 + \left( \int_0^t s^2\|q\|^2 ds \right)^{\frac{1}{2}} \leq C\varepsilon \quad \forall t \in [0, T_0].$$

holds.

The following subsection treats the transfer of the results that have been derived for the linear case to the non linear one.

**4.2. Error estimates for the nonlinear perturbed problem.** Let  $(\mathbf{u}; p)$  be the solution of the system (1.1) and let  $(\mathbf{v}; \gamma)$  be the solution of the perturbed non linearly problem:

$$\mathbf{v}_t - \nu\Delta\mathbf{v} + \nabla\gamma = f - B(\mathbf{u}, \mathbf{u}) - a|\mathbf{u}|^\alpha\mathbf{v} \quad \text{in } \Omega \times [0, T] \quad (4.18)$$

$$\nabla \cdot \mathbf{v} + \varepsilon\gamma = 0, \quad \text{in } \Omega \times [0, T] \quad \mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{v}(0) = \mathbf{v}_0, \quad (4.19)$$

Taking  $\xi = \mathbf{v} - \mathbf{u}$ ,  $\phi = \gamma - p$ , and subtracting (4.18)-(4.19) from (1.1), we get

$$\xi_t - \nu\Delta\xi + \nabla\phi = 0 \quad (4.20)$$

$$\nabla \cdot \xi + \varepsilon\phi = -\varepsilon p, \quad \xi(0) = 0.$$

The next Lemma describes the control of the errors through the non-linearities.

**Lemma 4.4.** *We suppose that (S1) and (S2) are valid. Then we have*

$$\left( \int_0^t \|\xi(s)\|^2 ds \right)^{\frac{1}{2}} + \sqrt{t} \|\xi(t)\| + t \|\xi(t)\|_1 + \left( \int_0^t s^2 \|\phi(s)\|^2 ds \right)^{\frac{1}{2}} \leq C\varepsilon \quad \forall t \in [0, T_0].$$

*Proof.* The lemma is a consequence of Lemma.4.1 and Theorem.4.1 applied to the problem (4.20).  $\square$

We now take  $\eta = \mathbf{u}^\varepsilon - \mathbf{v}$ ,  $q = p^\varepsilon - \gamma$ , and subtracting (4.18)-(4.19) from (3.1)-(3.2) to obtain

$$\eta_t - \nu \Delta \eta + \tilde{B}(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon) - \tilde{B}(\mathbf{u}, \mathbf{u}) + a(|\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon - |\mathbf{u}|^\alpha \mathbf{u}) + \nabla q = 0 \quad (4.21)$$

$$\nabla \cdot \eta + \varepsilon q = 0, \quad \eta(0) = 0 \quad (4.22)$$

Or

$$\begin{aligned} \tilde{B}(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon) - \tilde{B}(\mathbf{u}, \mathbf{u}) &= \tilde{B}(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon - \mathbf{u}) + \tilde{B}(\mathbf{u}^\varepsilon - \mathbf{u}, \mathbf{u}^\varepsilon) \\ &= \tilde{B}(\mathbf{u}^\varepsilon, \xi + \eta) + \tilde{B}(\xi + \eta, \mathbf{u}^\varepsilon) \end{aligned} \quad (4.23)$$

We rewrite the equation (4.21) as

$$\eta_t + A_\varepsilon \eta + \tilde{B}(\mathbf{u}^\varepsilon, \xi + \eta) + \tilde{B}(\xi + \eta, \mathbf{u}^\varepsilon) + a(|\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon - |\mathbf{u}|^\alpha \mathbf{u}) = 0 \quad (4.24)$$

The following theorem states a general priori estimate for the errors  $\mathbf{e} = \mathbf{u} - \mathbf{u}^\varepsilon$  and  $q = p - p^\varepsilon$ ; with  $(\mathbf{u}, p)$  are solutions of the state problem (1.1), and  $(\mathbf{u}^\varepsilon, p^\varepsilon)$  are solutions of the penalized problem (3.1).

**Theorem 4.2.** *Assume that (S1) and (S2) hold. Then we have*

$$\sqrt{t} \|\mathbf{u}(t) - \mathbf{u}^\varepsilon(t)\| + \sqrt{\nu t} \|\mathbf{u}(t) - \mathbf{u}^\varepsilon(t)\|_1 + \left( \int_0^t s^2 \|p(t) - p^\varepsilon(t)\|^2 ds \right)^{\frac{1}{2}} \leq C\varepsilon \quad \forall t \in [0, T_0].$$

*Proof.* Multiplying (4.24) by  $A_\varepsilon^{-1} \eta$ , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|A_\varepsilon^{\frac{1}{2}} \eta\|^2 + \nu \|\eta\|^2 &= -\tilde{b}(\mathbf{u}^\varepsilon, \xi + \eta, A_\varepsilon^{-1} \eta) - \tilde{b}(\xi + \eta, \mathbf{u}, A_\varepsilon^{-1} \eta) - a(|\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon - |\mathbf{u}|^\alpha \mathbf{u}, A_\varepsilon^{-1} \eta) \\ &= I_1 + I_2 + I_3 \end{aligned}$$

Using (2.6), (2.7), and Cauchy Schwartz's inequality we derive that

$$\begin{aligned} I_1 &\leq C \|\mathbf{u}^\varepsilon\|_2 \|\xi + \eta\| \|A_\varepsilon^{-1} \eta\|_1 \leq C \|\mathbf{u}^\varepsilon\|_2 \|\xi + \eta\| \|\nabla A_\varepsilon^{-1} \eta\| \\ &\leq C \|\mathbf{u}^\varepsilon\|_2 (\|\xi\| + \|\eta\|) \|A_\varepsilon^{-\frac{1}{2}} \eta\| \leq \frac{\nu}{4} \|\eta\|^2 + C \|\xi\|^2 + C \|\mathbf{u}^\varepsilon\|_2^2 \|A_\varepsilon^{-\frac{1}{2}} \eta\|^2 \end{aligned}$$

With the same argument, we have

$$I_2 \leq \frac{\nu}{4} \|\eta\|^2 + C \|\xi\|^2 + C \|\mathbf{u}\|_2^2 \|A_\varepsilon^{-\frac{1}{2}} \eta\|^2$$

For the third term we proceed similarly

$$\begin{aligned} I_3 &\leq C \|\mathbf{u}^\varepsilon - \mathbf{u}\|_2 \|A_\varepsilon^{-\frac{1}{2}} \eta\| \\ &\leq C (\|\mathbf{u}^\varepsilon\|_2^2 + \|\mathbf{u}\|_2^2) + C \|A_\varepsilon^{-\frac{1}{2}} \eta\|^2 \end{aligned}$$

Adding the above inequalities we get

$$\frac{d}{dt} \|A_\varepsilon^{-\frac{1}{2}} \eta\|^2 + \nu \|\eta\|^2 \leq C \|\xi\|^2 + C(1 + \|\mathbf{u}^\varepsilon\|_2^2 + \|\mathbf{u}\|_2^2) \|A_\varepsilon^{-\frac{1}{2}} \eta\|^2 + C(\|\mathbf{u}^\varepsilon\|_2^2 + \|\mathbf{u}\|_2^2) \quad (4.25)$$

Thanks to lemma 2.5,  $\int_0^{T_0} (\|\mathbf{u}^\varepsilon\|_2^2 + \|\mathbf{u}\|_2^2) dt \leq C$ , we can apply the Gronwall lemma to (4.25), using lemma 2.8 to obtain

$$\|A_\varepsilon^{-\frac{1}{2}} \eta(t)\|^2 + \nu \int_0^t \|\eta(s)\|^2 ds \leq C \int_0^t \|\xi(s)\|^2 ds \leq C\varepsilon^2 \quad \forall t \in [0, T_0]. \quad (4.26)$$

We now multiply the equation (4.21) by  $t\eta$  and (4.22) by  $tq$  and we sum them up, then using (2.6) and Cauchy Schwartz's inequality one gets

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} t \|\eta\|^2 + \nu t \|\nabla \eta\|^2 + \varepsilon t \|q\|^2 &= \frac{1}{2} \|\eta\|^2 - t \tilde{b}(\mathbf{u}^\varepsilon, \xi + \eta, \eta) - t \tilde{b}(\xi + \eta, \mathbf{u}, \eta) \\
&\quad - at(|\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon - |\mathbf{u}|^\alpha \mathbf{u}, \eta) \\
&\leq \frac{1}{2} \|\eta\|^2 + Ct \|\mathbf{u}^\varepsilon\|_2 \|\xi + \eta\|_1 \|\eta\| + Ct \|\mathbf{u}\|_2 \|\xi + \eta\|_1 \|\eta\| + Ct \|\mathbf{u}^\varepsilon - \mathbf{u}\|_1 \|\eta\| \\
&\leq \frac{1}{2} \|\eta\|^2 + Ct \|\mathbf{u}^\varepsilon\|_2 \|\xi + \eta\|_1 \|\eta\| + Ct \|\mathbf{u}\|_2 \|\xi + \eta\|_1 \|\eta\| + Ct \|(\mathbf{u}^\varepsilon - \mathbf{v}) - (\mathbf{u} - \mathbf{v})\|_1 \|\eta\| \\
&\leq \frac{1}{2} \|\eta\|^2 + Ct \|\mathbf{u}^\varepsilon\|_2 \|\xi + \eta\|_1 \|\eta\| + Ct \|\mathbf{u}\|_2 \|\xi + \eta\|_1 \|\eta\| + Ct \|\eta - \xi\|_1 \|\eta\| \\
&\leq \frac{1}{2} \|\eta\|^2 + \frac{\nu t}{2} \|\nabla \eta\|^2 + Ct \|\xi\|_1^2 + Ct (\|\mathbf{u}^\varepsilon\|^2 + \|\mathbf{u}\|^2) \|\eta\|^2
\end{aligned}$$

We integrate from 0 to  $t$  and use (4.26), Lemma 2.8 and the Gronwall lemma we get

$$t \|\eta(t)\|^2 + \nu \int_0^t s \|\nabla \eta(s)\|^2 ds + \varepsilon \int_0^t s \|q(s)\|^2 ds \leq C\varepsilon^2 \quad (4.27)$$

We now derive one time with respect to the equation (4.22)

$$\nabla \cdot \eta_t + \varepsilon q_t = 0 \quad (4.28)$$

Multiplying the equation (4.21) by  $t^2 \eta_t$ , and (4.28) by  $t^2 q$ , summing them up to get

$$\begin{aligned}
t^2 \|\eta_t\|^2 + \frac{\nu}{2} \frac{d}{dt} t^2 \|\nabla \eta\|^2 + \frac{\varepsilon}{2} \frac{d}{dt} t^2 \|q\|^2 &= \nu t \|\nabla \eta\|^2 + \varepsilon t \|q\|^2 \\
&\quad - t^2 \tilde{b}(\mathbf{u}^\varepsilon, \xi + \eta, \eta_t) - t^2 \tilde{b}(\xi + \eta, \mathbf{u}, \eta_t) - at^2 (|\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon - |\mathbf{u}|^\alpha \mathbf{u}, \eta_t)
\end{aligned} \quad (4.29)$$

For the first nonlinear term in the right hand side, we proceed as follow

$$\begin{aligned}
t^2 \tilde{b}(\mathbf{u}^\varepsilon, \xi + \eta, \eta_t) &\leq t^2 \|\mathbf{u}^\varepsilon\|_2 \|\xi + \eta\|_1 \|\eta_t\| \\
&\leq \frac{t^2}{6} \|\eta_t\|^2 + Ct^2 \|\mathbf{u}^\varepsilon\|_2^2 (\|\xi\|_1^2 + \|\eta\|_1^2) \\
&\leq \frac{t^2}{6} \|\eta_t\|^2 + C\varepsilon^2 \|\mathbf{u}^\varepsilon\|_2^2 + Ct^2 \|\mathbf{u}^\varepsilon\|_2^2 \|\nabla \eta\|^2
\end{aligned}$$

With the same manner we obtain

$$t^2 \tilde{b}(\xi + \eta, \mathbf{u}, \eta_t) \leq \frac{t^2}{6} \|\eta_t\|^2 + C\varepsilon^2 \|\mathbf{u}\|_2^2 + Ct^2 \|\mathbf{u}\|_2^2 \|\nabla \eta\|^2$$

For the last non linear term we have

$$\begin{aligned}
at^2 (|\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon - |\mathbf{u}|^\alpha \mathbf{u}, \eta_t) &\leq at^2 \|\eta_t\| \|\mathbf{u}^\varepsilon - \mathbf{u}\|_1 \\
&\leq \frac{t^2}{6} \|\eta_t\|^2 + Ct^2 \|\eta - \xi\|_1^2
\end{aligned}$$

Summing the above inequalities into (4.29) to get

$$\begin{aligned}
t^2 \|\eta_t\|^2 + \nu \frac{d}{dt} t^2 \|\nabla \eta\|^2 + \varepsilon \frac{d}{dt} t^2 \|q\|^2 &\leq \nu t \|\nabla \eta\|^2 + \varepsilon t \|q\|^2 \\
&\quad + C(\varepsilon^2 + t^2 \|\nabla \eta\|^2) (\|\mathbf{u}^\varepsilon\|_2^2 + \|\mathbf{u}\|_2^2) + Ct^2 \|\eta - \xi\|_1^2
\end{aligned}$$

We integrate from 0 to  $t$ , using (4.27) and the Gronwall lemma we obtain

$$\int_0^t s^2 \|\eta_t\|^2 ds + \nu t^2 \|\nabla \eta(t)\|^2 + \varepsilon t^2 \|q(t)\|^2 \leq C\varepsilon^2 \quad (4.30)$$

Using (2.6), we have

$$\begin{aligned}
\|\tilde{B}(\mathbf{u}^\varepsilon, \xi + \eta)\|_{-1} &\leq C(\|\mathbf{u}^\varepsilon\|_1 \|\xi + \eta\|_1) \leq C(\|\mathbf{u}^\varepsilon\|_1 \|\xi\|_1 + \|\eta\|_1) \\
\|\tilde{B}(\xi + \eta, \mathbf{u})\|_{-1} &\leq C(\|\mathbf{u}\|_1 \|\xi + \eta\|_1) \leq C(\|\mathbf{u}\|_1 \|\xi\|_1 + \|\eta\|_1) \\
a(|\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon - |\mathbf{u}|^\alpha \mathbf{u}) &\leq C(\|\xi\|_1 + \|\eta\|_1)
\end{aligned}$$

From (4.21) and (4.23) we obtain

$$\nabla q = -\eta_t + \nu \Delta \eta - \tilde{B}(\mathbf{u}^\varepsilon, \xi + \eta) - \tilde{B}(\xi + \eta, \mathbf{u}) - a(|\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon - |\mathbf{u}|^\alpha \mathbf{u})$$

consequently, by applying the above estimates on the previous equation we get

$$\int_0^{T_0} s^2 \|q\|^2 ds \leq \int_0^{T_0} s^2 \|\nabla q\|_{-1}^2 ds \leq C\varepsilon^2$$

which end the proof of the theorem 4.2.2. □

## 5. OPTIMAL SHAPE DESIGN

The goal of a shape design problem is to find an optimal shape domain in a set of admissible domains when a cost functional achieves a minimum on this domain. In gradient-based optimization techniques, the objective is to minimize a suitable cost function with respect to a set of design variables. One of the key ingredients to compute efficiently the cost function gradient is the use of the adjoint system techniques to simplify the computation of gradients.

**5.1. Adjoint equation.** In this section we derive an adjoint equation related to the fluid flow system

$$\left\{ \begin{array}{ll}
\mathbf{u}_t^\varepsilon - \nu \Delta \mathbf{u}^\varepsilon + (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{u}^\varepsilon + \frac{1}{2} (\nabla \cdot \mathbf{u}^\varepsilon) \mathbf{u}^\varepsilon + a |\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon + \nabla p^\varepsilon = \mathbf{f} & \text{in } \Omega \times [0, T], \\
\nabla \cdot \mathbf{u}^\varepsilon + \varepsilon p^\varepsilon = 0 & \text{in } \Omega \times [0, T], \\
\mathbf{u}^\varepsilon = 0 & \text{on } \partial\Omega \times [0, T], \\
\mathbf{u}^\varepsilon|_{t=0} = \mathbf{u}_0 & \text{in } \Omega.
\end{array} \right. \quad (5.1)$$

The aim is to obtain a design in which the velocity is close to a typical suitable velocity and flow turbulence as low as possible. According to this, we define the objective function as

$$J(\Omega) = J_1(\Omega) + J_2(\Omega), \quad (5.2)$$

where

$$J_1(\Omega) = \frac{1}{2} \int_0^T \int_\Omega |\mathbf{u}^\varepsilon - \mathbf{u}_d|^2 dt dx, \text{ and } J_2(\Omega) = \frac{\sigma}{2} \int_0^T \int_\Omega |\text{curl}(\mathbf{u}^\varepsilon)|^2 dt dx.$$

The generic optimization problem can be succinctly stated as follows

$$\begin{aligned}
\min_{\Omega \in Q_{ad}} J(\Omega) &= \frac{1}{2} \int_0^T \int_\Omega |\mathbf{u}^\varepsilon - \mathbf{u}_d|^2 dt dx + \frac{\sigma}{2} \int_0^T \int_\Omega |\text{curl}(\mathbf{u}^\varepsilon)|^2 dt dx \\
&\text{such that } (\mathbf{u}^\varepsilon, p^\varepsilon) \text{ is solution of (5.1)}
\end{aligned} \quad (5.3)$$

and the corresponding adjoint state system satisfy the following theorem.

**Theorem 5.1.** *Let  $(\mathbf{u}^\varepsilon, p^\varepsilon) \in H^2(0, T; \Omega) \times Y$  be solution of the penalized problem (5.1). Then the adjoint equation associated to the equation (5.1) takes the form*

$$\left\{ \begin{array}{ll} -\mathbf{v}_t - \nu \Delta \mathbf{v} + (\nabla \mathbf{u}^\varepsilon)^T \cdot \mathbf{v} - (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{v} - \frac{1}{2} \nabla (\mathbf{u}^\varepsilon \cdot \mathbf{v}) + \frac{1}{2} (\nabla \cdot \mathbf{u}^\varepsilon) \mathbf{v} + a |\mathbf{u}^\varepsilon|^{\alpha-2} (|\mathbf{u}^\varepsilon|^2 \mathbf{v} + \alpha (\mathbf{u}^\varepsilon \cdot \mathbf{v}) \mathbf{u}^\varepsilon) \\ \quad + \nabla q = (\mathbf{u}^\varepsilon - \mathbf{u}_d) - \sigma \vec{curl}(curl(\mathbf{u}^\varepsilon)) & \text{in } \Omega \times [0, T] \\ \nabla \cdot \mathbf{v} + \varepsilon q = 0 & \text{in } \Omega \times [0, T] \\ \nu \frac{\partial \mathbf{v}}{\partial n} - nq = \sigma curl(\mathbf{u}^\varepsilon) \cdot \boldsymbol{\tau} & \text{on } \partial \Omega \times [0, T] \\ \mathbf{v}(T) = 0 & \end{array} \right. \quad (5.4)$$

*Proof.* We define also the following function

$$\begin{aligned} F(\Omega, \mathbf{u}^\varepsilon, p^\varepsilon, \mathbf{v}, q) &= \int_0^T \int_\Omega \left( \mathbf{u}_t^\varepsilon + \nu \nabla \mathbf{u}^\varepsilon : \nabla \mathbf{v} + (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{u}^\varepsilon \cdot \mathbf{v} + \frac{1}{2} (\nabla \cdot \mathbf{u}^\varepsilon) \mathbf{u}^\varepsilon \cdot \mathbf{v} + a |\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon \cdot \mathbf{v} \right) dt dx \\ &\quad - \int_0^T \int_\Omega f \cdot \mathbf{v} dt dx - \int_0^T \int_\Omega \left( p^\varepsilon \nabla \cdot \mathbf{v} - q \nabla \cdot \mathbf{u}^\varepsilon - \varepsilon q p^\varepsilon \right) dt dx. \end{aligned}$$

Let  $L(\Omega, \mathbf{u}^\varepsilon, p^\varepsilon, \mathbf{v}, q)$  be a Lagrangian functional defined by

$$L(\Omega, \mathbf{u}^\varepsilon, p^\varepsilon, \mathbf{v}, q) = J(\Omega) - F(\Omega, \mathbf{u}^\varepsilon, p^\varepsilon, \mathbf{v}, q), \quad (5.5)$$

we first derive  $L$  with respect to the state variable  $p$  in any direction  $\tilde{p} \in M(\Omega)$ , we obtain

$$\frac{\partial L}{\partial p}(\Omega, \mathbf{u}^\varepsilon, p^\varepsilon, \mathbf{v}, q) \cdot \tilde{p} = \int_0^T \int_\Omega (\tilde{p} \nabla \cdot \mathbf{v} + \varepsilon \tilde{p} q) dt dx,$$

the variation  $\tilde{p}$  is arbitrary, we get

$$\nabla \cdot \mathbf{v} + \varepsilon q = 0 \quad \text{on } \Omega. \quad (5.6)$$

Next, we derive  $L$  with respect of the state variable  $\mathbf{u}^\varepsilon$  in the arbitrary direction  $\tilde{\mathbf{u}} \in V_0(\Omega)$  we have

$$\begin{aligned} 0 &= \frac{\partial L}{\partial \mathbf{u}^\varepsilon}(\Omega, \mathbf{u}^\varepsilon, p^\varepsilon, \mathbf{v}, q) \cdot \tilde{\mathbf{u}}, \\ &= \int_0^T \int_\Omega \left( (\mathbf{u}^\varepsilon - \mathbf{u}_d) \cdot \tilde{\mathbf{u}} - \sigma curl(\mathbf{u}^\varepsilon) \cdot curl(\tilde{\mathbf{u}}) \right) dt dx \\ &\quad - \int_0^T \int_\Omega \left( -\mathbf{v}_t \tilde{\mathbf{u}} + \nu \nabla \tilde{\mathbf{u}} : \nabla \mathbf{v} + (\tilde{\mathbf{u}} \cdot \nabla) \mathbf{u}^\varepsilon \cdot \mathbf{v} + (\mathbf{u}^\varepsilon \cdot \nabla) \tilde{\mathbf{u}} \cdot \mathbf{v} + \frac{1}{2} (\nabla \cdot \tilde{\mathbf{u}}) \mathbf{u}^\varepsilon \cdot \mathbf{v} + (\nabla \cdot \mathbf{u}^\varepsilon) \tilde{\mathbf{u}} \cdot \mathbf{v} \right) dt dx \\ &\quad + \int_0^T \int_\Omega \left( q \nabla \cdot \tilde{\mathbf{u}} - (a |\mathbf{u}^\varepsilon|^\alpha \tilde{\mathbf{u}} + a \alpha |\mathbf{u}^\varepsilon|^{\alpha-2} (\mathbf{u}^\varepsilon \cdot \tilde{\mathbf{u}}) \mathbf{u}^\varepsilon) \cdot \mathbf{v} \right) dt dx, \\ &= \int_0^T \int_\Omega \left( (\mathbf{u}^\varepsilon - \mathbf{u}_d) - \sigma \vec{curl}(curl(\mathbf{u}^\varepsilon)) + \mathbf{v}_t + \nu \Delta \mathbf{u}^\varepsilon - (\nabla \mathbf{u}^\varepsilon)^T \cdot \mathbf{v} + (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{v} \right. \\ &\quad \left. + \frac{1}{2} \nabla (\mathbf{u}^\varepsilon \cdot \mathbf{v}) - \frac{1}{2} (\nabla \cdot \mathbf{u}^\varepsilon) \mathbf{v} - (a |\mathbf{u}^\varepsilon|^\alpha \mathbf{v} + a \alpha |\mathbf{u}^\varepsilon|^{\alpha-2} (\mathbf{u}^\varepsilon \cdot \mathbf{v}) \mathbf{u}^\varepsilon - \nabla q) \cdot \tilde{\mathbf{u}} \right) dt dx \\ &\quad + \int_0^T \int_{\partial \Omega} \left( \sigma curl(\mathbf{u}^\varepsilon) \boldsymbol{\tau} - \left( \nu \frac{\partial \mathbf{v}}{\partial n} - nq \right) \right) \cdot \tilde{\mathbf{u}} dt dx + \int_\Omega [\mathbf{v} \cdot \tilde{\mathbf{u}}]_0^T dx. \end{aligned}$$

Considering an arbitrary direction  $\tilde{\mathbf{u}}$  which vanishes in neighborhood of the boundary  $\partial \Omega$ , we obtain

$$-\mathbf{v}_t - \nu \Delta \mathbf{v} + (\nabla \mathbf{u}^\varepsilon)^T \cdot \mathbf{v} - (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{v} - \frac{1}{2} \nabla (\mathbf{u}^\varepsilon \cdot \mathbf{v}) + \frac{1}{2} (\nabla \cdot \mathbf{u}^\varepsilon) \mathbf{v} + a |\mathbf{u}^\varepsilon|^{\alpha-2} (|\mathbf{u}^\varepsilon|^2 \mathbf{v} + \alpha (\mathbf{u}^\varepsilon \cdot \mathbf{v}) \mathbf{u}^\varepsilon) + \nabla q = (\mathbf{u}^\varepsilon - \mathbf{u}_d) - \sigma \vec{curl}(curl(\mathbf{u}^\varepsilon)),$$

and an arbitrary  $\tilde{\mathbf{u}}$  in  $\partial\Omega$  gives

$$\nu \frac{\partial \mathbf{v}}{\partial n} - nq = \sigma \operatorname{curl}(\mathbf{u}^\varepsilon) \cdot \boldsymbol{\tau} \quad \text{on } \partial\Omega$$

Finally we obtain the adjoint equation

$$\left\{ \begin{array}{ll} -\mathbf{v}_t - \nu \Delta \mathbf{v} + (\nabla \mathbf{u}^\varepsilon)^T \cdot \mathbf{v} - (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{v} - \frac{1}{2} \nabla (\mathbf{u}^\varepsilon \cdot \mathbf{v}) + \frac{1}{2} (\nabla \cdot \mathbf{u}^\varepsilon) \mathbf{v} + a |\mathbf{u}^\varepsilon|^{\alpha-2} (|\mathbf{u}^\varepsilon|^2 \mathbf{v} + \alpha (\mathbf{u}^\varepsilon \cdot \mathbf{v}) \mathbf{u}^\varepsilon) \\ \quad + \nabla q = (\mathbf{u}^\varepsilon - \mathbf{u}_d) - \sigma \overrightarrow{\operatorname{curl}}(\operatorname{curl}(\mathbf{u}^\varepsilon)) & \text{in } \Omega \times [0, T] \\ \nabla \cdot \mathbf{v} + \varepsilon q = 0 & \text{in } \Omega \times [0, T] \\ \nu \frac{\partial \mathbf{v}}{\partial n} - nq = \sigma \operatorname{curl}(\mathbf{u}^\varepsilon) \cdot \boldsymbol{\tau} & \text{on } \partial\Omega \times [0, T] \\ \mathbf{v}(T) = 0 & \end{array} \right. \quad (5.7)$$

□

**5.2. Shape gradient.** In the following, we derive the structures of shape gradients for time-dependent cost functionals by using the the velocity method (see C ea [1]) and its associated adjoint state.

Let  $\Omega$  be a reference domain in  $\mathbb{R}^2$ , the perturbation of  $\Omega$  by the velocity method is described as the velocity field

$$\mathbf{V}(t)(x) = \mathbf{V}(t, x), \forall x \in \Omega, t \geq 0.$$

It can generate transformations

$$T_t(\mathbf{V})\mathbf{X} = x(t, \mathbf{X}), t \geq 0, \mathbf{X} \in \Omega,$$

the flow defined by the initial value problem

$$\left\{ \begin{array}{ll} \frac{d\chi}{dt}(t, \mathbf{X}) & = \mathbf{V}(t, \chi(t)), \\ \chi(0, \mathbf{X}) & = \mathbf{X}, \end{array} \right. \quad (5.8)$$

with the initial value  $X$  given. Let  $\Omega_t$  be a perturbation domain of  $\Omega$  and  $J(\Omega)$  be a functional associated to  $\Omega_t$ . The shape derivative of the functional  $J(\Omega_t)$  at  $\partial\Omega$  in the direction of the deformation field  $\mathbf{V}$  is written as

$$dJ(\Omega; \mathbf{V}) = \lim_{t \rightarrow 0} \frac{J(\Omega_t) - J(\Omega)}{t}.$$

If  $dJ(\Omega; \mathbf{V})$  exists for all  $\mathbf{V} \in \mathcal{C}([0, T]; (\mathcal{D}^k(\mathbb{R}^2))^2)$ , for small positive constant  $T$ , the functional  $J$  is called shape differentiable at  $\Omega$  and its shape gradient verifies

$$dJ(\Omega; \mathbf{V}) = (\nabla J, \mathbf{V})_{((\mathcal{D}^k(\mathbb{R}^2))^2)' \times (\mathcal{D}^k(\mathbb{R}^2))^2}$$

We define the velocity field admissible domain as follow

$$V_{ad} = \{ \mathbf{V} \in \mathcal{C}^0(0, \tau; (\mathcal{C}^2(\mathbb{R}^2))^2) \mid \mathbf{V} = 0 \quad \text{on } \partial\Omega \}.$$

The use of the velocity  $\mathbf{V}$  for  $t \geq 0$ , implies the transformation of the domain  $\Omega$  into  $\Omega_t = T_t(\Omega)$  by the velocity method with formulation (5.8). Let us find an expression of the derivative of the saddle point problem  $j(t)$  with respect to  $\mathbf{t}$  where

$$j(t) = \min_{(\mathbf{u}_t^\varepsilon, p_t) \in V_\sigma(\Omega_t) \times \hat{Y}(\Omega_t)} \max_{(\mathbf{v}_t, q_t) \in V_0(\Omega_t) \times \hat{Y}(\Omega_t)} L(\Omega_t, \mathbf{u}_t^\varepsilon, p_t^\varepsilon, \mathbf{v}_t, q_t)$$

with  $(\mathbf{u}, p)$  and  $(\mathbf{v}, q)$  are solutions of (1.1) and (5.4) in the perturbed domain  $\Omega_t$ , respectively.

We consider the Hilbert spaces which depend on the parameter  $\mathbf{t}$  defined by

$$\begin{aligned} V_0(\Omega_t) &= \{ \mathbf{u}^\varepsilon \circ T_t^{-1} : \mathbf{u}^\varepsilon \in V_0(\Omega) \} \\ \hat{Y}(\Omega_t) &= \{ p^\varepsilon \circ T_t^{-1} : p^\varepsilon \in \hat{Y}(\Omega) \} \end{aligned}$$

since  $T_t$  and  $T_t^{-1}$  are diffeomorphisms, the parametrization do not influence  $j(t)$ . And we have

$$j(t) = \min_{(\mathbf{u}^\varepsilon, p^\varepsilon) \in V_g(\Omega) \times \hat{Y}(\Omega)} \max_{(\mathbf{v}, q) \in V_0(\Omega) \times \hat{Y}(\Omega)} L(\Omega_t, \mathbf{u}^\varepsilon \circ T_t^{-1}, p^\varepsilon \circ T_t^{-1}, \mathbf{v} \circ T_t^{-1}, q \circ T_t^{-1})$$

we define the following functions which depend on the parameter  $t$

$$\begin{aligned} l_1(t) &= \frac{1}{2} \int_0^T \int_{\Omega_t} |\mathbf{u}^\varepsilon \circ T_t^{-1} - \mathbf{u}_d|^2 dx + \frac{\sigma}{2} \int_0^T \int_{\Omega_t} |\text{curl}(\mathbf{u}^\varepsilon \circ T_t^{-1})|^2 dt dx, \\ l_2(t) &= \int_0^T \int_{\Omega_t} [\nu \nabla(\mathbf{u}^\varepsilon \circ T_t^{-1}) : \nabla(\mathbf{v} \circ T_t^{-1}) + ((\mathbf{u}^\varepsilon \circ T_t^{-1}) \cdot \nabla)(\mathbf{u}^\varepsilon \circ T_t^{-1}) \cdot (\mathbf{v} \circ T_t^{-1}) \\ &\quad + a |\mathbf{u}^\varepsilon \circ T_t^{-1}|^\alpha (\mathbf{u}^\varepsilon \circ T_t^{-1}) \cdot (\mathbf{v} \circ T_t^{-1}) - (p^\varepsilon \circ T_t^{-1}) \nabla \cdot (\mathbf{v} \circ T_t^{-1}) - (f \circ T_t^{-1}) \cdot (\mathbf{v} \circ T_t^{-1})] dt dx \\ &\quad - \int_0^T \int_{\Omega_t} (q \circ T_t^{-1}) \nabla \cdot (\mathbf{u}^\varepsilon \circ T_t^{-1}) dt dx - \varepsilon \int_0^T \int_{\Omega_t} (q \circ T_t^{-1}) (p^\varepsilon \circ T_t^{-1}) dt dx \end{aligned}$$

The Lagrangian functional writes

$$L(\Omega_t, \mathbf{u}^\varepsilon \circ T_t^{-1}, p^\varepsilon \circ T_t^{-1}, \mathbf{v} \circ T_t^{-1}, q \circ T_t^{-1}) = l_1(t) - l_2(t)$$

If  $\Phi : [0, \tau] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is sufficiently smooth, we have the following Hadamard formula

$$\frac{d}{dt} \int_0^T \int_{\Omega_t} \Phi(t, x) dx \Big|_{t=0} = \int_0^T \int_{\Omega} \frac{\partial \Phi}{\partial t}(0, x) dx + \int_0^T \int_{\partial \Omega} \Phi(0, x) \mathbf{v}(0, X) \cdot \mathbf{n} ds \quad (5.9)$$

We consider  $\mathbf{V}(0, X) \in V_{ad}$ , and observe that  $\mathbf{V}(0, X) = \mathbf{V}$ . Therefore we can derive the shape gradient using the formula (5.8)

$$\frac{d}{dt} L(\Omega_t, \mathbf{u}^\varepsilon \circ T_t^{-1}, p^\varepsilon \circ T_t^{-1}, \mathbf{v} \circ T_t^{-1}, q \circ T_t^{-1}) \Big|_{t=0} = l'_1(0) - l'_2(0) \quad (5.10)$$

where

$$\begin{aligned} l'_1(0) &= \int_0^T \int_{\Omega} (\mathbf{u}^\varepsilon - \mathbf{u}_d) \cdot (-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) dx + \sigma \int_0^T \int_{\Omega} \text{curl}(\mathbf{u}^\varepsilon) \text{curl}(-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) dx \\ &\quad + \frac{1}{2} \int_0^T \int_{\partial \Omega} (|\mathbf{u}^\varepsilon - \mathbf{u}_d|^2 \mathbf{V} \cdot \mathbf{n}) ds + \frac{\sigma}{2} \int_0^T \int_{\partial \Omega} (|\text{curl}(\mathbf{u}^\varepsilon)|^2 \mathbf{V} \cdot \mathbf{n}) ds \end{aligned} \quad (5.11)$$

$$\begin{aligned} l'_2(0) &= \int_0^T \int_{\Omega} [(-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V})_t v + (-\nabla \mathbf{v} \cdot \mathbf{V})_t \mathbf{u}^\varepsilon + \nu \nabla(-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) : \nabla \mathbf{v} + \nu \nabla \mathbf{u}^\varepsilon : \nabla(-\nabla \mathbf{v} \cdot \mathbf{V}) \\ &\quad + ((-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) \cdot \nabla \mathbf{u}^\varepsilon) \cdot \mathbf{v} + (\mathbf{u}^\varepsilon \cdot \nabla(-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V})) \cdot \mathbf{v} + (\mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon) \cdot (-\nabla \mathbf{v} \cdot \mathbf{V}) \\ &\quad + \frac{1}{2} \nabla \cdot (-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) \mathbf{u}^\varepsilon \cdot \mathbf{v} + \frac{1}{2} \nabla \cdot \mathbf{u}^\varepsilon (-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) \cdot \mathbf{v} + \frac{1}{2} \nabla \cdot \mathbf{u}^\varepsilon \mathbf{u}^\varepsilon \cdot (-\nabla \mathbf{v} \cdot \mathbf{V}) \\ &\quad + a \alpha |\mathbf{u}^\varepsilon|^{\alpha-2} (-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V} \cdot \mathbf{u}^\varepsilon) (\mathbf{u}^\varepsilon \cdot \mathbf{v}) + a |\mathbf{u}^\varepsilon|^\alpha (-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) \cdot \mathbf{v} \\ &\quad + a |\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon \cdot (-\nabla \mathbf{v} \cdot \mathbf{V}) - (f \cdot \mathbf{V}) \cdot v - f \cdot ((-\nabla v) \cdot \mathbf{V}) - (\nabla p^\varepsilon \cdot \mathbf{V}) \nabla \cdot \mathbf{v} \\ &\quad - p^\varepsilon \nabla \cdot (-\nabla \mathbf{v} \cdot \mathbf{V}) - q \nabla \cdot (-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) - (\nabla \cdot \mathbf{u}^\varepsilon) (-\nabla q \cdot \mathbf{V})] dx \end{aligned}$$

Using Green's formula and the condition  $\mathbf{u}^\varepsilon = 0$  on  $\partial\Omega$ , we obtain

$$\begin{aligned}
l'_2(0) = & - \int_0^T \int_\Omega [(\mathbf{u}_t^\varepsilon - \nu \Delta \mathbf{u}^\varepsilon + (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{u}^\varepsilon + \frac{1}{2} \nabla \cdot \mathbf{u}^\varepsilon \mathbf{u}^\varepsilon + a |\mathbf{u}^\varepsilon|^{\alpha} \mathbf{u}^\varepsilon + \nabla p^\varepsilon - f) \cdot (\nabla \mathbf{v} \cdot \mathbf{V})] dt dx \\
& + \int_0^T \int_\Omega (\nabla \cdot \mathbf{u}^\varepsilon) (\nabla q \cdot \mathbf{V}) dt dx + \int_0^T \int_\Omega (\nabla \cdot \mathbf{v}) (\nabla p^\varepsilon \cdot \mathbf{V}) dt dx + \int_0^T \int_\Omega [-\mathbf{v}_t - \nu \Delta \mathbf{v} + (\nabla \mathbf{u}^\varepsilon)^T \cdot \mathbf{v} \\
& - (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{v} - \frac{1}{2} \nabla (\mathbf{u}^\varepsilon \cdot \mathbf{v}) + \frac{1}{2} (\nabla \cdot \mathbf{u}^\varepsilon) \mathbf{v} + a |\mathbf{u}^\varepsilon|^{\alpha-2} (|\mathbf{u}^\varepsilon|^2 \mathbf{v} + \alpha (\mathbf{u}^\varepsilon \cdot \mathbf{v}) \mathbf{u}^\varepsilon) - \nabla q] \cdot (\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) dt dx \\
& - \int_0^T \int_{\partial\Omega} [\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - nq] \cdot (\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) dt ds - \int_0^T \int_{\partial\Omega} [\frac{\partial \mathbf{u}^\varepsilon}{\partial \mathbf{n}} - nq] \cdot (\nabla \mathbf{v} \cdot \mathbf{V}) dt ds \\
& + \int_0^T \int_{\partial\Omega} [\nu \nabla \mathbf{u}^\varepsilon : \nabla \mathbf{v} + (\mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon) \cdot \mathbf{v} + a |\mathbf{u}^\varepsilon|^{\alpha} \mathbf{u}^\varepsilon \cdot \mathbf{v} - p \nabla \cdot \mathbf{v} - q \nabla \cdot \mathbf{u}^\varepsilon] \mathbf{V} \cdot \mathbf{n} dt ds
\end{aligned} \tag{5.12}$$

Replacing (5.11) and (5.12) in (5.10) and using the fact that  $(\mathbf{u}^\varepsilon, p^\varepsilon)$  is solution of (1.2) and  $(\mathbf{v}, q)$  is solution of (5.4) respectively yields

$$\begin{aligned}
dJ(\Omega; \mathbf{V}) &= \frac{d}{dt} L|_{t=0} \\
&= \frac{1}{2} \int_0^T \int_{\partial\Omega} (|\mathbf{u}^\varepsilon - \mathbf{u}_d|^2 + \sigma |\text{curl}(\mathbf{u}^\varepsilon)|^2) \mathbf{V} \cdot \mathbf{n} ds + \int_0^T \int_{\partial\Omega} [\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - nq] \cdot (\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) dt ds \\
&\quad + \int_0^T \int_{\partial\Omega} [\nu \frac{\partial \mathbf{u}^\varepsilon}{\partial \mathbf{n}} - n p^\varepsilon] \cdot (\nabla \mathbf{v} \cdot \mathbf{V}) dt ds - \int_0^T \int_{\partial\Omega} [(\nu \nabla \mathbf{u}^\varepsilon : \nabla \mathbf{v}) \mathbf{V} \cdot \mathbf{n} dt ds \\
&\quad - \sigma \int_0^T \int_{\partial\Omega} (\frac{\partial \mathbf{u}^\varepsilon}{\partial \mathbf{n}} \cdot (\text{curl}(\mathbf{u}^\varepsilon) \wedge \mathbf{n})) \mathbf{V} \cdot \mathbf{n} dt ds
\end{aligned} \tag{5.13}$$

Note that  $\mathbf{u}^\varepsilon = 0$  and  $\mathbf{v} = 0$  in  $\partial\Omega$  we have

$$\mathbf{n} \cdot (\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) = \nabla \mathbf{u}^\varepsilon \cdot (\mathbf{n} \otimes \mathbf{n}) \cdot \mathbf{V} \cdot \mathbf{n} = \nabla \mathbf{u}^\varepsilon \cdot \mathbf{n} \cdot \mathbf{n} (\mathbf{V} \cdot \mathbf{n}) = (\nabla \cdot \mathbf{u}^\varepsilon) (\mathbf{V} \cdot \mathbf{n}) = 0, \quad \forall x \in \partial\Omega \tag{5.14}$$

$$\frac{\partial \mathbf{v}}{\partial \mathbf{n}} \cdot (\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) = \nabla \mathbf{u}^\varepsilon \cdot (\mathbf{n} \otimes \mathbf{n}) \cdot \mathbf{V} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{n}} = \frac{\partial \mathbf{u}^\varepsilon}{\partial \mathbf{n}} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{n}} (\mathbf{V} \cdot \mathbf{n}) = (\nabla \mathbf{u}^\varepsilon : \nabla \mathbf{v}) \mathbf{V} \cdot \mathbf{n} \tag{5.15}$$

Similarly we obtain

$$\mathbf{n} \cdot (\nabla \mathbf{v} \cdot \mathbf{V}) = 0, \quad \frac{\partial \mathbf{u}^\varepsilon}{\partial \mathbf{n}} \cdot (\nabla \mathbf{v} \cdot \mathbf{V}) = \frac{\partial \mathbf{u}^\varepsilon}{\partial \mathbf{n}} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{n}} (\mathbf{V} \cdot \mathbf{n}) = (\nabla \mathbf{u}^\varepsilon : \nabla \mathbf{v}) \mathbf{V} \cdot \mathbf{n} \tag{5.16}$$

Substituting (5.14), (5.15) and (5.16) in (5.13) the shape derivative takes the form

$$dJ(\Omega; \mathbf{V}) = \int_0^T \int_{\partial\Omega} \left[ \frac{1}{2} |\mathbf{u}^\varepsilon - \mathbf{u}_d|^2 + \frac{\sigma}{2} |\text{curl}(\mathbf{u}^\varepsilon)|^2 + \frac{\partial \mathbf{u}^\varepsilon}{\partial \mathbf{n}} \cdot \left( \nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - \sigma \text{curl}(\mathbf{u}^\varepsilon) \wedge \mathbf{n} \right) \right] \mathbf{V} \cdot \mathbf{n} dt ds \tag{5.17}$$

Consequently, the shape gradient writes

$$\nabla J = \left[ \frac{1}{2} |\mathbf{u}^\varepsilon - \mathbf{u}_d|^2 + \frac{\sigma}{2} |\text{curl}(\mathbf{u}^\varepsilon)|^2 + \frac{\partial \mathbf{u}^\varepsilon}{\partial \mathbf{n}} \cdot \left( \nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - \sigma \text{curl}(\mathbf{u}^\varepsilon) \wedge \mathbf{n} \right) \right] \mathbf{n} \tag{5.18}$$

## 6. DISCRETE FINITE ELEMENT METHOD

**6.1. Preliminaries.** In this section we write some results and notations which will be used in this work. We define the following forms

$$\begin{aligned}
a(\mathbf{u}, \mathbf{v}) &= \nu (A^{1/2} \mathbf{u}, A^{1/2} \mathbf{v}), \quad a_\varepsilon(\mathbf{u}, \mathbf{v}) = \nu (A_\varepsilon^{1/2} \mathbf{u}, A_\varepsilon^{1/2} \mathbf{v}), \quad c_\alpha(\mathbf{u}, \mathbf{v}) = (a |\mathbf{u}|^\alpha \mathbf{u}, \mathbf{v}) \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{X} \\
d(\mathbf{v}, q) &= (\nabla \cdot \mathbf{v}, q), \quad \forall \mathbf{v} \in \mathbf{X}, \quad \forall q \in M
\end{aligned}$$

The following assumptions on the datum  $\mathbf{u}_0$  and  $\mathbf{f}$  will be needed throughout this section

$$\begin{aligned} &\text{The initial velocity } \mathbf{u}_0 \in D(A) \text{ with } \nabla \cdot \mathbf{u}_0 = 0 \\ &\text{the forcing function } \mathbf{f} \in H^{1,\infty}(0, T; Y) \text{ with } \|A\mathbf{u}_0\| + \sup_{t \in [0, T]} \{ \|\mathbf{f}(t)\| + \|\mathbf{f}_t(t)\| \} \leq C. \end{aligned} \quad (\text{A1})$$

We recall the two weak formulations concerning the initial problem and its penalized.

$$\begin{aligned} &\text{Find } (\mathbf{u}, p) \in L^\infty(0, T; Y) \cap L^2(0, T; \mathbf{X}) \times L^2(0, T; M) \text{ such that} \\ &(\mathbf{u}_t, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) - d(\mathbf{v}, p) + d(\mathbf{u}, q) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) + c_\alpha(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \forall (\mathbf{v}, q) \in \mathbf{X} \times M \end{aligned} \quad (\text{6.1})$$

and

$$\begin{aligned} &\text{Find } (\mathbf{u}_\varepsilon, p_\varepsilon) \in L^\infty(0, T; Y) \cap L^2(0, T; \mathbf{X}) \times L^2(0, T; M) \text{ such that} \\ &(\mathbf{u}_{\varepsilon t}, \mathbf{v}) + a(\mathbf{u}_\varepsilon, \mathbf{v}) - d(\mathbf{v}, p_\varepsilon) + d(\mathbf{u}_\varepsilon, q) + \varepsilon(p_\varepsilon, q) + \tilde{b}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{v}) + c_\alpha(\mathbf{u}_\varepsilon, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \\ &\text{for all } (\mathbf{v}, q) \in \mathbf{X} \times M \end{aligned} \quad (\text{6.2})$$

with the initial data  $\mathbf{u}(0) = \mathbf{u}_0$  and  $\mathbf{u}_\varepsilon(0) = \mathbf{u}_0$  respectively.

Now, we define the time discretization of the penalized weak formulation (6.2) by the backward Euler scheme

$$(d_t \mathbf{u}_\varepsilon^n, \mathbf{v}) + a(\mathbf{u}_\varepsilon^n, \mathbf{v}) - d(\mathbf{v}, p_\varepsilon^n) + d(\mathbf{u}_\varepsilon^n, q) + \varepsilon(p_\varepsilon^n, q) + \tilde{b}(\mathbf{u}_\varepsilon^n, \mathbf{u}_\varepsilon^n, \mathbf{v}) + c_\alpha(\mathbf{u}_\varepsilon^n, \mathbf{v}) = (f(t_n), \mathbf{v}) \quad (\text{6.3})$$

for all  $(\mathbf{v}, q) \in \mathbf{X} \times M$  and  $1 \leq n \leq N$ , where  $0 < \Delta t < 1$  is the time step size,  $t_n = n\Delta$ ,  $t_N = T$ ,  $(\mathbf{u}_\varepsilon^0, p_\varepsilon^0) = (\mathbf{u}_0, 0)$ ,  $d_t \mathbf{u}_\varepsilon^n = \frac{\mathbf{u}_\varepsilon^n - \mathbf{u}_\varepsilon^{n-1}}{\Delta t}$  for  $1 \leq n \leq N$  and  $d_t \mathbf{u}_\varepsilon^0$  satisfies the following equation

$$(d_t \mathbf{u}_\varepsilon^0, \mathbf{v}) = a(\mathbf{u}_0, \mathbf{v}) + ((\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0, \mathbf{v}) + c_\alpha(\mathbf{u}_0, \mathbf{v}) = (\mathbf{f}(0), \mathbf{v}), \forall \mathbf{v} \in \mathbf{X} \text{ with } \operatorname{div} \mathbf{v} = 0$$

By using the inequality (2.11), we infer that

$$\begin{aligned} \|d_t \mathbf{u}_\varepsilon^0\| &\leq \nu \|A\mathbf{u}_0\| + \|(\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0\| + \|A^{1/2} \mathbf{u}_0\| + \|\mathbf{f}(0)\| \\ &\leq c_\nu \|A\mathbf{u}_0\| + c \|\mathbf{u}_0\| \|A^{1/2} \mathbf{u}_0\|^2 + \|\mathbf{f}(0)\| \end{aligned} \quad (\text{6.4})$$

We begin with a general regularity result, on the perturbed solution, useful for the establishment of an error bound associated with the finite elements discretization.

**Theorem 6.1.** *Under the assumptions (A1) and  $C\varepsilon \leq 1$ , there is a constant  $k_0 > 0$  such that if  $k_0 \Delta t \leq 1$ , then the following estimates hold*

$$\|A_\varepsilon^{1/2} \mathbf{u}_\varepsilon^m\|^2 + \Delta t \sum_{n=1}^m (\|d_t \mathbf{u}_\varepsilon^n\|^2 + \|A_\varepsilon \mathbf{u}_\varepsilon^n\|^2 + \|p_\varepsilon^n\|^2 + a \|\mathbf{u}_\varepsilon^n\|_{L^{\alpha+2}}^{\alpha+2}) \leq C, \quad (\text{6.5})$$

$$\|d_t \mathbf{u}_\varepsilon^m\|^2 + \|A_\varepsilon \mathbf{u}_\varepsilon^m\|^2 + \|p_\varepsilon^m\|_1^2 + \Delta t \sum_{n=1}^m \|A_\varepsilon^{1/2} d_t \mathbf{u}_\varepsilon^n\|^2 \leq C, \quad (\text{6.6})$$

for all  $1 \leq m \leq N$ .

*Proof.* Taking  $(\mathbf{v}, q) = 2(\mathbf{u}_\varepsilon^n, p_\varepsilon^n) \Delta t$  in (6.3), using (6.3) and the relation

$$2(\mathbf{u} - \mathbf{v}, \mathbf{u}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2, \quad \forall \mathbf{u}, \mathbf{v} \in Y \quad (\text{6.7})$$

to have

$$\|\mathbf{u}_\varepsilon^n\|^2 - \|\mathbf{u}_\varepsilon^{n-1}\|^2 + 2\nu \|A^{1/2} \mathbf{u}_\varepsilon^n\|^2 \Delta t + a \Delta t \sum_{n=1}^m \|\mathbf{u}_\varepsilon^n\|_{L^{\alpha+2}}^{\alpha+2} \leq 2\|\mathbf{f}(t_n)\| \|\mathbf{u}_\varepsilon^n\| \Delta t$$

We sum this inequality from 1 to  $m$  and using (2.9) and the Young's inequality, to obtain

$$\|\mathbf{u}_\varepsilon^m\|^2 + \nu\Delta t \sum_{n=1}^m \|A^{1/2}\mathbf{u}_\varepsilon^n\|^2 + a\Delta t \sum_{n=1}^m \|\mathbf{u}_\varepsilon^n\|_{L^{\alpha+2}}^{\alpha+2} \leq \|\mathbf{u}_0\|^2 + \nu^{-1}c\Delta t \sum_{n=1}^m \|\mathbf{f}(t_n)\|^2 \leq C \quad (6.8)$$

We can establish from (6.3) that

$$d_t\mathbf{u}_\varepsilon^n + \nu A_\varepsilon\mathbf{u}_\varepsilon^n + B(\mathbf{u}_\varepsilon^n, \mathbf{u}_\varepsilon^n) + a|\mathbf{u}_\varepsilon^n|^\alpha\mathbf{u}_\varepsilon^n = \mathbf{f}(t_n) \quad (6.9)$$

Taking the scalar product of (6.9) with  $(\nu^{-1}d_t\mathbf{u}_\varepsilon^n + A_\varepsilon\mathbf{u}_\varepsilon^n)\Delta t$  in  $Y$  and using the relation

$$2(A_\varepsilon^{1/2}(\mathbf{u} - \mathbf{v}), A_\varepsilon^{1/2}\mathbf{u}) = \|A_\varepsilon^{1/2}\mathbf{u}\|^2 - \|A_\varepsilon^{1/2}\mathbf{v}\|^2 + \|A_\varepsilon^{1/2}(\mathbf{u} - \mathbf{v})\|^2, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{X} \quad (6.10)$$

we obtain

$$\begin{aligned} \|A_\varepsilon^{1/2}\mathbf{u}_\varepsilon^n\|^2 &- \|A_\varepsilon^{1/2}\mathbf{u}_\varepsilon^{n-1}\|^2 + \nu^{-1}\|d_t\mathbf{u}_\varepsilon^n\|^2\Delta t + \nu\|A_\varepsilon\mathbf{u}_\varepsilon^n\|^2\Delta t \\ &+ c_\alpha(\mathbf{u}_\varepsilon^n, \nu^{-1}d_t\mathbf{u}_\varepsilon^n + A_\varepsilon\mathbf{u}_\varepsilon^n)\Delta t + \tilde{b}(\mathbf{u}_\varepsilon^n, \mathbf{u}_\varepsilon^n, \nu^{-1}d_t\mathbf{u}_\varepsilon^n + A_\varepsilon\mathbf{u}_\varepsilon^n)\Delta t \\ &= (\mathbf{f}(t_n), \nu^{-1}d_t\mathbf{u}_\varepsilon^n + A_\varepsilon\mathbf{u}_\varepsilon^n)\Delta t \end{aligned} \quad (6.11)$$

Using (2.11)-(2.12), we get

Firstly

$$\begin{aligned} &|\tilde{b}(\mathbf{u}_\varepsilon^n, \mathbf{u}_\varepsilon^n, \nu^{-1}d_t\mathbf{u}_\varepsilon^n + A_\varepsilon\mathbf{u}_\varepsilon^n)| \\ &\leq c\|A^{1/2}\mathbf{u}_\varepsilon^n\| \|\mathbf{u}_\varepsilon^n\|_{L^\infty} \|\nu^{-1}d_t\mathbf{u}_\varepsilon^n + A_\varepsilon\mathbf{u}_\varepsilon^n\| \\ &\leq \frac{1}{6\nu}\|d_t\mathbf{u}_\varepsilon^n\|^2 + \frac{\nu}{6}\|A_\varepsilon\mathbf{u}_\varepsilon^n\|^2 + \nu^{-1}c\|\mathbf{u}_\varepsilon^n\|^2\|A^{1/2}\mathbf{u}_\varepsilon^n\|^2\|A_\varepsilon^{1/2}\mathbf{u}_\varepsilon^n\|^2 \end{aligned}$$

Secondly

$$\begin{aligned} |c_\alpha(\mathbf{u}_\varepsilon^n, \nu^{-1}d_t\mathbf{u}_\varepsilon^n + A_\varepsilon\mathbf{u}_\varepsilon^n)| &\leq \frac{1}{6\nu}\|d_t\mathbf{u}_\varepsilon^n\|^2 + \frac{\nu}{6}\|A_\varepsilon\mathbf{u}_\varepsilon^n\|^2 \\ &+ \nu^{-1}c\|A^{1/2}\mathbf{u}_\varepsilon^n\|^2\|A_\varepsilon^{1/2}\mathbf{u}_\varepsilon^n\|^2 \end{aligned}$$

Finally

$$|(\mathbf{f}(t_n), \nu^{-1}d_t\mathbf{u}_\varepsilon^n + A_\varepsilon\mathbf{u}_\varepsilon^n)| \leq \frac{1}{6\nu}\|d_t\mathbf{u}_\varepsilon^n\|^2 + \frac{\nu}{6}\|A_\varepsilon\mathbf{u}_\varepsilon^n\|^2 + \nu^{-1}c\|\mathbf{f}(t_n)\|^2$$

Combining these inequalities with (6.11) to obtain

$$\begin{aligned} \|A_\varepsilon^{1/2}\mathbf{u}_\varepsilon^n\|^2 &- \|A_\varepsilon^{1/2}\mathbf{u}_\varepsilon^{n-1}\|^2 + \frac{1}{2}(\nu^{-1}\|d_t\mathbf{u}_\varepsilon^n\|^2 + \nu\|A_\varepsilon\mathbf{u}_\varepsilon^n\|^2)\Delta t \\ &+ c_\alpha(\mathbf{u}_\varepsilon^n, \nu^{-1}d_t\mathbf{u}_\varepsilon^n + A_\varepsilon\mathbf{u}_\varepsilon^n) \leq d_n\|A_\varepsilon^{1/2}\mathbf{u}_\varepsilon^n\|^2\Delta t + \nu^{-1}c\|\mathbf{f}(t_n)\|^2\Delta t \end{aligned}$$

with  $d_n = \nu^{-1}c(1 + \nu^{-2}\|\mathbf{u}_\varepsilon^n\|^2)\|A^{1/2}\mathbf{u}_\varepsilon^n\|^2$ . Summing this inequality from 1 to  $m$  and observing that  $A_\varepsilon^{1/2}\mathbf{u}_0 = A^{1/2}\mathbf{u}_0$ , we get

$$\begin{aligned} \|A_\varepsilon^{1/2}\mathbf{u}_\varepsilon^m\|^2 &+ \frac{1}{2}\Delta t \sum_{n=1}^m (\nu^{-1}\|d_t\mathbf{u}_\varepsilon^n\|^2 + \nu\|A_\varepsilon\mathbf{u}_\varepsilon^n\|^2) + \Delta t \sum_{n=1}^m c_\alpha(\mathbf{u}_\varepsilon^n, \nu^{-1}d_t\mathbf{u}_\varepsilon^n + A_\varepsilon\mathbf{u}_\varepsilon^n) \\ &\leq \|A^{1/2}\mathbf{u}_0\|^2 + \Delta t \sum_{n=1}^m d_n\|A_\varepsilon^{1/2}\mathbf{u}_\varepsilon^n\|^2 + \nu^{-1}c\Delta t \sum_{n=1}^m \|\mathbf{f}(t_n)\|^2 \\ &\leq C + \Delta t \sum_{n=1}^m d_n\|A_\varepsilon^{1/2}\mathbf{u}_\varepsilon^n\|^2 \end{aligned} \quad (6.12)$$

If we take  $\Delta t$  such that  $d_n \Delta t \leq \frac{1}{2}$ , by applying Lemma 2.4 to (6.12) we infer

$$\|A_\varepsilon^{1/2} \mathbf{u}_\varepsilon^m\|^2 + \frac{1}{2} \Delta t \sum_{n=1}^m (\nu^{-1} \|d_t \mathbf{u}_\varepsilon^n\|^2 + \nu \|A_\varepsilon \mathbf{u}_\varepsilon^n\|^2) + \Delta t \sum_{n=1}^m c_\alpha(\mathbf{u}_\varepsilon^n, \nu^{-1} d_t \mathbf{u}_\varepsilon^n + A_\varepsilon \mathbf{u}_\varepsilon^n) \leq C \exp(2\Delta t \sum_{n=1}^m d_n) \quad (6.13)$$

for all  $1 \leq m \leq N$ . By using (2.12), (6.8) and (6.12) there exists a constant  $C_0 > 0$  such that

$$2\Delta t \sum_{n=1}^m d_n \leq C \text{ and } d_m = \nu^{-1} c(1 + \nu^{-2} \|\mathbf{u}_\varepsilon^m\|^2) \|A_\varepsilon^{1/2} \mathbf{u}_\varepsilon^m\|^2 \leq \frac{1}{2} C_0 \quad (6.14)$$

for all  $1 \leq m \leq N$ .

Next, using (6.3) to obtain

$$\begin{aligned} & (d_{tt} \mathbf{u}_\varepsilon^n, \mathbf{v}) + a_\varepsilon(d_t \mathbf{u}_\varepsilon^n, \mathbf{v}) + \tilde{b}(d_t \mathbf{u}_\varepsilon^n, \mathbf{u}_\varepsilon^n, \mathbf{v}) + \tilde{b}(\mathbf{u}_\varepsilon^{n-1}, d_t \mathbf{u}_\varepsilon^n, \mathbf{v}) \\ & + a(d_t(|\mathbf{u}_\varepsilon^n|^\alpha \mathbf{u}_\varepsilon^n), \mathbf{v}) = \left( \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \mathbf{f}_t(t) dt, \mathbf{v} \right), \quad \forall \mathbf{v} \in X \end{aligned} \quad (6.15)$$

We take  $\mathbf{v} = 2d_t \mathbf{u}_\varepsilon^n \Delta t$  in (6.15) and we use (2.4) and (6.7) to obtain

$$\begin{aligned} & \|d_t \mathbf{u}_\varepsilon^n\|^2 - \|d_t \mathbf{u}_\varepsilon^{n-1}\|^2 + 2\nu \|A_\varepsilon^{1/2} d_t \mathbf{u}_\varepsilon^n\|^2 \Delta t + 2\tilde{b}(d_t \mathbf{u}_\varepsilon^n, \mathbf{u}_\varepsilon^n, d_t \mathbf{u}_\varepsilon^n) \Delta t \\ & + 2a(d_t(|\mathbf{u}_\varepsilon^n|^\alpha \mathbf{u}_\varepsilon^n), d_t \mathbf{u}_\varepsilon^n) \Delta t \leq 2 \left( \int_{t_{n-1}}^{t_n} \mathbf{f}_t(t) dt, d_t \mathbf{u}_\varepsilon^n \right) \end{aligned} \quad (6.16)$$

Since  $2a(d_t(|\mathbf{u}_\varepsilon^n|^\alpha \mathbf{u}_\varepsilon^n), d_t \mathbf{u}_\varepsilon^n) \geq 0$  thanks to (2.13). We use (2.9) and (2.12) to get

Firstly

$$\begin{aligned} 2|\tilde{b}(d_t \mathbf{u}_\varepsilon^n, \mathbf{u}_\varepsilon^n, d_t \mathbf{u}_\varepsilon^n)| & \leq c \|d_t \mathbf{u}_\varepsilon^n\| \|A_\varepsilon^{1/2} d_t \mathbf{u}_\varepsilon^n\| \|A_\varepsilon^{1/2} \mathbf{u}_\varepsilon^n\| \\ & + c \|d_t \mathbf{u}_\varepsilon^n\|^{1/2} \|A_\varepsilon^{1/2} d_t \mathbf{u}_\varepsilon^n\|^{3/2} \|A_\varepsilon^{1/2} \mathbf{u}_\varepsilon^n\|^{1/2} \|\mathbf{u}_\varepsilon^n\|^{1/2} \\ & \leq \frac{\nu}{3} \|A_\varepsilon d_t \mathbf{u}_\varepsilon^n\|^2 + d_n \|d_t \mathbf{u}_\varepsilon^n\|^2 \end{aligned}$$

Secondly

$$\left| \left( \int_{t_{n-1}}^{t_n} \mathbf{f}_t(t) dt, d_t \mathbf{u}_\varepsilon^n \right) \right| \leq \frac{\nu}{3} \|A_\varepsilon d_t \mathbf{u}_\varepsilon^n\|^2 + \nu^{-1} c \int_{t_{n-1}}^{t_n} \|\mathbf{f}(t)\|^2 dt$$

Adding these inequalities and replacing them in (6.16) we have

$$\begin{aligned} & \|d_t \mathbf{u}_\varepsilon^n\|^2 - \|d_t \mathbf{u}_\varepsilon^{n-1}\|^2 + \nu \|A_\varepsilon^{1/2} d_t \mathbf{u}_\varepsilon^n\|^2 \Delta t \\ & \leq (\nu^{-1} c + d_n) \|d_t \mathbf{u}_\varepsilon^n\|^2 \Delta t + \nu^{-1} c \int_{t_{n-1}}^{t_n} \|\mathbf{f}(t)\|^2 dt \end{aligned} \quad (6.17)$$

Summing this inequality from 1 to  $m$  and using (6.4), (6.13) and (6.14) we obtain

$$\|d_t \mathbf{u}_\varepsilon^m\|^2 + \nu \Delta t \sum_{n=1}^m \|A_\varepsilon^{1/2} d_t \mathbf{u}_\varepsilon^n\|^2 \leq C, \quad \forall 1 \leq m \leq N \quad (6.18)$$

Finally, we can establish from (6.3), (6.15) and the inf sup condition [19] that

$$\nu \|A_\varepsilon \mathbf{u}_\varepsilon^n\| \leq \|d_t \mathbf{u}_\varepsilon^n\| + \|\tilde{B}(\mathbf{u}_\varepsilon^n, \mathbf{u}_\varepsilon^n)\| + \| |\mathbf{u}_\varepsilon^n|^\alpha \mathbf{u}_\varepsilon^n \| + \|\mathbf{f}(t_n)\| \quad (6.19)$$

$$\|p_\varepsilon^n\|_1 \leq c \|d_t \mathbf{u}_\varepsilon^n\| + \nu c \|A_\varepsilon \mathbf{u}_\varepsilon^n\| + c \|\tilde{B}(\mathbf{u}_\varepsilon^n, \mathbf{u}_\varepsilon^n)\| + c \| |\mathbf{u}_\varepsilon^n|^\alpha \mathbf{u}_\varepsilon^n \| + c \|\mathbf{f}(t_n)\| \quad (6.20)$$

Applying (2.11)-(2.12), we get

$$\begin{aligned} \|\tilde{B}(\mathbf{u}_\varepsilon^n, \mathbf{u}_\varepsilon^n)\| &\leq c\|A^{1/2}\mathbf{u}_\varepsilon^n\| \|\mathbf{u}_\varepsilon^n\|_{L^\infty} \leq c\|A^{1/2}\mathbf{u}_\varepsilon^n\| \|A_\varepsilon\mathbf{u}_\varepsilon^n\|^{1/2} \|\mathbf{u}_\varepsilon^n\|^{1/2} \\ &\leq \frac{\nu}{4} \|A_\varepsilon\mathbf{u}_\varepsilon^n\| + \nu^{-1}c \|\mathbf{u}_\varepsilon^n\| \|A^{1/2}\mathbf{u}_\varepsilon^n\|^2 \end{aligned}$$

and

$$\|\mathbf{u}_\varepsilon^n\|^\alpha \|\mathbf{u}_\varepsilon^n\| \leq c\|A_\varepsilon\mathbf{u}_\varepsilon^n\|$$

Replacing these inequalities in (6.19) and (6.20) and using (2.12), we obtain

$$\nu\|A_\varepsilon\mathbf{u}_\varepsilon^n\|^2 \leq \nu^{-1}\|d_t\mathbf{u}_\varepsilon^n\|^2 + \nu^{-3}c\|\mathbf{u}_\varepsilon^n\|^2\|A_\varepsilon^{1/2}\mathbf{u}_\varepsilon^n\|^4 + \nu^{-1}c\|f(t_n)\|^2 \quad (6.21)$$

$$\|p_\varepsilon^n\|_1^2 \leq c(\|d_t\mathbf{u}_\varepsilon^n\|^2 + \|A_\varepsilon\mathbf{u}_\varepsilon^n\|^2) + c\nu^{-2}\|\mathbf{u}_\varepsilon^n\|^2\|A_\varepsilon^{1/2}\mathbf{u}_\varepsilon^n\|^4 + \|f(t_n)\|^2 \quad (6.22)$$

$$\begin{aligned} \Delta t \sum_{n=1}^m \|p_\varepsilon^n\|_1^2 &\leq c\Delta t \sum_{n=1}^m (\|d_t\mathbf{u}_\varepsilon^n\|^2 + \|A_\varepsilon\mathbf{u}_\varepsilon^n\|^2) \\ &\quad + c\Delta t \sum_{n=1}^m (\nu^{-2}\|\mathbf{u}_\varepsilon^n\|^2\|A_\varepsilon^{1/2}\mathbf{u}_\varepsilon^n\|^4 + \|f(t_n)\|^2) \end{aligned} \quad (6.23)$$

Combining the inequalities (6.21)-(6.23) with (6.13)-(6.14) and (6.18), we end the proof of the Theorem 6.1  $\square$

**Theorem 6.2.** *Under the assumptions of Theorem 6.1 we have the following estimate*

$$\begin{aligned} \tau(t_m)\|A^{1/2}d_t\mathbf{u}_\varepsilon^m\|^2 + \Delta t \sum_{n=1}^m \tau(t_n) (\|d_{tt}\mathbf{u}_\varepsilon^n\|^2 + \|A_\varepsilon d_t\mathbf{u}_\varepsilon^n\|^2 + \|d_t p_\varepsilon^n\|_1^2) + \\ 2\nu^{-1}\Delta t \sum_{n=1}^m a(d_t(|\mathbf{u}_\varepsilon^n|^\alpha \mathbf{u}_\varepsilon^n), d_{tt}\mathbf{u}_\varepsilon^n) \leq C, \end{aligned}$$

for all  $1 \leq m \leq N$ .

*Proof.* Taking  $\mathbf{v} = 2\nu^{-1}d_{tt}\mathbf{u}_\varepsilon^n\Delta t$  in (6.15) yields

$$\begin{aligned} 2\left(A_\varepsilon^{1/2}d_t\mathbf{u}_\varepsilon^n, A_\varepsilon^{1/2}(d_t\mathbf{u}_\varepsilon^n - d_t\mathbf{u}_\varepsilon^{n-1})\right) + 2\nu^{-1}\|d_{tt}\mathbf{u}_\varepsilon^n\|^2\Delta t + 2\nu^{-1}\tilde{b}(d_t\mathbf{u}_\varepsilon^n, \mathbf{u}_\varepsilon^n, d_{tt}\mathbf{u}_\varepsilon^n)\Delta t \\ + 2\nu^{-1}\tilde{b}(\mathbf{u}_\varepsilon^n, d_t\mathbf{u}_\varepsilon^n, d_{tt}\mathbf{u}_\varepsilon^n)\Delta t + 2\nu^{-1}a(d_t(|\mathbf{u}_\varepsilon^n|^\alpha \mathbf{u}_\varepsilon^n), d_{tt}\mathbf{u}_\varepsilon^n)\Delta t \\ = 2\nu^{-1}\left(\int_{t_{n-1}}^{t_n} \mathbf{f}_t(t)dt, d_{tt}\mathbf{u}_\varepsilon^n\right) \end{aligned} \quad (6.24)$$

We use (2.9)-(2.12) to raise

$$\begin{aligned} 2\nu^{-1}|\tilde{b}(d_t\mathbf{u}_\varepsilon^n, \mathbf{u}_\varepsilon^n, d_{tt}\mathbf{u}_\varepsilon^n)| &+ 2\nu^{-1}|\tilde{b}(\mathbf{u}_\varepsilon^n, d_t\mathbf{u}_\varepsilon^n, d_{tt}\mathbf{u}_\varepsilon^n)| \\ &\leq c\nu^{-1}\|A^{1/2}d_t\mathbf{u}_\varepsilon^n\| \|A_\varepsilon\mathbf{u}_\varepsilon^n\| \|d_{tt}\mathbf{u}_\varepsilon^n\| \\ &\leq \frac{1}{6\nu}\|d_{tt}\mathbf{u}_\varepsilon^n\|^2 + \nu^{-1}c\|A_\varepsilon\mathbf{u}_\varepsilon^n\|^2\|A_\varepsilon^{1/2}d_t\mathbf{u}_\varepsilon^n\|^2 \\ 2\nu^{-1}a(d_t(|\mathbf{u}_\varepsilon^n|^\alpha \mathbf{u}_\varepsilon^n), d_{tt}\mathbf{u}_\varepsilon^n) &\leq \frac{1}{6\nu}\|d_{tt}\mathbf{u}_\varepsilon^n\|^2 + \nu^{-1}c\|A_\varepsilon^{1/2}d_t\mathbf{u}_\varepsilon^n\|^2 \\ 2\nu^{-1}\left|\left(\int_{t_{n-1}}^{t_n} \mathbf{f}_t(t)dt, d_{tt}\mathbf{u}_\varepsilon^n\right)\right| &\leq \frac{1}{6\nu}\|d_{tt}\mathbf{u}_\varepsilon^n\|^2\Delta t + \nu^{-1}c\int_{t_{n-1}}^{t_n} \|\mathbf{f}_t(t)\|^2 dt \end{aligned}$$

Replacing these inequalities in (6.24) yields

$$\begin{aligned} 2\tau(t_n)(A_\varepsilon^{1/2}d_t\mathbf{u}_\varepsilon^n, A_\varepsilon^{1/2}(d_t\mathbf{u}_\varepsilon^n - d_t\mathbf{u}_\varepsilon^{n-1})) + \nu^{-1}\tau(t_n)\|d_{tt}\mathbf{u}_\varepsilon^n\|^2\Delta t \\ \leq \nu^{-1}c(1 + \|A_\varepsilon\mathbf{u}_\varepsilon^n\|^2)\|A_\varepsilon^{1/2}d_t\mathbf{u}_\varepsilon^n\|^2 + \nu^{-1}c\int_{t_{n-1}}^{t_n}\|\mathbf{f}_t(t)\|^2dt \end{aligned} \quad (6.25)$$

We observe that

$$\begin{aligned} 2\tau(t_n)(A_\varepsilon^{1/2}d_t\mathbf{u}_\varepsilon^n, A_\varepsilon^{1/2}(d_t\mathbf{u}_\varepsilon^n - d_t\mathbf{u}_\varepsilon^{n-1})) \geq \tau(t_n)\|A_\varepsilon^{1/2}d_t\mathbf{u}_\varepsilon^n\|^2 - \tau(t_{n-1})\|A_\varepsilon^{1/2}d_t\mathbf{u}_\varepsilon^{n-1}\|^2 \\ - \|A_\varepsilon^{1/2}d_t\mathbf{u}_\varepsilon^{n-1}\|^2\Delta t, \quad 2 \leq n \leq N \end{aligned}$$

$$\begin{aligned} 2\tau(t_1)(A_\varepsilon^{1/2}d_t\mathbf{u}_\varepsilon^1, A_\varepsilon^{1/2}(d_t\mathbf{u}_\varepsilon^1 - d_t\mathbf{u}_\varepsilon^0)) \geq \tau(t_1)\|A_\varepsilon^{1/2}d_t\mathbf{u}_\varepsilon^1\|^2 \\ - \|A_\varepsilon\mathbf{u}_\varepsilon^1 - A\mathbf{u}_0\|\|d_t\mathbf{u}_\varepsilon^0\| \end{aligned}$$

We sum (6.25) in  $1 \leq n \leq m$  to obtain

$$\begin{aligned} \tau(t_m)\|A_\varepsilon^{1/2}d_t\mathbf{u}_\varepsilon^m\|^2 + \nu^{-1}\Delta t\sum_{n=1}^m\tau(t_n)\|d_{tt}\mathbf{u}_\varepsilon^n\|^2 + 2\nu^{-1}\Delta t\sum_{n=1}^ma(d_t(|\mathbf{u}_\varepsilon^n|^\alpha\mathbf{u}_\varepsilon^n), d_{tt}\mathbf{u}_\varepsilon^n) \\ \leq \|A_\varepsilon\mathbf{u}_\varepsilon^1 - A\mathbf{u}_0\|\|d_t\mathbf{u}_\varepsilon^0\| + \Delta t\sum_{n=1}^m\|A_\varepsilon^{1/2}d_t\mathbf{u}_\varepsilon^n\|^2 \\ + \nu^{-1}c\Delta t\sum_{n=1}^m(1 + \|A_\varepsilon\mathbf{u}_\varepsilon^n\|^2)\|A_\varepsilon^{1/2}d_t\mathbf{u}_\varepsilon^n\|^2 + \nu^{-1}c\int_{t_{n-1}}^{t_n}\|\mathbf{f}_t(t)\|^2dt \end{aligned} \quad (6.26)$$

Combining (6.26) with (6.6) yields

$$\tau(t_m)\|A_\varepsilon^{1/2}d_t\mathbf{u}_\varepsilon^m\|^2 + \nu^{-1}\Delta t\sum_{n=1}^m\tau(t_n)\|d_{tt}\mathbf{u}_\varepsilon^n\|^2 \leq C, \quad \forall 1 \leq m \leq N \quad (6.27)$$

Next, we derive from (6.3) and the inf-sup condition [20] that

$$\nu\|A_\varepsilon d_t\mathbf{u}_\varepsilon^n\| \leq \|d_{tt}\mathbf{u}_\varepsilon^n\| + \|\tilde{B}(d_t\mathbf{u}_\varepsilon^n, \mathbf{u}_\varepsilon^n)\| + \|\tilde{B}(\mathbf{u}_\varepsilon^n, d_t\mathbf{u}_\varepsilon^n)\| + \|d_t(|\mathbf{u}_\varepsilon^n|^\alpha\mathbf{u}_\varepsilon^n)\| + \|\mathbf{f}(t_n)\| \quad (6.28)$$

$$\begin{aligned} \|d_t p_\varepsilon^n\|_1 \leq c\|d_{tt}\mathbf{u}_\varepsilon^n\| + \nu c\|A d_t\mathbf{u}_\varepsilon^n\| + c\|\tilde{B}(d_t\mathbf{u}_\varepsilon^n, \mathbf{u}_\varepsilon^n)\| + c\|\tilde{B}(\mathbf{u}_\varepsilon^n, d_t\mathbf{u}_\varepsilon^n)\| \\ + c\|d_t(|\mathbf{u}_\varepsilon^n|^\alpha\mathbf{u}_\varepsilon^n)\| + c\|\mathbf{f}(t_n)\| \end{aligned} \quad (6.29)$$

Due to (2.9)-(2.12), we get

$$\|\tilde{B}(d_t\mathbf{u}_\varepsilon^n, \mathbf{u}_\varepsilon^n)\| + \|\tilde{B}(\mathbf{u}_\varepsilon^n, d_t\mathbf{u}_\varepsilon^n)\| \leq c\|A_\varepsilon\mathbf{u}_\varepsilon^n\|\|A_\varepsilon^{1/2}d_t\mathbf{u}_\varepsilon^n\|$$

Combining this inequality with (6.28) and (6.29), we obtain

$$\begin{aligned} \nu^2\Delta t\sum_{n=1}^m\tau(t_n)\|A_\varepsilon d_t\mathbf{u}_\varepsilon^n\|^2 \leq c\Delta t\sum_{n=1}^m\tau(t_n)\|d_{tt}\mathbf{u}_\varepsilon^n\|^2 \\ + c\Delta t\sum_{n=1}^m(1 + \|A_\varepsilon\mathbf{u}_\varepsilon^n\|^2)\|A_\varepsilon^{1/2}d_t\mathbf{u}_\varepsilon^n\|^2 + c\Delta t\sum_{n=1}^m\|\mathbf{f}_t(t_n)\|^2 \end{aligned} \quad (6.30)$$

$$\begin{aligned} \Delta t\sum_{n=1}^m\tau(t_n)\|d_t p_\varepsilon^n\|_1^2 \leq c\Delta t\sum_{n=1}^m\tau(t_n)(\|d_{tt}\mathbf{u}_\varepsilon^n\|^2 + \nu^2\|A_\varepsilon d_t\mathbf{u}_\varepsilon^n\|^2) \\ + c\Delta t\sum_{n=1}^m\left((1 + \|A_\varepsilon\mathbf{u}_\varepsilon^n\|^2)\|A_\varepsilon^{1/2}d_t\mathbf{u}_\varepsilon^n\|^2 + \|\mathbf{f}_t(t_n)\|^2\right) \end{aligned} \quad (6.31)$$

Combining the inequalities (6.30) and (6.31) with (6.27), this complete the proof of the Theorem 6.2  $\square$

**6.2. Finite element penalty method.** Let us consider a real positive parameter  $h$ . We denote by  $(\mathbf{X}_h, M_h)$  the finite element pair of  $(\mathbf{X}, M)$ . It is a discretization of these domain into triangles  $\tau$ .

We introduce the  $L^2$ -orthogonal projections:  $\pi_h : M \rightarrow M_h$  defined by

$$(\pi_h q, q_h) = (q, q_h), \quad \forall q \in M, \quad q_h \in M_h$$

We shall make standing the assumption for the finite element pair  $(\mathbf{X}_h, M_h)$

$$\text{There exists a mapping } \mathbf{r}_h : D(A) \cap \mathbf{X} \rightarrow \mathbf{X}_h \text{ such that} \quad (\text{A2})$$

$$(\text{div}(\mathbf{u} - \mathbf{r}_h \mathbf{u}), q_h) = 0, \quad \forall q \in M, \quad q_h \in M_h \quad (\text{6.32})$$

$$\|A^{1/2}(\mathbf{u} - \mathbf{r}_h \mathbf{u})\| \leq ch \|Au\|, \|p - \pi_h p\| \leq ch \|p\|_1, \quad \forall p \in H^1(\Omega) \cap M \quad (\text{6.33})$$

Moreover, we suppose the inverse inequality

$$\|A^{1/2} v_h\| \leq ch^{-1} \|\mathbf{v}_h\|, \quad \forall \mathbf{v}_h \in \mathbf{X}_h \quad (\text{6.34})$$

and the inf-sup condition

$$\|q\| \leq c \sup_{\mathbf{v}_h \in \mathbf{X}_h} \frac{(\text{div}(\mathbf{v}_h), q_h)}{\|A^{1/2} \mathbf{v}_h\|}, \quad \forall q_h \in M_h \quad (\text{6.35})$$

Let us introduce the following finite element discretization of the formulation (6.3)

$$\begin{aligned} (d_t \mathbf{u}_{\varepsilon h}^n, \mathbf{v}_h) + a(\mathbf{u}_{\varepsilon h}^n, \mathbf{v}_h) - d(\mathbf{v}_h, p_{\varepsilon h}^n) + d(\mathbf{u}_{\varepsilon h}^n, q_h) + \varepsilon(p_{\varepsilon h}^n, q_h) + \tilde{b}(\mathbf{u}_{\varepsilon h}^n, \mathbf{u}_{\varepsilon h}^n, \mathbf{v}_h) \\ + c_\alpha(\mathbf{u}_{\varepsilon h}^n, \mathbf{v}_h) = (f(t_n), \mathbf{v}_h), \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{X}_h \times M_h \end{aligned} \quad (\text{6.36})$$

where  $\mathbf{u}_{\varepsilon h}^0 = \mathbf{r}_h \mathbf{u}_0$ ,  $p_{\varepsilon h}^0 = 0$ ,  $\{\mathbf{u}_{\varepsilon h}^n\}_{n=1}^N \subset \mathbf{X}_h$  and  $\{p_{\varepsilon h}^n\}_{n=1}^N \subset M_h$  are the finite element approximations of  $\{\mathbf{u}_\varepsilon^n\}_{n=1}^N \subset \mathbf{X}$  and  $\{p_\varepsilon^n\}_{n=1}^N \subset M$  respectively.

**Theorem 6.3.** *Suppose (A1), (A2) and  $\varepsilon c_0 \leq 1$ ,  $\Delta t k_0 \leq 1$  are valid. Then the following proprieties holds*

$$\|\mathbf{u}_{\varepsilon h}^n\|^2 + \Delta t \sum_{n=1}^m \|A^{1/2} \mathbf{u}_{\varepsilon h}^n\|^2 + a \Delta t \sum_{n=1}^m \|\mathbf{u}_{\varepsilon h}^n\|_{L^{\alpha+2}}^{\alpha+2} \leq C, \quad (\text{6.37})$$

$$\|A^{1/2} \mathbf{u}_{\varepsilon h}^n\|^2 + \Delta t \sum_{n=1}^m \|d_t \mathbf{u}_{\varepsilon h}^n\|^2 \leq C + Ch^{-2} \Delta t \sum_{n=1}^m \|A^{1/2}(\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n)\|^2 \|\mathbf{u}_{\varepsilon h}^n\|^2, \quad (\text{6.38})$$

$$+ Ch^{-2} \Delta t \sum_{n=1}^m \|\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n\|^2 \|A^{1/2} \mathbf{u}_{\varepsilon h}^n\|^2,$$

for all  $1 \leq m \leq N$ .

*Proof.* In a similar manner as the estimation (6.8) in the Theorem 6.1, we can derive (6.37).

Next we can establish from (6.36) that

$$\begin{aligned} (d_t \mathbf{u}_{\varepsilon h}^n, \mathbf{v}_h) + a(\mathbf{u}_{\varepsilon h}^n, \mathbf{v}_h) - d(\mathbf{v}_h, p_{\varepsilon h}^n) + d(d_t \mathbf{u}_{\varepsilon h}^n, q_h) + \varepsilon(d_t p_{\varepsilon h}^n, q_h) + \tilde{b}(\mathbf{u}_{\varepsilon h}^n, \mathbf{u}_{\varepsilon h}^n, \mathbf{v}_h) \\ + c_\alpha(\mathbf{u}_{\varepsilon h}^n, \mathbf{v}_h) = (f(t_n), \mathbf{v}_h), \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{X}_h \times M_h \end{aligned} \quad (\text{6.39})$$

Setting  $(\mathbf{v}_h, q_h) = 2(d_t \mathbf{u}_{\varepsilon h}^n, p_{\varepsilon h}^n) \Delta t$  in (6.39) and using (6.10), we get

$$\begin{aligned} 2\|d_t \mathbf{u}_{\varepsilon h}^n\|^2 \Delta t + \nu \|A^{1/2} \mathbf{u}_{\varepsilon h}^n\|^2 - \nu \|A^{1/2} \mathbf{u}_{\varepsilon h}^{n-1}\|^2 + \varepsilon(\|p_{\varepsilon h}^n\|^2 - \|p_{\varepsilon h}^{n-1}\|^2) \\ + 2c_\alpha(\mathbf{u}_{\varepsilon h}^n, d_t \mathbf{u}_{\varepsilon h}^n) \Delta t + 2\tilde{b}(\mathbf{u}_{\varepsilon h}^n, \mathbf{u}_{\varepsilon h}^n, d_t \mathbf{u}_{\varepsilon h}^n) \Delta t = 2(f(t_n), d_t \mathbf{u}_{\varepsilon h}^n) \Delta t \end{aligned} \quad (\text{6.40})$$

Applying (2.9)-(2.11), (2.4) and (6.34), we obtain

$$2|\tilde{b}(\mathbf{u}_{\varepsilon h}^n, \mathbf{u}_{\varepsilon h}^n, d_t \mathbf{u}_{\varepsilon h}^n)| \leq 2|\tilde{b}(\mathbf{u}_{\varepsilon h}^n - \mathbf{u}_\varepsilon^n, \mathbf{u}_{\varepsilon h}^n, d_t \mathbf{u}_{\varepsilon h}^n)| + 2|\tilde{b}(\mathbf{u}_\varepsilon^n, \mathbf{u}_{\varepsilon h}^n, d_t \mathbf{u}_{\varepsilon h}^n)|$$

$$\begin{aligned}
2|\tilde{b}(\mathbf{u}_\varepsilon^n, \mathbf{u}_{\varepsilon h}^n, d_t \mathbf{u}_{\varepsilon h}^n)| &\leq c\|\mathbf{A}\mathbf{u}_\varepsilon^n\| \|A^{1/2}\mathbf{u}_{\varepsilon h}^n\| \|d_t \mathbf{u}_{\varepsilon h}^n\| \\
&\leq \frac{1}{4}\|d_t \mathbf{u}_{\varepsilon h}^n\|^2 + c\|\mathbf{A}\mathbf{u}_\varepsilon^n\|^2 \|A^{1/2}\mathbf{u}_{\varepsilon h}^n\|^2 \\
2|\tilde{b}(\mathbf{u}_{\varepsilon h}^n - \mathbf{u}_\varepsilon^n, \mathbf{u}_{\varepsilon h}^n, d_t \mathbf{u}_{\varepsilon h}^n)| &\leq c\|\mathbf{u}_{\varepsilon h}^n - \mathbf{u}_\varepsilon^n\|_{L^4} \|\mathbf{u}_{\varepsilon h}^n\|_{L^4} \|A^{1/2}d_t \mathbf{u}_{\varepsilon h}^n\| \\
&\quad + c\|A^{1/2}(\mathbf{u}_{\varepsilon h}^n - \mathbf{u}_\varepsilon^n)\|^2 \|\mathbf{u}_{\varepsilon h}^n\|_{L^4} \|A^{1/2}d_t \mathbf{u}_{\varepsilon h}^n\|_{L^4} \\
&\leq \frac{1}{4}\|d_t \mathbf{u}_{\varepsilon h}^n\|^2 + ch^{-2}\|A^{1/2}(\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n)\|^2 \|\mathbf{u}_{\varepsilon h}^n\|^2 \\
&\quad + ch^{-2}\|\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n\|^2 \|A^{1/2}\mathbf{u}_{\varepsilon h}^n\|^2 \\
2|c_\alpha(\mathbf{u}_{\varepsilon h}^n, d_t \mathbf{u}_{\varepsilon h}^n)| &\leq \frac{1}{4}\|d_t \mathbf{u}_{\varepsilon h}^n\|^2 + c\|A^{1/2}\mathbf{u}_{\varepsilon h}^n\|^2 \\
2|(\mathbf{f}(t_n), d_t \mathbf{u}_{\varepsilon h}^n)| &\leq \frac{1}{4}\|d_t \mathbf{u}_{\varepsilon h}^n\|^2 + c\|\mathbf{f}(t_n)\|^2
\end{aligned}$$

Replacing these inequalities in (6.40). with some sample calculation we obtain

$$\begin{aligned}
\nu\|A^{1/2}\mathbf{u}_{\varepsilon h}^n\|^2 - \nu\|A^{1/2}\mathbf{u}_{\varepsilon h}^n\|^2 + \|d_t \mathbf{u}_{\varepsilon h}^n\|^2 \Delta t + \varepsilon(\|p_{\varepsilon h}^n\|^2 - \|p_{\varepsilon h}^{n-1}\|^2) \\
\leq c\|\mathbf{A}\mathbf{u}_\varepsilon^n\|^2 \|A^{1/2}\mathbf{u}_{\varepsilon h}^n\|^2 \Delta t + ch^{-2}\|A^{1/2}(\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n)\|^2 \|\mathbf{u}_{\varepsilon h}^n\|^2 \Delta t \\
+ \|\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n\|^2 \|A^{1/2}\mathbf{u}_{\varepsilon h}^n\| \Delta t + c\|A^{1/2}\mathbf{u}_{\varepsilon h}^n\|^2 + c\|\mathbf{f}(t_n)\|^2
\end{aligned} \tag{6.41}$$

Summing these inequality from 1 to  $m$  and using Theorem 6.1 and (2.12) we get (6.38)  $\square$

In order to establish the error estimates of the finite element penalty method, we introduce the following Galerkin projection  $R_h; (\mathbf{X}, M) \rightarrow \mathbf{X}_h$ ,  $Q_h : (\mathbf{X}, M) \rightarrow M_h$  defined by

$$\begin{aligned}
a(\mathbf{u} - R_h(\mathbf{u}, p), \mathbf{v}_h) - d(\mathbf{v}_h, p - Q_h(\mathbf{u}, p)) + d(\mathbf{u} - R_h(\mathbf{u}, p), q_h) + \varepsilon(p - Q_h(\mathbf{u}, p), q_h) = 0 \\
\forall(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times M_h,
\end{aligned} \tag{6.42}$$

for all  $(\mathbf{u}, p) \in \mathbf{X} \times M$  with  $\operatorname{div} \mathbf{u} + \varepsilon p = 0$ .

**Lemma 6.1.** *Under the assumptions (A1), (A2),  $c_0\varepsilon \leq 1$  and  $k_0\Delta t \leq 1$ , the Galerkin projection  $(R_h, Q_h)$  verifies*

- $$\begin{aligned}
\|\mathbf{u} - R_h(\mathbf{u}, p)\| + h\|A^{1/2}(\mathbf{u} - R_h(\mathbf{u}, p))\| + h\|p - Q_h(\mathbf{u}, p)\| \\
\leq c_\nu h(\|A^{1/2}\mathbf{u}\| + \|p\|), \quad \forall(\mathbf{u}, p) \in \mathbf{X} \times M
\end{aligned} \tag{6.43}$$

- $$\begin{aligned}
\|\mathbf{u} - R_h(\mathbf{u}, p)\| + h\|A^{1/2}(\mathbf{u} - R_h(\mathbf{u}, p))\| + h\|p - Q_h(\mathbf{u}, p)\| \\
\leq C_\nu h^2(\|\mathbf{A}\mathbf{u}\| + \|p\|_1), \quad \forall(\mathbf{u}, p) \in D(A) \times (H^1(\Omega) \cap M)
\end{aligned} \tag{6.44}$$

with  $\operatorname{div} \mathbf{u} + \varepsilon p = 0$ , where  $c_\nu$  is a general positive constant depending only on  $\Omega$  and  $\nu$ .

*Proof.* The Galerkin projection has its stability from (6.35) and (6.42), and we have

$$\nu\|A^{1/2}R_h(\mathbf{u}, p)\|^2 \leq \frac{\nu}{2}\|A^{1/2}R_h(\mathbf{u}, p)\|^2 + c_\nu \left( \|A^{1/2}\mathbf{u}\|^2 + \|p\|^2 \right) \tag{6.45}$$

$$\|Q_h(\mathbf{u}, p)\| \leq c_\nu(\|A^{1/2}R_h(\mathbf{u}, p)\| + \|A^{1/2}\mathbf{u}\| + \|p\|) \tag{6.46}$$

Using (6.45)-(6.46) and the triangular inequality, yields to

$$\|A^{1/2}(\mathbf{u} - R_h(\mathbf{u}, p))\| + \|p - Q_h(\mathbf{u}, p)\| \leq c_\nu \left( \|A^{1/2}\mathbf{u}\| + \|p\| \right) \tag{6.47}$$

for all  $(\mathbf{u}, p) \in \mathbf{X} \times M$  with  $\operatorname{div} \mathbf{u} + \varepsilon p = 0$ .

Let consider the dual problem: find  $(\Phi, \Psi) \in \mathbf{X} \times M$  such that for all  $(\mathbf{v}, q) \in \mathbf{X} \times M$

$$a(\mathbf{v}, \Phi) + d(\mathbf{v}, \Psi) - d(\Phi, q) + \varepsilon(q, \Psi) = (\mathbf{v}, \mathbf{u} - R_h(\mathbf{u}p)) \quad (6.48)$$

By using the Lemma 4.2, the problem (6.48) has a unique solution  $(\Phi, \Psi)$  satisfying

$$\|A\mathbf{u}\| + \|\Psi\|_1 \leq c_\nu \|\mathbf{u} - R_h(\mathbf{u}, p)\| \quad (6.49)$$

Taking  $(\mathbf{v}, q) = (\mathbf{u} - R_h(\mathbf{u}, p), p - Q_h(\mathbf{u}, p))$  in (6.48) and  $(\mathbf{v}_h, q_h) = (\mathbf{r}_h \Phi, \pi_h \Psi)$  in (6.42) and using (6.33) and (6.49) we infer that

$$\begin{aligned} |u - R_h(\mathbf{u}, p)|^2 &= a(\mathbf{u} - R_h(\mathbf{u}, p), \Phi - \mathbf{r}_h \Phi) + d(\mathbf{u} - R_h(\mathbf{u}, p), \Psi - \pi_h \Psi) \\ &\quad - d(\Phi - \mathbf{r}_h \Phi, p - Q_h(\mathbf{u}, p)) + \varepsilon(p - Q_h(\mathbf{u}, p), \Psi - \pi_h \Psi) \\ &\leq c_\nu h (\|A^{1/2}(\mathbf{u} - R_h(\mathbf{u}, p))\| + \|p - Q_h(\mathbf{u}, p)\|) (\|A\Phi\| + \|\psi\|_1) \\ &\leq c_\nu h (\|A^{1/2}(\mathbf{u} - R_h(\mathbf{u}, p))\| + \|p - Q_h(\mathbf{u}, p)\|) \|\mathbf{u} - R_h(\mathbf{u}, p)\| \end{aligned} \quad (6.50)$$

The estimation (6.43) is a consequence of (6.50) and (6.47).

We consider  $(\mathbf{u}, p) \in D(A) \times (H^1(\Omega) \cap M)$  with  $\operatorname{div} \mathbf{u} + \varepsilon p = 0$ . based on (6.32)-(6.33) and (6.42), we deduce that

$$\begin{aligned} \|A^{1/2}(\mathbf{r}_h \mathbf{u} - R_h(\mathbf{u}, p))\| + \|\pi_h p - Q_h(\mathbf{u}, p)\| \\ \leq c_\nu (\|A^{1/2}(\mathbf{u} - \mathbf{r}_h \mathbf{u})\| + \|\pi_h p - p\|) \\ \leq c_\nu h (\|A\mathbf{u}\| + \|p\|_1) \end{aligned}$$

Therefore, using the triangles inequality and (6.33) yields to

$$\|A^{1/2}(\mathbf{u} - R_h(\mathbf{u}, p))\| + \|p - Q_h(\mathbf{u}, p)\| \leq c_\nu h (\|A\mathbf{u}\| + \|p\|_1) \quad (6.51)$$

Combining (6.51) and (6.50) to obtain

$$\|\mathbf{u} - R_h(\mathbf{u}, p)\| \leq c_\nu h^2 (\|A\mathbf{u}\| + \|p\|_1) \quad (6.52)$$

Combining (6.51)-(6.52), we obtain the estimation (6.44)  $\square$

### 6.3. Error estimates.

In this subsection we are focused on the error estimates of  $(\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n, p_\varepsilon^n - p_{\varepsilon h}^n)$  and then derive the optimal bound of the error  $(\mathbf{u}(t_n) - \mathbf{u}_{\varepsilon h}^n, p(t_n) - p_{\varepsilon h}^n)$ .

**Lemma 6.2.** *Under the assumption of theorem 6.3, we have the following estimate*

$$\|\mathbf{u}_\varepsilon^m - \mathbf{u}_{\varepsilon h}^m\|^2 + \Delta t \sum_{n=1}^m \|A^{1/2}(\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n)\|^2 \leq Ch^2 \quad (6.53)$$

for all  $1 \leq m \leq N$

*Proof.* Subtracting (6.36) from (6.3) with  $(\mathbf{v}, q) = (\mathbf{v}_h, q_h)$  and applying (6.32), we get

$$\begin{aligned}
& ((I - \mathbf{r}_h)d_t \mathbf{u}_\varepsilon^n + d_t e^n, \mathbf{v}_h) + a((I - \mathbf{r}_h)\mathbf{u}_\varepsilon^n + e^n, \mathbf{v}_h) \\
& - d(\mathbf{v}_h, (I - \pi_h)p_\varepsilon^n + \eta^n) + d(e^n, q_h) \\
& + \tilde{b}((I - \mathbf{r}_h)\mathbf{u}_\varepsilon^n + e^n, \mathbf{u}_{\varepsilon h}^n, \mathbf{v}_h) + \tilde{b}(\mathbf{u}_{\varepsilon h}^n, (I - \mathbf{r}_h)\mathbf{u}_\varepsilon^n + e^n, \mathbf{v}_h) \\
& + c_\alpha(\mathbf{u}_\varepsilon^n, \mathbf{v}_h) - c_\alpha(\mathbf{u}_{\varepsilon h}^n, \mathbf{v}_h) \\
& + \varepsilon((I - \pi_h)p_\varepsilon^n + \eta^n, q_h) = 0, \quad \forall (\mathbf{v}_h, q_h) \in X_h \times M_h
\end{aligned} \tag{6.54}$$

where  $(e^n, \eta^n) = (\mathbf{r}_h \mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n, \pi_h p_\varepsilon^n - p_{\varepsilon h}^n)$ . Setting  $(\mathbf{v}_h, q_h) = 2(e^n, \eta^n)\Delta t$  in (6.54) and using (2.4) and (6.7), we obtain

$$\begin{aligned}
& \|e^n\|^2 - \|e^{n-1}\|^2 + 2\nu\|A^{1/2}e^n\|^2\Delta t + \varepsilon\|\eta^n\|^2\Delta t + 2\tilde{b}(e^n, \mathbf{u}_\varepsilon^n, e^n)\Delta t \\
& + 2\tilde{b}((I - \mathbf{r}_h)\mathbf{u}_\varepsilon^n, \mathbf{u}_\varepsilon^n, e^n)\Delta t + 2\tilde{b}(\mathbf{u}_{\varepsilon h}^n, (I - \mathbf{r}_h)\mathbf{u}_\varepsilon^n, e^n)\Delta t \\
& + 2c_\alpha(\mathbf{u}_\varepsilon^n, e^n)\Delta t - 2c_\alpha(\mathbf{r}_h \mathbf{u}_\varepsilon^n, e^n)\Delta t + 2c_\alpha(\mathbf{u}_h \mathbf{u}_\varepsilon^n, e^n)\Delta t - 2c_\alpha(\mathbf{u}_{\varepsilon h}^n, e^n)\Delta t \\
& \leq \|(I - \pi_h)p_\varepsilon^n\|^2\Delta t + 2d(e^n, (I - \pi_h)p_\varepsilon^n)\Delta t \\
& - 2a((I - \mathbf{r}_h)\mathbf{u}_\varepsilon^n, e^n)\Delta t - 2((I - \mathbf{r}_h)d_t \mathbf{u}_\varepsilon^n, e^n)\Delta t
\end{aligned} \tag{6.55}$$

Applying (2.9) and (6.33), we infer that

$$\begin{aligned}
2|\tilde{b}(e^n, \mathbf{u}_\varepsilon^n, e^n)| & \leq c\|e^n\|_{L^4}^2\|A^{1/2}\mathbf{u}_\varepsilon^n\| + c\|A^{1/2}e^n\|\|\mathbf{u}_\varepsilon^n\|_{L^4}\|e^n\|_{L^4} \\
& \leq \frac{\nu}{4}\|A^{1/2}e^n\|^2 + d_n\|e^n\|^2 \\
2|\tilde{b}((I - \mathbf{r}_h)\mathbf{u}_\varepsilon^n, \mathbf{u}_\varepsilon^n, e^n)| & + 2|\tilde{b}(\mathbf{u}_{\varepsilon h}^n, (I - \mathbf{r}_h)\mathbf{u}_\varepsilon^n, e^n)| \\
& \leq c\|A^{1/2}(I - \mathbf{r}_h)\mathbf{u}_\varepsilon^n\| (\|A^{1/2}\mathbf{u}_\varepsilon^n\| + \|A^{1/2}\mathbf{u}_{\varepsilon h}^n\|) \|A^{1/2}e^n\| \\
& \leq \frac{\nu}{4}\|A^{1/2}e^n\|^2 + c_\nu h^2\|\mathbf{A}\mathbf{u}_\varepsilon^n\|^2 (\|A^{1/2}\mathbf{u}_\varepsilon^n\|^2 + \|A^{1/2}\mathbf{u}_{\varepsilon h}^n\|^2) \\
\varepsilon\|(I - \pi_h)p_\varepsilon^n\|^2 + 2|d(e^n, (I - \pi_h)p_\varepsilon^n)| & \leq \frac{\nu}{3}\|A^{1/2}e^n\|^2 + c_\nu h^2\|p_\varepsilon^n\|_1^2 \\
2|a((I - \mathbf{r}_h)\mathbf{u}_\varepsilon^n, e^n)| & \leq \frac{\nu}{12}\|A^{1/2}e^n\| + c_\nu h^2\|\mathbf{A}\mathbf{u}_\varepsilon^n\|^2 \\
2|((I - \mathbf{r}_h)d_t \mathbf{u}_\varepsilon^n, e^n)| & \leq \frac{\nu}{12}\|A^{1/2}e^n\|^2 + c_\nu h^2\|A^{1/2}d_t \mathbf{u}_\varepsilon^n\|^2 \\
2|c_\alpha(\mathbf{u}_\varepsilon^n, e^n) - c_\alpha(\mathbf{r}_h \mathbf{u}_\varepsilon^n, e^n)| & \leq \frac{\nu}{12}\|A^{1/2}e^n\|^2 + c_\nu h^2\|\mathbf{A}\mathbf{u}_\varepsilon^n\|^2
\end{aligned}$$

Replacing these inequalities in (6.55) yields to

$$\begin{aligned}
\|e^n\|^2 - \|e^{n-1}\|^2 + \nu\|A^{1/2}e^n\|^2\Delta t & \leq d_n\|e^n\|^2\Delta t \\
& + c_\nu h^2 (\|A^{1/2}\mathbf{u}_\varepsilon^n\|^2 + \|A^{1/2}\mathbf{u}_{\varepsilon h}^n\|^2) \|\mathbf{A}\mathbf{u}_\varepsilon^n\|^2\Delta t \\
& + c_\nu h^2 (2\|\mathbf{A}\mathbf{u}_\varepsilon^n\|^2 + \|A^{1/2}d_t \mathbf{u}_\varepsilon^n\|^2 + \|p_\varepsilon^n\|_1^2) \Delta t, \quad \forall 1 \leq n \leq N
\end{aligned} \tag{6.56}$$

Summing this inequality from 1 to  $m$  and using theorem 6.1, theorem 6.3 and (2.12) we deduce

$$\|e^m\|^2 + \nu\Delta t \sum_{n=1}^m \|A^{1/2}e^n\|^2 \leq \Delta t \sum_{n=1}^m d_n\|e^n\|^2 + kh^2 \tag{6.57}$$

Using (6.14) and discrete Gronwall's Lemma to obtain

$$\|e^m\|^2 + \nu \Delta t \sum_{n=1}^m \|A^{1/2} e^n\|^2 \leq Ch^2 \quad (6.58)$$

By using (2.12), (6.33) and theorem 6.1 we get

$$\begin{aligned} \|(I - \mathbf{r}_h) \mathbf{u}_\varepsilon^m\|^2 + \nu \Delta t \sum_{n=1}^m \|A^{1/2} (I - \mathbf{r}_h) \mathbf{u}_\varepsilon^n\|^2 &\leq ch^2 \left( \|A^{1/2} \mathbf{u}_\varepsilon^m\|^2 + \nu \Delta t \sum_{n=1}^m \|\mathbf{A} \mathbf{u}_\varepsilon^n\|^2 \right) \\ &\leq Ch^2 \end{aligned} \quad (6.59)$$

We end the proof of lemma 6.1 by combining (6.58) and (6.59).  $\square$

**Lemma 6.3.** *Under the assumptions of Theorem 6.3, the following error estimate*

$$\|A^{1/2} (\mathbf{u}_\varepsilon^m - \mathbf{u}_{\varepsilon h}^m)\|^2 + \Delta t \sum_{n=1}^m \|p_\varepsilon^n - p_{\varepsilon h}^n\|^2 \leq Ch^2, \quad \forall 1 \leq m \leq N \quad (6.60)$$

holds

*Proof.* Subtracting (6.36) from (6.3) with  $(\mathbf{v}, q) = (\mathbf{v}_h, q_h)$  and taking  $e^n = R_h^n - \mathbf{u}_{\varepsilon h}^n$ ,  $\eta^n = Q_h^n - p_{\varepsilon h}^n$ ,  $R_h^n = R_h(\mathbf{u}_\varepsilon^n, p_\varepsilon^n)$ ,  $Q_h^n = Q_h(\mathbf{u}_\varepsilon^n, p_\varepsilon^n)$ , we get

$$\begin{aligned} (d_t e^n, \mathbf{v}_h) &+ a(e^n, \mathbf{v}_h) - d(\mathbf{v}_h, \eta^n) + d(d_t e^n, q_h) + \varepsilon(d_t \eta^n, q_h) \\ &+ \tilde{b}(\mathbf{u}_\varepsilon^n - R_h^n + e^n, \mathbf{u}_\varepsilon^n, \mathbf{v}_h) + \tilde{b}(\mathbf{u}_\varepsilon^n, \mathbf{u}_\varepsilon^n - R_h^n + e^n, \mathbf{v}_h) \\ &- \tilde{b}(\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n, \mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n, \mathbf{v}_h) + a_\alpha(\mathbf{u}_\varepsilon, \mathbf{v}_h) \\ &- c_\alpha(\mathbf{u}_{\varepsilon h}^n, \mathbf{v}_h) = -(d_t \mathbf{u}_\varepsilon^n - R_h(d_t \mathbf{u}_\varepsilon^n, d_t p_\varepsilon^n), \mathbf{v}_h) \end{aligned} \quad (6.61)$$

for all  $(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times M_h$ , with  $1 \leq n \leq N$ . Setting  $(\mathbf{v}_h, q_h) = 2(d_t e^n, \eta^n) \Delta t$  in (6.61), we obtain

$$\begin{aligned} 2\|d_t e^n\|^2 \Delta t &+ \nu (\|A^{1/2} e^n\|^2 - \|A^{1/2} e^{n-1}\|^2) + \varepsilon (\|\eta^n\|^2 - \|\eta^{n-1}\|^2) \\ &+ 2\tilde{b}(\mathbf{u}_\varepsilon^n - R_h^n + e^n, \mathbf{u}_\varepsilon^n, d_t e^n) + 2\tilde{b}(\mathbf{u}_{\varepsilon h}^n, \mathbf{u}_\varepsilon^n - R_h^n + e^n, d_t e^n) \Delta t \\ &- 2\tilde{b}(\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n, \mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n, d_t e^n) \Delta t + c_\alpha(\mathbf{u}_\varepsilon^n, d_t e^n) \Delta t - c_\alpha(R_h^n, d_t e^n) \Delta t \\ &+ c_\alpha(R_h^n, d_t e^n) \Delta t - c_\alpha(\mathbf{u}_{\varepsilon h}^n, d_t e^n) \Delta t = -2(d_t \mathbf{u}_\varepsilon^n - R_h(d_t \mathbf{u}_\varepsilon^n, d_t p_\varepsilon^n), d_t e^n) \Delta t \end{aligned} \quad (6.62)$$

Using the fact that  $(\mathbf{u}_\varepsilon^n, p_\varepsilon^n) = (\mathbf{u}_0, 0)$ , we establish from (6.33) and (6.40) that

$$\|A^{1/2} (\mathbf{u}_\varepsilon^0 - R_h(\mathbf{u}_\varepsilon^0, p_\varepsilon^n))\| + \|p_\varepsilon^0 - Q_h(\mathbf{u}_\varepsilon^0, p_\varepsilon^n)\| \leq c_\nu h \|\mathbf{A} \mathbf{u}_0\| \quad (6.63)$$

$$\|A^{1/2} e^n\| + \|\eta^n\| \leq \|A^{1/2} (\mathbf{u}_0 - \mathbf{r}_h \mathbf{u}_\varepsilon^0)\| + \|A^{1/2} (\mathbf{u}_\varepsilon^0 - R_h(\mathbf{u}_\varepsilon^0, p_\varepsilon^n))\| \quad (6.64)$$

$$\|Q_h(\mathbf{u}_\varepsilon^0, p_\varepsilon^n)\| \leq c_\nu h \|\mathbf{A} \mathbf{u}_0\|$$

Applying ones more (2.9)-(2.11) and (6.34), to have

$$\begin{aligned} 2|\tilde{b}(e^n, \mathbf{u}_\varepsilon^n, d_t e^n)| + 2|\tilde{b}(\mathbf{u}_\varepsilon^n, e^n, d_t e^n)| &\leq c \|\mathbf{A} \mathbf{u}_\varepsilon^n\| \|A^{1/2} e^n\| \|d_t e^n\| \\ &\leq \frac{1}{4} \|d_t e^n\|^2 + c \|\mathbf{A} \mathbf{u}_\varepsilon^n\|^2 \|A^{1/2} e^n\|^2 \\ 2|\tilde{b}(\mathbf{u}_\varepsilon^n - R_h^n, \mathbf{u}_\varepsilon^n, d_t e^n)| + 2|\tilde{b}(\mathbf{u}_\varepsilon^n, \mathbf{u}_\varepsilon^n - R_h^n, d_t e^n)| \\ &\leq c \|\mathbf{A} \mathbf{u}_\varepsilon^n\| \|A^{1/2} (\mathbf{u}_\varepsilon^n - R_h^n)\| \|d_t \mathbf{u}_\varepsilon^n\| \\ &\leq \frac{1}{4} \|d_t e^n\|^2 + c \|\mathbf{A} \mathbf{u}_\varepsilon^n\|^2 \|A^{1/2} (\mathbf{u}_\varepsilon^n - R_h^n)\|^2 \end{aligned}$$

$$\begin{aligned}
2|\tilde{b}(\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n, \mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n, d_t e^n)| &\leq c(h^{-1} \|A^{1/2}(\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n)\|) \| \mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n \| \\
&\quad + \|A^{1/2}(\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n)\|^2 \|d_t e^n\| \\
&\leq \frac{1}{8} \|d_t e^n\|^2 + c(h^{-2} \|A^{1/2}(\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n)\|^2) \| \mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n \|^2 \\
&\quad + \|A^{1/2}(\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n)\|^4
\end{aligned}$$

$$2|c_\alpha(\mathbf{u}_\varepsilon^n, d_t e^n) - c_\alpha(R_h^n, d_t e^n)| \leq \frac{1}{8} \|d_t e^n\|^2 + c \|A^{1/2}(\mathbf{u}_\varepsilon^n - R_h^n)\|^2$$

$$2|c_\alpha(\mathbf{u}_{\varepsilon h}^n, d_t e^n) - c_\alpha(R_h^n, d_t e^n)| \leq \frac{1}{8} \|d_t e^n\|^2 + c \|A^{1/2}(\mathbf{u}_{\varepsilon h}^n - R_h^n)\|^2$$

$$2|(d_t \mathbf{u}_\varepsilon^n - R_h(d_t \mathbf{u}_\varepsilon^n, d_t p_\varepsilon^n), d_t e^n)| \leq \frac{1}{8} \|d_t e^n\|^2 + \|d_t \mathbf{u}_\varepsilon^n - R_h(d_t \mathbf{u}_\varepsilon^n, d_t p_\varepsilon^n)\|^2$$

Substituting these inequalities in (6.62) and using Lemma 6.1 results in

$$\begin{aligned}
\nu \|A^{1/2} e^n\|^2 &- \nu \|A^{1/2} e^{n-1}\|^2 + \|d_t e^n\|^2 \Delta t + \varepsilon (\|\eta^n\|^2 - \|\eta^{n-1}\|^2) \\
&\leq c \|A \mathbf{u}_\varepsilon^n\|^2 (\|A^{1/2} e^n\|^2 + h^2 \|A \mathbf{u}_\varepsilon^n\|^2 + h^2 \|p_\varepsilon^n\|_1^2) \Delta t \\
&\quad + ch^4 (\|A d_t \mathbf{u}_\varepsilon^n\|^2 + \|A d_t \mathbf{u}_{\varepsilon h}^n\|^2 + \|d_t p_\varepsilon^n\|^2) \Delta t \\
&\quad + c(h^{-2} \|A^{1/2}(\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n)\|^2) \| \mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n \|^2 + \|A^{1/2}(\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n)\|^4 \Delta t
\end{aligned} \tag{6.65}$$

Using theorem 6.1 and 6.3, Lemma 6.2 and the propriety  $h^2 \leq c_1 \Delta t \leq c_1 \tau(t_n)$ , we get

$$\begin{aligned}
\nu \|A^{1/2} e^n\|^2 &- \nu \|A^{1/2} e^{n-1}\|^2 + \|d_t e^n\|^2 \Delta t + \varepsilon (\|\eta^n\|^2 - \|\eta^{n-1}\|^2) \\
&\leq k (\|A^{1/2} e^n\|^2 + h^2 \|A \mathbf{u}_\varepsilon^n\|^2 + h^2 \|p_\varepsilon^n\|_1^2) \Delta t \\
&\quad + ch^2 \tau(t_n) (\|A d_t \mathbf{u}_\varepsilon^n\|^2 + \|A d_t \mathbf{u}_{\varepsilon h}^n\|^2 + \|d_t p_\varepsilon^n\|^2) \Delta t \\
&\quad + k (\|A^{1/2}(\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n)\|^2 + \|A^{1/2}(\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n)\|^4) \Delta t
\end{aligned} \tag{6.66}$$

Summing this inequality from 1 to  $m$  and using Theorem 6.1 and 6.2, lemma 6.3 and (6.64) yields to

$$\nu \|A^{1/2} e^m\|^2 + \Delta t \sum_{n=1}^m \|d_t e^n\|^2 \leq kh^2, \quad \forall 1 \leq m \leq N \tag{6.67}$$

By applying Lemma 6.1, Theorem 6.1, Theorem 6.2 and  $h^2 \leq c_1 \tau(t_n)$ , we obtain

$$\begin{aligned}
\nu \|A^{1/2}(\mathbf{u}_\varepsilon^n - R_h(\mathbf{u}_\varepsilon^n, p_\varepsilon^n))\|^2 &+ \Delta t \sum_{n=1}^m \|d_t \mathbf{u}_\varepsilon^n - R_h(d_t \mathbf{u}_\varepsilon^n, d_t p_\varepsilon^n)\|^2 \\
&\leq c_\nu h^2 (\|A \mathbf{u}_\varepsilon^n\|^2 + \|p_\varepsilon^n\|_1^2) + c_\nu h^2 \Delta t \sum_{n=1}^m \tau(t_n) (\|A d_t \mathbf{u}_\varepsilon^n\|^2 + \|d_t p_\varepsilon^n\|_1^2) \\
&\leq kh^2, \quad \forall 1 \leq m \leq N
\end{aligned} \tag{6.68}$$

Combining (6.67) with (6.68) leads to

$$\|A^{1/2}(\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n)\|^2 + \Delta t \sum_{n=1}^m \|d_t \mathbf{u}_\varepsilon^n - d_t \mathbf{u}_{\varepsilon h}^n\|^2 \leq kh^2, \quad \forall 1 \leq m \leq N \tag{6.69}$$

Next, by using (6.54) , (6.35) and (2.9) we get

$$\begin{aligned} \|\pi_h p_\varepsilon^n - p_{\varepsilon h}^n\|^2 &\leq c \|d_t \mathbf{u}_\varepsilon^n - d_t \mathbf{u}_{\varepsilon h}^n\| + c\nu \|A^{1/2}(\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n)\| \\ &\quad + c \|A^{1/2}(\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n)\| (\|A^{1/2} \mathbf{u}_\varepsilon^n\| + \|A^{1/2} \mathbf{u}_{\varepsilon h}^n\|) \\ &\quad + c \|(I - \pi_h) p_\varepsilon^n\| \end{aligned} \quad (6.70)$$

Applying theorem 6.1, theorem 6.3 and lemma 6.1 results for establish from (6.70) that

$$\begin{aligned} \|p_\varepsilon^n - p_{\varepsilon h}^n\|^2 \Delta t &\leq c \|d_t \mathbf{u}_\varepsilon^n - d_t \mathbf{u}_{\varepsilon h}^n\|^2 \Delta t + k \|A^{1/2}(\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon h}^n)\|^2 \Delta t \\ &\quad + c \|(I - \pi_h) p_\varepsilon^n\|^2 \Delta t \end{aligned} \quad (6.71)$$

Summing this inequality from 1 to  $m$  and using (6.69), (6.33) and Theorem 6.1 we infer that

$$\Delta t \sum_{n=1}^m \|p_\varepsilon^n - p_{\varepsilon h}^n\|^2 \leq Ch^2, \quad \forall 1 \leq m \leq N \quad (6.72)$$

Adding the inequalities (6.72) and (6.69) gives (6.60).  $\square$

Combining Lemma 6.3 with Theorem 6.3 gives the following optimal error estimate

**Theorem 6.4.** *Under the assumption of Theorem 6.3, we have the following error estimate*

$$\begin{aligned} \tau^2(t_m) \|A^{1/2}(\mathbf{u}(t_m) - \mathbf{u}_{\varepsilon h}^m)\|^2 &+ \Delta \sum_{n=1}^m \tau^2(t_n) \|p(t_n) - p_{\varepsilon h}^n\|^2 \\ &\leq C(\varepsilon^2 + \Delta t^2 + h^2), \quad \forall 1 \leq m \leq N \end{aligned} \quad (6.73)$$

## 7. NUMERICAL APPROXIMATION AND SOLUTION ALGORITHMS

From now on we will restrict ourselves to the approximation of optimization problems involving fish-passes.

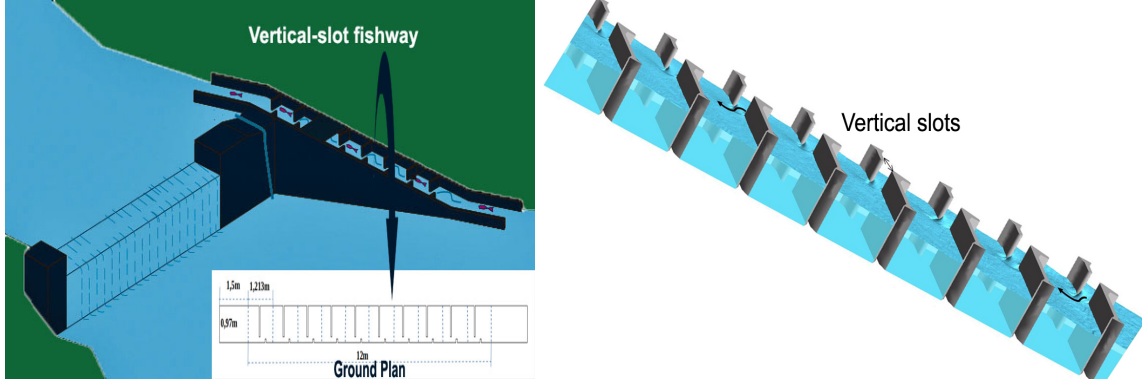


FIGURE 1. Schematic description of fishway structure

The geometry of the vertical slot based on the use of guide elements to lead smooth hydraulic flow into the next slot. The positioning of the guide elements was carried out at two different locations,  $a$  and  $b$ , which configure the shape of the fish ladder  $\Omega$  (Figure 2). The design variables  $a$  and  $b$  are subject to constraints in order to ensure a positive influence on the flow in the individual pools.

These constraints are formulated as

$$\begin{aligned} \frac{1}{4} 1.213 &\leq y_1, y_3 \leq \frac{3}{4} 1.213 \\ 0 &\leq y_2, y_4 \leq \frac{1}{4} 0.97 \end{aligned} \quad (7.1)$$

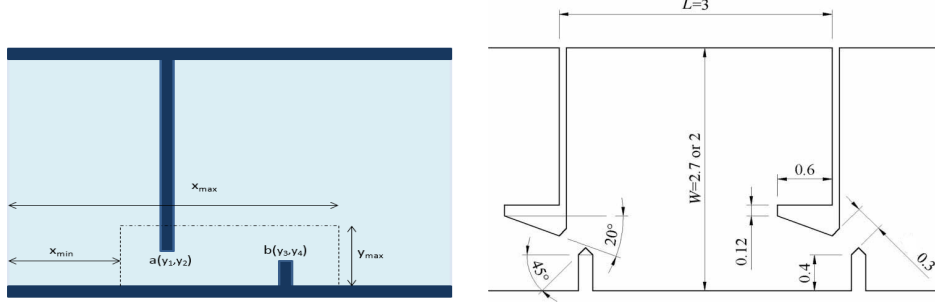


FIGURE 2. Fish pass geometry I shaped baffle (left) and L shaped baffle (right) for the central pool

In order to provide a comfort conditions during the fish passage and permit to a maximum number of fishes to pass to the river upstream, the following constraints are introduced

$$y_3 - y_1 \geq d_1 = 0.1 \quad (7.2)$$

$$y_2 - y_4 \geq d_2 = 0.05$$

**7.1. Spectral Projected Gradient.** We recall the approach of spectral projected gradient for solving the shape optimization problem. This method use the projection into a closed and convex  $\Omega_0 \subset \mathbb{R}^4$  defined as

$$\Omega_0 = \{y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4 : a_i \leq y_i \leq b_i, i = 1, \dots, 4, y_3 - y_1 \geq d_1, y_2 - y_4 \geq d_2\} \quad (7.3)$$

where  $a_1 = a_3 = \frac{1}{4}1.213$ ,  $b_1 = b_3 = \frac{3}{4}1.213$ ,  $a_2 = a_4 = 0$ ,  $b_2 = b_4 = \frac{1}{4}0.97$ . The optimization problem is then reduced to finding the positions of guidelines minimizing the cost function

$$\min_{y \in \Omega_0} J(y). \quad (7.4)$$

The optimal position  $y \in \Omega_0$  is found through the following algorithm using the computation of the cost function gradient and the projection functions.

- Begin with  $\bar{y} \in \Omega$  and  $\varepsilon > 0$  a positive tolerance.
- Compute the projection  $d = P_{\Omega_0}(\bar{y} - \eta \nabla J(\bar{y})) - \bar{y}$ , where  $\nabla J$  is the gradient of the cost function with  $\eta$  is a positive constant chosen according to particular conditions.
- Stop if  $d = 0$  (in practice,  $\|d\| < \varepsilon$ ).
- Define  $\tilde{y} = \bar{y} + \xi d$ , with  $\xi$  is the descent step. Go to (a) with  $\bar{y} = \tilde{y}$ .

The value of  $J$  in this algorithm is given by the ensuing discrete formula

$$\bar{\Phi}(y) = \frac{\Delta t}{2} \sum_{n=1}^N \sum_{e \in T_i} \left[ \int_e \|\mathbf{u}^{n,i} - \vec{v}\|^2 + \sigma \int_e |\text{curl}(\mathbf{u}^{n,i})|^2 \right]. \quad (7.5)$$

$y = P_{\Omega_0}(z)$  is the projection of  $z \in \mathbb{R}^4$  in  $\Omega$  onto  $\Omega_0$ . It is computed by minimizing a quadratic function of the distance of  $z$  to  $\Omega$  as follow

$$\min_{y \in \Omega} \frac{1}{2} \|y - z\|^2 = \min_{y \in \Omega} \frac{1}{2} z^T z - z^T y + \frac{1}{2} y^T y \quad (7.6)$$

which is equivalent to

$$\left\{ \begin{array}{l} \min_{(y_1, y_2, y_3, y_4)} \sum_{i=1}^4 \left\{ \frac{1}{2} z_i^2 - z_i y_i + \frac{1}{2} y_i^2 \right\} \\ \text{subject to } a_i \leq y_i \leq b_i, i = 1, \dots, 4 \\ y_3 - y_1 \geq d_1, \quad y_2 - y_4 \geq d_2 \end{array} \right.$$

We separate the previous quadratic problem into two quadratic problems the first one for the abscesses, the second for the ordinates and we write

$$\begin{array}{l} \min_{(y_1, y_3)} \quad \frac{1}{2}(z_1^2 + z_3^2) - (z_1 y_1 + z_3 y_3) + \frac{1}{2}(y_1^2 + y_3^2) \\ \text{subject to } \quad a_i \leq y_i \leq b_i, i = 1, 3 \\ y_3 - y_1 \geq d_1, \end{array}$$

and

$$\begin{array}{l} \min_{(y_2, y_4)} \quad \frac{1}{2}(z_2^2 + z_4^2) - (z_2 y_2 + z_4 y_4) + \frac{1}{2}(y_2^2 + y_4^2) \\ \text{subject to } \quad a_i \leq y_i \leq b_i, i = 2, 4 \\ y_2 - y_4 \geq d_2, \end{array}$$

A Karush-Kuhn-Tucker (KKT) technique is used to solve the two quadratic problems above. We begin by the first one and we choose  $a = a_1 = a_3$ , and  $b = b_1 = b_3$ . We kept only three constraints and write the optimization problem to be solved as

$$\begin{array}{l} \min_{(x_1, x_2)} \quad c - (l_1 x_1 + l_2 x_2) + \frac{1}{2}(x_1^2 + x_2^2) \\ \text{subject to } \quad x_2 \leq b, \\ x_1 \geq a, \\ x_2 - x_1 \geq d_1, \end{array}$$

where  $a, b, d_1, c$ , and  $l_i, i=1,2$ , are real numbers.

We choose the following matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -b \\ a \\ d_1 \end{pmatrix},$$

Then the linear constraints write

$$Ax - v = b, \quad v \geq 0,$$

where  $v = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)^T$  is the slack variables. The optimization problem has an unique solution due to the convexity of the cost function in  $\mathbb{R}^2$ . This solution satisfies the KKT conditions

$$\begin{array}{l} v = Ax - b \\ -l + x = A^T \lambda, \\ v \geq 0, \quad \lambda \geq 0, \\ v^T \lambda = 0, \end{array}$$

where  $l = (l_1, l_2, l_3)$ , and  $\lambda = (\lambda_1, \lambda_2, \lambda_3)$  is the vector of the Lagrange multipliers associated to the three constraints above. The unique optimal solution writes

$$x = l + A^T \lambda$$

which gives

$$\begin{cases} x_1 = l_1 + \lambda_2 - \lambda_3 \\ x_2 = l_2 - \lambda_1 + \lambda_3 \end{cases}$$

For computing the Lagrange multipliers  $\lambda_i$ , we solve the following linear complementarity problem

$$\begin{aligned} v &= (-b + Al) + AA^T \lambda \\ v &\geq 0, \quad \lambda \geq 0 \\ v^T \lambda &= 0 \end{aligned}$$

**7.2. Numerical results.** In this subsection we present some results about the optimization of a fishway using finite element method investigated in the previous sections to solve the state problem combined with a projected gradient algorithm for the resolution of the optimization problem. The numerical implementation is carried out in the finite element package FreeFem++[8], using the sparse linear solver for the linear systems. The computational grids are uniform, unstructured.  $\mathbb{P}_2$ - $\mathbb{P}_1$  finite elements have been used for the spatial discretization.

The domain fluid is considered as a rectangular channel consisting of ten pools. Inside each pool, two baffles in form of I or L are built and they are perpendicular to the lateral boundary. The length and width of baffles vary according to the nature of the studied fishway structures. In figure 3, we give a description of the fishway structure.



FIGURE 3. ground plan  $\Omega$ : each pool is designed by dashed lines

We conduct two 3-D tests of fishway design mostly used for the modeling of fishway passage. Firstly, we begin by a 3-D channel with I shaped baffle (Figure 2 (left)). Next, we consider a L shaped baffle design (Figure 2 (right)).

The numerical simulation for the two tests is done with the same physical and numerical conditions. All initial and boundary conditions are taken constants. the inflow velocity is  $(0.1, 0, 0)$ , the target velocity is  $\mathbf{u}_d = (0.8, 0, 0)$  to be as close as possible to a typical horizontal velocity suitable for fish leaping and swimming capabilities. The vorticity parameter is  $\sigma = 0.3$ . The viscosity is  $\nu = 0.01$ . The exterior function reflecting the gravity effect is  $\mathbf{f} = (0, 0, 9.81)$ . The non-slip boundary condition was applied to all walls (bottom, lateral walls and baffles).

**7.2.1. Slot Fishway (3-D "I" shaped baffle-rectangular slots).** The state penalized problem (3.1)-(3.2) is solved by the finite element method (6.36) combined with a spectral projected gradient algorithm for the resolution of the shape optimization problem (5.3).

This test consists to consider a 3-D geometry of pools with two rectangular baffles as shown in the Figure 4.

The coming results gives a description of the velocity for ten pools related to the natures of the studied shape of fishway structures.

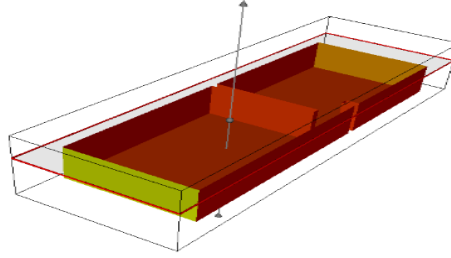
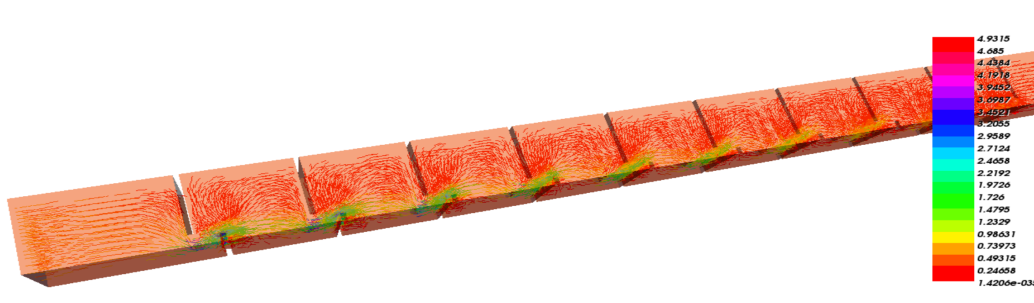
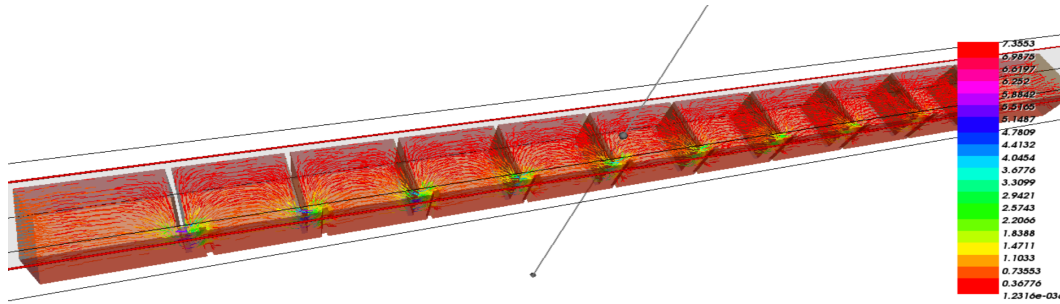


FIGURE 4. Configuration of central pool

FIGURE 5. Initial and random shape and corresponding velocity for ten pools ( $T=20$ )FIGURE 6. Optimal shape and corresponding velocity for ten pools ( $T=20$ )

We observe that the recirculation areas near of slots are removed for the optimal shape (Figure 6) compared to the initial random structure (Figure 5). The velocity is almost null in the areas near from the boundaries behind slots for the optimal shape which provides a rest zones for fish before to begin the passage from the current pool to the next one. These results appears clearly in the following examples for the central pool (Figure 7 (right))

**7.2.2. Slot Fishway (3-D "L" shaped baffle-oblique slots).** In this subsection we give results of numerical simulations in 3-D dimension with a L shape baffle. We consider a channel of ten pools. Each pool has two vertical slots. The small one is oblique and the longest is in the form of L and they are vertical to the lateral boundary. The different values corresponding to the geometry of pools are shown in the figure 2) (left). An example of the first pool geometry is given in the Figure 8.

The following results represent the velocity for ten pools in the initial random shape (Figure 9) and the optimal shape (Figure 10). The nature of the studied structure takes part of the minimization of turbulence by the use of the form L of baffles which are considered as a break waves of water.

The velocity is almost close from the uniform target velocity  $\mathbf{u}_d$  for the optimal shape (Figure 10).

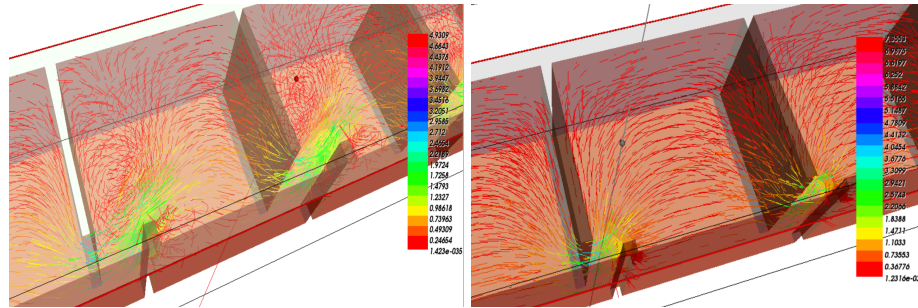


FIGURE 7. Initial and random shape (left) Optimal shape (right) and corresponding velocity for the central pool

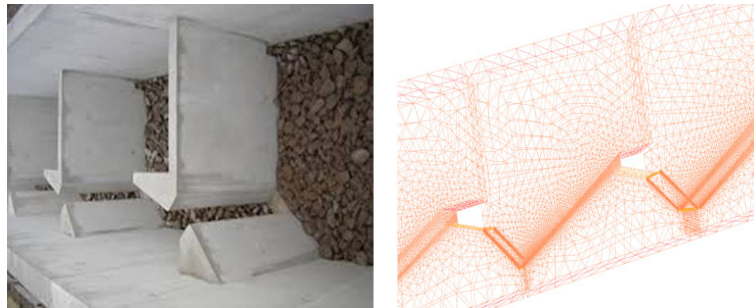


FIGURE 8. 3-D example of fishway with porous medium (left), Mesh of the central pool (right)

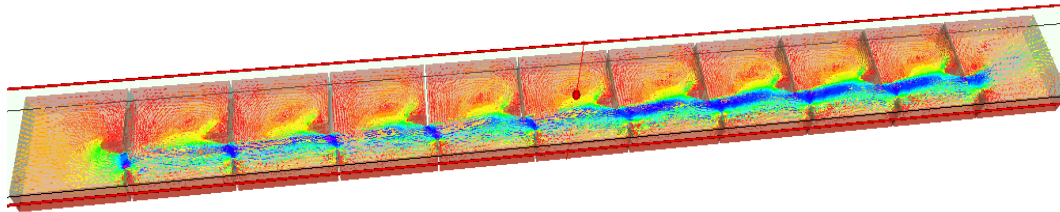


FIGURE 9. Non optimal shape and corresponding velocity for ten pools ( $T=20$ )

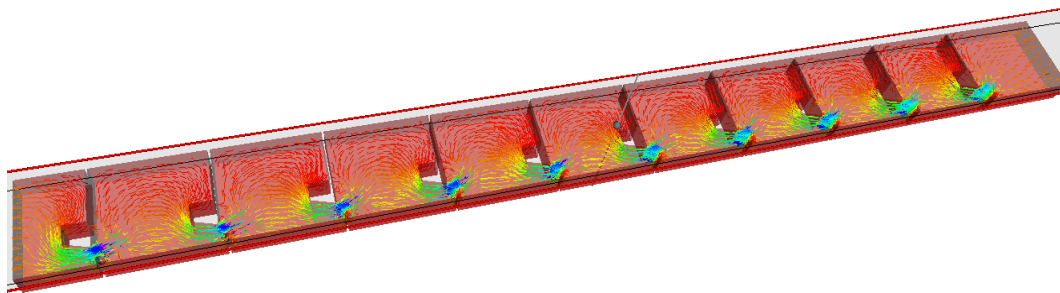


FIGURE 10. Optimal shape and corresponding velocity for ten pools ( $T=20$ )

We give numerical results for the central pool in the ensuing figures. Turbulence appears clearly in the areas near from slots in the figure 11 (left). For the optimal shape (Figure 11(right)), the nature of velocity permits a suitable passage for fish and provides for them a rest in the zones where the velocity is

almost null.

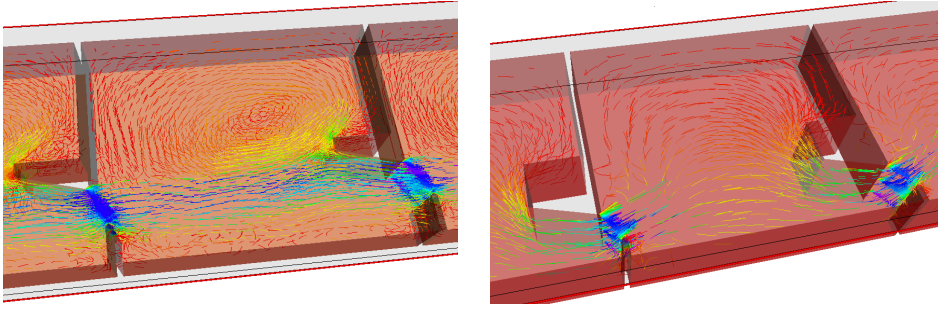


FIGURE 11. Initial and random shape (left) Optimal shape (right) and corresponding velocity for the central pool

Figure 12 represents the fast convergence of the cost functional for the various Forchheimer coefficients for the L shaped baffle design.

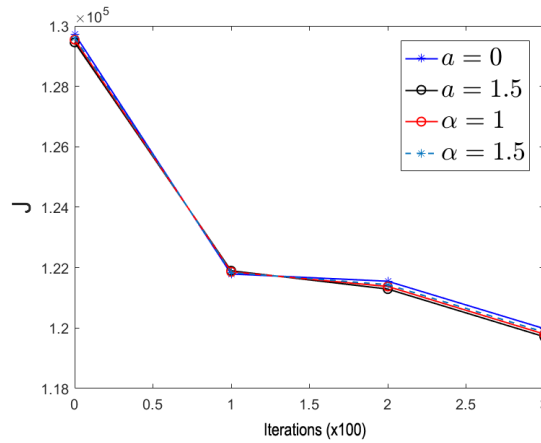


FIGURE 12. Convergence history

## 8. CONCLUSION

The problem of the optimal shape design of vertical slot fishways has been addressed. For this purpose, a complete penalty method has been introduced and different properties were derived. A finite element method has been used to discretize the fluid domain and obtain an error estimate for the velocity and the pressure in term of  $H^1$  and  $L^2$  norms respectively. The suitability and computational efficiency of the proposed optimization methodology has been successfully demonstrated for 3-D fishways structures.

### APPENDIX A. CONTROL PROBLEM OF MIXED BOUNDARY CONDITIONS

For some numerical simulations and in the case where the both Dirichlet-type and Neumann-type boundary conditions are considered, we give the description of the shape derivative. We express the shape gradients of the cost functionals by introducing the corresponding linear adjoint state systems.

A.1. **Adjoint equation.** In this section we derive an adjoint equation related to the equation

$$\left\{ \begin{array}{ll} \mathbf{u}_t^\varepsilon - \nu \Delta \mathbf{u}^\varepsilon + (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{u}^\varepsilon + \frac{1}{2} (\nabla \cdot \mathbf{u}^\varepsilon) \mathbf{u}^\varepsilon + a |\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon + \nabla p^\varepsilon = \mathbf{f} & \text{in } \Omega \times [0, T], \\ \nabla \cdot \mathbf{u}^\varepsilon + \varepsilon p^\varepsilon = 0 & \text{in } \Omega \times [0, T], \\ \mathbf{u}^\varepsilon = 0 & \text{on } \Gamma_1 \times [0, T], \\ \nu \frac{\partial \mathbf{u}^\varepsilon}{\partial \mathbf{n}} - p^\varepsilon \mathbf{n} = 0 & \text{on } \Gamma_2 \times [0, T], \\ \mathbf{u}^\varepsilon|_{t=0} = \mathbf{u}_0 & \text{in } \Omega. \end{array} \right. \quad (\text{A.1})$$

where  $\partial\Omega = \Gamma_1 \cup \Gamma_2$ . We keep the same definition of the cost function (5.2). The optimization problem can be expressed as

$$\min_{\Omega \in Q_{ad}} J(\Omega) = \frac{1}{2} \int_0^T \int_\Omega |\mathbf{u}^\varepsilon - \mathbf{u}_d|^2 dt dx + \frac{\sigma}{2} \int_0^T \int_\Omega |\text{curl}(\mathbf{u}^\varepsilon)|^2 dt dx \quad (\text{A.2})$$

*such that  $(\mathbf{u}^\varepsilon, p^\varepsilon)$  is solution of (A.1)*

**Theorem A.1.** *Let  $(\mathbf{u}^\varepsilon, p^\varepsilon) \in H^2(0, T; \Omega) \times Y$  be solution of the penalized problem (A.1). Then the adjoint equation associated to the equation (A.1) takes the form*

$$\left\{ \begin{array}{ll} -\mathbf{v}_t - \nu \Delta \mathbf{v} + (\nabla \mathbf{u}^\varepsilon)^T \cdot \mathbf{v} - (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{v} - \frac{1}{2} \nabla (\mathbf{u}^\varepsilon \cdot \mathbf{v}) + \frac{1}{2} (\nabla \cdot \mathbf{u}^\varepsilon) \mathbf{v} + a |\mathbf{u}^\varepsilon|^{\alpha-2} (|\mathbf{u}^\varepsilon|^2 \mathbf{v} + \alpha (\mathbf{u}^\varepsilon \cdot \mathbf{v}) \mathbf{u}^\varepsilon) \\ \quad + \nabla q = (\mathbf{u}^\varepsilon - \mathbf{u}_d) - \sigma \vec{\text{curl}}(\text{curl}(\mathbf{u}^\varepsilon)) & \text{in } \Omega \times [0, T] \\ \nabla \cdot \mathbf{v} + \varepsilon q = 0 & \text{in } \Omega \times [0, T] \\ \mathbf{v} = 0 & \text{on } \Gamma_1 \times [0, T] \\ \nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} + 2(\mathbf{u}^\varepsilon \cdot \mathbf{n}) \mathbf{v} - nq = \sigma \text{curl}(\mathbf{u}^\varepsilon) \cdot \boldsymbol{\tau} & \text{on } \Gamma_2 \times [0, T] \\ \mathbf{v}(T) = 0 & \end{array} \right. \quad (\text{A.3})$$

*Proof.* We define also the following function

$$F(\Omega, \mathbf{u}^\varepsilon, p^\varepsilon, \mathbf{v}, q) = \int_0^T \int_\Omega \left( \mathbf{u}_t^\varepsilon + \nu \nabla \mathbf{u}^\varepsilon : \nabla \mathbf{v} + (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{u}^\varepsilon \cdot \mathbf{v} + \frac{1}{2} (\nabla \cdot \mathbf{u}^\varepsilon) \mathbf{u}^\varepsilon \cdot \mathbf{v} + a |\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon \cdot \mathbf{v} - f \cdot \mathbf{v} \right) dt dx \\ - \int_0^T \int_\Omega \left( p^\varepsilon \nabla \cdot \mathbf{v} - q \nabla \cdot \mathbf{u}^\varepsilon - \varepsilon q p^\varepsilon \right) dt dx.$$

Let  $L(\Omega, \mathbf{u}^\varepsilon, p^\varepsilon, \mathbf{v}, q)$  be a Lagrangian functional defined by

$$L(\Omega, \mathbf{u}^\varepsilon, p^\varepsilon, \mathbf{v}, q) = J(\Omega) - F(\Omega, \mathbf{u}^\varepsilon, p^\varepsilon, \mathbf{v}, q), \quad (\text{A.4})$$

we first derive  $L$  with respect to the state variable  $p$  in any direction  $\tilde{p} \in M(\Omega)$ , we obtain

$$\frac{\partial L}{\partial p}(\Omega, \mathbf{u}^\varepsilon, p^\varepsilon, \mathbf{v}, q) \cdot \tilde{p} = \int_0^T \int_\Omega (\tilde{p} \nabla \cdot \mathbf{v} + \varepsilon \tilde{p} q) dt dx,$$

the variation  $\tilde{p}$  is arbitrary, we get

$$\nabla \cdot \mathbf{v} + \varepsilon q = 0 \quad \text{on } \Omega. \quad (\text{A.5})$$

Next, we derive  $L$  with respect of the state variable  $\mathbf{u}^\varepsilon$  in the arbitrary direction  $\tilde{\mathbf{u}} \in V_0(\Omega)$  we have

$$\begin{aligned}
0 &= \frac{\partial L}{\partial \mathbf{u}^\varepsilon}(\Omega, \mathbf{u}^\varepsilon, p^\varepsilon, \mathbf{v}, q) \cdot \tilde{\mathbf{u}}, \\
&= \int_0^T \int_\Omega \left( (\mathbf{u}^\varepsilon - \mathbf{u}_d) \cdot \tilde{\mathbf{u}} \sigma \operatorname{curl}(\mathbf{u}^\varepsilon) \cdot \operatorname{curl}(\tilde{\mathbf{u}}) \right) dt dx \\
&\quad - \int_0^T \int_\Omega \left( -\mathbf{v}_t \tilde{\mathbf{u}} + \nu \nabla \tilde{\mathbf{u}} : \nabla \mathbf{v} + (\tilde{\mathbf{u}} \cdot \nabla) \mathbf{u}^\varepsilon \cdot \mathbf{v} + (\mathbf{u}^\varepsilon \cdot \nabla) \tilde{\mathbf{u}} \cdot \mathbf{v} + \frac{1}{2} (\nabla \cdot \tilde{\mathbf{u}}) \mathbf{u}^\varepsilon \cdot \mathbf{v} + (\nabla \cdot \mathbf{u}^\varepsilon) \tilde{\mathbf{u}} \cdot \mathbf{v} \right) dt dx \\
&\quad + \int_0^T \int_\Omega \left( q \nabla \cdot \tilde{\mathbf{u}} - (a |\mathbf{u}^\varepsilon|^\alpha \tilde{\mathbf{u}} + a \alpha |\mathbf{u}^\varepsilon|^{\alpha-2} (\mathbf{u}^\varepsilon \cdot \tilde{\mathbf{u}}) \mathbf{u}^\varepsilon) \cdot \mathbf{v} \right) dt dx, \\
&= \int_0^T \int_\Omega \left( \mathbf{u}^\varepsilon - \mathbf{u}_d \right) - \sigma \vec{\operatorname{curl}}(\operatorname{curl}(\mathbf{u}^\varepsilon)) + \mathbf{v}_t + \nu \Delta \mathbf{u}^\varepsilon - (\nabla \mathbf{u}^\varepsilon)^T \cdot \mathbf{v} + (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{v} \\
&\quad + \frac{1}{2} \nabla (\mathbf{u}^\varepsilon \cdot \mathbf{v}) - \frac{1}{2} (\nabla \cdot \mathbf{u}^\varepsilon) \mathbf{v} - (a |\mathbf{u}^\varepsilon|^\alpha \mathbf{v} + a \alpha |\mathbf{u}^\varepsilon|^{\alpha-2} (\mathbf{u}^\varepsilon \cdot \mathbf{v}) \mathbf{u}^\varepsilon - \nabla q) \cdot \tilde{\mathbf{u}} dt dx \\
&\quad + \int_0^T \int_{\Gamma_2} \left( \sigma \operatorname{curl}(\mathbf{u}^\varepsilon) \tau - \left( \nu \frac{\partial \mathbf{v}}{\partial n} + 2(\mathbf{u}^\varepsilon \cdot n) \mathbf{v} - n q \right) \right) \cdot \tilde{\mathbf{u}} dt dx + \int_\Omega [\mathbf{v} \cdot \tilde{\mathbf{u}}]_0^T dx.
\end{aligned}$$

Considering an arbitrary direction  $\tilde{\mathbf{u}}$  which vanishes in neighborhood of the boundary  $\Gamma_2$ , we obtain

$$-\mathbf{v}_t - \nu \Delta \mathbf{v} + (\nabla \mathbf{u}^\varepsilon)^T \cdot \mathbf{v} - (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{v} - \frac{1}{2} \nabla (\mathbf{u}^\varepsilon \cdot \mathbf{v}) + \frac{1}{2} (\nabla \cdot \mathbf{u}^\varepsilon) \mathbf{v} + a |\mathbf{u}^\varepsilon|^{\alpha-2} (|\mathbf{u}^\varepsilon|^2 \mathbf{v} + \alpha (\mathbf{u}^\varepsilon \cdot \mathbf{v}) \mathbf{u}^\varepsilon) + \nabla q = (\mathbf{u}^\varepsilon - \mathbf{u}_d) - \sigma \vec{\operatorname{curl}}(\operatorname{curl}(\mathbf{u}^\varepsilon)),$$

and an arbitrary  $\tilde{\mathbf{u}}$  in  $\Gamma_2$  gives

$$\nu \frac{\partial \mathbf{v}}{\partial n} + 2(\mathbf{u} \cdot n) \mathbf{v} - n q = \sigma \operatorname{curl}(\mathbf{u}^\varepsilon) \cdot \tau \quad \text{on } \Gamma_2$$

Finally we obtain the adjoint equation

$$\left\{ \begin{array}{ll}
-\mathbf{v}_t - \nu \Delta \mathbf{v} + (\nabla \mathbf{u}^\varepsilon)^T \cdot \mathbf{v} - (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{v} - \frac{1}{2} \nabla (\mathbf{u}^\varepsilon \cdot \mathbf{v}) + \frac{1}{2} (\nabla \cdot \mathbf{u}^\varepsilon) \mathbf{v} + a |\mathbf{u}^\varepsilon|^{\alpha-2} (|\mathbf{u}^\varepsilon|^2 \mathbf{v} + \alpha (\mathbf{u}^\varepsilon \cdot \mathbf{v}) \mathbf{u}^\varepsilon) \\
\quad + \nabla q = (\mathbf{u}^\varepsilon - \mathbf{u}_d) - \sigma \vec{\operatorname{curl}}(\operatorname{curl}(\mathbf{u}^\varepsilon)) & \text{in } \Omega \times [0, T] \\
\nabla \cdot \mathbf{v} + \varepsilon q = 0 & \text{in } \Omega \times [0, T] \\
\mathbf{v} = 0 & \text{on } \Gamma_1 \times [0, T] \\
\nu \frac{\partial \mathbf{v}}{\partial n} + 2(\mathbf{u}^\varepsilon \cdot n) \mathbf{v} - n q = \sigma \operatorname{curl}(\mathbf{u}^\varepsilon) \cdot \tau & \text{on } \Gamma_2 \times [0, T] \\
\mathbf{v}(T) = 0 &
\end{array} \right. \quad (\text{A.6})$$

□

**A.2. Shape gradient.** In this subsection, we will try to express the shape gradient using the velocity method (see C ea [1]), via the state and adjoint problems.

Let us find an expression of the derivative of the saddle point problem  $j(t)$  with respect to  $\mathbf{t}$  where

$$j(t) = \min_{(\mathbf{u}^\varepsilon, p^\varepsilon) \in V_g(\Omega_t) \times \hat{Y}(\Omega_t)} \max_{(\mathbf{v}, q) \in V_0(\Omega_t) \times \hat{Y}(\Omega_t)} L(\Omega_t, \mathbf{u}^\varepsilon, p^\varepsilon, \mathbf{v}, q)$$

with  $(\mathbf{u}, p)$  and  $(\mathbf{v}, q)$  are solutions of (1.1) and (A.3) in the perturbed domain  $\Omega_t$ , respectively.

We consider the Hilbert spaces which depend on the parameter  $t$  defined by

$$V_0(\Omega_t) = \{ \mathbf{u}^\varepsilon \circ T_t^{-1} : \mathbf{u}^\varepsilon \in V_0(\Omega) \}$$

$$\hat{Y}(\Omega_t) = \{ p^\varepsilon \circ T_t^{-1} : p^\varepsilon \in \hat{Y}(\Omega) \}$$

since  $T_t$  and  $T_t^{-1}$  are diffeomorphisms, the parametrisation do not influence  $j(t)$ . And we have

$$j(t) = \min_{(\mathbf{u}^\varepsilon, p^\varepsilon) \in V_g(\Omega) \times \hat{Y}(\Omega)} \max_{(\mathbf{v}, q) \in V_0(\Omega) \times \hat{Y}(\Omega)} L(\Omega_t, \mathbf{u}^\varepsilon \circ T_t^{-1}, p^\varepsilon \circ T_t^{-1}, \mathbf{v} \circ T_t^{-1}, q \circ T_t^{-1})$$

we define the following functions which depend on the parameter  $t$

$$\begin{aligned}
l_1(t) &= \frac{1}{2} \int_0^T \int_{\Omega_t} |\mathbf{u}^\varepsilon \circ T_t^{-1} - \mathbf{u}_d|^2 dx + \frac{\sigma}{2} \int_0^T \int_{\Omega_t} |\operatorname{curl}(\mathbf{u}^\varepsilon \circ T_t^{-1})|^2 dt dx, \\
l_2(t) &= \int_0^T \int_{\Omega_t} [\nu \nabla(\mathbf{u}^\varepsilon \circ T_t^{-1}) : \nabla(\mathbf{v} \circ T_t^{-1}) + ((\mathbf{u}^\varepsilon \circ T_t^{-1}) \cdot \nabla)(\mathbf{u}^\varepsilon \circ T_t^{-1}) \cdot (\mathbf{v} \circ T_t^{-1}) \\
&\quad + a |\mathbf{u}^\varepsilon \circ T_t^{-1}|^\alpha (\mathbf{u}^\varepsilon \circ T_t^{-1}) \cdot (\mathbf{v} \circ T_t^{-1}) - (p^\varepsilon \circ T_t^{-1}) \nabla \cdot (\mathbf{v} \circ T_t^{-1}) - (f \circ T_t^{-1}) \cdot (\mathbf{v} \circ T_t^{-1})] dt dx \\
&\quad - \int_0^T \int_{\Omega_t} (q \circ T_t^{-1}) \nabla \cdot (\mathbf{u}^\varepsilon \circ T_t^{-1}) dt dx - \varepsilon \int_0^T \int_{\Omega_t} (q \circ T_t^{-1}) (p^\varepsilon \circ T_t^{-1}) dt dx
\end{aligned}$$

The Lagrangian functional writes

$$L(\Omega_t, \mathbf{u}^\varepsilon \circ T_t^{-1}, p^\varepsilon \circ T_t^{-1}, \mathbf{v} \circ T_t^{-1}, q \circ T_t^{-1}) = l_1(t) - l_2(t)$$

We consider  $\mathbf{V}(0, X) \in V_{ad}$ , and observe that  $\mathbf{V}(0, X) = \mathbf{V}$ . Therefore we can derive the shape gradient using the formula (5.8)

$$\frac{d}{dt} L(\Omega_t, \mathbf{u}^\varepsilon \circ T_t^{-1}, p^\varepsilon \circ T_t^{-1}, \mathbf{v} \circ T_t^{-1}, q \circ T_t^{-1})|_{t=0} = l'_1(0) - l'_2(0) \quad (\text{A.7})$$

where

$$\begin{aligned}
l'_1(0) &= \int_0^T \int_{\Omega} (\mathbf{u}^\varepsilon - \mathbf{u}_d) \cdot (-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) dx + \sigma \int_0^T \int_{\Omega} \operatorname{curl}(\mathbf{u}^\varepsilon) \operatorname{curl}(-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) dx \\
&\quad + \frac{1}{2} \int_0^T \int_{\Gamma_1} (|\mathbf{u}^\varepsilon - \mathbf{u}_d|^2 \mathbf{V} \cdot \mathbf{n}) ds + \frac{\sigma}{2} \int_0^T \int_{\Gamma_1} (|\operatorname{curl}(\mathbf{u}^\varepsilon)|^2 \mathbf{V} \cdot \mathbf{n}) ds
\end{aligned} \quad (\text{A.8})$$

$$\begin{aligned}
l'_2(0) &= \int_0^T \int_{\Omega} [(-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V})_t v + (-\nabla \mathbf{v} \cdot \mathbf{V})_t \mathbf{u}^\varepsilon + \nu \nabla(-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) : \nabla \mathbf{v} + \nu \nabla \mathbf{u}^\varepsilon : \nabla(-\nabla \mathbf{v} \cdot \mathbf{V}) \\
&\quad + ((-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) \cdot \nabla \mathbf{u}^\varepsilon) \cdot \mathbf{v} + (\mathbf{u}^\varepsilon \cdot \nabla(-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V})) \cdot \mathbf{v} + (\mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon) \cdot (-\nabla \mathbf{v} \cdot \mathbf{V}) \\
&\quad + \frac{1}{2} \nabla \cdot (-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) \mathbf{u}^\varepsilon \cdot \mathbf{v} + \frac{1}{2} \nabla \cdot \mathbf{u}^\varepsilon (-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) \cdot \mathbf{v} + \frac{1}{2} \nabla \cdot \mathbf{u}^\varepsilon \mathbf{u}^\varepsilon \cdot (-\nabla \mathbf{v} \cdot \mathbf{V}) \\
&\quad + a \alpha |\mathbf{u}^\varepsilon|^{\alpha-2} (-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) \cdot \mathbf{u}^\varepsilon (\mathbf{u}^\varepsilon \cdot \mathbf{v}) + a |\mathbf{u}^\varepsilon|^\alpha (-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) \cdot \mathbf{v} \\
&\quad + a |\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon \cdot (-\nabla \mathbf{v} \cdot \mathbf{V}) - (f \cdot \mathbf{V}) \cdot v - f \cdot ((-\nabla v) \cdot \mathbf{V}) - (\nabla p^\varepsilon \cdot \mathbf{V}) \nabla \cdot \mathbf{v} \\
&\quad - p^\varepsilon \nabla \cdot (-\nabla \mathbf{v} \cdot \mathbf{V}) - q \nabla \cdot (-\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) - (\nabla \cdot \mathbf{u}^\varepsilon) (-\nabla q \cdot \mathbf{V})] dx
\end{aligned}$$

Using Green's formula and the condition  $\mathbf{u}^\varepsilon = 0$  on  $\Gamma_1$ , we obtain

$$\begin{aligned}
l'_2(0) = & - \int_0^T \int_\Omega [(\mathbf{u}_t^\varepsilon - \nu \Delta \mathbf{u}^\varepsilon + (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{u}^\varepsilon + \frac{1}{2} \nabla \cdot \mathbf{u}^\varepsilon \mathbf{u}^\varepsilon + a |\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon + \nabla p^\varepsilon - f) \cdot (\nabla \mathbf{v} \cdot \mathbf{V})] dt dx \\
& + \int_0^T \int_\Omega (\nabla \cdot \mathbf{u}^\varepsilon) (\nabla q \cdot \mathbf{V}) dt dx + \int_0^T \int_\Omega (\nabla \cdot \mathbf{v}) (\nabla p \cdot \mathbf{V}) dt dx + \int_0^T \int_\Omega [-\mathbf{v}_t - \nu \Delta \mathbf{v} + (\nabla \mathbf{u}^\varepsilon)^T \cdot \mathbf{v} \\
& - (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{v} - \frac{1}{2} \nabla (\mathbf{u}^\varepsilon \cdot \mathbf{v}) + \frac{1}{2} (\nabla \cdot \mathbf{u}^\varepsilon) \mathbf{v} + a |\mathbf{u}^\varepsilon|^{\alpha-2} (|\mathbf{u}^\varepsilon|^2 \mathbf{v} + \alpha (\mathbf{u}^\varepsilon \cdot \mathbf{v}) \mathbf{u}^\varepsilon) - \nabla q] \cdot (\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) dt dx \\
& - \int_0^T \int_{\Gamma_1} [\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} + 2(\mathbf{u}^\varepsilon \cdot \mathbf{n}) \mathbf{v} - nq] \cdot (\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) dt ds - \int_0^T \int_{\Gamma_1} [\frac{\partial \mathbf{u}^\varepsilon}{\partial \mathbf{n}} - np^\varepsilon] \cdot (\nabla \mathbf{v} \cdot \mathbf{V}) dt ds \\
& + \int_0^T \int_{\Gamma_1} [\nu \nabla \mathbf{u}^\varepsilon : \nabla \mathbf{v} + (\mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon) \cdot \mathbf{v} + a |\mathbf{u}^\varepsilon|^\alpha \mathbf{u}^\varepsilon \cdot \mathbf{v} - p^\varepsilon \nabla \cdot \mathbf{v} - q \nabla \cdot \mathbf{u}^\varepsilon] \mathbf{V} \cdot \mathbf{n} dt ds
\end{aligned} \tag{A.9}$$

Replacing (A.8) and (A.9) in (A.7) and using the fact that  $(\mathbf{u}^\varepsilon, p^\varepsilon)$  is solution of (1.2) and  $(\mathbf{v}, q)$  is solution of (A.3) respectively yields

$$\begin{aligned}
dJ(\Omega; \mathbf{V}) &= \frac{d}{dt} L|_{t=0} \\
&= \frac{1}{2} \int_0^T \int_{\Gamma_1} (|\mathbf{u}^\varepsilon - \mathbf{u}_d|^2 + \sigma |\text{curl}(\mathbf{u}^\varepsilon)|^2) \mathbf{V} \cdot \mathbf{n} ds + \int_0^T \int_{\Gamma_1} [\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - nq] \cdot (\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) dt ds \\
&\quad + \int_0^T \int_{\Gamma_1} [\nu \frac{\partial \mathbf{u}^\varepsilon}{\partial \mathbf{n}} - np^\varepsilon] \cdot (\nabla \mathbf{v} \cdot \mathbf{V}) dt ds - \int_0^T \int_{\Gamma_1} [(\nu \nabla \mathbf{u}^\varepsilon : \nabla \mathbf{v}) \mathbf{V} \cdot \mathbf{n} dt ds \\
&\quad - \sigma \int_0^T \int_{\Gamma_1} (\frac{\partial \mathbf{u}^\varepsilon}{\partial \mathbf{n}} \cdot (\text{curl}(\mathbf{u}^\varepsilon) \wedge \mathbf{n})) \mathbf{V} \cdot \mathbf{n} dt ds
\end{aligned} \tag{A.10}$$

Noting  $\mathbf{u}^\varepsilon = 0$  and  $\mathbf{v} = 0$  in  $\Gamma_1$  to have

$$\mathbf{n} \cdot (\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) = \nabla \mathbf{u}^\varepsilon \cdot (\mathbf{n} \otimes \mathbf{n}) \cdot \mathbf{V} \cdot \mathbf{n} = \nabla \mathbf{u}^\varepsilon \cdot \mathbf{n} \cdot \mathbf{n} (\mathbf{V} \cdot \mathbf{n}) = (\nabla \cdot \mathbf{u}^\varepsilon) (\mathbf{V} \cdot \mathbf{n}) = 0, \quad \forall x \in \Gamma_1 \tag{A.11}$$

$$\frac{\partial \mathbf{v}}{\partial \mathbf{n}} \cdot (\nabla \mathbf{u}^\varepsilon \cdot \mathbf{V}) = \nabla \mathbf{u}^\varepsilon \cdot (\mathbf{n} \otimes \mathbf{n}) \cdot \mathbf{V} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{n}} = \frac{\partial \mathbf{u}^\varepsilon}{\partial \mathbf{n}} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{n}} (\mathbf{V} \cdot \mathbf{n}) = (\nabla \mathbf{u}^\varepsilon : \nabla \mathbf{v}) \mathbf{V} \cdot \mathbf{n} \tag{A.12}$$

Similarly we obtain

$$\mathbf{n} \cdot (\nabla \mathbf{v} \cdot \mathbf{V}) = 0, \quad \frac{\partial \mathbf{u}^\varepsilon}{\partial \mathbf{n}} \cdot (\nabla \mathbf{v} \cdot \mathbf{V}) = \frac{\partial \mathbf{u}^\varepsilon}{\partial \mathbf{n}} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{n}} (\mathbf{V} \cdot \mathbf{n}) = (\nabla \mathbf{u}^\varepsilon : \nabla \mathbf{v}) \mathbf{V} \cdot \mathbf{n} \tag{A.13}$$

Substituting (A.11), (A.12) and (A.13) in (5.13) the shape derivative takes the form

$$dJ(\Omega; \mathbf{V}) = \int_0^T \int_{\Gamma_1} \left[ \frac{1}{2} |\mathbf{u}^\varepsilon - \mathbf{u}_d|^2 + \frac{\sigma}{2} |\text{curl}(\mathbf{u}^\varepsilon)|^2 + \frac{\partial \mathbf{u}^\varepsilon}{\partial \mathbf{n}} \cdot \left( 2\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - \sigma \text{curl}(\mathbf{u}^\varepsilon) \wedge \mathbf{n} \right) \right] \mathbf{V} \cdot \mathbf{n} dt ds \tag{A.14}$$

Consequently, the shape gradient writes

$$\nabla J = \left[ \frac{1}{2} |\mathbf{u}^\varepsilon - \mathbf{u}_d|^2 + \frac{\sigma}{2} |\text{curl}(\mathbf{u}^\varepsilon)|^2 + \frac{\partial \mathbf{u}^\varepsilon}{\partial \mathbf{n}} \cdot \left( 2\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - \sigma \text{curl}(\mathbf{u}^\varepsilon) \wedge \mathbf{n} \right) \right] \mathbf{n} \tag{A.15}$$

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