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Multi-quadratic p -rational Number Fields

Y. Benmerieme and A. Movahhedi

ABSTRACT

For each odd prime p , we prove the existence of infinitely many real quadratic fields which are p -rational. Explicit imaginary and real bi-quadratic p -rational fields are also given for each prime p . Using a recent method developed by Greenberg, we deduce the existence of Galois extensions of \mathbf{Q} with Galois group isomorphic to an open subgroup of $GL_n(\mathbf{Z}_p)$, for $n = 4$ and $n = 5$ and at least for all the primes $p < 192.699.943$.

Keywords: p -rational number field, Galois representation

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1. Introduction

Let F be a number field and p be a prime number. Denote by S the set of primes lying above p and by F_S the maximal p -extension of F which is unramified outside the primes above S . The structure of the Galois group $G(F_S/F)$ reflects the arithmetic properties of the base field F and it has long been studied. The number field F is called p -rational if the Galois group $G(F_S/F)$ is a free pro- p -group. The terminology comes from the fact that p -rational fields behave in the same way as the field of rational numbers \mathbf{Q} , with respect to the prime p . Recently, p -rational fields have been revisited by Greenberg [14] to study some representations of the absolute Galois group $G_{\mathbf{Q}}$ of \mathbf{Q} with large image by a new fundamental (non-geometric, but essentially p -adic) method. This work has been continued by Cornut and Ray [10]. In doing so, Greenberg proposed a new conjecture according to which for any odd prime p , there exist arbitrary large multi-quadratic p -rational number fields. If this conjecture holds true for a prime p with complex multi-quadratic fields, then there exists continuous representations $G_{\mathbf{Q}} \rightarrow GL_n(\mathbf{Z}_p)$ on the general linear group of the ring of p -adic integers \mathbf{Z}_p with open image for all degrees $n \geq 4$. According to Katz, these representations are “spectacularly non-motivic” [18, Introduction]. For some (motivic) representations $\rho : G_{\mathbf{Q}} \rightarrow GL_n(\mathbf{Q}_p)$ on general linear groups of the field of p -adic numbers with open image for $n \geq 6$ even and primes $p \equiv 1 \pmod{3}$ or $\pmod{4}$, we refer the reader to [18].

In Section 2, we briefly recall the notion of p -rationality and provide various equivalences for an arbitrary number field to be p -rational with special attention to the case when the number field is totally real. In particular, alternative proofs for the necessary and sufficient conditions for a quadratic field to be p -rational are given. When $p = 3$, a quadratic field $F = \mathbf{Q}(\sqrt{d})$ with $d \equiv -3 \pmod{9}$ is not 3-rational except when $d = -3$. When $d \not\equiv -3 \pmod{9}$, the field F is 3-rational precisely when the 3-part of the class group of its mirror field $F' = \mathbf{Q}(\sqrt{-3d})$ is trivial (Proposition 2.9). In addition, if the field F is imaginary and its class number is not divisible by 3, then it is 3-rational. Using these results, we show the existence of infinitely many

imaginary bi-quadratic 3-rational fields (Corollary 2.10). We end this section by proving that for any prime $p \geq 7$ which does not divide the order of the tame kernel $K_2(o_F)$ of a real multi-quadratic p -rational field F , there exist infinitely many totally imaginary quadratic extensions of F which are p -rational.

For a prime $p > 3$, an imaginary quadratic field F is p -rational as soon as p does not divide the class number of F . On the other hand a result of Hartung dating back to the seventies [15] ensures the existence of infinitely many such number fields. There therefore exist infinitely many imaginary quadratic fields which are p -rational (even for $p = 2$ and $p = 3$). The real case is rather more involved. According to Greenberg's conjecture, there exist infinitely many real quadratic p -rational fields. In Section 3, we prove the existence of such an infinite family (Theorem 3.4). The proof appeals, in particular, to a formula due to Coates expressing the order of the \mathbf{Z}_p -torsion of the abelianized factor group $G(F_S/F)^{ab}$ in terms of the invariants of the totally real number field F on the ground level such as class number and discriminant (Formula (2.1) below), as well as a result of Ono on the existence of some infinitely many fundamental discriminants (Theorem 3.3).

In Section 4, for each odd prime p , we provide an imaginary and a real bi-quadratic number field which are p -rational. Namely, the imaginary bi-quadratic field $\mathbf{Q}(\sqrt{p(p+2)}, \sqrt{-p})$ and the real bi-quadratic field $\mathbf{Q}(\sqrt{p(p+2)}, \sqrt{p(p-2)})$ are p -rational as soon as $p > 3$. Finally, adding the square root of $-p$ to the above real bi-quadratic field yields continuous representations $\rho : G_{\mathbf{Q}} \rightarrow GL_n(\mathbf{Z}_p)$ with open image for both $n = 4$ and $n = 5$ and at least for all the primes $p < 192.699.943$.

We list here the notations used in the text for any number field F . They are mostly the usual ones.

o_F	ring of integers of F ;
$o'_F = o_F[1/p]$	ring of p -integers of F ;
d_F	discriminant of F/\mathbf{Q} ;
h_F	the class number of F ;
r_1, r_2	number of real (resp. non-conjugate complex) embeddings of F ;
$\mu_{p^n}, \mu_{p^\infty}$	group of p^n -th (resp. all p -primary) roots of unity;
$\mu_p(F) = \mu_p \cap F$	group of all p -th roots of unity contained in F ;
U_F	group of units in F ;
$\bar{U}_F = \varprojlim (U_F/p^m)$	pro- p -completion of U_F ;
F_v	completion of F at a place v in F ;
U_v	group of units in the local field F_v ;
$U_v^{(i)}$	i -th higher unit group in the local field F_v ;
$\bar{U}_v = \varprojlim (U_v/p^m)$	pro- p -completion of U_v ;
S	set of all places of F lying above p ;
A_F	p -primary part of the class group of F ;
A'_F	p -primary part of the (p) -class group of F ;
F_S	maximal pro- p -extension of F unramified outside S ;
\hat{F}	maximal abelian pro- p -extension of F unramified outside the p -adic places;
\tilde{F}	compositum of all the \mathbf{Z}_p -extensions of F ;
$G_S(F) := \text{Gal}(F_S/F)$	Galois group over F of the maximal S -ramified pro- p -extension of F ;
$\mathcal{G}_S(F)$	Galois group over F of the maximal S -ramified extension of F ;
$X_F := \text{Gal}(\hat{F}/F)$	Galois group over F of the maximal S -ramified abelian pro- p -extension of F ;
$T_F := \text{Tor}_{\mathbf{Z}_p}(X_F)$	finite torsion subgroup of X_F .

We will also need the following notations and conventions: for any group M , the notation M/p denotes the cokernel of the raising to the p -th power map $x \mapsto x^p$ on M . For a quadratic field F with discriminant d_F , we denote by χ_F or χ_{d_F} the associated quadratic character. By a multi-quadratic field we mean a composition of quadratic fields. Namely, a Galois extension F of the field of rational numbers \mathbf{Q} such that $\text{Gal}(F/\mathbf{Q})$ is isomorphic to $(\mathbf{Z}/2)^t$ for some integer $t \geq 1$. The cardinality of a finite set A is denoted by $|A|$.

2. p -rationality

The notion of p -rationality was introduced and studied in [30, 28, 29], notably in order to produce infinitely many non abelian number fields satisfying Leopoldt’s Conjecture at the prime p . A number field F is called p -rational when the Galois group $G_S(F)$ is a free pro- p -group. In other words, the second cohomology group $H^2(G_S(F), \mathbf{Z}/p)$ vanishes. Such a field satisfies the Leopoldt conjecture at the prime p since one of the equivalent statements for this conjecture is the vanishing of the second cohomology group $H^2(G_S(F), \mathbf{Q}_p/\mathbf{Z}_p)$. The terminology takes into account the fact that the field of rationals \mathbf{Q} is p -rational for all p and in some sense a p -rational field behaves like \mathbf{Q} with respect to p .

Let R_2F be the p -primary part of the “regular kernel” (see [12, Sections I & II] and also [13, Introduction]). If p is odd or if F has no real embedding then R_2F is the same as the p -primary part of the so called tame kernel $K_2(o_F)$, whereas when $p = 2$ and F possesses a real embedding then the difference between the 2-part of $K_2(o_F)$ and R_2F lies in the fact that the Hilbert symbols at real places are considered as being tame. A number field F with trivial R_2F is called p -regular by Gras and Jaulent [13]. The terminology comes from the fact that the cyclotomic field $\mathbf{Q}(\mu_p)$ is p -regular exactly when p is a regular prime. For an odd prime p , using Tate’s result on K_2 and Galois cohomology, Soulé showed [37, Lemme 10] that the Chern character realises an isomorphism

$$K_2(o_F)/p \cong H^2(\text{Spec } o'_F, \mathbf{Z}/p(2)) \cong H^2(\mathcal{G}_S(F), \mathbf{Z}/p(2)),$$

where $\mathbf{Z}/p(2)$ denotes the 2-fold Tate twist of \mathbf{Z}/p . Hence the p -regularity of a number field F can be interpreted by the vanishing of the second Galois cohomology group $H^2(\mathcal{G}_S(F), \mathbf{Z}/p(2))$.

Therefore, the two notions of p -regularity and p -rationality are of different nature since they correspond to different twists “à la Tate”. Nevertheless, when the maximal real subfield $\mathbf{Q}(\mu_p)^+$ of $\mathbf{Q}(\mu_p)$ is contained in F , then F is p -rational precisely when F is p -regular. In particular, F is 3-rational precisely when 3 does not divide the order of the tame kernel $K_2(o_F)$.

By the Euler-Poincaré characteristic

$$\chi(G_S(F)) = 1 - d(G_S(F)) + r(G_S(F)) = -r_2,$$

where $-r_2$ is the number of complex places of F . Hence the pro- p -group $G_S(F)$ is of rank $1 + r_2$ when it is free and the p -rationality of F is equivalent to the fact that the abelianized factor group $X_F := G_S(F)^{ab} = \text{Gal}(\hat{F}/F)$ is a free \mathbf{Z}_p -module of rank $1 + r_2$:

$$X_F \simeq \mathbf{Z}_p^{1+r_2}.$$

In other words a number field F is p -rational precisely when Leopoldt’s conjecture holds for F at p and the \mathbf{Z}_p -torsion module $T_F := \text{Tor}_{\mathbf{Z}_p}(X_F)$ is trivial.

An interesting genus formula for this finite module T_F “à la Chevalley” is obtained in [12, Proposition 6] under Leopoldt’s conjecture. For a different proof see [30, appendice]. As a consequence, Gras obtains all abelian p -extensions F of \mathbf{Q} such that $T_F = 1$ for $p = 2$ and for $p = 3$ [12, Corollary to Theorem 2]. Since an abelian extension of \mathbf{Q} satisfies Leopoldt’s conjecture, the above fields F turn out to be those abelian p -extensions of \mathbf{Q} which are p -rational (for $p = 2$ or for $p = 3$). For an arbitrary odd prime p , Galois p -extensions of \mathbf{Q} (or

an arbitrary totally real number field) which are p -rational are characterised in [28, Chapitre III, Proposition 2, p.41] by means of ramification: an arbitrary Galois p -extension F of \mathbf{Q} is p -rational precisely when at most one non- p -adic prime ℓ ramifies in F and ℓ is inert in the cyclotomic \mathbf{Z}_p -extension of \mathbf{Q} , in other words $\ell \not\equiv 1 \pmod{p^2}$. Note that for ℓ to be ramified in a p -extension F/\mathbf{Q} , it is necessary that $\ell \equiv 1 \pmod{p}$.

When the number field F is totally real, Leopoldt's conjecture for F at p is expressed by the non vanishing of the p -adic regulator $R_{p,F}$ and the following remarkable formula

linking the order of T_F with $R_{p,F}$ was proved by Coates [8, Appendix, Lemma 8] (see also [32, Proposition 2.1]): assume $R_{p,F} \neq 0$, then

$$|T_F| \sim |\mu_{p^\infty} \cap F(\mu_p)| \frac{h_F R_{p,F}}{\sqrt{d_F}} \prod_{v|p} (1 - (Nv)^{-1}), \tag{2.1}$$

where p is any odd prime, \sim means that both sides have the same p -adic valuation and $N := N_{F/\mathbf{Q}}$ is the norm in F/\mathbf{Q} . The above formula theoretically provides a way to test p -rationality of totally real number fields. For instance, suppose the totally real field F to be quadratic. Then, the above formula becomes

$$|T_F| \sim h_F R_{p,F} / p^{1/e}, \tag{2.2}$$

where $e := 2$ or 1 according as p ramifies in F or not. Denote by ε the fundamental unit of F so that $R_{p,F} = \log_p(\varepsilon)$, where \log_p is the p -adic logarithm. Then the p -adic valuation of $R_{p,F}$ is the same as that of $(\varepsilon^{q-1} - 1)$, where $q := Nv$ and v is a p -adic place of F . Hence, the real quadratic field F is p -rational precisely when p does not divide the class number of F and $\varepsilon^{q-1} \in U_v^{(1)} \setminus U_v^{(1+e)}$. Now, when $p > 3$ or $p = 3$ and is unramified in F , then $e < p - 1$ and raising to the p -th power realises an isomorphism between $U_v^{(1)}$ and $U_v^{(1+e)}$ [36, Prop 9, page 219]. Hence $\varepsilon^{q-1} \in U_v^{(1+e)}$ precisely when ε^{q-1} or ε is a p -th power in the completion F_v . Therefore, when $p > 3$ or $p = 3$ and is unramified in F , then the real quadratic field F is p -rational precisely when p does not divide the class number of F and the fundamental unit of F is not a p -th power in F_v . A different proof of this last equivalence was given in [14, Proposition 4.1] and we shall provide another alternative proof below (see Corollary 2.6).

Reconsider the case of a general totally real number field F of degree r . Let $\zeta_{F,p}(s)$ be the p -adic ζ -function of F . The residue of $\zeta_{F,p}$ at $s = 1$ is given by [9]

$$\lim_{s \rightarrow 1} (s - 1) \zeta_{F,p}(s) = \frac{2^{r-1} h_F R_{p,F}}{\sqrt{d_F}} \prod_{v|p} (1 - (Nv)^{-1})$$

and the Leopoldt conjecture for F at p (which asserts the non vanishing of $R_{p,F}$) is valid precisely when $\zeta_{F,p}$ has a simple pole at $s = 1$.

Under Leopoldt's conjecture for F at the prime p , the above formula (2.1) becomes

$$|T_F| \sim |\mu_{p^\infty} \cap F(\mu_p)| \lim_{s \rightarrow 1} (s - 1) \zeta_{F,p}(s). \tag{2.3}$$

Hence, we have the following interpretation of p -rationality in terms of the residue of the p -adic ζ -function:

PROPOSITION 2.1. *Let p be an odd prime number and F be any totally real number field. Let $\mu_{p^m} := \mu_{p^\infty} \cap F(\mu_p)$. Then F is p -rational precisely when the p -adic valuation of the residue of the p -adic ζ -function of F at 1 is $-m$:*

$$\lim_{s \rightarrow 1} (s - 1) \zeta_{F,p}(s) \sim p^{-m}.$$

When the totally real number field F is abelian corresponding to a group of Dirichlet characters X , then

$$\lim_{s \rightarrow 1} (s-1)\zeta_{F,p}(s) = \lim_{s \rightarrow 1} (s-1)L_p(s, \chi_0) \prod_{\chi_0 \neq \chi \in X} L_p(1, \chi),$$

where χ_0 is the trivial character and $L_p(\cdot, \chi)$ is the p -adic L -function associated to χ . Since the residue of $L_p(s, \chi_0)$ at $s = 1$ is $(1 - \frac{1}{p})$, according to the above proposition, F is p -rational precisely when

$$\prod_{\chi_0 \neq \chi \in X} L_p(1, \chi) \sim p^{-m+1}. \tag{2.4}$$

Therefore, we have the following special case which will be used further on to prove the existence of infinitely many real quadratic p -rational fields.

COROLLARY 2.2. *Let p be an odd prime. A real quadratic number field F is p -rational precisely when $L_p(1, \chi_F)$ is a p -adic unit.*

This Corollary (2.2) together with the above relation (2.4) immediately yields the fact that a real multi-quadratic field is p -rational precisely when it is the case of all its quadratic subfields. For a more general result see [14, Proposition 3.6]. Another consequence of the characterisation of the p -rationality by the above relation (2.4) is the fact that the maximal real subfield $\mathbf{Q}(\mu_p)^+$ of the p -th cyclotomic field is p -rational precisely when p is a regular prime (see also [1, Example (ii)a, page 503]).

On the completely algebraic side, we have the following proposition noticed by Nguyen Quang Do [33, Appendix].

PROPOSITION 2.3. *A number field F is p -rational precisely when the following three conditions are satisfied:*

- (i) *The p -Hilbert class field $H \subseteq \tilde{F}$;*
- (ii) *The natural map $\mu_p(F) \rightarrow \oplus_{v|p} \mu_p(F_v)$ is an isomorphism;*
- (iii) *The natural map $U_F/p \rightarrow \oplus_{v|p} U_v/p$ is injective.*

Proof. By definition, the Leopoldt kernel D_F is the kernel of the diagonal map $U_F \rightarrow \oplus_{v|p} U_v$ once tensored with \mathbf{Z}_p and we have a four-term exact sequence provided by global class field theory

$$0 \rightarrow D_F \rightarrow \bar{U}_F \rightarrow \oplus_{v|p} \bar{U}_v \rightarrow \text{Gal}(\hat{F}/H) \rightarrow 0.$$

First assume F to be p -rational. Then obviously $H \subseteq \tilde{F}$ and $D_F = 0$. So applying the snake lemma to the above exact sequence for raising to the p -th power maps shows the other two conditions (ii) and (iii). Now assume the three conditions to be fulfilled. Consider the commutative diagram where the vertical maps consist of raising to the p -th power:

$$\begin{array}{ccccccccc} 0 & \rightarrow & D_F & \rightarrow & \bar{U}_F & \rightarrow & \oplus_{v|p} \bar{U}_v & \rightarrow & \text{Gal}(\hat{F}/H) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & D_F & \rightarrow & \bar{U}_F & \rightarrow & \oplus_{v|p} \bar{U}_v & \rightarrow & \text{Gal}(\hat{F}/H) & \rightarrow & 0. \end{array}$$

A diagram chasing shows that under condition (ii), D_F/p is contained in the kernel of $U_F/p \rightarrow \oplus_{v|p} U_v/p$. This latter map is, in turn, injective by assumption (iii). Consequently $D_F = 0$ and, once again, we are faced with a short exact sequence. Applying once more the

snake lemma to the diagram obtained by the raising to the p -th power maps shows that $\text{Gal}(\hat{F}/H)$ is torsion free. Hence F is p -rational. \square

Consider here a totally real number field F , which is Galois over \mathbf{Q} and satisfies Leopoldt's conjecture at p . If the ramification index e of the prime p in F/\mathbf{Q} is prime to p , then Condition (i) in the above Proposition 2.3 is equivalent to $p \nmid h_F$. Also, if $p - 1 \nmid e$, then Condition (ii) is fulfilled. Therefore, the p -rationality of F can be read through the class number and the units in the following way:

PROPOSITION 2.4. *Let F be a totally real number field which is Galois over \mathbf{Q} and satisfies Leopoldt's conjecture at the prime p . Suppose neither p nor $p - 1$ divide the ramification index of the prime p in F/\mathbf{Q} . Then F is p -rational precisely when the following two conditions are satisfied:*

- (i) *The prime p does not divide the class number h_F and*
- (ii) *The natural map $U_F/p \rightarrow \bigoplus_{v|p} U_v/p$ is injective.*

Consider now a cyclic extension F of odd prime degree n over \mathbf{Q} . If $n = p$, as explained before, F is p -rational precisely when (apart from p which may or may not be ramified in F) at most one non- p -adic prime $\ell \not\equiv 1 \pmod{p^2}$ ramifies in F . For $n \neq p$, the cyclic field F satisfies the hypotheses of the above Proposition 2.4 and we have

COROLLARY 2.5. *Let F be a cyclic extension of \mathbf{Q} of odd prime degree $n \neq p$. Then F is p -rational precisely when the following two conditions are satisfied:*

- (i) *The prime p does not divide the class number h_F and*
- (ii) *The natural map $U_F/p \rightarrow \bigoplus_{v|p} U_v/p$ is injective.*

When the totally real field F is quadratic, special attention must be paid to the above condition, $p - 1 \nmid e$ when $p = 3$ and is ramified in F , which we will treat a little further on. Otherwise, the hypotheses of the above Proposition 2.4 are fulfilled leading to an alternative proof of the following corollary [14, Proposition 4.1].

COROLLARY 2.6. *Let F be a real quadratic number field. Suppose either $p \geq 5$ or $p = 3$ and is unramified in F/\mathbf{Q} . Then F is p -rational precisely when the following two conditions are satisfied:*

- (i) *The prime p does not divide the class number h_F and*
- (ii) *The fundamental unit of F is not a p -th power in the completion F_v of F at a p -adic place v .*

If instead the quadratic number field F is imaginary then, the p -Hilbert class field H being Galois over \mathbf{Q} , the first condition $H \subseteq \tilde{F}$ in Proposition 2.3 is equivalent to H being contained in the anticyclotomic \mathbf{Z}_p -extension of F , since the layers of the cyclotomic \mathbf{Z}_p -extension of F are necessarily ramified at the prime(s) above p . Therefore, we also get an alternative proof of the following corollary [14, Proposition 4.1].

COROLLARY 2.7. *Let F be an imaginary quadratic number field. Suppose either $p \geq 5$ or $p = 3$ and is unramified in F/\mathbf{Q} . Then F is p -rational precisely when the p -Hilbert class field H is contained in the anticyclotomic \mathbf{Z}_p -extension of F . In particular:*

- (i) If p does not divide the class number h_F , then F is p -rational.
- (ii) If F is p -rational then A_F , the p -primary part of the class group of F , is at most cyclic.

Many examples exist where the imaginary quadratic number field F is p -rational with a non-trivial p -class group. For instance $\mathbf{Q}(\sqrt{-23})$, whose class group is of order 3, turns out to be 3-rational. Another example is provided by $\mathbf{Q}(\sqrt{-47})$ which is 5-rational and whose class group is of order 5.

Consider now the remaining case when $p = 3$ and is ramified in the quadratic field $F = \mathbf{Q}(\sqrt{d})$ (thus the integer d is square-free and a multiple of 3). Then locally, we have $\mathbf{Q}_3(\sqrt{d}) = \mathbf{Q}_3(\sqrt{-3})$, precisely when $-d/3$ is a principal unit in the local field \mathbf{Q}_3 which is equivalent to $d \equiv -3 \pmod{9}$. So in this case, again because of Condition (ii) in Proposition 2.3, the field F is 3-rational if and only if $F = \mathbf{Q}(\sqrt{-3})$. If instead $d \not\equiv -3 \pmod{9}$, then Condition (ii) in Proposition 2.3 is automatically valid and we obtain the same result as before:

COROLLARY 2.8. *Let $F = \mathbf{Q}(\sqrt{d})$ be a quadratic field, with $d \neq -3$ square-free divisible by 3.*

- (i) *If $d \equiv -3 \pmod{9}$ then F is not 3-rational.*
- (ii) *If $d \not\equiv -3 \pmod{9}$ and $d > 0$, then F is 3-rational precisely when $3 \nmid h_F$ and the fundamental unit of F is not a third power locally at a 3-adic place v .*
- (iii) *If $d \not\equiv -3 \pmod{9}$ and $d < 0$, then F is 3-rational precisely when the 3-Hilbert class field H is contained in the anticyclotomic \mathbf{Z}_3 -extension of F . In particular, if 3 does not divide the class number h_F , then F is 3-rational. Also, as before, if F is 3-rational then A_F , the 3-primary part of the class group of F , is at most cyclic.*

Let $F = \mathbf{Q}(\sqrt{d})$ be a quadratic field, with $d \neq -3$ square-free and $F' = \mathbf{Q}(\sqrt{-3d})$. Assume that $d \not\equiv -3 \pmod{9}$ which is equivalent to 3 not splitting in F' . Suppose first that $d > 0$, we have

$$L_3(1, \chi_F) \equiv L_3(1 - 1, \chi_F) = (1 - \chi_{F'}(3))L(1 - 1, \chi_{F'}) = (1 - \chi_{F'}(3))\frac{2}{w}h_{F'} \pmod{3},$$

where $L(\cdot, \chi_{F'})$ is the Dirichlet L-function associated to $\chi_{F'}$ and $w = 2$ except when $d = 3$, in which case $w = 4$. Since 3 does not split in F' , the factor $(1 - \chi_{F'}(3))$ does not vanish and, by Corollary 2.2, F is 3-rational precisely when $3 \nmid h_{F'}$ (this last equivalence can also be seen using [6, Theorem 2] since, as mentioned before, F is 3-rational precisely when 3 does not divide the order of the tame kernel $K_2(\mathcal{O}_F)$). Now if $d < 0$, the 3-rationality of F is equivalent to the triviality of $A'_{F'}$, the 3-primary part of the 3-class group of F' [11, Theorem 4.1]. Finally, the non splitting of 3 in F' also implies that $A'_{F'} = A_{F'}$ and we obtain the following interesting relationship between the 3-rationality of F and the class number of its mirror quadratic field (see also [14, Corollary 4.2] and the subsequent paragraph).

PROPOSITION 2.9. *Let $F = \mathbf{Q}(\sqrt{d})$ be a quadratic field, with $d \neq -3$ square-free and $F' = \mathbf{Q}(\sqrt{-3d})$.*

- (i) *If $d \equiv -3 \pmod{9}$ then F is not 3-rational.*
- (ii) *If $d \not\equiv -3 \pmod{9}$, then F is 3-rational precisely when $A_{F'} = 0$.*

According to [40, Theorem A], there exist infinitely many imaginary quadratic fields $\mathbf{Q}(\sqrt{-d})$ in which 3 is inert and with class number prime to 3. In particular, these fields $\mathbf{Q}(\sqrt{-d})$ are 3-rational. Since $3d \not\equiv -3 \pmod{9}$, the fields $\mathbf{Q}(\sqrt{3d})$ (and then also $\mathbf{Q}(\sqrt{-3}, \sqrt{-d})$) are 3-rational by Proposition 2.9. Hence, we have the following

COROLLARY 2.10. *There exist infinitely many imaginary bi-quadratic 3-rational number fields.*

Now a multi-quadratic field is p -rational precisely when all its quadratic subfields are [14, Proposition 3.6]. Therefore combining Corollary 2.7 (i), Corollary 2.8 (iii) and Proposition 2.9, we obtain the following characterisation of multi-quadratic (hence bi-quadratic) 3-rational fields containing $\sqrt{-3}$.

COROLLARY 2.11. *Let F be a real multi-quadratic 3-rational field. Then $F(\sqrt{-3})$ is 3-rational precisely when F does not contain any \sqrt{d} with $d \equiv 1 \pmod{3}$. In particular a multi-quadratic 3-rational field which contains $\sqrt{-3}$ is at most bi-quadratic.*

An example of such a bi-quadratic field is simply $\mathbf{Q}(\sqrt{-1}, \sqrt{-3})$.

Finally, to be thorough, let us briefly treat the case of $p = 2$. As far as the 2-rationality of a quadratic field F is concerned, the situation is completely different from the odd prime case. Recall first that if the p -th roots of unity $\mu_p \subset F$, then F is p -rational precisely when F contains only one p -adic place and the p -primary part of the (p) -class group A'_F is trivial [30, Théorème et Définition 2.1]. For $p = 2$, the narrow class group A'^+_F has to replace A'_F .

Therefore p -rationality for $p = 2$ amounts to having one dyadic prime and the vanishing of the 2-primary part of the narrow (2)-class group A'^+_F . By the genus formula for the class groups “à la Chevalley”, it should be possible to draw the list of 2-rational quadratic fields. But much more is known. In fact, for $p = 2$, the above two conditions are also equivalent to the vanishing of the positive étale cohomology group $H^2_+(o'_F, \mathbf{Z}_2(i))$ for any integer $i \geq 2$ (see [3, Proposition 4.8] whose proof follows the same ideas as those in the proof of [20, Proposition 2.6] where the case of real number fields is dealt with). This vanishing is characterised for arbitrary finite Galois 2-extensions of \mathbf{Q} [3, Proposition 6.4]:

PROPOSITION 2.12. *Let F be a finite Galois 2-extension of \mathbf{Q} and $i \geq 2$. Then the positive cohomology group $H^2_+(o'_F, \mathbf{Z}_2(i))$ vanishes (in other words, F is 2-rational) exactly when F/\mathbf{Q} is unramified outside a set of primes $\{2, \infty, \ell\}$ with $\ell \equiv \pm 3 \pmod{8}$.*

So, the quadratic 2-rational fields are $\mathbf{Q}(\sqrt{2})$, $\mathbf{Q}(\sqrt{-1})$, $\mathbf{Q}(\sqrt{-2})$ as well as those of the form $\mathbf{Q}(\sqrt{\ell})$, $\mathbf{Q}(\sqrt{2\ell})$, $\mathbf{Q}(\sqrt{-\ell})$ and $\mathbf{Q}(\sqrt{-2\ell})$ where ℓ is any odd prime $\equiv \pm 3 \pmod{8}$. Therefore, obviously the only real multi-quadratic 2-rational fields turn out to be bi-quadratic of the form $\mathbf{Q}(\sqrt{2}, \sqrt{\ell})$ with $\ell \equiv \pm 3 \pmod{8}$ and the only tri-quadratic imaginary 2-rational fields are of the form $\mathbf{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{\ell})$ with $\ell \equiv \pm 3 \pmod{8}$. Note that the same list can also be obtained from [12, Corollary to Theorem 2].

By contrast, we have the following conjecture of Greenberg for odd primes p , which motivated this study:

CONJECTURE 2.13. *([14, Conjecture 4.8]) For each odd prime number p , and each positive integer t , there exists a p -rational number field F which is Galois over \mathbf{Q} with $\text{Gal}(F/\mathbf{Q}) \cong (\mathbf{Z}/2)^t$.*

As explained by Greenberg [14, Remark 6.8], if the above conjecture 2.13 is true for imaginary multi-quadratic fields for a given odd prime p , then there would exist continuous representations $\rho : G_{\mathbf{Q}} \rightarrow GL_n(\mathbf{Z}_p)$ with open image for all degrees $n \geq 4$.

In fact, we know when exactly p -rationality is inherited by an arbitrary Galois p -extension ([30, 28, 29]) and this was the strategy to obtain infinitely many non abelian number fields satisfying Leopoldt’s conjecture at a prime p . For the above Conjecture 2.13, it would be interesting to investigate going up properties of p -rationality along quadratic (and more generally prime-to- p degrees) extensions.

PROPOSITION 2.14. *Let $p > 2$ and F be a totally real p -rational number field and L a totally imaginary field which is a quadratic extension of F , a so-called C.M. field. Suppose that $p \nmid h_L$ and $\mu_p \not\subset L_w$ for any p -adic place w of L . Then L is also p -rational.*

Proof. Since p is odd, the inclusion $U_F \subset U_L$ induces a natural injective map $U_F/p \hookrightarrow U_L/p$. Since $\mu_p \not\subset L$ by hypothesis, U_F/p and U_L/p both have the same \mathbf{Z}/p -rank $[F : \mathbf{Q}] - 1$. Therefore, the above natural injective map is in fact an isomorphism. In other words, the Galois group $\text{Gal}(L/F)$ acts trivially on U_L/p . Consider now the following commutative diagram

$$\begin{array}{ccc} U_F/p & \longrightarrow & \bigoplus_{v|p} U_v/p \\ \downarrow \simeq & & \downarrow \\ U_L/p & \longrightarrow & \bigoplus_{v|p} \bigoplus_{w|v} U_w/p. \end{array}$$

Since F is supposed p -rational, the top horizontal map is injective by Proposition 2.3, therefore the bottom horizontal map is also injective and the p -rationality of L follows by the same Proposition 2.3. □

In the above Proposition 2.14, the condition $p \nmid h_L$ can be replaced by the less restrictive condition that the p -Hilbert class field of L is contained in \tilde{L} . We also note that, for $p > 3$, if the C.M. field L in the above Proposition 2.14 is supposed to be multi-quadratic, then it is p -rational as soon as $p \nmid h_L$. The same result holds for $p = 3$, provided that L does not contain any \sqrt{d} , where $d < 0$ and $\equiv -3 \pmod{9}$.

Let $w_2(F)$ be the order of the Galois cohomology group $H^0(F, \mathbf{Q}/\mathbf{Z}(2))$ with coefficients the 2-fold Tate-twist of the module \mathbf{Q}/\mathbf{Z} . The p -part of $w_2(F)$ is the maximal power p^m , such that the Galois group $\text{Gal}(F(\mu_{p^m})/F)$ has exponent 2. If F is totally real and p is odd, the latter condition is equivalent to the fact that F contains the maximal real subfield $\mathbf{Q}(\mu_{p^m})^+$ of $\mathbf{Q}(\mu_{p^m})$. Then we have the following

COROLLARY 2.15. *Let $p \geq 7$ and F be a totally real multi-quadratic p -rational number field. Assume that $p \nmid w_2(F)\zeta_F(-1)$, where ζ_F is the Dedekind ζ -function of F . Then there exist infinitely many totally imaginary quadratic extensions L_i of F which are p -rational.*

Proof. Since F is assumed to be multi-quadratic, the odd prime p ramifies in the layers of the cyclotomic \mathbf{Z}_p -extension of F . Therefore, by Proposition 2.3 (i), we have $p \nmid h_F$. Now by [31, Theorem], there exist infinitely many totally imaginary quadratic extensions L_i of F whose class numbers are primes to p . Since p is supposed to be greater than 7, the completion of L_i at any p -adic place does not contain the p -th roots of unity and the Corollary follows from Proposition 2.14. □

The above Corollary 2.15 also holds for $p = 5$ as soon as F does not contain any quadratic field $\mathbf{Q}(\sqrt{d})$, where $d \equiv \pm 5 \pmod{25}$, since then the completion F_v of F at any p -adic place v would not contain $\sqrt{5}$ and therefore none of its quadratic extensions could contain μ_5 .

When F is totally real and abelian over \mathbf{Q} , then

$$w_2(F)\zeta_F(-1) = \pm |K_2(o_F)| \tag{2.5}$$

(cf. [21, Appendix A, Corollary 0.9]), as a consequence of the Main conjecture in Iwasawa theory proved by Wiles [39]. Hence the above Corollary 2.15 in fact applies whenever the totally real multi-quadratic field F is both p -rational and p -regular.

3. The quadratic case

There exist infinitely many imaginary quadratic fields whose class numbers are not divisible by a given prime p . This was a conjecture of Chowla proved by Hartung [15]. Since then, many papers have dealt with the non divisibility by p of the class number of imaginary quadratic fields F under different conditions. For instance, the same result is proved in [17] or in [16] imposing the behaviour of the prime p or finite sets of primes in the imaginary quadratic fields in question. In [19], the authors even show the existence of a lower bound for the number of imaginary quadratic fields with discriminant $d > -X$ for a large integer X having class numbers prime to p . For more recent treatments see [40, 5].

So, according to Corollary 2.7, for each prime number p there are infinitely many imaginary quadratic fields which are p -rational. Also, we note that the nine imaginary quadratic fields with class number 1 are p -rational for each odd prime p (in fact they are also 2-rational except for $\mathbf{Q}(\sqrt{-7})$) and it would be interesting to list the (imaginary quadratic) number fields which are p -rational for all odd primes p .

As far as real quadratic fields F are concerned, the situation is much harder to handle. This is due to the fact that, on one side, in the Dirichlet's class number formula

$$h_F = \frac{L(1, \chi_F)\sqrt{d_F}}{2R_F}, \tag{3.1}$$

the regulator $R_F = \log(\varepsilon)$ intervenes so knowledge of the fundamental unit ε of F is required and, on the other hand, for the real quadratic field to be p -rational, the fundamental unit must not be a p -th power locally at a p -adic place (Corollaries 2.6 and 2.8).

When the odd prime $p \neq 5$, the p -rationality of $\mathbf{Q}(\sqrt{5})$ can be connected to Fibonacci numbers in a particularly pleasant form as explained by Greenberg in [14, Corollary 4.5]. Let

$$q := \begin{cases} p & \text{if } p \equiv \pm 1 \pmod{5}, \\ p^2 & \text{if } p \equiv \pm 2 \pmod{5}. \end{cases}$$

Then $\mathbf{Q}(\sqrt{5})$ is p -rational precisely when the q -th Fibonacci number $F_q \not\equiv 1 \pmod{p^2}$. The Fibonacci numbers are given by the Binet formula

$$F_q := \frac{\varepsilon_0^q - \bar{\varepsilon}_0^q}{\varepsilon_0 - \bar{\varepsilon}_0},$$

where $\varepsilon_0 := (1 + \sqrt{5})/2$ is the fundamental unit of $\mathbf{Q}(\sqrt{5})$. Greenberg's proof is extendable similarly to any real quadratic field F in which the odd prime p is not ramified. Indeed let ε be the fundamental unit of F . Introduce $q := p^f$, where f denotes the residue degree of p in F . Then ε^{q-1} is a principal unit in the completion F_v at a p -adic place v and is locally a p -th power precisely when it belongs to $U_v^{(2)}$ [36, Prop 9, page 219]. Therefore, ε is locally a p -th power precisely when $\varepsilon^q \equiv \varepsilon \pmod{\mathfrak{p}_v^2}$ where \mathfrak{p}_v is the prime ideal of F associated to v . The

same holds for the conjugate $\bar{\varepsilon}$ which coincides with the inverse ε^{-1} . Denote by

$$\mathcal{F}_n := \frac{\varepsilon^n - \bar{\varepsilon}^n}{\varepsilon - \bar{\varepsilon}}$$

the general Fibonacci-type of two-term linear recurrence

$$\mathcal{F}_{n+2} := (\varepsilon + \bar{\varepsilon})\mathcal{F}_{n+1} - (\varepsilon\bar{\varepsilon})\mathcal{F}_n,$$

with initial terms $\mathcal{F}_0 = 0$ and $\mathcal{F}_1 = 1$. Accordingly, it is now clear that we have the following corollary which is a consequence of Corollary 2.6 and the above discussion inspired by Greenberg's proof of [14, Corollary 4.5] in the special case of $\mathbf{Q}(\sqrt{5})$.

COROLLARY 3.1. *Let F be a real quadratic field in which the odd prime p does not ramify. Then F is p -rational precisely when the following two conditions are satisfied:*

- (i) *The prime p does not divide the class number h_F and*
- (ii) *The generalised Fibonacci number $\mathcal{F}_q \not\equiv 1 \pmod{p^2}$.*

This provides an algorithm for testing p -rationality of a real quadratic field (and therefore also a real multi-quadratic field).

Now, if we fix the prime p , we have a real p -rational quadratic field which is explicit:

PROPOSITION 3.2. *For each prime $p \neq 3$, the real quadratic field $F = \mathbf{Q}(\sqrt{p(p+2)})$ is p -rational. For $p = 3$ the corresponding field $F = \mathbf{Q}(\sqrt{15})$ is not 3-rational.*

Proof. For $p = 2$, the field in question is $\mathbf{Q}(\sqrt{2})$, the first layer of the cyclotomic \mathbf{Z}_2 -extension of \mathbf{Q} , which is 2-rational. For $p = 3$, the quadratic field $\mathbf{Q}(\sqrt{15})$ is not 3-rational (Corollary 2.8). In fact this is because once completed at the 3-adic place it coincides with $\mathbf{Q}_3(\sqrt{-3})$, hence condition (ii) of Proposition 2.3 is not fulfilled.

We may therefore assume $p > 3$. Denote by d the square-free part of $p(p+2)$. Since $d \equiv 3 \pmod{4}$, the fundamental unit is of the form $x + y\sqrt{d}$, where x and y are strictly positive integers. We claim that $\varepsilon := p + 1 + \sqrt{p(p+2)}$ is the fundamental unit. Since otherwise there would exist $n \geq 2$ such that:

$$2(p+1) = \text{Tr}(\varepsilon) = \text{Tr}((x + y\sqrt{d})^n) \geq 2x^n + 2x^{n-2}y^2d,$$

where $\text{Tr} := \text{Tr}_{F/\mathbf{Q}}$ is the trace map in F/\mathbf{Q} . Since $p \mid d$, we would necessarily have $x = y = 1$ and $d = p$. But this is impossible since then $x + y\sqrt{d} = 1 + \sqrt{p}$ which is not even a unit.

Now, the fundamental unit $\varepsilon = p + 1 + \sqrt{p(p+2)}$ is obviously a principal unit of the completion F_v of F at the p -adic place v and we have

$$\varepsilon \in U_v^{(1)} \setminus U_v^{(2)},$$

where $U_v^{(i)}$ denotes the i -th higher unit group in the local field F_v . Therefore $\varepsilon \notin F_v^p$ since raising to the p -th power yields an isomorphism [36, Prop 9, page 219]

$$\begin{array}{ccc} \mu_{p-1}U_v^{(1)} & \longrightarrow & \mu_{p-1}U_v^{(3)} \\ \zeta x & \mapsto & \zeta x^p. \end{array}$$

It remains to show that p does not divide the class number h_F . Since 2 ramifies in F , we also have [25, Corollary 2]

$$L(1, \chi_F) \leq \frac{\log d_F + \kappa_2}{4},$$

where $\kappa_2 := 2 + \gamma - \log(\pi) = 1.432\dots$, and $\gamma = 0.577\dots$ denotes Euler's constant.

For our quadratic field F , the discriminant $d_F = 4d$ and $R_F = \log(p + 1 + \sqrt{p(p+2)})$ is the regulator. Consequently, by class number formula (3.1), we successively have:

$$\begin{aligned} h_F &< \frac{\log(4p(p+2)) + 2}{4} \frac{\sqrt{4p(p+2)}}{2 \log(p + 1 + \sqrt{p(p+2)})} \\ &< \frac{\log(4p(p+2)) + 2}{4} \frac{\sqrt{p(p+2)}}{\log(2\sqrt{p(p+2)})} \\ &< \frac{\log(4p(p+2)) + 2}{2 \log(4p(p+2))} \sqrt{p(p+2)} \\ &< \left(\frac{1}{2} + \frac{1}{3}\right) \sqrt{p(p+2)} \\ &< p \end{aligned}$$

The above inequality $h_F < p$ can also be obtained using [35, Corollary 3]. Hence the prime p does not divide h_F and the proof is complete. \square

Now fix a prime number $p > 3$ and let $d \neq (-1)^{\frac{p-1}{2}} p$ be a square-free integer such that $(-1)^{\frac{p-1}{2}} d > 0$. Introduce the real quadratic field $F := \mathbf{Q}(\sqrt{(-1)^{\frac{p-1}{2}} pd})$ as well as the quadratic field $K := \mathbf{Q}(\sqrt{d})$. Then χ_κ and $\frac{p-1}{2}$ are of the same parity so that $L(1 - \frac{p-1}{2}, \chi_\kappa)$ is a non-zero integer. We also have [38, Chapter 5]:

$$\begin{aligned} L_p(1, \chi_F) &\equiv L_p(1 - \frac{p-1}{2}, \chi_F) \pmod{p} \\ &\equiv L(1 - \frac{p-1}{2}, \chi_F \omega^{\frac{p-1}{2}}) \pmod{p} \\ &\equiv L(1 - \frac{p-1}{2}, \chi_\kappa) \pmod{p} \end{aligned} \tag{3.2}$$

where ω is the Teichmüller character. Thus, according to Corollary 2.2, the quadratic field F is p -rational precisely when $L(1 - \frac{p-1}{2}, \chi_\kappa)$ is a p -adic unit. Now take for d the square-free part of $(-1)^{\frac{p-1}{2}} (p+2)$. Notice that if $p+2$ is a square then $p \equiv 3 \pmod{4}$ so that d is never a square, whatever the prime p . Then, for this choice of d , the field F is p -rational (Proposition 3.2) as soon as $p > 3$ and accordingly

$$L(1 - \frac{p-1}{2}, \chi_\kappa)$$

(so also the generalised Bernoulli number $B_{\frac{p-1}{2}, \chi_\kappa}$) is a p -adic unit. Therefore the hypotheses of the following theorem are fulfilled with D_0 being the discriminant of $K = \mathbf{Q}(\sqrt{d})$:

THEOREM 3.3. [34, Theorem 2]. *Let $p > 3$ be prime, and suppose there is a fundamental discriminant D_0 co-prime to p for which*

- (i) $(-1)^{\frac{p-1}{2}} D_0 > 0$
- (ii) $|B_{\frac{p-1}{2}, \chi_{D_0}}|_p = 1$.

Then there is an arithmetic progression $r_p \pmod{t_p}$ with $(r_p, t_p) = 1$ and a constant $\kappa(p)$ such that for each prime $\ell \equiv r_p \pmod{t_p}$ there is an integer $1 \leq d_\ell \leq \kappa(p)\ell$, for which

- (i) $D_\ell := d_\ell \ell p$ is a fundamental discriminant,
- (ii) $h_{\mathbf{Q}(\sqrt{D_\ell})} \not\equiv 0 \pmod{p}$,
- (iii) $\left| \frac{R_p(D_\ell)}{\sqrt{D_\ell}} \right|_p = 1$.

In the above theorem of Ono, $R_p(D_\ell) := R_{p, \mathbf{Q}(\sqrt{D_\ell})}$ is the p -adic regulator of $\mathbf{Q}(\sqrt{D_\ell})$ and $|\cdot|_p$ is the p -adic absolute value normalised by $|p|_p = 1/p$.

Now according to Formula (2.2), the real quadratic number fields $\mathbf{Q}(\sqrt{D_\ell})$ with the fundamental discriminants D_ℓ intervening in the above theorem turn out to be all p -rational. Summarising, we proved the following:

THEOREM 3.4. *For each prime $p > 3$, there exist infinitely many (with positive density) real quadratic p -rational fields in which p ramifies.*

What we have just explained is inspired by a paper of Byeon [7, Theorem 1.1]. He does not consider the same quadratic field F and his proof is longer and more complicated. For $p = 3$, there also exist infinitely many real quadratic 3-rational fields in which 3 ramifies (Corollary 2.10).

REMARK 3.5. 1. A proof of the existence of infinitely many real quadratic 5-rational fields using half-integral weight modular forms can be found in [2].
 2. On the other hand, for any fixed real quadratic field F and any sufficiently large integer X , the generalised abc -conjecture for F implies that the number of primes $p \leq X$ such that F is p -rational is at least $c \log(X)$ for a constant $c > 0$ depending on F [26, Corollary].

We finish this section by making a few observations about the special case of $p = 5$.

PROPOSITION 3.6. *Let $d \neq 5$ be a positive square-free integer. Then $\mathbf{Q}(\sqrt{5d})$ is 5-rational precisely when $\mathbf{Q}(\sqrt{d})$ is 5-regular.*

Proof. Let $F = \mathbf{Q}(\sqrt{5d})$ and $K = \mathbf{Q}(\sqrt{d})$. By definition, K is 5-regular precisely when 5 does not divide the order of the tame kernel $K_2(o_K)$. Since 2 and 3 are the only prime divisors of $w_2(K)$ and $\zeta_{\mathbf{Q}}(-1) = \frac{-1}{12}$, the last condition means, by the equality (2.5), that $\zeta_K(-1)$ or equivalently $L(-1, \chi_K)$ is a 5-unit. Now, by the congruences (3.2), the last condition is, in turn, equivalent to $L_5(1, \chi_F)$ being a 5-unit. Finally Corollary 2.2 completes the proof. \square

EXAMPLE 3.7. *The quadratic field $\mathbf{Q}(\sqrt{35})$ is both 5-rational (Proposition 3.2) and 5-regular (Proposition 3.6). Therefore, by the discussion following the proof of Corollary 2.15, there exist infinitely many imaginary 5-rational number fields of degree 4 over \mathbf{Q} containing $\mathbf{Q}(\sqrt{35})$.*

Let F/k be a bi-quadratic extension of totally real number fields with quadratic subfields k_1, k_2, k_3 , such that F/\mathbf{Q} is abelian. Then we have the following Brauer relation for the tame kernels [22, Proposition 1.1]

$$|K_2(o_k)|^2 |K_2(o_F)| = |K_2(o_{k_1})| |K_2(o_{k_2})| |K_2(o_{k_3})|.$$

Now, let F be a real multi-quadratic field of degree $n = 2^m$. Since the tame kernel $K_2(\mathbf{Z})$ of \mathbf{Q} is of order 2 [27, Corollary 10.2], an induction on the integer m readily yields the following formula

$$2^{n-2}|K_2(o_F)| = \prod |K_2(o_k)|,$$

where the product is taken over all quadratic subfields of F . In particular, the real multi-quadratic field F is p -regular for an odd prime p precisely when it is the case of all its quadratic subfields. Recall that the same holds for the p -rationality of F . Therefore the above discussion, together with Proposition 3.6 and the fact that $\mathbf{Q}(\sqrt{5})$ is 5-rational, yields the following:

COROLLARY 3.8. *Let F be a real multi-quadratic field which is both 5-rational and 5-regular. Then $F(\sqrt{5})$ is 5-rational (and 5-regular).*

4. The bi-quadratic case

In this section, we will prove that for each prime p there exist imaginary and real bi-quadratic p -rational fields. Recall that a multi-quadratic field is p -rational precisely when all its quadratic subfields are [14, Proposition 3.6].

4.1. Imaginary bi-quadratic case

The class number of an imaginary quadratic field can be expressed by means of the Kronecker symbol $\left(\frac{\cdot}{\cdot}\right)$:

THEOREM 4.1. [23, Theorem 219] *Let d_F be the fundamental discriminant of an imaginary quadratic field F . Then*

$$h_F = \frac{w}{2[2 - (\frac{d_F}{2})]} \sum_r \left(\frac{d_F}{r}\right),$$

where $w = 2, 4$ or 6 stands for the number of roots of unity in F . The sum runs over all integers r from 1 to $-d_F/2$.

As an immediate consequence of the above class number formula, we see that p does not divide the class number of the imaginary quadratic fields $\mathbf{Q}(\sqrt{-p})$ and $\mathbf{Q}(\sqrt{-p-2})$, which are therefore p -rational for any $p > 3$ (Corollary 2.7). Now, as an immediate consequence of [14, Proposition 3.6] and Proposition 3.2, we obtain

PROPOSITION 4.2. *For any prime $p > 3$, the imaginary bi-quadratic field $\mathbf{Q}(\sqrt{p(p+2)}, \sqrt{-p})$ is p -rational.*

On the other hand, $\mathbf{Q}(\sqrt{2}, \sqrt{-1}) = \mathbf{Q}(\mu_8)$ is 2 and 3-rational. J. Ray kindly informed us that another imaginary p -rational bi-quadratic field is to be found in [4, Theorem 2.4]. We also note that when $p+2$ is a square, then the field in Proposition 4.2 is just $\mathbf{Q}(\sqrt{p}, \sqrt{-1})$, which turns out to be p -rational. The same holds when $p-2$ is a square (Proposition 4.4) and also when $p-1$ or $p-4$ is a square since our methods also lead to the p -rationality of $\mathbf{Q}(\sqrt{p(p-1)})$ and $\mathbf{Q}(\sqrt{p(p-4)})$ in a similar manner. One may then ask the following:

QUESTION 4.3. *Is $\mathbf{Q}(\sqrt{p})$ always p -rational?*

We do not know of a single prime p for which $\mathbf{Q}(\sqrt{p})$ is not p -rational. By the same upper bound as in Proposition 4.4, p does not divide the class number of $\mathbf{Q}(\sqrt{p})$ and in some sense the above question concerns its fundamental unit. We also note that, since the cyclotomic field $\mathbf{Q}(\mu_p)$ is p -rational for a regular prime, it is also the case of all its subfields. Therefore, if such a prime p exists, it is either irregular or $\equiv 3 \pmod{4}$.

4.2. Real bi-quadratic case

PROPOSITION 4.4. *For each prime $p > 3$, the real bi-quadratic field $\mathbf{Q}(\sqrt{p(p+2)}, \sqrt{p(p-2)})$ is p -rational.*

Proof. The difference $p^2 - 4$ can never be a square so we are really dealing with a bi-quadratic field. The proof for the p -rationality of $\mathbf{Q}(\sqrt{p(p-2)})$ goes along the same line as that of Proposition 3.2 except that the non divisibility of its class number by p is easier to establish since $p(p-2) < p^2$ (see for instance [24, Theorem (a)]). Note that the quadratic field $\mathbf{Q}(\sqrt{p(p-2)})$ has been considered in [7].

Let us now prove the p -rationality of the third quadratic subfield $k := \mathbf{Q}(\sqrt{p^2 - 4})$. Introduce

$$q := \begin{cases} p & \text{if } p \equiv 1 \pmod{4}, \\ p^2 & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

and the unit $\varepsilon := \frac{1}{2}(p + \sqrt{p^2 - 4})$. Then,

$$\begin{aligned} (2\varepsilon)^{q-1} &= (p + \sqrt{p^2 - 4})^{q-1} \\ &\equiv (p^2 - 4)^{\frac{q-1}{2}} + p(q-1)(p^2 - 4)^{\frac{q-3}{2}} \sqrt{p^2 - 4} && \pmod{p^2 o_k} \\ &\equiv (-4)^{\frac{q-1}{2}} - (-4)^{\frac{q-3}{2}} p \sqrt{p^2 - 4} && \pmod{p^2 o_k} \\ &\equiv 2^{q-1} + 2^{q-3} p \sqrt{p^2 - 4} && \pmod{p^2 o_k} \end{aligned}$$

where o_k denotes the ring of integers of k . Accordingly $\varepsilon \in U_v^{(1)} \setminus U_v^{(2)}$ where again $U_v^{(i)}$ denotes the i -th higher unit group in the local field k_v for a p -adic place v of k . This guarantees the fact that our unit ε (hence, also the principal unit) is not a p -th power locally at v . Let us now look at the class number h_k . It is possible to prove $h_k < p$ using again [25, Corollary 2] but here we have a better upper bound. Namely, by [24, Theorem (a)] or [35, Corollary 2], we have the following inequalities

$$h_k \leq \frac{1}{2} \sqrt{d_k} \leq \frac{1}{2} p.$$

Hence the prime p does not divide h_F and the field k is p -rational by Corollary 2.6. The proof is complete. \square

For $p = 3$, the above corresponding bi-quadratic field $\mathbf{Q}(\sqrt{3}, \sqrt{5})$ is not 3-rational since as noticed before $\mathbf{Q}(\sqrt{15})$ is not 3-rational. The field $\mathbf{Q}(\sqrt{2}, \sqrt{5})$ is both 2 and 3-rational (Proposition 2.12 and Corollary 2.6).

EXAMPLE 4.5. *By the above Proposition 4.4, the bi-quadratic field $\mathbf{Q}(\sqrt{15}, \sqrt{35})$ is 5-rational. On the other hand, the three quadratic subfields $\mathbf{Q}(\sqrt{15})$, $\mathbf{Q}(\sqrt{21})$ and $\mathbf{Q}(\sqrt{35})$ are 5-regular by Proposition 3.6, hence $\mathbf{Q}(\sqrt{15}, \sqrt{35})$ is also 5-regular [22, Proposition 1.1]. Therefore, by the discussion at the end of Section 2, there exist infinitely many 5-rational C.M. fields having the same maximal real subfield $\mathbf{Q}(\sqrt{15}, \sqrt{35})$.*

Let α be a prime-to- p positive integer and consider the real bi-quadratic field

$$K_\alpha := \mathbf{Q}(\sqrt{\alpha p(p+2)}, \sqrt{\alpha p(p-2)}).$$

The same arguments as those in the proof of Proposition 3.2 and the above Proposition 4.4 show that the fundamental unit of each subfield of K_α is not locally a p th-power at the p -adic places. So K_α is p -rational as soon as $p \nmid h_{K_\alpha}$. Though there exist α such that $p \mid h_{K_\alpha}$ (for instance with $p = 5$ and $\alpha = 17$), it would be interesting to find an infinite family of integers α for which $p \nmid h_{K_\alpha}$.

4.3. A step further

Having the real bi-quadratic p -rational field $\mathbf{Q}(\sqrt{p(p+2)}, \sqrt{p(p-2)})$ at hand, the first question which springs to mind is how to add an imaginary square root in order to obtain a tri-quadratic p -rational field. We already noticed that the imaginary quadratic fields $\mathbf{Q}(\sqrt{-p})$ and $\mathbf{Q}(\sqrt{-p-2})$ are p -rational. Likewise, $\mathbf{Q}(\sqrt{-p+2})$ is p -rational since p does not divide its class number according to the above Theorem 4.1. Therefore, by Proposition 4.4, the tri-quadratic number field

$$L := \mathbf{Q}(\sqrt{p(p+2)}, \sqrt{p(p-2)}, \sqrt{-p}) = \mathbf{Q}(\sqrt{-p-2}, \sqrt{-p}, \sqrt{-p+2})$$

is p -rational precisely when the imaginary quadratic field $\mathbf{Q}(\sqrt{-p(p^2-4)})$ is p -rational. Using PARI/GP, the smallest prime number dividing its class number is $p_1 := 192.699.943$. Thus, at least for every prime $p < p_1$, the above field L is p -rational. It would be interesting to investigate its p -rationality for an arbitrary prime p . It is also worth mentioning that the existence of the above tri-quadratic p -rational field together with the method developed by Greenberg [14, Proposition 6.1 and Remark 6.8] guarantee the existence of continuous representations

$$\rho : G_{\mathbf{Q}} \rightarrow GL_n(\mathbf{Z}_p)$$

of the absolute Galois group $G_{\mathbf{Q}}$ of \mathbf{Q} with open image for both $n = 4$ and $n = 5$ and at least for all $p < 192.699.943$.

Another natural choice is to add the square root of -1 to the real bi-quadratic p -rational field $\mathbf{Q}(\sqrt{p(p+2)}, \sqrt{p(p-2)})$. Then the p -rationality of the tri-quadratic field

$$\mathbf{Q}(\sqrt{p(p+2)}, \sqrt{p(p-2)}, \sqrt{-1})$$

is equivalent to that of the three imaginary quadratic subfields other than $\mathbf{Q}(\sqrt{-1})$ which is also interesting to investigate.

In any case, we believe that for each odd prime $p > 3$, there exists a square-free positive integer d_p such that the relative class number of

$$\mathbf{Q}(\sqrt{p(p+2)}, \sqrt{p(p-2)}, \sqrt{-d_p})$$

is not divisible by p . Such a tri-quadratic field is then p -rational by Propositions 4.4 and 2.14. Therefore, we would have the above representations

$$\rho : G_{\mathbf{Q}} \rightarrow GL_n(\mathbf{Z}_p),$$

for both $n = 4$ and $n = 5$ and any prime number p .

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