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# A simple characterization for sustained growth\*

Thai Ha-Huy<sup>†</sup>, Nhat-Thien Tran<sup>‡</sup>

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## ABSTRACT

*This article considers an inter-temporal optimisation problem in a general form and gives conditions ensuring the convergence to infinity of the economy. These conditions can be easily verified and applied for a large class of problems in the literature. Some applications for different economies are given as illustrative examples.*

*Keywords:* Sustained growth, non-convex dynamic programming, super-modularity

*JEL classification:* C61, O40, O41

## 1 INTRODUCTION

Initiated by Bellman [4], the dynamic programming literature has rapidly become a workhorse of economic dynamic analysis. The traditional approach, culminating

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in Stokey and Lucas (with Prescott) in [26], gives a good explanation for and prediction of many economic phenomena. The theory of dynamic programming described in Stokey and Lucas (with Prescott) [26] is based on a relatively strong structure of convexity. One of its implications is that in general, the economy converges to a steady state independently of the initial state.

Many studies have shown configurations where this strong convex structure is not satisfied. Clark [7], Majumdar and Mitra [21], Majumdar and Nermuth [20], and Skiba [25] consider economies where production functions exhibit an early phase of increasing returns, usually known as *convex-concave* functions. Dechert and Nishimura [10] extend their works to a general non-concave production function. These works prove the existence of a critical level of capital stock, usually named the "*Dechert-Nishimura-Skiba*" point<sup>1</sup>. Beginning with a level capital stock under the *Dechert-Nishimura-Skiba* point, the economy shrinks and collapses to zero, otherwise it increases to a steady state<sup>2</sup>.

Kamihigashi and Roy [16] extend the analysis to a larger class of production function, by assuming only the upper-semi continuity. They characterise the critical point below which the economy collapses in the long run and above which survival (bounded away from zero) is possible.

Another line of the literature studies conditions allowing the convergence to infinity of the economy. Jones and Manuelli [12], [13] work with concave production function which keeps sufficiently high productivity even with a large accumulation of capital. Under this condition, the economy always converges to infinity.

Kamihigashi and Roy [17] relax not only the concavity but also the continuity of production, and prove that under the condition that the productivity is sufficiently high for a large accumulation of capital stock, if the initial state of the economy is higher than a critical level, it will increase to infinity<sup>3</sup>.

Majumdar and Nermuth [21], Dechert and Nishimura [10], Mitra and Ray [22],

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<sup>1</sup>For a more detail survey, see Akao *et al* [1].

<sup>2</sup>For an analysis in continuous time, see Akao *et al* [2].

<sup>3</sup>In order to cover the case where the production function is not continuous, their condition is stated on the properties of upper and under derivatives of this function.

and Kamihigashi and Roy [17] use the notion of *net gain function*, representing the discounted net returns on investment. They prove that the economy always evolves to increase the value of *net gain function*. It is interesting and surprising to see how the use of this notion provides such rich results, and it gives us deep insights in economic dynamics.

Roy [24] studies an economy with *wealth effects*, where the utility depends not only on the consumption but also on the capital level. He proves that if the sum of the marginal rate of substitution between capital-consumption and the productivity overcomes the discount rate, beginning with a sufficiently high level of capital accumulation, the economy does not stop accumulating and, hence, converges to infinity.

In this article, we consider the same question about conditions ensuring sustained growth, in the generalised case, *i.e.* where the dynamics of the economy can be characterised as a solution of

$$\max \left[ \sum_{s=0}^{\infty} \delta^s V(x_s, x_{s+1}) \right],$$

where  $\delta \in (0, 1)$  is the discount factor and  $V$  denotes the payoffs function.

Under mild conditions, the following condition is sufficient for characterising sustained growth:

$$V_2(x, x) + \delta V_1(x, x) > 0, \tag{1.1}$$

for every  $x$  large enough<sup>4</sup>.

The intuition for (1.1) is that, if choosing between saving and remaining in *status quo*, the saving choice always prevails, then sustained growth is possible.

The results in this article allow us to gather a large class of cases studied in the literature under the same viewpoint. It can also be applied to situations where Kamihigashi and Roy's [17] techniques for one-sector economy can not be

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<sup>4</sup>The notations  $V_1(x, y)$  and  $V_2(x, y)$  denote respectively the partial derivatives corresponding to the first and the second arguments.

used, for example, two-sector economies, the economy with *wealth effects* presented by Roy [24] and the *capitalism spirit* of Kamihigashi [15], or an economy with accumulation of human capital, presented in this article.

The article is organised as follows. Section 2 presents the fundamentals of the model. Under the *tail-insensitivity* condition, an optimal solution exists and, under the *super-modularity* condition, its monotonicity is ensured. Section 3 studies the conditions ensuring sustained growth, with the main one being (1.1). Section 4 concludes. Examples and proofs are given in the Appendix.

## 2 FUNDAMENTALS

### 2.1 THE MODEL

Time is discrete:  $s = 0, 1, 2, \dots$ . The discount factor is  $0 < \delta < 1$ . The technology of this economy is characterised by a correspondence  $\Gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ . For  $x_0 \geq 0$ , denote by  $\Pi(x_0)$  the set of feasible paths  $\{x_s\}_{s=0}^\infty$  satisfying  $x_{s+1} \in \Gamma(x_s)$  for all  $s \geq 0$ .

Given capital stocks at some consecutive dates  $x_s$  and  $x_{s+1}$ , the corresponding payoff utility level at date  $s$  is  $V(x_s, x_{s+1})$ , where  $V$  is a real function whose domain of definition is the graph of  $\Gamma$ : the set  $(x, y)$ , such that  $y \in \Gamma(x)$ .

For a given  $x_0 \geq 0$ , the economy solves the following inter-temporal optimisation problem

$$\begin{aligned} \max & \left[ \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}) \right], \\ \text{s.c } & x_{s+1} \in \Gamma(x_s), \forall s \geq 0. \end{aligned}$$

Denote by  $v$  the value function of this problem:

$$v(x_0) = \sup_{\Pi(x_0)} \left[ \sum_{s=0}^{\infty} \delta^s V(x_s, x_{s+1}) \right].$$

## 2.2 EXISTENCE OF SOLUTION AND THE BELL-MAN FUNCTIONAL EQUATION

Assumption **A1** establishes standard conditions ensuring the existence of a solution for the maximisation problem. For the detailed comments about these conditions, curious readers can refer to Le Van and Morhaim [18].

**Assumption A1.** i) *The correspondence  $\Gamma$  is non-empty, convex, compact valued, ascending<sup>5</sup> and upper semi-continuous.*

ii) *The function  $V$  is continuous on  $\text{Graph}(\Gamma)$  and differentiable on its interior, strictly increasing with respect to the first argument and strictly decreasing with respect to the second one.*

iii) *Non-triviality: for every  $x_0 > 0$ , there exists  $\{x_s\}_{s=0}^\infty \in \Pi(x_0)$  such that*

$$\sum_{s=0}^{\infty} \delta^s V(x_s, x_{s+1}) > -\infty.$$

iv) *Tail-insensitivity: Fixed  $x_0 > 0$ , for every  $\epsilon > 0$ , there exists  $T_0$ , a neighbourhood  $\mathcal{V}$  of  $x_0$ , such that for every  $x'_0 \in \mathcal{V}$ , any  $\{x'_s\}_{s=0}^\infty \in \Pi(x'_0)$ , any  $T \geq T_0$ :*

$$\sum_{s=T}^{\infty} \delta^s V(x'_s, x'_{s+1}) < \epsilon.$$

The conditions (i), (ii) and (iii) are usual in the literature, characterising the main properties of the technology, the trade-off between consume today and invest tomorrow, and ensuring that the problem is not trivial.

The most important condition is the *tail-insensitivity* one. This condition not only states that the value function should be finite, but also allows the satisfaction of *upper semi-continuity*, which is important for the existence of a solution.

Under **A1**, the value function is strictly increasing and upper-semi continuous.

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<sup>5</sup>For  $x \leq x'$ ,  $y \in \Gamma(x)$ ,  $y' \in \Gamma(x')$ , we have  $\min\{y, y'\} \in \Gamma(x)$  and  $\max\{y, y'\} \in \Gamma(x')$ .

This continuity ensures the existence of a solution for the optimisation problem. For the details about the proof of Proposition 2.1, curious readers are referred to Stokey and Lucas (with Prescott) [26], chapter 4.

PROPOSITION 2.1. *Assume **A1**. Then:*

- i) *The value function  $v$  is strictly increasing and upper-semi continuous.*
- ii) *A solution exists.*
- iii) *The value function satisfies the Bellman equation:*

$$v(x_0) = \max_{x_1 \in \Gamma(x_0)} [V(x_0, x_1) + \delta v(x_1)].$$

- iv) *A sequence  $\{x_s\}_{s=0}^{\infty}$  is a optimal if and only if for all  $s \geq 0$ ,*

$$v(x_s) = V(x_s, x_{s+1}) + \delta v(x_{s+1}).$$

From now on, for  $x_0 \geq 0$ , we denote by  $\phi$  the optimal policy correspondence:

$$\phi(x_0) = \operatorname{argmax}_{x_1 \in \Gamma(x_0)} [V(x_0, x_1) + \delta v(x_1)].$$

Proposition 2.1 has a consequence that  $\phi(x_0)$  is a non-empty, compact valued correspondence<sup>6</sup>.

### 2.3 SUPER-MODULARITY AND MONOTONICITY

In this section, we will study the monotonicity of optimal path and optimal policy correspondence. It is intuitive to assume super-modularity, a property stating the complementarity of capital accumulations.

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<sup>6</sup>As in Dechert and Nishimura [10], under super-modularity condition, for  $x_1 \in \phi(x_0)$ , the set  $\phi(x_1)$  is single-valued. Moreover, the value function  $v$  is differentiable at  $x_1$ . This implies that almost everywhere, the correspondence  $\phi$  is single-valued and the value function is differentiable. A generalisation of this result for configurations with uncertainty is given in Nishimura *et al* [23].

**Assumption A2.** *The payoff function  $V$  is strictly super modular.*<sup>7</sup>

Under the super-modularity property, the optimal policy correspondence is "increasing", as stated in Proposition 2.1. This is an important result helping the understanding of optimal paths' behaviour. The super-modularity implies that every optimal path is strictly monotonic. The result and proof of Lemma 2.1 are similar to the one-sector configuration studied in Dechert and Nishimura [10], and Amir [3].

**LEMMA 2.1.** *Assume A1 and A2. Then*

- i) *For every  $x_0 < x'_0$ , and  $x_1 \in \phi(x_0)$ ,  $x'_1 \in \phi(x'_0)$ , we have  $x_1 < x'_1$ .*
- ii) *Every optimal path is either strictly monotonic or constant.*

A direct consequence of Lemma 2.1 is that every optimal path converges either to some real value, or to infinity. Moreover, Lemma 2.1 allows us to characterise a general feature of optimal paths, stated in Proposition 2.2. If, for some initial state  $x_0$ , the optimal path converges to infinity, then the same property is also satisfied for every greater initial level of capital stock, thanks to the monotonicity of optimal policy correspondence. If such an initial state  $x_0$  does not exist, every optimal path is bounded from above.

**PROPOSITION 2.2.** *Assume A1 and A2. Then only one of the following holds:*

- i) *There exists  $\bar{x} \geq 0$ , such that for every  $x_0 \geq \bar{x}$ , every optimal path beginning from  $x_0$  is strictly increasing and converges to infinity.*
- ii) *for every  $x_0 \geq 0$ , every optimal path beginning from  $x_0$  is bounded from above.*

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<sup>7</sup>The (strict) super-modularity is defined as: for every  $(x, x')$  and  $(y, y')$  that belong to  $\text{Graph}(\Gamma)$ ,  $V(x, y) + V(x', y')(>) \geq V(x', y) + V(x, y')$  is verified whenever  $(x', y')(>) \geq (x, y)$ . When  $V$  is twice differentiable, (strict) super modularity sums up to positive cross derivatives:  $V_{12}(x, y>(>)) \geq 0$  for every  $x, y$ . See Amir [3].



### 3 THE SUSTAINED GROWTH CONDITION

Under Assumption **A1**, a solution exists. Under Assumption **A2**, optimal paths are monotonic. Hence, each optimal path converges either to a steady state, or to infinity. If, for some  $x_0$ , there is an optimal path beginning from  $x_0$  that converges to infinity, then this property is verified for every optimal path beginning from  $x'_0 > x_0$ .

In this section, we discuss the condition ensuring the possibility of sustained growth, *i.e.*, the convergence to infinity of the economy. To fix ideas and to simplify the exposition, we also assume that the payoff function  $V$  is bounded from below. The case where  $V$  is unbounded from below will be discussed in Section 3.3.

#### 3.1 THE CONDITION

The main idea runs as follows: staying in the *status quo* is better than diminishing the capital accumulation and remaining unchanging after that. Under this situation, the economic agents prefer to save and the capital increases.

DEFINITION 3.1. *The state  $x > 0$  satisfies condition  $\mathcal{C}$  if  $x \in \text{int}(\Gamma(x))$  and*

$$V_2(x, x) + \delta V_1(x, x) > 0. \tag{3.1}$$

The significance of condition  $\mathcal{C}$  is intuitive: for every capital accumulation level  $x$ , between the choice of staying in *status quo* and saving a little, the economy prefers the latter.

Proposition 3.1 states our first main result. The idea runs as follows<sup>8</sup>. Assume that condition  $\mathcal{C}$  is satisfied for every  $x > 0$ . Suppose there is a strictly decreasing

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<sup>8</sup>The idea is inspired by similar consideration in Cao and Werming [6].

optimal path from some  $x_0 > 0$ . For every  $T$ ,

$$\begin{aligned}
v(x_0) &\geq \sum_{s=0}^{\infty} \delta^s V(x_0, x_0) \\
&= \frac{V(x_0, x_0)}{1 - \delta} \\
&> V(x_0, x_1) + \frac{\delta V(x_1, x_1)}{1 - \delta} \\
&> V(x_0, x_1) + \delta V(x_1, x_2) + \frac{\delta^2 V(x_2, x_2)}{1 - \delta} \\
&\dots \\
&> \sum_{s=0}^T \delta^s V(x_s, x_{s+1}) + \frac{\delta^{T+1} V(x_{T+1}, x_{T+1})}{1 - \delta}.
\end{aligned}$$

Let  $T$  converges to infinity. As  $V$  is bounded from below, the right-hand-side of the inequality converges to  $v(x_0)$ , which leads us to a contradiction.

### 3.2 THE SUSTAINED GROWTH

In this subsection, we prove that if the condition  $\mathcal{C}$  is satisfied for every  $x > 0$ , then the sustained growth is ensured.

**PROPOSITION 3.1.** *Assume **A1** and **A2**. Assume that every  $x > 0$  satisfies condition  $\mathcal{C}$ .*

*Then every optimal path beginning from  $x_0 > 0$  is strictly increasing and converges to infinity.*

In opposition to the condition for sustained growth, we can also characterise the one under which the economy is always bounded.

**PROPOSITION 3.2.** *Assume **A1** and **A2**. Suppose that there exists some  $\tilde{x}$  such that for every  $x > \tilde{x}$ , either  $x \leq \min \Gamma(x)$ , or  $x \in \text{int}(\Gamma(x))$  and*

$$V_2(x, x) + \delta V_1(x, x) < 0.$$

*Then for every  $x_0$ , every optimal path beginning from  $x_0$  is bounded.*

### 3.3 NO UNBOUNDED PAYOFF FUNCTION

By the importance of logarithmic function, in many studies, the payoff function  $V$  is assumed to be unbounded from below. Moreover, it is possible that, for example in economies with a *convex-concave* production function, the condition  $\mathcal{C}$  is satisfied only for a sufficiently large level of capital accumulation. This section provides a partial response to these concerns. First, we relax the condition that  $V$  is bounded from below.

Under a suitable conditions, without the boundeness hypothesis of payoff function, the condition  $\mathcal{C}$  ensures sustained growth. Condition (3.2) states that although the payoff function is unbounded from below, the speed of convergence to minus infinity when  $x$  converges to zero is not too fast<sup>9</sup>. Under this condition, extinction does not occur along optimal paths.

PROPOSITION 3.3. *Assume **A1** and **A2**.*

- i) *Suppose that for every  $x$  that is sufficiently small, the condition  $\mathcal{C}$  is satisfied. Moreover,*

$$\lim_{x \rightarrow 0} [V(x, 0) - V(x, x)] = 0. \quad (3.2)$$

*Then every optimal path beginning from  $x_0 > 0$  is bounded away from zero.*

- ii) *Assume further that for every  $x > 0$ , the condition  $\mathcal{C}$  is satisfied. Then every optimal path beginning from  $x_0 > 0$  is strictly increasing and converges to infinity.*

Part (ii) in Proposition 3.3 is a direct consequence of (i). As optimal paths are bounded away from zero, they can not be decreasing. Hence, under condition  $\mathcal{C}$ , every optimal path converges to infinity.

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<sup>9</sup>For example, in the one sector economy in Kamihigashi and Roy [17], (3.2) is satisfied for logarithmic utility function and a production function satisfying Inada condition at zero.

COROLLARY 3.1. *Assume **A1** and **A2**. Suppose that for every  $x > 0$ , the function  $V(x, \cdot)$  is concave in respect to the second argument. Moreover,*

$$\lim_{x \rightarrow 0} xV_2(x, x) = 0. \quad (3.3)$$

*Then every optimal path beginning from  $x_0 > 0$  is bounded away from zero.*

Corollary 3.1 follows from the result that under the concavity hypothesis, (3.3) implies (3.2).

In the economic literature, it is possible that for some small value of capital stock, the condition  $\mathcal{C}$  is not satisfied. Proposition 3.4 establishes conditions ensuring sustained growth for cases where the economy begins with a sufficiently large level of capital.

PROPOSITION 3.4. *Assume **A1** and **A2**. Suppose that for every  $x$  sufficiently large, the condition  $\mathcal{C}$  is satisfied. Moreover, there exists  $x > 0$  such that*

$$\int_x^\infty (V_2(y, y) + \delta V_1(y, y)) dy = \infty.$$

*Then there exists  $\bar{x}$  such that every optimal path beginning from  $x_0 > \bar{x}$  is strictly increasing and converges to infinity.*

Corollary 3.2 follows from Proposition 3.4.

COROLLARY 3.2. *Assume **A1**, **A2**. Suppose that,*

$$\lim_{x \rightarrow \infty} V(x, x) = \infty.$$

*Suppose that for some  $\epsilon > 0$ , the following inequality is satisfied for every  $x$  that is sufficiently large:*

$$\frac{-V_2(x, x)}{\delta V_1(x, x)} \leq 1 - \epsilon.$$

*Then there exists  $\bar{x}$  such that every optimal path beginning from  $x_0 > \bar{x}$  is strictly increasing and converges to infinity.*

## 4 CONCLUSIONS

This article establishes conditions ensuring sustained growth. The threshold beyond which the economy converges to infinity is characterised. The conditions, in our subjective opinion, are simple and easy to verify. Conditions under which extinction does not occur along optimal paths are precise. We can apply them in a large class of inter-temporal optimisation problems.

The *strict* super-modularity in condition **A2** is not only for technical convenience. If the utility function satisfies only the super-modularity (but not strict), the optimal paths exhibit complicated behaviours. For example, in Kamihigashi and Roy [16], the instantaneous utility function is linear, and the optimal path reaches one steady state in a finite time and can jump between different steady states afterwards<sup>10</sup>. A careful consideration for this case is interesting, but that must be the subject of another work.

## 5 APPENDIX

### 5.1 APPLICATIONS

In this section, we discuss some applications. For the details of the models, see Kamihigashi and Roy [17], Roy [24], Dana and Le Van [9] and Crettez *et al* [8]. The detailed proofs can be found in the online version of this article [11].

#### 5.1.1 ONE SECTOR ECONOMY

For the details, see Kamihigashi and Roy [17]. The payoff function is defined as  $V(x, y) = u(f(x) - y)$ .

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<sup>10</sup>The monotonicity is not verified.

We have

$$\begin{aligned} V_2(x, x) + \delta V_1(x, x) &= -u(f(x) - x) + \delta u(f(x) - x) f'(x) \\ &= u'(f(x) - x) (\delta f(x) - 1). \end{aligned}$$

The condition  $\mathcal{C}$  is equivalent to  $f'(x) > \frac{1}{\delta}$ . This is the same condition in Kamihigashi and Roy [17].

PROPOSITION 5.1. *Assume that*

- i)  $\liminf_{x \rightarrow \infty} f'(x) > \frac{1}{\delta}$ .
- ii) *The utility function is unbounded from above.*

*Then there exists  $\bar{x} \geq 0$ , such that for every  $x_0 \geq \bar{x}$ , every optimal path beginning from  $x_0$  is strictly increasing and converges to infinity.*

Proposition 5.1 is a direct consequence of Proposition 3.4. It adds a complementary feature to the result of Kamihigashi and Roy [17], which requires that  $\lim_{c \rightarrow \infty} u'(c)c < \infty$  and hence rules out the constant elasticity and constant elasticity of marginal utility functions.

### 5.1.2 A TWO-SECTOR ECONOMY

Consider the two-sectors economy in Dana and Le Van<sup>11</sup> [9]. One sector produces consumption good, characterised by function  $f$ , and the other one produces capital good, characterised by function  $g$ . Both functions are concave.

Define  $\zeta(x) = g^{-1}(x)$ , the inverse function of  $g$ . This function is convex. The payoff function is:

$$V(x, y) = u[f(x - \zeta(y))].$$

The condition  $\mathcal{C}$  is equivalent to  $\zeta'(x) < \delta$ . Proposition 5.2 is a consequence of Proposition 3.4.

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<sup>11</sup>Chapter 4, page 92.

PROPOSITION 5.2. *Assume that  $\delta(1 + \lambda) > 1$ . Then, for every  $x > 0$ ,  $\zeta'(x) < \delta$  and the optimal path beginning from  $x$  is strictly increasing, converges to infinity.*

### 5.1.3 THE ECONOMY WITH WEALTH EFFECTS

Consider the model of economic growth with wealth effects, presented in Kamihi-gashi [15] and Roy [24]. In this set up, the utility function depends on consumption level and capital stock level.

The indirect function is  $V(x, y) = u(f(x) - y, x)$ . This function is concave. The *Euler condition* is equivalent to

$$f'(x) + \frac{u_x(f(x) - x, x)}{u_c(f(x) - x, x)} > \frac{1}{\delta}.$$

This is the same condition as in Roy [24]. By Roy [24], for  $x$  that is sufficiently large, the optimal policy function  $h$  satisfies  $h(x) \geq x$ . We weaken the condition **U1** of Roy by the following one: either  $u_{cx}(c, x) \geq 0$  for every  $(c, x)$ , or  $u_{cx}(c, x) \leq 0$  for every  $(c, x)$ . By Proposition 3.1, we obtain the same result in Roy [24].

PROPOSITION 5.3. *Denote by  $x^*$  the biggest steady state (if steady state does not exist, let  $x^* = 0$ ). Assume also that for some  $x > x^*$  we have*

$$f'(x) + \frac{u_x(f(x) - x, x)}{u_c(f(x) - x, x)} > \frac{1}{\delta}.$$

*Then every optimal path beginning from  $x_0 > x^*$  is strictly increasing and converges to infinity.*

### 5.1.4 OPTIMAL GROWTH WITH INVESTMENT ENHANCING LABOUR

We consider the optimisation problem presented by Crettez et al [8], considering an economy with investment enhancing labour. The labour force is divided in one part to the production sector, and in other part to enhancing labour, for example the labour allocated to the financial sector.

The payoff function is defined as

$$V(x, y) = \max u(F(x, y)),$$

where

$$F(x, y) = \max_{0 \leq z \leq 1} \left( f(x, 1 - z) - \frac{y}{\phi(z)} \right).$$

The condition  $\mathcal{C}$  is equivalent to

$$\phi(z(x, x)) f_1(x, 1 - z(x, x)) > \frac{1}{\delta}.$$

Proposition 5.4 is a consequence of Propositions 3.1 and 3.4.

PROPOSITION 5.4. *Assume that  $u$  is unbounded from above and*

$$\liminf_{x \rightarrow \infty} \left( -\frac{1}{\phi(z(x, x))} + \delta f_1(x, 1 - z(x, x)) \right) > 0.$$

*Then for a large enough  $x$ , every optimal path beginning from  $x$  is strictly increasing and converges to infinity.*

Although the productivity is low, sustained growth may occur if the financial sector is sufficiently efficient.

### 5.1.5 OPTIMAL GROWTH WITH HUMAN CAPITAL

In this section, we consider a economy where investing in human capital may yield a sustainable economic growth, presented in Tran [27]. There is no physical capital. The economic agent divides the production in consumption and investment in



human capital, and solves the inter-temporal maximization:

$$\begin{aligned} & \max \left[ \sum_{t=0}^{\infty} \delta^t u(c_t) \right], \\ & \text{s.c } c_t + s_{t+1} \leq f(h_t), \\ & \frac{h_{t+1}}{h_t} = \varphi(s_{t+1}). \end{aligned}$$

The quantities  $c_t, s_t$  are respectively the consumption and the saving at period  $t$  and  $h_t$  is the human capital at the same period. The function  $\varphi$  represents the technology of human capital formation. It is supposed to be continuous and strictly increasing.

Let  $\psi(s) = \varphi^{-1}(s)$ . It is the inverse function of  $\phi$ . The payoff function is defined as

$$V(x, y) = u \left( f(x) - \psi \left( \frac{y}{x} \right) \right).$$

The condition  $\mathcal{C}$  is equivalent to

$$u' \left( f(x) - \psi(1) \right) \left( \delta f'(x) - (1 - \delta) \frac{\psi'(1)}{x} \right) > 0.$$

PROPOSITION 5.5. i) *Suppose that  $f'(x) > \frac{1-\delta}{\delta} \frac{\psi'(1)}{x}$ , for every  $x > 0$ . For every initial level of human capital, the economy converges to infinity.*

ii) *Suppose that the utility function  $u$  is unbounded from above. Assume that*

$$\liminf_{x \rightarrow \infty} f'(x) > 0.$$

*Then, there exists  $\bar{h} \geq 0$  such that for all  $h_0 > \bar{h}$ , every optimal path beginning from  $h_0$  is strictly increasing and converges to infinity.*

## 5.2 PROOFS

### 5.2.1 PROOF OF LEMMA 2.1

This proof is in line with Dechert and Nishimura [10].

Suppose that there exists  $x_0 < x'_0$ ,  $x_1 \in \phi(x_0)$ ,  $x'_1 \in \phi(x'_0)$  and  $x_1 \geq x'_1$ . Hence  $x'_1 \in \Gamma(x_0)$  and in the same definition as Amir [3]:

$$(x'_0, x_1) = (x_0, x_1) \vee (x'_0, x'_1),$$

$$(x_0, x'_1) = (x_0, x_1) \wedge (x'_0, x'_1).$$

We have

$$V(x_0, x_1) + \delta v(x_1) \geq V(x_0, x'_1) + \delta v(x'_1),$$

$$V(x'_0, x'_1) + \delta v(x'_1) \geq V(x'_0, x_1) + \delta v(x_1).$$

Combining these two equations we obtain

$$V(x_0, x_1) + V(x'_0, x'_1) \geq V(x_0, x'_1) + V(x'_0, x_1),$$

which is contradictory to the super-modularity assumption **A2**.

The monotonicity of optimal paths is a direct consequence of the monotonicity of the optimal policy correspondence.

### 5.2.2 PROOF OF PROPOSITION 2.2

Assume that for some  $\bar{x}$ , there exists an optimal path  $\{\bar{x}_s\}_{s=0}^\infty$  beginning from  $\bar{x}$  that converges to infinity. Then by induction, using Lemma 2.1, for all  $x_0 > \bar{x}$ , every optimal path  $\{x_s\}_{s=0}^\infty$  beginning from  $x_0$  satisfies  $x_s > \bar{x}_s$  for every  $s \geq 0$ . Hence, the sequence  $\{x_s\}_{s=0}^\infty$  is strictly increasing and  $\lim_{s \rightarrow \infty} x_s = \infty$ .

### 5.2.3 PROOF OF PROPOSITION 3.1

Consider  $x_0 > 0$  and an optimal path  $\{x_s\}_{s=0}^\infty$  beginning from  $x_0$ . By Proposition 2.1, the sequence  $\{x_s\}_{s=0}^\infty$  is strictly increasing, decreasing or constant. Suppose that this sequence is strictly decreasing:  $x_s > x_{s+1}$  for all  $t \geq 0$ .

First, we prove that for all  $s \geq 0$ ,

$$\frac{V(x_s, x_s)}{1 - \delta} > V(x_s, x_{s+1}) + \frac{\delta V(x_{s+1}, x_{s+1})}{1 - \delta}.$$

Indeed, for  $x_{s+1} < y < x_s$ , consider the function:

$$h(y) = V(x, y) + \frac{\delta}{1 - \delta} V(y, y).$$

Since  $x_s \in \text{int}(\Gamma(x_s))$ , by the continuity and the compact-valued properties of the correspondence  $\Gamma$ ,  $x_s \in \text{int}(\Gamma(x_{s+1}))$ . The function  $h$  is well defined and differentiable.

We have,

$$\begin{aligned} h'(y) &= V_2(x, y) + \frac{\delta}{1 - \delta} (V_1(y, y) + V_2(y, y)) \\ &\geq V_2(y, y) + \frac{\delta}{1 - \delta} (V_1(y, y) + V_2(y, y)) \\ &= \frac{1}{1 - \delta} (V_2(y, y) + \delta V_1(y, y)) \\ &> 0. \end{aligned}$$

Hence,  $h(x_s) > h(x_{s+1})$ , which implies

$$\frac{V(x_s, x_s)}{1 - \delta} > V(x_s, x_{s+1}) + \frac{\delta V(x_{s+1}, x_{s+1})}{1 - \delta}.$$

As the constant sequence  $(x_0, x_0, \dots)$  belongs to  $\Pi(x_0)$ , we have:

$$\begin{aligned}
v(x_0) &\geq \frac{V(x_0, x_0)}{1 - \delta} \\
&> V(x_0, x_1) + \frac{\delta}{1 - \delta} V(x_1, x_1) \\
&> V(x_0, x_1) + \delta V(x_1, x_2) + \frac{\delta^2}{1 - \delta} V(x_2, x_2) \\
&\dots \\
&> \sum_{s=0}^T \delta^s V(x_s, x_{s+1}) + \frac{\delta^{T+1}}{1 - \delta} V(x_{T+1}, x_{T+1}).
\end{aligned}$$

Letting  $T$  converge to infinity, the right-hand-side of the inequality converges to  $v(x_0)$ : a contradiction.

Therefore the sequence  $\{x_s\}_{s=0}^\infty$  is either strictly increasing or constant. Suppose that this sequence does not converge to infinity, then  $\lim_{s \rightarrow \infty} x_s = \tilde{x}$ . By the upper semi-continuity of the value function  $v$ , we have  $\tilde{x} \in \phi(\tilde{x})$ : the limit value  $\tilde{x}$  is a steady state. By the Euler equation, we have

$$V_2(\tilde{x}, \tilde{x}) + \delta V_1(\tilde{x}, \tilde{x}) = 0,$$

a contradiction. The proof is completed.

#### 5.2.4 PROOF OF PROPOSITION 3.2

Fix a sufficiently large  $x_0$ . Suppose that there is an strictly increasing optimal path beginning from  $x_0$ . As, for a sufficiently large  $x$ , we have  $V_2(x, x) + \delta V_1(x, x) < 0$ , the function  $h(y) = V(x, y) + \frac{\delta V(y, y)}{1 - \delta}$  is decreasing in  $[x, \infty)$ . Using the same argument in the proof of Proposition 3.1, we obtain:

$$\begin{aligned}
v(x_0) &\geq \frac{V(x_0, x_0)}{1 - \delta} \\
&> \sum_{s=0}^T \delta^s V(x_s, x_{s+1}) + \frac{\delta^{T+1} V(x_{T+1}, x_{T+1})}{1 - \delta},
\end{aligned}$$

which converges to  $v(x_0)$  when  $T$  converges to infinity: a contradiction.

### 5.2.5 PROOF OF PROPOSITION 3.3

(i) Fix  $x_0 > 0$  and an optimal path  $\{x_s\}_{s=0}^{\infty}$  beginning from  $x_0$ . Assume that this path converges to zero. This implies that for all  $0 < x < x_0$ , every optimal path beginning from  $x$  is strictly decreasing and converges to zero.

First, we prove the following claim: under (3.2),

$$\lim_{x \rightarrow \infty} \left[ v(x) - \frac{V(x, x)}{1 - \delta} \right] = 0.$$

Indeed, fix any  $\epsilon > 0$ . By (3.2), for a sufficiently small  $x$ , we have

$$V(x, 0) < V(x, x) + (1 - \delta)\epsilon.$$

Consider an optimal path  $\{x'_s\}_{s=0}^{\infty}$  beginning from  $x$ . We have

$$\begin{aligned} v(x) &= \sum_{s=0}^{\infty} \delta^s V(x'_s, x'_{s+1}) \\ &\leq \sum_{s=0}^{\infty} \delta^s V(x, 0) \\ &< \frac{V(x, x)}{1 - \delta} + \epsilon. \end{aligned}$$

As  $\epsilon$  is chosen arbitrarily, the claim is proven.

We prove the second claim: for every  $0 < x < x_0$ , the following inequality is satisfied:

$$v(x_0) - v(x) < \int_x^{x_0} V_1(y, y) dy.$$

Consider  $T$  such that  $x_T > x \geq x_{T+1}$ .

For  $0 \leq s \leq T - 1$  we have

$$\begin{aligned}
v(x_s) - v(x_{s+1}) &= V(x_s, x_{s+1}) + \delta v(x_{s+1}) - v(x_{s+1}) \\
&\leq V(x_s, x_{s+1}) + \delta v(x_{s+1}) - V(x_{s+1}, x_{s+1}) - \delta v(x_{s+1}) \\
&= V(x_s, x_{s+1}) - V(x_{s+1}, x_{s+1}) \\
&= \int_{x_{s+1}}^{x_s} V_1(y, x_{s+1}) dy \\
&\leq \int_{x_{s+1}}^{x_s} V_1(y, y) dy.
\end{aligned}$$

The last inequality comes from the super-modularity: since  $V$  satisfies the increasing differences property,  $V_1(y, y) \geq V_1(y, x_{s+1})$  for  $y \geq x_{s+1}$ .

For  $s = T$ , observe that by the *ascending* property and the continuity of  $\Gamma$ ,  $x_{T+1} \in \Gamma(x)$ . We then have

$$\begin{aligned}
v(x_T) - v(x) &\leq V(x_T, x_{T+1}) + \delta v(x_{T+1}) - V(x, x_{T+1}) - \delta v(x_{T+1}) \\
&= V(x_T, x_{T+1}) - V(x, x_{T+1}) \\
&= \int_x^{x_T} V_1(y, x_{T+1}) dy \\
&\leq \int_x^{x_T} V_1(y, y) dy.
\end{aligned}$$

This implies

$$\begin{aligned}
v(x_0) - v(x) &= \sum_{s=0}^{T-1} (v(x_s) - v(x_{s+1})) + (v(x_T) - v(x)) \\
&\leq \sum_{s=0}^{T-1} \int_{x_{s+1}}^{x_s} V_1(y, y) dy + \int_x^{x_T} V_1(y, y) dy \\
&= \int_x^{x_0} V_1(y, y) dy.
\end{aligned}$$

The second claim is proven.

Now, we chose a sufficiently small  $x$  such that

$$v(x) - \frac{V(x, x)}{1 - \delta} < \frac{1}{1 - \delta} \int_x^{x_0} (V_2(y, y) + \delta V_1(y, y)) dy.$$

Let  $\epsilon = v(x) - \frac{V(x, x)}{1 - \delta}$ .

We have

$$\begin{aligned} V(x_0, x_0) - V(x, x) &\leq (1 - \delta)v(x_0) - (1 - \delta)v(x) + (1 - \delta)\epsilon \\ &\leq (1 - \delta)(v(x_0) - v(x)) + (1 - \delta)\epsilon \\ &\leq (1 - \delta) \int_x^{x_0} V_1(y, y) dy + (1 - \delta)\epsilon. \end{aligned}$$

This implies

$$\int_x^{x_0} (V_1(y, y) + V_2(y, y)) dy \leq (1 - \delta) \int_x^{x_0} V_1(y, y) dy + (1 - \delta)\epsilon,$$

which is equivalent to

$$\int_{x^*}^x (V_2(y, y) + \delta V_1(y, y)) dy \leq (1 - \delta)\epsilon,$$

a contradiction.

The sequence  $\{x_s\}_{s=0}^\infty$  is bounded away from zero.

(ii) Fix any  $x_0 > 0$ . Suppose that there exists an optimal path  $\{x_s\}_{s=0}^\infty$  beginning from  $x_0$  that is decreasing or constant. In the former case, by part (i), this path is bounded away from zero. Using the same arguments in the proof of Proposition 3.1, we have  $\lim_{T \rightarrow \infty} \delta^T V(x_T, x_T) = 0$ , which implies  $V(x_0, x_0) > (1 - \delta)v(x_0)$ : a contradiction. In the later case, since  $x_0 \in \text{int}(\Gamma(x_0))$ , the Euler equation give  $V_2(x_0, x_0) + \delta V_1(x_0, x_0) = 0$ : a contradiction.

### 5.2.6 PROOF OF COROLLARY 3.1

By the concavity of  $V(x, \cdot)$  with respect to the second argument, we have

$$V(x, 0) - V(x, x) \leq -xV_2(x, x).$$

Then (3.3) implies (3.2).

### 5.2.7 PROOF OF PROPOSITION 3.4

Fix any  $x > 0$  such that every  $x_0 > x$  satisfies condition  $\mathcal{C}$ . Fix  $\bar{x} > x$  such that

$$v(x) - \frac{V(x, x)}{1 - \delta} < \int_x^{\bar{x}} (V_2(y, y) + \delta V_1(y, y)) dy.$$

Consider some  $x_0 > \bar{x}$  and an optimal path  $\{x_s\}_{s=0}^\infty$  beginning from it. If there exists  $T$  such that  $x_T \geq x > x_{T+1}$ , using the same arguments as in the proof of Proposition 3.3, we obtain

$$(1 - \delta) \int_x^{\bar{x}} (V_2(y, y) + \delta V_1(y, y)) \leq (1 - \delta)v(x) - V(x, x),$$

a contradiction.

Hence, the path  $\{x_s\}_{s=0}^\infty$  is bounded from below by  $x$ . Using the same arguments as in the proof of Proposition 3.1, the sequence  $\{x_s\}_{s=0}^\infty$  must be strictly increasing and converges to infinity.

### 5.2.8 PROOF OF COROLLARY 3.2

Obviously, the condition in the statement implies that  $V_2(x, x) + \delta V_1(x, x) > 0$  for  $x > x^*$ . As  $V_2(y, y) \leq 0$  for all  $y$  and

$$V(x, x) - V(x^*, x^*) = \int_{x^*}^x (V_1(y, y) + V_2(y, y)) dy,$$



we have

$$\int_{x^*}^{\infty} V_1(y, y) dy = \infty.$$

The condition in the statement also implies that for  $x > x^*$ ,

$$V_2(x, x) + \delta V_1(x, x) \geq \epsilon V_1(x, x).$$

Hence

$$\begin{aligned} \int_{x^*}^{\infty} (V_2(y, y) + \delta V_1(y, y)) dy &\geq \int_{x^*}^{\infty} V_1(y, y) dy \\ &= \infty. \end{aligned}$$

The proof is completed by applying Proposition 3.4.

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