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► **To cite this version:**

Romain Blanchard, Laurence Carassus. No-arbitrage with multiple-priors in discrete time. Stochastic Processes and their Applications, 2020, 130, pp.6657 - 6688. 10.1016/j.spa.2020.06.006 . hal-03493513

HAL Id: hal-03493513

<https://hal.science/hal-03493513v1>

Submitted on 17 Oct 2022

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No-arbitrage with multiple-priors in discrete time

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Abstract

In a discrete time and multiple-priors setting, we propose a new characterisation of the condition of quasi-sure no-arbitrage which has become a standard assumption. We show that it is equivalent to the existence of a subclass of priors having the same polar sets as the initial class and such that the uni-prior no-arbitrage holds true for all priors in this subset. This characterisation shows that it is indeed a well-chosen condition being equivalent to several previously used alternative notions of no-arbitrage and allowing the proof of important results in mathematical finance. We also revisit the geometric and quantitative no-arbitrage conditions and explicit two important examples where all these concepts are illustrated.

Keywords: No-arbitrage; Knightian uncertainty; multiple-priors; non-dominated model.

AMS 2000 subject classification: Primary 91B70, 91B30, 28B20.

1. Introduction

The concept of no-arbitrage is fundamental in the modern theory of mathematical finance. Roughly speaking, it means that one cannot hope to make a profit without taking some risk. In a classical uni-prior setting, the Fundamental Theorem of Asset Pricing (FTAP in short) makes the link between an appropriate notion
5 of no-arbitrage and the existence of equivalent risk-neutral probability measures. This result is essential for pricing issues, namely for the super-replication price which is for a given claim the minimum selling price needed to superreplicate it by trading in the market. The FTAP was initially formalised in [1] and [2] while [3] established it in a general discrete-time setting and [4] in continuous time models. The literature on the subject is huge and we refer to [5] and [6] for a general overview.

10 However, the reliance on a single probability measure has long been questioned in the economic literature and is often referred to as Knightian uncertainty, in reference to [7]. In a financial context, it is called model-risk and also has a long history. The financial crisis together with the evolution of the structure and behaviour of financial markets, have made these issues even more acute for both academics and practitioners. In particular, this has motivated further research to find good notions of no-arbitrage allowing to extend the FTAP

^{*}The authors would like to thank two anonymous Referees and the Associate Editor for their perceptive reading and valuable comments which have led to an improved version of this manuscript.

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15 and the super-replication price characterisation while accounting for model uncertainty. A typical example of such endeavor, directly motivated by concrete situations, is to find no-arbitrage prices for some exotic derivative products (such as barrier options, lookback options, double digit options,...) using as input the prices of actively traded european options, without making any assumptions on the dynamic of the underlying. This is the so-called model-independent approach, pioneered in [8]. Importantly, [9] have shown that the expected
20 dichotomy between the existence of a suitable martingale measure and the existence of a model-independent arbitrage might not hold. [10] have also established a FTAP in a model-independent framework under a fairly weak notion of no-arbitrage¹, but assuming the existence of a traded option with a super-linearly growing payoff-function.

An alternative way of modeling uncertainty is to replace the single probability measure of the classical setting
25 with a set of priors representing all the possible models: This is the so-called quasi-sure or multiple-priors approach. As the set can vary between a singleton and all the probability measures on a given space, this formulation encompasses a wide range of settings, including the classical one. As the set of priors is not assumed to be dominated, this has raised challenging mathematical questions and has lead to the development of innovative tools such as quasi-sure stochastic analysis, non-linear expectations and G-Brownian motions.
30 On these topics, we refer among others to [11], [12] and [13].

Following this approach, [14] have introduced in a discrete-time setting with finite time horizon T , a no-arbitrage condition called the $NA(Q^T)$ condition (where Q^T represents all the possible models). It states that if the terminal value of a trading strategy is non-negative Q^T -quasi-surely, then it always equals 0 Q^T -quasi-surely (see Definition 3.1). This is a natural extension of the classical uni-prior where almost sure equality
35 and inequality are replaced with their quasi-sure pendant. [14] established a generalisation of the FTAP together with a Superhedging Theorem. This framework has also been used to study a large range of related problems (FTAP with portfolio constraints, american options, worst-case optimal investment, ...) and we refer among others to [15], [16], [17] and [18].

Finally, the so-called pathwise approach is an other fruitful modeling approach: In this setting, uncertainty
40 is introduced by describing a subset of relevant events or scenarii without references to any probability measure and without specifying their relative weight. In a discrete-time setting, [19] introduces a set of scenarii \mathcal{S} representing the agent beliefs and an Arbitrage de la Classe \mathcal{S} is a trading strategy leading to a terminal value that is always non-negative for all the events in \mathcal{S} and positive for a least one event in \mathcal{S} . A corresponding FTAP is then obtained. Note that by choosing different sets \mathcal{S} , different definitions of no-arbitrage
45 can be considered and in particular the model independent approach previously mentioned can be recovered by choosing the whole space for \mathcal{S} . Importantly, [20] have recently unified the quasi-sure and the pathwise approaches showing that under technical assumptions both approaches are actually equivalent (see Metatheorem 1.1, see also Remark 3.33).

50 In this paper we follow the multiple-priors approach of [14]. Despite its success, one might still wonder

¹An arbitrage is a strategy with a strictly positive terminal payoff in all states of the world.

if the $NA(Q^T)$ condition is the “right” one. Indeed, at first sight at least, under this condition it is not even clear if there exists a model $P \in Q^T$ satisfying the uni-prior no-arbitrage condition $NA(P)$. Theorem 3.29 will prove that there exists such P . But as Lemmata 3.5 and 4.4 show, Q^T might still contain some models that are not arbitrage free. This means that an agent may not be able to delta-hedge a simple vanilla option using different levels of volatility in an arbitrage free way. So instead of $NA(Q^T)$ one may assume that every model in Q^T is arbitrage free. We call this $sNA(Q^T)$ for strong no-arbitrage, see Definition 3.3. This alternative condition has appeared in recent results on robust utility maximisation of unbounded functions, see for instance [17] and [21]. Our main result provides a characterisation of the $NA(Q^T)$ condition that gives some kind of definitive answer to these questions and confirms that the $NA(Q^T)$ condition is indeed the “right” condition in the quasi-sure setting. More precisely, Theorem 3.6 shows that the $NA(Q^T)$ condition is equivalent to the existence of a subclass of priors $\mathcal{P}^T \subset Q^T$ such that \mathcal{P}^T and Q^T have the same polar sets and such that $NA(P)$ holds true for all $P \in \mathcal{P}^T$ i.e. $sNA(\mathcal{P}^T)$ holds true. In addition to enable a better economic comprehension of $NA(Q^T)$, Theorem 3.6 also provides several interesting results. First, it allows for a short proof of a refinement of the FTAP of [14] using the classical Dalang-Morton-Willinger Theorem (see Corollary 3.10 and [15, Theorem 2.1]). Then, Theorem 3.6 provides tractable theorems for the existence of solutions in the problem of robust utility maximisation. Indeed it allows to prove the equivalence between $NA(Q^T)$ and the no-arbitrage condition introduced in [22] which states that for every prior $Q \in Q^T$ there exists $P \in Q^T$ such that $Q \ll P$ and $NA(P)$ holds true (see Corollary 3.10). It is also similar to the condition used by [21] that requires the existence of a price model which is arbitrage free and such that its redundant assets are also redundant for all other price processes (see Theorem 3.29, Remark 3.34 and also [15] in a one period setup). Finally, Theorem 3.6 allows to show that the set \mathcal{P}^T is dense in the set Q^T in a very strong sense (see Lemma 3.9) and that one may replace Q^T by \mathcal{P}^T in the problem of maximisation of robust expected utility without changing the value function (see Proposition 3.12).

We then introduce local characterisations of the $NA(Q^T)$ condition called the geometric and the quantitative conditions (see Definitions 3.18, 3.19 and Theorem 3.23). The geometric condition goes back in the uni-prior setup to [23, Theorem 3 g)] and provides some geometric intuition. Theorem 3.23 generalises the preceding result to the quasi-sure setting. The geometric condition is an important tool in the multiple-priors literature. It has been used in different setups by [19] and by [20]. It is also efficient to prove concretely that the $NA(Q^T)$ condition holds true. The quantitative no-arbitrage goes back to [24, Proposition 3.3] and is used to solve optimisation problems using the dynamic programming principle. For example, it provides explicit bounds on the optimal strategies in the problem of maximisation of expected utility, see Remark 3.21. Again Theorem 3.23 generalises [24, Proposition 3.3] to the quasi-sure setting. Together with Propositions 3.27 and 3.35, this fills a gap opened in [17, Proposition 2.3], proving difficult measurability results and opening the possibility to solve, in the setting of [14], the problem of multi-prior optimal investment for unbounded utility function defined on the whole real-line (see Remark 3.28).

Finally, Proposition 3.37 explicits the relation between the different notions of no-arbitrage in the dominated case while Proposition 3.39 is used to build examples of sets of probability measures Q^T which are not dominated.

In previous studies on the dual characterisation of the robust super-replication price, it is often assumed that there exist some additional assets available only for static trading (buy and hold), see for instance [14, Theorem 5.1]. This raises the mathematical difficulties as, roughly speaking, it breaks the dynamic consistency between time zero and future times and might prevent from obtaining a dynamic programming principle. In this paper, we are not interested in the dual representation of the super-replication price but in a primal characterisation of $NA(Q^T)$. This is why in our setting all assets are dynamically traded but some of them may be derivatives products. Obviously the level of uncertainty regarding the behaviors of each asset might depend on its nature and this will be reflected in the set of priors \mathcal{Q}^T . This follows the spirit of the original approach developed in [8] where the prices of actively traded options are taken as input. Furthermore, from a pure practical point of view, we think that additional financial assets which provide useful informations for pricing should be traded at least on a daily basis.

The proofs follow the same idea: We first study a one-period problem with deterministic initial data where we rely on separation theorem and elementary geometric consideration in finite dimension. Then we extend the results to the multi-period setting relying on advanced measurable selections arguments. The proof of Proposition 3.39 relies also on relatively recent topological results.

Finally, these theoretical results are complemented by two concrete and useful examples. The first one proposes a multiple-priors binomial model and the second one a generic way of introducing uncertainty for the discretised dynamics of a diffusion process. In both cases, we show that the $NA(Q^T)$ conditions holds true and provide explicit expressions for the parameters introduced in the geometric and quantitative versions of the $NA(Q^T)$ condition and for the set \mathcal{P}^T .

The paper is structured as follows: Section 2 presents the framework and notations needed in the paper. Different definitions of conditional support which are at the heart of our study are introduced and important measurability results established. Section 3 contains the different definitions of no-arbitrage together with our main result. In Section 4 we propose two detailed examples illustrating the previous results and also how to build sets of probability measures which are not dominated. Finally, Section 5 collects the missing proofs.

2. The Model

This section presents our multiple-priors framework and gives introductory definitions.

2.1. Uncertainty modeling

The construction of the global probability space is based on a product of the local (between time t and $t + 1$) ones using measurable selection under Assumption 2.2 below. This is tailor made for the dynamic programming approach.

We fix a time horizon $T \in \mathbb{N}$ and introduce a sequence $(\Omega_t)_{1 \leq t \leq T}$ of Polish spaces. Each Ω_{t+1} contains all possible scenarii between time t and $t + 1$. For some $1 \leq t \leq T$, we set $\Omega^t := \Omega_1 \times \dots \times \Omega_t$ (with the convention that Ω^0 is reduced to a singleton), $\mathcal{B}(\Omega^t)$ its Borel sigma-algebra and $\mathfrak{P}(\Omega^t)$ the set of all probability measures on $(\Omega^t, \mathcal{B}(\Omega^t))$. An element of Ω^t will be denoted by $\omega^t = (\omega_1, \dots, \omega_t) = (\omega^{t-1}, \omega_t)$ for $(\omega_1, \dots, \omega_t) \in \Omega_1 \times \dots \times \Omega_t$. We also introduce the universal sigma-algebra $\mathcal{B}_c(\Omega^t)$ as the intersection of all possible completions of $\mathcal{B}(\Omega^t)$.

125 Let $S := \{S_t, 0 \leq t \leq T\}$ be a \mathbb{R}^d -valued process where for all $0 \leq t \leq T$, $S_t = (S_t^i)_{1 \leq i \leq d}$ represents the price of d risky securities at time t . We assume that there is a riskless asset whose price is constant and equals 1. We also make the following assumptions already stated in [14] to which we refer for further details and motivations on the framework.

Assumption 2.1 The process S is $(\mathcal{B}(\Omega^t))_{0 \leq t \leq T}$ -adapted.

130 Trading strategies are represented by $(\mathcal{B}_c(\Omega^{t-1}))_{1 \leq t \leq T}$ -measurable and d -dimensional processes $\phi := \{\phi_t, 1 \leq t \leq T\}$ where for all $1 \leq t \leq T$, $\phi_t = (\phi_t^i)_{1 \leq i \leq d}$ represents the investor's holdings in each of the d assets at time t . The set of all such trading strategies is denoted by Φ . The notation $\Delta S_t := S_t - S_{t-1}$ will often be used. If $x, y \in \mathbb{R}^d$ then the concatenation xy stands for their scalar product. The symbol $|\cdot|$ denotes the Euclidean norm on \mathbb{R}^d (or on \mathbb{R}). Trading is assumed to be self-financing and the value at time t of a portfolio ϕ starting from initial capital $x \in \mathbb{R}$ is given by $V_t^{x, \phi} = x + \sum_{s=1}^t \phi_s \Delta S_s$.

We construct the set \mathcal{Q}^T of all possible priors in the market. For all $0 \leq t \leq T-1$, let² $\mathcal{Q}_{t+1} : \Omega^t \rightarrow \mathfrak{P}(\Omega_{t+1})$ where $\mathcal{Q}_{t+1}(\omega^t)$ can be seen as the set of all possible priors for the t -th period given the state ω^t until time t .

Assumption 2.2 For all $0 \leq t \leq T-1$, \mathcal{Q}_{t+1} is a non-empty and convex valued random set such that $\text{graph}(\mathcal{Q}_{t+1}) := \{(\omega^t, p) \in \Omega^t \times \mathfrak{P}(\Omega_{t+1}), p \in \mathcal{Q}_{t+1}(\omega^t)\}$ is an analytic set.

140 Let X be a Polish space. An analytic set of X is the continuous image of some Polish space, see [25, Theorem 12.24 p447]. We denote by $\mathcal{A}(X)$ the set of analytic sets of X and recall some key properties that will often be used without further reference in the rest of the paper. The projection of an analytic set is an analytic set (see [26, Proposition 7.39 p165]), a countable union or intersection of analytic sets is an analytic set (see [26, Corollary 7.35.2 p160]), the Cartesian product of analytic sets is an analytic set (see [26, Proposition 7.38 p165]), the image or pre-image of an analytic set is an analytic set (see [26, Proposition 7.40 p165]) and $\mathcal{B}(X) \subset \mathcal{A}(X) \subset \mathcal{B}_c(X)$ holds true (see [26, Proposition 7.36 p161, Corollary 7.42.1 p169]). However the complement of an analytic set does not need to be an analytic set.

We will also use without further references a particular case of the Projection Theorem (see [27, Theorem 3.23 p75]) and of the Auman's Theorem (see [28, Corollary 1]) which we recall for sake of completeness. Let (X, \mathcal{T}) be a measurable space and Y be some Polish space. If $G \in \mathcal{T} \otimes \mathcal{B}(Y)$, then $\text{Proj}_X(G)$, the projection of G on X , belongs to $\mathcal{T}_c(X)$, the completion of \mathcal{T} with respect to any probability measures on (X, \mathcal{T}) . Let $\Gamma : X \rightarrow Y$ be such that $\text{graph}(\Gamma) \in \mathcal{T} \otimes \mathcal{B}(Y)$. Then there exists a $\mathcal{T}_c(X) - \mathcal{B}(Y)$ measurable selector $\sigma : X \rightarrow Y$ such that $\sigma(x) \in \Gamma(x)$ for all $x \in \{\Gamma \neq \emptyset\}$.

150 From the Jankov-von Neumann Theorem (see [26, Proposition 7.49 p182]) and Assumption 2.2, there exists a $\mathcal{B}_c(\Omega^t)$ -measurable $q_{t+1} : \Omega^t \rightarrow \mathfrak{P}(\Omega_{t+1})$ such that for all $\omega^t \in \Omega^t$, $q_{t+1}(\cdot, \omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$ (recall that for all $\omega^t \in \Omega^t$, $\mathcal{Q}_{t+1}(\omega^t) \neq \emptyset$). For all $1 \leq t \leq T$ let $\mathcal{Q}^t \subset \mathfrak{P}(\Omega^t)$ be defined by

$$\mathcal{Q}^t := \{Q_1 \otimes q_2 \otimes \cdots \otimes q_t, Q_1 \in \mathcal{Q}_1, q_{s+1} \in \mathcal{SK}_{s+1}, q_{s+1}(\cdot, \omega^s) \in \mathcal{Q}_{s+1}(\omega^s), \forall \omega^s \in \Omega^s, \forall 1 \leq s \leq t-1\}, \quad (1)$$

²The notation \rightarrow stands for set-valued mapping.

where $Q^t := Q_1 \otimes q_2 \otimes \cdots \otimes q_t$ denotes the t -fold application of Fubini's Theorem (see [26, Proposition 7.45 p175]) which defines a measure on $\mathfrak{P}(\Omega^t)$ and \mathcal{SK}_{t+1} is the set of universally-measurable stochastic kernel on Ω_{t+1} given Ω^t (see [26, Definition 7.12 p134, Lemma 7.28 p174]).

Apart from Assumption 2.2, no specific assumptions on the set of priors are made: Q^T is neither assumed to be dominated by a given probability measure nor to be weakly compact. This setting allows for various general definitions of the sets Q^T . Section 4 presents some concrete examples of non-dominated settings. We refer also to [18] for other examples.

2.2. Multiple-priors conditional supports

The following definitions are at the heart of our study.

Definition 2.3 Let $P \in \mathfrak{P}(\Omega^T)$ with the fixed disintegration $P := Q_1 \otimes q_2 \otimes \cdots \otimes q_T$ where $q_t \in \mathcal{SK}_t$ for all $1 \leq t \leq T$. For all $0 \leq t \leq T-1$, the random sets $E^{t+1} : \Omega^t \times \mathfrak{P}(\Omega_{t+1}) \rightarrow \mathbb{R}^d$, D^{t+1} , $D_P^{t+1} : \Omega^t \rightarrow \mathbb{R}^d$ are defined for $\omega^t \in \Omega^t$, $p \in \mathfrak{P}(\Omega_{t+1})$ by

$$\begin{aligned} E^{t+1}(\omega^t, p) &:= \bigcap \{A \subset \mathbb{R}^d, \text{ closed}, p(\Delta S_{t+1}(\omega^t, \cdot) \in A) = 1\}, \\ D^{t+1}(\omega^t) &:= \bigcap \{A \subset \mathbb{R}^d, \text{ closed}, p(\Delta S_{t+1}(\omega^t, \cdot) \in A) = 1, \forall p \in \mathcal{Q}_{t+1}(\omega^t)\}, \\ D_P^{t+1}(\omega^t) &:= \bigcap \{A \subset \mathbb{R}^d, \text{ closed}, q_{t+1}(\Delta S_{t+1}(\omega^t, \cdot) \in A, \omega^t) = 1\}. \end{aligned} \quad (2)$$

Remark 2.4 As \mathbb{R}^d is second countable, $p(\Delta S_{t+1}(\omega^t, \cdot) \in E^{t+1}(\omega^t, p)) = 1$, see [25, Theorem 12.14] and $p(\Delta S_{t+1}(\omega^t, \cdot) \in D^{t+1}(\omega^t)) = 1$ for all $p \in \mathcal{Q}_{t+1}(\omega^t)$, see [14, Lemma 4.2].

Remark 2.5 It is easy to verify that for all $\omega^t \in \Omega^t$ and all $p \in \mathcal{Q}_{t+1}(\omega^t)$, $E^{t+1}(\omega^t, p) \subset D^{t+1}(\omega^t)$. Recall that any probability $P \in \mathfrak{P}(\Omega^T)$ can be decomposed using Borel-measurable stochastic kernel, see for instance [26, Corollary 7.27.2 p139]. But in the paper, most of the time, we work directly on the stochastic kernels and not on the probability measure. Moreover, these kernels are usually only universally measurable (see for example the definition of Q^T in (1)). So from now when we consider $P \in \mathfrak{P}(\Omega^T)$, we precise the fixed disintegration for which the required result holds true.

Let $P \in Q^T$ with the fixed disintegration $P := Q_1 \otimes q_2 \otimes \cdots \otimes q_T$ where $q_t \in \mathcal{SK}_t$ for all $1 \leq t \leq T$. Then for all $0 \leq t \leq T-1$ and all $\omega^t \in \Omega^t$, as $q_t(\cdot, \omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$, we get that (see (1))

$$D_P^{t+1}(\omega^t) = E^{t+1}(\omega^t, q_{t+1}(\cdot, \omega^t)) \subset D^{t+1}(\omega^t). \quad (3)$$

The following lemma establishes some important measurability properties of the random sets previously introduced and uses the following notations. For some $R \subset \mathbb{R}^d$, let

$$\begin{aligned} \text{Aff}(R) &:= \bigcap \{A \subset \mathbb{R}^d, \text{ affine subspace}, R \subset A\}, \\ \text{Conv}(R) &:= \bigcap \left\{ C \subset \mathbb{R}^d, \text{ convex}, R \subset C = \left\{ \sum_{i=1}^n \lambda_i p_i, n \geq 1, p_i \in R, \sum_{i=1}^n \lambda_i = 1, \lambda_i \geq 0 \right\} \right\}, \\ \overline{\text{Conv}}(R) &:= \bigcap \{C \subset \mathbb{R}^d, \text{ closed convex}, R \subset C\} = \overline{\text{Conv}(R)}, \end{aligned}$$

165 recall [29, Theorem 2.3 p12]. For a given random set $R : \Omega \rightarrow \mathbb{R}^d$, $\overline{\text{Conv}}(R)$ and $\text{Aff}(R)$ are the random sets defined by $\overline{\text{Conv}}(R)(\omega) := \overline{\text{Conv}}(R(\omega))$ and $\text{Aff}(R)(\omega) := \text{Aff}(R(\omega))$ for all $\omega \in \Omega$.

Lemma 2.6 *Let Assumptions 2.1 and 2.2 hold true and let $0 \leq t \leq T - 1$ be fixed. Let $P \in \mathcal{Q}^T$ with a fixed disintegration $P := Q_1 \otimes q_2 \otimes \cdots \otimes q_T$ where $q_t \in SK_t$ for all $1 \leq t \leq T$.*

- *The random sets E^{t+1} , $\overline{\text{Conv}}(E^{t+1})$, $\text{Aff}(E^{t+1})$ are non-empty, closed valued and $\mathcal{B}(\Omega^t) \otimes \mathcal{B}(\mathfrak{P}(\Omega_{t+1}))$ -measurable³ with graphs in $\mathcal{B}(\Omega^t) \otimes \mathcal{B}(\mathfrak{P}(\Omega_{t+1})) \otimes \mathcal{B}(\mathbb{R}^d)$.*
- *The random sets D^{t+1} , D_P^{t+1} , $\overline{\text{Conv}}(D^{t+1})$, $\overline{\text{Conv}}(D_P^{t+1})$, $\text{Aff}(D^{t+1})$ and $\text{Aff}(D_P^{t+1})$ are non-empty, closed valued and $\mathcal{B}_c(\Omega^t)$ -measurable. Furthermore their graphs belong to $\mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$.*

Proof. The measurability of D^{t+1} follows from [17, Lemma 2.2]. Fix some open set $O \subset \mathbb{R}^d$. Assumption 2.1 and [26, Proposition 7.29 p144] imply that $(\omega^t, p) \mapsto p(\Delta S_{t+1}(\omega^t, \cdot) \in O)$ is $\mathcal{B}(\Omega^t) \otimes \mathcal{B}(\mathfrak{P}(\Omega_{t+1}))$ -measurable. The measurability of E^{t+1} and D_P^{t+1} follows from

$$\begin{aligned} \{(\omega^t, p), E^{t+1}(\omega^t, p) \cap O \neq \emptyset\} &= \{(\omega^t, p), p(\Delta S_{t+1}(\omega^t, \cdot) \in O) > 0\} \in \mathcal{B}(\Omega^t) \otimes \mathcal{B}(\mathfrak{P}(\Omega_{t+1})), \\ \{\omega^t, D_P^{t+1}(\omega^t) \cap O \neq \emptyset\} &= \{\omega^t, \exists q \in \mathfrak{P}(\Omega_{t+1}), q_{t+1}(\cdot, \omega^t) = q, E^{t+1}(\omega^t, q) \cap O \neq \emptyset\} \\ &= \text{Proj}_{\Omega^t} \{(\omega^t, q), q_{t+1}(\cdot, \omega^t) = q, E^{t+1}(\omega^t, q) \cap O \neq \emptyset\} \in \mathcal{B}_c(\Omega^t), \end{aligned}$$

where we have used Assumption 2.2 and the Projection Theorem as $(\omega^t, q) \mapsto q_{t+1}(\cdot, \omega^t) - q$ is $\mathcal{B}_c(\Omega^t) \otimes \mathfrak{P}(\Omega_{t+1})$ -measurable. Then, [30, Proposition 14.2, Exercise 14.12] implies that $\overline{\text{Conv}}(E^{t+1})$, $\text{Aff}(E^{t+1})$ are $\mathcal{B}(\Omega^t) \otimes \mathcal{B}(\mathfrak{P}(\Omega_{t+1}))$ -measurable and that $\overline{\text{Conv}}(D^{t+1})$, $\overline{\text{Conv}}(D_P^{t+1})$, $\text{Aff}(D^{t+1})$ and $\text{Aff}(D_P^{t+1})$ are $\mathcal{B}_c(\Omega^t)$ -measurable. Finally, [30, Theorem 14.8] implies that the graphs of E^{t+1} , $\overline{\text{Conv}}(E^{t+1})$ and $\text{Aff}(E^{t+1})$ belong to $\mathcal{B}(\Omega^t) \otimes \mathcal{B}(\mathfrak{P}(\Omega_{t+1})) \otimes \mathcal{B}(\mathbb{R}^d)$ while the graphs of D^{t+1} , D_P^{t+1} , $\overline{\text{Conv}}(D^{t+1})$, $\overline{\text{Conv}}(D_P^{t+1})$, $\text{Aff}(D^{t+1})$ and $\text{Aff}(D_P^{t+1})$ belong to $\mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$. \square

3. No-arbitrage characterisations

3.1. Global no-arbitrage condition and main result

In the uni-prior case, for any $P \in \mathfrak{P}(\Omega^T)$, the no-arbitrage $NA(P)$ condition holds true if $V_T^{0, \phi} \geq 0$ P -a.s. for some $\phi \in \Phi$ implies that $V_T^{0, \phi} = 0$ P -a.s. In the multiple-priors setting, the no-arbitrage condition $NA(\mathcal{Q}^T)$, also referred as quasi-sure no-arbitrage, was introduced in [14]. Our main message will be that it is indeed a well-chosen assumption.

Definition 3.1 The $NA(\mathcal{Q}^T)$ condition holds true if $V_T^{0, \phi} \geq 0$ \mathcal{Q}^T -q.s. for some $\phi \in \Phi$ implies that $V_T^{0, \phi} = 0$ \mathcal{Q}^T -q.s.

Recall that for a given $\mathcal{P} \subset \mathfrak{P}(\Omega^T)$, a set $N \subset \Omega^T$ is called a \mathcal{P} -polar set if for all $P \in \mathcal{P}$, there exists some $A_P \in \mathcal{B}(\Omega^T)$ such that $P(A_P) = 0$ and $N \subset A_P$. A property holds true \mathcal{P} -quasi-surely (q.s.), if it is true outside a \mathcal{P} -polar set. Finally a set is of \mathcal{P} -full measure if its complement is a \mathcal{P} -polar set.

³See [30, Definition 14.1].

[14] proves that Definition 3.1 allows a FTAP generalisation. The $NA(Q^T)$ is equivalent to the following: For all $Q \in \mathcal{Q}^T$, there exists some $P \in \mathcal{R}^T$ such that $Q \ll P$ where

$$\mathcal{R}^T := \{P \in \mathfrak{P}(\Omega^T), \exists Q' \in \mathcal{Q}^T, P \ll Q' \text{ and } P \text{ is a martingale measure}\}. \quad (4)$$

190 The next result is straightforward.

Lemma 3.2 *Let \mathcal{P} and \mathcal{M} be two sets of probability measures on $\mathfrak{P}(\Omega^T)$ such that \mathcal{P} and \mathcal{M} have the same polar sets. Then the $NA(\mathcal{P})$ and the $NA(\mathcal{M})$ conditions are equivalent.*

Nevertheless, it is not true that under the $NA(Q^T)$ condition, the $NA(P)$ condition holds true for all $P \in \mathcal{Q}^T$, see Lemma 3.5 below. This condition is called the “strong no-arbitrage” or $sNA(Q^T)$. In the spirit of
195 the model-dependent arbitrage of [9] we also introduce the notion of “weak no-arbitrage”.

Definition 3.3 The $sNA(Q^T)$ condition holds true if the $NA(P)$ condition holds true for all $P \in \mathcal{Q}^T$. The $wNA(Q^T)$ condition holds true if there exists some $P \in \mathcal{Q}^T$ such that the $NA(P)$ condition holds true.

Remark 3.4 The $sNA(Q^T)$ is a strong condition. But it is related to practical situations in finance: If it does not hold true, there exist a model $P \in \mathcal{Q}^T$ and a strategy $\phi \in \Phi$ such that $V_T^{0,\phi} \geq 0$ P -a.s. and $P(V_T^{0,\phi} > 0) > 0$
200 and an agent having sold some derivative product may not be able to use different arbitrage free models to manage the resulting position (think for instance of different volatility levels to delta-hedge a vanilla option). The $sNA(Q^T)$ condition is also useful to obtain tractable theorems on multiple-priors expected utility maximisation for unbounded function, see [17, Theorem 3.6] and [21, Theorem 3.11]. Finally, this definition seems also relevant in a continuous time setting for studying the no-arbitrage characterisation, see [31, Definition
205 2.1, Theorem 3.4].

We illustrate now the obvious relations between the three no-arbitrage conditions introduced (see also Figure 1). The more subtle ones will be addressed in Theorems 3.6 and 3.29. This last theorem shows in particular that the $NA(Q^T)$ condition implies the $wNA(Q^T)$ one.

- Lemma 3.5**
1. Assume that $\mathcal{Q}^T = \{P\}$ for some $P \in \mathfrak{P}(\Omega^T)$. Then the $NA(Q^T)$, $sNA(Q^T)$, $wNA(Q^T)$ and
210 $NA(P)$ conditions are equivalent.
 2. The $sNA(Q^T)$ condition implies the $wNA(Q^T)$ and the $NA(Q^T)$ but the converse does not hold true.
 3. The $wNA(Q^T)$ condition does not imply the $NA(Q^T)$ condition.

Proof. The first item is clear. We now prove item 2. It is clear that the $sNA(Q^T)$ condition implies the $wNA(Q^T)$. If the $NA(Q^T)$ condition fails, there exist some $\phi \in \Phi$ and $P \in \mathcal{Q}^T$ such that $V_T^{0,\phi} \geq 0$ Q^T -q.s.
215 and $P(V_T^{0,\phi} > 0) > 0$: The $sNA(Q^T)$ condition also fails. Now consider a one-period model with one risky asset $S_0 = 0$, $S_1 : \Omega \rightarrow \mathbb{R}$ (for some Polish space Ω). Let P_1 such that $P_1(\pm \Delta S_1 > 0) > 0$ and P_2 such that $P_2(\Delta S_1 \geq 0) = 1$ and $P_2(\Delta S_1 > 0) > 0$ and set $\mathcal{Q} = \{\lambda P_1 + (1 - \lambda)P_2, 0 < \lambda \leq 1\}$. Then $NA(P_1)$ (and also $wNA(\mathcal{Q})$) and $NA(\mathcal{Q})$ hold true while $NA(P_2)$ (and also $sNA(\mathcal{Q})$) fail. Finally for item 3, consider a one period model with two risky assets $S_0^1 = S_0^2 = 0$ and $S_1^{1,2} : \Omega \rightarrow \mathbb{R}$. Let P_1 be such that $P_1(\Delta S_1^1 \geq 0) = 1$,

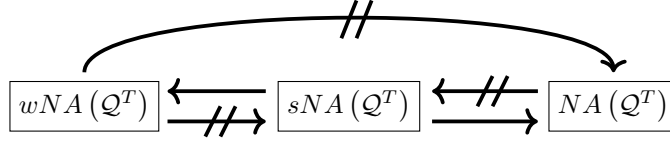


Figure 1: Relations between the no-arbitrage definitions, see Lemma 3.5.

220 $P_1(\Delta S_1^1 > 0) > 0$ and P_2 such that $P_2(\Delta S_1^1 = 0) = 1$, $P_2(\pm \Delta S_1^2 > 0) > 0$ and set $\mathcal{Q} = \{\lambda P_1 + (1-\lambda)P_2, 0 < \lambda \leq 1\}$. Then the $NA(P_2)$ and thus the $wNA(\mathcal{Q})$ conditions are clearly verified. But the $NA(\mathcal{Q})$ condition does not hold true. Indeed, let $h = (1, 0)$. Then $h\Delta S_1 \geq 0$ \mathcal{Q} -q.s. but $P_1(h\Delta S_1 > 0) > 0$. Note that $\text{Aff}(D) = \mathbb{R}^2$ while $\text{Aff}(D_{P_2}) = \{0\} \times \mathbb{R}$. \square

The following theorem is our main result.

225 **Theorem 3.6** *Assume that Assumptions 2.1 and 2.2 hold true. The following conditions are equivalent.*

- *The $NA(\mathcal{Q}^T)$ condition holds true.*
- *There exists some $\mathcal{P}^T \subset \mathcal{Q}^T$ such that \mathcal{P}^T and \mathcal{Q}^T have the same polar sets and such that the $sNA(\mathcal{P}^T)$ condition holds true i.e. $NA(P)$ holds true for all $P \in \mathcal{P}^T$.*

Proof. See Section 5.2.4. \square

Remark 3.7 Let P^* as in Theorem 3.29 below with the fix disintegration $P^* := P_1^* \otimes p_2^* \otimes \dots \otimes p_T^*$. Then, it follows from (25) that $P \in \mathcal{P}^T$ if and only if there exist some $Q \in \mathcal{Q}^T$ with the fixed disintegration $Q := Q_1 \otimes q_2 \otimes \dots \otimes q_T$ and some $(\lambda_1, \dots, \lambda_T) \in (0, 1]^T$ such that

$$P = (\lambda_1 P_1^* + (1 - \lambda_1)Q_1) \otimes (\lambda_2 p_2^* + (1 - \lambda_2)q_2) \otimes \dots \otimes (\lambda_T p_T^* + (1 - \lambda_T)q_T). \quad (5)$$

230 Theorem 3.6 shows that the $NA(P)$ condition holds true and thus proves that any (kind of) convex combination of P^* with some $Q \in \mathcal{Q}^T$ yields to an arbitrage free model.

Remark 3.8 [19, Theorem 4] delivers a similar message but in a completely different setup which does not rely on a set of priors and under the no open-arbitrage assumption defined therein. The set \mathcal{P}^T is replaced by the set of probability measures with full support.

235 The next lemma shows that the set \mathcal{P}^T is dense in \mathcal{Q}^T in a very strong sense. This allows one to switch between \mathcal{Q}^T and \mathcal{P}^T in Proposition 3.12 where not only polar sets but probability measures are involved.

Lemma 3.9 *Let $P^* \in \mathcal{Q}^T$ with the fix disintegration $P^* := P_1^* \otimes p_2^* \otimes \dots \otimes p_T^*$ as in Theorem 3.29. Let $Q \in \mathcal{Q}^T$ with the fix disintegration $Q := Q_1 \otimes q_2 \otimes \dots \otimes q_T$. Then for all $1 \leq t \leq T$, there exist some $(R_k^t)_{0 \leq k \leq t-1} \subset \mathcal{Q}^t$ such that for all $n > 1$,*

$$P_n^t := \left(1 - \frac{1}{n}\right)^t Q^t + \frac{1}{n^t} \sum_{k=0}^{t-1} (n-1)^k R_k^t \in \mathcal{P}^t. \quad (6)$$

Proof. See Section 5.2.4. □

We now propose two applications of Theorem 3.6 which show how useful it is. The first application establishes the equivalence between $NA(\mathcal{Q}^T)$ and the no-arbitrage condition introduced by [22] which studies the problem of robust maximisation of expected utility using medial limits. It also proves the robust FTAP from the classical one. Our proof uses the one-period arguments of [15, Theorem 2.1] adapted to the multi-period setting. Let $\mathcal{K}^T := \{P \in \mathfrak{P}(\Omega^T), \exists Q' \in \mathcal{P}^T, P \sim Q' \text{ and } P \text{ is a martingale measure}\} \subset \mathcal{R}^T$, see (4).

Corollary 3.10 *Assume that Assumptions 2.1 and 2.2 hold true. The following conditions are equivalent*

1. *The $NA(\mathcal{Q}^T)$ condition holds true.*
2. *For all $Q \in \mathcal{Q}^T$, there exists some $Q' \in \mathcal{P}^T$ such that $Q \ll Q'$ and such that $NA(Q')$ holds true.*
3. *For all $Q \in \mathcal{Q}^T$, there exists some $Q' \in \mathcal{Q}^T$ such that $Q \ll Q'$ and such that $NA(Q')$ holds true.*
4. *For all $Q \in \mathcal{Q}^T$, there exists some $P \in \mathcal{K}^T$ such that $Q \ll P$.*
5. *For all $Q \in \mathcal{Q}^T$, there exists some $P \in \mathcal{R}^T$ such that $Q \ll P$.*

We actually obtain a refinement of the FTAP of [14] as we have more information about the measure P in item 4.

Proof. Assume that 1. holds true and choose some $Q \in \mathcal{Q}^T$ with the fixed disintegration $Q := Q_1 \otimes q_2 \otimes \dots \otimes q_T$.

Let Q' be given by (5) with $\lambda_1 = \dots = \lambda_T = 1/2$. Then $Q \ll Q'$ and $Q' \in \mathcal{P}^T$. Now, Theorem 3.6 implies that the $NA(Q')$ condition holds true and 2. is proved. As $\mathcal{P}^T \subset \mathcal{Q}^T$, 2. implies 3.

If 2. (resp 3.) holds true, for all $Q \in \mathcal{Q}^T$ there exists some $Q' \in \mathcal{P}^T$ (resp. $Q' \in \mathcal{Q}^T$) such that $Q \ll Q'$ and such that $NA(Q')$ holds true. Now the classical FTAP (see [3]) establishes the existence of some $P \sim Q'$ such that P is a martingale measure. Thus $P \in \mathcal{K}^T$ (resp. $P \in \mathcal{R}^T$). As $Q \ll P$, 4. (resp 5.) holds true.

As $\mathcal{K}^T \subset \mathcal{R}^T$, 4. implies 5. Assume now that 5. holds true and let $\phi \in \Phi$ such that $V_T^{0,\phi} \geq 0$ \mathcal{Q}^T -q.s. Fix some $Q \in \mathcal{Q}^T$. Then there exist $P \in \mathfrak{P}(\Omega^T)$ and $Q' \in \mathcal{Q}^T$ such that $Q \ll P$, $P \ll Q'$ and P is a martingale measure. As $V_T^{0,\phi} \geq 0$ Q' -a.s and thus P -a.s. and $E_P(V_T^{0,\phi}) = 0$, we get that $V_T^{0,\phi} = 0$ P -a.s and also Q -a.s. As this is true for all $Q \in \mathcal{Q}^T$, we obtain that $V_T^{0,\phi} = 0$ \mathcal{Q}^T -q.s. and the proof is complete. □

Secondly, Theorem 3.6 allows to obtain a tractable theorem on maximisation of expected utility under the $NA(\mathcal{Q}^T)$ condition avoiding the difficult [17, Assumption 2.1]. Indeed the density property (see (6)) allows to replace the set \mathcal{Q}^T with the set \mathcal{P}^T (where $NA(P)$ holds true for all $P \in \mathcal{P}^T$) under the additional Assumption 3.13 (see Remark 3.14). Note also that the no-arbitrage condition is indeed related to the utility maximisation problem in the uni-prior case (see for instance [32]). In the robust case, it is not clear whether a similar approach could work. This is the subject of further research.

A random utility U is a function defined on $\Omega^T \times (0, \infty)$ taking values in $\mathbb{R} \cup \{-\infty\}$ such that for every $x \in \mathbb{R}$, $U(\cdot, x)$ is $\mathcal{B}(\Omega^T)$ -measurable and for every $\omega^T \in \Omega^T$, $U(\omega^T, \cdot)$ is proper⁴, non-decreasing and concave on $(0, +\infty)$. We extend U by (right) continuity in 0 and set $U(\cdot, x) = -\infty$ if $x < 0$.

Fix some $x \geq 0$. For $P \in \mathfrak{P}(\Omega^T)$ fixed, we denote by $\Phi(x, U, P)$ the set of all strategies $\phi \in \Phi$ such that

⁴There exists $x_0 \in (0, +\infty)$ such that $U(\omega^T, x_0) > -\infty$ and $U(\omega^T, x) < +\infty$ for all $x \in (0, +\infty)$.

$V_T^{x,\phi}(\cdot) \geq 0$ P -a.s. and such that either $E_P U^+(\cdot, V_T^{x,\phi}(\cdot)) < \infty$ or $E_P U^-(\cdot, V_T^{x,\phi}(\cdot)) < \infty$. Then $\Phi(x, U, \mathcal{Q}^T) := \bigcap_{P \in \mathcal{Q}^T} \Phi(x, U, P)$. The set $\Phi(x, U, \mathcal{P}^T)$ is defined similarly changing \mathcal{Q}^T by \mathcal{P}^T . The multiple-priors portfolio problem with initial wealth $x \geq 0$ is

$$u(x) := \sup_{\phi \in \Phi(x, U, \mathcal{Q}^T)} \inf_{P \in \mathcal{Q}^T} E_P U(\cdot, V_T^{x,\phi}(\cdot)). \quad (7)$$

We also define $u^{\mathcal{P}}(x)$ changing \mathcal{Q}^T by \mathcal{P}^T in (7).

Let $\mathcal{W}_t := \bigcap_{r>0} \{X : \Omega^t \rightarrow \mathbb{R} \cup \{\pm\infty\}, \mathcal{B}(\Omega^t)\text{-measurable}, \sup_{P \in \mathcal{Q}^t} E_P |X|^r < \infty\}$ for all $1 \leq t \leq T$.

Assumption 3.11 We have that $U^+(\cdot, 1), U^-(\cdot, \frac{1}{4}) \in \mathcal{W}_T$ and $\Delta S_t, 1/\alpha_t^P \in \mathcal{W}_t$ for all $1 \leq t \leq T$ and $P \in \mathcal{P}^t$ (see Remark 3.26 for the definition of α_t^P).

The first lemma shows the equality between both value functions.

Proposition 3.12 Assume that the $NA(\mathcal{Q}^T)$ condition and Assumptions 2.1 and 2.2 hold true. Furthermore, assume that U is either bounded from above or that Assumption 3.11 holds true. Then $u(x) = u^{\mathcal{P}}(x)$ for all $x \geq 0$.

Proof. Fix $x \geq 0$. Theorem 3.6 will be in force. Let P^* be given by Theorem 3.29 with the fixed disintegration $P^* := P_1^* \otimes p_2^* \otimes \dots \otimes p_T^*$. First we show that $\Phi(x, U, \mathcal{Q}^T) = \Phi(x, U, \mathcal{P}^T)$. The first inclusion follows from $\mathcal{P}^T \subset \mathcal{Q}^T$. As \mathcal{P}^T and \mathcal{Q}^T have the same polar sets, $V_T^{x,\phi}(\cdot) \geq 0$ \mathcal{Q}^T -q.s. and $V_T^{x,\phi}(\cdot) \geq 0$ \mathcal{P}^T -q.s. are equivalent. So to prove the reverse inequality it is enough to show that for $\phi \in \Phi(x, U, \mathcal{P}^T)$ $E_Q U^+(\cdot, V_T^{x,\phi}(\cdot)) < \infty$ or $E_Q U^-(\cdot, V_T^{x,\phi}(\cdot)) < \infty$ for any $Q \in \mathcal{Q}^T$. It is obviously true if U is bounded from above. Assume now that Assumption 3.11 holds true. Let $Q \in \mathcal{Q}^T$ with the fixed disintegration $Q := Q_1 \otimes q_2 \otimes \dots \otimes q_T$ and let $R \in \mathcal{P}^T$ be given by (5) with $\lambda_1 = \dots = \lambda_T = 1/2$. Assume that $E_R U^+(\cdot, V_T^{x,\phi}(\cdot)) < \infty$ (the same argument applies to U^-).

$$\frac{1}{2^T} E_Q U^+(\cdot, V_T^{x,\phi}(\cdot)) \leq E_R U^+(\cdot, V_T^{x,\phi}(\cdot)) < \infty.$$

Thus $u(x) = \sup_{\phi \in \Phi(x, U, \mathcal{P}^T)} \inf_{P \in \mathcal{Q}^T} E_P U(\cdot, V_T^{x,\phi}(\cdot))$. Next we show that for all $x \geq 0$ and $\phi \in \Phi(x, U, \mathcal{P}^T)$

$$u(x, \phi) := \inf_{P \in \mathcal{Q}^T} E_P U(\cdot, V_T^{x,\phi}(\cdot)) = \inf_{P \in \mathcal{P}^T} E_P U(\cdot, V_T^{x,\phi}(\cdot)) =: u^{\mathcal{P}}(x, \phi). \quad (8)$$

As $\mathcal{P}^T \subset \mathcal{Q}^T$, $u^{\mathcal{P}}(x, \phi) \geq u(x, \phi)$. Let $Q \in \mathcal{Q}^T$ with the fixed disintegration $Q := Q_1 \otimes q_2 \otimes \dots \otimes q_T$. Using Lemma 3.9, there exist $(R_k^T)_{0 \leq k \leq T-1} \subset \mathcal{Q}^T$ such that for all $n > 1$ $P_n := P_n^T$ defined in (6) belongs to \mathcal{P}^T and

$$u^{\mathcal{P}}(x, \phi) \leq E_{P_n} U(\cdot, V_T^{x,\phi}(\cdot)) = \frac{1}{n^T} \sum_{k=0}^{T-1} (n-1)^k E_{R_k^T} U(\cdot, V_T^{x,\phi}(\cdot)) + \left(1 - \frac{1}{n}\right)^T E_Q U(\cdot, V_T^{x,\phi}(\cdot)). \quad (9)$$

We first prove that $E_{P_n} U^+(\cdot, V_T^{x,\phi}(\cdot)) < \infty$. If U is bounded from above this is immediate. Else assume that Assumption 3.11 holds true. Theorem 3.6 implies that $sNA(\mathcal{P}^T)$ and also $NA(P^n)$ hold true. Note that

$\phi \in \phi(x, U, P^n)$. Then [33, (31)] together with Assumption 3.11 show that for P^n -almost all $\omega^T \in \Omega^T$,

$$|V_T^{x,\phi}(\omega^T)| \leq x \prod_{s=1}^T \left(1 + \frac{|\Delta S_s(\omega^s)|}{\alpha_{s-1}^{P^n}(\omega^{s-1})} \right) =: \frac{\lambda_n(\omega^T)}{2} \in \mathcal{W}_T. \quad (10)$$

Suppose that $x \geq 1$ else by monotonicity of U^+ , one may replace x by 1. Then Assumption 3.11 and [17, Proposition 3.24] (as $\lambda_n(\cdot) \geq 1$) imply that

$$E_{P^n} U^+(\cdot, V_T^{x,\phi}(\cdot)) \leq E_{P^n} U^+ \left(\cdot, \lambda_n(\cdot) \frac{1}{2} \right) \leq 2E_{P^n} \left(\lambda_n(\cdot) \left(U^+(\cdot, 1) + U^- \left(\cdot, \frac{1}{4} \right) \right) \right) < \infty. \quad (11)$$

270 Now if $E_{P^n} U^-(\cdot, V_T^{x,\phi}(\cdot)) = +\infty$, as $P^n \in \mathcal{P}^T$, we get that $u(x, \phi) \leq u^{\mathcal{P}}(x, \phi) = -\infty$. Thus $u^{\mathcal{P}}(x, \phi) = u(x, \phi)$. Assume that $E_{P^n} U^-(\cdot, V_T^{x,\phi}(\cdot)) < +\infty$. Using (11), the equality in (9) (for U^+ and U^-) implies that for all $0 \leq k \leq T-1$ $E_{R_k^T} U^\pm(\cdot, V_T^{x,\phi}(\cdot)) < +\infty$ and $E_Q U^\pm(\cdot, V_T^{x,\phi}(\cdot)) < +\infty$. Letting n go to infinity in (9) we obtain that $u^{\mathcal{P}}(x, \phi) \leq E_Q U(\cdot, V_T^{x,\phi}(\cdot))$ and taking the infimum over all $Q \in \mathcal{Q}^T$, $u^{\mathcal{P}}(x, \phi) \leq u(x, \phi)$: (8) is proved. Finally taking in (8) the supremum over all $\phi \in \Phi(x, U, \mathcal{P}^T)$, we get that $u(x) = u^{\mathcal{P}}(x)$. \square

275

To state the corollary on the existence of an optimal solution for (7), we need two additional assumptions.

Assumption 3.13 The process S is uniformly bounded from below and for all $0 \leq t \leq T-1$, $\text{graph}(\mathcal{P}_{t+1})$ is an analytic set, see (24) for the definition of \mathcal{P}_{t+1} .

Remark 3.14 As \mathcal{Q}^T (see (1)), the set \mathcal{P}^T is constructed from a sequence of random sets $(\mathcal{P}_{t+1})_{0 \leq t \leq T-1}$ (see 280 (5), (24) and (25)). In the general case, the graph of \mathcal{P}_{t+1} might not be an analytic set and Assumption 2.2 might fail for \mathcal{P}_{t+1} . This is the reason why we need the second part of Assumption 3.13 in Corollary 3.16. However, for the examples of Section 4 this technical condition is automatically verified (under the additional assumption that p_{t+1}^0 is Borel-measurable for the discretized d -dimensional diffusion case). Finally, we point out that if $\text{graph}(\mathcal{Q}_{t+1})$ belongs to the set of all nuclei of Suslin Scheme on $\mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathfrak{F}(\Omega_{t+1}))$ (see [26, 285 Definition 7.15 p157]), it can be proved that the same holds true for $\text{graph}(\mathcal{P}_{t+1})$. Considering this extended class instead of analytic sets in Assumption 2.2 and proving the results of [14] and of the present paper is left for further research.

Assumption 3.15 For all $r \in \mathbb{Q}$, $r > 0$, $\sup_{P \in \mathcal{Q}^T} E_P U^-(\cdot, r) < +\infty$.

Corollary 3.16 Assume that the $NA(\mathcal{Q}^T)$ condition and Assumptions 2.1, 2.2, 3.13 and 3.15 hold true. Furthermore, assume that U is either bounded from above or that Assumption 3.11 holds true. Let $x \geq 0$. Then, 290 there exists some optimal strategy $\phi^* \in \Phi(x, U, \mathcal{Q}^T)$ such that $u(x) = \inf_{P \in \mathcal{Q}^T} E_P U(\cdot, V_T^{x,\phi^*}(\cdot)) < \infty$.

Proof. Fix some $x \geq 0$. Theorem 3.6 implies that $sNA(\mathcal{P}^T)$ holds true. So [17, Theorem 3.6] gives the existence of an optimal strategy for $u^{\mathcal{P}}(x)$. Proposition 3.12 allows to conclude since $u(x) = u^{\mathcal{P}}(x)$. \square

3.2. Local no-arbitrage conditions and further results

295 We now turn to local conditions which are at the heart of the proofs due to the structure of the model. We recall the first part of [14, Theorem 4.5] which establishes the essential link between the global version

$NA(\mathcal{Q}^T)$ and its local version.

Theorem 3.17 *Under Assumptions 2.1 and 2.2, the following statements are equivalent.*

1. *The $NA(\mathcal{Q}^T)$ condition hold true.*

300 2. *For all $0 \leq t \leq T - 1$, there exists a \mathcal{Q}^t -full measure set $\Omega_{NA}^t \in \mathcal{B}_c(\Omega^t)$ such that for all $\omega^t \in \Omega_{NA}^t$, $h\Delta S_{t+1}(\omega^t, \cdot) \geq 0$ $\mathcal{Q}_{t+1}(\omega^t)$ -q.s. for some $h \in \mathbb{R}^d$ implies that $h\Delta S_{t+1}(\omega^t, \cdot) = 0$ $\mathcal{Q}_{t+1}(\omega^t)$ -q.s.*

We present two other local definitions of no-arbitrage and establish their equivalence with the $NA(\mathcal{Q}^T)$ conditions in Theorem 3.23 which is an analogous of Theorem 3.17.

305 The first definition proposes a geometric view of the no-arbitrage. Theorem 3.23 extends the uni-prior result of [23, Theorem 3g)], see also [6, Proposition 2.1.6]. Note that the geometric no-arbitrage has appeared in different multiple-priors contexts, see [20, Proposition 6.4] and [19, Corollary 21]. A similar idea was already exploited in [14, Lemma 3.3]. Theorem 3.23 will also allow us to prove Proposition 3.27 and Theorem 3.29.

310 Recall that for a convex set $C \subset \mathbb{R}^d$, the relative interior of C (see [29, Section 6]) is $\text{Ri}(C) = \{y \in C, \exists \varepsilon > 0, \text{Aff}(C) \cap B(y, \varepsilon) \subset C\}$ where $B(y, \varepsilon)$ is the open ball in \mathbb{R}^d centered in y with radius ε . Moreover for a convex-valued random set R , $\text{Ri}(R)$ is the random set defined by $\text{Ri}(R)(\omega) := \text{Ri}(R(\omega))$ for $\omega \in \Omega$.

Definition 3.18 The geometric no-arbitrage condition holds true if for all $0 \leq t \leq T - 1$, there exists some \mathcal{Q}^t -full measure set $\Omega_{gNA}^t \in \mathcal{B}_c(\Omega^t)$ such that for all $\omega^t \in \Omega_{gNA}^t$, $0 \in \text{Ri}(\text{Conv}(D^{t+1}))(\omega^t)$. In this case for all $\omega^t \in \Omega_{gNA}^t$, there exists $\varepsilon_t(\omega^t) > 0$ such that

$$B(0, \varepsilon_t(\omega^t)) \cap \text{Aff}(D^{t+1})(\omega^t) \subset \text{Conv}(D^{t+1})(\omega^t). \quad (12)$$

The geometric (local) no-arbitrage condition is indeed practical: Together with Theorem 3.23 it allows to check whether the (global) $NA(\mathcal{Q}^T)$ condition holds true or not. As for all $0 \leq t \leq T - 1$, \mathcal{Q}_{t+1} and ΔS_{t+1} are given one gets $\text{Ri}(\text{Conv}(D^{t+1}))$ and it is easy to check whether 0 is in it or not (see Section 4 for examples of such a reasoning).

315 Secondly, in the spirit of [24, Proposition 3.3] (see also [17, Proposition 2.3]), we introduce the so-called quantitative no-arbitrage condition.

Definition 3.19 The quantitative no-arbitrage condition holds true if for all $0 \leq t \leq T - 1$, there exists some \mathcal{Q}^t -full measure set $\Omega_{qNA}^t \in \mathcal{B}_c(\Omega^t)$ such that for all $\omega^t \in \Omega_{qNA}^t$, there exist $\beta_t(\omega^t), \kappa_t(\omega^t) \in (0, 1]$ such that for all $h \in \text{Aff}(D^{t+1})(\omega^t)$, $h \neq 0$ there exists $p_h \in \mathcal{Q}_{t+1}(\omega^t)$ satisfying

$$p_h(h\Delta S_{t+1}(\omega^t, \cdot) < -\beta_t(\omega^t)|h|) \geq \kappa_t(\omega^t). \quad (13)$$

320 *Remark 3.20* Definition 3.19 is the direct adaptation to the multiple-priors set-up of [24, Proposition 3.3]: The probability measure depends of the strategy. In the case where there is only one risky asset and one period, (13) is interpreted as follows : There exist a prior p^+ for which the price of the risky asset increases enough and an other one p^- for which it decreases i.e. $p^\mp(\pm\Delta S(\cdot) < -\beta) \geq \kappa$ where $\beta, \kappa \in (0, 1]$. For an agent buying or selling some quantity of risky assets, there is always a prior in which she is exposed to a potential loss. Proposition 3.35 will show that one can in fact choose in Definition 3.19 a common prior for all strategies.

Remark 3.21 Theorem 3.6 and Proposition 3.35 are precious for solving the problem of maximisation of expected utility. For example when the utility function U is defined on $(0, \infty)$ β_t and κ_t provide natural bounds for the one step strategies or for $U(V_T^{x, \Phi})$, see (10) and [17, Lemma 3.11 and (44)]. This is used to prove the existence of the optimal strategy but it could also be used to compute it numerically. We propose in Section 4 explicit values for β_t and κ_t .

Remark 3.22 In (13), the number $\kappa_t(\omega^t)$ serves as a measure of the gain/loss probability and the number $\beta_t(\omega^t)$ of their size. Thus, $\beta_t(\omega^t)$ provides information on $D^{t+1}(\omega^t)$ while $\kappa_t(\omega^t)$ provides information on $\mathcal{Q}_{t+1}(\omega^t)$. But we may replace in (13) $\kappa_t(\omega^t)$ and $\beta_t(\omega^t)$ by a common number $\alpha_t(\omega^t) \in (0, 1]$ choosing for example $\alpha_t(\omega^t) = \min(\kappa_t(\omega^t), \beta_t(\omega^t))$.

Theorem 3.23 Assume that Assumptions 2.1 and 2.2 hold true. Then the $NA(\mathcal{Q}^T)$ condition (see Definition 3.1), the geometric no-arbitrage (see Definition 3.18) and the quantitative no-arbitrage (see Definition 3.19) are equivalent and one can choose $\Omega_{NA}^t = \Omega_{qNA}^t = \Omega_{gNA}^t$ for all $0 \leq t \leq T - 1$. Furthermore, one can choose $\beta_t = \varepsilon_t/2$ in (13) (for ε_t introduced in (12)).

Proof. See Section 5.2.2. □

Remark 3.24 Under Assumptions 2.1 and 2.2 and any of the no-arbitrage condition, $0 \in \text{Conv}(D^{t+1})(\omega^t)$ and $\text{Aff}(D^{t+1})(\omega^t)$ is a vector space for all $\omega^t \in \Omega_{NA}^t$.

The next proposition is [23, Theorem 3] but could also be obtained as a direct application of Theorem 3.23 together with [26, Lemma 7.28 p174] and [25, Theorem 12.28] in the specific setting where $\mathcal{Q}^T := \{P_1 \otimes p_2 \otimes \dots \otimes p_T\}$ where $p_t \in \mathcal{SK}_t$ for all $1 \leq t \leq T$. Indeed, Theorem 3.23 does not apply directly as $\text{graph}(p_t)$ belongs a priori to $\mathcal{B}_c(\Omega^t \times \mathfrak{P}(\Omega_{t+1}))$ and not to $\mathcal{A}(\Omega^t \times \mathfrak{P}(\Omega_{t+1}))$, and one needs to build some Borel-measurable version of p_t . Proposition 3.25 will be used in the sequel to prove that the $NA(P)$ condition holds true.

Proposition 3.25 Assume that Assumption 2.1 holds true and let $P \in \mathfrak{P}(\Omega^T)$ with the fixed disintegration $P := P_1 \otimes p_2 \otimes \dots \otimes p_T$ where $p_t \in \mathcal{SK}_t$ for all $1 \leq t \leq T$. Then the $NA(P)$ condition holds true if and only if $0 \in \text{Ri}(\text{Conv}(D_P^{t+1}))(\cdot)$ P^t -a.s. for all $0 \leq t \leq T - 1$.

Remark 3.26 Similarly, under the assumption of Proposition 3.25, one can show that the $NA(P)$ condition holds true if and only if the quantitative no-arbitrage condition holds true for $\mathcal{Q}^T = \{P\}$ which is exactly [24, Proposition 3.3]. In this case, we denote α_t in Remark 3.22 by α_t^P .

We now establish some tricky measurability properties.

Proposition 3.27 Assume that Assumptions 2.1 and 2.2 hold true. Under one of the no-arbitrage conditions (see Definitions 3.1, 3.18 and 3.19) one can choose a $\mathcal{B}_c(\Omega^t)$ -measurable version of ε_t (in (12)) and β_t (in (13)).

Proof. See Section 5.2.2. □

Remark 3.28 The measurability of κ_t cannot be directly inferred from the one of ε_t but will be obtained in Proposition 3.35 as a consequence of Theorem 3.29. The measurability of κ_t is useful to solve the problem of multi-priors optimal investment for unbounded utility function defined on the whole real-line since the bounds on the optimal strategies depend on κ_t see for instance [24, (17)] in a non-robust setting and [21, Proof of Lemma 3.3] in the robust context.

The next theorem is crucial. It is a first step towards Theorem 3.6: It gives the existence of a measure P^* which allows to build the set \mathcal{P}^T (see (5)). But it is also of own interest since it gives the equivalence between $NA(\mathcal{Q}^T)$ and a stronger form of $wNA(\mathcal{Q}^T)$.

Theorem 3.29 *Assume that Assumptions 2.1 and 2.2 hold true. The $NA(\mathcal{Q}^T)$ condition holds true if and only if there exists some $P^* \in \mathcal{Q}^T$ with the fix disintegration $P^* := P_1^* \otimes p_2^* \otimes \cdots \otimes p_T^*$ such that $\text{Aff}(D_{P^*}^{t+1})(\omega^t) = \text{Aff}(D^{t+1})(\omega^t)$ and $0 \in \text{Ri}(\text{Conv}(D_{P^*}^{t+1}))(\omega^t)$ for all $0 \leq t \leq T-1$ and all $\omega^t \in \Omega_{NA}^t$ ⁵.*

Proof. See Section 5.2.3. □

Remark 3.30 Theorem 3.29 was proved in a one period setting in [15, Lemma 2.2].

Remark 3.31 The probability measure P^* of Theorem 3.29 is not unique. In fact, under $NA(\mathcal{Q}^T)$, all $P \in \mathcal{P}^T$ satisfy $\text{Aff}(D_P^{t+1})(\omega^t) = \text{Aff}(D^{t+1})(\omega^t)$ and $0 \in \text{Ri}(\text{Conv}(D_P^{t+1}))(\omega^t)$ for all $0 \leq t \leq T-1$ and all $\omega^t \in \Omega_{NA}^t$, see proof of Theorem 3.6 step 2 *iii*).

Remark 3.32 The main (and difficult) point in Theorem 3.29 is that $P^* \in \mathcal{Q}^T$. Thus any \mathcal{Q}^t -null set is also a P^* -null set and Ω_{NA}^t is of P^* -full measure. So $0 \in \text{Ri}(\text{Conv}(D_{P^*}^{t+1}))(\cdot)$ P^{*t} -a.s. and the $NA(P^*)$ condition holds true (see Proposition 3.25). The counter example of the last item in Lemma 3.5 illustrates why the condition $\text{Aff}(D_{P^*}^{t+1})(\cdot) = \text{Aff}(D^{t+1})(\cdot)$ \mathcal{Q}^t -q.s. is needed in Theorem 3.29. It is also easy to find a counterexample showing why $0 \in \text{Ri}(\text{Conv}(D_{P^*}^{t+1}))(\cdot)$ P^{*t} -a.s. is not enough.

Remark 3.33 Theorem 3.29 is related and complements [20, Theorem 3.1]. Indeed, in both cases the main issue is to find some $p_{t+1}^*(\cdot, \omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$ such that $0 \in \text{Ri}(\text{Conv}(D_{P^*}^{t+1}))(\omega^t) \subset \text{Ri}(\text{Conv}(D^{t+1}))(\omega^t)$ (recall (3)). This is used in [20] to make the link with the quasi-sure setting and for us to establish Theorem 3.6.

Remark 3.34 In [21] an alternative framework for model uncertainty is proposed : All possible dynamics of the stock prices are represented by a collection \mathcal{S} of stochastic processes on the same filtered probability space, rather than by a family of probability measures. The conditions verified by P^* in Theorem 3.29 are similar to [21, Assumption 2.1] which asserts that there exists at least a $S^* \in \mathcal{S}$ which is an arbitrage free model and such that the affine space generated by its conditional support is maximum (for the inclusion) over \mathcal{S} . [21, Assumption 2.1] is used to establish existence in the problem of maximisation of worst case expected utility for functions defined on $(0, \infty)$ or on the whole real line.

⁵The set Ω_{NA}^t was introduced in Theorem 3.17.

385 The following result provides an answer to the measurability issue raised in Remark 3.28 and also provides a common prior for all strategies.

Proposition 3.35 *Assume that Assumptions 2.1 and 2.2 as well as the $NA(\mathcal{Q}^T)$ condition hold true. Then for all $0 \leq t \leq T - 1$ there exist some $\mathcal{B}_c(\Omega^t)$ -measurable random variables $\beta_t(\cdot), \kappa_t(\cdot) \in (0, 1)$ such that for all $\omega^t \in \Omega_{NA}^t$ and $h \in \text{Aff}(D^{t+1})(\omega^t)$, $h \neq 0$, $p_{t+1}^*(h\Delta S_{t+1}(\omega^t, \cdot) < -\beta_t(\omega^t)|h|, \omega^t) \geq \kappa_t(\omega^t)$, where $p_{t+1}^*(\cdot, \omega^t)$ is*
 390 *defined in Theorem 3.29 with the fix disintegration $P^* := P_1^* \otimes p_2^* \otimes \dots \otimes p_T^*$.*

Proof. See Section 5.2.5. □

Remark 3.36 We have that $\beta_t(\omega^t) = \kappa_t(\omega^t) = 1$ if and only if $D_{P^*}^{t+1}(\omega^t) = \{0\}$. Indeed if $\beta_t(\omega^t) = \kappa_t(\omega^t) = 1$ and $D_{P^*}^{t+1}(\omega^t) \neq \{0\}$, then for all $h \in \text{Aff}(D^{t+1})(\omega^t)$ with $|h| = 1$, we have that $p_{t+1}^*(h\Delta S_{t+1}(\omega^t, \cdot) < -1, \omega^t) = 1$. Fix such a h and let $F_h := \{y \in \mathbb{R}^d, hy \leq -1\}$. Then $p_{t+1}^*(\Delta S_{t+1}(\omega^t, \cdot) \in F_{\pm h}, \omega^t) = 1$ and $D_{P^*}^{t+1}(\omega^t) = E^{t+1}(\omega^t, p_{t+1}^*(\cdot, \omega^t)) \subset F_{-h} \cap F_{+h} = \emptyset$, see Remark 2.5. Note that it is not easy to obtain this result for
 395 Theorem 3.23 as the prior in (13) depends on h .

3.3. Dominated versus non-dominated set of probability measures

This last subsection focuses first on the specific case of dominated set of priors. We then propose local condition in order to have a non-dominated set of priors. The proofs which are reported to Section 5.3 are
 400 quite involved. Note that the reverse implication in Proposition 3.39 seems intuitive but raises challenging technical issues.

Proposition 3.37 *Assume that Assumptions 2.1 and 2.2 hold true. Assume furthermore that there exists some dominating measure $\hat{P} \in \mathcal{Q}^T$. Then the $NA(\hat{P})$ and the $NA(\mathcal{Q}^T)$ conditions are equivalent. Suppose that one of these two conditions is satisfied. Then, for all $0 \leq t \leq T - 1$, $D_{\hat{P}}^{t+1}(\cdot) = D^{t+1}(\cdot)$ and*
 405 *$0 \in \text{Ri}\left(\text{Conv}(D_{\hat{P}}^{t+1})\right)(\cdot)$ \mathcal{Q}^t -q.s.*

Remark 3.38 One can choose $P^* = \hat{P}$ in Proposition 3.35 changing Ω_{NA}^t by the full-measure set where the properties of Proposition 3.37 hold true. Moreover, \mathcal{P}^T (see (5)) in Theorem 3.6 can be constructed from \hat{P} .

Proposition 3.39 *Assume that Assumption 2.2 holds true and that there exist some $\tilde{P} \in \mathcal{Q}^T$, some $0 \leq t \leq T - 1$ and some $\Omega_N^t \in \mathcal{B}_c(\Omega^t)$ such that $\tilde{P}^t(\Omega_N^t) > 0$ and such that $\mathcal{Q}_{t+1}(\omega^t)$ is not dominated for all $\omega^t \in \Omega_N^t$.*
 410 *Then the set \mathcal{Q}^T is not dominated.*

4. Examples

This section proposes concrete examples of multiple-priors setting illustrating our results. We also use these examples to build sets of probability measures which are not dominated.

4.1. Robust Binomial model

Suppose that $T \geq 1$, $d = 1$ and $\Omega_t = \mathbb{R}$ (or $(0, \infty)$) for all $1 \leq t \leq T$. The risky asset $(S_t)_{0 \leq t \leq T}$ is such that $S_0 = 1$ and $S_{t+1} = S_t Y_{t+1}$ where Y_{t+1} is a real-valued and $\mathcal{B}(\Omega_{t+1})$ -measurable random variable such that $Y_{t+1}(\Omega_{t+1}) = (0, \infty)$ for all $0 \leq t \leq T - 1$ (if $\Omega_t = (0, \infty)$ you can think of $Y_t(\omega_t) = \omega_t$). The positivity of Y_t implies that $S_t(\omega^t) > 0$ for all $\omega^t \in \Omega^t$. Assumption 2.1 is clearly verified. For $0 \leq t \leq T - 1$ and $\omega^t \in \Omega^t$ let

$$\begin{aligned} \mathcal{B}_{t+1}(\omega^t) &:= \{\pi\delta_u + (1 - \pi)\delta_d, \pi_t(\omega^t) \leq \pi \leq \Pi_t(\omega^t), u_t(\omega^t) \leq u \leq U_t(\omega^t), d_t(\omega^t) \leq d \leq D_t(\omega^t)\}, \\ \tilde{\mathcal{Q}}_{t+1}(\omega^t) &:= \{q \in \mathfrak{P}(\Omega_{t+1}), q(Y_{t+1} \in \cdot) \in \mathcal{B}_{t+1}(\omega^t)\} \quad \text{and} \quad \mathcal{Q}_{t+1}(\omega^t) := \text{Conv}(\tilde{\mathcal{Q}}_{t+1}(\omega^t)), \end{aligned}$$

415 where $\pi_t, \Pi_t, u_t, U_t, d_t, D_t$ are real-valued $\mathcal{B}(\Omega^t)$ -measurable random variables such that $0 \leq \pi_t(\omega^t) \leq \Pi_t(\omega^t) \leq 1$, $u_t(\omega^t) \leq U_t(\omega^t)$, $d_t(\omega^t) \leq D_t(\omega^t)$ for all $\omega^t \in \Omega^t$ ⁶ and $q(Y_{t+1} \in \cdot)$ is the law of Y_{t+1} under q . In words, at each step, the risky asset can go up or down and there is uncertainty not only on the probability of the jumps but also on their sizes.

Assumption 4.1 We have that $\pi_t(\omega^t) < 1$, $\Pi_t(\omega^t) > 0$ and $0 < d_t(\omega^t) < 1 < U_t(\omega^t)$ for all $0 \leq t \leq T - 1$ and $\omega^t \in \Omega^t$.
420

Remark 4.2 The usual binomial model (see [34]) corresponds to $\pi_t = \Pi_t = \pi$, $u_t = U_t = u$ and $d_t = D_t = d$ where $0 < \pi < 1$, $d < 1 < u$.

Lemma 4.3 Under Assumption 4.1, Assumption 2.2 holds true.

Proof. First, \mathcal{Q}_{t+1} is convex valued by definition. Since $Y_{t+1}(\Omega_{t+1}) = (0, \infty)$, $\tilde{\mathcal{Q}}_{t+1}(\omega^t) \neq \emptyset$, hence $\mathcal{Q}_{t+1}(\omega^t) \neq \emptyset$ for all $\omega^t \in \Omega^t$. We show successively that $\text{graph}(\mathcal{B}_{t+1})$, $\text{graph}(\tilde{\mathcal{Q}}_{t+1})$ and $\text{graph}(\mathcal{Q}_{t+1})$ are analytic sets. For $\omega^t \in \Omega^t$, let $E(\omega^t) := [u_t(\omega^t), U_t(\omega^t)] \times [d_t(\omega^t), D_t(\omega^t)] \times [\pi_t(\omega^t), \Pi_t(\omega^t)]$ and $F(\omega^t, u, d, \pi) := (\omega^t, \pi\delta_u + (1 - \pi)\delta_d)$ for $(\omega^t, u, d, \pi) \in \Omega^t \times \mathbb{R}^3$. Then F is Borel-measurable (see [26, Corollary 7.21.1 p130]) and $\text{graph}(E) \in \mathcal{B}(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^3)$ as $\pi_t, \Pi_t, u_t, U_t, d_t$ and D_t are Borel-measurable. We conclude that $\text{graph}(\mathcal{B}_{t+1}) = F(\text{graph}(E))$ is analytic. Let $\Phi : \mathfrak{P}(\Omega_{t+1}) \rightarrow \mathfrak{P}(\mathbb{R})$ be defined by $\Phi(q) := q(Y_{t+1} \in \cdot)$. Using [26, Propositions 7.29 p144 and 7.26 p134], Φ is a Borel-measurable stochastic kernel on \mathbb{R} given $\mathfrak{P}(\Omega_{t+1})$. So $\hat{\Phi}(\omega^t, q) := (\omega^t, \Phi(q))$ is also Borel-measurable and $\text{graph}(\tilde{\mathcal{Q}}_{t+1}) = \hat{\Phi}^{-1}(\text{graph}(\mathcal{B}_{t+1}))$ is analytic. Then one can show as in [18, Proofs for Section 2.3] that $\text{graph}(\mathcal{Q}_{t+1})$ is analytic since \mathcal{Q}_{t+1} is the convex hull of $\tilde{\mathcal{Q}}_{t+1}$. \square
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Lemma 4.4 Under Assumption 4.1, the $NA(\mathcal{Q}^T)$ condition holds true and the $sNA(\mathcal{Q}^T)$ condition might fails.

Proof. It is clear that for all $0 \leq t \leq T - 1$, all $\omega^t \in \Omega^t$, $\text{Conv}(D^{t+1})(\omega^t) = [S_t(\omega^t)(d_t(\omega^t) - 1), S_t(\omega^t)(U_t(\omega^t) - 1)]$. So the $NA(\mathcal{Q}^T)$ condition holds true as $0 \in \text{Ri}(\text{Conv}(D^{t+1}))(\omega^t)$ for all $\omega^t \in \Omega^t$ (see Theorem 3.23). Under Assumption 4.1, one may have that $u_t(\omega^t) < 1$ for all $\omega^t \in \Omega^t$, $0 \leq t \leq T - 1$ and find some $a_t(\omega^t) \in [u_t(\omega^t), 1)$. For all $0 \leq t \leq T - 1$ and $\omega^t \in \Omega^t$, let $q_{t+1}(Y_{t+1} \in \cdot, \omega^t) := r_t(\omega^t)\delta_{a_t(\omega^t)}(\cdot) + (1 - r_t(\omega^t))\delta_{d_t(\omega^t)}(\cdot)$, where $r_t(\omega^t) \in$

⁶This could be generalised by setting $\mathcal{B}_{t+1}(\omega^t) := \{\pi\delta_u + (1 - \pi)\delta_d, \pi \in \mathcal{S}_t(\omega^t), u \in \mathcal{U}_t(\omega^t), d \in \mathcal{D}_t(\omega^t)\}$, where $\mathcal{S}_t, \mathcal{U}_t, \mathcal{D}_t$ are Borel-measurable random sets $\Omega^t \rightarrow \mathbb{R}$.

$[\pi_t(\omega^t), \Pi_t(\omega^t)]$. Set $Q := Q_1 \otimes Q_2 \otimes \cdots \otimes Q_T \in \mathcal{Q}^T$. As $\text{Conv}(D_Q^{t+1})(\omega^t) = [S_t(\omega^t)(d_t(\omega^t) - 1), S_t(\omega^t)(a_t(\omega^t) - 1)]$, $0 \notin \text{Conv}(D_Q^{t+1})(\omega^t)$ for all $\omega^t \in \Omega^t$ and Proposition 3.25 implies that the $NA(Q)$ and thus the $sNA(\mathcal{Q}^T)$ conditions fail. \square

We now provide some explicit expressions for ε_t , β_t and κ_t of (12) and (13) and exhibit a candidate for the measure P^* of Theorem 3.29.

Lemma 4.5 *Assume that Assumption 4.1 holds true. For all $0 \leq t \leq T - 1$, all $\omega^t \in \Omega^t$ let*

$$\begin{aligned} \bar{\pi}_t(\omega^t) &:= \frac{\pi_t(\omega^t) + \Pi_t(\omega^t)}{2} \in (0, 1), \quad \bar{d}_t(\omega^t) = \frac{d_t(\omega^t) + 1}{2} \in (d_t, 1), \quad \bar{U}_t(\omega^t) = \frac{U_t(\omega^t) + 1}{2} \in (1, U_t) \\ \frac{\varepsilon_t(\omega^t)}{2} = \beta_t(\omega^t) &:= \frac{S_t(\omega^t)}{N} \min(\bar{U}_t(\omega^t) - 1, 1 - \bar{d}_t(\omega^t)) > 0, \quad \kappa_t(\omega^t) := \frac{1}{M} \min(\bar{\pi}_t(\omega^t), 1 - \bar{\pi}_t(\omega^t)) > 0, \\ a_t^-(\omega^t) &:= \max(u_t(\omega^t), \bar{U}_t(\omega^t)) + \frac{1}{N} > 1, \quad a_t^+(\omega^t) := U_t(\omega^t) > 1, \\ b_t^-(\omega^t) &:= d_t(\omega^t) < 1, \quad b_t^+(\omega^t) := \min(D_t(\omega^t), \bar{d}_t(\omega^t)) - \frac{1}{N} < 1, \\ r_{t+1}^\pm(\cdot, \omega^t) &:= \bar{\pi}_t(\omega^t) \delta_{a_t^\pm(\omega^t)}(\cdot) + (1 - \bar{\pi}_t(\omega^t)) \delta_{b_t^\pm(\omega^t)}(\cdot) \in \mathcal{B}_{t+1}(\omega^t), \\ r_{t+1}^*(\cdot, \omega^t) &:= \frac{1}{2} (r_{t+1}^+(\cdot, \omega^t) + r_{t+1}^-(\cdot, \omega^t)) \in \mathcal{B}_{t+1}(\omega^t), \quad p_{t+1}^*(Y_{t+1} \in \cdot, \omega^t) := r_{t+1}^*(\cdot, \omega^t) \in \mathcal{Q}_{t+1}(\omega^t), \end{aligned}$$

where $N > 1$ and $M > 1$ are fixed and allows to get sharper values for $\varepsilon_t(\omega^t)$, $\beta_t(\omega^t)$ and $\kappa_t(\omega^t)$. Then

$$p_{t+1}^*(\pm \Delta S_{t+1}(\omega^t, \cdot) < -\beta_t(\omega^t), \omega^t) \geq \kappa_t(\omega^t), \quad (14)$$

(13) is satisfied with the common prior p_{t+1}^* and $\Omega_{NA}^t = \Omega^t$; (12) also holds true. Moreover, let $P^* := P_0^* \otimes p_1^* \cdots \otimes p_T^* \in \mathcal{Q}^T$. Then P^* satisfies the conditions of Theorem 3.29.

Finally, assume that for some $0 \leq t \leq T - 1$ and some $\omega^t \in \Omega^t$, $u_t(\omega^t) < U_t(\omega^t)$ or $d_t(\omega^t) < D_t(\omega^t)$. Then the set $\mathcal{Q}_{t+1}(\omega^t)$ is not dominated and one can construct sets \mathcal{Q}^T which are not dominated.

Remark 4.6 Note that P^* is not unique. The (Borel) measurability of ε_t , β_t and κ_t are clear. Similarly they will inherit any integrability conditions imposed on S_t , π_t , Π_t , d_t , D_t , u_t and U_t . For instance if they belong to \mathcal{W}_t for all $1 \leq t \leq T$ so do ε_t , β_t and κ_t .

Proof. Fix some $0 \leq t \leq T - 1$, $\omega^t \in \Omega^t$. Let $q_{t+1}^\pm(Y_{t+1} \in \cdot, \omega^t) := r_{t+1}^\pm(\cdot, \omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$. Then

$$\begin{aligned} q_{t+1}^\pm(\Delta S_{t+1}(\omega^t, \cdot) < -\beta_t(\omega^t), \omega^t) &\geq q_{t+1}^\pm(Y_{t+1}(\cdot) < \bar{d}_t(\omega^t), \omega^t) = r_{t+1}^\pm((-\infty, \bar{d}_t(\omega^t)), \omega^t) \geq 1 - \bar{\pi}_t(\omega^t) \geq \kappa_t(\omega^t) \\ q_{t+1}^\pm(\Delta S_{t+1}(\omega^t, \cdot) > \beta_t(\omega^t), \omega^t) &\geq q_{t+1}^\pm(Y_{t+1}(\cdot) > \bar{U}_t(\omega^t), \omega^t) = r_{t+1}^\pm((\bar{U}_t(\omega^t), +\infty), \omega^t) \geq \bar{\pi}_t(\omega^t) \geq \kappa_t(\omega^t) \end{aligned}$$

and (14) follows. Note that $\Omega_{NA}^t = \Omega^t$. Theorem 3.23, Remark 3.26 and Proposition 3.25 imply (12), $NA(P^*)$ and that $0 \in \text{Ri}(\text{Conv}(D_{P^*}^{t+1}))(\omega^t)$ for all $\omega^t \in \Omega^t$. It is clear that for all $\omega^t \in \Omega^t$, $\text{Aff}(D_{P^*}^{t+1})(\omega^t) = \mathbb{R}$ and $\text{Aff}(D^{t+1})(\omega^t) = \mathbb{R}$ follows from (3).

For the last item, assume that for some $0 \leq t \leq T - 1$ and some $\omega^t \in \Omega^t$, $u_t(\omega^t) < U_t(\omega^t)$ and that the set $\mathcal{Q}_{t+1}(\omega^t)$ is dominated by some measure \hat{p} . For $x \in (0, \infty)$ let $A_x := \{Y_{t+1}^{-1}(\{x\})\} \neq \emptyset$ as $Y_{t+1}(\Omega^t) = (0, \infty)$.

Fix $x(\omega^t) \in (u_t(\omega^t), U_t(\omega^t))$ and choose $a(\omega^t) \in A_{x(\omega^t)}$ and $b(\omega^t) \in A_{d_t(\omega^t)}$. Let $r_x(\cdot, \omega^t) := \Pi_t(\omega^t) \delta_{a(\omega^t)} +$

$(1 - \Pi_t(\omega^t))\delta_{b(\omega^t)} \in \mathcal{B}_{t+1}(\omega^t)$ and $p_x(Y_{t+1} \in \cdot, \omega^t) := r_x(\cdot, \omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$. As $r_x(\{a(\omega^t)\}, \omega^t) = \Pi_t(\omega^t) > 0$, $\hat{p}(\{a(\omega^t)\}) > 0$, which leads to an uncountable number of atoms for \hat{p} .

Then, Proposition 3.39 allows to build examples of sets \mathcal{Q}^T which are not dominated. \square

4.2. Discretized d -dimensional diffusion

We provide now an example for the discretized dynamics of a multi-dimensional diffusion process in the spirit of [35, Example 8.2]. Fix a period $T \geq 1$ and $n \geq d$. Denote by M_n the set of real-valued matrix with n rows and n columns. Choose some constant $Y_0 \in \mathbb{R}^n$ and let Y_{t+1} be defined by the following difference equation for all $0 \leq t \leq T - 1$, $(\omega^t, \omega_{t+1}) \in \Omega^t \times \Omega_{t+1}$

$$Y_{t+1}(\omega^t, \omega_{t+1}) - Y_t(\omega^t) = \mu_{t+1}(Y_t(\omega^t), \omega^t, \omega_{t+1}) + \nu_{t+1}(Y_t(\omega^t), \omega^t) Z_{t+1}(\omega^t, \omega_{t+1}) \quad (15)$$

where $\mu_{t+1} : \mathbb{R}^n \times \Omega^t \times \Omega_{t+1} \rightarrow \mathbb{R}^n$, $\nu_{t+1} : \mathbb{R}^n \times \Omega^t \rightarrow M_n$, $Z_{t+1} : \Omega^t \times \Omega_{t+1} \rightarrow \mathbb{R}^n$ are assumed to be Borel-measurable.

Two cases will be studied: $S_t^i = Y_t^i$ and $S_t^i = e^{Y_t^i}$ for all $1 \leq i \leq d$. In a uni-prior setting if the law of Z_{t+1} is assumed to be normal, this corresponds to the popular normal and lognormal dynamic for the underlying assets. Note that in both cases if $d < n$ we may think that Y_t^i for $i > d$ represents some non-traded assets or the evolution of some economic factors that will influence the market.

Assume that some $P^0 \in \mathfrak{P}(\Omega^T)$ is given with fixed disintegration $P^0 := P_1^0 \otimes p_2^0 \otimes \dots \otimes p_T^0$, where $p_{t+1}^0 \in \mathcal{SK}_{t+1}$ for all $0 \leq t \leq T - 1$: P^0 could be an initial guess or estimate for the prior. For all $0 \leq t \leq T - 1$, let r_t and q_t be functions from Ω^t to $(0, \infty)$: r_t will be the bound on the drift while q_t guarantees that the diffusion is non-degenerated (in dimension one it is a lower bound on the volatility). We make the following assumptions on the dynamic of Y .

Assumption 4.7 For all $0 \leq t \leq T - 1$, r_t is $\mathcal{B}(\Omega^t)$ -measurable. For all $\omega^t \in \Omega^t$, $x \in \mathbb{R}^n$,

- $\nu_{t+1}(x, \omega^t) \in M_n^{q_t(\omega^t)}$ where $M_n^\delta := \{M \in M_n, \forall h \in \mathbb{R}^n, h^t M M^t h \geq \delta h^t h\}$ for $\delta > 0$.
- $Z_{t+1}(\omega^t, \cdot)$ and $\mu_{t+1}(Y_t(\omega^t), \omega^t, \cdot)$ are independent under $p_{t+1}^0(\cdot, \omega^t)$.
- $p_{t+1}^0(\mu_{t+1}(Y_t(\omega^t), \omega^t, \cdot) \in [-r_t(\omega^t), r_t(\omega^t)]^n, \omega^t) = 1$.
- $D_{Z_{t+1}}^{t+1}(\omega^t) = \mathbb{R}^n$, where $D_{Z_{t+1}}^{t+1}(\omega^t)$ is the support of $Z_{t+1}(\omega^t, \cdot)$ under $p_{t+1}^0(\cdot, \omega^t)$, see (2).

Let $\mathcal{Q}_{t+1}^1(\omega^t) := \{p \in \mathfrak{P}(\Omega_{t+1}), p(\mu_{t+1}(Y_t(\omega^t), \omega^t, \cdot) \in [-r_t(\omega^t), r_t(\omega^t)]^n) = 1\}$. Let for some $k \geq 1$, $F_t : \mathfrak{P}(\Omega_{t+1}) \times \Omega^t \rightarrow \mathbb{R}^k$ be a Borel-measurable function such that $F_t(p_{t+1}^0(\cdot, \omega^t), \omega^t) = 0$ for all $0 \leq t \leq T - 1$, $\omega^t \in \Omega^t$. Set $\mathcal{Q}_{t+1}^2(\omega^t) := \{p \in \mathfrak{P}(\Omega_{t+1}), F_t(p, \omega^t) = 0\}$ and $\mathcal{Q}_{t+1}(\omega^t) := \mathcal{Q}_{t+1}^1(\omega^t) \cap \text{Conv}(\mathcal{Q}_{t+1}^2(\omega^t))$. By assumption $p_{t+1}^0(\cdot, \omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$ for all $\omega^t \in \Omega^t$ and thus $P^0 \in \mathcal{Q}^T$. Note that for a given $p \in \mathcal{Q}_{t+1}(\omega^t)$ the law of $Z_{t+1}(\omega^t, \cdot)$ and $\mu_{t+1}(Y_t(\omega^t), \omega^t, \cdot)$ under p are not necessarily independent.

The financial interpretation is the following. The set $\mathcal{Q}_{t+1}^1(\omega^t)$ allows the drift of the diffusion to be not only stochastic but with an unknown distribution. It is only assumed to be bounded. If $F_t(p, \omega^t) =$

¹ $\text{dist}_t(p, p_{t+1}^0(\cdot, \omega^t)) \leq b_t(\omega^t) - 1$ with $b_t(\omega^t) > 0$ and dist_t some kind of distance function between probability measures, the set $\mathcal{Q}_{t+1}^2(\omega^t)$ contains models which are close enough from $p_{t+1}^0(\cdot, \omega^t)$. This could happen if the physical measure is not known but estimated from data at each step. A popular choice for dist_t is the Wasserstein distance. But one may also choose for the coordinate i of $F(p, \omega^t)$ (with $1 \leq i \leq k$) the difference between the moments of order i of $Z_{t+1}(\omega^t, \cdot)$ under p and under $p_{t+1}^0(\cdot, \omega^t)$ and incorporate all the models p such that the moments of $Z_{t+1}(\omega^t, \cdot)$ under p equal to the ones of $Z_{t+1}(\omega^t, \cdot)$ under $p_{t+1}^0(\cdot, \omega^t)$ up to order k .

Lemma 4.8 *Under Assumption 4.7, Assumptions 2.1 and 2.2 are satisfied.*

Proof. Assumption 2.1 follows from the Borel measurability of $\mu_{t+1}, \nu_{t+1}, Z_{t+1}$ and thus of Y_{t+1} . As the function $(\omega^t, p) \mapsto p(\mu_{t+1}(Y_t(\omega^t), \omega^t, \cdot) \in [-r_t(\omega^t), r_t(\omega^t)]^n)$ is Borel-measurable (see [26, Proposition 7.29 p144]), $\text{graph}(\mathcal{Q}_{t+1}^1)$ is analytic. The Borel-measurability of F_t implies that $\text{graph}(\mathcal{Q}_{t+1}^2)$ is an analytic set and so is $\text{graph}(\mathcal{Q}_{t+1})$ (see the proof of Lemma 4.3). As \mathcal{Q}_{t+1} is convex valued, Assumption 2.2 is proved. \square

Now we give explicit values for β_t and κ_t in (13) with $p_h = p_{t+1}^0(\cdot, \omega^t)$ and prove that the $NA(\mathcal{Q}^T)$ condition holds. Let K be the (finite) set of functions from $\{1, \dots, d\}$ to $\{-1, 1\}$ and let for all $\omega^t \in \Omega^t, 1 \leq t \leq T-1$ and $k \in K$

$$G_k(\omega^t) := \{k(i)\Delta Y_{t+1}^i(\omega^t, \cdot) < -\ln 2, 1 \leq i \leq d\} \quad \text{and} \quad \kappa_t(\omega^t) := \min_{k \in K} (p_{t+1}^0(G_k(\omega^t), \omega^t)). \quad (16)$$

Lemma 4.9 *Assume that Assumption 4.7 is satisfied. Then $D^{t+1}(\omega^t) = \mathbb{R}^d$ for all $\omega^t \in \Omega^t$ and $1 \leq t \leq T-1$ and the $NA(\mathcal{Q}^T)$ condition holds true.*

If for all $1 \leq i \leq d$ and all $0 \leq t \leq T$, $S_t^i = Y_t^i$, let $\beta_t(\omega^t) := \frac{\ln 2}{\sqrt{n}} > 0$, else if $S_t^i = e^{Y_t^i}$, let $\beta_t(\omega^t) := \frac{1}{2} \min\left(1, \frac{\min_{1 \leq i \leq d} S_t^i(\omega^t)}{\sqrt{n}}\right) > 0$ for all $\omega^t \in \Omega^t$. Then, for all $h \in \mathbb{R}^d$ with $|h| = 1$

$$p_{t+1}^0(h\Delta S_{t+1}(\omega^t, \cdot) < -\beta_t(\omega^t), \omega^t) \geq \kappa_t(\omega^t) > 0, \quad (17)$$

(13) is satisfied with the common prior $p_{t+1}^* = p_{t+1}^0, \Omega_{NA}^t = \mathbb{R}^t$. Moreover, we can choose $P^* = P^0$ in Theorem 3.29.

Proof. First, we show that for all $\omega^t \in \Omega^t, D^{t+1}(\omega^t) = \mathbb{R}^d$. To do that we prove first that

$$D_{Y, P_0}^{t+1}(\omega^t) := \bigcap \{A \subset \mathbb{R}^n, \text{closed}, p_{t+1}^0(\Delta Y_{t+1}(\omega^t, \cdot) \in A, \omega^t) = 1\} = \mathbb{R}^n. \quad (18)$$

Fix some $\omega^t \in \Omega^t$. For ease of reading, we adopt the following notations. Let $\Delta Y(\cdot) = \Delta Y_{t+1}(\omega^t, \cdot), R(\cdot) = \mu_{t+1}(Y_t(\omega^t), \omega^t, \cdot), X(\cdot) = \Delta Y(\cdot) - R(\cdot), M = \nu_{t+1}(Y_t(\omega^t), \omega^t), Z(\cdot) = Z_{t+1}(\omega^t, \cdot)$ and $p^0(\cdot) = p_{t+1}^0(\cdot, \omega^t)$. As $X(\cdot) = MZ(\cdot)$ (see (15)) and Z and R are independent under p^0, X and R are also independent under p^0 .

Fix some $x_0 \in \mathbb{R}^n, \varepsilon > 0$. By assumption M is an invertible matrix, $M^{-1}(B(x_0, \varepsilon))$ is a non-empty open set in \mathbb{R}^n and there exist some $y_0 \in \mathbb{R}^n, \alpha > 0$, such that $B(y_0, \alpha) \subset M^{-1}(B(x_0, \varepsilon))$. The forth item of Assumption

4.7 together with Lemma 5.2 imply that

$$\begin{aligned} p^0(X(\cdot) \in B(x_0, \varepsilon)) &= p^0(Z(\cdot) \in M^{-1}(B(x_0, \varepsilon))) \geq p^0(Z(\cdot) \in B(y_0, \alpha)) > 0 \\ p^0(\Delta Y(\cdot) \in B(x_0, \varepsilon)) &= p^0(X(\cdot) + R(\cdot) \in B(x_0, \varepsilon)) = \int_{\mathbb{R}} p^0(X(\cdot) \in B(x_0 - u, \varepsilon)) p_R^0(du) > 0, \end{aligned}$$

as X and R are independent under p^0 (with the notation $p_R^0(A) = p^0(R \in A)$ for all $A \in \mathcal{B}(\mathbb{R}^n)$). Now Lemma 5.2 implies that $D_{Y, P_0}^{t+1}(\omega^t) = \mathbb{R}^n$. If $S_t^i = Y_t^i$ for all $1 \leq i \leq d$ and all $1 \leq t \leq T$, then $D_{P_0}^{t+1}(\omega^t) = \mathbb{R}^d$.

495 Now we treat the case $S_t^i = e^{Y_t^i}$ for all $1 \leq i \leq d$ and all $1 \leq t \leq T$. Lemma 5.2 will imply that $D_{P_0}^{t+1}(\omega^t) = \mathbb{R}^d$ if for any open set O of \mathbb{R}^d , $p_{t+1}^0(\Delta S_{t+1}(\cdot, \omega^t) \in O, \omega^t) > 0$. Fix such an O and let $F_{\omega^t} : \mathbb{R}^n \rightarrow \mathbb{R}^d$ be defined by $F_{\omega^t}(x_1, \dots, x_n) := (e^{Y_t^1(\omega^t)}(e^{x_1} - 1), \dots, e^{Y_t^d(\omega^t)}(e^{x_d} - 1))$. As F_{ω^t} is continuous $F_{\omega^t}^{-1}(O)$ is an open set of \mathbb{R}^n . Then using (18) and Lemma 5.2 again, $p_{t+1}^0(\Delta e^{Y_{t+1}}(\cdot, \omega^t) \in O, \omega^t) = p_{t+1}^0(\Delta Y_{t+1}(\cdot, \omega^t) \in F_{\omega^t}^{-1}(O), \omega^t) > 0$.

So in both cases, for all $0 \leq t \leq T - 1$, $\omega^t \in \Omega^t$, $D_{P_0}^{t+1}(\omega^t) = \mathbb{R}^d$ and (3) implies that $D^{t+1}(\omega^t) = \mathbb{R}^d$. So $0 \in \text{Ri}(\text{Aff}(D^{t+1})(\omega^t))$ and Theorem 3.23 implies that the $NA(Q^T)$ condition is verified. Fix now some $\omega^t \in \Omega^t$ and $h \in \mathbb{R}^d$ with $|h| = 1$.

First we treat the case $S_t^i = Y_t^i$ for all $1 \leq i \leq d$ and all $1 \leq t \leq T$. As $D_{Y, P_0}^{t+1}(\omega^t) = \mathbb{R}^n$, for all $k \in K$, $\omega^t \in \Omega^t$

$$p_{t+1}^0(G_k(\omega^t), \omega^t) = p_{t+1}^0(\Delta Y_{t+1}(\omega^t, \cdot) \in \mathcal{O}_k, \omega^t) > 0, \quad (19)$$

where $\mathcal{O}_k := \{z \in \mathbb{R}^n, k(i)z_i < -\ln 2, \forall 1 \leq i \leq d\}$ is an open set of \mathbb{R}^n . Set $k^*(i) := \text{sign}(h_i)$ for all $1 \leq i \leq d$, then $k^* \in K$. As (19) implies that $G_{k^*}(\omega^t)$ is not empty, let $\omega_{t+1} \in G_{k^*}(\omega^t)$. For all $1 \leq i \leq d$, $h_i \Delta S_{t+1}^i(\omega^t, \omega_{t+1}) = |h_i| k^*(i) \Delta Y_{t+1}^i(\omega^t, \omega_{t+1}) \leq -\ln 2 |h_i| \leq 0$. As $|h| = 1$ there exists $1 \leq i^* \leq d$ such that $\frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{d}} \leq |h_{i^*}| \leq 1$ and

$$h \Delta S_{t+1}(\omega^t, \omega_{t+1}) < -\frac{\ln 2}{\sqrt{n}} + \sum_{i \neq i^*} h_i \Delta Y_{t+1}^i(\omega^t, \omega_{t+1}) \leq -\frac{\ln 2}{\sqrt{n}}.$$

Therefore $p_{t+1}^0(h \Delta S_{t+1}(\omega^t, \cdot) < -\ln 2 / \sqrt{n}, \omega^t) \geq \min_{k \in K} (p_{t+1}^0(G_k(\omega^t), \omega^t))$. Recalling (19), (17) is satisfied.

Now we treat the case $S_t^i = e^{Y_t^i}$ for all $1 \leq i \leq d$ and all $1 \leq t \leq T$. Let k^* as before and $\omega_{t+1} \in G_{k^*}(\omega^t)$. Then

$$h_i S_t^i(\omega^t) \left(e^{\Delta Y_{t+1}^i(\omega^t, \omega_{t+1})} - 1 \right) < \begin{cases} -\frac{|h_i| S_t^i(\omega^t)}{2} & \text{if } k^*(i) = 1 \\ -|h_i| S_t^i(\omega^t) & \text{if } k^*(i) = -1 \end{cases} \leq 0,$$

for all $1 \leq i \leq d$. As $|h| = 1$ there is a component h_{i^*} such that $\frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{d}} \leq |h_{i^*}| \leq 1$ and as $S_t^{i^*}(\omega^t) > 0$,

$$\begin{aligned} h \Delta S_{t+1}(\omega^t, \omega_{t+1}) &= \sum_{i=1}^d h_i S_t^i(\omega^t) \left(e^{\Delta Y_{t+1}^i(\omega^t, \omega_{t+1})} - 1 \right) \\ &< -\frac{S_t^{i^*}(\omega^t)}{2\sqrt{n}} + \sum_{i \neq i^*} h_i S_t^i(\omega^t) \left(e^{\Delta Y_{t+1}^i(\omega^t, \omega_{t+1})} - 1 \right) \leq -\frac{\min_{1 \leq i \leq d} S_t^i(\omega^t)}{2\sqrt{n}}. \end{aligned}$$

So, $p_{t+1}^0(h \Delta S_{t+1}(\omega^t, \cdot) < -\frac{1}{2} \min(1, \frac{\min_{1 \leq i \leq d} S_t^i(\omega^t)}{\sqrt{n}}), \omega^t) \geq \min_{k \in K} p_{t+1}^0(G_k(\omega^t), \omega^t)$, and using (19), (17) is sat-

500 isfied.

In both cases, $NA(P^0)$ holds true (see Remark 3.26) and Theorem 3.23 implies that $0 \in \text{Ri}(\text{Conv}(D_{P^0}^{t+1}))(\omega^t)$ for all $\omega^t \in \Omega^t$. Moreover $\text{Aff}(D_{P^0}^{t+1})(\omega^t) = \text{Aff}(D^{t+1})(\omega^t) = \mathbb{R}^d$ for all ω^t . \square

We now give a one dimension illustration of the previous setting where \mathcal{Q}^T is not dominated. Take $n = d = 1$ and $\Omega_t := \Omega$ for some Polish space Ω . Let Z be some real-valued random variable defined on Ω and $p_0 \in \mathfrak{P}(\Omega)$ be such that under p_0 , Z is normally distributed with mean 0 and standard deviation 1. Set $P^0 := p_0 \otimes \dots \otimes p_0$ and $Z_{t+1}(\omega^t, \omega_{t+1}) := Z(\omega_{t+1})$ for all $0 \leq t \leq T-1$ and $\omega^t \in \Omega^t$. Define $F : \mathfrak{P}(\Omega) \rightarrow \mathbb{R}^2$ by $F(p) := (E_p(Z), E_p(Z - E_p(Z))^2 - 1)$. Finally, set $\mathcal{Q}_{t+1}(\omega^t) := F^{-1}(\{0\}) =: \mathcal{Q}$ for all $0 \leq t \leq T-1$, $\omega^t \in \Omega^t$. For each ω^t , the law of the driving process Z for the next period is centered with variance 1 but not necessarily normally distributed. Assumption 4.7 on the dynamic of Y is verified if we choose $Y_0 := 1$ and for all $0 \leq t \leq T-1$, $x \in \mathbb{R}$, $(\omega^t, \omega_{t+1}) \in \Omega^t \times \Omega$, $r_t(\omega^t) := r$, $\mu_{t+1}(x, \omega^t, \omega_{t+1}) := r$, $\nu_{t+1}(x, \omega^t) := \sigma$ and $q_t(\omega^t) := \sigma^2$, for some $r \in \mathbb{R}$ and $\sigma > 0$ fixed.

As $\Delta Y_t = r + \sigma Z$ and Z is normally distributed with mean 0 and standard deviation 1 under p_0 , (16) implies that $\kappa_t = \kappa = \min(\Phi(-\frac{\ln 2+r}{\sigma}), 1 - \Phi(\frac{\ln 2-r}{\sigma}))$, where Φ is the cumulative distribution function of some normal law with mean 0 and standard deviation 1. Finally, the set \mathcal{Q}^T is not dominated. Indeed, we show that \mathcal{Q} is not dominated and conclude using Proposition 3.39. Assume that there is some $\hat{p} \in \mathfrak{P}(\Omega)$ which dominates \mathcal{Q} . For $x \neq 0$, let $q_x \in \mathfrak{P}(\Omega)$ such that $q_x(Z = x) = \frac{1}{2x^2}$, $q_x(Z = -x) = \frac{1}{2x^2}$, $q_x(Z = 0) = 1 - \frac{1}{x^2}$. Then $q_x \in \mathcal{Q}$ and $\{x \in \mathbb{R}, \hat{p}(\{Z = x\}) > 0\} = \mathbb{R} \setminus \{0\}$, a contradiction.

5. Proofs

The first section presents the one-period version of our problems with deterministic initial data. We will study the different notions of arbitrage and their equivalence (see Proposition 5.7). We also prove Proposition 5.8 that will be used in the proof of Theorem 3.29. In the second section the multi-period results are proved relying on the one-period results together with measurable selections technics. Finally, the third section presents the proof of Propositions 3.37 and 3.39.

5.1. One-period model

Let $(\bar{\Omega}, \mathcal{G})$ be a measured space, $\mathfrak{P}(\bar{\Omega})$ the set of all probability measures defined on \mathcal{G} and \mathcal{Q} a non-empty convex subset of $\mathfrak{P}(\bar{\Omega})$. For $P \in \mathcal{Q}$ fixed, E_P denotes the expectation under P . Let Y be a \mathcal{G} -measurable \mathbb{R}^d -valued random variable.

The following sets are the pendant in the one-period case of the ones introduced in Definition 2.3. Let $P \in \mathcal{Q}$

$$E(P) := \bigcap \{A \subset \mathbb{R}^d, \text{ closed}, P(Y(\cdot) \in A) = 1\} \quad D := \bigcap \{A \subset \mathbb{R}^d, \text{ closed}, P(Y(\cdot) \in A) = 1, \forall P \in \mathcal{Q}\}.$$

525 The next lemma will be used in the proof of Proposition 3.27.

Lemma 5.1 *Let C be a convex set of \mathbb{R}^d and fix some $\varepsilon > 0$. Then $B(0, \varepsilon) \cap \text{Aff}(C) \subset \bar{C}$ if and only if $B(0, \varepsilon) \cap \text{Aff}(C) \subset C$.*

Proof. The reverse implication is trivial. Assume that $B(0, \varepsilon) \cap \text{Aff}(C) \subset \overline{C}$ and let $x \in B(0, \varepsilon) \cap \text{Aff}(C)$. As $|x| < \varepsilon$, there exists some $\delta > 0$ such that $B(x, \delta) \cap \text{Aff}(C) \subset B(0, \varepsilon) \cap \text{Aff}(C) \subset \overline{C}$. Hence $x \in \text{Ri}(\overline{C}) = \text{Ri}(C) \subset C$ (see [29, Theorem 6.3 p46]). \square

This lemma provides an easy characterisation of the support.

Lemma 5.2 *Let $h \in \mathbb{R}^d$ and $P \in \mathfrak{P}(\overline{\Omega})$ be fixed. Then, $h \in E(P)$ if and only if $P(Y(\cdot) \in B(h, 1/n)) > 0$ for all $n \geq 1$. Similarly, $h \in D$ if and only if for all $n \geq 1$, there exists $P^n \in \mathcal{Q}$ such that $P^n(Y(\cdot) \in B(h, 1/n)) > 0$.*

Proof. Fix some $h \in \mathbb{R}^d$. By definition $h \notin E(P)$ if and only if there exists an open set $O \subset \mathbb{R}^d$ such that $h \in O$ and $P(Y(\cdot) \in O) = 0$ and the first item follows. Similarly, $h \notin D$ if and only if there exists an open set $O \subset \mathbb{R}^d$ such that $h \in O$ and $P(Y(\cdot) \in O) = 0$ for all $P \in \mathcal{Q}$ and the second item follows. \square

Now, we introduce the definitions of no-arbitrage in this one period setting.

Definition 5.3 The one-period no-arbitrage condition holds true if $hY(\cdot) \geq 0$ \mathcal{Q} -q.s. for some $h \in \mathbb{R}^d$ implies that $hY(\cdot) = 0$ \mathcal{Q} -q.s.

Definition 5.4 The one-period geometric no-arbitrage condition holds true if $0 \in \text{Ri}(\text{Conv}(D))$. This is equivalent to $0 \in \text{Conv}(D)$ and there exists some $\varepsilon > 0$ such that $B(0, \varepsilon) \cap \text{Aff}(D) \subset \text{Conv}(D)$.

Definition 5.5 The one-period quantitative no-arbitrage condition holds true if there exist some constants $\beta, \kappa \in (0, 1]$ such that for all $h \in \text{Aff}(D)$, $h \neq 0$ there exists $P_h \in \mathcal{Q}$ satisfying

$$P_h(hY(\cdot) < -\beta|h|) \geq \kappa. \quad (20)$$

Lemma 5.6 *If $0 \notin \text{Ri}(\text{Conv}(D))$ there exists some $h^* \in \text{Aff}(D)$, $h^* \neq 0$ such that $h^*y \geq 0$ for all $y \in D$. Moreover, for any $h \in \mathbb{R}^d \setminus \{0\}$,*

$$hY(\cdot) \geq 0 \text{ } \mathcal{Q}\text{-q.s.} \iff hy \geq 0, \forall y \in D. \quad (21)$$

Proof. The first assertion is a classical exercise relying on separation arguments in \mathbb{R}^d , see [29, Theorems 11.1, 11.3 p97] or [36, Proposition A.1]. We show the direct implication in (21). If there exists $y_0 \in D$ such that $hy_0 < 0$, then there exists some $\delta > 0$ such that $hy < 0$ for all $y \in B(y_0, \delta)$. But Lemma 5.2 implies the existence of some $P \in \mathcal{Q}$ such that $P(Y(\cdot) \in B(y_0, \delta)) > 0$, a contradiction. For the reverse implication, we use that $P(\{Y(\cdot) \in D\}) = 1$ for all $P \in \mathcal{Q}$ (see Remark 2.4). \square

Proposition 5.7 establishes the equivalence between the three preceding conditions.

Proposition 5.7 *Definitions 5.3, 5.4 and 5.5 are equivalent. Moreover, one can choose $\beta = \varepsilon/2$ in (20) where $\varepsilon > 0$ is such that $B(0, \varepsilon) \cap \text{Aff}(D) \subset \text{Conv}(D)$ in Definition 5.4.*

Proof. Step 1 : Definition 5.3 implies Definitions 5.4 and 5.5.

First we show by contradiction that for all $h \in \text{Aff}(D)$

$$hY(\cdot) \geq 0 \text{ } \mathcal{Q}\text{-q.s.} \Rightarrow h = 0. \quad (22)$$

Assume that there exists some $h \in \text{Aff}(D)$, $h \neq 0$ such that $hY(\cdot) \geq 0$ \mathcal{Q} -q.s. Definition 5.3 implies that $hY(\cdot) = 0$ \mathcal{Q} -q.s. and⁷ $h \in \{h \in \mathbb{R}^d, hy = 0 \text{ for all } y \in D\} = D^\perp = (\text{Aff}(D))^\perp$, see for instance [37, Proof of Lemma 2.6]. This implies that $h \in \text{Aff}(D) \cap (\text{Aff}(D))^\perp \subset \{0\}$, a contradiction.

Now we show that Definition 5.4 holds true. If $0 \notin \text{Ri}(\text{Conv}(D))$, Lemma 5.6 implies that there exists some $h^* \in \text{Aff}(D)$, $h^* \neq 0$ such that $h^*Y(\cdot) \geq 0$ \mathcal{Q} -q.s. which contradicts (22). Then, we prove that Definition 5.5 holds also true. For all $n \geq 1$, let

$$A_n := \left\{ h \in \text{Aff}(D), |h| = 1, P \left(hY(\cdot) < -\frac{1}{n} \right) < \frac{1}{n} \forall P \in \mathcal{Q} \right\} \quad n_0 := \inf \{ n \geq 1, A_n = \emptyset \}$$

with the convention that $\inf \emptyset = +\infty$. We have seen that Definition 5.4 holds true: $0 \in \text{Ri}(\text{Conv}(D)) \subset \text{Aff}(D)$ and $\text{Aff}(D)$ is a vector space. If $\text{Aff}(D) = \{0\}$, then $n_0 = 1 < \infty$. Assume now that $\text{Aff}(D) \neq \{0\}$. We prove by contradiction that $n_0 < \infty$. Assume that $n_0 = \infty$. For all $n \geq 1$, there exists some $h_n \in A_n$. By passing to a sub-sequence we can assume that h_n tends to some $h^* \in \text{Aff}(D)$ with $|h^*| = 1$. Let $B_n := \{h_n Y(\cdot) < -1/n\}$. Then $\{h^* Y(\cdot) < 0\} \subset \liminf_n B_n$ and Fatou's Lemma implies that for any $P \in \mathcal{Q}$

$$P(h^* Y(\cdot) < 0) \leq \int_{\Omega} 1_{\liminf_n B_n}(\omega) P(d\omega) \leq \liminf_n \int_{\Omega} 1_{B_n}(\omega) P(d\omega) = 0.$$

So $h^* Y(\cdot) \geq 0$ \mathcal{Q} -q.s. and (22) implies that $h^* = 0$ which contradicts $|h^*| = 1$. Thus $n_0 < \infty$ and we can set $\beta = \kappa = 1/n_0$. It is clear that $\beta, \kappa \in (0, 1]$ and by definition of A_{n_0} , (20) holds true.

Step 2 : Definition 5.5 implies Definition 5.4.

If Definition 5.4 does not hold true, Lemma 5.6 implies that there exists some $h^* \in \text{Aff}(D)$, $h^* \neq 0$ such that $h^* Y(\cdot) \geq 0$ \mathcal{Q} -q.s.: A contradiction with (20).

Step 3 : Definition 5.4 implies Definition 5.3.

Fix some $h \in \mathbb{R}^d$ such that $hY(\cdot) \geq 0$ \mathcal{Q} -q.s. Let $p(h)$ be the orthogonal projection of h on $\text{Aff}(D)$ (recall that $\text{Aff}(D)$ is a vector space since $0 \in \text{Ri}(\text{Conv}(D)) \subset \text{Aff}(D)$). Assume for a moment that $p(h) = 0$. Remark 2.4 shows that $P(\{Y(\cdot) \in D\}) = 1$ for all $P \in \mathcal{Q}$, $hY(\cdot) = p(h)Y(\cdot) = 0$ \mathcal{Q} -q.s. and Definition 5.3 is verified.

If $p(h) \neq 0$, as $0 \in \text{Ri}(\text{Conv}(D))$, there exists some $\varepsilon > 0$ such that $B(0, \varepsilon) \cap \text{Aff}(D) \subset \text{Conv}(D)$, $-\varepsilon p(h)/(2|p(h)|) \in \text{Conv}(D)$ and

$$-\varepsilon \frac{p(h)}{2|p(h)|} h = -\varepsilon \frac{p(h)}{2|p(h)|} p(h) < 0,$$

which contradicts (21) (note that by convex combinations $hy \geq 0$ for all $y \in \text{Conv}(D)$).

Step 4: If $B(0, \varepsilon) \cap \text{Aff}(D) \subset \text{Conv}(D)$ one can choose $\beta = \varepsilon/2$ in (20).

⁷ X^\perp stands for the orthogonal space of some set X .

555 This is similar to the proof of Step 1: A_n is modified in $A_n := \{h \in \text{Aff}(D), |h| = 1, P(hY(\cdot) < -\frac{\varepsilon}{2}) < \frac{1}{n}, \forall P \in \mathcal{Q}\}$. The same arguments as before apply and if $n_0 = \infty$ there exists some $h^* \in \text{Aff}(D)$, $|h^*| = 1$ such that $h^*Y \geq -\varepsilon/2$ \mathcal{Q}^T -q.s. We also get that $h^*y \geq -\varepsilon/2$ for all $y \in \text{Conv}(D)$, see (21). Choosing $y = -(2/3)\varepsilon h^* \in B(0, \varepsilon) \cap \text{Aff}(D) \subset \text{Conv}(D)$, we obtain a contradiction. So, (20) holds true with $\beta = \varepsilon/2$ and $\kappa = 1/n_0$ ⁸. \square

560 The next proposition follows from [15, Lemma 2.2] and will be used in the proof of Theorem 3.29.

Proposition 5.8 *Assume that the one-period no-arbitrage condition (see Definition 5.3) holds true. Then there exists some $P^* \in \mathcal{Q}$ such that $0 \in \text{Ri}(\text{Conv}(E(P^*)))$ and $\text{Aff}(E(P^*)) = \text{Aff}(D)$.*

Proof. [15, Lemma 2.2] gives the existence of some $P^* \in \mathcal{Q}$ such that $NA(P^*)$ holds true and $\text{Aff}(E(P^*)) = \text{Aff}(D)$. Note that the proof of [15, Lemma 2.2] relies on the convexity of \mathcal{Q} . Now Proposition 3.25 (for $T = 1$) shows that $0 \in \text{Ri}(\text{Conv}(E(P^*)))$. \square

5.2. Multi-period model

5.2.1. Proof of Theorem 3.23

Proof. Fix some $0 \leq t \leq T - 1$ and $\omega^t \in \Omega^t$. We say that $NA(\mathcal{Q}_{t+1}(\omega^t))$ holds true if $h\Delta S_{t+1}(\omega^t, \cdot) \geq 0$ $\mathcal{Q}_{t+1}(\omega^t)$ -q.s. for some $h \in \mathbb{R}^d$ implies that $h\Delta S_{t+1}(\omega^t, \cdot) = 0$ $\mathcal{Q}_{t+1}(\omega^t)$ -q.s. Proposition 5.7 implies that the $NA(\mathcal{Q}_{t+1}(\omega^t))$ condition is equivalent to (12) and (13) for any $\omega^t \in \Omega^t$. Then Theorem 3.17 shows that $NA(\mathcal{Q}^T)$ is equivalent to the fact that $\Omega_{NA}^t = \{\omega^t \in \Omega^t, NA(\mathcal{Q}_{t+1}(\omega^t)) \text{ holds true}\} \in \mathcal{B}_c(\Omega^t)$ is a \mathcal{Q}^t -full measure set for all $0 \leq t \leq T - 1$. Thus, for all $0 \leq t \leq T - 1$, one may choose $\Omega_{NA}^t = \Omega_{qNA}^t = \Omega_{gNA}^t$. Furthermore, Proposition 5.7 shows that one can take $\beta_t(\omega^t) = \varepsilon_t(\omega^t)/2$ for $\omega^t \in \Omega_{NA}^t$. \square

5.2.2. Proof of Proposition 3.27

Proof. Fix some $0 \leq t \leq T - 1$. We set $\Gamma^{t+1}(\omega^t) = \emptyset$ for $\omega^t \notin \Omega_{NA}^t$ and for all $\omega^t \in \Omega_{NA}^t$

$$\begin{aligned} \Gamma^{t+1}(\omega^t) &:= \{\varepsilon \in \mathbb{Q}, \varepsilon > 0, B(0, \varepsilon) \cap \text{Aff}(D^{t+1})(\omega^t) \subset \text{Conv}(D^{t+1})(\omega^t)\} \\ &= \{\varepsilon \in \mathbb{Q}, \varepsilon > 0, B(0, \varepsilon) \cap \text{Aff}(D^{t+1})(\omega^t) \subset \overline{\text{Conv}}(D^{t+1})(\omega^t)\}, \end{aligned}$$

where the equality comes from Lemma 5.1. Assume for a moment that graph $\Gamma^{t+1} \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$ has been proved. The Aumann Theorem implies the existence of a $\mathcal{B}_c(\Omega^t)$ -measurable selector $\varepsilon_t : \{\Gamma^{t+1} \neq \emptyset\} \rightarrow \mathbb{R}$ such that $\varepsilon_t(\omega^t) \in \Gamma^{t+1}(\omega^t)$ for every $\omega^t \in \{\Gamma^{t+1} \neq \emptyset\}$. Now, Theorem 3.23 and (12) imply that $\Omega_{NA}^t = \{\Gamma^{t+1}(\omega^t) \neq \emptyset\}$ (recall that $\Gamma^{t+1}(\omega^t) = \emptyset$ outside Ω_{NA}^t). Setting $\varepsilon_t = 1$ outside Ω_{NA}^t , ε_t is $\mathcal{B}_c(\Omega^t)$ -measurable and Proposition 3.27 is proved as we can choose $\beta_t = \varepsilon_t/2$ (see Theorem 3.23).

It remains to show that graph $\Gamma^{t+1} \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$. As graph $\Gamma^{t+1} = \bigcup_{\varepsilon \in \mathbb{Q}, \varepsilon > 0} A_\varepsilon \times \{\varepsilon\}$, where $A_\varepsilon := \{\omega^t \in \Omega_{NA}^t, B(0, \varepsilon) \cap \text{Aff}(D^{t+1})(\omega^t) \subset \overline{\text{Conv}}(D^{t+1})(\omega^t)\}$, it is enough to prove that $A_\varepsilon \in \mathcal{B}_c(\Omega^t)$. Let $h : \mathbb{R}^d \times \Omega^t \rightarrow \mathbb{R}$

⁸The same argument shows that one can set $\kappa = \inf_{h \in \text{Aff}(D), |h|=1} \sup_{P \in \mathcal{Q}} P(hY(\cdot) < -\frac{\varepsilon}{2}) > 0$ illustrating why the measurability of κ cannot be directly obtained, see Remark 3.28.

be defined by

$$h(x, \omega^t) := d(x, B(0, \varepsilon) \cap \text{Aff}(D^{t+1})(\omega^t)) - d\left(x, \overline{\text{Conv}}(D^{t+1})(\omega^t)\right),$$

575 where $d(\cdot, F)$ is the distance function on \mathbb{R}^d and is defined by $d(x, F) := \inf\{|x - f|, f \in F\}$ for any $x \in \mathbb{R}^d$ and any non-empty set $F \subset \mathbb{R}^d$, see for instance [25, p80]. Then [25, Theorem 18.5 p595] and Lemma 2.6 show that for all $x \in \mathbb{R}^d$, $h(x, \cdot)$ is $\mathcal{B}_c(\Omega^t)$ -measurable and that $h(\cdot, \omega^t)$ is continuous for all $\omega^t \in \Omega^t$. As $\overline{\text{Conv}}(D^{t+1})(\omega^t)$ is closed-valued, $A_\varepsilon = \{\omega^t \in \Omega_{NA}^t, h(x, \omega^t) \geq 0, \forall x \in \mathbb{R}^d\} = \bigcap_{q \in \mathbb{Q}^d} \{\omega^t \in \Omega_{NA}^t, h(q, \omega^t) \geq 0\} \in \mathcal{B}_c(\Omega^t)$. \square

5.2.3. Proof of Theorem 3.29

Proof. Reverse implication.

Fix some $0 \leq t \leq T - 1$ and $\omega^t \in \Omega_{NA}^t$. As $P^* \in \mathcal{Q}^T$, Remark 2.5 implies that $D_{P^*}^{t+1}(\omega^t) \subset D^{t+1}(\omega^t)$. As $\text{Aff}(D^{t+1})(\omega^t) = \text{Aff}(D_{P^*}^{t+1})(\omega^t)$ and $0 \in \text{Ri}(\text{Conv}(D_{P^*}^{t+1}))(\omega^t)$, there exists some $\varepsilon > 0$ such that

$$B(0, \varepsilon) \cap \text{Aff}(D^{t+1})(\omega^t) = B(0, \varepsilon) \cap \text{Aff}(D_{P^*}^{t+1})(\omega^t) \subset \text{Conv}(D_{P^*}^{t+1})(\omega^t) \subset \text{Conv}(D^{t+1})(\omega^t).$$

As $0 \in \text{Conv}(D_{P^*}^{t+1})(\omega^t) \subset \text{Conv}(D^{t+1})(\omega^t)$, $\text{NA}(\mathcal{Q}^T)$ follows from Theorem 3.23.

Direct implication.

For all $0 \leq t \leq T - 1$, let $\mathcal{E}_{t+1} : \Omega^t \rightarrow \mathfrak{P}(\Omega_{t+1})$ be defined by $\mathcal{E}_{t+1}(\omega^t) = \emptyset$ if $\omega^t \notin \Omega_{NA}^t$ and if $\omega^t \in \Omega_{NA}^t$

$$\mathcal{E}_{t+1}(\omega^t) := \{p \in \mathcal{Q}_{t+1}(\omega^t), 0 \in \text{Ri}(\text{Conv}(E^{t+1}))(\omega^t, p) \text{ and } \text{Aff}(E^{t+1})(\omega^t, p) = \text{Aff}(D^{t+1})(\omega^t)\}.$$

Theorem 3.17 and Proposition 5.8 show that $\Omega_{NA}^t = \{\mathcal{E}_{t+1} \neq \emptyset\}$. Assume for a moment that we have proved the existence of $p_{t+1}^* \in \mathcal{SK}_{t+1}$ such that $p_{t+1}^*(\cdot, \omega^t) \in \mathcal{E}_{t+1}(\omega^t)$ for all $\omega^t \in \Omega_{NA}^t$. Let $P^* := p_1^* \otimes \cdots \otimes p_T^*$. Then, $P^* \in \mathcal{Q}^T$ (see (1)), the equality in (3) implies that $\text{Aff}(D_{P^*}^{t+1})(\omega^t) = \text{Aff}(E^{t+1})(\omega^t, p_{t+1}^*(\cdot, \omega^t)) = \text{Aff}(D^{t+1})(\omega^t)$ and $0 \in \text{Ri}(\text{Conv}(E^{t+1}))(\omega^t, p_{t+1}^*(\cdot, \omega^t)) = \text{Ri}(\text{Conv}(D_{P^*}^{t+1}))(\omega^t)$ for all $\omega^t \in \Omega_{NA}^t$.

So it remains to prove the existence of p_{t+1}^* . Fix some $0 \leq t \leq T - 1$ and let

$$B := \{(\omega^t, p), \text{Ri}(\overline{\text{Conv}}(E^{t+1}))(\omega^t, p) \cap \{0\} \neq \emptyset\} \quad \text{and} \quad C := \{(\omega^t, p), \text{Aff}(E^{t+1})(\omega^t, p) = \text{Aff}(D^{t+1})(\omega^t)\}.$$

Lemma 2.6 and [30, Theorem 14.3] imply that $\{(\omega^t, p), \overline{\text{Conv}}(E^{t+1})(\omega^t, p) \cap F \neq \emptyset\} \in \mathcal{B}(\Omega^t) \otimes \mathcal{B}(\mathfrak{P}(\Omega_{t+1}))$ for all closed sets F of \mathbb{R}^d . Thus [38, Lemma 5.7] shows that the same holds true for $\text{Ri}(\overline{\text{Conv}}(E^{t+1}))$ and $B \in \mathcal{B}(\Omega^t) \otimes \mathcal{B}(\mathfrak{P}(\Omega_{t+1}))$ follows. Let $h : \Omega^t \times \mathfrak{P}(\Omega_{t+1}) \rightarrow \mathbb{R}$ be defined by

$$h(\omega^t, p) := d(\text{Aff}(E^{t+1})(\omega^t, p), \text{Aff}(D^{t+1})(\omega^t)) := \sup_{x \in \mathbb{R}^d} |d(x, \text{Aff}(E^{t+1})(\omega^t, p)) - d(x, \text{Aff}(D^{t+1})(\omega^t))|. \quad (23)$$

580 So here $d(F, G)$ is the Hausdorff distance between two non-empty sets $F, G \subset \mathbb{R}^d$, see for instance [25, Definition 3.70 p110 and Lemma 3.74 p111]. Note that $C = h^{-1}(\{0\})$. Then [25, Theorem 18.5 p595] and Lemma 2.6 show that for all $x \in \mathbb{R}^d$, $(\omega^t, p) \mapsto d(x, \text{Aff}(E^{t+1})(\omega^t, p))$ is $\mathcal{B}(\Omega^t) \otimes \mathcal{B}(\mathfrak{P}(\Omega_{t+1}))$ -measurable and $\omega^t \mapsto d(x, \text{Aff}(D^{t+1})(\omega^t))$ is $\mathcal{B}_c(\Omega^t)$ -measurable. The same results also show that $x \mapsto |d(x, \text{Aff}(E^{t+1})(\omega^t, p)) - d(x, \text{Aff}(D^{t+1})(\omega^t))|$ is continuous for all $(\omega^t, p) \in \Omega^t \times \mathfrak{P}(\Omega_{t+1})$. Thus one can replace \mathbb{R}^d by \mathbb{Q}^d in (23) and h

585 is $\mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathfrak{P}(\Omega_{t+1}))$ -measurable. It follows that $C \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathfrak{P}(\Omega_{t+1}))$.

[29, Theorem 6.3 p46], Assumption 2.2 and Lemma 5.9 below show that $\text{graph}(\mathcal{E}_{t+1}) = \text{graph}(\mathcal{Q}_{t+1}) \cap B \cap C \in \mathfrak{A}(\mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathfrak{P}(\Omega_{t+1})))$, where for some Polish space X and some paving \mathcal{J} (i.e. a non-empty collection of subsets of X containing the empty set), $\mathfrak{A}(\mathcal{J})$ denotes the set of all nuclei of Suslin Scheme on \mathcal{J} (see [26, Definition 7.15 p157]). Now [14, Lemma 4.11] (which relies on [39]) gives the existence of $p_{t+1}^* \in \mathcal{SK}_{t+1}$ such
590 that $p_{t+1}^*(\cdot, \omega^t) \in \mathcal{E}_{t+1}(\omega^t)$ for all $\omega^t \in \Omega_{NA}^t = \{\mathcal{E}_{t+1} \neq \emptyset\}$. The proof is complete. \square

Lemma 5.9 *Let X, Y be two Polish spaces. Let $\Gamma_1 \in \mathcal{A}(X \times Y)$ and $\Gamma_2 \in \mathcal{B}_c(X) \otimes \mathcal{B}(Y)$. Then $\Gamma_1 \cap \Gamma_2 \in \mathfrak{A}(\mathcal{B}_c(X) \otimes \mathcal{B}(Y))$.*

Proof. [26, Proposition 7.35 p158, Proposition 7.41 p166] imply that $\Gamma_1 \in \mathcal{A}(X \times Y) = \mathfrak{A}(\mathcal{B}(X) \otimes \mathcal{B}(Y)) \subset \mathfrak{A}(\mathcal{B}_c(X) \otimes \mathcal{B}(Y))$ and $\Gamma_2 \in \mathcal{B}_c(X) \otimes \mathcal{B}(Y) \subset \mathfrak{A}(\mathcal{B}_c(X) \otimes \mathcal{B}(Y))$ and thus $\Gamma_1 \cap \Gamma_2 \in \mathfrak{A}(\mathcal{B}_c(X) \otimes \mathcal{B}(Y))$. \square

595 5.2.4. Proofs of Theorem 3.6 and of Lemma 3.9

Proof. of Theorem 3.6. Step 1: Reverse implication.

Lemma 3.5 implies that the $NA(\mathcal{P}^T)$ condition holds true and Lemma 3.2 shows that the $NA(\mathcal{Q}^T)$ is satisfied.

Step 2: Direct implication.

Theorem 3.29 implies that there exists some $P^* \in \mathcal{Q}^T$ with the fixed disintegration $P^* := P_1^* \otimes p_2^* \otimes \dots \otimes p_T^*$ such that $\text{Aff}(D_{P^*}^{t+1})(\omega^t) = \text{Aff}(D^{t+1})(\omega^t)$ and $0 \in \text{Ri}(\text{Conv}(D_{P^*}^{t+1}))(\omega^t)$ for all $\omega^t \in \Omega_{NA}^t$ and all $0 \leq t \leq T-1$. The direct implication holds true if *i)*, *ii)* and *iii)* below are proved⁹. For all $1 \leq t \leq T-1$, let $\mathcal{P}_{t+1} : \Omega^t \rightarrow \mathfrak{P}(\Omega_{t+1})$ be defined for all $\omega^t \in \Omega^t$ by

$$\mathcal{P}_{t+1}(\omega^t) := \{\lambda p_{t+1}^*(\cdot, \omega^t) + (1-\lambda)q, q \in \mathcal{Q}_{t+1}(\omega^t), 0 < \lambda \leq 1\}. \quad (24)$$

Let \mathcal{P}^T be defined recursively as follows: $\mathcal{P}^1 := \{\lambda P_1^* + (1-\lambda)P, P \in \mathcal{Q}^1, 0 < \lambda \leq 1\}$ and for all $1 \leq t \leq T-1$

$$\mathcal{P}^{t+1} := \left\{ P \otimes q, P \in \mathcal{P}^t, q(\cdot, \omega^t) \in \mathcal{P}_{t+1}(\omega^t) \text{ for all } \omega^t \in \Omega^t \right\}. \quad (25)$$

i) $\mathcal{P}^t \subset \mathcal{Q}^t$ for all $1 \leq t \leq T$.

This follows by induction from (25), $p_{t+1}^*(\cdot, \omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$ and the convexity of $\mathcal{Q}_{t+1}(\omega^t)$.

ii) \mathcal{Q}^t and \mathcal{P}^t have the same polar-sets for all $1 \leq t \leq T$.

Fix some $1 \leq t \leq T$. As $\mathcal{P}^t \subset \mathcal{Q}^t$, it is clear that a \mathcal{Q}^t -polar set is also a \mathcal{P}^t -polar set. The other inclusion follows from (6) for $n = 2$. Indeed, for some $Q^t \in \mathcal{Q}^t$, we find that $P^t := \frac{1}{2^t}(Q^t + \sum_{k=0}^{t-1} R_k^t) \in \mathcal{P}^t$ and $Q^t \ll P^t$.

This proves that a \mathcal{P}^t -polar set is also a \mathcal{Q}^t -polar set.

iii) The $sNA(\mathcal{P}^T)$ condition holds true.

Fix some $P \in \mathcal{P}^T \subset \mathcal{Q}^T$, some $0 \leq t \leq T-1$ and $\omega^t \in \Omega_{NA}^t$. We establish that $0 \in \text{Ri}(\text{Conv}(D_P^{t+1}))(\omega^t)$. Then $P^t(\Omega_{NA}^t) = 1$ and Proposition 3.25 show that $NA(P)$ holds true and *iii)* follows. Remark 2.5 and (25) imply

⁹Note that *i)* and *ii)* are true if we only assume that $P^* \in \mathcal{Q}^T$.

that $D_{P^*}^{t+1}(\omega^t) \subset D_P^{t+1}(\omega^t) \subset D^{t+1}(\omega^t)$. Thus, $0 \in \text{Conv}(D_{P^*}^{t+1})(\omega^t) \subset \text{Conv}(D_P^{t+1})(\omega^t)$. We have that

$$\text{Aff}(D^{t+1})(\omega^t) = \text{Aff}(D_{P^*}^{t+1})(\omega^t) \subset \text{Aff}(D_P^{t+1})(\omega^t) \subset \text{Aff}(D^{t+1})(\omega^t).$$

As $0 \in \text{Ri}(\text{Conv}(D_{P^*}^{t+1}))(\omega^t)$, there exists some $\varepsilon > 0$ such that

$$B(0, \varepsilon) \cap \text{Aff}(D_{P^*}^{t+1})(\omega^t) = B(0, \varepsilon) \cap \text{Aff}(D_P^{t+1})(\omega^t) \subset \text{Conv}(D_{P^*}^{t+1})(\omega^t) \subset \text{Conv}(D_P^{t+1})(\omega^t).$$

□

Proof. of Lemma 3.9 Let $P^* \in \mathcal{Q}^T$ with the fix disintegration $P^* := P_1^* \otimes p_2^* \otimes \cdots \otimes p_T^*$ as in Theorem 3.29. Let $Q \in \mathcal{Q}^T$ with the fix disintegration $Q := Q_1 \otimes q_2 \otimes \cdots \otimes q_T$. We define P_n^t by recursion: $P_n^1 := \frac{1}{n}P_1^* + (1 - \frac{1}{n})Q_1$ and for all $t \geq 1$, $P_n^{t+1} := P_n^t \otimes (\frac{1}{n}p_{t+1}^* + (1 - \frac{1}{n})q_{t+1})$. Then $P_n^t \in \mathcal{P}^t$ for all $1 \leq t \leq T$, see (25). A simple computation shows that (6) holds true with $R_1^1 := Q_1$ and for all $t \geq 1$, $R_0^t := P^{*,t}$, $R_k^{t+1} := R_k^t \otimes p_{t+1}^* + R_{k-1}^t \otimes q_{t+1}$ for all $1 \leq k \leq t-1$ and $R_t^{t+1} := Q^t \otimes p_{t+1}^* + R_{t-1}^t \otimes q_{t+1}$. □

5.2.5. Proof of Proposition 3.35

Proof. Theorem 3.29 implies that there exists some $P^* \in \mathcal{Q}^T$ with the fixed disintegration $P^* := P_1^* \otimes p_2^* \otimes \cdots \otimes p_T^*$ such that $\text{Aff}(D_{P^*}^{t+1})(\omega^t) = \text{Aff}(D^{t+1})(\omega^t)$ and $0 \in \text{Ri}(\text{Conv}(D_{P^*}^{t+1}))(\omega^t)$ for all $\omega^t \in \Omega_{NA}^t$ and all $0 \leq t \leq T-1$. To find a $\mathcal{B}_c(\Omega^t)$ -measurable version of β_t and κ_t in (13) we follow the same idea as in [33, Proposition 3.7]. Fix some $0 \leq t \leq T-1$. Set $n_t(\omega^t) := \inf\{n \geq 1, A_n^{P^*}(\omega^t) = \emptyset\}$ where for all $n \geq 1$ $A_n^{P^*}(\omega^t) = \emptyset$ if $\omega^t \notin \Omega_{NA}^t$ and if $\omega^t \in \Omega_{NA}^t$,

$$A_n^{P^*}(\omega^t) := \left\{ h \in \text{Aff}(D_{P^*}^{t+1})(\omega^t), |h| = 1, p_{t+1}^* \left(h\Delta S_{t+1}(\omega^t, \cdot) < -\frac{1}{n}, \omega^t \right) < \frac{1}{n} \right\}. \quad (26)$$

For all $\omega^t \in \Omega^t$, as in the proof of Proposition 5.7, $n_t(\omega^t) < \infty$ and one may set $\kappa_t(\omega^t) = \beta_t(\omega^t) := 1/n_t(\omega^t) \in (0, 1]$. Then, by definition of $A_n^{P^*}$, (13) is true with $p_h(\cdot) = p_{t+1}^*(\cdot, \omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$ since $\text{Aff}(D_{P^*}^{t+1})(\omega^t) = \text{Aff}(D^{t+1})(\omega^t)$ for all $\omega^t \in \Omega_{NA}^t$.

To prove that $\kappa_t = \beta_t$ is $\mathcal{B}_c(\Omega^t)$ -measurable, we show that $\{A_n^{P^*} \neq \emptyset\} \in \mathcal{B}_c(\Omega^t)$ since for all $k \geq 1$, $\{n_t \geq k\} = \Omega_{NA}^t \cap (\cap_{1 \leq j \leq k-1} \{A_j^{P^*} = \emptyset\})$. Fix some $n \geq 1$. As p_{t+1}^* is only universally-measurable, we use Lemma 5.10 below to prove that $\{A_n^{P^*} \neq \emptyset\} \in \mathcal{B}_c(\Omega^t)$. Fix $P \in \mathfrak{P}(\Omega^t)$. First, applying [26, Lemma 7.28 p174], there exist p_{t+1}^P a Borel-measurable stochastic kernel on Ω_{t+1} given Ω^t and $\Omega_P^t \in \mathcal{B}(\Omega^t)$ such that $P(\Omega_P^t) = 1$ and $p_{t+1}^P(\cdot, \omega^t) = p_{t+1}^*(\cdot, \omega^t)$ for all $\omega^t \in \Omega_P^t$. Set A_n^P as in (26) replacing p_{t+1}^* with p_{t+1}^P if $\omega^t \in \Omega_{NA}^t$ (and $A_n^P(\omega^t) = \emptyset$ if $\omega^t \notin \Omega_{NA}^t$) while keeping $h \in \text{Aff}(D_{P^*}^{t+1})(\omega^t)$. Then $\{A_n^{P^*} \neq \emptyset\} \cap \Omega_P^t = \{A_n^P \neq \emptyset\} \cap \Omega_P^t$ and it remains to establish that $\{A_n^P \neq \emptyset\} \in \mathcal{B}_c(\Omega^t)$. Remark that

$$\text{graph}(A_n^P) = \text{graph}(\text{Aff}(D_{P^*}^{t+1})) \cap \left\{ (\omega^t, h), |h| = 1, p_{t+1}^P \left(h\Delta S_{t+1}(\omega^t, \cdot) < -\frac{1}{n}, \omega^t \right) < \frac{1}{n} \right\}.$$

Lemma 2.6 implies that $\text{graph}(\text{Aff}(D_{P^*}^{t+1})) \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$. As $(\omega^t, h, \omega_{t+1}) \mapsto h\Delta S_{t+1}(\omega^t, \omega_{t+1})$ and p_{t+1}^P

are Borel-measurable, [26, Proposition 7.29 p144] implies that $(\omega^t, h) \mapsto p_{t+1}^P(h\Delta S_{t+1}(\omega^t, \cdot) < -1/n, \omega^t)$ is $\mathcal{B}(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable. Thus, applying the Projection Theorem, $\text{Proj}_{\Omega^t}(\text{graph}(A_n^P)) = \{A_n^P \neq \emptyset\} \in \mathcal{B}_c(\Omega^t)$ and the proof is complete. \square

Lemma 5.10 *Let X be a Polish space. Let $A \subset X$. Assume that for all $P \in \mathfrak{P}(X)$ there exist some $A_P \in \mathcal{B}_c(X)$ and some P -full measure set $X_P \in \mathcal{B}(X)$ such that $A \cap X_P = A_P \cap X_P$. Then $A \in \mathcal{B}_c(X)$.*

Proof. Fix some $P \in \mathfrak{P}(X)$. We show that $A \in \mathcal{B}_P(X)$, the completion of $\mathcal{B}(X)$ with respect to P . As this is true for all $P \in \mathfrak{P}(X)$, $A \in \mathcal{B}_c(X)$ will follow.

There exist $A_P \in \mathcal{B}_c(X)$ and $X_P \in \mathcal{B}(X)$ such that $P(X_P) = 1$ and $A \cap X_P = A_P \cap X_P$. As $A_P \cap X_P \in \mathcal{B}_c(X) \subset \mathcal{B}_P(X)$ there exist a P -negligible set N_P and $\tilde{A}_P \in \mathcal{B}(X)$ such that $A_P \cap X_P = \tilde{A}_P \cup N_P$. Now, let $M_P := A \cap (X \setminus X_P) \subset X \setminus X_P$. As $X \setminus X_P \in \mathcal{B}(X)$ and $P(X \setminus X_P) = 0$, M_P is a P -negligible set and $A = (A \cap X_P) \cup (A \cap (X \setminus X_P)) = \tilde{A}_P \cup N_P \cup M_P \in \mathcal{B}_P(X)$. \square

5.3. Proof of Section 3.3

Proof of both propositions rely on Lemma 5.12 below.

5.3.1. Proof of Propositions 3.37 and 3.39

Proof. of Proposition 3.37. Lemma 3.2 implies that $NA(\hat{P})$ and $NA(\mathcal{Q}^T)$ are equivalent. Fix some disintegration of $\hat{P} \in \mathcal{Q}^T$, $\hat{P} := \hat{P}_1 \otimes \hat{p}_2 \otimes \cdots \otimes \hat{p}_T$ and some $1 \leq t \leq T$. As \hat{P}^t dominates \mathcal{Q}^t Proposition 3.25 implies that $0 \in \text{Ri}(\text{Conv}(D_{\hat{P}}^{t+1}))(\cdot)$ \mathcal{Q}^t -q.s. Now Lemma 5.12 shows that $\hat{p}_{t+1}(\cdot, \omega^t)$ dominates $\mathcal{Q}_{t+1}(\omega^t)$ for all ω^t in the \mathcal{Q}^t -full measure set $\Omega^t \setminus \Omega_{nd}^t$. Thus $D^{t+1}(\omega^t) \subset D_{\hat{P}}^{t+1}(\omega^t)$ and the equality follows from (3) as $\hat{P} \in \mathcal{Q}^T$. \square

Proof. of Proposition 3.39. The proof of Proposition 3.39 follows directly from Lemma 5.12. Indeed assume that the set \mathcal{Q}^T is dominated. As $\Omega_N^t \subset \Omega_{nd}^t$, Ω_N^t is a \mathcal{Q}^t -polar set which contradicts $\tilde{P}^t(\Omega_N^t) > 0$. \square

5.3.2. Lemma 5.12

The proof of Lemma 5.12 is fairly technical and needs the introduction of the Wijsman topology as well as Lemma 5.11. Let (X, d) be a Polish space and \mathcal{F} be the set of non-empty closed subsets of X . The Wijsman topology on \mathcal{F} denoted by \mathcal{T}_W is such that

$$F_n \xrightarrow[n \rightarrow +\infty]{\mathcal{T}_W} F \iff d(x, F_n) \xrightarrow[n \rightarrow +\infty]{} d(x, F) \text{ for all } x \in X,$$

where $d(x, F) := \inf\{d(x, f), f \in F\}$. Note that \mathcal{F} endowed with \mathcal{T}_W is a Polish space (see [40]).

Lemma 5.11 *The function $\mathcal{F} \times X \rightarrow \mathbb{R}$, $(F, x) \mapsto 1_F(x)$ is $\mathcal{B}(\mathcal{F}) \otimes \mathcal{B}(X)$ -measurable.*

Proof. The function $d : X \times \mathcal{F} \ni (x, F) \mapsto d(x, F)$ is separately continuous. Indeed for all $x \in X$, $d(x, \cdot)$ is continuous by definition of \mathcal{T}_W and [25, Theorem 3.16 p80] implies that $d(\cdot, F)$ is continuous for all $F \in \mathcal{F}$. Using [25, Lemma 4.51 p153] d is $\mathcal{B}(X) \otimes \mathcal{B}(\mathcal{F})$ -measurable. We conclude since $x \in F$ if and only if $d(x, F) = 0$. \square

Lemma 5.12 Assume that Assumption 2.2 holds true and that the set \mathcal{Q}^T is dominated by $\widehat{P} \in \mathfrak{P}(\Omega^T)$ with the fix disintegration $\widehat{P} := \widehat{P}_0 \otimes \widehat{p}_1 \otimes \cdots \otimes \widehat{p}_T$ where $\widehat{p}_t \in \mathcal{SK}_t$ for all $1 \leq t \leq T$. Then $\Omega_{nd}^t := \{\omega^t \in \Omega^t, \mathcal{Q}_{t+1}(\omega^t) \text{ is not dominated by } \widehat{p}_{t+1}(\cdot, \omega^t)\} \in \mathcal{B}_c(\Omega^t)$ and is a \mathcal{Q}^t -polar set for all $0 \leq t \leq T-1$.

Proof. Fix some $0 \leq t \leq T-1$. We proceed in two steps.

Step 1: $\Omega_{nd}^t \in \mathcal{B}_c(\Omega^t)$.

To prove Step 1, we use Lemma 5.10 and fix $R \in \mathfrak{P}(\Omega^t)$ with the fix disintegration $R := R_0 \otimes r_1 \otimes \cdots \otimes r_t$ where $r_s \in \mathcal{SK}_s$ for all $1 \leq s \leq t$. Applying [26, Lemma 7.28 p174], there exist p_{t+1}^R a Borel-measurable stochastic kernel on Ω_{t+1} given Ω^t and a R -full-measure set $\Omega_R^t \in \mathcal{B}(\Omega^t)$ such that

$$p_{t+1}^R(\cdot, \omega^t) = \widehat{p}_{t+1}(\cdot, \omega^t) \text{ for all } \omega^t \in \Omega_R^t. \quad (27)$$

Let \mathcal{F}_{t+1} be the set of non-empty and closed subsets of Ω_{t+1} and let $N_t^R : \Omega^t \rightarrow \mathfrak{P}(\Omega_{t+1}) \times \mathcal{F}_{t+1}$ be defined by

$$N_t^R(\omega^t) := \{(q, F) \in \mathfrak{P}(\Omega_{t+1}) \times \mathcal{F}_{t+1}, q \in \mathcal{Q}_{t+1}(\omega^t), p_{t+1}^R(F, \omega^t) = 0, q(F) > 0\}. \quad (28)$$

We first claim that $\Omega_{nd}^t \cap \Omega_R^t = \{N_t^R \neq \emptyset\} \cap \Omega_R^t$. Let $\omega^t \in \Omega_{nd}^t \cap \Omega_R^t$. As $\mathcal{Q}_{t+1}(\omega^t)$ is not dominated by $\widehat{p}_{t+1}(\cdot, \omega^t) = p_{t+1}^R(\cdot, \omega^t)$, there exist some $q \in \mathcal{Q}_{t+1}(\omega^t)$ and some $A \in \mathcal{B}(\Omega_{t+1})$ such that $p_{t+1}^R(A, \omega^t) = 0$ and $q(A) > 0$. As $q \in \mathfrak{P}(\Omega_{t+1})$ is inner-regular (see [25, Definition 12.2 p435, Theorem 12.7 p438, Lemma 12.3 p435]), there exists some $F \in \mathcal{F}_{t+1}$, $F \subset A$ such that $q(F) > 0$ and $(q, F) \in N_t^R(\omega^t)$ follows. The reverse inclusion is clear.

Thus Lemma 5.10 applies and Step 1 is completed if $\{N_t^R \neq \emptyset\} = \text{Proj}_{\Omega^t}(\text{graph}(N_t^R)) \in \mathcal{B}_c(\Omega^t)$. This will be implied by [26, Proposition 7.39 p165] if

$$\text{graph}(N_t^R) \in \mathcal{A}(\Omega^t \times \mathfrak{P}(\Omega_{t+1}) \times \mathcal{F}_{t+1}). \quad (29)$$

Let $A := \text{graph}(\mathcal{Q}_{t+1}) \times \mathcal{F}_{t+1} \in \mathcal{A}(\Omega^t \times \mathfrak{P}(\Omega_{t+1}) \times \mathcal{F}_{t+1})$ (see Assumption 2.2), $B := \{(\omega^t, q, F), p_{t+1}^R(F, \omega^t) = 0\}$ and $C := \{(\omega^t, q, F), q(F) > 0\}$. Then B and C belong to $\mathcal{B}(\Omega^t) \otimes \mathcal{B}(\mathfrak{P}(\Omega_{t+1})) \otimes \mathcal{B}(\mathcal{F}_{t+1})$. Indeed Lemma 5.11 together with [26, Proposition 7.29 p144] imply that $(\omega^t, q, F) \mapsto p_{t+1}^R(F, \omega^t)$ and $(\omega^t, q, F) \mapsto q(F)$ are Borel-measurable (recall that $p_{t+1}^R(d\omega_{t+1}|\omega^t, q, F) = p_{t+1}^R(d\omega_{t+1}, \omega^t)$ and $q(d\omega_{t+1}|\omega^t, q, F) = q(d\omega_{t+1})$ are Borel-measurable stochastic kernels). Then (29) follows from $\text{graph}(N_t^R) = A \cap B \cap C$.

Step 2: Ω_{nd}^t is a \mathcal{Q}^t -polar set.

We proceed by contradiction and assume that there exists some $\overline{P} \in \mathcal{Q}^T$ such that $\overline{P}^t(\Omega_{nd}^t) > 0$. We choose $R = \widehat{P}^t$ in (27) and (28) and we denote by $\Omega_{nd1}^t := \Omega_{nd}^t \cap \Omega_{\widehat{P}^t}^t = \{N_t^{\widehat{P}^t} \neq \emptyset\} \cap \Omega_{\widehat{P}^t}^t \in \mathcal{B}_c(\Omega^t)$, see Step 1. The Jankov-von Neumann Theorem and (29) give the existence of $q_{t+1}^{\widehat{P}}$ an universally-measurable stochastic kernel on Ω_{t+1} given Ω^t and an universally measurable function $F_{t+1}^{\widehat{P}} : \Omega^t \rightarrow \mathcal{F}_{t+1}$ such that $(q_{t+1}^{\widehat{P}}(\cdot, \omega^t), F_{t+1}^{\widehat{P}}(\omega^t)) \in N_t^{\widehat{P}^t}(\omega^t)$ for all $\omega^t \in \Omega_{nd1}^t$. For $\omega^t \notin \Omega_{nd1}^t$ we set $F_{t+1}^{\widehat{P}}(\omega^t) = \emptyset$ and $q_{t+1}^{\widehat{P}}(\cdot, \omega^t) = q_{t+1}(\cdot, \omega^t)$ where q_{t+1} is a given universally-measurable selector of \mathcal{Q}_{t+1} .

Note that as \widehat{P}^t dominates \mathcal{Q}^t , $1 = \widehat{P}^t(\Omega_{\widehat{P}^t}^t) = \overline{P}^t(\Omega_{\widehat{P}^t}^t)$ and $\overline{P}^t(\Omega_{nd1}^t) > 0$.

We now build some $\widehat{Q} \in \mathcal{Q}^T$, $E \in \mathcal{B}_c(\Omega^{t+1})$ such that $\widehat{P}^{t+1}(E) = 0$ but $\widehat{Q}^{t+1}(E) > 0$ which contradicts the fact that \widehat{P} dominates \mathcal{Q}^T . Let $\widehat{Q} := \overline{P}^t \otimes q_{t+1}^{\widehat{P}} \otimes \overline{p}_{t+2} \otimes \cdots \otimes \overline{p}_T \in \mathcal{Q}^T$, and for $\varphi(\omega^t, \omega_{t+1}) := 1_{F_{t+1}^{\widehat{P}}(\omega^t)}(\omega_{t+1})$, let

$$E := \left\{ (\omega^t, \omega_{t+1}) \in \Omega^t \times \Omega_{t+1}, \omega^t \in \Omega_{nd1}^t, \omega_{t+1} \in F_{t+1}^{\widehat{P}}(\omega^t) \right\} = \varphi^{-1}(\{1\}) \cap (\Omega_{nd1}^t \times \Omega_{t+1}).$$

Lemma 5.11 implies that $(F, \omega_{t+1}) \mapsto 1_{F(\omega_{t+1})}$ is $\mathcal{B}(\mathcal{F}_{t+1}) \otimes \mathcal{B}(\Omega_{t+1})$ -measurable and as $(\omega^t, \omega_{t+1}) \mapsto (F_{t+1}^{\widehat{P}}(\omega^t), \omega_{t+1})$ is $\mathcal{B}_c(\Omega^{t+1})$ -measurable, φ is $\mathcal{B}_c(\Omega^{t+1})$ -measurable by composition. Thus E belong to $\mathcal{B}_c(\Omega^{t+1})$. Let $(E)_{\omega^t} := \{\omega_{t+1} \in \Omega_{t+1}, (\omega^t, \omega_{t+1}) \in E\}$, then

$$\widehat{P}^{t+1}(E) = \int_{\Omega_{nd1}^t} \widehat{p}_{t+1}((E)_{\omega^t}, \omega^t) \widehat{P}^t(d\omega^t) = \int_{\Omega_{nd1}^t} \widehat{p}_{t+1}(F_{t+1}^{\widehat{P}}(\omega^t), \omega^t) \widehat{P}^t(d\omega^t) = 0$$

where we have used that for $\omega^t \notin \Omega_{nd1}^t$ $(E)_{\omega^t} = \emptyset$ and for $\omega^t \in \Omega_{nd1}^t$ $(E)_{\omega^t} = F_{t+1}^{\widehat{P}}(\omega^t)$ and that $\widehat{p}_{t+1}(F_{t+1}^{\widehat{P}}(\omega^t), \omega^t) = p_{t+1}^{\widehat{P}}(F_{t+1}^{\widehat{P}}(\omega^t), \omega^t) = 0$. But

$$\widehat{Q}^{t+1}(E) = \int_{\Omega_{nd1}^t} q_{t+1}^{\widehat{P}}((E)_{\omega^t}, \omega^t) \overline{P}^t(d\omega^t) = \int_{\Omega_{nd1}^t} q_{t+1}^{\widehat{P}}(F_{t+1}^{\widehat{P}}(\omega^t), \omega^t) \overline{P}^t(d\omega^t) > 0$$

since $\overline{P}^t(\Omega_{nd1}^t) > 0$ and $q_{t+1}^{\widehat{P}}(F_{t+1}^{\widehat{P}}(\omega^t), \omega^t) > 0$ for all $\omega^t \in \Omega_{nd1}^t$. This concludes the proof. \square

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