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ON THE COHOMOLOGY OF LINE BUNDLES OVER CERTAIN FLAG SCHEMES II

LINYUAN LIU AND PATRICK POLO

To Jens Carsten Jantzen on the occasion of his 70th birthday

ABSTRACT. Over a field K of characteristic p, let Z be the incidence variety in $\mathbb{P}^d \times (\mathbb{P}^d)^*$ and let $\mathscr{L}$ be the restriction to Z of the line bundle $\mathscr{O}(-n-d) \boxtimes \mathscr{O}(n)$, where $n = p + f$ with $0 \leq f \leq p-2$. We prove that $H^d(Z, \mathscr{L})$ is the simple GL_{d+1}-module corresponding to the partition $\lambda_f = (p-1+f, p-1, f+1)$. When $f = 0$, using the first author's description of $H^d(Z, \mathscr{L})$ and Jantzen's sum formula, we obtain as a by-product that the sum of the monomial symmetric functions m_λ, for all partitions λ of $2p-1$ less than $(p-1, p-1, 1)$ in the dominance order, is the alternating sum of the Schur functions $S_{p-1,p-1-i,1^{i+1}}$ for $i = 0, \ldots, p-2$.

Introduction. This paper is an addition to the first author's paper [Liu19b]. We now consider the group scheme $G = \mathrm{SL}_{d+1}$ over an arbitrary field K of characteristic $p > 0$. For $m, n \in \mathbb{N}$, set $\mu_{m,n} = m\omega_1 - (n+d)\omega_d$ and write simply μ_n instead of $\mu_{n,n}$. We describe the $H^i(\mu_{m,n})$ when $p > n$ (thus recovering and extending [Liu19b], Cor. 2) and for $n = p + f$ with $0 \leq f \leq p-2$ we prove that $H^d(\mu_n)$ is the simple G-module $L(\lambda_f) = L(f\omega_1 + (p-2-f)\omega_2 + (f+1)\omega_3)$. Further, when $f = 0$ we express the character of $L(\lambda_f)$, using Jantzen sum formula, as an alternating sum of Weyl characters. Comparing this with the character of $H^d(\mu_p)$ given in [Liu19b], Cor. 3, we obtain as a by-product that the sum of the monomial symmetric functions m_λ, for all partitions λ of $2p-1$ less than λ_f in the dominance order, is the alternating sum of the Schur functions $S_{p-1,p-1-i,1^{i+1}}$ for $i = 0, \ldots, p-2$.

1.1. **Notation.** We keep the notation of [Liu19b], except that we now consider the group scheme $G = \mathrm{SL}_{d+1}$ over an arbitrary field K of characteristic $p > 0$ and denote simply by P the maximal parabolic subgroup P_1. Let W_P be the Weyl group of (P, T) and define W_Q similarly. Let $\mathrm{ch}\, V$ denote the character of a T-module V. For each simple root α_i, let $s_i \in W$ be the corresponding simple reflection. Let w_0 (resp. w_P, resp. w_Q) be the longest element of W (resp. W_P, resp. W_Q) and set $N = \ell(w_Q)$. Then $\ell(w_P) = N + d - 1$.

Let ρ_Q (resp. ρ_P) denote the half-sum of the positive roots of Q (resp. P). Then one has:

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(1.1.1) $2\rho_Q = (2-d)(\omega_1+\omega_d)+2\sum_{i=2}^{d-1}\omega_i, \qquad 2\rho_P = (1-d)\omega_1+2\sum_{i=2}^{d}\omega_i.$

Recall that, since $2\rho-2\rho_Q = d(\omega_1+\omega_d)$, the dualizing sheaf on $Z = G/Q$ is $\mathscr{L}(-d\omega_1 - d\omega_d)$. Hence, by Serre duality on G/Q, one has

$$H^d(\mu_{m,n}) \simeq H^{d-1}(-(m+d)\omega_1 + n\omega_d)^*. \tag{1.1.2}$$

Further, let τ be the (involutive) automorphism of (G,T) induced by the automorphism of the Dynkin diagram which swaps α_i and α_{d+1-i} for $i = 1,\dots,d$. Note that τ also acts on $X(T)$ and preserves $X(T)^+$. For any G-module V, let ${}^\tau V$ denote the corresponding module twisted by τ. For example, for a Weyl module $V(\lambda)$ (resp. a simple module $L(\lambda)$), one has ${}^\tau V(\lambda) \simeq V(\tau\lambda)$ (resp. ${}^\tau L(\lambda) \simeq L(\tau\lambda)$). Then, (1.1.2) can be rewrited as:

$$H^d(\mu_{m,n}) \simeq H^{d-1}(\tau\mu_{n,m})^*. \tag{1.1.3}$$

Recall (see [Jan03], II.4.6 and its proof) that if $P' \subset P''$ are parabolic subgroups containing B and if V is a P'-module, one has for all $i \geq 0$:

$$H^i(P''/B, V) \simeq H^i(P''/P', V). \tag{1.1.4}$$

This will be used several times, without always mentioning it.

1.2. **A description of $H^i(\mu_{m,n})$.** For each $i \in \mathbb{N}$, we denote by $H^i_Q(-)$ the functor $H^i(Q/B,-)$ and define $H^i_P(-)$ similarly. Since $-\mu_{m,n}$ is trivial and $\mu_{m,n}-2\rho_Q$ is anti-dominant with respect to the Levi subgroup of Q one has:

$$H^i_Q(-\mu_{m,n}) \simeq \begin{cases} -\mu_{m,n} & \text{if } i = 0, \\ 0 & \text{if } i > 0 \end{cases} \tag{1.2.1}$$

and, using Serre duality on Q/B:

$$H^i_Q(\mu_{m,n} - 2\rho_Q) \simeq \begin{cases} 0 & \text{if } i < N, \\ H^0_Q(-\mu_{m,n})^* \simeq \mu_{m,n} & \text{if } i = N. \end{cases} \tag{1.2.2}$$

Consider induction from B to P. Thanks to (1.2.2), the spectral sequence of composite functors (see [Jan03], I.4.5) degenerates and gives (together with (1.1.4) applied to $P' = Q$ and $P'' = P$) isomorphisms for each $i \geq 0$:

$$H^i_P(\mu_{m,n}) \simeq H^{i+N}_P(\mu_{m,n} - 2\rho_Q) \tag{1.2.3}$$

and since $\mu_{m,n}-2\rho_Q = (m+d-2)\omega_1-2\sum_{j=2}^{d-1}\omega_j-(n+2)\omega_d$ is anti-dominant with respect to P, the latter group is zero unless $i+N = \dim(P/B)$, i.e. $i = d-1$. Moreover, by Serre duality on P/B, one has:

$$\begin{aligned} H^{N+d-1}_P(\mu_{m,n} - 2\rho_Q) &\simeq H^0_P(-\mu_{m,n} + 2\rho_Q - 2\rho_P)^* \\ &= H^0_P((1-m)\omega_1 + n\omega_d)^*. \end{aligned}$$

Set $\nu_{m,n} = (1-m)\omega_1 + n\omega_d$ and $\pi_{m,n} = -w_P\nu_{m,n}$. Since $w_P = s_2\cdots s_d w_Q$, one has

$$\pi_{m,n} = (m-n-1)\omega_1 + n\omega_2 \tag{1.2.4}$$

and since $H^0_P(\nu_{m,n})^*$ is isomorphic with $V_P(\pi_{m,n})$, the Weyl module for P with highest weight $\pi_{m,n}$, one obtains:

Lemma 1.2.1. *For each $m, n \in \mathbb{N}$ and $i \in \mathbb{N}$, one has*

$$H^i_P(\mu_{m,n}) \simeq \begin{cases} V_P(\pi_{m,n}) & \text{if } i = d-1, \\ 0 & \text{if } i \neq d-1. \end{cases}$$

Now, consider induction from B to G. Thanks to the lemma, the spectral sequence of composite functors degenerates and gives (together with (1.1.4) applied to $P' = P$ and $P'' = G$) isomorphisms $H^i(\mu_{m,n}) \simeq H^{i-d+1}(V_P(\pi_{m,n}))$ for each $i \geq 0$. Since the former is zero for $i \notin \{d-1, d\}$ (see [Liu19b], Section 2), this gives:

Proposition 1.2.2. *For each $m, n \in \mathbb{N}$ and $i \in \mathbb{N}$, one has*

$$H^i(\mu_{m,n}) \simeq \begin{cases} H^0(V_P(\pi_{m,n})) & \text{if } i = d-1, \\ H^1(V_P(\pi_{m,n})) & \text{if } i = d, \\ 0 & \text{if } i \neq d-1, d. \end{cases}$$

This allows us to recover and extend the result of [Liu19b], Corollary 2:

Corollary 1.2.3. *Suppose that $p > n$.*

(i) *If $m \geq n$, then $H^d(\mu_{m,n}) = 0$ and $H^{d-1}(\mu_{m,n}) \simeq H^0(\pi_{m,n})$. In particular, $H^i(\mu_n) = 0$ for all $i \geq 0$.*

(ii) *If $n \geq m$, then $H^{d-1}(\mu_{m,n}) = 0$ and*

$$H^d(\mu_{m,n}) \simeq H^1(\pi_{m,n}) \simeq H^0(\pi_{n,m}).$$

(iii) *Furthermore, if $p > m \geq n$ then $H^{d-1}(\mu_{m,n}) \simeq L(\pi_{m,n}) \simeq H^d(\mu_{n,m})$.*

Proof. Suppose $p > n$. Then $V_P(\pi_{m,n})$ is irreducible as a P-module, being isomorphic as a module over the Levi subgroup of P to the GL_d-module $S^n K^d$ which is irreducible since $n < p$. Therefore, $V_P(\pi_{m,n}) \simeq H^0_P(\pi_{m,n})$. Thus, the proposition gives for all $i \geq 0$ that

$$H^i(\mu_{m,n}) \simeq H^{i-d+1}(\pi_{m,n}).$$

If $m \geq n$ then $\pi_{m,n}$ belongs to $\mathscr{C} := X(T)^+ - \rho$ and hence $H^j(\pi_{m,n}) = 0$ for $j > 0$ (and also for $j = 0$ if $m = n$). This proves (i).

If $m < n$, then $H^0(\pi_{m,n}) = 0$. Further, $\pi_{n,m} = (n-m-1)\omega_1 + m\omega_2$ belongs to $\mathscr{C}$ and one has $\pi_{m,n} = s_{\alpha_1} \cdot \pi_{n,m}$. Since $0 \leq (\pi_{n,m} + \rho, \alpha_1^\vee) = n - m < p$, one has $H^1(\pi_{m,n}) \simeq H^0(\pi_{n,m})$ (see [Jan03], II.5.4), which proves (ii).

Suppose now that $p > m \geq n$. By (i) and Serre duality on G/Q, one has

$$H^0(\pi_{m,n}) \simeq H^{d-1}(\mu_{m,n}) \simeq H^d(-(m+d)\omega_1 + n\omega_d)^* = H^d(\tau\mu_{n,m})^*.$$

Now, using the automorphism τ of (G, T), one deduces from (ii) that

$$H^d(\tau\mu_{n,m}))^* \simeq H^0(\tau\pi_{m,n})^*.$$

Since the latter is the Weyl module $V(\pi_{m,n})$, one obtains that $H^0(\pi_{m,n}) \simeq L(\pi_{m,n})$. This proves (iii). □

Our goal in the next subsection is to determine $H^d(\mu_p)$. Since for $d = 2$ the SL_3-modules $H^2(m, -n-2)$ have been described in [Liu19a], where it is proved in particular (see [Liu19a], Th. 3) that $H^2(p, -p-2)$ is the Weyl module $V(0, p-2)$, which is simple, we will henceforth assume that $d \geq 3$.

1.3. **Computation of $H^d(\mu_n)$ for $p \leq n \leq 2p-1$.** Suppose now that $d \geq 3$ and $n = p + f$ with $0 \leq f \leq p-1$. For each weight ν which is dominant (resp. dominant with respect to P), denote by $L(\nu)$ (resp. $L_P(\nu)$) the irreducible G-module (resp. P-module) with highest weight ν. Let

$$\lambda_f = \begin{cases} f\omega_1 + (p-2-f)\omega_2 + (f+1)\omega_3 & \text{if } f \leq p-2, \\ (p-1)\omega_1 + (p-2)\omega_3 + \omega_4 & \text{if } f = p-1 \end{cases} \tag{1.3.1}$$

(with the convention $\omega_4 = 0$ if $d = 3$), and set $L_f = L(\lambda_f)$ and $N_f = L_P(\lambda_f)$.

Proposition 1.3.1. *One has $H^{d-1}(\mu_n) \simeq L_f \simeq H^d(\mu_n)$.*

Remark 1.3.2. This generalizes Corollary 3 in [Liu19b], which was the case $f = 0$. Note that if $f > 0$ and $d > p-f+1$ then μ_{p+f} does not belong to the closure of the facet containing μ_p. Indeed, denoting by $\alpha_1, \ldots, \alpha_d$ the simple roots and setting $\beta = \alpha_1 + \cdots + \alpha_{p-f+1}$, one has $\langle \mu_p + \rho, \beta^\vee \rangle = 2p - f + 1$ whereas $\langle \mu_{p+f} + \rho, \beta^\vee \rangle = 2p+1$. Hence the result for μ_{p+f} cannot be deduced from the one for μ_p by applying a functor of translation.

Proof. Set $\pi_n = -\omega_1 + n\omega_2$ and $M = V_P(\pi_n)$. According to Doty [Dot85], §§2.3–2.4, one has exact sequences of P-modules:

$$0 \longrightarrow N_f \longrightarrow M \longrightarrow C \longrightarrow 0 \tag{1.3.2}$$

and

$$0 \longrightarrow C \longrightarrow H^0_P(\pi_n) \longrightarrow N_f \longrightarrow 0 \tag{1.3.3}$$

where $C = L_P(\pi_n)$. Applying the functor H^0 to (1.3.3) and using that:

$$H^i(H^0_P(\pi_n)) = H^i(\pi_n) = 0 \tag{1.3.4}$$

for all $i \geq 0$, one obtains $H^0(C) = 0$ and isomorphisms $H^i(C) \simeq H^{i-1}(N_f)$ for all $i \geq 1$. Taking this and Proposition 1.2.2 into account and applying the functor H^0 to (1.3.2), one obtains isomorphisms:

$$H^{d-1}(\mu_n) \simeq H^0(M) \simeq H^0(N_f), \tag{1.3.5}$$

an exact sequence:

(1.3.6)

$$0 \longrightarrow H^1(N_f) \longrightarrow H^d(\mu_n) \longrightarrow H^0(N_f) \longrightarrow H^2(N_f) \longrightarrow 0$$

and isomorphisms

$$H^i(N_f) \simeq H^{i+1}(C) \simeq H^{i+2}(N_f) \tag{1.3.7}$$

for $i \geq 1$. Since $H^i(N_f) = 0$ for $i > |R^+|$, one obtains $H^i(N_f) = 0$ for all $i \geq 1$. Together with (1.3.6) and (1.3.5), this gives isomorphisms:

$$H^d(\mu_n) \simeq H^0(N_f) \simeq H^{d-1}(\mu_n). \tag{1.3.8}$$

On the other hand, by [Liu19b], Cor. 4, λ_f has multiplicity 1 in $H^d(\mu_n)$, which is therefore non-zero. Thus $H^d(\mu_n) \simeq H^{d-1}(\mu_n) \simeq H^0(N_f)$ is a non-zero submodule of $H^0(H^0_P(\lambda_f)) = H^0(\lambda_f)$.

Now, using the automorphism τ of (G, T), one obtains that $H^d(\mu_n) \simeq H^{d-1}(\tau\mu_n)^*$ is a quotient of $H^0(\tau\lambda_f)^* \simeq V(\lambda_f)$. Since any non-zero morphism $V(\lambda_f) \to H^0(\lambda_f)$ has image L_f (see [Jan03] II.6.16 *Remark*), then (1.3.8) gives that $H^d(\mu_n) \simeq L_f \simeq H^{d-1}(\mu_n)$. □

Remark 1.3.3. [1] Seitz has shown ([Sei87], Prop. 6.1) that if a simple SL_{d+1}-module $L(\mu)$ with μ p-restricted has one-dimensional weight spaces then either μ is a fundamental weight ω_i or a multiple of ω_1 or ω_d, or $\mu = a\omega_i + (p-1-a)\omega_{i+1}$ for some $i \in \{1, \dots, d-1\}$ and $a \in \{0, \dots, p-1\}$. Since λ_f is not in that list when $f > 0$, it follows that $H^d(\mu_{p+f}) \simeq L(\lambda_f)$ never has all its weight spaces of dimension 1 when $f > 0$. This improves on Remark 3 (2) of [Liu19b].

1.4. **Jantzen sum formula and consequences.** Now, consider the case $f = 0$, i.e. $n = p$. Then $\lambda_0 = (p-2)\omega_2 + \omega_3$. We shall use Jantzen sum formula ([Jan03], II.8.19) to express $\operatorname{ch} L_0$ in terms of Weyl characters. Set $r = \min(d, p)$. In addition to λ_0, define for $i = 1, \dots, r-2$ the dominant weights: [2]

$$\lambda_i = i\omega_1 + (p-2-i)\omega_2 + \omega_{3+i} \tag{1.4.1}$$

(with the convention $\omega_{d+1} = 0$) and set $L_i = L(\lambda_i)$ and $N_i = L_P(\lambda_i)$. In other words, if $d > p$ the sequence ends with $\lambda_{p-2} = (p-2)\omega_1 + \omega_{p+1}$ whilst if $d \leq p$ it ends with

$$\lambda_{d-2} = (d-2)\omega_1 + (p-d)\omega_2 + \omega_{d+1} = (d-2)\omega_1 + (p-d)\omega_2.$$

Lemma 1.4.1. *Consider* SL_{n+1} *for some* $n \geq 2$ *and for* $k = 1, \dots, n$ *consider the weight* $\theta_k = \omega_1 - k\omega_k + (k-1)\omega_{k+1}$. *Note that* $\theta_1 = 0$.

(i) *For* $k \geq 2$, *one has* $s_k \cdot \theta_k = \theta_{k-1}$.

(ii) *For* $k \geq 2$, *one has* $\omega_1 - k\omega_k + k\omega_{k+1} = s_k \cdots s_2 \cdot \omega_{k+1}$.

Proof. One has $s_k \cdot \theta_k = \theta_k + (k-1)\alpha_k = \theta_{k-1}$. Thus $s_2 \cdots s_k \cdot \theta_k = \theta_1 = 0$, whence $\theta_k = s_k \cdots s_2 \cdot 0$. Next, for any weight λ and $w \in W$, one has $w \cdot \lambda = w\lambda + w \cdot 0$, which equals $\lambda + w \cdot 0$ if $w\lambda = \lambda$. Applying this to $w = s_k \cdots s_2$ and $\lambda = \omega_{k+1}$ gives assertion (ii). □

For $\alpha \in R^+$ let $\alpha^\vee$ be the corresponding coroot. Then, for $m \in \mathbb{Z}$, let $s_{\alpha,mp}$ be the affine reflection defined for all $\lambda \in X(T)$ by $s_{m,\alpha}(\lambda) = \lambda - ((\lambda, \alpha^\vee) - m)\alpha$. Further, one sets $s_{\alpha,m} \cdot \lambda = s_{m,\alpha}(\lambda + \rho) - \rho$.

For each Weyl module $V(\lambda)$, Jantzen has defined a decreasing filtration $V(\lambda) \supset V(\lambda)_1 \supset V(\lambda)_2 \supset \cdots$ and one has the following character formula ([Jan03], II.8.19):

$$\sum_{i \geq 1} \operatorname{ch} V(\lambda)_i = \sum_{\alpha \in R^+} \sum_{\substack{m \\ 0 < mp < (\lambda+\rho, \alpha^\vee)}} v_p(m) \chi(s_{\alpha,mp} \cdot \lambda)$$

where v_p denotes the p-adic valuation and χ is the Weyl character. Recall that $\chi(\mu) = 0$ if μ is singular for the dot action of W, i.e. if there exists $\alpha \in R^+$ such that $(\mu + \rho, \alpha^\vee) = 0$, and otherwise there exists a unique couple $(w, \mu^+) \in W \times X(T)^+$ such that $\mu = w \cdot \mu^+$ and then $\chi(\mu) = (-1)^{\ell(w)} \operatorname{ch} V(\mu^+)$. In particular, if the right-hand side, to which we shall refer as "Jantzen's sum" (relative to λ), equals $\operatorname{ch} L$ for some simple module L (resp. equals 0), then one has $\operatorname{ch} V(\lambda) = \operatorname{ch} L(\lambda) + \operatorname{ch} L$ (resp. $V(\lambda) = L(\lambda)$).

[1] We are grateful to one of the referees for this remark.

[2] These are different from the weights λ_f considered in section 1.3.

Proposition 1.4.2. *For $i = 0, \dots, r-2$ one has the equality:*

$$\sum_{\ell \geq 1} \operatorname{ch} V(\lambda_i)_\ell = \sum_{j=i+1}^{r-2} (-1)^{j-i-1} \operatorname{ch} V(\lambda_j) \tag{1.4.2}$$

and the exact sequences:

$$0 \longrightarrow L_i \longrightarrow H^0(\lambda_i) \longrightarrow L_{i+1} \longrightarrow 0 \tag{1.4.3}$$

and

$$0 \longrightarrow N_i \longrightarrow H^0_P(\lambda_i) \longrightarrow N_{i+1} \longrightarrow 0 \tag{1.4.4}$$

with the convention $L_{r-1} = 0 = N_{r-1}$. Therefore, one has exact sequences:

$$0 \to L(\lambda_0) \to H^0(\lambda_0) \to H^0(\lambda_1) \to \cdots \to H^0(\lambda_{r-2}) \to 0 \tag{1.4.5}$$

and

$$0 \to L_P(\lambda_0) \to H^0_P(\lambda_0) \to H^0_P(\lambda_1) \to \cdots \to H^0_P(\lambda_{r-2}) \to 0. \tag{1.4.6}$$

Proof. Fix $i \in \{0, \dots, r-2\}$ and for $1 \leq j \leq k \leq d$, set $\alpha_{j,k} = \alpha_j + \cdots + \alpha_k$ and $c_{i,j,k} = (\lambda_i + \rho, \alpha^\vee_{j,k})$. We are going to prove that all terms in Jantzen's sum (for both G and P) are zero, except the ones given in (1.4.2). Fix $j \leq k$ such that $c_{i,j,k} > p$, let $m \in \mathbb{N}^*$ such that $mp < c_{i,j,k}$ and set $t = c_{i,j,k} - mp$ and $\nu_m = s_{\alpha_{j,k}, mp} \cdot \lambda_i = \lambda_i - t\alpha_{j,k}$. There are four cases to consider.

Case 1: $c_{i,j,k} = k - j + 1$ (this occurs only $j > i + 3$ and $k \geq j + p$). Then the expression of ν_m in terms of the fundamental weights contains the "sequence" $-t\omega_j - t\omega_k$, the coefficients of the ω_ℓ for $j < \ell < k$ being 0, and hence $\nu_m + \rho$ is orthogonal to both $\alpha^\vee_{j,j+t-1}$ and $\alpha^\vee_{k-t+1,k}$. Therefore ν_m gives no contribution to Jantzen's sum, neither for G nor for P.

Case 2: $c_{i,j,k} = k - j + 2$ (this occurs only if $j \leq i + 3 \leq k$, including the case $k = i + 3$, $j = 2$ and $i = p - 2$). Assume first that $j < i + 3 < k$. Then the expression of ν_m in terms of the fundamental weights contains the "sequence" $-t\omega_j + \omega_{i+3} - t\omega_k$, the coefficients of the ω_ℓ for $j < \ell < k$ and $\ell \neq i + 3$ being 0.

For $s = 0, \dots, i + 2 - j, i + 3 - j, \dots, k - j - 1$, $(\nu_m + \rho, \alpha^\vee_{j,j+s})$ takes all the values from $1 - t$ to $k - j + 1 - t$, except $i - j + 4 - t$. Since $1 - t \leq 0$ and $k - j + 1 - t > 0$ (since $t \leq k - j + 2 - p \leq k - j$), the value 0 is obtained unless $t = i - j + 4$.

Similarly, for $s = 0, \dots, k - i - 4, k - i - 3, \dots, k - j - 1$, $(\nu_m + \rho, \alpha^\vee_{k-s,k})$ takes all the values from $1 - t$ to $k - j + 1 - t$, except $k - i - 2 - t$. Hence the value 0 is obtained, unless $t = k - i - 2$. This shows that $\nu_m + \rho$ is singular, except possibly if $t = i - j + 4 = k - i - 2$. But in this case one has $2t = k - j + 2$ and hence:

$$(\nu_m + \rho, \alpha^\vee_{j,k}) = k - j + 2(1 - t) = 0.$$

Consider now the "boundary" cases $j = i + 3$ or $k = i + 3$. If $j = i + 3$ then, since $k - t + 1 > j$, one has $k - t + 1 > i + 3$ and hence $(\nu_m + \rho, \alpha^\vee_{k-t+1,k}) = 0$. If $k = i + 3$, then $j + t - 1 < k = i + 3$ and hence $(\nu_m + \rho, \alpha^\vee_{j,j+t-1}) = 0$. Thus, in any case ν_m gives no contribution to Jantzen's sum, neither for G nor for P.

To close this case, note that since $i+3 \leq p+1 \leq k-j+2$, the case $k = i+3$ can occur only if $j \leq 2$, in which case $c_{i,j,k} = k-j+2$ implies $i = p-2$.

Case 3: $j = 2$ and $c_{i,j,k} = k+p-2-i$ and $i < p-2$. (Note that $c_{i,2,i+2} = p-1$ hence the hypothesis $c_{i,2,k} > p$ implies $k \geq i+3$.) Then one has $\nu_m = (i+t)\omega_1 + (p-2-i-t)\omega_2 + \omega_{i+3} - t\omega_k + t\omega_{k+1}$.

For $s = 0, \ldots, i, i+1, \ldots, k-3$, $(\nu_m + \rho, \alpha^\vee_{2,2+s})$ takes all the values from $p-i-1-t$ to $p+k-i-3-t$, except $p-t$. Since the last value taken is $\geq p-1 > 0$, the value 0 is obtained except if $t = p$ or if the initial value $p-i-1-t$ is > 0, i.e. $t \leq p-i-2$.

Similarly, for $s = 0, \ldots, k-i-4, k-i-3, \ldots, k-3$, $(\nu_m + \rho, \alpha^\vee_{k-s,k})$ takes all the values from $1-t$ to $k-1-t$, except $k-i-2-t$. Moreover one has $1-t \leq 0 < k-1-t$, hence the value 0 occurs unless $t = k-i-2$.

Thus, $\nu_m + \rho$ is singular except possibly if $t = k-i-2$ belongs to $\{1, \ldots, p-i-2\}$ or if $t = k-i-2 = p$. In the latter case, one has $2t = p+k-i-2$ and hence

$$(\nu_m + \rho, \alpha^\vee_{2,k}) = k-2+p-i-2t = 0.$$

In the former case, one has $k = i+2+t$ with $t = 1, \ldots, p-i-2$, whence $m = 1$. In fact, since $k = i+2+t$ is $\leq d$, we have $t \in \{1, \ldots, r-i-2\}$, recalling that $r = \min(p, d)$. For $t = 1, \ldots, r-i-2$, set

$$\theta'_t = s_{\alpha_{2,i+2+t},p} \cdot \lambda_i = (i+t)\omega_1 + (p-2-i-t)\omega_2 + \omega_{i+3} - t\omega_{i+2+t} + t\omega_{i+3+t}.$$

Using Lemma 1.4.1 with a shift of $i+2$ in the indices, one obtains that $\theta'_1 = \lambda_{i+1}$ and that $\theta'_t = s_{i+2+t} \cdots s_{i+4} \cdot \lambda_{i+t}$ for $t \geq 2$.

Denote by G_P the Levi subgroup of P containing T and recall ([Jan03], II.5.21) that $V_P(\lambda_i)$ is just the corresponding Weyl module for G_P on which the unipotent radical of P acts trivially. Therefore, applying Jantzen's sum formula for G_P, one already obtains the equality:

$$\sum_{\ell \geq 1} \operatorname{ch} V_P(\lambda_i)_\ell = \sum_{t=1}^{r-i-2} (-1)^{t-1} \operatorname{ch} V_P(\lambda_{i+t}). \tag{1.4.7}$$

To prove the analogous equality for G we must consider the last case, where $j = 1$.

Case 4: $j = 1$. Note that the assumption $c_{i,1,k} > p$ implies $k \geq 3$. If $k \leq i+3$ then $c_{i,1,k} = p+k-2+\delta_{k,i+3}$ is $\leq 2p$ (since $i \leq p-2$), hence $m = 1$ and $t = k-2+\delta_{k,i+3}$ and the expression of ν_m in terms of the fundamental weights contains the "sequence" $(p-i-2)\omega_2 - (k-2)\omega_k$, the coefficients of the ω_ℓ for $2 < \ell < k$ being 0. Then $(\nu_m + \rho, \alpha^\vee_{3,k}) = 0$ hence ν_m gives no contribution to Jantzen's sum.

Suppose now that $k > i+3$. Then $c_{i,1,k} = k-1+p$ and

$$\nu_m = (i-t)\omega_1 + (p-2-i)\omega_2 + \omega_{i+3} - t\omega_k + t\omega_{k+1}.$$

For $s = 0, \ldots, k-i-4, k-i-3, \ldots, k-3$, $(\nu_m + \rho, \alpha^\vee_{k-s,k})$ takes all the values from $1-t$ to $k-1-t$, except $k-i-2-t$. Moreover one has $1-t \leq 0 \leq k-1-t$, hence the value 0 occurs unless $t = k-i-2$.

Let us assume henceforth that $t = k-i-2$. Then $t > p$, for otherwise one would have $k-i-2 \leq p$ hence $k \leq 2p$ (since $i \leq p-2$) whence $m \leq 2$;

but $m=1$ gives $k-i-2=t=c_{i,1,k}-p=k-1$, a contradiction, whereas $m=2$ gives $k-i-2=t=c_{i,1,k}-2p=k-p-1$, a contradiction too, since $i\leq p-2$.

Now, for $s=0,\dots,i,i+1,\dots,k-3$, $(\nu_m+\rho,\alpha^\vee_{1,2+s})$ takes all the values from $p-t$ to $p+k-2-t$, except $p+i+1-t$. Moreover since $p<t\leq k-1$, the initial term is <0 and the final term >0, hence the value 0 occurs unless $t=p+i+1$. Now, if $t=k-i-2=p+i+1$ then $2t=p+k-1$ and hence

$$(\nu_m+\rho,\alpha^\vee_{1,k})=p+k-1-2t=0.$$

Thus, in any case ν_m gives no contribution to Jantzen's sum. This proves (1.4.2).

It follows from (1.4.2) that $H^0(\lambda_{r-2})=L_{r-2}$; then for λ_{r-3} the Jantzen sum equals $\operatorname{ch}L_{r-2}$ hence $\operatorname{ch}H^0(\lambda_{r-3})=\operatorname{ch}L_{r-3}+\operatorname{ch}L_{r-2}$. By decreasing induction one obtains that $\operatorname{ch}H^0(\lambda_i)=\operatorname{ch}L_i+\operatorname{ch}L_{i+1}$ for $i=r-3,\dots,0$, whence the exact sequences (1.4.3). Similarly, using (1.4.7) one obtains that $H^0_P(\lambda_{r-2})=N_{r-2}$ and $\operatorname{ch}H^0_P(\lambda_i)=\operatorname{ch}N_i+\operatorname{ch}N_{i+1}$ for $i=r-3,\dots,0$, whence the exact sequences (1.4.4). This completes the proof of Proposition 1.4.2. □

Let us derive the following corollary (which is not used in the sequel).

Corollary 1.4.3. *For $i=0,\dots,r-2$, one has $H^0(N_i)=L_i$ and $H^j(N_i)=0$ for $j>0$.*

Proof. Applying the functor H^0 to each exact sequence (1.4.4) gives an exact sequence:

$$0\longrightarrow H^0(N_i)\longrightarrow H^0(\lambda_i)\longrightarrow H^0(N_{i+1})\longrightarrow H^1(N_i)\longrightarrow 0$$

and isomorphisms $H^j(N_{i+1})\simeq H^{j+1}(N_i)$ for $j\geq 1$. Since $H^j(N_0)=0$ for $j\geq 1$, one obtains $H^j(N_i)=0$ for all $i\geq 0$ and $j\geq 1$, hence the previous exact sequence becomes:

$$0\longrightarrow H^0(N_i)\longrightarrow H^0(\lambda_i)\longrightarrow H^0(N_{i+1})\longrightarrow 0\ . \tag{1.4.8}$$

Since $H^0(N_0)\simeq L_0$, the exact sequences (1.4.3) then imply, by induction on i, that $H^0(N_i)\simeq L_i$ for $i=0,\dots,r-2$. □

On the other hand, in [Liu19b], Cor. 3 and 4, the first author proved, using a result of Suprunenko pointed out by one of the referees, that the dominant weights of $L(\lambda_0)$ (resp. of $L(\lambda'_0)=L((p-2)\omega_1+\omega_2)$) are exactly the dominant weights $\leq\lambda_0$ (resp. λ'_0), each occuring with multiplicity one. Let us now switch to representations of GL_{d+1} and identify each λ_i (resp. $\lambda'_i=(p-2-i)\omega_1+\omega_{2+i}$) with the partition $(p-1,p-1-i,1^{i+1})$ (resp. $(p-1-i,1^{i+1})$). Recall that (see for example [McD95], Chap. I) for a dominant weight λ of GL_{d+1}, identified with a partition with at most $d+1$ parts, the Weyl character $\operatorname{ch}V(\lambda)$ (resp. the orbit sum $\sum_{\nu\in W\lambda}e^\nu$) corresponds to the Schur function S_λ (resp. the monomial symmetric function m_λ). Let us assume that $d\geq 2p-2$, then the dominant weights smaller than λ_0 correspond to all partitions of $2p-1$ which are smaller than $(p-1,p-1,1)$ in the dominance order. Thus, one deduces from Propositions 1.4.2 and 1.3.1 (or Corollary 3 in [Liu19b]) the following:

Corollary 1.4.4. *For each prime number p, one has the equality:*

$$\sum_{\lambda \leq (p-1,p-1,1)} m_\lambda = \sum_{i=0}^{p-2} (-1)^i \, S_{(p-1,p-1-i,1^{i+1})}$$

and

$$\sum_{\lambda \leq (p-1,1)} m_\lambda = \sum_{i=0}^{p-2} (-1)^i \, S_{(p-1-i,1^{i+1})}$$

where the sum on the left hand-side of the first equality (resp. second equality) is taken over all partitions λ of $2p-1$ (resp. p) such that $\lambda \leq (p-1,p-1,1)$ (resp. $\lambda \leq (p-1,1)$) in the dominance order.

Remark 1.4.5. One may conjecture that the previous equality holds for *any* integer $n \geq 2$ (not only for prime numbers). Using a Sage program, we have checked this up to $n = 23$. [3]

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[3] On August 23, 2019 (the day after the first version of this article was posted on arXiv), a proof of the first equality of this conjecture was kindly sent to us by Darij Grinberg, see [Gri20].