



HAL
open science

On the cohomology of line bundles over certain flag schemes II

Linyuan Liu, Patrick Polo

► **To cite this version:**

Linyuan Liu, Patrick Polo. On the cohomology of line bundles over certain flag schemes II. Journal of Combinatorial Theory, Series A, 2021, 178, pp.105352 -. 10.1016/j.jcta.2020.105352 . hal-03493489

HAL Id: hal-03493489

<https://hal.science/hal-03493489>

Submitted on 7 Nov 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

ON THE COHOMOLOGY OF LINE BUNDLES OVER CERTAIN FLAG SCHEMES II

LINYUAN LIU AND PATRICK POLO

To Jens Carsten Jantzen on the occasion of his 70th birthday

ABSTRACT. Over a field K of characteristic p , let Z be the incidence variety in $\mathbb{P}^d \times (\mathbb{P}^d)^*$ and let \mathcal{L} be the restriction to Z of the line bundle $\mathcal{O}(-n-d) \boxtimes \mathcal{O}(n)$, where $n = p + f$ with $0 \leq f \leq p - 2$. We prove that $H^d(Z, \mathcal{L})$ is the simple GL_{d+1} -module corresponding to the partition $\lambda_f = (p - 1 + f, p - 1, f + 1)$. When $f = 0$, using the first author's description of $H^d(Z, \mathcal{L})$ and Jantzen's sum formula, we obtain as a by-product that the sum of the monomial symmetric functions m_λ , for all partitions λ of $2p - 1$ less than $(p - 1, p - 1, 1)$ in the dominance order, is the alternating sum of the Schur functions $S_{p-1, p-1-i, 1^{i+1}}$ for $i = 0, \dots, p - 2$.

Introduction. This paper is an addition to the first author's paper [Liu19b]. We now consider the group scheme $G = \mathrm{SL}_{d+1}$ over an arbitrary field K of characteristic $p > 0$. For $m, n \in \mathbb{N}$, set $\mu_{m,n} = m\omega_1 - (n + d)\omega_d$ and write simply μ_n instead of $\mu_{n,n}$. We describe the $H^i(\mu_{m,n})$ when $p > n$ (thus recovering and extending [Liu19b], Cor. 2) and for $n = p + f$ with $0 \leq f \leq p - 2$ we prove that $H^d(\mu_n)$ is the simple G -module $L(\lambda_f) = L(f\omega_1 + (p - 2 - f)\omega_2 + (f + 1)\omega_3)$. Further, when $f = 0$ we express the character of $L(\lambda_f)$, using Jantzen sum formula, as an alternating sum of Weyl characters. Comparing this with the character of $H^d(\mu_p)$ given in [Liu19b], Cor. 3, we obtain as a by-product that the sum of the monomial symmetric functions m_λ , for all partitions λ of $2p - 1$ less than λ_f in the dominance order, is the alternating sum of the Schur functions $S_{p-1, p-1-i, 1^{i+1}}$ for $i = 0, \dots, p - 2$.

1.1. Notation. We keep the notation of [Liu19b], except that we now consider the group scheme $G = \mathrm{SL}_{d+1}$ over an arbitrary field K of characteristic $p > 0$ and denote simply by P the maximal parabolic subgroup P_1 . Let W_P be the Weyl group of (P, T) and define W_Q similarly. Let $\mathrm{ch} V$ denote the character of a T -module V . For each simple root α_i , let $s_i \in W$ be the corresponding simple reflection. Let w_0 (resp. w_P , resp. w_Q) be the longest element of W (resp. W_P , resp. W_Q) and set $N = \ell(w_Q)$. Then $\ell(w_P) = N + d - 1$.

Let ρ_Q (resp. ρ_P) denote the half-sum of the positive roots of Q (resp. P). Then one has:

Date: August 22, 2020.

1991 Mathematics Subject Classification. 05E05, 05E10, 14L15, 20G05.

Key words and phrases. cohomology, line bundles, flag schemes, Weyl modules, symmetric functions.

$$(1.1.1) \quad 2\rho_Q = (2-d)(\omega_1 + \omega_d) + 2 \sum_{i=2}^{d-1} \omega_i, \quad 2\rho_P = (1-d)\omega_1 + 2 \sum_{i=2}^d \omega_i.$$

Recall that, since $2\rho - 2\rho_Q = d(\omega_1 + \omega_d)$, the dualizing sheaf on $Z = G/Q$ is $\mathcal{L}(-d\omega_1 - d\omega_d)$. Hence, by Serre duality on G/Q , one has

$$(1.1.2) \quad H^d(\mu_{m,n}) \simeq H^{d-1}(-(m+d)\omega_1 + n\omega_d)^*.$$

Further, let τ be the (involutive) automorphism of (G, T) induced by the automorphism of the Dynkin diagram which swaps α_i and α_{d+1-i} for $i = 1, \dots, d$. Note that τ also acts on $X(T)$ and preserves $X(T)^+$. For any G -module V , let ${}^\tau V$ denote the corresponding module twisted by τ . For example, for a Weyl module $V(\lambda)$ (resp. a simple module $L(\lambda)$), one has ${}^\tau V(\lambda) \simeq V(\tau\lambda)$ (resp. ${}^\tau L(\lambda) \simeq L(\tau\lambda)$). Then, (1.1.2) can be rewritten as:

$$(1.1.3) \quad H^d(\mu_{m,n}) \simeq H^{d-1}(\tau\mu_{n,m})^*.$$

Recall (see [Jan03], II.4.6 and its proof) that if $P' \subset P''$ are parabolic subgroups containing B and if V is a P' -module, one has for all $i \geq 0$:

$$(1.1.4) \quad H^i(P''/B, V) \simeq H^i(P''/P', V).$$

This will be used several times, without always mentioning it.

1.2. A description of $H^i(\mu_{m,n})$. For each $i \in \mathbb{N}$, we denote by $H_Q^i(-)$ the functor $H^i(Q/B, -)$ and define $H_P^i(-)$ similarly. Since $-\mu_{m,n}$ is trivial and $\mu_{m,n} - 2\rho_Q$ is anti-dominant with respect to the Levi subgroup of Q one has:

$$(1.2.1) \quad H_Q^i(-\mu_{m,n}) \simeq \begin{cases} -\mu_{m,n} & \text{if } i = 0, \\ 0 & \text{if } i > 0 \end{cases}$$

and, using Serre duality on Q/B :

$$(1.2.2) \quad H_Q^i(\mu_{m,n} - 2\rho_Q) \simeq \begin{cases} 0 & \text{if } i < N, \\ H_Q^0(-\mu_{m,n})^* \simeq \mu_{m,n} & \text{if } i = N. \end{cases}$$

Consider induction from B to P . Thanks to (1.2.2), the spectral sequence of composite functors (see [Jan03], I.4.5) degenerates and gives (together with (1.1.4) applied to $P' = Q$ and $P'' = P$) isomorphisms for each $i \geq 0$:

$$(1.2.3) \quad H_P^i(\mu_{m,n}) \simeq H_P^{i+N}(\mu_{m,n} - 2\rho_Q)$$

and since $\mu_{m,n} - 2\rho_Q = (m+d-2)\omega_1 - 2 \sum_{j=2}^{d-1} \omega_j - (n+2)\omega_d$ is anti-dominant with respect to P , the latter group is zero unless $i+N = \dim(P/B)$, i.e. $i = d-1$. Moreover, by Serre duality on P/B , one has:

$$\begin{aligned} H_P^{N+d-1}(\mu_{m,n} - 2\rho_Q) &\simeq H_P^0(-\mu_{m,n} + 2\rho_Q - 2\rho_P)^* \\ &= H_P^0((1-m)\omega_1 + n\omega_d)^*. \end{aligned}$$

Set $\nu_{m,n} = (1-m)\omega_1 + n\omega_d$ and $\pi_{m,n} = -w_P\nu_{m,n}$. Since $w_P = s_2 \cdots s_d w_Q$, one has

$$(1.2.4) \quad \pi_{m,n} = (m-n-1)\omega_1 + n\omega_2$$

and since $H_P^0(\nu_{m,n})^*$ is isomorphic with $V_P(\pi_{m,n})$, the Weyl module for P with highest weight $\pi_{m,n}$, one obtains:

Lemma 1.2.1. *For each $m, n \in \mathbb{N}$ and $i \in \mathbb{N}$, one has*

$$H_P^i(\mu_{m,n}) \simeq \begin{cases} V_P(\pi_{m,n}) & \text{if } i = d - 1, \\ 0 & \text{if } i \neq d - 1. \end{cases}$$

Now, consider induction from B to G . Thanks to the lemma, the spectral sequence of composite functors degenerates and gives (together with (1.1.4) applied to $P' = P$ and $P'' = G$) isomorphisms $H^i(\mu_{m,n}) \simeq H^{i-d+1}(V_P(\pi_{m,n}))$ for each $i \geq 0$. Since the former is zero for $i \notin \{d-1, d\}$ (see [Liu19b], Section 2), this gives:

Proposition 1.2.2. *For each $m, n \in \mathbb{N}$ and $i \in \mathbb{N}$, one has*

$$H^i(\mu_{m,n}) \simeq \begin{cases} H^0(V_P(\pi_{m,n})) & \text{if } i = d - 1, \\ H^1(V_P(\pi_{m,n})) & \text{if } i = d, \\ 0 & \text{if } i \neq d - 1, d. \end{cases}$$

This allows us to recover and extend the result of [Liu19b], Corollary 2:

Corollary 1.2.3. *Suppose that $p > n$.*

- (i) *If $m \geq n$, then $H^d(\mu_{m,n}) = 0$ and $H^{d-1}(\mu_{m,n}) \simeq H^0(\pi_{m,n})$. In particular, $H^i(\mu_n) = 0$ for all $i \geq 0$.*
- (ii) *If $n \geq m$, then $H^{d-1}(\mu_{m,n}) = 0$ and*

$$H^d(\mu_{m,n}) \simeq H^1(\pi_{m,n}) \simeq H^0(\pi_{n,m}).$$

- (iii) *Furthermore, if $p > m \geq n$ then $H^{d-1}(\mu_{m,n}) \simeq L(\pi_{m,n}) \simeq H^d(\mu_{n,m})$.*

Proof. Suppose $p > n$. Then $V_P(\pi_{m,n})$ is irreducible as a P -module, being isomorphic as a module over the Levi subgroup of P to the GL_d -module $S^n K^d$ which is irreducible since $n < p$. Therefore, $V_P(\pi_{m,n}) \simeq H_P^0(\pi_{m,n})$. Thus, the proposition gives for all $i \geq 0$ that

$$H^i(\mu_{m,n}) \simeq H^{i-d+1}(\pi_{m,n}).$$

If $m \geq n$ then $\pi_{m,n}$ belongs to $\mathcal{C} := X(T)^+ - \rho$ and hence $H^j(\pi_{m,n}) = 0$ for $j > 0$ (and also for $j = 0$ if $m = n$). This proves (i).

If $m < n$, then $H^0(\pi_{m,n}) = 0$. Further, $\pi_{n,m} = (n-m-1)\omega_1 + m\omega_2$ belongs to \mathcal{C} and one has $\pi_{m,n} = s_{\alpha_1} \cdot \pi_{n,m}$. Since $0 \leq (\pi_{n,m} + \rho, \alpha_1^\vee) = n - m < p$, one has $H^1(\pi_{m,n}) \simeq H^0(\pi_{n,m})$ (see [Jan03], II.5.4), which proves (ii).

Suppose now that $p > m \geq n$. By (i) and Serre duality on G/Q , one has

$$H^0(\pi_{m,n}) \simeq H^{d-1}(\mu_{m,n}) \simeq H^d(-(m+d)\omega_1 + n\omega_d)^* = H^d(\tau\mu_{n,m})^*.$$

Now, using the automorphism τ of (G, T) , one deduces from (ii) that

$$H^d(\tau\mu_{n,m})^* \simeq H^0(\tau\pi_{m,n})^*.$$

Since the latter is the Weyl module $V(\pi_{m,n})$, one obtains that $H^0(\pi_{m,n}) \simeq L(\pi_{m,n})$. This proves (iii). \square

Our goal in the next subsection is to determine $H^d(\mu_p)$. Since for $d = 2$ the SL_3 -modules $H^2(m, -n - 2)$ have been described in [Liu19a], where it is proved in particular (see [Liu19a], Th. 3) that $H^2(p, -p - 2)$ is the Weyl module $V(0, p - 2)$, which is simple, we will henceforth assume that $d \geq 3$.

1.3. Computation of $H^d(\mu_n)$ for $p \leq n \leq 2p - 1$. Suppose now that $d \geq 3$ and $n = p + f$ with $0 \leq f \leq p - 1$. For each weight ν which is dominant (resp. dominant with respect to P), denote by $L(\nu)$ (resp. $L_P(\nu)$) the irreducible G -module (resp. P -module) with highest weight ν . Let

$$(1.3.1) \quad \lambda_f = \begin{cases} f\omega_1 + (p-2-f)\omega_2 + (f+1)\omega_3 & \text{if } f \leq p-2, \\ (p-1)\omega_1 + (p-2)\omega_3 + \omega_4 & \text{if } f = p-1 \end{cases}$$

(with the convention $\omega_4 = 0$ if $d = 3$), and set $L_f = L(\lambda_f)$ and $N_f = L_P(\lambda_f)$.

Proposition 1.3.1. *One has $H^{d-1}(\mu_n) \simeq L_f \simeq H^d(\mu_n)$.*

Remark 1.3.2. This generalizes Corollary 3 in [Liu19b], which was the case $f = 0$. Note that if $f > 0$ and $d > p - f + 1$ then μ_{p+f} does not belong to the closure of the facet containing μ_p . Indeed, denoting by $\alpha_1, \dots, \alpha_d$ the simple roots and setting $\beta = \alpha_1 + \dots + \alpha_{p-f+1}$, one has $\langle \mu_p + \rho, \beta^\vee \rangle = 2p - f + 1$ whereas $\langle \mu_{p+f} + \rho, \beta^\vee \rangle = 2p + 1$. Hence the result for μ_{p+f} cannot be deduced from the one for μ_p by applying a functor of translation.

Proof. Set $\pi_n = -\omega_1 + n\omega_2$ and $M = V_P(\pi_n)$. According to Doty [Dot85], §§2.3–2.4, one has exact sequences of P -modules:

$$(1.3.2) \quad 0 \longrightarrow N_f \longrightarrow M \longrightarrow C \longrightarrow 0$$

and

$$(1.3.3) \quad 0 \longrightarrow C \longrightarrow H_P^0(\pi_n) \longrightarrow N_f \longrightarrow 0$$

where $C = L_P(\pi_n)$. Applying the functor H^0 to (1.3.3) and using that:

$$(1.3.4) \quad H^i(H_P^0(\pi_n)) = H^i(\pi_n) = 0$$

for all $i \geq 0$, one obtains $H^0(C) = 0$ and isomorphisms $H^i(C) \simeq H^{i-1}(N_f)$ for all $i \geq 1$. Taking this and Proposition 1.2.2 into account and applying the functor H^0 to (1.3.2), one obtains isomorphisms:

$$(1.3.5) \quad H^{d-1}(\mu_n) \simeq H^0(M) \simeq H^0(N_f),$$

an exact sequence:

$$(1.3.6) \quad 0 \longrightarrow H^1(N_f) \longrightarrow H^d(\mu_n) \longrightarrow H^0(N_f) \longrightarrow H^2(N_f) \longrightarrow 0$$

and isomorphisms

$$(1.3.7) \quad H^i(N_f) \simeq H^{i+1}(C) \simeq H^{i+2}(N_f)$$

for $i \geq 1$. Since $H^i(N_f) = 0$ for $i > |R^+|$, one obtains $H^i(N_f) = 0$ for all $i \geq 1$. Together with (1.3.6) and (1.3.5), this gives isomorphisms:

$$(1.3.8) \quad H^d(\mu_n) \simeq H^0(N_f) \simeq H^{d-1}(\mu_n).$$

On the other hand, by [Liu19b], Cor. 4, λ_f has multiplicity 1 in $H^d(\mu_n)$, which is therefore non-zero. Thus $H^d(\mu_n) \simeq H^{d-1}(\mu_n) \simeq H^0(N_f)$ is a non-zero submodule of $H^0(H_P^0(\lambda_f)) = H^0(\lambda_f)$.

Now, using the automorphism τ of (G, T) , one obtains that $H^d(\mu_n) \simeq H^{d-1}(\tau\mu_n)^*$ is a quotient of $H^0(\tau\lambda_f)^* \simeq V(\lambda_f)$. Since any non-zero morphism $V(\lambda_f) \rightarrow H^0(\lambda_f)$ has image L_f (see [Jan03] II.6.16 *Remark*), then (1.3.8) gives that $H^d(\mu_n) \simeq L_f \simeq H^{d-1}(\mu_n)$. \square

Remark 1.3.3. ¹ Seitz has shown ([Sei87], Prop. 6.1) that if a simple SL_{d+1} -module $L(\mu)$ with μ p -restricted has one-dimensional weight spaces then either μ is a fundamental weight ω_i or a multiple of ω_1 or ω_d , or $\mu = a\omega_i + (p-1-a)\omega_{i+1}$ for some $i \in \{1, \dots, d-1\}$ and $a \in \{0, \dots, p-1\}$. Since λ_f is not in that list when $f > 0$, it follows that $H^d(\mu_{p+f}) \simeq L(\lambda_f)$ never has all its weight spaces of dimension 1 when $f > 0$. This improves on Remark 3 (2) of [Liu19b].

1.4. Jantzen sum formula and consequences. Now, consider the case $f = 0$, i.e. $n = p$. Then $\lambda_0 = (p-2)\omega_2 + \omega_3$. We shall use Jantzen sum formula ([Jan03], II.8.19) to express $\mathrm{ch} L_0$ in terms of Weyl characters. Set $r = \min(d, p)$. In addition to λ_0 , define for $i = 1, \dots, r-2$ the dominant weights: ²

$$(1.4.1) \quad \lambda_i = i\omega_1 + (p-2-i)\omega_2 + \omega_{3+i}$$

(with the convention $\omega_{d+1} = 0$) and set $L_i = L(\lambda_i)$ and $N_i = L_P(\lambda_i)$. In other words, if $d > p$ the sequence ends with $\lambda_{p-2} = (p-2)\omega_1 + \omega_{p+1}$ whilst if $d \leq p$ it ends with

$$\lambda_{d-2} = (d-2)\omega_1 + (p-d)\omega_2 + \omega_{d+1} = (d-2)\omega_1 + (p-d)\omega_2.$$

Lemma 1.4.1. *Consider SL_{n+1} for some $n \geq 2$ and for $k = 1, \dots, n$ consider the weight $\theta_k = \omega_1 - k\omega_k + (k-1)\omega_{k+1}$. Note that $\theta_1 = 0$.*

(i) *For $k \geq 2$, one has $s_k \cdot \theta_k = \theta_{k-1}$.*

(ii) *For $k \geq 2$, one has $\omega_1 - k\omega_k + k\omega_{k+1} = s_k \cdots s_2 \cdot \omega_{k+1}$.*

Proof. One has $s_k \cdot \theta_k = \theta_k + (k-1)\alpha_k = \theta_{k-1}$. Thus $s_2 \cdots s_k \cdot \theta_k = \theta_1 = 0$, whence $\theta_k = s_k \cdots s_2 \cdot 0$. Next, for any weight λ and $w \in W$, one has $w \cdot \lambda = w\lambda + w \cdot 0$, which equals $\lambda + w \cdot 0$ if $w\lambda = \lambda$. Applying this to $w = s_k \cdots s_2$ and $\lambda = \omega_{k+1}$ gives assertion (ii). \square

For $\alpha \in R^+$ let α^\vee be the corresponding coroot. Then, for $m \in \mathbb{Z}$, let $s_{\alpha, mp}$ be the affine reflection defined for all $\lambda \in X(T)$ by $s_{\alpha, mp}(\lambda) = \lambda - ((\lambda, \alpha^\vee) - m)\alpha$. Further, one sets $s_{\alpha, m} \cdot \lambda = s_{\alpha, mp}(\lambda + \rho) - \rho$.

For each Weyl module $V(\lambda)$, Jantzen has defined a decreasing filtration $V(\lambda) \supset V(\lambda)_1 \supset V(\lambda)_2 \supset \cdots$ and one has the following character formula ([Jan03], II.8.19):

$$\sum_{i \geq 1} \mathrm{ch} V(\lambda)_i = \sum_{\alpha \in R^+} \sum_{0 < mp < (\lambda + \rho, \alpha^\vee)}^m v_p(m) \chi(s_{\alpha, mp} \cdot \lambda)$$

where v_p denotes the p -adic valuation and χ is the Weyl character. Recall that $\chi(\mu) = 0$ if μ is singular for the dot action of W , i.e. if there exists $\alpha \in R^+$ such that $(\mu + \rho, \alpha^\vee) = 0$, and otherwise there exists a unique couple $(w, \mu^+) \in W \times X(T)^+$ such that $\mu = w \cdot \mu^+$ and then $\chi(\mu) = (-1)^{\ell(w)} \mathrm{ch} V(\mu^+)$. In particular, if the right-hand side, to which we shall refer as ‘‘Jantzen’s sum’’ (relative to λ), equals $\mathrm{ch} L$ for some simple module L (resp. equals 0), then one has $\mathrm{ch} V(\lambda) = \mathrm{ch} L(\lambda) + \mathrm{ch} L$ (resp. $V(\lambda) = L(\lambda)$).

¹We are grateful to one of the referees for this remark.

²These are different from the weights λ_f considered in section 1.3.

Proposition 1.4.2. *For $i = 0, \dots, r-2$ one has the equality:*

$$(1.4.2) \quad \sum_{\ell \geq 1} \text{ch } V(\lambda_i)_\ell = \sum_{j=i+1}^{r-2} (-1)^{j-i-1} \text{ch } V(\lambda_j)$$

and the exact sequences:

$$(1.4.3) \quad 0 \longrightarrow L_i \longrightarrow H^0(\lambda_i) \longrightarrow L_{i+1} \longrightarrow 0$$

and

$$(1.4.4) \quad 0 \longrightarrow N_i \longrightarrow H_P^0(\lambda_i) \longrightarrow N_{i+1} \longrightarrow 0$$

with the convention $L_{r-1} = 0 = N_{r-1}$. Therefore, one has exact sequences:

$$(1.4.5) \quad 0 \rightarrow L(\lambda_0) \rightarrow H^0(\lambda_0) \rightarrow H^0(\lambda_1) \rightarrow \dots \rightarrow H^0(\lambda_{r-2}) \rightarrow 0$$

and

$$(1.4.6) \quad 0 \rightarrow L_P(\lambda_0) \rightarrow H_P^0(\lambda_0) \rightarrow H_P^0(\lambda_1) \rightarrow \dots \rightarrow H_P^0(\lambda_{r-2}) \rightarrow 0.$$

Proof. Fix $i \in \{0, \dots, r-2\}$ and for $1 \leq j \leq k \leq d$, set $\alpha_{j,k} = \alpha_j + \dots + \alpha_k$ and $c_{i,j,k} = (\lambda_i + \rho, \alpha_{j,k}^\vee)$. We are going to prove that all terms in Jantzen's sum (for both G and P) are zero, except the ones given in (1.4.2). Fix $j \leq k$ such that $c_{i,j,k} > p$, let $m \in \mathbb{N}^*$ such that $mp < c_{i,j,k}$ and set $t = c_{i,j,k} - mp$ and $\nu_m = s_{\alpha_{j,k}, mp} \cdot \lambda_i = \lambda_i - t\alpha_{j,k}$. There are four cases to consider.

Case 1: $c_{i,j,k} = k - j + 1$ (this occurs only $j > i + 3$ and $k \geq j + p$). Then the expression of ν_m in terms of the fundamental weights contains the "sequence" $-t\omega_j - t\omega_k$, the coefficients of the ω_ℓ for $j < \ell < k$ being 0, and hence $\nu_m + \rho$ is orthogonal to both $\alpha_{j,j+t-1}^\vee$ and $\alpha_{k-t+1,k}^\vee$. Therefore ν_m gives no contribution to Jantzen's sum, neither for G nor for P .

Case 2: $c_{i,j,k} = k - j + 2$ (this occurs only if $j \leq i + 3 \leq k$, including the case $k = i + 3$, $j = 2$ and $i = p - 2$). Assume first that $j < i + 3 < k$. Then the expression of ν_m in terms of the fundamental weights contains the "sequence" $-t\omega_j + \omega_{i+3} - t\omega_k$, the coefficients of the ω_ℓ for $j < \ell < k$ and $\ell \neq i + 3$ being 0.

For $s = 0, \dots, i + 2 - j, i + 3 - j, \dots, k - j - 1$, $(\nu_m + \rho, \alpha_{j,j+s}^\vee)$ takes all the values from $1 - t$ to $k - j + 1 - t$, except $i - j + 4 - t$. Since $1 - t \leq 0$ and $k - j + 1 - t > 0$ (since $t \leq k - j + 2 - p \leq k - j$), the value 0 is obtained unless $t = i - j + 4$.

Similarly, for $s = 0, \dots, k - i - 4, k - i - 3, \dots, k - j - 1$, $(\nu_m + \rho, \alpha_{k-s,k}^\vee)$ takes all the values from $1 - t$ to $k - j + 1 - t$, except $k - i - 2 - t$. Hence the value 0 is obtained, unless $t = k - i - 2$. This shows that $\nu_m + \rho$ is singular, except possibly if $t = i - j + 4 = k - i - 2$. But in this case one has $2t = k - j + 2$ and hence:

$$(\nu_m + \rho, \alpha_{j,k}^\vee) = k - j + 2(1 - t) = 0.$$

Consider now the "boundary" cases $j = i + 3$ or $k = i + 3$. If $j = i + 3$ then, since $k - t + 1 > j$, one has $k - t + 1 > i + 3$ and hence $(\nu_m + \rho, \alpha_{k-t+1,k}^\vee) = 0$. If $k = i + 3$, then $j + t - 1 < k = i + 3$ and hence $(\nu_m + \rho, \alpha_{j,j+t-1}^\vee) = 0$. Thus, in any case ν_m gives no contribution to Jantzen's sum, neither for G nor for P .

To close this case, note that since $i + 3 \leq p + 1 \leq k - j + 2$, the case $k = i + 3$ can occur only if $j \leq 2$, in which case $c_{i,j,k} = k - j + 2$ implies $i = p - 2$.

Case 3: $j = 2$ and $c_{i,j,k} = k + p - 2 - i$ and $i < p - 2$. (Note that $c_{i,2,i+2} = p - 1$ hence the hypothesis $c_{i,2,k} > p$ implies $k \geq i + 3$.) Then one has $\nu_m = (i + t)\omega_1 + (p - 2 - i - t)\omega_2 + \omega_{i+3} - t\omega_k + t\omega_{k+1}$.

For $s = 0, \dots, i, i + 1, \dots, k - 3$, $(\nu_m + \rho, \alpha_{2,2+s}^\vee)$ takes all the values from $p - i - 1 - t$ to $p + k - i - 3 - t$, except $p - t$. Since the last value taken is $\geq p - 1 > 0$, the value 0 is obtained except if $t = p$ or if the initial value $p - i - 1 - t$ is > 0 , i.e. $t \leq p - i - 2$.

Similarly, for $s = 0, \dots, k - i - 4, k - i - 3, \dots, k - 3$, $(\nu_m + \rho, \alpha_{k-s,k}^\vee)$ takes all the values from $1 - t$ to $k - 1 - t$, except $k - i - 2 - t$. Moreover one has $1 - t \leq 0 < k - 1 - t$, hence the value 0 occurs unless $t = k - i - 2$.

Thus, $\nu_m + \rho$ is singular except possibly if $t = k - i - 2$ belongs to $\{1, \dots, p - i - 2\}$ or if $t = k - i - 2 = p$. In the latter case, one has $2t = p + k - i - 2$ and hence

$$(\nu_m + \rho, \alpha_{2,k}^\vee) = k - 2 + p - i - 2t = 0.$$

In the former case, one has $k = i + 2 + t$ with $t = 1, \dots, p - i - 2$, whence $m = 1$. In fact, since $k = i + 2 + t$ is $\leq d$, we have $t \in \{1, \dots, r - i - 2\}$, recalling that $r = \min(p, d)$. For $t = 1, \dots, r - i - 2$, set

$$\theta'_t = s_{\alpha_{2,i+2+t,p}} \cdot \lambda_i = (i + t)\omega_1 + (p - 2 - i - t)\omega_2 + \omega_{i+3} - t\omega_{i+2+t} + t\omega_{i+3+t}.$$

Using Lemma 1.4.1 with a shift of $i + 2$ in the indices, one obtains that $\theta'_1 = \lambda_{i+1}$ and that $\theta'_t = s_{i+2+t} \cdots s_{i+4} \cdot \lambda_{i+t}$ for $t \geq 2$.

Denote by G_P the Levi subgroup of P containing T and recall ([Jan03], II.5.21) that $V_P(\lambda_i)$ is just the corresponding Weyl module for G_P on which the unipotent radical of P acts trivially. Therefore, applying Jantzen's sum formula for G_P , one already obtains the equality:

$$(1.4.7) \quad \sum_{\ell \geq 1} \text{ch } V_P(\lambda_i)_\ell = \sum_{t=1}^{r-i-2} (-1)^{t-1} \text{ch } V_P(\lambda_{i+t}).$$

To prove the analogous equality for G we must consider the last case, where $j = 1$.

Case 4: $j = 1$. Note that the assumption $c_{i,1,k} > p$ implies $k \geq 3$. If $k \leq i + 3$ then $c_{i,1,k} = p + k - 2 + \delta_{k,i+3}$ is $\leq 2p$ (since $i \leq p - 2$), hence $m = 1$ and $t = k - 2 + \delta_{k,i+3}$ and the expression of ν_m in terms of the fundamental weights contains the "sequence" $(p - i - 2)\omega_2 - (k - 2)\omega_k$, the coefficients of the ω_ℓ for $2 < \ell < k$ being 0. Then $(\nu_m + \rho, \alpha_{3,k}^\vee) = 0$ hence ν_m gives no contribution to Jantzen's sum.

Suppose now that $k > i + 3$. Then $c_{i,1,k} = k - 1 + p$ and

$$\nu_m = (i - t)\omega_1 + (p - 2 - i)\omega_2 + \omega_{i+3} - t\omega_k + t\omega_{k+1}.$$

For $s = 0, \dots, k - i - 4, k - i - 3, \dots, k - 3$, $(\nu_m + \rho, \alpha_{k-s,k}^\vee)$ takes all the values from $1 - t$ to $k - 1 - t$, except $k - i - 2 - t$. Moreover one has $1 - t \leq 0 \leq k - 1 - t$, hence the value 0 occurs unless $t = k - i - 2$.

Let us assume henceforth that $t = k - i - 2$. Then $t > p$, for otherwise one would have $k - i - 2 \leq p$ hence $k \leq 2p$ (since $i \leq p - 2$) whence $m \leq 2$;

but $m = 1$ gives $k - i - 2 = t = c_{i,1,k} - p = k - 1$, a contradiction, whereas $m = 2$ gives $k - i - 2 = t = c_{i,1,k} - 2p = k - p - 1$, a contradiction too, since $i \leq p - 2$.

Now, for $s = 0, \dots, i, i + 1, \dots, k - 3$, $(\nu_m + \rho, \alpha_{1,2+s}^\vee)$ takes all the values from $p - t$ to $p + k - 2 - t$, except $p + i + 1 - t$. Moreover since $p < t \leq k - 1$, the initial term is < 0 and the final term > 0 , hence the value 0 occurs unless $t = p + i + 1$. Now, if $t = k - i - 2 = p + i + 1$ then $2t = p + k - 1$ and hence

$$(\nu_m + \rho, \alpha_{1,k}^\vee) = p + k - 1 - 2t = 0.$$

Thus, in any case ν_m gives no contribution to Jantzen's sum. This proves (1.4.2).

It follows from (1.4.2) that $H^0(\lambda_{r-2}) = L_{r-2}$; then for λ_{r-3} the Jantzen sum equals $\text{ch } L_{r-2}$ hence $\text{ch } H^0(\lambda_{r-3}) = \text{ch } L_{r-3} + \text{ch } L_{r-2}$. By decreasing induction one obtains that $\text{ch } H^0(\lambda_i) = \text{ch } L_i + \text{ch } L_{i+1}$ for $i = r - 3, \dots, 0$, whence the exact sequences (1.4.3). Similarly, using (1.4.7) one obtains that $H_P^0(\lambda_{r-2}) = N_{r-2}$ and $\text{ch } H_P^0(\lambda_i) = \text{ch } N_i + \text{ch } N_{i+1}$ for $i = r - 3, \dots, 0$, whence the exact sequences (1.4.4). This completes the proof of Proposition 1.4.2. \square

Let us derive the following corollary (which is not used in the sequel).

Corollary 1.4.3. *For $i = 0, \dots, r - 2$, one has $H^0(N_i) = L_i$ and $H^j(N_i) = 0$ for $j > 0$.*

Proof. Applying the functor H^0 to each exact sequence (1.4.4) gives an exact sequence:

$$0 \longrightarrow H^0(N_i) \longrightarrow H^0(\lambda_i) \longrightarrow H^0(N_{i+1}) \longrightarrow H^1(N_i) \longrightarrow 0$$

and isomorphisms $H^j(N_{i+1}) \simeq H^{j+1}(N_i)$ for $j \geq 1$. Since $H^j(N_0) = 0$ for $j \geq 1$, one obtains $H^j(N_i) = 0$ for all $i \geq 0$ and $j \geq 1$, hence the previous exact sequence becomes:

$$(1.4.8) \quad 0 \longrightarrow H^0(N_i) \longrightarrow H^0(\lambda_i) \longrightarrow H^0(N_{i+1}) \longrightarrow 0.$$

Since $H^0(N_0) \simeq L_0$, the exact sequences (1.4.3) then imply, by induction on i , that $H^0(N_i) \simeq L_i$ for $i = 0, \dots, r - 2$. \square

On the other hand, in [Liu19b], Cor. 3 and 4, the first author proved, using a result of Suprunenko pointed out by one of the referees, that the dominant weights of $L(\lambda_0)$ (resp. of $L(\lambda'_0) = L((p-2)\omega_1 + \omega_2)$) are exactly the dominant weights $\leq \lambda_0$ (resp. λ'_0), each occurring with multiplicity one. Let us now switch to representations of GL_{d+1} and identify each λ_i (resp. $\lambda'_i = (p-2-i)\omega_1 + \omega_{2+i}$) with the partition $(p-1, p-1-i, 1^{i+1})$ (resp. $(p-1-i, 1^{i+1})$). Recall that (see for example [McD95], Chap. I) for a dominant weight λ of GL_{d+1} , identified with a partition with at most $d+1$ parts, the Weyl character $\text{ch } V(\lambda)$ (resp. the orbit sum $\sum_{\nu \in W\lambda} e^\nu$) corresponds to the Schur function S_λ (resp. the monomial symmetric function m_λ). Let us assume that $d \geq 2p - 2$, then the dominant weights smaller than λ_0 correspond to all partitions of $2p - 1$ which are smaller than $(p-1, p-1, 1)$ in the dominance order. Thus, one deduces from Propositions 1.4.2 and 1.3.1 (or Corollary 3 in [Liu19b]) the following:

Corollary 1.4.4. *For each prime number p , one has the equality:*

$$\sum_{\lambda \leq (p-1, p-1, 1)} m_\lambda = \sum_{i=0}^{p-2} (-1)^i S_{(p-1, p-1-i, 1^{i+1})}$$

and

$$\sum_{\lambda \leq (p-1, 1)} m_\lambda = \sum_{i=0}^{p-2} (-1)^i S_{(p-1-i, 1^{i+1})}$$

where the sum on the left hand-side of the first equality (resp. second equality) is taken over all partitions λ of $2p-1$ (resp. p) such that $\lambda \leq (p-1, p-1, 1)$ (resp. $\lambda \leq (p-1, 1)$) in the dominance order.

Remark 1.4.5. One may conjecture that the previous equality holds for any integer $n \geq 2$ (not only for prime numbers). Using a Sage program, we have checked this up to $n = 23$.³

Acknowledgements. We thank one of the referees for several very useful comments.

REFERENCES

- [Dot85] Stephen R. Doty, The submodule structure of certain Weyl modules for groups of type A_n , J. Algebra **95** (1985), no. 2, 373-383.
- [Gri20] Darij Grinberg, Petrie symmetric functions, arXiv:2004.11194.
- [Jan03] Jens Carsten Jantzen, Representations of algebraic groups (2nd ed.), Amer. Math. Soc, 2003.
- [Liu19a] Linyuan Liu, Cohomologie des fibrés en droites sur SL_3/B en caractéristique positive: deux filtrations et conséquences, submitted (see also arXiv:1903.08758).
- [Liu19b] Linyuan Liu, On the cohomology of line bundles over certain flag schemes, submitted [to this journal].
- [McD95] Ian G. Macdonald, Symmetric Functions and Hall Polynomials (2nd ed.), Oxford Univ. Press, 1995.
- [Sei87] Gary Seitz, The maximal subgroups of classical algebraic groups, Mem. Amer. Math. Soc. **67** (1987), no. 365.

INSTITUT DE MATHÉMATIQUES DE JUSSIEU-PARIS RIVE GAUCHE, SORBONNE UNIVERSITÉ – CAMPUS PIERRE ET MARIE CURIE, 4, PLACE JUSSIEU – BOÎTE COURRIER 247, F-75252 PARIS CEDEX 05, FRANCE

Email address: linyuan.liu@imj-prg.fr patrick.polo@imj-prg.fr

³On August 23, 2019 (the day after the first version of this article was posted on arXiv), a proof of the first equality of this conjecture was kindly sent to us by Darij Grinberg, see [Gri20].