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ON THE COHOMOLOGY OF LINE BUNDLES OVER CERTAIN FLAG SCHEMES II

LINYUAN LIU AND PATRICK POLO

To Jens Carsten Jantzen on the occasion of his 70th birthday

ABSTRACT. Over a field K of characteristic p , let Z be the incidence variety in $\mathbb{P}^d \times (\mathbb{P}^d)^*$ and let \mathcal{L} be the restriction to Z of the line bundle $\mathcal{O}(-n-d) \boxtimes \mathcal{O}(n)$, where $n = p + f$ with $0 \leq f \leq p - 2$. We prove that $H^d(Z, \mathcal{L})$ is the simple GL_{d+1} -module corresponding to the partition $\lambda_f = (p - 1 + f, p - 1, f + 1)$. When $f = 0$, using the first author's description of $H^d(Z, \mathcal{L})$ and Jantzen's sum formula, we obtain as a by-product that the sum of the monomial symmetric functions m_λ , for all partitions λ of $2p - 1$ less than $(p - 1, p - 1, 1)$ in the dominance order, is the alternating sum of the Schur functions $S_{p-1, p-1-i, 1^{i+1}}$ for $i = 0, \dots, p - 2$.

Introduction. This paper is an addition to the first author's paper [Liu19b]. We now consider the group scheme $G = \mathrm{SL}_{d+1}$ over an arbitrary field K of characteristic $p > 0$. For $m, n \in \mathbb{N}$, set $\mu_{m,n} = m\omega_1 - (n + d)\omega_d$ and write simply μ_n instead of $\mu_{n,n}$. We describe the $H^i(\mu_{m,n})$ when $p > n$ (thus recovering and extending [Liu19b], Cor. 2) and for $n = p + f$ with $0 \leq f \leq p - 2$ we prove that $H^d(\mu_n)$ is the simple G -module $L(\lambda_f) = L(f\omega_1 + (p - 2 - f)\omega_2 + (f + 1)\omega_3)$. Further, when $f = 0$ we express the character of $L(\lambda_f)$, using Jantzen sum formula, as an alternating sum of Weyl characters. Comparing this with the character of $H^d(\mu_p)$ given in [Liu19b], Cor. 3, we obtain as a by-product that the sum of the monomial symmetric functions m_λ , for all partitions λ of $2p - 1$ less than λ_f in the dominance order, is the alternating sum of the Schur functions $S_{p-1, p-1-i, 1^{i+1}}$ for $i = 0, \dots, p - 2$.

1.1. Notation. We keep the notation of [Liu19b], except that we now consider the group scheme $G = \mathrm{SL}_{d+1}$ over an arbitrary field K of characteristic $p > 0$ and denote simply by P the maximal parabolic subgroup P_1 . Let W_P be the Weyl group of (P, T) and define W_Q similarly. Let $\mathrm{ch} V$ denote the character of a T -module V . For each simple root α_i , let $s_i \in W$ be the corresponding simple reflection. Let w_0 (resp. w_P , resp. w_Q) be the longest element of W (resp. W_P , resp. W_Q) and set $N = \ell(w_Q)$. Then $\ell(w_P) = N + d - 1$.

Let ρ_Q (resp. ρ_P) denote the half-sum of the positive roots of Q (resp. P). Then one has:

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$$(1.1.1) \quad 2\rho_Q = (2-d)(\omega_1 + \omega_d) + 2 \sum_{i=2}^{d-1} \omega_i, \quad 2\rho_P = (1-d)\omega_1 + 2 \sum_{i=2}^d \omega_i.$$

Recall that, since $2\rho - 2\rho_Q = d(\omega_1 + \omega_d)$, the dualizing sheaf on $Z = G/Q$ is $\mathcal{L}(-d\omega_1 - d\omega_d)$. Hence, by Serre duality on G/Q , one has

$$(1.1.2) \quad H^d(\mu_{m,n}) \simeq H^{d-1}(-(m+d)\omega_1 + n\omega_d)^*.$$

Further, let τ be the (involutive) automorphism of (G, T) induced by the automorphism of the Dynkin diagram which swaps α_i and α_{d+1-i} for $i = 1, \dots, d$. Note that τ also acts on $X(T)$ and preserves $X(T)^+$. For any G -module V , let ${}^\tau V$ denote the corresponding module twisted by τ . For example, for a Weyl module $V(\lambda)$ (resp. a simple module $L(\lambda)$), one has ${}^\tau V(\lambda) \simeq V(\tau\lambda)$ (resp. ${}^\tau L(\lambda) \simeq L(\tau\lambda)$). Then, (1.1.2) can be rewritten as:

$$(1.1.3) \quad H^d(\mu_{m,n}) \simeq H^{d-1}(\tau\mu_{n,m})^*.$$

Recall (see [Jan03], II.4.6 and its proof) that if $P' \subset P''$ are parabolic subgroups containing B and if V is a P' -module, one has for all $i \geq 0$:

$$(1.1.4) \quad H^i(P''/B, V) \simeq H^i(P''/P', V).$$

This will be used several times, without always mentioning it.

1.2. A description of $H^i(\mu_{m,n})$. For each $i \in \mathbb{N}$, we denote by $H_Q^i(-)$ the functor $H^i(Q/B, -)$ and define $H_P^i(-)$ similarly. Since $-\mu_{m,n}$ is trivial and $\mu_{m,n} - 2\rho_Q$ is anti-dominant with respect to the Levi subgroup of Q one has:

$$(1.2.1) \quad H_Q^i(-\mu_{m,n}) \simeq \begin{cases} -\mu_{m,n} & \text{if } i = 0, \\ 0 & \text{if } i > 0 \end{cases}$$

and, using Serre duality on Q/B :

$$(1.2.2) \quad H_Q^i(\mu_{m,n} - 2\rho_Q) \simeq \begin{cases} 0 & \text{if } i < N, \\ H_Q^0(-\mu_{m,n})^* \simeq \mu_{m,n} & \text{if } i = N. \end{cases}$$

Consider induction from B to P . Thanks to (1.2.2), the spectral sequence of composite functors (see [Jan03], I.4.5) degenerates and gives (together with (1.1.4) applied to $P' = Q$ and $P'' = P$) isomorphisms for each $i \geq 0$:

$$(1.2.3) \quad H_P^i(\mu_{m,n}) \simeq H_P^{i+N}(\mu_{m,n} - 2\rho_Q)$$

and since $\mu_{m,n} - 2\rho_Q = (m+d-2)\omega_1 - 2 \sum_{j=2}^{d-1} \omega_j - (n+2)\omega_d$ is anti-dominant with respect to P , the latter group is zero unless $i+N = \dim(P/B)$, i.e. $i = d-1$. Moreover, by Serre duality on P/B , one has:

$$\begin{aligned} H_P^{N+d-1}(\mu_{m,n} - 2\rho_Q) &\simeq H_P^0(-\mu_{m,n} + 2\rho_Q - 2\rho_P)^* \\ &= H_P^0((1-m)\omega_1 + n\omega_d)^*. \end{aligned}$$

Set $\nu_{m,n} = (1-m)\omega_1 + n\omega_d$ and $\pi_{m,n} = -w_P\nu_{m,n}$. Since $w_P = s_2 \cdots s_d w_Q$, one has

$$(1.2.4) \quad \pi_{m,n} = (m-n-1)\omega_1 + n\omega_2$$

and since $H_P^0(\nu_{m,n})^*$ is isomorphic with $V_P(\pi_{m,n})$, the Weyl module for P with highest weight $\pi_{m,n}$, one obtains:

Lemma 1.2.1. *For each $m, n \in \mathbb{N}$ and $i \in \mathbb{N}$, one has*

$$H_P^i(\mu_{m,n}) \simeq \begin{cases} V_P(\pi_{m,n}) & \text{if } i = d - 1, \\ 0 & \text{if } i \neq d - 1. \end{cases}$$

Now, consider induction from B to G . Thanks to the lemma, the spectral sequence of composite functors degenerates and gives (together with (1.1.4) applied to $P' = P$ and $P'' = G$) isomorphisms $H^i(\mu_{m,n}) \simeq H^{i-d+1}(V_P(\pi_{m,n}))$ for each $i \geq 0$. Since the former is zero for $i \notin \{d-1, d\}$ (see [Liu19b], Section 2), this gives:

Proposition 1.2.2. *For each $m, n \in \mathbb{N}$ and $i \in \mathbb{N}$, one has*

$$H^i(\mu_{m,n}) \simeq \begin{cases} H^0(V_P(\pi_{m,n})) & \text{if } i = d - 1, \\ H^1(V_P(\pi_{m,n})) & \text{if } i = d, \\ 0 & \text{if } i \neq d - 1, d. \end{cases}$$

This allows us to recover and extend the result of [Liu19b], Corollary 2:

Corollary 1.2.3. *Suppose that $p > n$.*

- (i) *If $m \geq n$, then $H^d(\mu_{m,n}) = 0$ and $H^{d-1}(\mu_{m,n}) \simeq H^0(\pi_{m,n})$. In particular, $H^i(\mu_n) = 0$ for all $i \geq 0$.*
- (ii) *If $n \geq m$, then $H^{d-1}(\mu_{m,n}) = 0$ and*

$$H^d(\mu_{m,n}) \simeq H^1(\pi_{m,n}) \simeq H^0(\pi_{n,m}).$$

- (iii) *Furthermore, if $p > m \geq n$ then $H^{d-1}(\mu_{m,n}) \simeq L(\pi_{m,n}) \simeq H^d(\mu_{n,m})$.*

Proof. Suppose $p > n$. Then $V_P(\pi_{m,n})$ is irreducible as a P -module, being isomorphic as a module over the Levi subgroup of P to the GL_d -module $S^n K^d$ which is irreducible since $n < p$. Therefore, $V_P(\pi_{m,n}) \simeq H_P^0(\pi_{m,n})$. Thus, the proposition gives for all $i \geq 0$ that

$$H^i(\mu_{m,n}) \simeq H^{i-d+1}(\pi_{m,n}).$$

If $m \geq n$ then $\pi_{m,n}$ belongs to $\mathcal{C} := X(T)^+ - \rho$ and hence $H^j(\pi_{m,n}) = 0$ for $j > 0$ (and also for $j = 0$ if $m = n$). This proves (i).

If $m < n$, then $H^0(\pi_{m,n}) = 0$. Further, $\pi_{n,m} = (n-m-1)\omega_1 + m\omega_2$ belongs to \mathcal{C} and one has $\pi_{m,n} = s_{\alpha_1} \cdot \pi_{n,m}$. Since $0 \leq (\pi_{n,m} + \rho, \alpha_1^\vee) = n - m < p$, one has $H^1(\pi_{m,n}) \simeq H^0(\pi_{n,m})$ (see [Jan03], II.5.4), which proves (ii).

Suppose now that $p > m \geq n$. By (i) and Serre duality on G/Q , one has

$$H^0(\pi_{m,n}) \simeq H^{d-1}(\mu_{m,n}) \simeq H^d(-(m+d)\omega_1 + n\omega_d)^* = H^d(\tau\mu_{n,m})^*.$$

Now, using the automorphism τ of (G, T) , one deduces from (ii) that

$$H^d(\tau\mu_{n,m})^* \simeq H^0(\tau\pi_{m,n})^*.$$

Since the latter is the Weyl module $V(\pi_{m,n})$, one obtains that $H^0(\pi_{m,n}) \simeq L(\pi_{m,n})$. This proves (iii). \square

Our goal in the next subsection is to determine $H^d(\mu_p)$. Since for $d = 2$ the SL_3 -modules $H^2(m, -n-2)$ have been described in [Liu19a], where it is proved in particular (see [Liu19a], Th. 3) that $H^2(p, -p-2)$ is the Weyl module $V(0, p-2)$, which is simple, we will henceforth assume that $d \geq 3$.

1.3. Computation of $H^d(\mu_n)$ for $p \leq n \leq 2p - 1$. Suppose now that $d \geq 3$ and $n = p + f$ with $0 \leq f \leq p - 1$. For each weight ν which is dominant (resp. dominant with respect to P), denote by $L(\nu)$ (resp. $L_P(\nu)$) the irreducible G -module (resp. P -module) with highest weight ν . Let

$$(1.3.1) \quad \lambda_f = \begin{cases} f\omega_1 + (p-2-f)\omega_2 + (f+1)\omega_3 & \text{if } f \leq p-2, \\ (p-1)\omega_1 + (p-2)\omega_3 + \omega_4 & \text{if } f = p-1 \end{cases}$$

(with the convention $\omega_4 = 0$ if $d = 3$), and set $L_f = L(\lambda_f)$ and $N_f = L_P(\lambda_f)$.

Proposition 1.3.1. *One has $H^{d-1}(\mu_n) \simeq L_f \simeq H^d(\mu_n)$.*

Remark 1.3.2. This generalizes Corollary 3 in [Liu19b], which was the case $f = 0$. Note that if $f > 0$ and $d > p - f + 1$ then μ_{p+f} does not belong to the closure of the facet containing μ_p . Indeed, denoting by $\alpha_1, \dots, \alpha_d$ the simple roots and setting $\beta = \alpha_1 + \dots + \alpha_{p-f+1}$, one has $\langle \mu_p + \rho, \beta^\vee \rangle = 2p - f + 1$ whereas $\langle \mu_{p+f} + \rho, \beta^\vee \rangle = 2p + 1$. Hence the result for μ_{p+f} cannot be deduced from the one for μ_p by applying a functor of translation.

Proof. Set $\pi_n = -\omega_1 + n\omega_2$ and $M = V_P(\pi_n)$. According to Doty [Dot85], §§2.3–2.4, one has exact sequences of P -modules:

$$(1.3.2) \quad 0 \longrightarrow N_f \longrightarrow M \longrightarrow C \longrightarrow 0$$

and

$$(1.3.3) \quad 0 \longrightarrow C \longrightarrow H_P^0(\pi_n) \longrightarrow N_f \longrightarrow 0$$

where $C = L_P(\pi_n)$. Applying the functor H^0 to (1.3.3) and using that:

$$(1.3.4) \quad H^i(H_P^0(\pi_n)) = H^i(\pi_n) = 0$$

for all $i \geq 0$, one obtains $H^0(C) = 0$ and isomorphisms $H^i(C) \simeq H^{i-1}(N_f)$ for all $i \geq 1$. Taking this and Proposition 1.2.2 into account and applying the functor H^0 to (1.3.2), one obtains isomorphisms:

$$(1.3.5) \quad H^{d-1}(\mu_n) \simeq H^0(M) \simeq H^0(N_f),$$

an exact sequence:

$$(1.3.6) \quad 0 \longrightarrow H^1(N_f) \longrightarrow H^d(\mu_n) \longrightarrow H^0(N_f) \longrightarrow H^2(N_f) \longrightarrow 0$$

and isomorphisms

$$(1.3.7) \quad H^i(N_f) \simeq H^{i+1}(C) \simeq H^{i+2}(N_f)$$

for $i \geq 1$. Since $H^i(N_f) = 0$ for $i > |R^+|$, one obtains $H^i(N_f) = 0$ for all $i \geq 1$. Together with (1.3.6) and (1.3.5), this gives isomorphisms:

$$(1.3.8) \quad H^d(\mu_n) \simeq H^0(N_f) \simeq H^{d-1}(\mu_n).$$

On the other hand, by [Liu19b], Cor. 4, λ_f has multiplicity 1 in $H^d(\mu_n)$, which is therefore non-zero. Thus $H^d(\mu_n) \simeq H^{d-1}(\mu_n) \simeq H^0(N_f)$ is a non-zero submodule of $H^0(H_P^0(\lambda_f)) = H^0(\lambda_f)$.

Now, using the automorphism τ of (G, T) , one obtains that $H^d(\mu_n) \simeq H^{d-1}(\tau\mu_n)^*$ is a quotient of $H^0(\tau\lambda_f)^* \simeq V(\lambda_f)$. Since any non-zero morphism $V(\lambda_f) \rightarrow H^0(\lambda_f)$ has image L_f (see [Jan03] II.6.16 *Remark*), then (1.3.8) gives that $H^d(\mu_n) \simeq L_f \simeq H^{d-1}(\mu_n)$. \square

Remark 1.3.3. ¹ Seitz has shown ([Sei87], Prop. 6.1) that if a simple SL_{d+1} -module $L(\mu)$ with μ p -restricted has one-dimensional weight spaces then either μ is a fundamental weight ω_i or a multiple of ω_1 or ω_d , or $\mu = a\omega_i + (p-1-a)\omega_{i+1}$ for some $i \in \{1, \dots, d-1\}$ and $a \in \{0, \dots, p-1\}$. Since λ_f is not in that list when $f > 0$, it follows that $H^d(\mu_{p+f}) \simeq L(\lambda_f)$ never has all its weight spaces of dimension 1 when $f > 0$. This improves on Remark 3 (2) of [Liu19b].

1.4. Jantzen sum formula and consequences. Now, consider the case $f = 0$, i.e. $n = p$. Then $\lambda_0 = (p-2)\omega_2 + \omega_3$. We shall use Jantzen sum formula ([Jan03], II.8.19) to express $\mathrm{ch} L_0$ in terms of Weyl characters. Set $r = \min(d, p)$. In addition to λ_0 , define for $i = 1, \dots, r-2$ the dominant weights: ²

$$(1.4.1) \quad \lambda_i = i\omega_1 + (p-2-i)\omega_2 + \omega_{3+i}$$

(with the convention $\omega_{d+1} = 0$) and set $L_i = L(\lambda_i)$ and $N_i = L_P(\lambda_i)$. In other words, if $d > p$ the sequence ends with $\lambda_{p-2} = (p-2)\omega_1 + \omega_{p+1}$ whilst if $d \leq p$ it ends with

$$\lambda_{d-2} = (d-2)\omega_1 + (p-d)\omega_2 + \omega_{d+1} = (d-2)\omega_1 + (p-d)\omega_2.$$

Lemma 1.4.1. *Consider SL_{n+1} for some $n \geq 2$ and for $k = 1, \dots, n$ consider the weight $\theta_k = \omega_1 - k\omega_k + (k-1)\omega_{k+1}$. Note that $\theta_1 = 0$.*

(i) *For $k \geq 2$, one has $s_k \cdot \theta_k = \theta_{k-1}$.*

(ii) *For $k \geq 2$, one has $\omega_1 - k\omega_k + k\omega_{k+1} = s_k \cdots s_2 \cdot \omega_{k+1}$.*

Proof. One has $s_k \cdot \theta_k = \theta_k + (k-1)\alpha_k = \theta_{k-1}$. Thus $s_2 \cdots s_k \cdot \theta_k = \theta_1 = 0$, whence $\theta_k = s_k \cdots s_2 \cdot 0$. Next, for any weight λ and $w \in W$, one has $w \cdot \lambda = w\lambda + w \cdot 0$, which equals $\lambda + w \cdot 0$ if $w\lambda = \lambda$. Applying this to $w = s_k \cdots s_2$ and $\lambda = \omega_{k+1}$ gives assertion (ii). \square

For $\alpha \in R^+$ let α^\vee be the corresponding coroot. Then, for $m \in \mathbb{Z}$, let $s_{\alpha, mp}$ be the affine reflection defined for all $\lambda \in X(T)$ by $s_{\alpha, mp}(\lambda) = \lambda - ((\lambda, \alpha^\vee) - m)\alpha$. Further, one sets $s_{\alpha, m} \cdot \lambda = s_{\alpha, mp}(\lambda + \rho) - \rho$.

For each Weyl module $V(\lambda)$, Jantzen has defined a decreasing filtration $V(\lambda) \supset V(\lambda)_1 \supset V(\lambda)_2 \supset \cdots$ and one has the following character formula ([Jan03], II.8.19):

$$\sum_{i \geq 1} \mathrm{ch} V(\lambda)_i = \sum_{\alpha \in R^+} \sum_{0 < mp < (\lambda + \rho, \alpha^\vee)}^m v_p(m) \chi(s_{\alpha, mp} \cdot \lambda)$$

where v_p denotes the p -adic valuation and χ is the Weyl character. Recall that $\chi(\mu) = 0$ if μ is singular for the dot action of W , i.e. if there exists $\alpha \in R^+$ such that $(\mu + \rho, \alpha^\vee) = 0$, and otherwise there exists a unique couple $(w, \mu^+) \in W \times X(T)^+$ such that $\mu = w \cdot \mu^+$ and then $\chi(\mu) = (-1)^{\ell(w)} \mathrm{ch} V(\mu^+)$. In particular, if the right-hand side, to which we shall refer as ‘‘Jantzen’s sum’’ (relative to λ), equals $\mathrm{ch} L$ for some simple module L (resp. equals 0), then one has $\mathrm{ch} V(\lambda) = \mathrm{ch} L(\lambda) + \mathrm{ch} L$ (resp. $V(\lambda) = L(\lambda)$).

¹We are grateful to one of the referees for this remark.

²These are different from the weights λ_f considered in section 1.3.

Proposition 1.4.2. *For $i = 0, \dots, r-2$ one has the equality:*

$$(1.4.2) \quad \sum_{\ell \geq 1} \text{ch } V(\lambda_i)_\ell = \sum_{j=i+1}^{r-2} (-1)^{j-i-1} \text{ch } V(\lambda_j)$$

and the exact sequences:

$$(1.4.3) \quad 0 \longrightarrow L_i \longrightarrow H^0(\lambda_i) \longrightarrow L_{i+1} \longrightarrow 0$$

and

$$(1.4.4) \quad 0 \longrightarrow N_i \longrightarrow H_P^0(\lambda_i) \longrightarrow N_{i+1} \longrightarrow 0$$

with the convention $L_{r-1} = 0 = N_{r-1}$. Therefore, one has exact sequences:

$$(1.4.5) \quad 0 \rightarrow L(\lambda_0) \rightarrow H^0(\lambda_0) \rightarrow H^0(\lambda_1) \rightarrow \dots \rightarrow H^0(\lambda_{r-2}) \rightarrow 0$$

and

$$(1.4.6) \quad 0 \rightarrow L_P(\lambda_0) \rightarrow H_P^0(\lambda_0) \rightarrow H_P^0(\lambda_1) \rightarrow \dots \rightarrow H_P^0(\lambda_{r-2}) \rightarrow 0.$$

Proof. Fix $i \in \{0, \dots, r-2\}$ and for $1 \leq j \leq k \leq d$, set $\alpha_{j,k} = \alpha_j + \dots + \alpha_k$ and $c_{i,j,k} = (\lambda_i + \rho, \alpha_{j,k}^\vee)$. We are going to prove that all terms in Jantzen's sum (for both G and P) are zero, except the ones given in (1.4.2). Fix $j \leq k$ such that $c_{i,j,k} > p$, let $m \in \mathbb{N}^*$ such that $mp < c_{i,j,k}$ and set $t = c_{i,j,k} - mp$ and $\nu_m = s_{\alpha_{j,k}, mp} \cdot \lambda_i = \lambda_i - t\alpha_{j,k}$. There are four cases to consider.

Case 1: $c_{i,j,k} = k - j + 1$ (this occurs only $j > i + 3$ and $k \geq j + p$). Then the expression of ν_m in terms of the fundamental weights contains the “sequence” $-t\omega_j - t\omega_k$, the coefficients of the ω_ℓ for $j < \ell < k$ being 0, and hence $\nu_m + \rho$ is orthogonal to both $\alpha_{j,j+t-1}^\vee$ and $\alpha_{k-t+1,k}^\vee$. Therefore ν_m gives no contribution to Jantzen's sum, neither for G nor for P .

Case 2: $c_{i,j,k} = k - j + 2$ (this occurs only if $j \leq i + 3 \leq k$, including the case $k = i + 3$, $j = 2$ and $i = p - 2$). Assume first that $j < i + 3 < k$. Then the expression of ν_m in terms of the fundamental weights contains the “sequence” $-t\omega_j + \omega_{i+3} - t\omega_k$, the coefficients of the ω_ℓ for $j < \ell < k$ and $\ell \neq i + 3$ being 0.

For $s = 0, \dots, i + 2 - j, i + 3 - j, \dots, k - j - 1$, $(\nu_m + \rho, \alpha_{j,j+s}^\vee)$ takes all the values from $1 - t$ to $k - j + 1 - t$, except $i - j + 4 - t$. Since $1 - t \leq 0$ and $k - j + 1 - t > 0$ (since $t \leq k - j + 2 - p \leq k - j$), the value 0 is obtained unless $t = i - j + 4$.

Similarly, for $s = 0, \dots, k - i - 4, k - i - 3, \dots, k - j - 1$, $(\nu_m + \rho, \alpha_{k-s,k}^\vee)$ takes all the values from $1 - t$ to $k - j + 1 - t$, except $k - i - 2 - t$. Hence the value 0 is obtained, unless $t = k - i - 2$. This shows that $\nu_m + \rho$ is singular, except possibly if $t = i - j + 4 = k - i - 2$. But in this case one has $2t = k - j + 2$ and hence:

$$(\nu_m + \rho, \alpha_{j,k}^\vee) = k - j + 2(1 - t) = 0.$$

Consider now the “boundary” cases $j = i + 3$ or $k = i + 3$. If $j = i + 3$ then, since $k - t + 1 > j$, one has $k - t + 1 > i + 3$ and hence $(\nu_m + \rho, \alpha_{k-t+1,k}^\vee) = 0$. If $k = i + 3$, then $j + t - 1 < k = i + 3$ and hence $(\nu_m + \rho, \alpha_{j,j+t-1}^\vee) = 0$. Thus, in any case ν_m gives no contribution to Jantzen's sum, neither for G nor for P .

To close this case, note that since $i + 3 \leq p + 1 \leq k - j + 2$, the case $k = i + 3$ can occur only if $j \leq 2$, in which case $c_{i,j,k} = k - j + 2$ implies $i = p - 2$.

Case 3: $j = 2$ and $c_{i,j,k} = k + p - 2 - i$ and $i < p - 2$. (Note that $c_{i,2,i+2} = p - 1$ hence the hypothesis $c_{i,2,k} > p$ implies $k \geq i + 3$.) Then one has $\nu_m = (i + t)\omega_1 + (p - 2 - i - t)\omega_2 + \omega_{i+3} - t\omega_k + t\omega_{k+1}$.

For $s = 0, \dots, i, i + 1, \dots, k - 3$, $(\nu_m + \rho, \alpha_{2,2+s}^\vee)$ takes all the values from $p - i - 1 - t$ to $p + k - i - 3 - t$, except $p - t$. Since the last value taken is $\geq p - 1 > 0$, the value 0 is obtained except if $t = p$ or if the initial value $p - i - 1 - t$ is > 0 , i.e. $t \leq p - i - 2$.

Similarly, for $s = 0, \dots, k - i - 4, k - i - 3, \dots, k - 3$, $(\nu_m + \rho, \alpha_{k-s,k}^\vee)$ takes all the values from $1 - t$ to $k - 1 - t$, except $k - i - 2 - t$. Moreover one has $1 - t \leq 0 < k - 1 - t$, hence the value 0 occurs unless $t = k - i - 2$.

Thus, $\nu_m + \rho$ is singular except possibly if $t = k - i - 2$ belongs to $\{1, \dots, p - i - 2\}$ or if $t = k - i - 2 = p$. In the latter case, one has $2t = p + k - i - 2$ and hence

$$(\nu_m + \rho, \alpha_{2,k}^\vee) = k - 2 + p - i - 2t = 0.$$

In the former case, one has $k = i + 2 + t$ with $t = 1, \dots, p - i - 2$, whence $m = 1$. In fact, since $k = i + 2 + t$ is $\leq d$, we have $t \in \{1, \dots, r - i - 2\}$, recalling that $r = \min(p, d)$. For $t = 1, \dots, r - i - 2$, set

$$\theta'_t = s_{\alpha_{2,i+2+t,p}} \cdot \lambda_i = (i + t)\omega_1 + (p - 2 - i - t)\omega_2 + \omega_{i+3} - t\omega_{i+2+t} + t\omega_{i+3+t}.$$

Using Lemma 1.4.1 with a shift of $i + 2$ in the indices, one obtains that $\theta'_1 = \lambda_{i+1}$ and that $\theta'_t = s_{i+2+t} \cdots s_{i+4} \cdot \lambda_{i+t}$ for $t \geq 2$.

Denote by G_P the Levi subgroup of P containing T and recall ([Jan03], II.5.21) that $V_P(\lambda_i)$ is just the corresponding Weyl module for G_P on which the unipotent radical of P acts trivially. Therefore, applying Jantzen's sum formula for G_P , one already obtains the equality:

$$(1.4.7) \quad \sum_{\ell \geq 1} \text{ch } V_P(\lambda_i)_\ell = \sum_{t=1}^{r-i-2} (-1)^{t-1} \text{ch } V_P(\lambda_{i+t}).$$

To prove the analogous equality for G we must consider the last case, where $j = 1$.

Case 4: $j = 1$. Note that the assumption $c_{i,1,k} > p$ implies $k \geq 3$. If $k \leq i + 3$ then $c_{i,1,k} = p + k - 2 + \delta_{k,i+3}$ is $\leq 2p$ (since $i \leq p - 2$), hence $m = 1$ and $t = k - 2 + \delta_{k,i+3}$ and the expression of ν_m in terms of the fundamental weights contains the "sequence" $(p - i - 2)\omega_2 - (k - 2)\omega_k$, the coefficients of the ω_ℓ for $2 < \ell < k$ being 0. Then $(\nu_m + \rho, \alpha_{3,k}^\vee) = 0$ hence ν_m gives no contribution to Jantzen's sum.

Suppose now that $k > i + 3$. Then $c_{i,1,k} = k - 1 + p$ and

$$\nu_m = (i - t)\omega_1 + (p - 2 - i)\omega_2 + \omega_{i+3} - t\omega_k + t\omega_{k+1}.$$

For $s = 0, \dots, k - i - 4, k - i - 3, \dots, k - 3$, $(\nu_m + \rho, \alpha_{k-s,k}^\vee)$ takes all the values from $1 - t$ to $k - 1 - t$, except $k - i - 2 - t$. Moreover one has $1 - t \leq 0 \leq k - 1 - t$, hence the value 0 occurs unless $t = k - i - 2$.

Let us assume henceforth that $t = k - i - 2$. Then $t > p$, for otherwise one would have $k - i - 2 \leq p$ hence $k \leq 2p$ (since $i \leq p - 2$) whence $m \leq 2$;

but $m = 1$ gives $k - i - 2 = t = c_{i,1,k} - p = k - 1$, a contradiction, whereas $m = 2$ gives $k - i - 2 = t = c_{i,1,k} - 2p = k - p - 1$, a contradiction too, since $i \leq p - 2$.

Now, for $s = 0, \dots, i, i + 1, \dots, k - 3$, $(\nu_m + \rho, \alpha_{1,2+s}^\vee)$ takes all the values from $p - t$ to $p + k - 2 - t$, except $p + i + 1 - t$. Moreover since $p < t \leq k - 1$, the initial term is < 0 and the final term > 0 , hence the value 0 occurs unless $t = p + i + 1$. Now, if $t = k - i - 2 = p + i + 1$ then $2t = p + k - 1$ and hence

$$(\nu_m + \rho, \alpha_{1,k}^\vee) = p + k - 1 - 2t = 0.$$

Thus, in any case ν_m gives no contribution to Jantzen's sum. This proves (1.4.2).

It follows from (1.4.2) that $H^0(\lambda_{r-2}) = L_{r-2}$; then for λ_{r-3} the Jantzen sum equals $\text{ch } L_{r-2}$ hence $\text{ch } H^0(\lambda_{r-3}) = \text{ch } L_{r-3} + \text{ch } L_{r-2}$. By decreasing induction one obtains that $\text{ch } H^0(\lambda_i) = \text{ch } L_i + \text{ch } L_{i+1}$ for $i = r - 3, \dots, 0$, whence the exact sequences (1.4.3). Similarly, using (1.4.7) one obtains that $H_P^0(\lambda_{r-2}) = N_{r-2}$ and $\text{ch } H_P^0(\lambda_i) = \text{ch } N_i + \text{ch } N_{i+1}$ for $i = r - 3, \dots, 0$, whence the exact sequences (1.4.4). This completes the proof of Proposition 1.4.2. \square

Let us derive the following corollary (which is not used in the sequel).

Corollary 1.4.3. *For $i = 0, \dots, r - 2$, one has $H^0(N_i) = L_i$ and $H^j(N_i) = 0$ for $j > 0$.*

Proof. Applying the functor H^0 to each exact sequence (1.4.4) gives an exact sequence:

$$0 \longrightarrow H^0(N_i) \longrightarrow H^0(\lambda_i) \longrightarrow H^0(N_{i+1}) \longrightarrow H^1(N_i) \longrightarrow 0$$

and isomorphisms $H^j(N_{i+1}) \simeq H^{j+1}(N_i)$ for $j \geq 1$. Since $H^j(N_0) = 0$ for $j \geq 1$, one obtains $H^j(N_i) = 0$ for all $i \geq 0$ and $j \geq 1$, hence the previous exact sequence becomes:

$$(1.4.8) \quad 0 \longrightarrow H^0(N_i) \longrightarrow H^0(\lambda_i) \longrightarrow H^0(N_{i+1}) \longrightarrow 0.$$

Since $H^0(N_0) \simeq L_0$, the exact sequences (1.4.3) then imply, by induction on i , that $H^0(N_i) \simeq L_i$ for $i = 0, \dots, r - 2$. \square

On the other hand, in [Liu19b], Cor. 3 and 4, the first author proved, using a result of Suprunenko pointed out by one of the referees, that the dominant weights of $L(\lambda_0)$ (resp. of $L(\lambda'_0) = L((p - 2)\omega_1 + \omega_2)$) are exactly the dominant weights $\leq \lambda_0$ (resp. λ'_0), each occurring with multiplicity one. Let us now switch to representations of GL_{d+1} and identify each λ_i (resp. $\lambda'_i = (p - 2 - i)\omega_1 + \omega_{2+i}$) with the partition $(p - 1, p - 1 - i, 1^{i+1})$ (resp. $(p - 1 - i, 1^{i+1})$). Recall that (see for example [McD95], Chap. I) for a dominant weight λ of GL_{d+1} , identified with a partition with at most $d + 1$ parts, the Weyl character $\text{ch } V(\lambda)$ (resp. the orbit sum $\sum_{\nu \in W\lambda} e^\nu$) corresponds to the Schur function S_λ (resp. the monomial symmetric function m_λ). Let us assume that $d \geq 2p - 2$, then the dominant weights smaller than λ_0 correspond to all partitions of $2p - 1$ which are smaller than $(p - 1, p - 1, 1)$ in the dominance order. Thus, one deduces from Propositions 1.4.2 and 1.3.1 (or Corollary 3 in [Liu19b]) the following:

Corollary 1.4.4. *For each prime number p , one has the equality:*

$$\sum_{\lambda \leq (p-1, p-1, 1)} m_\lambda = \sum_{i=0}^{p-2} (-1)^i S_{(p-1, p-1-i, 1^{i+1})}$$

and

$$\sum_{\lambda \leq (p-1, 1)} m_\lambda = \sum_{i=0}^{p-2} (-1)^i S_{(p-1-i, 1^{i+1})}$$

where the sum on the left hand-side of the first equality (resp. second equality) is taken over all partitions λ of $2p-1$ (resp. p) such that $\lambda \leq (p-1, p-1, 1)$ (resp. $\lambda \leq (p-1, 1)$) in the dominance order.

Remark 1.4.5. One may conjecture that the previous equality holds for any integer $n \geq 2$ (not only for prime numbers). Using a Sage program, we have checked this up to $n = 23$.³

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³On August 23, 2019 (the day after the first version of this article was posted on arXiv), a proof of the first equality of this conjecture was kindly sent to us by Darij Grinberg, see [Gri20].