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Sampled-Data Observers for Delay Systems and Hyperbolic PDE-ODE Loops

Tarek Ahmed-Ali*[a], Iasson Karafyllis[b] and Fouad Giri[c]

Abstract

This paper studies the problem of designing sampled-data observers and observer-based, sampled-data, output feedback stabilizers for systems with both discrete and distributed, state and output time-delays. The obtained results can be applied to time delay systems of strict-feedback structure, transport Partial Differential Equations (PDEs) with nonlocal terms, and feedback interconnections of Ordinary Differential Equations with a transport PDE. The proposed design approach consists in exploiting any existing observer that features robust exponential convergence of the error when continuous-time output measurements are available. The (continuous-time) observer is then modified, mainly by adding an inter-sample output predictor, to compensate for the effect of data-sampling. Using small-gain analysis, we show that robust exponential stability of the error is preserved, provided the sampling period is not too large. The results are applied to a chemical reactor and to the class of triangular globally Lipschitz delay systems.

Key words: Nonlinear observers, sampled-data observers, delay systems, inter-sample predictor.

1. Introduction

With the growing penetration of network technology in control systems, the compensation of the effects of time-delay has become a major issue in control theory [19,25,26,27,33]. A great deal of interest has recently been paid to the problem of designing state observers for linear and nonlinear systems with measurement delays. The dominant design approach consists in starting with the design of an exponentially convergent observer for the delay-free system, which is described by Ordinary Differential Equations (ODEs), and modifying it mainly by adding predictors: static predictors (see [27]) or dynamic (chain) predictors (see [1,4,11,13,15,20,30]). In parallel to this research activity, which takes into account the time-delay explicitly in the model, a separate activity, based on Partial Differential Equations (PDEs) has been initiated in [31]. This consists in modeling time-delays by means of first-order hyperbolic PDEs, leading to a representation of the delayed system in the form of an ODE-PDE cascade (see also the recent work [3], where a PDE-based chain-observer is constructed).

Most existing results on observer design for delayed systems have been established assuming the measurement delay to be of discrete nature. So far, only a few studies have investigated the case of distributed measurement delays. The PDE-based observer developed in [9] and the recent observer developed in [6], are notable exceptions.

The nowadays-digital implementation of observers entails sampling in time of all system signals needed by the observer. Consequently, the design of sampled-data observers is a major issue. Sampled-data observers for ODE systems can be classified in four main categories:
1) observers where data-sampling is accounted for through a standard Zero-Order-Hold (ZOH) sampling of the output estimation error; see for example [2,37],
2) observers designed on approximate discrete-time models (see [7,12]),
3) continuous-discrete time observers where correction terms are employed at the sampling times; see for instance [34], and
4) sampled-data observers, where the time-varying delay effect (caused by output sampling) is compensated by using inter-sample output predictors; see [23].
The use of inter-sample output predictors was extended to systems with asynchronous measurements (see [32]) and systems described by parabolic PDEs (see [29]).

The combination of time-delay and data-sampling effects necessarily makes the problem of observer design more complex. Indeed, not only data-sampling introduces a time-varying delay but it also entails information lost. The case of discrete measurement delays, in conjunction with data

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sampling, has been investigated in [4,5,25,27,37]. Results on observer-based output feedback stabilization of delay systems with sampled measurements have been recently given in [16,36] (see also [35]).

In the present work, we extend for the first time the use of inter-sample predictors to the case of time-delay systems with state and output (discrete and/or distributed) delays. Moreover, we provide observer-based output feedback stabilization results for delay systems with sampled measurements under appropriate assumptions. More specifically, we consider nonlinear time-delay systems of the form:

$$\dot{x} = f(x_t, u, d) \quad , \quad y = h(x_t) + \xi$$
$$(x, u, d) \in \Re^n \times U \times D, y, \xi \in \Re^k \qquad (1)$$

where $U \subseteq \Re^m$, $D \subseteq \Re^q$ are convex sets with $0 \in U$, $0 \in D$, $f : C^0([-r,0];\Re^n) \times U \times D \to \Re^n$, $h : C^0([-r,0];\Re^n) \to \Re^k$ are continuous mappings with $f(0,0,0) = 0$, $h(0) = 0$. The input u is assumed to be available, but the inputs d, ξ are unknown and represent possible modeling errors and measurement noise, respectively. The signal x_t denotes the "r-history" of x, see definition in the notation subsection at the end of this section. The proposed sampled-data observer design approach consists in starting with any existing observer, that features *robust exponential convergence* when continuous-time output measurements are available (see Definition 2.1 for the precise meaning of the notion of "robust exponential convergence"). The available observer, based on continuous-time measurements, is then modified, mainly by adding an inter-sample output predictor, to compensate for the effect of data-sampling. Using small gain analysis, we show that the robust exponential stability feature is preserved, provided that the sampling period is sufficiently small (Theorem 2.2). The sampled-data observer can be used in a straightforward way for the design of observer-based output feedback stabilizers (Corollary 2.4) under certain assumptions.

The second contribution of the paper is that it provides a framework where sampled-data observer and feedback design for ODE-PDE loops can be converted to a similar problem for a time-delay system. More specifically, we consider feedback interconnections of ODEs with a first-order, hyperbolic PDE with non-local terms of the form

$$\frac{d\bar{x}}{dt}(t) = \tilde{F}(\bar{x}(t), v[t], u(t)) \text{, for } t \geq 0 \qquad (2)$$

$$\frac{\partial v}{\partial t}(t,z) + c\frac{\partial v}{\partial z}(t,z) = a(z)v(t,z) + g(z,\bar{x}(t))$$
$$+\sum_{i=1}^{N_1} b_i(z)v(t,z_i) + \sum_{i=1}^{N_2} \beta_i(z)\int_0^1 \gamma_i(s)v(t,s)ds$$
$$\text{for all } (t,z) \in \Re_+ \times [0,1] \qquad (3)$$
$$v(t,0) = 0 \text{, for } t \geq 0 \qquad (4)$$

where $c > 0$ is a constant (the transport speed), $z_i \in (0,1]$ $(i = 1,...,N_1)$, $\bar{x}(t) \in \Re^{\bar{n}}$, $v(t,z) \in \Re$ are the states, $u \in C^0\left(\Re_+;\Re^m\right)$ is an external input and the mappings $a : [0,1] \to \Re$, $b_i : [0,1] \to \Re$ $(i = 1,...,N_1)$, $\beta_i, \gamma_i : [0,1] \to \Re$ $(i = 1,...,N_2)$, $g : [0,1] \times \Re^{\bar{n}} \to \Re$, $\tilde{F} : \Re^{\bar{n}} \times C^0\left([0,1]\right) \times \Re^m \to \Re^{\bar{n}}$ are given continuous mappings. For such systems the output $y(t) = (y_1(t),...,y_k(t))^T \in \Re^k$ is given by equations of the following form for $j = 1,...,k$:

$$y_j(t) = \sum_{i=1}^{N_1} \bar{b}_{j,i} v(t,z_i)$$
$$+\sum_{i=1}^{N_2} \bar{\beta}_{j,i}\int_0^1 \gamma_i(s)v(t,s)ds + q_j^T \bar{x}(t) \qquad (5)$$

where $q_j \in \Re^{\bar{n}}$ $(j = 1,...,k)$ are constant vectors, $\bar{b}_{j,i} \in \Re$ $(j = 1,...,k, i = 1,...,N_1)$ and $\bar{\beta}_{j,i} \in \Re$ $(j = 1,...,k, i = 1,...,N_2)$ are constants. Hyperbolic PDE-ODE loops have been studied recently in [3,8,10,17,18,28,38]. For the class of systems (2), (3), (4), (5), we provide sampled-data observers and observer-based, sampled-data, output feedback stabilizers (Theorem 3.2, Corollary 3.4).

It should be noted that in all cases the results are global. Moreover, we are in a position to consider uncertain sampling schedules (i.e., the sampling times are not a priori known) and guarantee robustness with respect to measurement noise. Finally, in the absence of measurement noise and unknown disturbances, exponential convergence of the observer error is achieved. The fact that the proposed sampled-data observer design approach with an inter-sample output predictor can indifferently be applied to time-delay systems, to transport PDEs, and to interconnections of ODEs with a transport PDE, provides the approach with a strong unifying feature.

The present paper provides illustrative applications of the theoretical results. In Section 4, we consider the ODE-PDE loop that is used for the mathematical description of a chemical reactor with an exothermic chemical reaction taking place in it and a cooling jacket. A sampled-data observer is designed for this system when we only measure the temperature of the cooling medium at the exit of the jacket. The observer guarantees exponential convergence of the observer error in the noise-free case, no matter what the transport speed inside the reactor is. In Section 5, we deal with uncertain, triangular, globally Lipschitz delay systems of the form

$$\begin{aligned}
\dot{x}_1(t) &= f_1(x_{1,t}, u(t)) + x_2(t) + d_1(t)\\
&\vdots\\
\dot{x}_{n-1}(t) &= f_{n-1}(x_{1,t},...,x_{n-1,t}, u(t)) + x_n(t) + d_{n-1}(t)\\
\dot{x}_n(t) &= f_n(x_{1,t},...,x_{n,t}, u(t)) + d_n(t)\\
y(t) &= x_1(t)
\end{aligned} \qquad (6)$$

where $x(t) = (x_1(t),...,x_n(t)) \in \Re^n$ is the state vector, x_t denotes the r-history of x with $r > 0$ being the maximum system delay (see the notation subsection), $u(t) \in \Re^m$ is a

known input, $d(t)=(d_1(t),...,d_n(t))\in\Re^n$ is a vector of disturbances (or unknown inputs) and $f_i:C^0([-r,0];\Re^i)\times\Re^m\to\Re$ $(i=1,...,n)$ with $f_i(0)=0$ $(i=1,...,n)$ are globally Lipschitz functionals. The use of inter-sample predictors in the design of sampled-data observers for (6) guaranteees exponential convergence of the observer error, no matter how large the maximum delay $r>0$ is (Theorem 5.1). The design is based on the high-gain observer design for ODEs, proposed in [21].

Notation. Throughout this paper, we adopt the following notation.

* $\Re_+=[0,+\infty)$. Let $u:\Re_+\times[0,1]\to\Re$ be given. We use the notation $u[t]$ to denote the profile at $t\geq 0$, i.e., $(u[t])(z)=u(t,z)$ for $z\in[0,1]$. For $u\in C^0([0,1])$ we define $\|u\|_\infty=\sup_{0\leq z\leq 1}\left(|u(z)|\right)$.
* Let $S\subseteq\Re^n$ be an open set and let $A\subseteq\Re^n$ be a set that satisfies $S\subseteq A\subseteq cl(S)$. By $C^0(A;\Omega)$, we denote the class of continuous functions on A with values in $\Omega\subseteq\Re^m$. By $C^k(A;\Omega)$, where $k\geq 1$ is an integer, we denote the class of functions on $A\subseteq\Re^n$ with values in $\Omega\subseteq\Re^m$ and has continuous derivatives of order k. When $\Omega=\Re$ then we write $C^0(A)$ or $C^k(A)$.
* For a vector $x\in\Re^n$ we denote by $|x|$ its usual Euclidean norm and by x^T its transpose. By $|A|:=\sup\{|Ax|;x\in\Re^n,|x|=1\}$ we denote the induced norm of a matrix $A\in\Re^{m\times n}$ and I denotes the identity matrix. By $B=diag(b_1,...,b_n)$ we denote the diagonal matrix $B\in\Re^{n\times n}$ with $b_1,...,b_n$ in its diagonal. For $x\in C^0([-r,0];\Re^n)$ we define $\|x\|:=\max_{\theta\in[-r,0]}\left(|x(\theta)|\right)$.
* Let $x:[a-r,b)\to\Re^n$ be a continuous mapping with $b>a>-\infty$ and $r>0$. By x_t we denote the "r-history" of x at time $t\in[a,b)$, i.e., $(x_t)(\theta):=x(t+\theta)$; $\theta\in[-r,0]$. Notice that $x_t\in C^0([-r,0];\Re^n)$.
* By K we denote the set of increasing, continuous functions $\rho:\Re_+\to\Re_+$ with $\rho(0)=0$. We say that a function $\rho\in K$ is of class K_∞ if $\lim_{s\to+\infty}\rho(s)=+\infty$.
* Let $D\subseteq\Re^l$ be a non-empty set and $I\subseteq\Re_+$ an interval. By $L^\infty_{loc}(I;D)$ we denote the class of measurable and locally bounded mappings $d:I\to D$. Notice that by $\sup_{\tau\in I}\left(|d(\tau)|\right)$ we do not mean the essential supremum of d on I but the actual supremum of d on I

2. Main Results for Delay Systems

In the present work we study systems of the form (1) under the following assumptions:

(H1) *The mappings* $f:C^0([-r,0];\Re^n)\times U\times D\to\Re^n$, $h:C^0([-r,0];\Re^n)\to\Re^k$ *where* $U\subseteq\Re^m$, $D\subseteq\Re^q$ *are convex sets with* $0\in U$, $0\in D$, *are continuous and satisfy the following properties, (i)* $f(0,0,0)=0$, $h(0)=0$, *(ii) for every bounded* $\Omega\subset C^0([-r,0];\Re^n)\times U\times D$ *the image set* $f(\Omega)\subset\Re^n$ *is bounded, (iii) for every bounded* $\Omega\subset C^0([-r,0];\Re^n)$ *the image set* $h(\Omega)\subset\Re^k$ *is bounded, and (iv) for every bounded* $S\subset C^0([-r,0];\Re^n)\times U\times D$, *there exists a constant* $L_S\geq 0$ *such that*

$$\left(x(0)-\bar{x}(0)\right)^T\left(f(x,u,d)-f(\bar{x},u,d)\right)\leq L_S\left\|x-\bar{x}\right\|^2$$
$$\forall(x,u,d)\in S,\ \forall(\bar{x},u,d)\in S$$

(H2) *System (1) is Forward Complete, i.e., for every* $x_0\in C^0([-r,0];\Re^n)$ *and for every* $u\in L^\infty_{loc}(\Re_+;U)$, $d\in L^\infty_{loc}(\Re_+;D)$ *the solution of (1) with initial condition* x_0, *corresponding to inputs* u,d, *exists for all* $t\geq 0$.

Assumption (H1) is a standard assumption for time-delay systems that guarantees existence and uniqueness of solutions for system (1), i.e. guarantees that for every $x_0\in C^0([-r,0];\Re^n)$ and for every $u\in L^\infty_{loc}(\Re_+;U)$, $d\in L^\infty_{loc}(\Re_+;D)$ there exists $t_{\max}\in(0,+\infty]$ and a unique continuous mapping $x:[-r,t_{\max})\to\Re^n$ which is absolutely continuous on $[0,t_{\max})$ and satisfies $x(\theta)=x_0(\theta)$ for $\theta\in[-r,0]$ and (1) for $t\in[0,t_{\max})$ a.e.. This mapping $x:[-r,t_{\max})\to\Re^n$ is the solution of (1) with initial condition x_0, corresponding to inputs u,d. Assumption (H2) guarantees that $t_{\max}=+\infty$ for every $x_0\in C^0([-r,0];\Re^n)$, $u\in L^\infty_{loc}(\Re_+;U)$ and $d\in L^\infty_{loc}(\Re_+;D)$. Assumption (H2) is usual in observer studies. Indeed, if an observer is expected to achieve asymptotic estimation of the state then the solution of (1) must exist for all times. On the other hand, the (possible) state and measurement delays are respectively accounted for through x_t in the quantities $f(x_t,u,d)$ and $h(x_t)$.

The following assumption plays a crucial role.

(H3) *There exists a continuous mapping* $R:C^0([-r,0];\Re^n)\times U\times D\to\Re^k$ *with the following property: for every* $x_0\in C^0([-r,0];\Re^n)$, $u\in L^\infty_{loc}(\Re_+;U)$, $d\in L^\infty_{loc}(\Re_+;D)$ *the unique solution* x *of (1) with initial condition* x_0, *corresponding to inputs* u,d, *satisfies for* $t\geq 0$ *a.e. the following equation:*

$$\frac{d}{dt}\big(h(x_t)\big) = R(x_t, u(t), d(t)) \tag{7}$$

Moreover, there exists a constant $L \geq 0$ *and a function* $\kappa \in K$ *such that the following inequality holds for all* $x, \bar{x} \in C^0([-r,0];\Re^n)$, $(u,d) \in U \times D$ *:*

$$\big|R(\bar{x},u,0) - R(x,u,d)\big| \leq L\|\bar{x} - x\| + \kappa\big(|d|\big) \tag{8}$$

Assumption (H3) requires that the derivative of the output of system (1) exists and is expressed by the globally Lipschitz (with respect to x) mapping R. Not every nonlinear time-delay system satisfies (H3). Nevertheless, the class of systems satisfying Assumption (H3) is very wide, including systems of practical interest (see [6]), like globally Lipschitz delay systems with linear delay-free outputs.

The notion of the Robust Exponential Observer (REO) for system (1) is crucial to the development of the results of the paper and it is given in the following definition.

Definition 2.1 (Robust Exponential Observer): *Consider the following system*

$$\begin{aligned} \dot{z} &= F(z_t, y, u), z \in \Re^l \\ \hat{x}_t &= \Psi(z_t), \hat{x} \in \Re^n \end{aligned} \tag{9}$$

where $F : C^0([-r,0];\Re^l) \times \Re^k \times U \to \Re^l$, $\Psi : C^0([-r,0];\Re^l) \to C^0([-r,0];\Re^n)$ *are continuous mappings with* $F(0,0,0) = 0$, $\Psi(0) = 0$. *Suppose that the mapping* F *is such that, for every bounded* $\Omega \subset C^0([-r,0];\Re^l) \times \Re^k \times U$ *the image set* $F(\Omega) \subset \Re^l$ *is bounded and there exists a constant* $L_\Omega \geq 0$ *such that*

$$\begin{aligned} &\big(z(0) - \bar{z}(0)\big)^T \big(F(z,y,u) - F(\bar{z},y,u)\big) \leq L_\Omega \|z - \bar{z}\|^2 \\ &\forall (z,y,u) \in \Omega, \forall (\bar{z},y,u) \in \Omega \end{aligned}$$

System (9) is called a ***Robust Exponential Observer (REO)*** *for system (1), if there exist constants* $\gamma, \sigma > 0$ *and functions* $a, g \in K_\infty$ *such that for all* $u \in L^\infty_{loc}(\Re_+;U)$, $(x_0, z_0) \in C^0([-r,0];\Re^n) \times C^0([-r,0];\Re^l)$, $d \in L^\infty_{loc}(\Re_+;D)$, $\xi \in L^\infty_{loc}(\Re_+;\Re^k)$ *the solution* $(x(t), z(t))$ *of*

$$\begin{aligned} \dot{x} &= f(x_t, u, d) \\ \dot{z} &= F(z_t, h(x_t) + \xi, u) \\ \hat{x}_t &= \Psi(z_t) \end{aligned} \tag{10}$$

with initial condition (x_0, z_0), *corresponding to inputs* u, d, ξ, *exists for all* $t \geq 0$ *and satisfies the following estimate for all* $t \geq 0$ *:*

$$\begin{aligned} \|\hat{x}_t - x_t\| &\leq \exp(-\sigma t) a\big(\|x_0\| + \|z_0\|\big) \\ &+ \gamma \sup_{0 \leq s \leq t}\big(|\xi(s)| \exp(-\sigma(t-s))\big) + \sup_{0 \leq s \leq t}\big(g(|d(s)|)\big) \end{aligned} \tag{11}$$

At this point, it should be noticed that the way the inputs d and ξ enter the Input-to-Output (IOS) Stability estimate (11) is different. While the input d comes in estimate (11) through a (possibly) nonlinear gain function, the input ξ appears in estimate (11) with a linear gain and with a fading memory effect (see [24]). This difference is important and allows, in what follows, the construction of sampled-data observers. Besides the fact that Definition 2.1 introduces the notion of REO for systems with state delays, there are important differences between this notion of a REO in Definition 2.1 and similar notions in the literature (for systems described by ODEs; see [4,23]):

1) In Definition 2.1, the effect of disturbances is explicitly taken into account (see the term $\sup_{0 \leq s \leq t} g(|d(s)|)$ in estimate (11)), while in other similar notions in the literature no disturbances are assumed to act on the system.

2) In Definition 2.1, the IOS estimate (11) is assumed to hold uniformly for inputs $u \in L^\infty_{loc}(\Re_+;U)$, while in other notions in the literature either there is no control input u or the sup-norm of u appears in the corresponding observer error estimate. This difference is important when the observer is to be used in conjunction with a state feedback control law.

The existence of a REO for a control system is a strict requirement which does not hold for all nonlinear systems. A discussion of classes of finite-dimensional systems (which may be considered as a special case of delay systems) which admit a REO can be found in [23] (linear detectable systems, globally Lipschitz systems). It should be noticed that Definition 2.1 allows (in general) arbitrarily large measurement delays (included in h) because the REO may include continuous-time predictors (see [4]).

We are now in a position to state our main result.

Theorem 2.2 (Sampled-Data Observer Design): *Consider system (1) under (H1), (H2), (H3) and suppose that system (9) is a REO for system (1). Moreover, suppose that for every bounded* $S \subset C^0([-r,0];\Re^n) \times U$, *there exists a constant* $L_S \geq 0$ *such that*

$$\begin{aligned} &\big|R(\Psi(z),u,0) - R(\Psi(\bar{z}),u,0)\big| \leq L_S \|z - \bar{z}\|, \\ &\forall (z,u) \in S, \forall (\bar{z},u) \in S \end{aligned} \tag{12}$$

Let $\delta > 0$ *and* $\omega \in (0, \sigma]$ *be constants that satisfy*

$$\delta < \omega^{-1} \ln\big(1 + (\gamma L)^{-1} \omega\big) \tag{13}$$

where $L > 0$ *is the constant appering in (8) and* $\sigma, \gamma > 0$ *are the constants appering in (11). Then for every sampling sequence* $\{\tau_i\}_{i=0}^\infty$ *with* $\tau_0 = 0$, $\lim(\tau_i) = +\infty$, $0 < \tau_{i+1} - \tau_i \leq \delta$ *for* $i = 0, 1, \ldots$, *for every* $(x_0, z_0) \in C^0([-r,0];\Re^n) \times C^0([-r,0];\Re^l)$ *and* $u \in L^\infty_{loc}(\Re_+;U)$, $d \in L^\infty_{loc}(\Re_+;D)$, $\xi \in L^\infty_{loc}(\Re_+;\Re^k)$, *the solution* $(x(t), z(t), w(t))$ *of (1) with*

$$\begin{aligned} \dot{z}(t) &= F(z_t, w(t), u(t)), t \geq 0 \\ \dot{w}(t) &= R(\Psi(z_t), u(t), 0), t \in [\tau_i, \tau_{i+1}) \\ \hat{x}_t &= \Psi(z_t), t \geq 0 \end{aligned} \tag{14}$$

$$w(\tau_i) = h(x_{\tau_i}) + \xi(\tau_i) \tag{15}$$

initial condition (x_0, z_0), *corresponding to inputs* u, d, ξ, *exists for all* $t \geq 0$ *and satisfies the following estimate*

$$\|\hat{x}_t - x_t\| \leq (1-B)^{-1} \exp(-\omega t) a\left(\|x_0\| + \|z_0\|\right) + \sup_{0\leq s\leq t}\left(\tilde{g}(|d(s)|)\right)$$
$$+(1-B)^{-1}\gamma \exp(\omega\delta) \sup_{0\leq s\leq t}\left(|\xi(s)| \exp(-\omega(t-s))\right) \quad (16)$$

where $\tilde{g}(s) := (1-B)^{-1}(g(s) + \gamma\delta\kappa(s))$ *and* $B := \gamma L\left(\exp(\omega\delta) - 1\right)/\omega < 1$.

Remark 2.3: (a) The observer (14), (15) is the REO (9) with the unavailable output signal replaced by the signal produced by the inter-sample output predictor

$$\dot{w}(t) = R(\Psi(z_t), u(t), 0)\,,\ t \in [\tau_i, \tau_{i+1})$$
$$w(\tau_i) = h(x_{\tau_i}) + \xi(\tau_i)$$

(b) Notice that (16) guarantees the IOS property for the output map $Y = \hat{x}_t - x_t$ from the inputs d, ξ, i.e. from the inputs expressing the effect of modeling errors and measurement noise, respectively. However, a comparison of (11) and (16) shows that the input gains are higher for the sampled-data observer (14), (15) than the continuous-time REO (9). It is clear that sampling makes the observer more sensitive to modeling errors and measurement noise.

(c) Despite the hybrid nature of the observer (14)-(15), the trajectory of the estimated state features continuity. The proof of Theorem 2.2 is based on a small-gain argument (see Section 5). It is therefore expected that the observer error estimate (16) and the upper bound for the diameter of the sampling sequence $\delta > 0$ given by (13) are conservative. However, formulas (13), (16) are useful because they indicate which parameters affect the performance of the observer and (qualitatively) how the upper bound for the diameter of the sampling sequence depends on the parameters of the system.

(d) Since the mapping $(0, \sigma] \ni \omega \to \omega^{-1}\ln\left(1 + (\gamma L)^{-1}\omega\right)$ is decreasing with $\lim_{\omega\to 0^+}\left(\omega^{-1}\ln\left(1+(\gamma L)^{-1}\omega\right)\right) = (\gamma L)^{-1}$, it is clear from (13) that: (i) Theorem 2.2 requires sampling sequences with diameter $\delta > 0$ being less than $1/\gamma L$, and (ii) the smaller the diameter $\delta > 0$ of the sampling sequence is, the larger the constant $\omega > 0$ is, in the absence of modeling errors and noise (recall (16)).

For the design of observed-based output feedback, we need a stabilizability assumption.

(H4) *The equalities* $f(0,0,0) = 0$ *and* $U = \Re^m$ *hold. Moreover there exist a function* $\bar{\kappa} \in K$, *constants* $\sigma, M > 0$ *and a functional* $\tilde{k} : C^0([-r, 0]; \Re^n) \to \Re^m$ *with* $\tilde{k}(0) = 0$ *and a constant* $\bar{L} > 0$ *such that the inequalities*

$$\left|\tilde{k}(\bar{x}) - \tilde{k}(x)\right| + \left|f(\bar{x}, \bar{u}, 0) - f(x, u, 0)\right| \leq \bar{L}\|\bar{x} - x\| + \bar{L}|\bar{u} - u|$$
$$\left|f(x, u, d) - f(x, u, 0)\right| \leq \bar{\kappa}(|d|)$$

hold for $x, \bar{x} \in C^0([-r, 0]; \Re^n)$, $\bar{u}, u \in \Re^m$, $d \in D$ *and such that for all* $x_0 \in C^0([-r, 0]; \Re^n)$, *the solution* $x(t)$ *of*

$$\dot{x}(t) = f(x_t, u(t), 0)$$
$$u(t) = \tilde{k}(x_t) \quad (17)$$

with initial condition x_0 *exists for all* $t \geq 0$ *and satisfies the following estimate*

$$\|x_t\| \leq M \exp(-\sigma t)\|x_0\|\,,\ \forall t \geq 0 \quad (18)$$

When Assumption (H4) holds then we obtain the following stabilization result.

Corollary 2.4 (Global Stabilization by Means of Observer-Based Sampled-Data Output Feedback): *Consider system (1) under (H1), (H2), (H3), (H4) and suppose that system (9) is a REO for system (1). Moreover, suppose that for every bounded* $S \subset C^0([-r, 0]; \Re^n) \times U$, *there exists a constant* $L_S \geq 0$ *such that (12) holds. Then there exist constants* $\bar{\delta}, \omega, \hat{\gamma} > 0$ *and functions* $\hat{g}, \hat{a} \in K$ *such that for every sampling sequence* $\{\tau_i\}_{i=0}^{\infty}$ *with* $\tau_0 = 0$, $\lim(\tau_i) = +\infty$, $0 < \tau_{i+1} - \tau_i \leq \bar{\delta}$ *for* $i = 0, 1, \ldots$, *for every* $(x_0, z_0) \in C^0([-r, 0]; \Re^n) \times C^0([-r, 0]; \Re^l)$, $\xi \in L^{\infty}_{loc}(\Re_+; \Re^k)$, $d \in L^{\infty}_{loc}(\Re_+; D)$, *the solution* $x(t)$ *of (1) with (14), (15) and*

$$u(t) = \tilde{k}\left(\hat{x}_{\tau_i}\right), t \in [\tau_i, \tau_{i+1}) \quad (19)$$

initial condition (x_0, z_0), *corresponding to inputs* d, ξ, *exists for all* $t \geq 0$ *and satisfies the following estimate*

$$\|x_t\| + \|\hat{x}_t\| \leq \exp(-\omega t)\hat{a}\left(\|x_0\| + \|z_0\|\right)$$
$$+\hat{\gamma}\sup_{0\leq s\leq t}\left(|\xi(s)|\right) + \sup_{0\leq s\leq t}\left(\hat{g}(|d(s)|)\right) \quad (20)$$

Moreover, if $a \in K_{\infty}$ *(the function involved in (11)) is linear then* $\hat{a}$ *is linear too.*

Example 2.5: To briefly illustrate the results of this section, let us consider the following quite simple example:

$$\dot{x}(t) = -\sigma x(t) + y(t) + u(t) + d(t),\ y(t) = b\int_{t-r}^{t} x(s)ds \quad (21)$$

with $\sigma > 0, b \in \Re$, $b \neq 0$. Clearly, this system is of the form (1) which meets all requirements of Theorem 2.2. Consider the continuous-time observer $\dot{z}(t) = -\sigma z(t) + y(t) + u(t)$, which is a REO. Indeed, it is straightforward to show that (11) holds with $\gamma = 1/\sigma$. Moreover, assumption (H3) holds with $R(x, u, d) = bx(0) - bx(-r)$ and (8) holds with $L = 2|b|$. The sampled-data version of this observer is:

$$\dot{z}(t) = -\sigma z(t) + w(t) + u(t)\,,\ t \geq 0$$
$$\dot{w}(t) = bz(t) - bz(t-r)\,,\ t \in [\tau_i, \tau_{i+1})$$
$$\hat{x}_t = z_t\,,\ t \geq 0 \quad (22)$$

$$w(\tau_i) = b\int_{\tau_i - r}^{\tau_i} x(s)ds + \xi(\tau_i) \quad (23)$$

By virtue of Theorem 2.2, the sampled-data observer (22), (23) preserves the exponential convergence feature of the (continuous-time) REO, provided that the upper diameter $\delta > 0$ of the sampling schedule satisfies the inequality

$\delta < \sigma / (2|b|)$. On the other hand, we consider the following (continuous-time) linearizing state-feedback:

$$u = \tilde{k}(x) = -b\int_{-r}^{0} x(s)ds \tag{24}$$

It is readily seen that the feedback law (24) meets all requirements of Corollary 2.4. It turns out that the observer-based, sampled-data feedback $u = \tilde{k}(\hat{x})$, with $\hat{x}$ being online generated by the sampled-data observer (22), (23), is an exponentially globally stabilizing regulator. ◁

3. Hyperbolic PDE-ODE Interconnections

3.1. The General Case

When a plant is interconnected with a transport process then we can obtain a system of the form (1) with distributed delays. This is the reason that in this section we consider initial-boundary value problems of the form (2), (3), (4) with initial condition

$$v[0] = v_0\,,\ \bar{x}(0) = \bar{x}_0 \tag{25}$$

where $c > 0$ is a constant, $\bar{x}(t) \in \Re^{\bar{n}}$, $v(t,z) \in \Re$ are the states, $u \in C^0\left(\Re_+;\Re^m\right)$ is an external input, $a \in C^0([0,1])$, $b_i \in C^1([0,1])$ $(i = 1,...,N_1)$, $\gamma_i \in C^0([0,1]\times[0,1])$, $\beta_i \in C^1([0,1])$ $(i = 1,...,N_2)$, $\bar{x}_0 \in \Re^n$, $v_0 \in C^1\left([0,1]\right)$ satisfy the compatibility conditions $v_0(0) = 0$ and

$$cv_0'(0) = g(0,\bar{x}_0) + \sum_{i=1}^{N_1} b_i(0)v_0(z_i) + \sum_{i=1}^{N_2}\beta_i(0)\int_0^1 \gamma_i(s)v_0(s)ds \tag{26}$$

and the following assumption holds for the mappings $\tilde{F}: \Re^{\bar{n}} \times C^0\left([0,1]\right)\times\Re^m \to \Re^{\bar{n}}$, $g: [0,1]\times\Re^{\bar{n}} \to \Re$:

(A1) *$\tilde{F}: \Re^{\bar{n}} \times C^0\left([0,1]\right)\times\Re^m \to \Re^{\bar{n}}$, $g \in C^1([0,1]\times\Re^{\bar{n}};\Re)$ are continuous mappings with $\tilde{F}(0,0,0) = 0$, $g(z,0) = 0$ for all $z \in [0,1]$, for which there exists a constant $L > 0$ such that the inequalities*

$$\left|\tilde{F}(x,v,u) - \tilde{F}(y,w,u)\right| \le L\left\|v - w\right\|_\infty + L|x - y|,$$

$$\max_{0\le z\le 1}\left(\left|g(z,x) - g(z,y)\right|\right) \le L|x - y|$$

hold for all $v,w \in C^0([0,1])$, $x,y \in \Re^{\bar{n}}$, $u \in \Re^m$.

Under (A1), Theorem 2.2 on page 22 in [28] shows that for every $v_0 \in C^1\left([0,1]\right)$, $\bar{x}_0 \in \Re^{\bar{n}}$, $u \in C^0\left(\Re_+;\Re^m\right)$ with $v_0(0) = 0$ satisfying (26), there exist unique mappings $v \in C^1(\Re_+\times[0,1])$ and $\bar{x} \in C^1(\Re_+;\Re^{\bar{n}})$ satisfying (2), (3), (4), (25). The solution satisfies the formula

$$v(t,z) = v_0\left(\max(0,z-ct)\right)\exp\left(c^{-1}\int_{\max(0,z-ct)}^{z} a(w)dw\right) +$$

$$c^{-1}\int_{\max(0,z-ct)}^{z}\exp\left(c^{-1}\int_l^z a(w)dw\right)g(l,\bar{x}(t+c^{-1}(l-z)))dl +$$

$$c^{-1}\sum_{i=1}^{N_1}\int_{\max(0,z-ct)}^{z}\exp\left(c^{-1}\int_l^z a(w)dw\right)b_i(l)v(t+c^{-1}(l-z),z_i)dl +$$

$$c^{-1}\sum_{i=1}^{N_2}\int_{\max(0,z-ct)}^{z}\exp\left(c^{-1}\int_l^z a(w)dw\right)\beta_i(l)\int_0^1\gamma_i(s)v(t+c^{-1}(l-z),s)dsdl$$

$$\text{for } (t,z) \in \Re_+\times[0,1] \tag{27}$$

For such systems, the output $y(t) = (y_1(t),...,y_k(t))^T \in \Re^k$ is given by (5), where $q_j \in \Re^{\bar{n}}$ $(j = 1,...,k)$ are constant vectors, $\bar{b}_{j,i} \in \Re$ $(j = 1,...,k,\ i = 1,...,N_1)$ and $\bar{\beta}_{j,i} \in \Re$ $(j = 1,...,k,\ i = 1,...,N_2)$ are constants. Therefore, for $t \ge r$, where

$$r = 1/c \tag{28}$$

we obtain from (27), (28) and (5), system (2) with

$$v(t,z) = \int_{t-rz}^{t}\frac{C(z)}{C(z-c(t-p))}\bar{g}(z-c(t-p),\bar{x}(p),\eta(p),\lambda(p))dp$$

$$\text{for } t \ge r\,,\ z \in [0,1] \tag{29}$$

with

$$C(z) := \exp\left(r\int_0^z a(s)ds\right) \tag{30}$$

$$\bar{g}(z,\bar{x},\eta,\lambda) := g(z,\bar{x}) + \sum_{i=1}^{N_1} b_i(z)\eta_i + \sum_{i=1}^{N_2}\beta_i(z)\lambda_i \tag{31}$$

$$\eta_i(t) = v(t,z_i) = \int_{t-rz_i}^{t}\frac{C(z_i)\bar{g}(z_i - c(t-p),\bar{x}(p),\eta(p),\lambda(p))}{C(z_i - c(t-p))}dp$$

$$\text{for } t \ge r\,,\ i = 1,...,N_1 \tag{32}$$

$$\lambda_i(t) = \int_0^1\gamma_i(z)v(t,z)dz =$$

$$\int_0^1\int_{t-rz}^{t}\frac{\gamma_i(z)C(z)\bar{g}(z-c(t-p),\bar{x}(p),\eta(p),\lambda(p))}{C(z-c(t-p))}dpdz$$

$$\text{for } t \ge r\,,\ i = 1,...,N_2 \tag{33}$$

and output given by

$$y_j(t) = q_j^T\bar{x}(t) + \sum_{i=1}^{N_1}\bar{b}_{j,i}\eta_i(t) + \sum_{i=1}^{N_2}\bar{\beta}_{j,i}\lambda_i(t)\,,$$

$$\text{for } t \ge r\,,\ j = 1,...,k \tag{34}$$

At this point, we need the following technical assumption:

(A2) *There exists a constant $G > 0$ such that for every $v_0 \in C^1\left([0,1]\right)$, $\bar{x}_0 \in \Re^{\bar{n}}$ with $v_0(0) = 0$ satisfyng (26), there exist $\bar{x} \in C^0\left([-r,0];\Re^{\bar{n}}\right)$, $\eta \in C^0\left([-r,0];\Re^{N_1}\right)$, $\lambda \in C^0\left([-r,0];\Re^{N_2}\right)$ with $\bar{x}(0) = \bar{x}_0$, $\eta_i(0) = v_0(z_i)$*

$(i=1,...,N_1)$, $\lambda_i(0)=\int_0^1 \gamma_i(z)v_0(z)dz$ $(i=1,...,N_2)$ *and* $\|\bar{x}\|+\|\eta\|+\|\lambda\|\le G\left(\|v_0\|_\infty+\|v_0'\|_\infty+|\bar{x}_0|\right)$ *that satisfy*

$$v_0(s)=\int_{-rs}^{0}\frac{C(s)}{C(cp+s)}\bar{g}(cp+s,\bar{x}(p),\eta(p),\lambda(p))dp\,,$$
$$\text{for all } s\in[0,1] \tag{35}$$

If Assumption (A2) holds then equations (29), (32), (33), (34) are valid for all $(t,z)\in\Re_+\times[0,1]$. Moreover, in this case and if the output is sampled, then Theorem 2.2 can be used for sampled-data observer design. We define

$$x=\left(\bar{x}^T,\eta^T,\lambda^T\right)^T\in\Re^n,\ \text{with } n=\bar{n}+N_1+N_2 \tag{36}$$
$$f_2(x):=\left(f_{2,1}(x),...,f_{2,N_1}(x)\right)^T,$$
$$f_3(x):=\left(f_{3,1}(x),...,f_{3,N_2}(x)\right)^T$$
$$f(x,u,d):=\left((f_1(x,u,d))^T,(f_2(x))^T,(f_3(x))^T\right)^T \tag{37}$$

for $x\in C^0\left([-r,0];\Re^n\right)$, $u\in\Re^m$ and $d\in\Re^q$ by means of the following equations:

$$f_1(x,u,d):=\tilde{F}(\bar{x}(0),v,u)+d\ ,\ \text{with}$$
$$v(z)=\int_{-rz}^{0}\frac{C(z)\bar{g}(cp+z,x(p))}{C(cp+z)}dp\ ,\ z\in[0,1] \tag{38}$$

$$f_{2,i}(x):=\bar{g}(z_i,x(0))-C(z_i)\bar{g}(0,x(-rz_i))+\int_{-rz_i}^{0}\frac{C(z_i)\tilde{g}(z_i+cp,x(p))}{C(z_i+cp)}dp\,,$$
$$\text{for } i=1,...,N_1 \tag{39}$$

$$f_{3,i}(x):=\int_0^1\gamma_i(z)\left(\bar{g}(z,x(0))-C(z)\bar{g}(0,x(-rz))\right)dz$$
$$+\int_0^1\int_{-rz}^{0}\frac{\gamma_i(z)C(z)}{C(z+cp)}\tilde{g}(z+cp,x(p))dpdz$$
$$\text{for } i=1,...,N_2 \tag{40}$$

$$h_j(x):=q_j^T\bar{x}(0)+\sum_{i=1}^{N_1}\bar{b}_{j,i}\eta_i(0)+\sum_{i=1}^{N_2}\bar{\beta}_{j,i}\lambda_i(0)\,,\ j=1,...,k \tag{41}$$

$$\tilde{g}(z,\bar{x},\eta,\lambda):=a(z)g(z,\bar{x})-c\frac{\partial g}{\partial z}(z,\bar{x})+\sum_{j=1}^{N_1}\tilde{b}_j(z_i)\eta_j+\sum_{j=1}^{N_2}\tilde{\beta}_j(z_i)\lambda_j \tag{42}$$

$$\tilde{b}_j(z):=a(z)b_j(z)-cb_j'(z)\ ,\ i=1,...,N_1 \tag{43}$$

$$\tilde{\beta}_j(z):=a(z)\beta_j(z)-c\beta_j'(z)\ ,\ i=1,...,N_2 \tag{44}$$

The proof of the following lemma is trivial and is omitted.

Lemma 3.1: *Suppose that Assumptions (A1), (A2) hold for system (2), (3), (4). Moreover, suppose that:*

(A3) *There exists a constant* $L>0$ *such that the inequality* $\max_{0\le z\le1}\left(\left|\frac{\partial g}{\partial z}(z,x)-\frac{\partial g}{\partial z}(z,y)\right|\right)\le L|x-y|$ *holds for all* $x,y\in\Re^n$.

Consider system (1) with $D=\{0\}$, *where* f,h *are defined by (37), (38), (39), (40), (41). Then Assumptions (H1), (H2), (H3) for system (1) hold with* $U=\Re^m$ *and* $R:C^0([-r,0];\Re^n)\times\Re^m\times D\to\Re^k$ *defined for all* $(x,u,d)\in C^0([-r,0];\Re^n)\times\Re^m\times D$ *by*

$$R(x,u,d)=\begin{pmatrix}q_1^Tf_1(x,u,d)+\sum_{i=1}^{N_1}\bar{b}_{1,i}f_{2,i}(x)+\sum_{i=1}^{N_2}\bar{\beta}_{1,i}f_{3,i}(x)\\ \vdots\\ q_k^Tf_1(x,u,d)+\sum_{i=1}^{N_1}\bar{b}_{k,i}f_{2,i}(x)+\sum_{i=1}^{N_2}\bar{\beta}_{k,i}f_{3,i}(x)\end{pmatrix} \tag{45}$$

Based on Theorem 2.2 and Lemma 3.1, we are in a position to show the following result.

Theorem 3.2 (Sampled-Data Observer for Hyperbolic PDE-ODE Loops): *Suppose that Assumptions (A1), (A2), (A3) hold for system (2), (3), (4). Consider system (1) with* $D=\{0\}$, *where* f,h *are defined by (37), (38), (39), (40), (41). Suppose that system (9) is a REO for system (1). Moreover, suppose that for every bounded* $S\subset C^0([-r,0];\Re^n)\times\Re^m$, *there exists a constant* $L_S\ge0$ *such that (12) holds with* R *being defined by (45). Then there exist constants* $\delta,P,\omega>0$ *and* $\bar{a}\in K_\infty$ *such that for every sequence* $\{\tau_i\}_{i=0}^{\infty}$ *with* $\tau_0=0$, $\lim(\tau_i)=+\infty$, $0<\tau_{i+1}-\tau_i\le\delta$ *for* $i=0,1,...$, *for every* $v_0\in C^1([0,1])$, $\bar{x}_0\in\Re^{\bar{n}}$ *with* $v_0(0)=0$ *satisfying (26),* $z_0\in C^0([-r,0];\Re^l)$, $u\in C^0\left(\Re_+;\Re^m\right)$, $\xi\in L^\infty_{loc}(\Re_+;\Re^k)$, *the solution* $(x(t),v[t],z(t),w(t))$ *of (2), (3), (4) together with (14) and*

$$w_j(\tau_i)=\xi(\tau_i)+q_j^T\bar{x}(\tau_i)+\sum_{l=1}^{N_1}\bar{b}_{j,l}v(\tau_i,z_l)+\sum_{l=1}^{N_2}\bar{\beta}_{j,l}\int_0^1\gamma_l(s)v(\tau_i,s)ds\,,$$
$$j=1,...,k \tag{46}$$

$$\hat{v}(t,z)=\int_{t-rz}^{t}\frac{C(z)\bar{g}(z-c(t-p),\hat{x}(p))}{C(z-c(t-p))}dp\,,$$
$$\text{for } t\ge0\,,\ z\in[0,1] \tag{47}$$

initial condition $(\bar{x}_0,v_0,z_0)$, *corresponding to inputs* u,ξ, *exists for all* $t\ge0$ *and satisfies the following estimate:*

$$|\bar{x}(t)-H\hat{x}(t)|+\|v[t]-\hat{v}[t]\|_\infty$$
$$\le\exp(-\omega t)\bar{a}\left(\|v_0\|_\infty+\|v_0'\|_\infty+|\bar{x}_0|+\|z_0\|\right)$$
$$+P\sup_{0\le s\le t}\left(|\xi(s)|\exp(-\omega(t-s))\right) \tag{48}$$

where $H\in\Re^{\bar{n}\times n}$ *is the matrix for which the relation*

$$\bar{x}=H\left(\bar{x}^T,\eta^T,\lambda^T\right)^T \tag{49}$$

holds for $\bar{x}\in\Re^{\bar{n}}$, $\eta\in\Re^{N_1}$, $\lambda\in\Re^{N_2}$. *Moreover, if* $a\in K_\infty$ *(involved in (11)) is linear then* $\bar{a}$ *is linear.*

Theorem 3.2 can be used for the construction of sampled-data observers for PDE-ODE loops. It transforms the sampled-data observer design problem for a PDE-ODE loop to the construction of a REO for a delay system.

Remark 3.3: It should be noted that system (2), (3), (4) is *not* equivalent to the system

$$\frac{d\bar{x}}{dt}(t) = f_1(\bar{x}_t, \eta_t, \lambda_t, u(t), d(t))$$
$$\frac{d\eta}{dt}(t) = f_2(\bar{x}_t, \eta_t, \lambda_t)$$
$$\frac{d\lambda}{dt}(t) = f_3(\bar{x}_t, \eta_t, \lambda_t) \qquad (50)$$

Indeed, for every solution of system (2), (3), (4) there exists a solution of system (50) for which (29), (32), (33) hold. However, not every solution of system (50) provides a solution of system (2), (3), (4) by means of (29), (32), (33). For such a thing, the initial condition of the solution $(\bar{x}_t, \eta_t, \lambda_t)$ of system (50) has to satisfy $\eta_i(0) = \int_{-rz_i}^{0} \frac{C(z_i)\bar{g}(z_i+cp, \bar{x}(p), \eta(p), \lambda(p))}{C(z_i+cp)} dp$, for $i = 1,...,N_1$ and $\lambda_i(0) = \int_0^1 \int_{-rz}^{0} \frac{\gamma_i(z)C(z)}{C(z+cp)} \bar{g}(z+cp, \bar{x}(p), \eta(p), \lambda(p))dpdz$, for $i = 1,...,N_2$. In other words, system (2), (3), (4) is *immersed into* system (50).

3.2. Stabilization of Hyperbolic PDE-ODE Loops

For stabilization purposes, we assume (A1), (A2), (A3) as well as the following (stabilizability) assumption.

(A4) *There exists* $\bar{k} : \Re^{\bar{n}} \times C^0([0,1]) \to \Re^m$ *such that (H4) holds for (1) with* $D = \{0\}$, *where* f *is defined by (37), (38), (39), (40), for* $\tilde{k} : C^0([-r,0];\Re^n) \to \Re^m$ *given by*

$$\tilde{k}(\bar{x}, \eta, \lambda) = \bar{k}(\bar{x}_0, v_0) \qquad (51)$$

for every $v_0 \in C^1([0,1])$, $\bar{x}_0 \in \Re^{\bar{n}}$ *with* $v_0(0) = 0$ *satisfying (26), where* $\bar{x} \in C^0([-r,0];\Re^{\bar{n}})$, $\eta \in C^0([-r,0];\Re^{N_1})$, $\lambda \in C^0([-r,0];\Re^{N_2})$ *with* $\bar{x}(0) = \bar{x}_0$, $\eta_i(0) = v_0(z_i)$ $(i = 1,...,N_1)$, $\lambda_i(0) = \int_0^1 \gamma_i(z)v_0(z)dz$ $(i = 1,...,N_2)$ *are the functions involved in (35).*

Under (A4), we obtain the following stabilization result.

Corollary 3.4 (Global Stabilization of Hyperbolic PDE-ODE Loops with Observer-Based Sampled-Data Feedback): *Suppose that (A1), (A2), (A3), (A4) hold for system (2), (3), (4). Consider system (1) with* $D = \{0\}$, *where* f, h *are defined by (37), (38), (39), (40), (41). Suppose that system (9) is a REO for system (1). Moreover, suppose that for every bounded* $S \subset C^0([-r,0];\Re^n) \times \Re^m$, *there exists a constant* $L_S \geq 0$ *such that (12) holds with* R *given by (45). Then there exist constants* $\delta, P, \omega > 0$ *and a function* $\bar{a} \in K_\infty$ *such that for every sampling sequence* $\{\tau_i\}_{i=0}^{\infty}$ *with* $\tau_0 = 0$, $\lim(\tau_i) = +\infty$, $0 < \tau_{i+1} - \tau_i \leq \delta$ *for* $i = 0,1,...$, *for every* $z_0 \in C^0([-r,0];\Re^l)$, $\xi \in L^\infty_{loc}(\Re_+;\Re^k)$, $v_0 \in C^1([0,1])$, $\bar{x}_0 \in \Re^{\bar{n}}$ *with* $v_0(0) = 0$ *satisfying (26), the solution of (2), (3), (4), (14), (46), (47) and*

$$u(t) = \bar{k}(H\hat{x}(\tau_i), \hat{v}[\tau_i]), \text{ for } t \in [\tau_i, \tau_{i+1}) \qquad (52)$$

initial condition $(\bar{x}_0, v_0, z_0)$, *corresponding to input* ξ, *exists for all* $t \geq 0$ *and satisfies the following estimate:*

$$|\bar{x}(t)| + |H\hat{x}(t)| + \|v[t]\|_\infty + \|\hat{v}[t]\|_\infty$$
$$\leq \exp(-\omega t)\bar{a}\left(\|v_0\|_\infty + \|v_0'\|_\infty + |\bar{x}_0| + \|z_0\|\right)$$
$$+ P \sup_{0 \leq s \leq t}\left(|\xi(s)|\exp(-\omega(t-s))\right) \qquad (53)$$

where H *is the matrix involved in (49).*

4. Application to Chemical Reactor

The model of a chemical reactor with an exothermic chemical reaction taking place in it and a cooling jacket with negligible axial heat conduction of the cooling medium can be found in Chapter 2 and Chapter 8 of [28]. It consists of the following ODE and PDE:

$$\frac{d\bar{x}}{dt}(t) = \Theta(\bar{x}(t)) - (\mu+1)\bar{x}(t) + \mu\int_0^1 v(t,z)dz \qquad (54)$$

$$\frac{\partial v}{\partial t}(t,z) + c\frac{\partial v}{\partial z}(t,z) = -\zeta\, v(t,z) + \zeta\, \bar{x}(t) \qquad (55)$$

with boundary condition (4), where $\mu, \zeta > 0$ are real constants, $\bar{x}(t)$ is the (dimensionless) reactor outlet temperature, $v(t,z)$ is the (dimensionless) temperature of the cooling medium at position $z \in [0,1]$ in the jacket ($z = 0$ is the entrance of the jacket and $z = 1$ is its exit) and $\Theta : \Re \to \Re$ is a C^1 globally Lipschitz function with $\Theta(0) = 0$. When the reactor inlet temperature is not constant then equation (54) is modified as follows:

$$\frac{d\bar{x}}{dt}(t) = \Theta(\bar{x}(t)) - (\mu+1)\bar{x}(t) + \mu\int_0^1 v(t,z)dz + u(t) \qquad (56)$$

where $u(t) \in \Re$ is the input that expresses the variation of the reactor inlet temperature. Moreover, it is indeed the case that the measured temperature is the temperature of the cooling medium at the exit of the jacket, i.e., the measured output is $y(t) = v(t,1)$. The reactor model (4), (55), (56), is a system of the form (2), (3), (4), (5) with $a(z) \equiv -\zeta$, $\bar{n} = 1$, $N_1 = 1$, $z_1 = 1$, $k = 1$, $b_1(z) \equiv 0$, $\tilde{F}(\bar{x}, v, u) = \Theta(\bar{x}) - (\mu+1)\bar{x} + \mu\int_0^1 v(z)dz + u$, $g(z, \bar{x}) = \zeta\bar{x}$, $q_1 = 0$, $\bar{b}_{1,1} = 1$ (N_2, γ_i, β_i, $\bar{\beta}_{1,i}$ are irrelevant). Assumptions (A1), (A3) are automatically verified. Assumption (A2) also holds, since for every $v_0 \in C^1([0,1])$, $\bar{x}_0 \in \Re$ with $cv_0'(0) = \zeta\bar{x}_0$, $v_0(0) = 0$, there exists $\bar{x} \in C^0([-r,0];\Re)$

with $\bar{x}(0)=\bar{x}_0$ that satisfies $v_0(s)=\zeta\int_{-rs}^{0}\exp(\zeta p)\bar{x}(p)dp$, where

$$r=1/c \tag{57}$$

More specifically, $\bar{x}\in C^0\left([-r,0];\Re\right)$ is given by

$$\bar{x}(-rz)=(r\zeta)^{-1}\exp(r\zeta z)v_0'(z)\text{, for } z\in[0,1] \tag{58}$$

for arbitrary $\eta\in C^0\left([-r,0];\Re^{N_1}\right)$ with $\eta(0)=v_0(1)$. Selecting the constant function $\eta(s)=v_0(1)$ for $s\in[-r,0]$, we guarantee that $\|\bar{x}\|+\|\eta\|\le G\left(\|v_0\|_\infty+\|v_0'\|_\infty+|\bar{x}_0|\right)$ holds with $G=1+(r\zeta)^{-1}\exp(r\zeta)$. Using the formulas of the previous section, we relate system (4), (55), (56) with output $y(t)=v(t,1)$ to the delay system

$$\begin{aligned}\dot{x}_1(t)&=\zeta x_2(t)-\zeta x_2(t-r)\exp(-\zeta r)-\zeta x_1(t)\\ \dot{x}_2(t)&=\Theta(x_2(t))-(\mu+1)x_2(t)+u(t)\\&\quad+\mu\zeta\int_0^1\int_{t-rl}^{t}x_2(s)\exp(-\zeta(t-s))dsdl\\ y(t)&=x_1(t)\end{aligned} \tag{59}$$

with initial condition that satisfies $x_1(0)=\zeta\int_{-r}^{0}\exp(\zeta s)x_2(s)ds$. Model (4), (55), (56) is *not* equivalent to system (59): the solutions of the reactor model (4), (55), (56) correspond to the solutions of (59) only when $x_1(0)=\zeta\int_{-r}^{0}\exp(\zeta s)x_2(s)ds$. In this case, we get $x_1(t)=\zeta\int_{t-r}^{t}\exp(\zeta(s-t))x_2(s)ds$ for $t\ge 0$ and the reactor model (4), (55), (56) turns out to be equivalent to the following system with distributed delays:

$$\begin{aligned}\frac{d\bar{x}}{dt}(t)&=\Theta(\bar{x}(t))-(\mu+1)\bar{x}(t)+u(t)\\&\quad+\mu\zeta\int_0^1\int_{t-rl}^{t}\bar{x}(s)\exp(-\zeta(t-s))dsdl\\ y(t)&=\zeta\int_{t-r}^{t}\exp(\zeta(s-t))\bar{x}(s)ds\end{aligned}$$

Exploiting Theorem 3.2, we can prove the following result.

Theorem 4.1 (Sample-Data Observer for the Chemical Reactor): *There exist constants* $k_1,k_2,\delta,P,M,\omega>0$ *such that for every sampling sequence* $\{\tau_i\}_{i=0}^{\infty}$ *with* $\tau_0=0$, $\lim(\tau_i)=+\infty$, $0<\tau_{i+1}-\tau_i\le\delta$ *for* $i=0,1,\ldots$, *for every* $v_0\in C^1([0,1])$, $\bar{x}_0\in\Re$ *with* $cv_0'(0)=\zeta\bar{x}_0$, $v_0(0)=0$, $z_0\in C^0([-r,0];\Re^2)$, $u\in C^0(\Re_+;\Re)$, $\xi\in L^\infty_{loc}(\Re_+;\Re)$, *the solution* $(\bar{x}(t),v[t],z(t),w(t))$ *of (4), (55), (56) with*

$$\begin{aligned}\dot{z}_1(t)&=\zeta z_2(t)-\zeta z_2(t-r)\exp(-\zeta r)-\zeta z_1(t)-k_1\left(z_1(t)-w(t)\right)\\ \dot{z}_2(t)&=\Theta(z_2(t))-(\mu+1)z_2(t)+u(t)-k_2\left(z_1(t)-w(t)\right)\\&\quad+\mu\zeta\int_0^1\int_{t-rl}^{t}z_2(s)\exp(-\zeta(t-s))dsdl\\ \dot{w}(t)&=\zeta z_2(t)-\zeta z_2(t-r)\exp(-\zeta r)-\zeta z_1(t),\quad t\in[\tau_i,\tau_{i+1})\\ \hat{x}_t&=z_t\end{aligned} \tag{60}$$

$$w_j(\tau_i)=\xi(\tau_i)+v(\tau_i,1) \tag{61}$$

$$\hat{v}(t,z)=\zeta\int_{t-rz}^{t}\exp(-\zeta(t-p))\hat{x}_2(p)dp \tag{62}$$

for $t\ge 0$, $z\in[0,1]$, *initial condition* $(\bar{x}_0,v_0,z_0)$, *corresponding to* u,ξ, *exists for all* $t\ge 0$ *and satisfies*

$$\begin{aligned}&|\bar{x}(t)-\hat{x}_2(t)|+\|v[t]-\hat{v}[t]\|_\infty\\&\le\exp(-\omega t)M\left(\|v_0\|_\infty+\|v_0'\|_\infty+|\bar{x}_0|+\|z_0\|\right)\\&\quad+P\sup_{0\le s\le t}\left(|\xi(s)|\exp(-\omega(t-s))\right)\end{aligned} \tag{63}$$

The proof of Theorem 4.1 is constructive and provides formulae for the observer gains k_1,k_2 (see formulae (103), (104), (105)), although other choices for the observer gains are possible. It should be noticed that there is *no restriction* in the speed $c>0$ of the transport process (or equivalently, in the delay $r>0$; see (57)).

We next evaluate numerically the sampled-data observer (60), (61), (62), together with the inter-sample predictor, considering the following parameter values:

$$\zeta=1/2,\ \mu=0.05,$$

$$\Theta(x)=10\left(\frac{\exp\left(-0.1(x+1)^{-1}\right)}{1+\exp\left(-0.1(x+1)^{-1}\right)}-\frac{\exp(-0.1)}{1+\exp(-0.1)}\right)\text{ for } x>-1,$$

$$\Theta(x)=-\frac{10\exp(-0.1)}{1+\exp(-0.1)}\text{ for } x\le-1,$$

$$k_1=10,\ k_2=30,\ r=0.1,\ u(t)=\sin(10t),\ \xi(t)\equiv 0,$$

$$\delta=0.4,\ \tau_i=i\delta,\text{ for } i=0,1,2,\ldots$$

$$v_0(z)=0.5\left(1-\exp(-rz)\right),\text{ for } z\in[0,1]$$

We also consider comparing of the novel sampled-data observer, defined by (60), (61) and (62), with the ZOH observer where the unknown output signal is simply replaced by its most recent measurement. Specifically, the ZOH observer consists of (61), (62) and

$$\begin{aligned}\dot{z}_1(t)&=\zeta z_2(t)-\zeta z_2(t-r)\exp(-\zeta r)-\zeta z_1(t)-k_1\left(z_1(t)-w(t)\right)\\ \dot{z}_2(t)&=\Theta(z_2(t))-(\mu+1)z_2(t)+u(t)-k_2\left(z_1(t)-w(t)\right)\\&\quad+\mu\zeta\int_0^1\int_{t-rl}^{t}z_2(s)\exp(-\zeta(t-s))dsdl\\ \dot{w}(t)&=0,\quad t\in[\tau_i,\tau_{i+1})\\ \hat{x}_t&=z_t\end{aligned} \tag{64}$$

The study is completed by comparing the observer (60), (61), (62) with the continuous-time observer that uses the continuous output signal $y(t)$, i.e., (62) and

$$\begin{aligned}\dot{z}_1(t) &= \zeta z_2(t) - \zeta z_2(t-r)\exp(-\zeta r) - \zeta z_1(t) - k_1\left(z_1(t)-y(t)\right)\\ \dot{z}_2(t) &= \Theta(z_2(t)) - (\mu+1)z_2(t) + u(t) - k_2\left(z_1(t)-y(t)\right)\\ &\quad + \mu\zeta\int_0^1\int_{t-rl}^{t} z_2(s)\exp\left(-\zeta(t-s)\right)dsdl\\ \hat{x}_t &= z_t\end{aligned} \tag{65}$$

All observers are launched with the initial conditions $z_1(0) = 0.2926$, $z_2(t) = 0.6$ for $t \in [-0.1, 0]$. The results are shown in Fig.1. We found that the state estimates provided by (60), (61), (62) with the inter-sample predictor were indistinguishable from those of the continuous-time observer (65), (62). Exponential convergence of the error to zero for observer (60), (61), (62), is apparent in Fig.1. On the other hand, the error for the ZOH observer (64), (61), (62) does not converge to zero but presents an oscillation with amplitude 0.02. Clearly, the inter-sample predictor compensates well for the effects of sampling. These results were confirmed for a wide range of parameter values that were tested.

Pushing further the study, we seek global stabilization of the reactor by observer-based sampled-data feedback. It is readily checked that (4), (55), (56) satisfies (A4), with

$$\bar{k}(\bar{x}, v) := -Q\bar{x} - \mu\int_0^1 v(z)dz \text{, for } \bar{x} \in \Re,\ v \in C^0([0,1]) \tag{66}$$

where $Q > 0$ is a sufficiently large constant. It follows from Theorem 2.1 and Corollary 3.4 that there exist constants $\delta, P, \omega > 0$ and a function $\bar{a} \in C^0(\Re_+;\Re_+)$ with $\bar{a}(0) = 0$ such that for every sequence $\{\tau_i\}_{i=0}^{\infty}$ with $\tau_0 = 0$, $\lim(\tau_i) = +\infty$, $0 < \tau_{i+1} - \tau_i \le \delta$ for $i = 0,1,\ldots$, for every $v_0 \in C^1([0,1])$, $\bar{x}_0 \in \Re$ with $c\,v_0'(0) = \zeta\bar{x}_0$, $v_0(0) = 0$, $z_0 \in C^0([-r,0];\Re^2)$, $\xi \in L^\infty_{loc}(\Re_+;\Re)$, the solution $(\bar{x}(t), v[t], z(t), w(t))$ of (4), (55), (56) together with (61), (60), (62) and

$$u(t) = -Q\hat{x}_2(\tau_i) - \mu\int_0^1 \hat{v}(\tau_i, z)dz \text{, for } t \in [\tau_i, \tau_{i+1}) \tag{67}$$

with initial condition $(\bar{x}_0, v_0, z_0)$, corresponding to input ξ, exists for all $t \ge 0$ and satisfies the estimate

$$\begin{aligned}&|\bar{x}(t)| + |\hat{x}(t)| + \|v[t]\|_\infty + \|\hat{v}[t]\|_\infty\\ &\le \exp(-\omega t)\bar{a}\left(\|v_0\|_\infty + \|v_0'\|_\infty + |\bar{x}_0| + \|z_0\|\right)\\ &\quad + P\sup_{0\le s\le t}\left(|\xi(s)|\exp(-\omega(t-s))\right)\end{aligned} \tag{68}$$

5. Triangular Globally Lipschitz Delay Systems

In this section we consider systems of the form (6), where we assume that there exists a constant $\tilde{L} \ge 0$ such that the following inequalities hold for $i = 1,\ldots,n$:

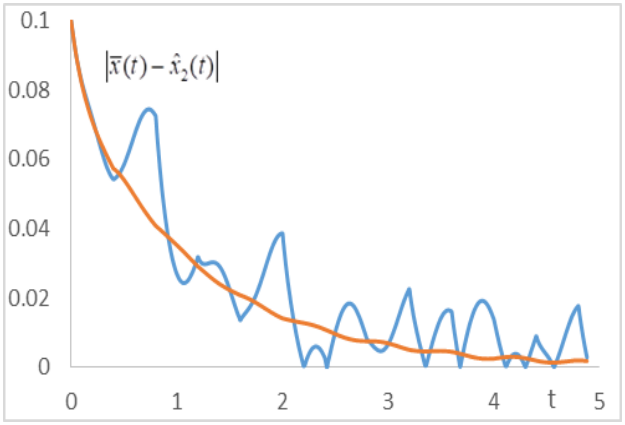

Fig. 1: The error $|\bar{x}(t) - \hat{x}_2(t)|$ for observer (60), (61), (62) with the inter-sample predictor (red line) and the ZOH observer (64), (61), (62) (blue line).

$$|f_i(x_1,\ldots,x_i,u) - f_i(z_1,\ldots,z_i,u)| \le \tilde{L}\sum_{j=1}^{i}\|x_j - z_j\|,$$

$$\text{for all } (x_1,\ldots,x_i),(z_1,\ldots,z_i) \in C^0([-r,0];\Re^i),\ u \in \Re^m \tag{69}$$

Notice that systems of the form (6) satisfying (69) are Forward Complete and satisfy Assumptions (H1), (H2) with $U = \Re^m$, $D = \Re^n$. Indeed, using (6), (69), we get that for all $x_0 \in C^0([-r,0];\Re^n)$ and $u \in L^\infty_{loc}(\Re_+;\Re^m)$, $d \in L^\infty_{loc}(\Re_+;\Re^n)$ the solution x of (6) with initial condition x_0, corresponding to u, d, satisfies the estimate

$$\|x_t\| \le \exp(n\tilde{L}t)\left(\|x_0\| + t\sup_{0\le s\le t}\left(|d(s)|\right) + t\sup_{0\le s\le t}\left(\sum_{i=1}^{n}|f_i(0,u(s))|\right)\right) \tag{70}$$

for $t \ge 0$. Assumption (H3) holds with $k = 1$, $L = \tilde{L} + 1$, $\kappa(s) = s$ for $s \ge 0$ and $R(x,u,d) := f_1(x_1,u) + d_1$ for $x \in C^0([-r,0];\Re^n)$, $(u,d) \in \Re^m \times \Re^n$. Define the matrix $A = \{a_{i,j} : i = 1,..,n, j = 1,...,n\} \in \Re^{n\times n}$ by the relations

$$a_{i,i+1} = 1 \text{, for } i = 1,...,n-1, \text{ and } a_{i,j} = 0 \text{ if otherwise} \tag{71}$$

and the vector

$$c := (1,0,...,0)^T \in \Re^n \tag{72}$$

Since the pair of matrices (A,c) is observable, there exists $K = (K_1,...,K_n)^T \in \Re^n$ so that the matrix $(A + Kc^T)$ is Hurwitz. Using Theorem 2.2, we are in a position to prove the following result.

Theorem 5.1: *There exist constants $\delta, \omega > 0$, $\theta, Q_1, Q_2, Q_3 \ge 1$ such that for every sampling sequence $\{\tau_i\}_{i=0}^{\infty}$ with $\tau_0 = 0$, $\lim(\tau_i) = +\infty$, $0 < \tau_{i+1} - \tau_i \le \delta$ for $i = 0,1,\ldots$, for every $(x_0, z_0) \in C^0([-r,0];\Re^n) \times C^0([-r,0];\Re^n)$, $u \in L^\infty_{loc}(\Re_+;\Re^m)$, $d \in L^\infty_{loc}(\Re_+;\Re^n)$, $\xi \in L^\infty_{loc}(\Re_+;\Re)$, the solution $(x(t), z(t), w(t))$ of (6) with*

$$\dot{z}_1(t) = f_i(z_{1,t}, u(t)) + z_2(t) + \theta K_1(c^T z(t) - w(t))$$
$$\vdots$$
$$\dot{z}_{n-1}(t) = f_i(z_{1,t}, ..., z_{n-1,t}, u(t)) + z_n(t) + \theta^{n-1} K_{n-1}(c^T z(t) - w(t))$$
$$\dot{z}_n(t) = f_n(z_{1,t}, ..., z_{n,t}, u(t)) + \theta^n K_n(c^T z(t) - w(t))$$
$$\dot{w}(t) = f_1(z_{1,t}, u(t)) + z_2(t)\,,\ t \in [\tau_i, \tau_{i+1})$$
$$\hat{x}_t = z_t \tag{73}$$
$$w(\tau_i) = x_1(\tau_i) + \xi(\tau_i) \tag{74}$$

initial condition (x_0, z_0)*, corresponding to inputs* u, d, ξ*, exists for all* $t \geq 0$ *and satisfies the following estimate*

$$\|\hat{x}_t - x_t\| \leq \exp(-\omega t) Q_1\left(\|x_0\| + \|z_0\|\right)$$
$$+ Q_2 \sup_{0\leq s\leq t}\left(|\xi(s)|\exp(-\omega(t-s))\right) + Q_3 \sup_{0\leq s\leq t}\left(|d(s)|\right) \tag{75}$$

Clearly, the construction of the observer (73), (74) consists in combining a high-gain observer with an inter-sample predictor. The proof of Theorem 5.1, presented in Section 6, makes use of Lyapunov analysis together and small-gain arguments. The parameter $\theta \geq 1$ depends on the maximum delay $r > 0$ (see inequality (122) in the proof of Theorem 5.1, which gives a formula for $\theta \geq 1$). Thus, the design of a sampled-data observer for system (6) is straightforward:

1) Select $K = (K_1, ..., K_n)^T \in \Re^n$ so that the matrix $(A + Kc^T)$ is Hurwitz.
2) Select $\theta \geq 1$ sufficiently large (so that inequality (122) in the proof of Theorem 5.1 holds).

The proof of Theorem 5.1 in conjunction with Theorem 2.2 allow us to prescribe the convergence rate $\omega > 0$. Recent works have studied this particular problem in the finite-dimensional case (see for instance [14]). Indeed, the proof of Theorem 5.1 shows that (11) holds with $\sigma = \theta\mu/(4|P|)$, where $\mu > 0$ is a constant and $P \in \Re^{n\times n}$ is a positive definite matrix that satisfies $P(A + Kc^T) + (A^T + cK^T)P + 2\mu I \leq 0$. By picking $\theta \geq 4\mu^{-1}\omega|P|$, where $\omega > 0$ is the prescribed rate of convergence, we can use (13) (with $L = \tilde{L} + 1$ and $\gamma > 0$ determined by estimate (125) in the proof of Theorem 5.1) to find the upper diameter $\delta > 0$ of the sampling schedule for which (75) holds with the prescribed convergence rate $\omega > 0$.

To numerically evaluate the relevance of the proposed sampled-data design, we consider the system

$$\dot{x}_1(t) = x_2(t) \quad ; \quad \dot{x}_2(t) = 2|x_1(t-r)| + u(t)$$
$$y(t) = x_1(t) \tag{76}$$

with

$$r = 0.1\,,\ u(t) = 0.5\sin(2t)\,,$$
$$x_1(t) = 0.5 \ \text{ for } t \in [-0.1, 0]\,,\ x_2(0) = -0.5$$

We next compare the sampled-data observer with inter-sample output predictor

$$\dot{z}_1(t) = z_2(t) - 3\left(z_1(t) - w(t)\right)$$
$$\dot{z}_2(t) = 2|z_1(t-r)| + u(t) - 9\left(z_1(t) - w(t)\right)$$
$$\dot{w}(t) = z_2(t) \quad ,\quad t \in [\tau_i, \tau_{i+1}), i = 0, 1, 2, ... \tag{77}$$
$$w(\tau_i) = y(\tau_i)\,,\ i = 0, 1, 2, ...$$
$$\hat{x}(t) = z(t)$$

with the ZOH sampled-data observer that uses the most recent measurement, i.e., the observer

$$\dot{z}_1(t) = z_2(t) - 3\left(z_1(t) - w(t)\right)$$
$$\dot{z}_2(t) = 2|z_1(t-r)| + u(t) - 9\left(z_1(t) - w(t)\right)$$
$$\dot{w}(t) = 0 \quad ,\quad t \in [\tau_i, \tau_{i+1}), i = 0, 1, 2, ... \tag{78}$$
$$w(\tau_i) = y(\tau_i)\,,\ i = 0, 1, 2, ...$$
$$\hat{x}(t) = z(t)$$

Both sampled-data observers will be compared to the continuous-time observer

$$\dot{z}_1(t) = z_2(t) - 3\left(z_1(t) - y(t)\right)$$
$$\dot{z}_2(t) = 2|z_1(t-r)| + u(t) - 9\left(z_1(t) - y(t)\right) \tag{79}$$
$$\hat{x}(t) = z(t)$$

All observers are initiated with $z_1(t) = 0$ for $t \in [-0.1, 0]$, $z_2(0) = 0$ and operated with the sampling schedule was $\tau_i = i/10$ for $i = 0, 1, 2, ...$. The results of comparison are shown in Fig. 2. Clearly, the ZOH sampled-data observer fails to estimate the state, unlike the observer involving the inter-sample predictor. The latter features a behavior that is comparable to the continuous-time system.

6. Proofs

In this section we provide the proofs of all main results.

Proof of Theorem 2.2: Let $i \geq 0$ be an integer for which z_{τ_i} exists. We show first that z_t exists for all $t \in [\tau_i, \tau_{i+1}]$. Due to the regularity assumptions of Definition 2.1 and due to (12) there exists $t_{\max} > 0$ such that the solution of

$$\dot{z}(t) = F(z_t, w(t), u(t))$$
$$\dot{w}(t) = R(\Psi(z_t), u(t), 0) \tag{80}$$

is defined on $t \in [\tau_i, t_{\max})$, where $t_{\max} > \tau_i$ is the maximal existence time of the solution of (80). If $t_{\max} > \tau_{i+1}$ then there is nothing to show. We next focus on the case $t_{\max} \leq \tau_{i+1}$. Define:

$$v(t) := w(t) - h(x_t) \tag{81}$$

The component of the solution z_t of (80) is a solution of

$$\dot{z}(t) = F(z_t, h(x_t) + v(t), u(t)) \tag{82}$$

for $t \in [0, t_{\max})$. Therefore, by virtue of (11) and since $\omega \leq \sigma$, the following estimate holds for all $t \in [0, t_{\max})$:

$$\|\hat{x}_t - x_t\| \leq \exp(-\omega t) a\left(\|x_0\| + \|z_0\|\right)$$
$$+ \gamma \sup_{0\leq s\leq t}\left(|v(s)|\exp(-\omega(t-s))\right) + \sup_{0\leq s\leq t}\left(g(|d(s)|)\right) \tag{83}$$

Using (7), (8), (14) and (15), we get for $t \in [0, t_{\max})$:

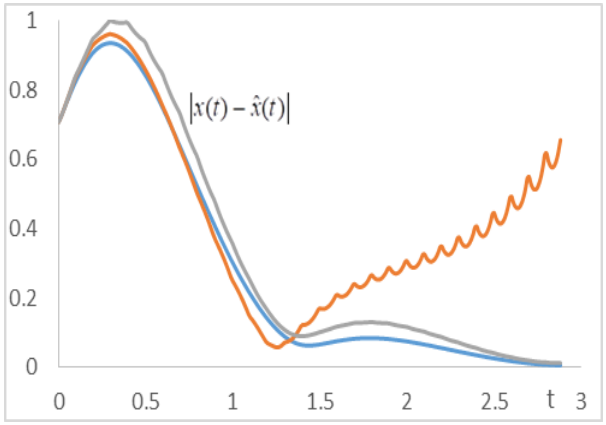

Fig. 2: The error $|x(t)-\hat{x}(t)|$ for observer (77) with the inter-sample predictor (grey line), the ZOH observer (78) (red line) and the continuous observer (79) (blue line).

$$|v(t)| \le |\xi(q(t))| + L\int_{q(t)}^{t} \|\hat{x}_s - x_s\| ds + (t-q(t)) \sup_{q(t)\le s\le t} \left(\kappa\left(|d(s)|\right)\right) \quad (84)$$

where

$$q(t) = \max\{\tau_i : \tau_i \le t\}. \quad (85)$$

Using (84) and the fact that $t-q(t)\le\delta$ (a consequence of definition (85) and the fact that $0<\tau_{i+1}-\tau_i\le\delta$ for $i=0,1,\dots$), we obtain for all $t\in[0,t_{\max})$:

$$\begin{aligned}|v(t)|\exp(\omega t) &- \delta\exp(\omega t)\sup_{0\le s\le t}\left(\kappa\left(|d(s)|\right)\right)\\ &\le |\xi(q(t))|\exp(\omega q(t))\exp(\omega\delta)\\ &+L\frac{\exp(\omega\delta)-1}{\omega}\sup_{q(t)\le s\le t}\left(\|\hat{x}_s-x_s\|\exp(\omega s)\right)\end{aligned}$$

which implies the following estimate for all $t\in[0,t_{\max})$:

$$\begin{aligned}|v(t)|\exp(\omega t) &\le \exp(\sigma\delta)\sup_{0\le s\le t}\left(|\xi(s)|\exp(\omega s)\right)\\ &+L\frac{\exp(\omega\delta)-1}{\omega}\sup_{0\le s\le t}\left(\|\hat{x}_s-x_s\|\exp(\omega s)\right)\\ &+\delta\exp(\omega t)\sup_{0\le s\le t}\left(\kappa\left(|d(s)|\right)\right) \quad (86)\end{aligned}$$

Combining estimates (83), (86) we get for $t\in[0,t_{\max})$:

$$\begin{aligned}\|\hat{x}_t-x_t\|\exp(\omega t) &\le a\left(\|x_0\|+\|z_0\|\right)\\ &+\gamma\exp(\omega\delta)\sup_{0\le s\le t}\left(|\xi(s)|\exp(\omega s)\right)\\ &+\gamma L\frac{\exp(\omega\delta)-1}{\omega}\sup_{0\le s\le t}\left(\|\hat{x}_s-x_s\|\exp(\omega s)\right)\\ &+\exp(\omega t)\left(\gamma\delta\sup_{0\le s\le t}\left(\kappa\left(|d(s)|\right)\right)+\sup_{0\le s\le t}\left(g(|d(s)|)\right)\right) \quad (87)\end{aligned}$$

Since $B:=\gamma L\left(\exp(\sigma\delta)-1\right)/\sigma<1$, it follows from (87) and definition $\tilde{g}(s):=(1-B)^{-1}(g(s)+\gamma\delta\kappa(s))$ that (16) holds for $t\in[0,t_{\max})$. Estimates (86) and (16) show that $|v(t)|$ is bounded on $[0,t_{\max})$. Extending $v(t)$ on the interval $[t_{\max},+\infty)$ in a way that $|v(t)|$ is bounded on $\Re_+$, it follows that the solution of (82) exists for $t\in[\tau_i,t_{\max}]$, which shows that $t_{\max}>\tau_i$ is not the maximal existence time of the solution of (80), a contradiction. Therefore, the case $t_{\max}\le\tau_{i+1}$ cannot arise and thus, $t_{\max}>\tau_i$. By induction and since $\lim(\tau_i)=+\infty$, it follows that z_t exists for all $t\ge0$. Moreover, (16) holds for all $t\ge0$. ◁

Proof of Corollary 2.4: Assumptions (H1), (H2), (H3), (H4) guarantee that (1) with (14), (15) and (19) is forward complete, i.e. its unique solution exists for $t\ge0$, for arbitrary initial conditions, inputs and arbitrary sequences $\{\tau_i\}_{i=0}^{\infty}$ with $\tau_0=0$, $\lim(\tau_i)=+\infty$, $0<\tau_{i+1}-\tau_i\le\delta$ for $i=0,1,\dots$, where $\delta>0$ is sufficiently small. Using a modification of the proof of Theorem 5 in [36] (applying a small-gain analysis instead of Halanay's inequality), (H4) guarantees the existence of constants $\tilde{\delta},\bar{\gamma},\bar{M},\omega>0$ and a function $\bar{g}\in K$ such that for every sequence $\{\tau_i\}_{i=0}^{\infty}$ with $\tau_0=0$, $\lim(\tau_i)=+\infty$, $0<\tau_{i+1}-\tau_i\le\tilde{\delta}$ for $i=0,1,\dots$, for every $x_0\in C^0([-r,0];\Re^n)$ and $\tilde{d}\in L^\infty_{loc}(\Re_+;\Re^m)$, $d\in L^\infty_{loc}(\Re_+;D)$, the solution $x(t)$ of

$$\begin{aligned}\dot{x}(t) &= f(x_t,u(t),d(t))\\ u(t) &= \tilde{k}\left(x_{\tau_i}\right)+\tilde{d}\left(\tau_i\right), t\in[\tau_i,\tau_{i+1}) \quad (88)\end{aligned}$$

with initial condition x_0, corresponding to inputs $\tilde{d},d$, exists for all $t\ge0$ and satisfies the following estimate

$$\begin{aligned}\|x_t\| &\le \exp(-\omega t)\bar{M}\|x_0\|\\ &+\bar{\gamma}\sup_{0\le s\le t}\left(\left|\tilde{d}(s)\right|\exp(-\omega(t-s))\right)+\sup_{0\le s\le t}\left(\bar{g}(|d(s)|)\right) \quad (89)\end{aligned}$$

Notice that the component $x(t)$ of the solution of system (1) with (14), (15) and (19) coincides with the solution of (88) with the same initial condition, same input d and input $\tilde{d}\in L^\infty_{loc}(\Re_+;\Re^m)$ defined by

$$\tilde{d}(t)=\tilde{k}\left(\hat{x}_{\tau_i}\right)-\tilde{k}\left(x_{\tau_i}\right), \text{ for } t\in[\tau_i,\tau_{i+1}) \text{ and } i\ge0 \quad (90)$$

It follows from (H4) and (90) that the inequality

$$\left|\tilde{d}(t)\right|\le\bar{L}\left\|\hat{x}_{\tau_i}-x_{\tau_i}\right\|, \text{ for } t\in[\tau_i,\tau_{i+1}) \text{ and } i\ge0 \quad (91)$$

holds. Selecting $\bar{\delta}>0$ sufficiently small so that (16) and (89) hold for sequences $\{\tau_i\}_{i=0}^{\infty}$ with $\tau_0=0$, $\lim(\tau_i)=+\infty$, $0<\tau_{i+1}-\tau_i\le\bar{\delta}$, for $i=0,1,\dots$, we get from (16) and (91) that the following estimate holds for $t\ge0$:

$$\begin{aligned}\left|\tilde{d}(t)\right|\exp(\omega t) &\le \bar{L}\exp(\omega t)\sup_{0\le s\le t}\left(\tilde{g}(|d(s)|)\right)\\ &+\bar{L}(1-B)^{-1}\gamma\exp(2\omega\bar{\delta})\sup_{0\le s\le t}\left(|\xi(s)|\exp(\omega s)\right)\\ &+\bar{L}(1-B)^{-1}\exp(\omega\bar{\delta})a\left(\|x_0\|+\|z_0\|\right) \quad (92)\end{aligned}$$

where $\tilde{g}(s) := (1-B)^{-1}(g(s)+\gamma\bar{\delta}\kappa(s))$ and $B := \gamma L\left(\exp(\omega\bar{\delta})-1\right)/\omega < 1$. Combining (16), (89), (92) and the inequality $\|\hat{x}_t\| \le \|x_t\| + \|\hat{x}_t - x_t\|$, we obtain (20) for appropriate $\hat{\gamma} > 0$ and $\hat{g}, \hat{a} \in K$. If $a \in K_\infty$ (the function involved in (11), (16), (92)) is linear then $\hat{a}$ is linear. ◁

Proof of Theorem 3.2: Under (A1), (A2), (A3), the solution $v \in C^1(\Re_+ \times [0,1])$ and $\bar{x} \in C^1(\Re_+; \Re^{\bar{n}})$ of system (2), (3), (4), (5) with initial conditions $v_0 \in C^1([0,1])$, $\bar{x}_0 \in \Re^{\bar{n}}$ with $v_0(0) = 0$ satisfying (26), corresponding $u \in C^0\left(\Re_+; \Re^m\right)$, is expressed by (29), where $x(t) = (\bar{x}(t), \eta(t), \lambda(t))$ is the solution of (1) with $D = \{0\}$, f, h being defined by (37), (38), (39), (40), (41), initial condition provided by (A2) and corresponding to the same input u. It follows that Theorem 3.2 is a consequence of Theorem 2.2, Lemma 3.1 and the fact that there exists $G > 0$ such that the initial condition of system (1) satisfies $\|x\| \le G\left(\|v_0\|_\infty + \|v_0'\|_\infty + |\bar{x}_0|\right)$ (recall assumption (A2) and (36)). Indeed, using (29), (31), (36), (47) and the fact that there exists $L > 0$ such that inequality $\max_{0 \le z \le 1}\left(|g(z,x) - g(z,y)|\right) \le L|x-y|$ holds (recall (A1)), we obtain the existence of $B > 0$ (that depends on $c > 0$, a, b_i ($i = 1,...,N_1$), γ_i, β_i ($i = 1,...,N_2$) and g) such that:

$$\|\hat{v}[t] - v[t]\|_\infty \le B\|\hat{x}_t - x_t\|, \text{ for } t \ge 0 \qquad (93)$$

Inequality (48) is obtained by combining (16), (93) and using the fact the initial condition of system (1) satisfies the inequality $\|x\| \le G\left(\|v_0\|_\infty + \|v_0'\|_\infty + |\bar{x}_0|\right)$. ◁

Proof of Corollary 3.4: Under (A1), (A2), (A3), (A4), the solution of (2), (3), (4), (14), (46), (47) and (52) is expressed by (29), (32), (33) using the solution $(x(t), z(t), w(t))$ of (1), (14), (15) and (19) with $D = \{0\}$, where f, h are defined by (37), (38), (39), (40), (41), with initial condition provided by assumption (A2). Following the proof of Theorem 3.2, we conclude that Corollary 3.4 is a consequence of Corollary 2.4 and the fact that there exists $G > 0$ such that the initial condition of (1) satisfies $\|x\| \le G\left(\|v_0\|_\infty + \|v_0'\|_\infty + |x_0|\right)$ (recall assumption (A2)). ◁

Proof of Theorem 4.1: By virtue of Theorem 3.2, it suffices to prove that for appropriate selection of the constants $k_1, k_2 \in \Re$, the system

$$\begin{aligned}
\dot{z}_1(t) &= \zeta z_2(t) - \zeta z_2(t-r)\exp(-\zeta r) - \zeta z_1(t) - k_1\left(z_1(t) - y(t)\right)\\
\dot{z}_2(t) &= \Theta(z_2(t)) - (\mu+1)z_2(t) + \mu\zeta\int_0^1\int_{t-rl}^{t} z_2(s)\exp\left(-\zeta(t-s)\right)dsdl\\
&\quad + u(t) - k_2\left(z_1(t) - y(t)\right)\\
\hat{x}_t &= z_{2,t}
\end{aligned} \qquad (94)$$

is a REO for system (59). We consider the functional

$$\begin{aligned}
V(t) &= Q\int_{t-r}^{t}\left(z_2(s) - x_2(s)\right)^2\exp\left(-\zeta(t-s)\right)ds\\
&\quad + R\left(z_1(t) - x_1(t)\right)^2/2 + \left(z_2(t) - x_2(t) - bz_1(t) + bx_1(t)\right)^2/2
\end{aligned} \qquad (95)$$

where $R, b, Q > 0$ are constants to be selected. For every $\xi \in L^\infty_{loc}(\Re_+; \Re)$, the time derivative of $V(t)$ along the trajectories of (94), (59) with $y(t) = x_1(t) + \xi(t)$ is

$$\begin{aligned}
\dot{V}(t) &= -\left((k_1+\zeta)R - R\zeta b - Qb^2\right)E_1^2(t)\\
&\quad -(\mu + 1 + b\zeta - Q)E_2^2(t) + \left(\Theta(z_2(t)) - \Theta(x_2(t))\right)E_2(t)\\
&\quad -\zeta Q\int_{t-r}^{t}\left(z_2(s) - x_2(s)\right)^2\exp\left(-\zeta(t-s)\right)ds\\
&\quad +\left(R\zeta + 2bQ + b(k_1+\zeta) - k_2 - (\mu+1+b\zeta)b\right)E_1(t)E_2(t)\\
&\quad -Q\left(z_2(t-r) - x_2(t-r)\right)^2\exp\left(-\zeta r\right)\\
&\quad +\left(bE_2(t) - RE_1(t)\right)\zeta\exp(-\zeta r)\left(z_2(t-r) - x_2(t-r)\right)\\
&\quad +\left(\left(k_2 - bk_1\right)E_2(t) + Rk_1E_1(t)\right)\xi(t)\\
&\quad +\mu\zeta E_2(t)\int_0^1\int_{t-rl}^{t}\exp\left(-\zeta(t-s)\right)\left(z_2(s) - x_2(s)\right)dsdl
\end{aligned} \qquad (96)$$

for $t \ge 0$ a.e., where

$$E_1(t) := z_1(t) - x_1(t) \qquad (97)$$

$$E_2(t) := z_2(t) - x_2(t) - bE_1(t) \qquad (98)$$

Since $\theta : \Re \to \Re$ is a globally Lipschitz function, there exists a constant $\Phi > 0$ such that

$$\left|\Theta(z_2(t)) - \Theta(x_2(t))\right| \le \Phi\left|E_2(t)\right| + \Phi b\left|E_1(t)\right| \qquad (99)$$

In the above inequality we have used the triangle inequality and (97), (98). By using the inequalities

$$\Phi b\left|E_1(t)\right|\left|E_2(t)\right| \le \Phi E_2^2(t) + \Phi b^2E_1^2(t)/4$$

$$2E_2(t)\left(z_2(t-r) - x(t-r)\right) \le E_2^2(t) + \left(z_2(t-r) - x(t-r)\right)^2$$

$$2\left|E_1(t)\right|\left|z_2(t-r) - x(t-r)\right| \le E_1^2(t) + \left(z_2(t-r) - x(t-r)\right)^2$$

$$\left|k_2 - bk_1\right|\left|E_2(t)\right|\left|\xi(t)\right| \le E_2^2(t) + \left|k_2 - bk_1\right|^2\xi^2(t)/4$$

$$\left|k_1\right|\left|E_1(t)\right|\left|\xi(t)\right| \le E_1^2(t) + k_1^2\xi^2(t)/4$$

we obtain from (96), (99) the inequality for $t \ge 0$ a.e.:

$$\begin{aligned}
\dot{V}(t) &\le -\left(\mu + b\zeta - Q - 2\Phi - b\zeta\exp(-\zeta r)/2\right)E_2^2(t)\\
&\quad -\left((k_1+\zeta)R - R\zeta b - Qb^2 - \frac{\Phi b^2}{4} - \frac{1}{2}R\zeta\exp(-\zeta r) - R\right)E_1^2(t)\\
&\quad +\left(R\zeta + 2bQ + b(k_1+\zeta) - k_2 - (\mu+1+b\zeta)b\right)E_1(t)E_2(t)\\
&\quad +\left((b+R)\zeta/2 - Q\right)\exp(-\zeta r)\left(z_2(t-r) - x(t-r)\right)^2\\
&\quad -\zeta Q\int_{t-r}^{t}\left(z_2(s) - x(s)\right)^2\exp\left(-\zeta(t-s)\right)ds\\
&\quad +\mu\zeta E_2(t)\int_0^1\int_{t-rl}^{t}\exp\left(-\zeta(t-s)\right)\left(z_2(s) - x_2(s)\right)dsdl\\
&\quad +\left(\left|k_2 - bk_1\right|^2 + Rk_1^2\right)\xi^2(t)/4
\end{aligned} \qquad (100)$$

Using the Cauchy-Schwarz inequality (twice), we get for the integral appearing on the right hand side of (100):

$$\left|\int_0^1\int_{t-rl}^{t}\exp\big(-\zeta(t-s)\big)\big(z_2(s)-x_2(s)\big)\,ds\,dl\right|$$
$$\le\left(\int_0^1\left(\int_{t-rl}^{t}\exp\big(-\zeta(t-s)\big)\big|z_2(s)-x_2(s)\big|ds\right)^2 dl\right)^{1/2}$$
$$\le\left(\int_{t-r}^{t}\exp\big(-\zeta(t-s)\big)ds\int_{t-r}^{t}\exp\big(-\zeta(t-s)\big)\big|z_2(s)-x_2(s)\big|^2 ds\right)^{1/2}$$
$$\le\left(\frac{1-\exp(-\zeta r)}{\zeta}\int_{t-r}^{t}\exp\big(-\zeta(t-s)\big)\big|z_2(s)-x_2(s)\big|^2 ds\right)^{1/2}$$

Using the above inequality in conjunction with estimate (100), we obtain the following inequality for $t\ge 0$ a.e.:

$$\dot V(t)\le-\big(\mu+b\zeta-Q-2\Phi-b\zeta\exp(-\zeta r)/2\big)E_2^2(t)$$
$$-\left((k_1+\zeta)R-R\zeta b-Qb^2-\Phi\frac{b^2}{4}-\frac{1}{2}R\zeta\exp(-\zeta r)-R\right)E_1^2(t)$$
$$+\big(R\zeta+2bQ+b(k_1+\zeta)-k_2-(\mu+1+b\zeta)b\big)E_1(t)E_2(t)$$
$$+\big((b+R)\zeta/2-Q\big)\exp(-\zeta r)\big(z_2(t-r)-x_2(t-r)\big)^2$$
$$-\zeta Q\int_{t-r}^{t}\big(z_2(s)-x_2(s)\big)^2\exp\big(-\zeta(t-s)\big)ds$$
$$+\mu\zeta\big|E_2(t)\big|\sqrt{\frac{1-\exp(-\zeta r)}{\zeta}\int_{t-r}^{t}\big(z_2(s)-x_2(s)\big)^2\exp\big(-\zeta(t-s)\big)ds}$$
$$+\left(\big|k_2-bk_1\big|^2+Rk_1^2\right)\xi^2(t)/4 \qquad (101)$$

Finally, using the inequality

$$\big|E_2(t)\big|\left(\frac{1-\exp(-\zeta r)}{\zeta}\int_{t-r}^{t}\big(z_2(s)-x_2(s)\big)^2\exp\big(-\zeta(t-s)\big)ds\right)^{1/2}$$
$$\le\frac{E_2^2(t)}{2}+\frac{1-\exp(-\zeta r)}{2\zeta}\int_{t-r}^{t}\big(z_2(s)-x_2(s)\big)^2\exp\big(-\zeta(t-s)\big)ds$$

in conjunction with estimate (101), we obtain the following inequality for $t\ge 0$ a.e.:

$$\dot V(t)\le-\big(\mu+b\zeta-Q-2\Phi-\zeta\big(b\exp(-\zeta r)+\mu\big)/2\big)E_2^2(t)$$
$$-\left((k_1+\zeta)R-R\zeta b-Qb^2-\Phi\frac{b^2}{4}-\frac{1}{2}R\zeta\exp(-\zeta r)-R\right)E_1^2(t)$$
$$+\big(R\zeta+2bQ+b(k_1+\zeta)-k_2-(\mu+1+b\zeta)b\big)E_1(t)E_2(t)$$
$$-\big(Q-(b+R)\zeta/2\big)\exp(-\zeta r)\big(z_2(t-r)-x_2(t-r)\big)^2$$
$$-\left(\zeta Q-\mu\frac{1-\exp(-\zeta r)}{2}\right)\int_{t-r}^{t}\big(z_2(s)-x_2(s)\big)^2\exp\big(-\zeta(t-s)\big)ds$$
$$+\left(\big|k_2-bk_1\big|^2+Rk_1^2\right)\xi^2(t)/4 \qquad (102)$$

By selecting

$$R=\frac{2\mu(1-\exp(-\zeta r))}{\zeta^2},\quad b=\frac{4\Phi+(\mu+R+1)\zeta}{\zeta\big(1-\exp(-\zeta r)\big)} \qquad (103)$$

$$Q=(b+R)\zeta/2,$$
$$k_2=R\zeta+b(b+R)\zeta+b(k_1+\zeta)-(\mu+1+b\zeta)b \qquad (104)$$
$$k_1=\zeta b+\frac{1}{2R}(b+R)b^2\zeta+\Phi\frac{b^2}{4R}+\frac{1}{2}\zeta\exp(-\zeta r)+1 \qquad (105)$$

we obtain from (102) and (95) the following differential inequality for $t\ge 0$ a.e.:

$$\dot V(t)\le-\zeta V(t)/2+\left(\big|k_2-bk_1\big|^2+Rk_1^2\right)\xi^2(t)/4 \qquad (106)$$

Applying Lemma 2.12 in [23] in (106), we get for $t\ge 0$:

$$V(t)\le\exp\big(-\zeta t/2\big)V(0)+$$
$$\frac{1}{4}\left(\big|k_2-bk_1\big|^2+Rk_1^2\right)\int_0^t\exp\big(-\zeta(t-s)/2\big)\xi^2(s)ds \qquad (107)$$

The quadratic form $S(x)=Rx_1^2/2+\big(x_2-bx_1\big)^2/2$ on $\Re^2$ is positive definite. Thus, there exists $K_1>0$ with $K_1|x|^2\le S(x)$ for all $x\in\Re^2$. Using this fact, (95) and bounding the integral in (107) for any $\sigma\in\big(0,\zeta/4\big)$

$$\int_0^t\exp\big(-\zeta(t-s)/2\big)\big|\xi(s)\big|^2 ds\le\frac{2\exp(-2\sigma t)}{\zeta-4\sigma}\sup_{0\le s\le t}\Big(\big|\xi(s)\big|^2\exp(2\sigma s)\Big)$$

we obtain from (107) for all $t\ge 0$:

$$\big|z(t)-x(t)\big|^2\le K_1^{-1}\exp(-2\sigma t)V(0)$$
$$+\frac{\big|k_2-bk_1\big|^2+Rk_1^2}{2K_1(\zeta-4\sigma)}\sup_{0\le s\le t}\Big(\xi^2(s)\exp(-2\sigma(t-s))\Big) \qquad (108)$$

Definition (95) implies that there exists $K_2\ge K_1$ (independent of z_t,x_t) such that $K_2\Big(\|x_t\|^2+\|z_t\|^2\Big)\ge V(t)$. Therefore, we obtain from (108) for $t\ge 0$:

$$\big|z(t)-x(t)\big|\le\sqrt{K_2/K_1}\exp(-\sigma t)\big(\|x_0\|+\|z_0\|\big)$$
$$+\sqrt{\frac{\big|k_2-bk_1\big|^2+Rk_1^2}{2K_1(\zeta-4\sigma)}}\sup_{0\le s\le t}\Big(\big|\xi(s)\big|\exp(-\sigma(t-s))\Big) \qquad (109)$$

Notice that since $K_2\ge K_1$, inequality (109) holds for all $t\ge -r$. Consequently, we obtain from (109) for $t\ge -r$:

$$\sup_{-r\le s\le t}\Big(\big|z(s)-x(s)\big|\exp(\sigma s)\Big)\le\sqrt{K_2/K_1}\big(\|x_0\|+\|z_0\|\big)$$
$$+\sqrt{\frac{\big|k_2-bk_1\big|^2+Rk_1^2}{2K_1(\zeta-4\sigma)}}\sup_{0\le s\le t}\Big(\big|\xi(s)\big|\exp(\sigma s)\Big) \qquad (110)$$

Using the fact that $\sup_{t-r\le s\le t}\Big(\big|z(s)-x(s)\big|\exp\big(\sigma s\big)\Big)\ge\exp\big(\sigma(t-r)\big)\|z_t-x_t\|$, we obtain from (110) for all $t\ge 0$:

$$\|z_t-x_t\|\le\sqrt{K_2/K_1}\exp\big(-\sigma(t-r)\big)\big(\|x_0\|+\|z_0\|\big)+$$
$$\sqrt{\frac{\big|k_2-bk_1\big|^2+Rk_1^2}{2K_1(\zeta-4\sigma)}}\exp(\sigma r)\sup_{0\le s\le t}\Big(\big|\xi(s)\big|\exp(-\sigma(t-s))\Big) \qquad (111)$$

Estimate (111) holds for all $\xi\in L^\infty_{loc}(\Re_+;\Re)$, $\sigma\in\big(0,\zeta/4\big)$ and shows that (94) is a REO for (59). ◁

Proof of Theorem 5.1: By virtue of Theorem 2.2, it suffices to show that the following system

$$\dot{z}_1(t) = f_i(z_{1,t}, u(t)) + z_2(t) + \theta K_1(c^T z(t) - y(t))$$
$$\vdots$$
$$\dot{z}_{n-1}(t) = f_{n-1}(z_{1,t}, ..., z_{n-1t}, u(t)) + z_n(t) + \theta^{n-1} K_{n-1}(c^T z(t) - y(t))$$
$$\dot{z}_n(t) = f_n(z_{1,t}, ..., z_{n,t}, u(t)) + \theta^n K_n(c^T z(t) - y(t))$$
$$\hat{x}_t = z_t \quad (112)$$

is a REO for (6). The proof is based on the Lyapunov function $V(t) := e^T(t)\Delta_\theta^{-1} P \Delta_\theta^{-1} e(t)$, where $e(t) := z(t) - x(t)$, $\Delta_\theta := diag(\theta, \theta^2, ..., \theta^n)$ and $P \in \Re^{n\times n}$ is a positive definite matrix that satisfies $P(A + Kc^T) + (A^T + cK^T)P + 2\mu I \le 0$ for a constant $\mu > 0$. The following inequalities hold:

$$\theta^{-i} \left| f_i(x_{1,t}, ..., x_{i,t}, u(t)) - f_i(z_{1,t}, ..., z_{i,t}, u(t)) \right| \le n\tilde{L} \left\| \varepsilon_t \right\|, \quad \text{for } i = 1, ..., n \text{ and } t \ge 0, \quad (113)$$

where

$$\varepsilon(t) = \Delta_\theta^{-1} e(t) \quad (114)$$

Indeed, inequalities (113) follow from (69), the fact that $\theta \ge 1$, the fact $\sum_{j=1}^{i} \left\| \varepsilon_{j,t} \right\| \le n \left\| \varepsilon_t \right\|$ and definition (114). For every $u \in L^\infty_{loc}(\Re_+; \Re^m)$, $d \in L^\infty_{loc}(\Re_+; \Re^n)$, $\xi \in L^\infty_{loc}(\Re_+; \Re)$ we obtain from (6) and (112) with $y(t) = x_1(t) + \xi(t)$:

$$\dot{e}(t) = (A + \Delta_\theta K c^T) e(t) + g(x_t, e_t, u(t)) - d(t) - \Delta_\theta K \xi(t), \quad \text{for } t \ge 0 \text{ a.e.} \quad (115)$$

where

$$g(x_t, e_t, u(t)) = (f_1(x_{1,t} + e_{1,t}, u(t)) - f_1(x_{1,t}, u(t)), ..., f_n(x_t + e_t, u(t)) - f_n(x_t, u(t)))^T$$

Therefore, using identities $\Delta_\theta^{-1} A = \theta A \Delta_\theta^{-1}$, $c^T = \theta c^T \Delta_\theta^{-1}$ and definition (114), we get for $t \ge 0$ a.e.:

$$\dot{V}(t) \le -2\theta\mu |\varepsilon(t)|^2 + 2|\varepsilon(t)||P| \left| \Delta_\theta^{-1} g(x_t, e_t, u(t)) \right| + 2|\varepsilon(t)||PK||\xi(t)| + 2|\varepsilon(t)| \left| P\Delta_\theta^{-1} \right| |d(t)| \quad (116)$$

Using (113), (116), the fact that $\left| \Delta_\theta^{-1} \right| \le \theta^{-1}$ in conjunction with the inequalities

$$2|\varepsilon(t)||PK||\xi(t)| \le \frac{\theta\mu}{3}|\varepsilon(t)|^2 + \frac{3}{\theta\mu}|PK|^2 |\xi(t)|^2,$$
$$2|\varepsilon(t)| \left| P\Delta_\theta^{-1} \right| |d(t)| \le \frac{\theta\mu}{3}|\varepsilon(t)|^2 + \frac{3}{\theta\mu} \left| P\Delta_\theta^{-1} \right|^2 |d(t)|^2,$$
$$2n\sqrt{n}|\varepsilon(t)||P|\tilde{L}\left\|\varepsilon_t\right\| \le \frac{\theta\mu}{3}|\varepsilon(t)|^2 + \frac{3n^3}{\theta\mu}|P|^2 \tilde{L}^2 \left\|\varepsilon_t\right\|^2,$$

we obtain for $t \ge 0$ a.e.:

$$\dot{V}(t) \le -\theta\mu|\varepsilon(t)|^2 + 3n^3|P|^2\tilde{L}^2\left\|\varepsilon_t\right\|^2/(\theta\mu) + 3|PK|^2|\xi(t)|^2/(\theta\mu) + 3|P|^2|d(t)|^2/(\theta^3\mu) \quad (117)$$

Setting $\varphi := \theta\mu/(2|P|)$ and using the fact that $V(t) \le |P||\varepsilon(t)|^2$, we obtain from (117) for $t \ge 0$ a.e.:

$$\dot{V}(t) \le -2\varphi V(t) + 3n^3|P|^2\tilde{L}^2\left\|\varepsilon_t\right\|^2/(\theta\mu) + 3|PK|^2|\xi(t)|^2/(\theta\mu) + 3|P|^2|d(t)|^2/(\theta^3\mu) \quad (118)$$

Applying Lemma 2.12 in [23] and (118), we get for $t \ge 0$:

$$V(t) \le \exp(-2\varphi t)V(0) + \frac{3n^3}{\theta\mu}|P|^2\tilde{L}^2\int_0^t \exp(-2\varphi(t-s))\left\|\varepsilon_s\right\|^2 ds + \frac{3}{\theta\mu}\int_0^t \exp(-2\varphi(t-s))\left(|PK|^2|\xi(s)|^2 + \frac{|P|^2}{\theta^2}|d(s)|^2\right) ds \quad (119)$$

Since $P \in \Re^{n\times n}$ is positive definite there exists $c_1 > 0$ such that $c_1|x|^2 \le x^T P x$ for all $x \in \Re^n$. Using the fact that $V(t) = \varepsilon^T(t) P \varepsilon(t) \le |P||\varepsilon(t)|^2$ and bounding the three integrals in the right hand side of (119) in the following way (similarly for the other two integrals) for $\sigma \in (0, \varphi)$

$$\int_0^t \exp(-2\varphi(t-s))|\xi(s)|^2 ds \le \frac{\exp(-2\sigma t)}{2(\varphi-\sigma)} \sup_{0\le s\le t}\left(|\xi(s)|^2\exp(2\sigma s)\right)$$

we obtain from (119) for all $t \ge 0$:

$$|\varepsilon(t)|^2\exp(2\sigma t) \le |P||\varepsilon(0)|^2/c_1 + \frac{3n^3|P|^2\tilde{L}^2}{2\theta\mu c_1(\varphi-\sigma)}\sup_{0\le s\le t}\left(\left\|\varepsilon_s\right\|^2\exp(2\sigma s)\right) + \frac{3|PK|^2}{2\theta\mu c_1(\varphi-\sigma)}\sup_{0\le s\le t}\left(|\xi(s)|^2\exp(2\sigma s)\right) + \frac{3|P|^2}{2\theta^3\mu c_1(\varphi-\sigma)}\sup_{0\le s\le t}\left(|d(s)|^2\exp(2\sigma s)\right) \quad (120)$$

Using the fact that $\sup_{t-r\le s\le t}\left(|\varepsilon(s)|^2\exp(2\sigma s)\right) \ge \exp(2\sigma(t-r))\sup_{t-r\le s\le t}\left(|\varepsilon(s)|^2\right)$, we obtain from (120) the following estimate for $t \ge 0$:

$$|\varepsilon(t)|\exp(\sigma t) \le \sqrt{|P|/c_1}\,|\varepsilon(0)| + \sqrt{\frac{3n^3|P|^2\tilde{L}^2\exp(2\sigma r)}{2\theta\mu c_1(\varphi-\sigma)}}\sup_{-r\le s\le t}\left(|\varepsilon(s)|\exp(\sigma s)\right) + \sqrt{\frac{3|PK|^2}{2\theta\mu c_1(\varphi-\sigma)}}\sup_{0\le s\le t}\left(|\xi(s)|\exp(\sigma s)\right) + \sqrt{\frac{3|P|^2}{2\theta^3\mu c_1(\varphi-\sigma)}}\sup_{0\le s\le t}\left(|d(s)|\exp(\sigma s)\right) \quad (121)$$

Selecting $\theta \ge 1$ so large so that

$$3n^3|P|^2\tilde{L}^2\exp(\varphi r)/(\theta\mu c_1\varphi) < 1 \quad (122)$$

and using the fact that

$$\sup_{-r\le s\le t}\left(|\varepsilon(s)|\exp(\sigma s)\right) \le \sup_{-r\le s\le 0}\left(|\varepsilon(s)|\exp(\sigma s)\right) + \sup_{0\le s\le t}\left(|\varepsilon(s)|\exp(\sigma s)\right)$$

we obtain from (121) that, for $\sigma = \varphi/2$ and for all $t \ge 0$:

$$(1-\Omega)\sup_{0\le s\le t}\left(|\varepsilon(s)|\exp(\sigma s)\right)\le\sqrt{|P|/c_1}\,|\varepsilon(0)|+$$
$$\sqrt{\frac{3}{\theta\mu c_1\varphi}}\left(|PK|\sup_{0\le s\le t}\left(|\xi(s)|\exp(\sigma s)\right)+\frac{|P|}{\theta}\sup_{0\le s\le t}\left(|d(s)|\exp(\sigma s)\right)\right)$$
$$+\Omega\sup_{-r\le s\le 0}\left(|\varepsilon(s)|\exp(\sigma s)\right) \qquad (123)$$

where $\Omega:=\sqrt{3n^3|P|^2\tilde{L}^2\exp(\varphi r)/(\theta\mu c_1\varphi)}<1$. Since $|P|\ge c_1$, it follows from (123) for $t\ge -r$:

$$(1-\Omega)\sup_{-r\le s\le t}\left(|\varepsilon(s)|\exp(\sigma s)\right)\le\left(\Omega+\sqrt{|P|/c_1}\right)\|\varepsilon_0\|+$$
$$\sqrt{\frac{3}{\theta\mu c_1\varphi}}\left(|PK|\sup_{0\le s\le t}\left(|\xi(s)|\exp(\sigma s)\right)+\frac{|P|}{\theta}\sup_{0\le s\le t}\left(|d(s)|\exp(\sigma s)\right)\right) \qquad (124)$$

Using the three inequalities $\sup_{t-r\le s\le t}\left(|\varepsilon(s)|\exp(\sigma s)\right)\ge\exp(\sigma(t-r))\|\varepsilon_t\|$, and $|e(t)|\le\theta^n|\varepsilon(t)|$ and $|\varepsilon(t)|\le\theta^{-1}|e(t)|$, we obtain from (124) the following estimate, for all $t\ge 0$:

$$\|e_t\|\le(1-\Omega)^{-1}\left(\Omega+\sqrt{|P|/c_1}\right)\theta^{n-1}\|e_0\|\exp(-\sigma(t-r))$$
$$+(1-\Omega)^{-1}\theta^n\sqrt{\frac{3|PK|^2}{\theta\mu c_1\varphi}}\exp(\sigma r)\sup_{0\le s\le t}\left(|\xi(s)|\exp(-\sigma(t-s))\right)$$
$$+(1-\Omega)^{-1}\theta^{n-1}\sqrt{\frac{3|P|^2}{\theta\mu c_1\varphi}}\exp(\sigma r)\sup_{0\le s\le t}\left(|d(s)|\right) \qquad (125)$$

Since $e_t=\hat{x}_t-x_t$ and $\|e_0\|\le\|x_0\|+\|z_0\|$, inequality (125) shows that (112) is a REO for system (6). ◁

7. Concluding Remarks

The present work studied the problem of designing sampled-data observers and observer-based, sampled-data, output feedback stabilizers for systems with both discrete and distributed, state and output time-delays. The obtained results are applicable to time delay systems of strict-feedback structure, transport PDEs with nonlocal terms, and feedback interconnections of ODEs with a transport PDE. The study constitutes a unified theoretical framework to deal with ODEs, delay systems and PDEs.

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