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# A boundary elements only solution method for 3D micropolar elasticity

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### 11 Abstract

Micropolar elasticity belongs to the class of so-called multi-fields problems. The numerical solution of the associated field equations by the pure boundary element method (BEM) is available only for some 2D geometries. A judicious combination of the local point interpolation method with the pure BEM leads to a pure BEM solution procedure of 3D problems. The effectiveness of the approach is demonstrated on some examples including indentation. A first analysis of the impact of the microstructure on the macroscopic response is presented.

- 18 Keywords: boundary element, local point interpolation, micropolar elasticity, indentation
- 19

# 20 **1- Introduction**

21

22 The concept of micropolar elasticity or Cosserat elasticity was first introduced by the Cosserat 23 brothers Eugène and François, (Cosserat and Cosserat, 1909) at the end of the 18th century, who 24 took into account the microstructure of materials in a quasi-homogeneous description. This 25 description is part of the generalized microcontinuum fields of Eringen ((Eringen, 1999a), Green 26 (Green et al., 1965), Mindlin (Mindlin and Tiersten, 1962) during the sixties. The materials that can 27 be described in the framework of generalized microcontinuum mechanics are numerous and various. As examples one can mention, granular materials such as minerals (Diepolder et al., 28 29 1991) or certain metals (Diebels, 2000), liquid crystals or the hard tissues that make up the body 30 of living beings (Park and Lakes, 1986).

- 31 At the present time the micropolar media are applied to the description of different media. In
- 32 the field of electromagnetism one can mention the work of (Maugin, 1988) (Eringen, 1990). The
- 33 model is also applied to porous and granular mediae.g. (Lakes, 1986) (Ehlers, 1997) (Diebels, 2002)

34 (Goddard, 2008) and even in the field of living tissues like bones e.g. (Park and Lakes, 1986)
35 (Ramezani et al., 2009) (Altenbach and Eremeyev, 2009).

36 In micropolar elasticity, the field equations associated with a given problem allow analytical 37 solutions only for "simple" geometries and stimuli. Generally, we will look for a numerical 38 solution. Different numerical methods have been developed to deal with solid mechanic 39 problems. The most common is the popular and powerful finite element method (FEM) which 40 has been applied to solve micropolar elasticity problems, mainly in the 2D context (Li and Xie, 41 2004; Ramezani et al., 2008). The development of special 3D finite element in this context has been recently been presented by Grbcic (Grbčić et al., 2018). As it is well known, in the case of 42 43 linear isotropic classical elasticity problems, the boundary element method (BEM) is an 44 alternative to the FEM. The main advantage of this method is the reduction by one of the dimension of the problem. It has already been successful for the solution of 2D micropolar 45 elasticity problem. Sládek brothers derived the fundamental solutions and the associated 46 traction for 3D isotropic micropolar elasticity problems (Sládek and Sládek, 1983). For 47 48 anisotropic solids, the BEM could still be applied, but its main advantage which is the reduction 49 of the problem dimension by one would be lost because the integral formulation will contain domain integrals. In recent years, a large number of researchers have invested in the 50 51 development of the so-called meshless or meshfree methods. Among the various meshless 52 approaches, the local point interpolation method is highly appealing on account of how simple 53 it is to implement. This approach falls in accuracy in the presence of Neumann boundary 54 conditions, which are almost an inevitability when solving solid mechanic problems. Liu et al. (Liu and Gu, 2001) have suggested a way to circumvent this difficulty by adopting the "weak-55 strong-form local point interpolation" method. Recently, Kouitat (Kouitat Njiwa, 2011a) 56 57 introduced a novel numerical approach which combines the bests of the BEM and the local point interpolation method. The method called the LPI-BEM (Local Point Interpolation-Boundary 58 59 Element Method) has already proven effective for the solution of various problems (e.g. 60 (Schwartz et al., 2012; Thurieau et al., 2012, 2014)

This paper will discuss the effectiveness of the LPI-BEM method for solving 3D isotropic micropolar elasticity problems. The extension of the approach to deal with nonhomogeneous and/or anisotropic solids is straightforward. At first we will discuss the method and then we will validate this description by comparing the outputs of the specifically developed tool to literature results. Finally, the response of micropolar elastic solid to indentation load will be analyzed.

67

## 68 **2- Governing Equations and method of solution**

- 69
- 70 2-1. Governing equations

Following Eringen (Eringen, 1999a), in the theory of micropolar medium occupying the domain  $\Omega$  with boundary  $\Gamma$ , the material point x is attached to a triad of directors that can rotate. Thus, the material point possesses six degrees of freedom: three degrees for the traditional translation (displacement) vector and three degrees for the micro-rotation vector. The field equations of such a medium under quasi-static evolution without external body load are: (lesan and Pompei,
1995), (Eringen, 1999a; Park and Lakes, 1986):

$$\sigma_{ji,j}(x) = 0 \tag{1}$$

$$m_{ji,j}(x) + \epsilon_{ijk}\sigma_{jk}(x) = 0 \tag{2}$$

77

In these equations,  $\sigma_{ij}(x)$  represents the stress tensor and  $m_{ij}(x)$  the moment stress tensor also called coupled-stress tensor.

80 With  $n_j$  as the outward normal vector on the boundary, the macro traction and the micro-torque 81 acting at a regular point on the boundary are given respectively by:

$$t_i = \sigma_{ji} n_j , \qquad m_i = m_{ji} n_j \tag{3}$$

82

85

In the case of homogeneous centro-symmetric linear isotropic material, the constitutiverelations are well documented and read:

$$\sigma_{ij} = \lambda \,\varepsilon_{rr} \delta_{ij} + 2(\mu + \kappa) \varepsilon_{ij} - \kappa \,u_{i,j} + \kappa \,\epsilon_{ijk} \varphi_k \tag{4}$$

$$m_{ij} = \alpha \,\omega_{rr} \delta_{ij} + (\beta + \gamma) \omega_{ij} + (\beta - \gamma) (\varphi_{i,j} - \varphi_{j,i})/2 \tag{5}$$

86 The fields  $\varepsilon_{ij}$  and  $\omega_{ij}$  in these relations are expressed in terms of displacement (*u*)and 87 microrotation ( $\varphi$ ) gradients as:

88 
$$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$$
  
89  $\omega_{ij} = (\varphi_{i,j} + \varphi_{j,i})/2$ 

90 Parameters  $\lambda$  and  $\mu$  are the Lamé constants and  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\kappa$  are the micropolar constants.

91 The consideration of stability of the thermodynamic state, which in the static case requires that

92 the strain energy is positive, has led to the following constraints on the parameters ((Eringen,

93 1999a)page 112)

94

$3\lambda + 2\mu + \kappa \ge 0$	$2\mu + \kappa \ge 0$	$\kappa = 0$	$3\alpha + \beta + \gamma \ge 0$
	$\beta + \gamma = 0$	$\gamma - \beta \ge 0$	

95

#### 96 2-2 Solution Method

97 We aim to solve the problem of the static deformation of a micropolar solid by a boundary 98 element based method. Let us remind the reader that, in the case of linear problems with well-99 established analytical fundamental solutions, the boundary element method has proven highly 100 efficient (see e.g.(Balaš et al., 1989; Bonnet, 1999; Brebbia and Dominguez, 1996). In the case of 101 the field equations we are dealing with, to our knowledge, no such fundamental solution exists 102 for the three dimensional case. Let us point out that fundamental solution and associated 103 traction have been derived for the plane strain case (e.g. (Atroshchenko et al., 2017; Liang and Huang, 1996; Shmoylova et al., 2007) ). The 3D-problem can still be solved by the boundary 104 105 element method, which is no longer pure since the formulation involves domain integrals and 106 the main appeal of the approach (reduction of the problem dimension by one) is tarnished. This 107 shortcoming can be ruled out by adopting one of the strategies of conversion of domain integrals 108 into boundary ones such as the dual reciprocity method (DRM)((Nardini and Brebbia, 1983) or 109 radial integration method (RIM)((Gao, 2002). In recent years, a large number of researchers 110 have invested in the development of so-called meshless or meshfree methods. Among the 111 various meshless approaches, the local point interpolation method is highly appealing of how 112 simple it is to implement. This approach falls in accuracy in the presence of Neumann type 113 boundary conditions, which are almost an inevitability when solving solid mechanic problems. 114 (LIU and GU, 2001) have suggested a way to circumvent this difficulty by adopting the "weak-115 strong-form local point interpolation" method. In a recent paper, (Kouitat Njiwa, 2011b) proposed a novel strategy that combines the best elements of both the conventional BEM and local point 116 117 interpolation methods. This LPI-BEM approach has proved efficient in the context of anisotropic 118 elasticity (Kouitat Njiwa, 2011b), piezoelectricity solids (Thurieau et al., 2012), multi-fields 119 problems (Thurieau et al., 2014), and nonlocal elasticity (Schwartz et al., 2012). We adopted this 120 method in our study, detailing below the principal steps followed in the context of a micropolar 121 medium. First, let us introduce the following fictitious shear modulus and Poisson ratios.

122 
$$\bar{G}_u = \mu + \kappa; \ \tilde{v}_u = \frac{\lambda}{2(\lambda + \mu + \kappa)}; \ \bar{G}_{\varphi} = \frac{1}{2}(\gamma + \beta); \ \tilde{v}_{\varphi} = \frac{\alpha}{\beta + \gamma + 2\alpha}$$

123 Note that we could equally adopt the usual shear modulus of micropolar material defined as 124  $\bar{G}_u = \mu + \kappa/2$ .

125 The field equations (1) and (2) take the forms:

$$\bar{G}_{u}\left[\Delta u_{i} + \frac{1}{1 - 2\tilde{v}_{u}}u_{j,ji}\right] - \kappa u_{j,ji} - \kappa \epsilon_{ijk}\varphi_{k,j} = 0$$

$$(6)$$

$$\bar{G}_{\varphi}\left[\Delta\varphi_{i} + \frac{1}{1 - 2\tilde{\nu}_{\varphi}}\varphi_{j,ji}\right] + (\beta - \gamma)\frac{(\varphi_{j,ij} - \varphi_{i,jj})}{2} + 2\kappa\varphi_{i} + \kappa\,\epsilon_{ijk}u_{k,j} = 0 \tag{7}$$

126

127 Let us assume that the primary kinematical fields are the sum of a complementary part and a 128 particular term. Namely:  $u_i = u_i^c + u_i^p$  and  $\varphi_i = \varphi_i^c + \varphi_i^p$ . The complementary parts are 129 assumed to satisfy Navier's type equations, that is:

$$\bar{G}_{u} \left[ \Delta u_{i}^{c} + \frac{1}{1 - 2\tilde{v}_{u}} u_{j,ji}^{c} \right] = 0$$
(8)

$$\bar{G}_{\varphi}\left[\Delta\varphi_{i}^{c} + \frac{1}{1 - 2\tilde{\nu}_{\varphi}}\varphi_{j,ji}^{c}\right] = 0$$
<sup>(9)</sup>

130 and accordingly, the particular fields solve:

$$\bar{G}_{u}\left[\Delta u_{i}^{P} + \frac{1}{1 - 2\tilde{v}_{u}}u_{j,ji}^{P}\right] - \kappa u_{j,ji} - \kappa \epsilon_{ijk}\varphi_{k,j} = 0$$

$$\tag{10}$$

$$\bar{G}_{\varphi}\left[\Delta\varphi_{i}^{P} + \frac{1}{1 - 2\tilde{\nu}_{\varphi}}\varphi_{j,ji}^{P}\right] + (\beta - \gamma)\frac{(\varphi_{j,ij} - \varphi_{i,jj})}{2} + 2\kappa\varphi_{i} + \kappa\epsilon_{ijk}u_{k,j} = 0$$
(11)

131

The Navier type equations 8 and 9 are solved by the conventional boundary element methodwhich produces systems of equations of the following form (see e.g. (Balaš et al., 1989)):

$$[H_u]\{u^C\} = [G_u]\{t^C\}, \qquad [H_{\varphi}]\{\varphi^C\} = [G_{\varphi}]\{m^C\}$$
(12)

135 In equations (12),  $\{u^{c}\}, \{\varphi^{c}\}, \{t^{c}\}, \{m^{c}\}$  represent respectively the vector of nodal displacement 136 , the vector of nodal microrotation, the vector of nodal tractions and the vector of nodal micro-137 torque.

The local point interpolation, applied to the strong form differential equation, is adopted for thesolution of equations (10) and (11).

140 In this method, a field v(x) is approximated as (LIU and GU, 2001):  $v(x) = \sum_{i=1}^{N} R_i(r)a_i + \sum_{j=1}^{M} p_j(x)b_j$  with the following constraints:  $\sum_{i=1}^{N} p_j(x)a_i = 0, j = 1 - M$  and i = 1 - N.

Here  $R_i(r)$  is the selected radial basis functions, *N* the number of nodes in the neighborhood (support domain) of point *x*, and *M* the number of monomial terms in the selected polynomial basis  $P_j(x)$ . Recall that *r* denotes the Euclidean distance between the point *x* and the collocation center  $x^i$ . Enforcing the approximation of v(x) to be satisfied at all centers in the support domain, coefficients  $a_i$  and  $b_j$  are determined by solving a system of equations of the form:

- 148  $\begin{cases} v_{/L} \\ 0 \end{cases} = \begin{bmatrix} R & P \\ P^T & 0 \end{bmatrix} \begin{pmatrix} a \\ b \end{cases}$
- 149 where  $\{v_{/L}\}$  denotes the vector of nodal values of v(x).
- 150 It can be shown that  $\{b\} = \left[ [P]^T [R]^{-1} [P] \right]^{-1} [P]^T [R]^{-1} \{v_{/L}\} = [F_b] \{v_{/L}\}$  and  $\{a\} =$

151 
$$[R]^{-1}[[I] - [P] [F_b]] \{v_{/L}\} = [F_a] \{v_{/L}\}$$

152 Then, the approximation of v(x) is now rewritten in the matrix form as v(x) =153  $[R_1 \ R_2 \ \dots \dots R_N] [F_a] \{v_{/L}\} + [P_1 \ P_2 \ \dots \dots P_m] [F_b] \{v_{/L}\}$ 

154 or more compactly as:

155 
$$v(x) = [\Phi(x)] \{ v_{/L} \}$$
 (13)

156 In the matrix form, equations (10) and (11) become:

$$[B(\nabla)]^T [C_u] [B(\nabla)] \{u^P\} + \nabla \nabla^T \{u\} + \kappa [D(\nabla)] \{\varphi\} = 0$$
(14)

$$[B(\nabla)]^{T} [A_{\varphi}] [B(\nabla)] \{\varphi^{P}\} + (\beta - \gamma) [\nabla \nabla^{T} + [I] \nabla^{T} \nabla] \{\varphi\} + \kappa \{\varphi\} - \kappa [D(\nabla)] \{u\} = 0$$
(15)

158 In the above, 
$$\{\nabla\} = \left(\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z}\right)^T$$
,  $\{z\} = (z_1 \quad z_2 \quad z_3)^T$ ,  $[B(z)] = \begin{bmatrix} z_1 & 0 & 0 & z_2 & z_3 & 0\\ 0 & z_2 & 0 & z_1 & 0 & z_3\\ 0 & 0 & z_3 & 0 & z_1 & z_2 \end{bmatrix}^T$ ,

159 
$$[D(z)] = \begin{bmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{bmatrix}$$
 and  $[I]$  is the identity tensor.

When adopting interpolation (13) for all kinematical fields, at a given collocation center, theparticular fields equations (14) and (15) have become:

$$[B(\nabla)]^{T}[C_{u}][B(\nabla)][\widetilde{\Phi}]\{u_{/L}^{P}\} + \kappa[\widetilde{\Phi}_{1}]\{u_{/L}\} + \kappa[\widetilde{\Phi}_{2}]\{\varphi_{/L}\} = \{0\}$$

$$(16)$$

$$[B(\nabla)]^{T}[A_{\varphi}][B(\nabla)][\widetilde{\Phi}]\{\varphi_{/L}^{P}\} + \kappa[\widehat{\Phi}_{0}]\{\varphi_{/L}\} + (\beta - \gamma)[\widehat{\Phi}_{1}]\{\varphi_{/L}\} + \kappa[\widehat{\Phi}_{3}]\{u_{/L}\} = \{0\}$$
(17)

163 Matrices  $C_u$  and  $A_{\varphi}$  are the Voigt representation of the isotropic like elasticity tensor 164 respectively with parameters  $\bar{G}_u$  and  $\tilde{v}_u$  n the one hand and  $\bar{G}_{\varphi}$  and  $\tilde{v}_{\varphi}$  on the other.

165 On collocation of the above equations (16) and (17) for all the internal collocation centers, 166 taking the assumption that the particular integrals are identically zero at all boundary points,

167 the following forms of systems of equations is obtained:

$$\{u^{P}\} = [A_{u}]\{u\} + [B_{u}]\{\varphi\}$$

$$\{\varphi^{P}\} = [A_{\varphi}]\{\varphi\} + [B_{\varphi}]\{u\}$$
(18)
(19)

168

169 The tractions at a regular point on the boundary is written as:

$$t_{i} = t_{i}^{C} + t_{i}^{P} + \delta t_{i} \text{ with } t_{i}^{A} = \left(\lambda \varepsilon_{rr}^{A} \delta_{ij} + 2(\mu + \kappa)\varepsilon_{ij}^{A}\right)n_{j} \quad (A = C \text{ or } P)$$
  
and  $\delta t_{i} = \left(-\kappa u_{i,i} + \kappa \varepsilon_{ijk}\varphi_{k}\right)n_{i}$  (20)

$$m_{i} = m_{i}^{C} + m_{i}^{P} + \delta m_{i} \text{ with } t_{i}^{A} = (\alpha \,\omega_{rr}^{A} \delta_{ij} + (\beta + \gamma) \omega_{ij}^{A}) n_{j} \quad (A = C \text{ or } P)$$
  
and  $\delta m_{i} = (\beta - \gamma) (\varphi_{j,i} - \varphi_{i,j}) n_{j}/2$  (21)

170

Following a similar strategy, the tractions at the boundary points could be written in the following forms:

$$\{t\} = \{t^c\} + [AK_{1u}]\{u^P\} + [AK_{uu}]\{u\} + [AK_{u\varphi}]\{\varphi\}$$
(22)

$$\{m\} = \{m^{c}\} + [AK_{1\varphi}]\{\varphi^{P}\} + [AK_{\varphi u}]\{u\} + [AK_{\varphi \varphi \phi}]\{\varphi\}$$
(23)

173

174 Introduce equations (18) and (19) in (22) and (23), considering the previous consideration  $u^c = u - u^p$  that we introduce in the system of equations (12) and after conducting some algebraic 176 manipulations, the final coupled systems of equations is of the following forms:

$$[\overline{H}_u]\{u\} + [H_{u\varphi}]\{\varphi\} = [G_u]\{t\}$$
(24)

$$\left[\overline{H}_{\varphi}\right]\{\varphi\} + \left[H_{\varphi u}\right]\{u\} = \left[G_{\varphi}\right]\{m\}$$
<sup>(25)</sup>

177

Particularly worthy of mention, the final equations contained similar boundary primary variables and internal kinematic unknowns to those of a pure BEM. Boundary conditions could be taken into account as in standard practice and the resulting system of equations was solved by a standard direct solver

181 by a standard direct solver.

### 182 **3- Numerical examples**

183 In our work, we used the generalized multi-quadrics radial basis functions  $R_i(r) = (r_i^2 + c^2)^q$ ,

184 where  $r_i = ||x - x_i||$  and c and q are parameters known as shape parameters. Shape parameter c185 was taken proportional to the minimum distance  $d_0$ , defined as the maximum value among the 186 minimum distances in the  $x_1$ ,  $x_2$  and  $x_3$  directions between collocation centers.

187 The first set of numerical results presented serve as validation of the proposed numerical188 approach.

189 3.1 Case of a bar under traction load

190 Let us first consider the case of a bar under a uniform tension load ( $\sigma_{33} = cte$ ). The bar with

height H = 5 units has a rectangular cross section with width L = 2 and thickness T = 1 units. (cf.

192 fig 1). The bar is simply supported at its lower end and uniformly loaded at its upper face. All the

remaining faces are free of traction. It is also assumed that all boundaries are free of micro-

torque. The boundary of the bar was subdivided into 88 nine-node quadrilaterals. The boundary

195 nodes were supplemented by 171 internal collocation centers. This case serves as a first

196 validation of the developed numerical approach.

197



198

#### 199 Figure 1 : Schematic of the bar under tension load

200 In this case, it can be established that, the axial strain is given in term of the applied traction by:

201 
$$\varepsilon_{33} = \frac{2\lambda + 2\mu_m + \kappa}{(3\lambda + 2\mu_m + \kappa)(2\mu_m + \kappa)}\sigma_{33}$$

202 were  $\mu_m = \mu + \frac{\kappa}{2}$  is the equivalent shear modulus for micropolar media and  $\mu$  the Lamé constant.

203 For our case, parameters are chosen as:  $\lambda = 1000 MPa$ ,  $\mu_m = 750 MPa$  and  $\kappa = 500 MPa$ .

The calculated axial elongation along the central line of the sample is compared to the analytical solution in fig 2. As can be observed accurate numerical results are obtained. As expected, the calculated microrotation was identically zero. Indeed, due to the sample geometry, the loading does note generate shear stress. These results were undisturbed when the radial basis shape parameters were varied in the ranges  $q \in [0.5; 2.5]$  and  $c \in [10^{-4}d_0; 10^{-1}d_0]$ .





210 Figure 2: Axial displacement along the axial direction at the center of the beam

211

212 3.2 Complex loading of a prismatic bar

Case of a prismatic bar in order to further validate the developed numerical tool, let us consider the following patch tests used in (Grbčić et al., 2018). in order to validate their finite element implementation for elastic micropolar materials. The displacement and microrotation fields in a loaded prismatic bar with height H = 0.24, width L = 0.12 and thickness T=0.06, as shown in figure 3, are known. The adopted system of Cartesian coordinates is such that  $0 \le x \le H$ ,  $0 \le y \le T$ ,  $0 \le z \le L$ . The known fields are given by:

$$U_x = (x + 0.5y + z) \cdot 10^{-3}$$
  $U_y = (x + y + 0.5z) \cdot 10^{-3}$   $U_z = (0.5x + y + z) \cdot 10^{-3}$ 

1.5

219 
$$\varphi_{\chi} = \varphi_{\chi} = \varphi_{z} = 0.25.10^{-3}$$

220

221 The stress field associated with such displacement and micro-rotation fields are given by:

222 
$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 5$$

223 
$$\sigma_{xy} = \sigma_{yz} = \sigma_{yz} = \sigma_{zy} = \sigma_{zx} = \sigma_{zx} = \sigma_{zx}$$

224 The micro-stress is identically zero.





#### Figure 3 : Representation of the distribution of the collocation center in the considering medium

228

For our calculation each face of the sample is considered as a nine-node quadrilateral element.

230 The boundary nodes are supplemented by 8 internal collocation centers shown in the figure.

Numerical results of our calculation are collected in the table below for all the internal 8 points:

232

	(x, y, z)	$\sigma_{yy}$	$\sigma_{\chi_Z}$	$m_{xx}$	m <sub>zx</sub>
1	(0.04,0.02,0.02)	4.9984	1.5000	3.4445E-06	1.7422E-06
2	(0.04,0.04,0.02)	4.9986	1.4999	2.7052E-06	1.9378E-06
3	(0.08,0.02,0.08)	4.9943	1.4990	-1.7573E-06	-1.3079E-06
4	(0.08,0.04,0.08)	4.9939	1.4986	-1.5173E-06	-1.2852E-06
5	(0.16,0.02,0.08)	4.9947	1.4984	-8.3675E-08	1.0612E-06
6	(0.16,0.04,0.08)	4.9956	1.4988	-2.3694E-07	1.0647E-06
7	(0.18,0.02,0.03)	4.9968	1.4993	1.7310E-07	-1.8554E-06
8	(0.18,0.04,0.03)	4.9960	1.4992	1.9306E-07	-1.6577E-06

233

#### 234 table 1: Stresses and micro-stresses at various points in the sample

As can be observed, results of our numerical tool are in excellent agreement with analyticalsolution. It can be concluded that our approach passed the test.

- 238 3.3 Torsion of a cylindrical specimen
- 239
- 240





Figure 4 : cylinder in torsion

243 We now consider a cylindrical specimen subjected to torsion load as shown in figure 4. With this 244 example we check the ability of the approach to deal with curved boundaries and natural 245 boundary conditions. Let us remind the reader that this case has been considered in a recent 246 work by Grbčić et al in a finite element context (Grbčić et al., 2018). They also derived the 247 corresponding analytical solution which is used here for comparison purposes. The boundary of 248 the cylinder with height 1 mm and radius 0.2 mm is subdivided into 96 nine-node quadrilateral 249 elements. The boundary nodes were supplemented by 567 internal collocation centers. As 250 shown in figure 4, each radius of the upper face of the cylinder is subjected to a linearly varying tangential traction ( $p_{\theta}=43.93\,r\,MPa\,with\,0~\leq r~\leq R$  ) simulating torsion. Each point of this 251 252 upper face is subjected to a constant microtorque around the z-axis ( $m_z = 3.64 N. mm^{-1}$ ). The 253 lower face of the sample is constrained against displacements and microrotation. The remaining 254 face of the cylinder is free of traction and microtorque. As in the work of Grbčić et al. we used the 255 following material parameters:  $\lambda$ =157 500 *MPa*,  $\mu$ m=8 750 *MPa*,  $\kappa$ =3 500 *MPa*,  $\alpha$ = $\beta$ =0 et  $\gamma = 210 N.$ 256

- 258 The tangential displacement along a radius of the top surface and the microrotation along a
- cylinder generator are compared to the analytical solution in table 2 and 3.
- 260

r	$U_{\theta}$ numerical (× 10 <sup>-3</sup> )	$U_{\theta}$ analytical (× 10 <sup>-3</sup> )	
0	0	0	
0.025	0.4360	0.4330	
0.05	0.8750	0.8660	
0.075	1.3146	1.2989	
0.1	1.7541	1.7318	
0.125	2.1909	2.1678	
0.15	2.6246	2.5977	
0.175	3.0566	3.0307	
0.1938	3.3806	3.3563	

table 2: Data of displacements  $u_{\theta}$  along the radius r on the upper face

And if we want to obtain values with more precision for the microstructure, it is necessary to

refine the internal mesh.

Ζ	$\varphi_z$ numerical (× 10 <sup>-2</sup> )	$\varphi_z$ analytical (× 10 <sup>-2</sup> )	
0.0312	0.0500	0.0540	
0.125	0.2161	0.2165	
0.25	0.4305	0.4330	
0.375	0.6644	0.6494	
0.5	0.8939	0.8659	
0.625	1.0926	1.0824	
0.75	1.2820	1.2989	
0.875	1.5166	1.5153	
0.9688	1.6693	1.6778	

264 table 3: Representation of microrotations  $\varphi_z$  along z on the lateral face

In the case of tangential displacement, the maximum relative error between computed and analytical results is less than 1.5%. In the case of microrotation, the maximum relative error (7%) is obtained near the constrained end. The further we get from this area, the smaller the relative error (less than 1.5%). It can be concluded that numerical results are in good agreement with analytical solutions.

As can be observed, in this case also sour numerical results are in excellent agreement withanalytical prediction.

272 3.5 Case of a plate with a hole

Let us consider a plate with a hole as shown in figure 5a. This could represent a bone plate
pierced to receive a screw. The plate has length 300 mm, width 200 mm and thickness 5 mm.
The plate is loaded uniformly in the x direction. Due to symmetry, only a quarter of the specimen
is considered for calculation (fig 5b).



Figure 5 : quater of aplate with a central circular hole of radius R

279 Following, symmetry boundary conditions are applied at the left and the bottom ends of the 280 geometry, the hole surface and the upper and lower faces of the plate are free of traction. 281 Microrotations are zero at the left and the bottom ends while the microtorques are zero at the 282 remaining surfaces. This example has already been considered by Eringen (Eringen, 1999b) and Rosenberg (Rosenberg and Cimrman, 2003). They introduced a characteristic length of the 283 material defined as  $c = \left(\frac{\gamma(\mu+\kappa)}{\kappa(2\mu+\kappa)}\right)^{1/2}$  analyzed they results with respect to the ratio R/c where R is the hole radius. The macroscopic material parameters ( $\lambda$  and  $\mu$ ) adopted for each value of 284 285 the ratio  $R/_{C}$  are determined by considering that  $b = \left(\frac{\gamma}{2(2\mu+\kappa)}\right)^{1/2}$ ,  $\gamma = 260$  N and R = 20 mm. 286 We take b/c = 0.4 and  $v = 0.2 = \lambda/(2\lambda + 2\mu + \kappa)$ . 287

Results from our simulations are analyzed through the evolution of the maximum tangential stress – the stress intensity factor – generated in the hole with respect to the ratio  $R/_c$ . This evolution is shown in the plot in figure 6 where it is compared to the case of a pure elastic material. The results, which agree well with those of Eringen (Eringen, 1999b) and Rosenberg (Rosenberg and Cimrman, 2003) show that the stress intensity factor gently increases through the pure elastic value as the material length scale decreases.





It is observed that the smaller this characteristic length *c* the closer we get to the elastic case.

Now let's look at the micro rotation. The presence of the hole induces shear stresses which do not exist in a plate without hole (cf exemple 1).



Figure 7: Representation of the - (a) tangent stress  $t_{xy}$  - (b) microrotation  $\varphi_z$  in the hole plate

The shear stresses in this case are maximum at the edge of the hole and they induce a noticeablemicro-rotations in the same place.

- 301 3.5 Localized load on a micropolar sample simulating indentation load
- 302

303 In this example we will simulate an indentation by a spherical punch of a cubic homogeneous 304 and isotropic specimen. To do this we will assume that the displacement profile under the 305 indenter is similar to the one obtained when indenting a pure elastic sample. The expression of

this displacement is known and can be find in the book by KL Johnson (Johnson, 1987).

307





#### 309

Figure 8: Representation of the domain with boundary

310

311 In our application, the cubic sample has size h = 10 mm and the non-deformable spherical 312 punch has radius  $R_{ind} = 10$  mm. Under a normal applied load P, the normal displacement beneath the indenter is given by  $U_z(r) = U_z(0) * (1 - (\frac{r}{2a})^2)$  where *r* stands for the position on 313 a radius of the contact disc with radius a.  $U_z(0)$  is the displacement of the first contact point (i.e. 314 315 the center of the circular contact area). In the case of an elastic infinite flat specimen with Young modulus E and Poisson ratio  $\nu$ , the mean pressure in the contact disc  $(P_m)$  is related to the 316 maximum contact stress (P<sub>0</sub>) by  $P_m = \frac{2}{3}P_0$  with  $P_o = \frac{2E^*a}{\pi R_{ind}}$ .  $E^*$  is an equivalent modulus given 317 by:  $E^* = \frac{E}{1-\nu^2}$ . 318

In the following, we compare the response of a pure elastic material to that of a micropolar one.
Both materials are selected to have the same macroscopic material parameters collected in table
4 below.

322

	E (GPa)	ν	к (GPa)	α (N)	β(N)	γ (N)	N <sub>c</sub>
Elastic case	85.2	0.36	/	/	/	/	/
Micropolar case	85.2	0.36	30	-50	50	260	0.57

#### 323 table 4: material parameter for classical and micropolar elastic medium

In table 4  $N_c$  is a coupling number for macro/microstructure (for micropolar media) defined as  $N = \sqrt{\frac{\kappa(1+\nu)}{E+\kappa(1+\nu)}}$ . *N* is a number that highlights the connectivity or the topology of the microtructure network, in other words it quantifies the degree of micropolarity exhibited by the material. A quadratic network microstructure should have a higher parameter than a hexagonal network microstructure and so on (McGregor and Wheel, 2014). A first set of numerical experiment has been performed for different values of contact arearadius (see table 5 below).

	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
$\frac{a}{h}$	0.025	0.05	0.075	0.1	0.125	0.15

table 5: Table summarizing the different cases according to the maximum insertion depths

332 The calculated maximum contact pressure  $(P_o)$  computed for the pure elastic material and the

micropolar elastic material are compared to the pure elastic analytical solution in figure 11.



# Figure 9: Representation of the indentation curves representing the penetration depth as a function of theapplied load

336 It can be observed that our numerical solution is in excellent agreement with analytical one in

337 the case of pure elastic material. Globally, the load required to achieve the same penetration

depth (the same contact area) is higher in the case of micropolar material. This is in agreement

with the results presented by (Salehi and Salehi, 2014).

Let us introduce *s* the stress tensor deviator define as  $s = \sigma - \frac{1}{3}tr(\sigma) * I$ , with  $\sigma$  the asymmetric

341 stress tensor and *m* the micro-stress tensor. Due to the asymmetric stress tensor (Eringen,

342 1999c), we can't use the usual von Mises stresses definition for an elastic classical medium. An

equivalent stress of the von Mises type has been introduced by Gombos (Gombos, 2007) and isgiven by:

345 
$$\sigma_{vm} = \sqrt{3\left(\frac{1}{4}(s:s+s:s^t) + \frac{1}{4}(m:m+m:m^t)\right)}$$

This stress is considered in the analysis of results of the specimen submitted to indentation typeloading.





# Figure 10: von Mises stress in a cross plane of the specimen (plane xy containing the symmetry axis); A) pure elastic material; B) micropolar elastic material.

The contour plots of the von Mises stresses in a vertical plane containing the axis of symmetry are shown in figure 10. The purely elastic case is shown for comparison purpose. It can be noted

353 that the subsurface stresses are higher in the case of micropolar elastic material. The maximum

value is located on the symmetry axis but closer to the contact compared to the pure elastic case.

Higher values of equivalent stresses are obtained at the free surface in a region enclosing the contact border.

Let us consider the effect of the coupling number  $N_c$  on the distribution of the normal stress and

358 the stress within the sample. The different values of the coupling number have been obtained by

359 modifying the value of parameter  $\kappa$  and the Young Modulus and Poisson coefficient stay the

360 same.

361 The profiles of normal contact load within the contact are shown in figure 11 below.



363 Figure 11: Representation of the constrained vector z dimensioned by p0 on the contact radius

Clearly, the higher the coupling number, the higher the values of the normal contact stress. This 364 365 result indicates that, for a given normal load P applied on a spherical punch, the higher the 366 degree of micropolarity, the smaller the radius of the contact area.

367 For a given radius of contact area, the contour plots of the von Mises stresses developed within 368 the sample are shown in figure 12. The stress levels increase with the value of the coupling 369 number. Moreover, the maximum value is obtained at the surface and in the material. For higher 370 values of the coupling number, the maximum value is located at the surface of the sample. This is

371 a clear difference with the case of classical elasticity



Figure 12: Representation of von Mises constraints in the xy plane passing through the center of the domain
 for the micropolar in function of the coupling number N

# 376 Conclusion

377 A pure boundary element method for the solution of 3D micropolar elasticity problems has been 378 presented. The approach is the so-called LPI-BEM which combines the advantages of the 379 boundary element method (reduction of the problem dimension by one) and of the local point 380 interpolation method (ease of implementation). The effectiveness of the approach in this context 381 has been proved by comparing our results with those available in the literature. The case of 382 indentation has also been considered and some first results highlighted. A forthcoming will be 383 dedicated to the indentation of micropolar elastic materials, focusing on the role of the known characteristic length and coupling number usually adopted for this material. 384

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