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A LOCAL TO GLOBAL PRINCIPLE FOR HIGHER ZERO-CYCLES

JOHANN HAAS AND MORTEN LÜDERS

Abstract. We study a local to global principle for certain higher zero-cycles over global fields. We thereby verify a conjecture of Colliot-Thélène for these cycles. Our main tool are the Kato conjectures proved by Jannsen, Kerz and Saito. Our approach also allows to reprove the ramified global class field theory of Kato and Saito. Finally, we apply the Kato conjectures to study the p-adic cycle class map over henselian discrete valuation rings of mixed characteristic and to deduce finiteness theorems for arithmetic schemes in low degree.

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1. Introduction

Let K be a number field, X a smooth projective geometrically integral variety over K and n a positive integer. Let P_K denote the set of places of K. The following conjecture was suggested by Kato and Saito in [22, Sec. 7] and Colliot-Thélène in [4, Conj. 1.5(c)] (see also [7] for the case of rational surfaces).

Conjecture 1.1. *The complex*

$$\varprojlim_n A_0(X)/n \to \prod_{v\in P_K} \varprojlim_n A_0(X_{K_v})/n \to \mathrm{Hom}(\mathrm{Br}(X)/\mathrm{Br}(K), \mathbb{Q}/\mathbb{Z})$$

is exact.

Conjecture 1.1 is known to hold if X is a curve and if the Tate-Shafarevich group of the Jacobian of X does not contain a non-zero element which is infinitely divisible (see [32, Sec. 7], [5, Sec. 3] and [42, Rem. 1.1(iv)] as well as Remark 5.8). For more details on this conjecture and results for higher dimensional schemes we refer to [42] and [14].

In [5], Colliot-Thélène considers a slightly weaker conjecture on the image of the complex (1.1) in étale cohomology. For our purposes we will generalize this conjecture to Bloch's higher Chow groups. In order to state the conjecture, we need to recall Saito's exact sequence generalizing the Tate-Poitou exact sequence (see Section 3) to schemes ([34]). For later purposes we state

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it in its most general form proven by Geisser and Schmidt ([12]). Let K be a global field. Let S be a non-empty and possibly infinite set of prime divisors of K containing the archimedean primes if K is a number field. Let $\mathcal{O}_S$ be the ring of elements in K which are integers at all primes $\mathfrak{p} \notin S$. Let $\mathcal{S} = \mathrm{Spec}\mathcal{O}_S$ and $\mathcal{X}$ a regular, flat and separated scheme of finite type of relative dimension d over $\mathcal{S}$. Let n be a positive integer invertible on $\mathcal{S}$ and let $\mathcal{F}$ be a locally constant constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathcal{X}$. Let $D(\mathcal{F}) = \mathrm{Hom}(\mathcal{F}, \mu_n^{\otimes d+1})$ be its dual. For an abelian group A, let $A^\vee = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$. Then there is an exact sequence

$$\ldots \to H^i_{\text{ét}}(\mathcal{X}, \mathcal{F}) \to \prod_{v\in S}{}' \hat{H}^i_{\text{ét}}(X_{K_v}, \mathcal{F}) \to H^{2d+2-i}_{\text{ét},c}(\mathcal{X}, D(\mathcal{F}))^\vee \to H^{i+1}_{\text{ét}}(\mathcal{X}, \mathcal{F}) \to \ldots \tag{1.1}$$

Here the groups $\hat{H}^i_{\text{ét}}(X_{K_v}, \mathcal{F})$ are the modified cohomology groups defined in [34, Def. 1.6]. If v is a non-archimedean place, then $\hat{H}^i_{\text{ét}}(X_{K_v}, \mathcal{F}) := H^i_{\text{ét}}(X_{K_v}, \mathcal{F})$. If v is a complex place or if v is a real place and n is prime to 2, then $\hat{H}^i_{\text{ét}}(X_{K_v}, \mathcal{F}) = 0$. Therefore if either K does not have any real places or n is odd, then we also write $H^i_{\text{ét}}(X_{K_v}, \mathcal{F})$ for $\hat{H}^i_{\text{ét}}(X_{K_v}, \mathcal{F})$. In the product we restrict to classes which are almost everywhere unramified. In Definition 2.1 we define an analogous restricted product of higher Chow groups $\prod'_{v\in S} \mathrm{CH}^i(X_{K_v} a, \mathbb{Z}/n\mathbb{Z})$ with coefficients in the ring $\mathbb{Z}/n\mathbb{Z}$. For the definition and properties of higher Chow groups for varieties over a field we refer to [2]. Let now $S = P_K$, $\mathcal{X} = X$ be smooth projective over $\mathrm{Spec}K$ and $\mathcal{F} = \mu_n^{\otimes i}$. By construction, cf. Remark 2.2, we then get a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^i(X, a, \mathbb{Z}/n\mathbb{Z}) \arrow[r] \arrow[d] & \prod'_{v\in P_K} \mathrm{CH}^i(X_{K_v}, a, \mathbb{Z}/n\mathbb{Z}) \arrow[d] & \\
H^{2i-a}_{\text{ét}}(X, \mu_n^{\otimes i}) \arrow[r] & \prod'_{v\in P_K} \hat{H}^{2i-a}_{\text{ét}}(X_{K_v}, \mu_n^{\otimes i}) \arrow[r] & H^{2d+2-2i+a}_{\text{ét}}(X, \mu_n^{\otimes d+1-i})^\vee.
\end{tikzcd} \tag{1.2}$$

Here if v is a real place, then the étale cycle class map is defined to be the composition $\mathrm{CH}^i(X_{K_v}, a, \mathbb{Z}/n\mathbb{Z}) \to H^{2i-a}_{\text{ét}}(X_{K_v}, \mu_n^{\otimes i}) \to \hat{H}^{2i-a}_{\text{ét}}(X_{K_v}, \mu_n^{\otimes i})$. By composing the right vertical and right horizontal map we get a map

$$\prod_{v\in P_K}{}' \mathrm{CH}^i(X_{K_v}, a, \mathbb{Z}/n\mathbb{Z}) \to H^{2d+2-2i+a}_{\text{ét}}(X, \mu_n^{\otimes d+1-i})^\vee$$

and equivalently particular a pairing

$$\prod_{v\in P_K}{}' \mathrm{CH}^i(X_{K_v}, a, \mathbb{Z}/n\mathbb{Z}) \times H^{2d+2-2i+a}_{\text{ét}}(X, \mu_n^{\otimes d+1-i}) \to \mathbb{Q}/\mathbb{Z}$$

both of which are zero on $\mathrm{CH}^i(X, a, \mathbb{Z}/n\mathbb{Z})$. The following conjecture is a version of the mentioned weaker conjecture of Coliot-Thélène extended to Bloch's higher Chow groups:

Conjecture 1.2. *(see* [5, Conj. 2]*) Let X/K be a smooth projective, geometrically integral, variety over a global field K and n be prime to* $\mathrm{ch}(K)$*. Let d be the dimension of X, and $i \geq 0$ an integer. Let $\{z_v\}_{v\in P_K} \in \prod'_{v\in P_K} \mathrm{CH}^i(X_{K_v}, a, \mathbb{Z}/n\mathbb{Z})$ and suppose that every class $\xi \in H^{2d+2-2i+a}_{\text{ét}}(X, \mu_n^{\otimes d+1-i})$ is orthogonal to the family $\{z_v\}_{v\in P_K}$. Then there exists a global cycle $z_n \in \mathrm{CH}^i(X, a)$ such that for every place v the class of z_n in $\hat{H}^{2i-a}_{\text{ét}}(X_{K_v}, \mu_n^{\otimes i})$ coincides with that of z_v.*

Furthermore, we can state the following conjecture analogous to Conjecture 1.1:

Conjecture 1.3. *Let $j = d + 1 - i$. The complex*

$$\mathrm{CH}^i(X, a, \mathbb{Z}/n\mathbb{Z}) \to \prod_{v\in P_K}{}' \mathrm{CH}^i(X_{K_v} a, \mathbb{Z}/n\mathbb{Z}) \to H^{2j+a}_{\text{ét}}(X, \mu_n^{\otimes j})^\vee$$

is exact.

Let $\mathcal{X}$ and S be as above. In this case we define Bloch's higher Chow groups (or motivic cohomology) by taking the Zariski-hypercohomology of Bloch's cycle complex $z^n(-,i)$ on $\mathcal{X}$, see [10, Sec. 3]. For $\mathcal{S}$ a semi-local Dedekind domain this coincides with taking the homology of Bloch's cycle complex due to the existence of a localisation sequence. For $\mathcal{S}$ a field this is due to Bloch and for $\mathcal{S}$ semi-local this is shown by Levine in [27] (see also [10]). We introduce the following notation:

$$Ш(\mathrm{CH}^{d+1}(\mathcal{X},a,\mathbb{Z}/n\mathbb{Z})) := \ker[\mathrm{CH}^{d+1}(\mathcal{X},a,\mathbb{Z}/n\mathbb{Z}) \to \prod_{v\in S}\mathrm{CH}^{d+1}(X_{K_v},a,\mathbb{Z}/n\mathbb{Z})]$$

and

$$Ш^{2d+2-a,d-1}(\mathcal{X}) := \ker[H^{2d+2-a}_{\text{ét}}(\mathcal{X},\mathbb{Z}/n\mathbb{Z}(d+1)) \to \prod_{v\in S}\hat{H}^{2d+2-a}_{\text{ét}}(X_{K_v},\mathbb{Z}/n\mathbb{Z}(d+1))]$$

These groups define analogues of the Tate-Shavarevich group for Chow groups and étale cohomology.

Throughout the article we will often need the following condition:

Condition 1.4. *Let K_v be a non-archimedean local field, V/K_v a smooth scheme.*

- ($\star$) *V is projective and admits a model over $\mathcal{O}_{K_v}$ whose special fiber is smooth except at finitely many points, where it has ordinary quadratic singularities.*
- ($\star\star$) *V is projective and has strictly semistable reduction.*

We are now able to state our main theorem:

Theorem 1.5 (Thm. 5.1)**.** *Let K be a global field, S a set of places of K and $n \in \mathbb{N}_{>1}$. Denote by $\mathcal{O}_S$ the ring of elements in K which are integers at all primes $\mathfrak{p} \notin S$ and let $\mathcal{S} = \mathrm{Spec}\mathcal{O}_S$. Suppose that*

(a) *$\mathcal{S}$ is either semi-local or an open of $\mathrm{Spec}(\mathcal{O}_K)$.*
(b) *n is invertible on $\mathcal{S}$.*
(c) *If K is a number field, S contains all archimedean places of K and either n is odd or K has no real places.*

Let $\mathcal{X}$ be regular, flat and projective of relative dimension d over $\mathcal{S}$ with smooth generic fiber X. Then the following statements hold:

(1) *There is an exact sequence*

$$\mathrm{CH}^{d+1}(\mathcal{X},a,\mathbb{Z}/n\mathbb{Z}) \to \prod_{v\in S}{}' \mathrm{CH}^{d+1}(X_{K_v},a,\mathbb{Z}/n\mathbb{Z}) \to H^a_{\text{ét}}(\mathcal{X},\mathbb{Z}/n\mathbb{Z})^\vee$$

(2) *For all a there is a natural surjection $Ш(\mathrm{CH}^{d+1}(\mathcal{X},a,\mathbb{Z}/n\mathbb{Z})) \to Ш^{2d+2-a,d+1}_{\text{ét}}(\mathcal{X})$.*
(3) *If K is a number field and condition ($\star$) holds for X_{K_v} if v divides n, then the group $Ш(\mathrm{CH}^{d+1}(\mathcal{X},1,\mathbb{Z}/n\mathbb{Z}))$ is finite.*
(4) *If K is a function field of one variable over a finite field and n is invertible in K, then $Ш(\mathrm{CH}^{d+1}(\mathcal{X},a,\mathbb{Z}/n\mathbb{Z}))$ is finite for arbitrary a.*

Note that $\mathrm{CH}^{d+1}(X,a) = 0$ for $a \leq 0$ by dimension reasons and that $\mathrm{CH}^s(X,2s-r) \cong H^r(X,\mathbb{Z}(s))$ for X smooth over a perfect field. Theorem 1.5(2) is related to a conjecture of Bloch (see [3, Conj. 3.16]). It means that the Tate-Shafarevich group $Ш^{2d+2-a,d-1}_{\text{ét}}(X)$ is - under the above assumptions - generated by algebraic cycles.

Corollary 1.6 (Cor. 5.5)**.** *Let the assumptions be as in Theorem 1.5. Let $S = P_K$ and therefore $X = \mathcal{X}$. Then the following statements hold:*

(1) *Conjecture 1.2 holds for $i \geq d+1$ and all a.*
(2) *Conjecture 1.3 holds for $i \geq d+1$ and all a.*

If $a = 1$, $S = P_K$ and $X = \mathcal{X}$, then Theorem 1.5 recovers the unramified class field theory of Bloch, Kato and Saito (see [1], [21, Thm. 3], [33, Sec. 5] and [20, Thm. 2.10]). For this note that

$$\mathrm{CH}^{d+1}(X,1) \cong \mathrm{coker}[\bigoplus_{x\in X^{d-1}} K_2^M k(x) \to \bigoplus_{x\in X^d} K_1^M k(x)] =: SK_1(X)$$

(see e.g. [29]). In Section 6 we extend this corollary to cover the ramified global class field theory of Kato and Saito:

Theorem 1.7 (Thm. 6.4)**.** *Let K be a global field. Let n be invertible in K. If K is a number field assume furthermore that either n is odd or that K has no real places. Let X be a smooth projective scheme over* $\mathrm{Spec}(K)$*. Let $D \subset X$ be an effective divisor on X and $j : U \hookrightarrow X$ be the inclusion. Then there is an isomorphism*

$$\mathrm{coker}[H^d_{\mathrm{Nis}}(X, \mathcal{K}^M_{d+1,X|D})/n \to \prod_{v\in P_K}{}' H^d_{\mathrm{Nis}}(X_{K_v}, \mathcal{K}^M_{d+1,X_{K_v}|D_{K_v}})/n] \to \pi_1^{\mathrm{ab}}(U)/n.$$

For the definition of the relative Milnor K-sheaves appearing in the cokernel on the left see Definition 6.1.

In Section 7 we show the following theorem:

Theorem 1.8 (Thm. 7.2)**.** *Let A be a henselian discrete valuation ring of characteristic zero with residue field of characteristic p and function field K. Let X be smooth and projective over* $\mathrm{Spec}(A)$*. Let X_1 denote the special and X_K the generic fiber of X. Let $d = \dim X - 1 \leq 2$. Then there is an isomorphism*

$$\mathrm{CH}^{d+1}(X, 1, \mathbb{Z}/p^r\mathbb{Z}) \to H^{2d+1}_{\text{ét}}(X, \mathcal{T}_r(d+1)),$$

where $\mathcal{T}_r(d+1)$ are the p-adic étale Tate twists defined in [35]*.*

The following corollary follows from [28, Prop. 1.4] and answers a question posed in *loc. cit.*, p. 582.

Corollary 1.9. *Let the situation be as in Theorem 1.8. Assume that $A = W(k)$ for a finite field k of characteristic $p > d + 2$. Let X_n denote the thickenings of the special fibre X_1 and $\mathcal{K}^M_{*,X_n}$ be the improved Milnor K-sheaf on X_n defined in* [23]*. Then there is an isomorphism of pro-abelian groups*

$$\mathrm{CH}^{d+1}(X, 1, \mathbb{Z}/p^r\mathbb{Z}) \to \text{"}\lim_n\text{"} H^d(X_1, \mathcal{K}^M_{d+1,X_n}/p^r).$$

In Section 8 we show a finiteness theorem in low degree for higher Chow groups of schemes over Dedekind domains. The idea of the proof can also be found in [11, Sec. 7.2].

We make a few remarks on the method and the outline of the article. The central method is the comparison of the Zariski and the étale motivic cohomology of a regular scheme X over a field or Dedekind domain by analysing the respective coniveau spectral sequences. The difference between the two spectral sequences may in some cases be measured by the Kato complexes which are the subject of the Kato conjectures. The Kato conjectures are a framework in which Kato generalises the Brauer-Hasse-Noether exact sequence to higher dimensional arithmetic schemes. To motivate these conjectures from the point of view of this article, we recall the Tate-Poitou exact sequence in Section 3. This sequence combines class field theory and the Brauer-Hasse-Noether exact sequence as well as Conjectures 1.1 and 1.3 in the zero dimensional case. The Kato conjectures have now to a large extend been proved by Jannsen, Kerz and Saito. We recall their results in Section 4. In Section 5 we apply these results and the above mentioned method to Diagram (1.2) in order to prove Theorem 1.5. In Section 6 we reprove the ramified global class field theory of Kato and Saito using the Kato conjectures. For this we use our method of Section 5 and an idea of Kerz and Zhao in their approach to ramified class field theory over local and finite fields. In Section 7 and 8 we apply what is known about the Kato conjectures to obtain the above mentioned results which are independent of the results obtained before.

2. Restricted products of Chow groups

We introduce restricted products of Chow groups over global fields. In the case of $i = d+1$ and $a = 1$ these were studied in arithmetic class field theory by Bloch, Kato and Saito.

Definition 2.1. *Let K be a global field, X/K a smooth projective geometrically integral variety, n a positive integer prime to $\mathrm{ch}(K)$ and $i, a \in \mathbb{N}$. Let $\mathcal{O}_K$ be the ring of integers of K, and $\mathcal{S} \subset \mathrm{Spec}(\mathcal{O}_K)$ a dense open over which X has a smooth projective model $\mathcal{X}$. Define*

$$\prod_{v \in P_K}' \mathrm{CH}^i(X_{K_v}, a, \mathbb{Z}/n\mathbb{Z}) \subset \prod_{v \in P_K} \mathrm{CH}^i(X_{K_v} a, \mathbb{Z}/n\mathbb{Z})$$

to be the subgroup of classes $(\alpha_v)_{v \in P_K}$ satisfying the condition that

$$\alpha_v \in \mathrm{im}(\mathrm{CH}^i(\mathcal{X}_{\mathcal{O}_{K_v}}, a, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}^i(X_{K_v}, a, \mathbb{Z}/n\mathbb{Z}))$$

for almost all $v \in \mathcal{S}$.

Remark 2.2.

(1) *This definition does not depend on the choice of $\mathcal{S}$ nor on that of $\mathcal{X}$ by standard spreading-out arguments.*

(2) *For $a = 0$, the restricted product of Chow groups agrees with the unrestricted product, since the restriction map $\mathrm{CH}^i(\mathcal{X}_{\mathcal{O}_{K_v}}) \to \mathrm{CH}^i(X_{K_v})$ is surjective. This does not hold for higher Chow groups, which is why we need to introduce the restricted product of Chow groups.*

(3) *Restricted products of étale cohomology groups are defined in an analogous manner. In particular, the étale cycle class map restricts to a morphism*

$$\prod_{v \in P_K}' \mathrm{CH}^i(X_{K_v}, a, \mathbb{Z}/n\mathbb{Z}) \to \prod_{v \in P_K}' \hat{H}^{2i-a}_{\text{ét}}(X_{K_v}, \mu_n^{\otimes i}).$$

(4) *By spreading out cycles, one sees that the pullback map*

$$\mathrm{CH}^i(X, a, \mathbb{Z}/n\mathbb{Z}) \to \prod_{v \in P_K} \mathrm{CH}^i(X_{K_v}, a, \mathbb{Z}/n\mathbb{Z})$$

factors over the inclusion of the restricted product.

In certain degrees, one can detect the restriced product fully on the level of étale cohomology:

Lemma 2.3. *For $i \geq d+1$, the diagram*

$$\begin{array}{ccc} \prod'_{v \in P_K} \mathrm{CH}^i(X_{K_v}, a, \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & \prod'_{v \in P_K} \hat{H}^{2i-a}_{\text{ét}}(X_{K_v}, \mu_n^{\otimes i}) \\ \downarrow & & \downarrow \\ \prod_{v \in P_K} \mathrm{CH}^i(X_{K_v}, a, \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & \prod_{v \in P_K} \hat{H}^{2i-a}_{\text{ét}}(X_{K_v}, \mu_n^{\otimes i}) \end{array}$$

is a pullback square of abelian groups.

Proof. Let $S \subset \mathrm{Spec}(\mathcal{O}_K)$ a dense open over which X has a smooth projective model $\mathcal{X}$ and over which n is invertible. For a place $v \in P_K$ lying in S, consider the commutative diagram of localization sequences

$$\begin{array}{ccccc} \mathrm{CH}^i(\mathcal{X}_{\mathcal{O}_{K_v}}, a, \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & \mathrm{CH}^i(X_{K_v}, a, \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & \mathrm{CH}^{i-1}(\mathcal{X}_{k_v}, a-1, \mathbb{Z}/n\mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow \\ H^{2i-a}_{\text{ét}}(\mathcal{X}_{\mathcal{O}_{K_v}}, \mu_n^{\otimes i}) & \longrightarrow & H^{2i-a}_{\text{ét}}(X_{K_v}, \mu_n^{\otimes i}) & \longrightarrow & H^{2i-a-1}_{\text{ét}}(\mathcal{X}_{k_v}, \mu_n^{\otimes i-1}). \end{array}$$

The localization sequence for higher Chow groups is proved in [27, Thm. 1.7]. The localization sequence for étale cohomology follows from the exact sequence for étale cohomology with support

and absolute purity since $\mathcal{X}_{k_v}$ is regular (see [9, Thm. 2.1.1]). Now by [25, Theorem 9.3] all vertical morphisms are isomorphisms for $a \geq 0$ and $i \geq d+1$. In particular, in this range an element of $\mathrm{CH}^i(X_{K_v}, a, \mathbb{Z}/n\mathbb{Z})$ is unramified iff its image in $H^{2i-a}_{\text{ét}}(X_{K_v}, \mu_n^{\otimes i})$ is unramified. □

3. The Tate-Poitou exact sequence

We recall results of Tate and Poitou (see [38, Thm. 3.1]). These will be generalised in many directions in higher dimension by the Kato conjectures, higher dimensional class field theory and local to global principles for (higher) Chow groups.

We introduce the following notation: let K be a global field. Let S be a non-empty and possibly infinite set of prime divisors of K containing the archimedean primes if K is a number field. Let $\mathcal{O}_S$ be the ring of elements in K which are integers at all primes $\mathfrak{p} \notin S$. Let K_S denote the maximal extension of K in K^s that is ramified over K only at primes in S. Let $G_S := \mathrm{Gal}(K_S/K)$. Let M be a finite G_S-module of order n which is invertible on K_S. Let $D(M) := \mathrm{Hom}(M, \mathbb{G}_m)$. For an abelian group A let $A^\vee := \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$.

Theorem 3.1. (1) *For $i \geq 3$ the natural map*

$$H^i(K_S, M) \to \prod_{v \in S} H^i(K_v, M)$$

is an isomorphism.

(2) *There is an exact nine-term sequence*

$$\begin{aligned} 0 \to H^0(K_S, M) \to \prod_{v\in S}{}' H^0(K_v, M) \to H^2(K_S, D(M))^\vee \to \\ H^1(K_S, M) \xrightarrow{\alpha_1} \prod_{v\in S}{}' H^1(K_v, M) \xrightarrow{\beta_1} H^1(K_S, D(M))^\vee \to \\ H^2(K_S, M) \xrightarrow{\alpha_2} \prod_{v\in S}{}' H^2(K_v, M) \xrightarrow{\beta_2} H^0(K_S, D(M))^\vee \to 0 \end{aligned}$$

Here $\prod'_{v\in S}$ denotes the restricted product with respect to the subgroups $H^i(\mathcal{O}_{K_v}, M)$. At the archimedean places we assume the cohomology groups to be the completed cohomology groups (see [37, Ch. VIII]). Note that the restricted product in the first line becomes a direct product since $H^0(\mathcal{O}_{K_v}, M) \to H^0(K_v, M)$ is surjective and that the restricted product in the third line becomes a direct sum since $H^2(\mathcal{O}_{K_v}, M) = 0$.

For $M = \mu_n$ the exact sequence of Theorem 3.1(2) encodes fundamental theorems in algebraic number theory. Let $S = P_K$. Since $H^1(K, \mu_n) \cong K^\times/n$ and $H^1(K, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Gal}(K^{\mathrm{ab}}/K)/n$ one recovers the class field theory isomorphism

$$C_K/n := \mathrm{coker}[K^\times/n \to \prod_{v\in P}{}' K_v^\times/n] \xrightarrow{\cong} \mathrm{Gal}(K^{\mathrm{ab}}/K)/n$$

from the second line since β_1 is surjective by the density of the Frobenii. Since $H^2(K, \mu_n) \cong \mathrm{Br}(K)[n]$, the third line recovers, after passing to the direct limit, the Brauer-Hasse-Noether exact sequence

$$0 \to \mathrm{Br}(K) \to \bigoplus_{v\in P_K} \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z} \to 0.$$

Noting that $\mathrm{CH}^a(K, a) \cong K_a^M(K)$ (see [2, Thm. 6.1], [31, Thm. 4.9] and [39]) and applying $\mathrm{Hom}(_, \mathbb{Q}/\mathbb{Z})$ to the Brauer-Hasse-Noether sequence, these may be interpreted as results about (higher) Chow groups. The second line then becomes

$$C_K/n := \mathrm{coker}[\mathrm{CH}^1(K, 1, \mathbb{Z}/n\mathbb{Z}) \to \prod_{v\in P}{}' \mathrm{CH}^1(K_v, 1, \mathbb{Z}/n\mathbb{Z})] \xrightarrow{\cong} \mathrm{Gal}(K^{\mathrm{ab}}/K)/n$$

asserting Conjecture 1.3 for $\mathrm{Spec}(K)$ and $i = a = 1$. The third line becomes

$$\varprojlim_n \mathrm{CH}_0(K)/n \to \varprojlim_n \prod_{v\in P}{}' \mathrm{CH}_0(K_v)/n \to \mathrm{Hom}(\mathrm{Br}(K), \mathbb{Q}/\mathbb{Z}) \tag{3.1}$$

which asserts Conjecture 1.1 for $\mathrm{Spec}(K)$ for any global field K (see also [42, Rem. 1.1]). For the latter statement one may also consider the Tate-Poitou exact sequence for $M = \mathbb{Z}/n\mathbb{Z}$. In this case the first line becomes

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \prod'_{v\in P_K} \mathbb{Z}/n\mathbb{Z} & \longrightarrow & H^2(K,\mu_n)^\vee = (\mathrm{Br}(X)[n])^\vee \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow = \\
0 & \longrightarrow & H^0(K,\mathbb{Z}/n) & \longrightarrow & \prod'_{v\in P_K} H^0(K_v,\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & H^2(K,\mu_n)^\vee
\end{array}$$

and taking the projective limit over all n gives (3.1).

4. The Kato conjectures

In this section we state some facts about the Kato conjectures. These conjectures concern the Kato complexes and generalise the Brauer-Hasse-Noether sequence (interpreted as a sequence of étale cohomology groups) to arbitrary dimensional schemes.

4.1. **Kato complexes and conjectures.** We introduce the following notation for Kato complexes:

Definition 4.1. (1) *For X a scheme over a finite field or the ring of integers in a number field or local field, we denote the complexes*

$$\ldots \to \bigoplus_{x\in X_a} H^{a+1}(k(x),\mathbb{Z}/n(a)) \to \ldots \to \bigoplus_{x\in X_{a-1}} H^a(k(x),\mathbb{Z}/n(a-1)) \to \ldots$$
$$\ldots \to \bigoplus_{x\in X_1} H^2(k(x),\mathbb{Z}/n(1)) \to \bigoplus_{x\in X_0} H^1(k(x),\mathbb{Z}/n)$$

by $KC^{(0)}(X,\mathbb{Z}/n\mathbb{Z})$. Here the term $\oplus_{x\in X_a} H^{a+1}(k(x),\mathbb{Z}/n(a))$ is placed in degree a. We set

$$KH_a^{(0)}(X,\mathbb{Z}/n\mathbb{Z}) := H_a(KC^{(0)}(X,\mathbb{Z}/n\mathbb{Z})).$$

The groups $H^{a+1}(k(x),\mathbb{Z}/n(a))$ are the étale cohomology groups of $\mathrm{Spec}k(x)$ *with coefficients in $\mathbb{Z}/n(a) := \mu_n^{\otimes a}$ if n is invertible on X and $\mathbb{Z}/n(a) := W_r\Omega^a_{X_1,\log}[-a]\oplus\mathbb{Z}/m(a)$ if $n = mp^r, (m,p) = 1$, is not invertible on X and X is smooth over a field of characteristic p.*

(2) *For X a scheme of finite type over a number field K or $K_v, v \in P_K$, we denote the complexes*

$$\ldots \to \bigoplus_{x\in X_a} H^{a+2}(k(x),\mathbb{Z}/n(a+1)) \to \ldots \to \bigoplus_{x\in X_{a-1}} H^{a+1}(k(x),\mathbb{Z}/n(a)) \to \ldots$$
$$\ldots \to \bigoplus_{x\in X_1} H^3(k(x),\mathbb{Z}/n(2)) \to \bigoplus_{x\in X_0} H^2(k(x),\mathbb{Z}/n(1))$$

by $KC^{(1)}(X,\mathbb{Z}/n\mathbb{Z})$ and set

$$KH_a^{(1)}(X,\mathbb{Z}/n\mathbb{Z}) := H_a(KC^{(1)}(X,\mathbb{Z}/n\mathbb{Z})).$$

(3) *Let K be a global field with ring of integers $\mathcal{O}_K$. Let $U \subset \mathrm{Spec}\mathcal{O}_K$ be a non-empty open subscheme and X of finite type over U. Then there is a natural restriction map*

$$KC^{(0)}(X,\mathbb{Z}/n\mathbb{Z})[1] \to KC^{(1)}(X_K,\mathbb{Z}/n\mathbb{Z}) \to KC^{(1)}(X_{K_v},\mathbb{Z}/n\mathbb{Z}).$$

We define

$$KC^{(0)}(X/U,\mathbb{Z}/n\mathbb{Z}) := \mathrm{cone}[KC^{(0)}(X,\mathbb{Z}/n\mathbb{Z})[1] \to \bigoplus_{v\in\sum_U} KC^{(1)}(X_{K_v},\mathbb{Z}/n\mathbb{Z})],$$

where $\sum_U$ denotes the set of places $v \in P_K$ which do not correspond to closed points of U. We set

$$KH_a^{(0)}(X/U,\mathbb{Z}/n\mathbb{Z}) := H_a(KC^{(0)}(X/U,\mathbb{Z}/n\mathbb{Z})).$$

Remark 4.2. *In* [20]*, Kato constructs the above complexes in greater generality. Let X be an excellent scheme, $n \in \mathbb{Z} - \{0\}, q, i \in \mathbb{Z}$ and assume that in the case $q = i+1$, for any prime divisor p of n and for any $x \in X_0$ such that $char(k(x)) = p$, we have $[k(x) : k(x)^p] \geq p^i$. Then there are complexes*

$$C_n^i(X) : ... \to \bigoplus_{x\in X_a} H^{i+a+1}(k(x),\mathbb{Z}/n(i+a)) \to ... \to \bigoplus_{x\in X_1} H^{i+2}(k(x),\mathbb{Z}/n(i+1)) \to \bigoplus_{x\in X_0} H^{i+1}(k(x),\mathbb{Z}/n(i))$$

Again the term $\oplus_{x\in X_a} H^{i+a+1}(k(x),\mathbb{Z}/n(i+a))$ is placed in degree a and the homology of $C_n^i(X)$ in degree a is denoted by $KH_a^{(i)}(X,\mathbb{Z}/n\mathbb{Z})$.

It is shown in [19]*, that these complexes coincide up to sign with the complexes arising from the appropriate homology theories via the niveau spectral sequence.*

Definition 4.1(3) is introduced in [25, p. 124] *as a variant of the above complexes.*

Note that if X is of finite type over the ring of integers $\mathcal{O}_{K_v}$ in a local field K_v, then by definition there is an exact triangle

$$KC^{(0)}(X,\mathbb{Z}/n\mathbb{Z})[1] \to KC^{(1)}(X_{K_v},\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\partial} KC^{(0)}(X_v,\mathbb{Z}/n\mathbb{Z}) \to KC^{(0)}(X,\mathbb{Z}/n\mathbb{Z})$$

which induces an exact sequence of homology groups

(4.1)

$$.. \to KH_{a+1}^{(0)}(X,\mathbb{Z}/n\mathbb{Z}) \to KH_a^{(1)}(X_{K_v},\mathbb{Z}/n\mathbb{Z}) \to KH_a^{(0)}(X_v,\mathbb{Z}/n\mathbb{Z}) \to KH_a^{(0)}(X,\mathbb{Z}/n\mathbb{Z}) \to ..$$

Let us state Kato's conjectures for the above complexes.

Conjecture 4.3. *(*[20, Conj. 0.3]*) Let X be a proper and smooth scheme over a finite field. Then*

$$KH_a^{(0)}(X,\mathbb{Z}/n\mathbb{Z}) = 0 \quad \text{for} \quad a > 0.$$

Conjecture 4.4. *(*[20, Conj. 5.1]*) Let X be a regular scheme proper and flat over* $\mathrm{Spec}(\mathcal{O}_K)$*, where $\mathcal{O}_K$ is the ring of integers in a local field. Then*

$$KH_a^{(0)}(X,\mathbb{Z}/n\mathbb{Z}) = 0 \quad \text{for} \quad a \geq 0.$$

Conjecture 4.5. *(*[20, Conj. 0.4]*) Let X be a proper and smooth scheme over a global field K. Then the map*

$$KH_a^{(1)}(X,\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} \bigoplus_{v\in P_K} KH_a^{(1)}(X_{K_v},\mathbb{Z}/n\mathbb{Z})$$

is an isomorphism for $a > 0$ and the sequence

$$0 \to KH_0^{(1)}(X,\mathbb{Z}/n\mathbb{Z}) \to \bigoplus_{v\in P_K} KH_0^{(1)}(X_{K_v},\mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z} \to 0$$

is exact.

Conjecture 4.6. *(*[20, Conj. 0.5]*) Let $\mathcal{X}$ be a regular scheme proper and flat with smooth generic fiber over a non-empty open subscheme $U \subset$* $\mathrm{Spec}(\mathcal{O}_K)$*, where $\mathcal{O}_K$ is the ring of integers in a global field. Then*

$$KH_a^{(0)}(\mathcal{X}/U,\mathbb{Z}/n\mathbb{Z}) = 0 \quad \text{for} \quad a \geq 0.$$

We add the following conjecture to the list:

Conjecture 4.7. *Let $\mathcal{O}_S$ be a regular semilocal subring of a global field K. Let $\mathcal{S} = \mathrm{Spec}\mathcal{O}_S$. Let $\mathcal{X}$ be a regular scheme proper and flat over $\mathcal{S}$ with smooth generic fiber X. Let*

$$KC^{(0)}(\mathcal{X}/S, \mathbb{Z}/n\mathbb{Z}) = \mathrm{coker}[KC^{(0)}(\mathcal{X}, \mathbb{Z}/n\mathbb{Z})[1] \to \bigoplus_{v \in \sum_S} KC^{(1)}(X_{K_v}, \mathbb{Z}/n\mathbb{Z})],$$

where $\sum_S$ denotes the set of places $v \in P_K$ which do not correspond to closed points of S. Then

$$KH_a^{(0)}(\mathcal{X}/S, \mathbb{Z}/n\mathbb{Z}) = 0 \quad \text{for} \quad a > 0.$$

The following is known about these conjectures:

Theorem 4.8. *(*[25, Thm. 8.1]*) Conjecture 4.3 holds if n is invertible on X. If n is not invertible on X, then it holds for $a \leq 4$.*

Theorem 4.9. *(*[25, Thm. 8.1]*) Conjecture 4.4 holds if n is invertible on X.*

Theorem 4.10. *(*[15, Thm. 0.9], [25, Thm. 8.3]*) Conjecture 4.5 holds if n is invertible on X.*

Theorem 4.11. *(*[25, Thm. 8.4]*) Conjecture 4.6 holds if n is invertible on U.*

Theorem 4.12. *Conjecture 4.7 holds if n is invertible on $\mathcal{S}$.*

Proof. The proof is analogous to the proof of Theorem 4.11 in *loc. cit.*. In fact,

$$KH_a^{(1)}(X, \mathbb{Z}/n\mathbb{Z}) \cong KH_a^{(0)}(\mathcal{X}/S, \mathbb{Z}/n\mathbb{Z})$$

since by Theorem 4.9 and (4.1) there is an isomorphism

$$KH_a^{(1)}(X_{K_v}, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\delta} KH_a^{(0)}(X_v, \mathbb{Z}/n\mathbb{Z})$$

for $v \in S$. □

4.2. **Applications.** We now introduce the central method of this article which is the comparison of the Zariski and the étale motivic cohomology of a regular scheme X over a field or Dedekind domain by analysing and comparing the respective coniveau spectral sequences. The difference may in some cases be measured by the Kato conjectures. This allows in some cases to use the finiteness of étale cohomology and the vanishing of the Kato homology to deduce finiteness results for Chow groups.

Lemma 4.13. *Let X be a regular irreducible scheme. Let A be a locally constant constructible sheaf such that the stalks are n-torsion abelian groups with n invertible on X. Then the coniveau spectral sequence for étale cohomology*

$$E_1^{p,q}(X, A) = \bigoplus_{x \in X^p} H_x^{p+q}(X_{\text{ét}}, A) \Rightarrow H_{\text{ét}}^{p+q}(X, A)$$

converges and

$$\bigoplus_{x \in X^p} H_x^{p+q}(X, A) \cong \bigoplus_{x \in X^p} H_{\text{ét}}^{q-p}(x, A(-p)).$$

Here $A(i) := A \otimes \mu_n^{\otimes i}$.

Proof. The existence of the spectral sequence is shown in this generality, in fact only assuming that X is equidimensional and noetherian, in [6]. The second statement follows from Gabber's absolute purity theorem (see [9]) since X is regular. □

Lemma 4.14. *(*[11, Prop. 2.1]*) Let X be essentially of finite type over a Dedekind domain S. Then the spectral sequence*

$${}^{\mathrm{CH}}E_1^{p,q}(X, r) = \bigoplus_{x \in X^p} \mathrm{CH}^{r-p}(x, -p-q) \Rightarrow \mathrm{CH}^r(X, -p-q)$$

converges.

Recall the following theorem of Voevodsky:

Theorem 4.15. *(Beilinson-Lichtenbaum conjecture, see* [41]*) Let X be a smooth scheme over a field k and let n be invertible on X. Then the étale cycle map*

$$\rho_X^{r,2r-s} : \mathrm{CH}^r(X, 2r-s, \mathbb{Z}/n\mathbb{Z}) \to H^s_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(r))$$

is an isomorphism for $s \leq r$.

We will need the following general lemma which was originally observed by Jannsen and Saito for schemes over finite fields in [16, Lem. 6.2].

Lemma 4.16. *Let $\mathcal{X}$ be a regular scheme of relative dimension d over the spectrum of a field $\mathcal{S} = \mathrm{Spec}(K)$ or a Dedekind scheme $\mathcal{S} = \mathrm{Spec}(\mathcal{O})$. Let n be invertible on $\mathcal{S}$. For a field k let $cd_n(k)$ denote the $\mathbb{Z}/n\mathbb{Z}$-cohomological dimension of k. Then the following statements hold:*

(1) *(*[40, Thm. 8]*) Let $\mathcal{S} = \mathrm{Spec}(K)$ be a field and $X := \mathcal{X}$. Assume that $c = cd_n(K) - 1$. Then the sequence*

$$\begin{aligned} ... \to KH^{(c)}_{q-c+2}(X, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}^{d+c}(X, q, \mathbb{Z}/n\mathbb{Z}) \to H^{2(d+c)-q}_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(d+c)) \\ \to KH^{(c)}_{q-c+1}(X, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}^{d+c}(X, q-1, \mathbb{Z}/n\mathbb{Z}) \to ... \end{aligned} \tag{4.2}$$

is exact.

(2) *Let $\mathcal{S} = \mathrm{Spec}(\mathcal{O})$ be of dimension 1. Assume that $c = cd_n(K) - 2$ and $c = cd_n(k(x)) - 1$ for $x \in \mathcal{S}_{(0)}$. Then the sequence*

$$\begin{aligned} ... \to KH^{(c)}_{q-c+2}(\mathcal{X}, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}^{d+1+c}(\mathcal{X}, q, \mathbb{Z}/n\mathbb{Z}) \to H^{2(d+1+c)-q}_{\text{ét}}(\mathcal{X}, \mathbb{Z}/n\mathbb{Z}(d+1+c)) \\ \to KH^{(c)}_{q-c+1}(\mathcal{X}, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}^{d+1+c}(\mathcal{X}, q-1, \mathbb{Z}/n\mathbb{Z}) \to ... \end{aligned} \tag{4.3}$$

is exact.

Proof. For the proof of (1) we refer to *loc. cit.*. The proof of (2) is analogous. In both cases one uses Theorem 4.15 to identify the spectral sequences of Lemma 4.13 and Lemma 4.14 in the range below the Kato complexes. □

Remark 4.17. *The cohomological dimension of a local field is* 2 *(see e.g.* [13, Exp. 10, Thm. 2.1]*). The cohomological dimension of a global field k is* 2 *in the following cases: (a) k is a number field and n is odd. (b) k is a number field and does not have any real places. (c) k is a function field of one variable over a finite field and n is prime to* $\mathrm{ch}(k)$*. This follows from Theorem 3.1(1). In all of these cases $c = 1$.*

We can determine the higher Chow groups of schemes over local fields appearing in the local to global principle of Theorem 1.5 (see also [11, Sec. 5.2]).

Proposition 4.18. *Let K_v be a non-archimedean local field with residue field k_v of characteristic p. Let X_{K_v} be a quasi-projective scheme over K_v. Let $d = \dim X_{K_v}$. Let $(n, p) = 1$. Then the groups $\mathrm{CH}^{d+1}(X_{K_v}, q, \mathbb{Z}/n\mathbb{Z})$ are finite for all q.*

Proof. Let X be a model of X_{K_v} over $\mathcal{O}_{K_v}$ with special fiber X_{k_v}. Consider the exact localization sequence

$$\begin{aligned} ... \to \mathrm{CH}^d(X_{k_v}, q, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}^{d+1}(X, q, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}^{d+1}(X_{K_v}, q, \mathbb{Z}/n\mathbb{Z}) \to \\ \mathrm{CH}^d(X_{k_v}, q-1, \mathbb{Z}/n\mathbb{Z}) \to ... \end{aligned}$$

from [27, Thm. 1.7]. By [25, Cor. 9.4] the groups $\mathrm{CH}^d(X_{k_v}, q, \mathbb{Z}/n\mathbb{Z})$ and $\mathrm{CH}^{d+1}(X_{K_v}, q, \mathbb{Z}/n\mathbb{Z})$ are finite. This implies the proposition. □

Lemma 4.19. *Let X be of finite type over a number field K. Let v be an infinite place of K and n odd or v complex. Then the complex*

$$KC^{(1)}(X_{K_v}, \mathbb{Z}/n\mathbb{Z})$$

is zero.

$$\begin{array}{lcccccc}
d+2 & H^{d+2}(k(\eta),\Lambda(d+1)) \to & \bigoplus_{x\in X^1} H^{d+1}(k(x),\Lambda(d)) \to & \dots \to & \bigoplus_{x\in X^d} H^2(k(x),\Lambda(1)) \to \\
d+1 & H^{d+1}(k(\eta),\Lambda(d+1)) \to & \bigoplus_{x\in X^1} H^{d}(k(x),\Lambda(d)) \to & \dots \to & \bigoplus_{x\in X^d} H^1(k(x),\Lambda(1)) \to \\
\dots & & \dots & & \\
1 & & \dots & & \\
0 & & & & \\
 & 0 & 1 & \dots & d
\end{array}$$

FIGURE 1. Table of $E^1_{p,q}(X,\Lambda(j))$ for X/S of relative dimension d, $j=d+1$ and $\Lambda := \mathbb{Z}/n\mathbb{Z}$.

Proof. All the groups $H^{a+1}(k(x),\mathbb{Z}/n(a))$ appearing in the complex $KC^{(1)}(X_{K_v},\mathbb{Z}/n\mathbb{Z})$ are zero. If v is a complex place, then its cohomological dimension is zero and the groups $H^{a+1}(k(x),\mathbb{Z}/n(a))$ vanish for all n. If v is a real place, then the groups $H^{a+1}(k(x),\mathbb{Z}/n(a))$ are 2-groups and vanish for all n prime to 2. □

Lemma 4.20. *Let K_v be a non-archimedean local field with residue field k_v of characteristic p. Let X_{K_v} be a proper smooth scheme over K_v. Let $d = \dim X_{K_v}$. Then the following statements hold:*

(1) *If n is prime to p, then the groups*
$$KH^{(1)}_a(X_{K_v},\mathbb{Z}/n\mathbb{Z})$$
are finite.

(2) *If X_{K_v} satisfies condition $(\star)$, then the groups*
$$KH^{(1)}_a(X_{K_v},\mathbb{Z}/p^n\mathbb{Z})$$
are finite for $a \le 2$. Furthermore, $KH^{(1)}_1(X_{K_v},\mathbb{Z}/p^n\mathbb{Z}) = 0$.

(3) *If $d=2$ and X_{K_v} satisfies condition $(\star\star)$, then the groups*
$$KH^{(1)}_a(X_{K_v},\mathbb{Z}/p^n\mathbb{Z})$$
are finite for $a \le 2$ (and hence for all a).

Proof. (1) This follows from the exact sequence (4.1) and Theorems 4.8 and 4.9.

(2) By Lemma 4.16 there is an exact sequence
$$\dots \to H^{2d}_{\text{ét}}(X_{K_v},\mathbb{Z}/p^n\mathbb{Z}(d+1)) \to KH^{(1)}_2(X_{K_v},\mathbb{Z}/p^n\mathbb{Z}) \to \mathrm{CH}^{d+1}(X_{K_v},1,\mathbb{Z}/p^n\mathbb{Z})$$
$$\xrightarrow{\cong} H^{2d+1}_{\text{ét}}(X_{K_v},\mathbb{Z}/p^n\mathbb{Z}(d+1)) \to KH^{(1)}_1(X_{K_v},\mathbb{Z}/p^n\mathbb{Z}) \to \mathrm{CH}^{d+1}(X_{K_v},0,\mathbb{Z}/p^n\mathbb{Z}) = 0$$
The isomorphism is shown in [18, Thm. 6] assuming condition $(\star)$. The finiteness of the étale cohomology groups for schemes over local fields (for a proof see [8, p. 773]) therefore implies the statement. Furthermore, since $\mathrm{CH}^{d+1}(X_{K_v},0,\mathbb{Z}/p^n\mathbb{Z}) = 0$ for dimension reasons, the group $KH^{(1)}_1(X_{K_v},\mathbb{Z}/p^n\mathbb{Z}) = 0$.

(3) Fix a strictly semistable model of X_{K_v} and denote the configuration complex of the special fiber X_v of X_{K_v} by Γ_{X_v}. Together with the Bloch-Kato conjecture, [25, Lemma 7.6] gives a short exact sequence
$$0 \to KH^{(1)}_{a+1}(X_{K_v},\mathbb{Q}_p/\mathbb{Z}_p)/p^n \to KH^{(1)}_a(X_{K_v},\mathbb{Z}/p^n\mathbb{Z}) \to KH^{(1)}_a(X_{K_v},\mathbb{Q}_p/\mathbb{Z}_p)[p^n] \to 0$$
where the left group vanishes for $a \ge 2$ by dimension reasons. Furthermore [17, Thm. 1.4] and [17, Thm. 1.6] yield isomorphisms
$$KH^{(1)}_a(X_{K_v},\mathbb{Q}_p/\mathbb{Z}_p) \cong KH^{(0)}_a(X_v,\mathbb{Q}_p/\mathbb{Z}_p) \cong H_a(\Gamma_{X_v},\mathbb{Q}_p/\mathbb{Z}_p)$$

which imply the finiteness results as the configuration complex of the special fiber Γ_{X_v} (see [17, p. 483]) is a finite simplicial complex. □

Remark 4.21. *For a regular and projective scheme X over a finite field of characteristic p it is shown in* [16] *that the Kato conjecture holds with $\mathbb{Z}/p^r$-coefficients for $a \leq 4$, i.e.*

$$KH_a(X, \mathbb{Z}/p^r) = \begin{cases} \mathbb{Z}/p^r & \text{for} \quad a = 0 \\ 0 & \text{otherwise.} \end{cases}$$

In [17, Sec. C]*, Jannsen and Saito define a suitable homology theory with $\mathbb{Z}/p^r$-coefficients for a scheme X over a discrete valuation ring A using p-adic étale Tate twists:*

$$KH_a(X/S, \mathbb{Z}/p^r(-1)) := H^{-a}(X_{et}, Rf^!\mathbb{Z}/p^r(1)_S)$$

with $S = \mathrm{Spec}(A)$ and

$$\mathbb{Z}/p^r(1)_S := \mathrm{cone}(Rj_*(\mathbb{Z}/p^r(1)_\eta) \to i_*(\mathbb{Z}/p^r)_s[-1])[-1].$$

Unfortunately this theory cannot be used as in the approach of [25, Sec. 3 (3.11)] *to show a Lefschetz theorem implying that $KH_a(X/S, \mathbb{Z}/p^r) = 0$ for $a \leq 4$ since there is no appropriate base change and and Artin vanishing for p-adic étale Tate twists. We are therefore obliged to use Lemma 4.20(2) which is implied by class field theory.*

5. Main theorem: a local to global principle

In this section we prove our our main Theorem 1.5.

Theorem 5.1. *Let K be a global field, S a set of places of K and $n \in \mathbb{N}_{>1}$. Denote by $\mathcal{O}_S$ the ring of elements in K which are integers at all primes $\mathfrak{p} \notin S$ and let $\mathcal{S} = \mathrm{Spec}\mathcal{O}_S$. Suppose that*

(a) *$\mathcal{S}$ is either semi-local or an open of $\mathrm{Spec}(\mathcal{O}_K)$.*
(b) *n is invertible on $\mathcal{S}$.*
(c) *If K is a number field, S contains all archimedean places of K and either n is odd or K has no real places.*

Let $\mathcal{X}$ be regular, flat and projective of relative dimension d over $\mathcal{S}$ with smooth generic fiber X. Then the following statements hold:

(1) *There is an exact sequence*

$$\mathrm{CH}^{d+1}(\mathcal{X}, a, \mathbb{Z}/n\mathbb{Z}) \to \prod_{v \in S}{}' \mathrm{CH}^{d+1}(X_{K_v}, a, \mathbb{Z}/n\mathbb{Z}) \to H^a_{\text{ét}}(\mathcal{X}, \mathbb{Z}/n\mathbb{Z})^\vee$$

(2) *For all a there is a natural surjection $Ш(\mathrm{CH}^{d+1}(\mathcal{X}, a, \mathbb{Z}/n\mathbb{Z})) \to Ш^{2d+2-a,d+1}_{\text{ét}}(\mathcal{X})$.*
(3) *If K is a number field and condition $(\star)$ holds for X_{K_v} if v divides n, then the group $Ш(\mathrm{CH}^{d+1}(\mathcal{X}, 1, \mathbb{Z}/n\mathbb{Z}))$ is finite.*
(4) *If K is a function field of one variable over a finite field and n is invertible in K, then $Ш(\mathrm{CH}^{d+1}(\mathcal{X}, a, \mathbb{Z}/n\mathbb{Z}))$ is finite for arbitrary a.*

Proof. If $\dim(\mathcal{S}) = 1$, consider the following commutative diagram:

$$\begin{array}{ccc}
KH^{(0)}_{a+2}(\mathcal{X}/S, \mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\cong} & \prod_{v \in S} KH^{(1)}_{a+1}(X_{K_v}, \mathbb{Z}/n\mathbb{Z}) \\
\downarrow & & \downarrow \\
\mathrm{CH}^{d+1}(\mathcal{X}, a, \mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\alpha} & \prod'_{v \in S} \mathrm{CH}^{d+1}(X_{K_v}, a, \mathbb{Z}/n\mathbb{Z}) \\
\downarrow & & \downarrow \\
H^{2d+2-a}_{\text{ét}}(\mathcal{X}, \mathbb{Z}/n\mathbb{Z}(d+1)) & \longrightarrow & \prod'_{v \in S} H^{2d+2-a}_{\text{ét}}(X_{K_v}, \mathbb{Z}/n\mathbb{Z}(d+1)) \\
\downarrow & & \downarrow \\
KH^{(0)}_{a+1}(\mathcal{X}/S, \mathbb{Z}n\mathbb{Z}) & \xrightarrow{\cong} & \prod_{v \in S} KH^{(1)}_{a}(X_{K_v}, \mathbb{Z}/n\mathbb{Z})
\end{array}$$

The isomorphisms in the first and the last row follow from Theorem 4.11 and 4.12. The collumns are exact by Lemma 4.16 and by the assumptions on K and n. The only place where this is not immediate is $\prod'_{v\in S} H^{2d+2-a}_{\text{ét}}(X_{K_v},\mathbb{Z}/n\mathbb{Z}(d+1))$. However, let

$$\prod_v a_v \in \ker[\prod_{v\in S}' H^{2d+2-a}_{\text{ét}}(X_{K_v},\mathbb{Z}/n\mathbb{Z}(d+1)) \to \prod_{v\in S} KH^{(1)}_a(X_{K_v},\mathbb{Z}/n\mathbb{Z})].$$

Then if $a_v \in H^{2d+2-a}_{\text{ét}}(\mathcal{X}_{\mathcal{O}_{K_v}},\mathbb{Z}/n\mathbb{Z}(d+1))$ for a place v at which $\mathcal{X}$ has good reduction and where n is invertible, then a_v comes from an element of $\mathrm{CH}^{d+1}(\mathcal{X}_{\mathcal{O}_{K_v}},a,\mathbb{Z}/n\mathbb{Z})$. This follows from the commutative diagram

$$\begin{array}{ccc}
\mathrm{CH}^{d+1}(\mathcal{X}_{\mathcal{O}_{K_v}},a,\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & \mathrm{CH}^{d+1}(X_{K_v},a,\mathbb{Z}/n\mathbb{Z}) \\
\downarrow\cong & & \downarrow \\
H^{2(d+1)-a}_{\text{ét}}(\mathcal{X}_{\mathcal{O}_{K_v}},\mathbb{Z}/n\mathbb{Z}(d+1)) & \longrightarrow & H^{2(d+1)-a}_{\text{ét}}(X_{K_v},\mathbb{Z}/n\mathbb{Z}(d+1))
\end{array}$$

(see Lemma 2.3). The third row is part of the generalisation of Saito's Poitou-Tate exact sequence

$$H^{2d+2-a}_{\text{ét}}(\mathcal{X},\mathbb{Z}/n\mathbb{Z}(d+1)) \to \prod_{v\in S}' H^{2d+2-a}_{\text{ét}}(X_{K_v},\mathbb{Z}/n\mathbb{Z}(d+1)) \to H^a_{\text{ét}}(\mathcal{X},\mathbb{Z}/n\mathbb{Z})^\vee \to \dots$$

by Geisser and Schmidt (see [12]). The two middle rows also fit into the commutative diagram

$$\begin{array}{ccccc}
Ш(\mathrm{CH}^{d+1}(\mathcal{X},a,\mathbb{Z}/n\mathbb{Z})) & \hookrightarrow & \mathrm{CH}^{d+1}(\mathcal{X},a,\mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\alpha} & \prod'_{v\in S}\mathrm{CH}^{d+1}(X_{K_v},a,\mathbb{Z}/n\mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow \\
Ш^{2d+2-a,d-1}_{\text{ét}}(\mathcal{X}) & \hookrightarrow & H^{2d+2-a}_{\text{ét}}(\mathcal{X},\mathbb{Z}/n\mathbb{Z}(d+1)) & \longrightarrow & \prod'_{v\in S} H^{2d+2-a}_{\text{ét}}(X_{K_v},\mathbb{Z}/n\mathbb{Z}(d+1)).
\end{array}$$

The statements (1) and (2) now follow from a diagram chase.

For the case of $S = P_K$, i.e. $\mathcal{S} = \mathrm{Spec}K$ and $\dim\mathcal{S} = 0$, note first that all relevant groups vanish for $a = 0$, so assume $a \geq 1$. The proof then procedes in the same way, except that we start with the commutative diagram

$$\begin{array}{ccc}
KH^{(1)}_{a+1}(X,\mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\cong} & \prod_{v\in P_K} KH^{(1)}_{a+1}(X_{K_v},\mathbb{Z}/n\mathbb{Z}) \\
\downarrow & & \downarrow \\
\mathrm{CH}^{d+1}(X,a,\mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\alpha} & \prod'_{v\in P_K}\mathrm{CH}^{d+1}(X_{K_v},a,\mathbb{Z}/n\mathbb{Z}) \\
\downarrow & & \downarrow \\
H^{2d+2-a}_{\text{ét}}(X,\mathbb{Z}/n\mathbb{Z}(d+1)) & \longrightarrow & \prod'_{v\in P_K} H^{2d+2-a}_{\text{ét}}(X_{K_v},\mathbb{Z}/n\mathbb{Z}(d+1)) \\
\downarrow & & \downarrow \\
KH^{(1)}_a(X,\mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\cong} & \prod_{v\in P_K} KH^{(1)}_a(X_{K_v},\mathbb{Z}/n\mathbb{Z}).
\end{array}$$

For (3) and (4) note first that the products $\prod'_{v\in P_K} KH^{(1)}_{a+1}(X_{K_v},\mathbb{Z}/n\mathbb{Z})$ and $\prod'_{v\in P_K} KH^{(1)}_a(X_{K_v},\mathbb{Z}/n\mathbb{Z})$ are products of finitely many groups since almost all X_{K_v} have good reduction. The statements now follow from Lemma 4.20 and the fact that $Ш^{2d+2-a,d-1}_{\text{ét}}(X)$ is finite. The latter follows from [34] (see also [12, Thm. A]). □

Remark 5.2. *An argument similar to the one in the above proof may already be found in* [20, Sec. 4].

Remark 5.3. *Inspecting the diagrams in the proof one finds an isomorphism between the kernel of*

$$Ш(\mathrm{CH}^{d+1}(\mathcal{X}, a, \mathbb{Z}/n\mathbb{Z})) \to Ш^{2d+2-a,d+1}_{\text{ét}}(\mathcal{X})$$

and the cokernel of

$$\prod_{v\in S}{}' \mathrm{CH}^{d+1}(X_{K_v}, a+1, \mathbb{Z}/n\mathbb{Z}) \to \ker\Big(H^{a+1}_{\text{ét}}(\mathcal{X}, \mu_n^{\otimes j})^\vee \to H^{2d+2-a}_{\text{ét}}(X_{K_v}, \mathbb{Z}/n\mathbb{Z}(d+1))\Big).$$

One might hence suspect that they fit together into a natural long exact sequence

$$\ldots \to \prod_{v\in S}{}' \mathrm{CH}^{d+1}(X_{K_v}, a+1, \mathbb{Z}/n\mathbb{Z}) \to H^{a+1}_{\text{ét}}(\mathcal{X}, \mu_n^{\otimes j})^\vee \to \mathrm{CH}^{d+1}(\mathcal{X}, a, \mathbb{Z}/n\mathbb{Z}) \to \ldots$$

but there seems to be a non-trivial extension problem one needs to solve in oder to prove this.

Remark 5.4. *For $b \geq d+2, a \geq 0$ and n odd, the map*

$$Ш(\mathrm{CH}^b(X, a, \mathbb{Z}/n\mathbb{Z})) \to Ш^{2b-a,d+1}_{\text{ét}}(X)$$

is an isomorphism and the sequence

$$\mathrm{CH}^b(X, a, \mathbb{Z}/n\mathbb{Z}) \to \prod_{v\in P_K}{}' \mathrm{CH}^b(X, a, \mathbb{Z}/n\mathbb{Z}) \to H^{2d+2-2b+a}_{\text{ét}}(X_{k_v}, \mu_n^{\otimes d+1-b})^\vee$$

is exact. This follows from cohomological dimension.

Corollary 5.5. *Let the assumptions be as in Theorem 1.5. Let $S = P_K$ and therefore $X = \mathcal{X}$. Then the following statements hold:*

(1) *Conjecture 1.2 holds for $i \geq d+1$ and all a.*
(2) *Conjecture 1.3 holds for $i \geq d+1$ and all a.*

Proof. In fact, (1) is implied by (2), and (2) follows from 5.1 and Remark 5.4. □

Corollary 5.6. *Let the notation be as in Corollary 5.5.*

(1) *If $a = 1$, then the map*

$$\mathrm{C}(X)/n := \mathrm{coker}[\mathrm{CH}^{d+1}(X, 1, \mathbb{Z}/n\mathbb{Z}) \to \prod_{v\in P_K}{}' \mathrm{CH}^{d+1}(X_{K_v}, 1, \mathbb{Z}/n\mathbb{Z})] \to \pi_1^{\mathrm{ab}}(X)/n$$

is an isomorphism.

(2) *Let $a \geq 2$. The map*

$$\mathrm{coker}[\mathrm{CH}^{d+1}(X, a, \mathbb{Z}/n\mathbb{Z}) \to \prod_{v\in P_K}{}' \mathrm{CH}^{d+1}(X_{K_v}, a, \mathbb{Z}/n\mathbb{Z})] \to H^a_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z})^\vee$$

is injective.

Proof. The injectivity follows in both cases from Theorem 5.1(1). The surjectivity of ρ in (1) follows from Chebotarev density. A different way to deduce it from the Kato conjectures is the following: consider the diagram with exact rows

$$\begin{tikzcd}
H^1(X, \mathbb{Z}/n\mathbb{Z})^\vee \arrow[r] & H^{2d+2}(X, \mu_n^{d+1}) \arrow[r] \arrow[d, "\cong"] & \prod{}' H^{2d+2}(X, \mu_n^{d+1}) \arrow[r] \arrow[d, "\cong"] & H^0(X, \mathbb{Z}/n\mathbb{Z})^\vee \arrow[d, "\cong"] & \\
0 \arrow[r] & KH_0^{(1)}(X, \mathbb{Z}/n\mathbb{Z}) \arrow[r] & \bigoplus_{v\in P_K} KH_0^{(1)}(X_{K_v}, \mathbb{Z}/n\mathbb{Z}) \arrow[r] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

The exactness of the second row follows from Theorem 4.10 and the exactness of the first row from Saito's exact sequence 1.1. The vertical maps can be shown to be isomorphisms using the spectral sequence of Lemma 4.13 or Lemma 4.16. The statement now follows from a diagram chase. □

Corollary 5.6(1) recovers the unramified class field theory of arithmetic schemes (see [33]). In the next section we strenthen the above argument to also cover the ramified case.

Question 5.7. *We would like to ask the following questions: can the group $H^a_{\text{ét}}(X,\mathbb{Z}/n\mathbb{Z})^\vee$ be interpreted as a higher homotopy group "$\pi_a(X)/n$" and can it be generated by algebraic cycles? The second question can be made more precise: is the map*

$$\operatorname{coker}[\mathrm{CH}^{d+1}(X,a,\mathbb{Z}/n\mathbb{Z}) \to \prod_{v\in P_K}{}' \mathrm{CH}^{d+1}(X_{K_v},a,\mathbb{Z}/n\mathbb{Z})] \to H^a_{\text{ét}}(X,\mathbb{Z}/n\mathbb{Z})^\vee$$

surjective, i.e. an isomorphism? Furthermore, is

$$Ш(\mathrm{CH}^{d+1}(X,a,\mathbb{Z}/n\mathbb{Z})) = 0?$$

Remark 5.8. *For X a scheme of finite type over a number field K or $K_v, v\in P_K$, we denote the complexes*

$$\dots \to \bigoplus_{x\in X_a} H^{a+1}(k(x),\mathbb{Z}/n\mathbb{Z}(a)) \to \dots \to \bigoplus_{x\in X_{a-1}} H^a(k(x),\mathbb{Z}/n\mathbb{Z}(a-1)) \to \dots$$
$$\dots \to \bigoplus_{x\in X_1} H^2(k(x),\mathbb{Z}/n\mathbb{Z}(1)) \to \bigoplus_{x\in X_0} H^1(k(x),\mathbb{Z}/n\mathbb{Z})$$

by $KC^{(0)}(X,\mathbb{Z}/n\mathbb{Z})$. Here the term $\oplus_{x\in X_a} H^{a+1}(k(x),\mathbb{Z}/n\mathbb{Z}(a))$ is placed in degree a. We set

$$KH^{(0)}_a(X,\mathbb{Z}/n\mathbb{Z}) := H_a(KC^{(0)}(X,\mathbb{Z}/n\mathbb{Z})).$$

In the case of zero-cycles, i.e. the case $i=d$ of Conjecture 1.1 and Conjecture 1.2, the difference between étale and Zariski motivic cohomology is measured by $KC^{(0)}(X,\mathbb{Z}/n\mathbb{Z})$ and one additional row in the coniveau spectral sequence converging to étale cohomology. The approach taken in the proof of Theorem 5.1 therefore only works in small dimensions.

As mentioned in the introduction, Conjecture 1.1 is known to hold if X is a curve and if the Tate-Shafarevich group of the Jacobian of X does not contain a non-zero element which is infinitely divisible (see [32, Sec. 7], [5, Sec. 3] *and* [42, Rem. 1.1(iv)]*). We recall the argument of* [5, Sec. 3]*, where it is shown that a version of Conjecture 1.2 with $\mathbb{Q}_\ell/\mathbb{Z}_\ell$-coefficients holds for such an X, using the above framework. Considering the map of coniveau spectral sequences*

$${}^{\mathrm{CH}}E_1^{p,q}(X,d) \to^{\text{ét}} E_1^{p,q}(X,\mathbb{Z}/\ell^n\mathbb{Z}(d))$$

gives the following commutative diagram with exact rows and collums:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
\mathrm{CH}^1(X,\mathbb{Z}/\ell^n\mathbb{Z}) \arrow[r] \arrow[d] & \prod_{v\in P_K} \mathrm{CH}^1(X_{K_v},\mathbb{Z}/\ell^n\mathbb{Z}) \arrow[d] & \\
H^2_{\text{ét}}(X,\mathbb{Z}/\ell^n\mathbb{Z}(1)) \arrow[r] \arrow[d] & \prod_{v\in P_K} H^2_{\text{ét}}(X_{K_v},\mathbb{Z}/\ell^n\mathbb{Z}(1)) \arrow[r] \arrow[d] & H^2_{\text{ét}}(X,\mu_{\ell^n}^{\otimes 1})^\vee \\
KH^{(0)}_1(X,\mathbb{Z}/\ell^n\mathbb{Z}) \arrow[r] \arrow[d] & \bigoplus_{v\in P_K} KH^{(0)}_1(X_{K_v},\mathbb{Z}/\ell^n\mathbb{Z}) \arrow[d] & \\
0 & 0 &
\end{tikzcd}$$

Now $KH^{(0)}_1(X,\mathbb{Z}/\ell^n\mathbb{Z}) \cong \mathrm{Br}(X)[\ell^n]$ and $KH^{(0)}_1(X,\mathbb{Z}/\ell^n\mathbb{Z}) \cong \mathrm{Br}(X_{K_v})[\ell^n]$. In fact, the collumns also arise from the exact sequence of sheaves $0\to\mu_{\ell^n}\to\mathcal{O}_X^\times \xrightarrow{\cdot\ell^n} \mathcal{O}_X^\times \to 0$. In [5],

Colliot-Thélène considers a family of cycles $\{z_v\} \in \prod_{v\in P_K} \mathrm{CH}^1(X_{K_v})$ *which is orthogonal to every class* $\xi \in H^2_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))$ *under the pairing*

$$\prod_{v\in P_K} \mathrm{CH}^1(X_{K_v}, \mathbb{Z}/\ell^n\mathbb{Z}) \times H^2_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \to \mathbb{Q}_\ell/\mathbb{Z}_\ell$$

and shows that its image in $\prod_{v\in P_K} H^2_{\text{ét}}(X_{K_v}, \mathbb{Z}/\ell^n\mathbb{Z}(1))$*, i.e. after passing to the limit over* n *in the above diagram, is still in the image of* $H^2_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))$ *(see loc. cit., Prop. 3.1). The assertion then follows from the fact that*

$$\ker[\varprojlim_{\ell^n} \mathrm{Br}(X)[\ell^n] \to \varprojlim_{\ell^n} \mathrm{Br}(X_{K_v})[\ell^n]] = 0$$

if the Tate-Shafarevich group of the Jacobian of X *does not contain a non-zero element which is infinitely divisible (see loc. cit., Lem. 3.6).*

6. Ramified global class field theory of Kato-Saito

In this section we generalise Corollary 5.6 to the ramified case. This is very similar to the treatment of class field theory over local and finite fields of Kerz and Zhao in [26].

In order to work with ramification one needs to use the relative Milnor K-sheaf. We adopt the following definition of Kato and Saito (see [22, p. 256]):

Definition 6.1. *Let* X *be a scheme and* $\mathcal{I} \subset \mathcal{O}_X$ *be a sheaf of ideals. We define*

$$K^M_q(\mathcal{O}_X, \mathcal{I}) := \ker(K^M_q(\mathcal{O}_X) \to K^M_q(\mathcal{O}_X/\mathcal{I})).$$

If $\mathcal{I}$ *corresponds to a divisor* D*, then we write* $K^M_{q,X|D}$ *for* $K^M_q(\mathcal{O}_X, \mathcal{I})$*.*

In order to relate Nisnevich to étale cohomology with modulus we need the following localized Chern class map. This was studied by Sato in [36, Thm. 1.2, Cor. 4.6] for regular schemes over a finite field.

Theorem 6.2. *Let* K *be a local or global field. Let* n *be invertible in* K*. If* K *is a number field assume furthermore that either* n *is odd or that* K *has no real places. Let* X *be a smooth projective scheme of dimension* d *over* $\mathrm{Spec}(K)$*. Let* $D \subset X$ *be an effective divisor on* X *and* $j : U \hookrightarrow X$ *be the inclusion. Let* $x \in X^a$*. Then there exists a canonical surjective map*

$$\mathrm{cl}^{d+1,\mathrm{loc}}_{X,D,x,n} : H^a_x(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D})/n \to H^{a+d+1}_x(X, j_!\mu_n^{\otimes d+1})$$

called the localized Chern class map. Furthermore, if $x \in X^d$*, then*

$$\mathrm{cl}^{d+1,\mathrm{loc}}_{X,D,x,n} : H^d_x(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D})/n \to H^{2d+1}_x(X, j_!\mu_n^{\otimes d+1})$$

is an isomorphism.

Proof. The proof is analogous to the proof of [36, Thm. 1.2, Cor. 4.6]. The main difference is that the cohomological dimension of the base field in *loc. cit.* is 1 and in our case 2. However, by working with $\mathcal{K}^M_{d+1,X|D}$ and not $\mathcal{K}^M_{d,X|D}$ we again win one cohomological dimension.

Let us nevertheless give an indication of how the map is constructed. First we may replace X by X^h_x. The construction works by induction on a. For $a = 0$ we have that $X = x$ and $D = 0$, and

$$\mathrm{cl}^{q,\mathrm{loc}}_{x,0,x,n} := h^q_{x,n} : K^M_q(x)/n \to H^q(x_{\text{ét}}, \mu_n^{\otimes q})$$

is the Galois symbol map. If $a = 1$, then X is the spectrum of a henselian discrete valuation ring $\mathcal{O}_F$ with function field F and generic point η. Then we get a commutative diagram with exact rows

$$\begin{tikzcd}
H^0_x(X_{\mathrm{Nis}}, \mathcal{K}^M_{q,X|D})/n \arrow[r] \arrow[d] & K^M_{q,X|D}(F)/n \arrow[r] \arrow[d, "h^q_{\eta,n}"] & H^1_x(X_{\mathrm{Nis}}, \mathcal{K}^M_{q,X|D})/n \arrow[r] \arrow[d, dotted, "\mathrm{cl}^{q,\mathrm{loc}}_{X,D,x,n}"] & 0 \\
H^q_{\text{ét}}(X, j_!\mu_n^{\otimes q}) \arrow[r, "\partial"] & H^q_{\text{ét}}(\eta, \mu_n^{\otimes d}) \arrow[r] & H^{q+1}_x(X_{\text{ét}}, j_!\mu_n^{\otimes q}) \arrow[r] & 0
\end{tikzcd}$$

in which the left vertical and the upper left horizontal maps zero if $D \neq 0$ by [36, Lem. 2.1] and otherwise the Galois symbol. The dotted map $\mathrm{cl}^{q,\mathrm{loc}}_{X,D,x,n}$ is induced by the other two vertical maps. If $a \geq 2$, then $\mathrm{cl}^{q,\mathrm{loc}}_{X,D,x,n}$ is induced by the diagram

$$\begin{tikzcd}
\bigoplus_{y\in X^{a-2}} H^{a-2}_y(X_{\mathrm{Nis}},\mathcal{K}^M_{q,X|D})/n \arrow[r] \arrow[d, "\mathrm{cl}^{q,\mathrm{loc}}_{X,D,y,n}"] & \bigoplus_{y\in X^{a-1}} H^{a-1}_y(X_{\mathrm{Nis}},\mathcal{K}^M_{q,X|D})/n \arrow[r, two heads] \arrow[d, "\mathrm{cl}^{q,\mathrm{loc}}_{X,D,y,n}"] & H^a_x(X_{\mathrm{Nis}},\mathcal{K}^M_{q,X|D})/n \arrow[r] \arrow[d, dotted, "\mathrm{cl}^{q,\mathrm{loc}}_{X,D,x,n}"] & 0 \\
\bigoplus_{y\in X^{a-2}} H^{q+a-2}_y(X_{\text{ét}}, j_!\mu_n^{\otimes q}) \arrow[r] & \bigoplus_{y\in X^{a-1}} H^{q+a-1}_y(X_{\text{ét}}, j_!\mu_n^{\otimes q}) \arrow[r] & H^{q+a}_x(X_{\text{ét}}, j_!\mu_n^{\otimes q}) &
\end{tikzcd}$$

The injectivity and surjectivity of $\mathrm{cl}^{d+1,\mathrm{loc}}_{X,D,x,n}$ now follow as in the proof of *loc. cit.* using the above remarks on cohomological dimension. □

We need the following proposition:

Proposition 6.3. *Let K be a local or global field. Let n be invertible in K. If K is a number field assume furthermore that either n is odd or that K has no real places. Let X be a smooth projective scheme over* $\mathrm{Spec}(K)$. *Let $D \subset X$ be an effective divisor on X and $j : U \hookrightarrow X$ be the inclusion. Let*

$$E_1^{p,q}(X,A) = \bigoplus_{x\in X^{(p)}} H^{p+q}_x(X_{\text{ét}},A) \Rightarrow H^{p+q}_{\text{ét}}(X,A)$$

be the coniveau spectral sequence of Lemma 4.13. Then

$$E_1^{\bullet,d+2}(X, j_!\mu_n^{\otimes d+1}) \cong E_1^{\bullet,d+2}(X,\mu_n^{\otimes d+1}).$$

In particular for X proper and smooth over a number field the map

$$KH^{(1)}_a(X, j_!\mu_n^{\otimes d+1}) \xrightarrow{\cong} \bigoplus_{v\in P_K} KH^{(1)}_a(X_{K_v}, j_!\mu_n^{\otimes d+1})$$

is an isomorphism for $a > 0$. Here

$$KH^{(1)}_a(X, j_!\mu_n^{\otimes d+1}) := E_1^{\bullet,d+2}(X, j_!\mu_n^{\otimes d+1})$$

and

$$KH^{(1)}_a(X_{K_v}, j_!\mu_n^{\otimes d+1}) := E_1^{\bullet,d+2}(X_{K_v}, j_!\mu_n^{\otimes d+1}).$$

In the latter case we let j denote the inclusion $D_{K_v} \hookrightarrow X_{K_v}$.

Proof. The proof is identical to the proof of [26, Prop. 2.2.1]. □

Theorem 6.4. *Let K be a global field. Let n be invertible in K. If K is a number field assume furthermore that either n is odd or that K has no real places. Let X be a smooth projective scheme over* $\mathrm{Spec}(K)$. *Let $D \subset X$ be an effective divisor on X and $j : U \hookrightarrow X$ be the inclusion. Then there is an isomorphism*

$$\mathrm{coker}[H^d_{\mathrm{Nis}}(X,\mathcal{K}^M_{d+1,X|D})/n \to \prod_{v\in P_K}{}' H^d_{\mathrm{Nis}}(X_{K_v},\mathcal{K}^M_{d+1,X_{K_v}|D_{K_v}})/n] \to \pi_1^{\mathrm{ab}}(U)/n.$$

Proof. There is a commutative diagram with exact rows

$$\begin{tikzcd}
\bigoplus_{x\in X_1} H^{d-1}_x(X_{\mathrm{Nis}},\mathcal{K}^M_{d+1,X|D})/n \arrow[r] \arrow[d, two heads] & \bigoplus_{x\in X_0} H^{d}_x(X_{\mathrm{Nis}},\mathcal{K}^M_{d+1,X|D})/n \arrow[r, two heads] \arrow[d, "\cong"] & H^d_{\mathrm{Nis}}(X,\mathcal{K}^M_{d+1,X|D})/n \arrow[d, "\cong"] \\
\bigoplus_{x\in X_1} H^{2d}_x(X_{\text{ét}}, j_!\mu_n^{\otimes d+1}) \arrow[r, "\partial"] & \bigoplus_{x\in X_0} H^{2d+1}_x(X_{\text{ét}}, j_!\mu_n^{\otimes d+1}) \arrow[r] & \mathrm{coker}\partial.
\end{tikzcd}$$

The upper exact row comes from the spectral sequence

$$E_1^{p,q}(X,\mathcal{K}^M_{d+1,X|D}) = \bigoplus_{x\in X^p} H_x^{p+q}(X,\mathcal{K}^M_{d+1,X|D}) \Rightarrow H^{p+q}(X,\mathcal{K}^M_{d+1,X|D})$$

and the fact that $E_1^{p,q}(X,\mathcal{K}^M_{d+1,X|D}) = 0$ for $q > d$ by Nisnevich cohomological dimension. The commutativity follows from the construction of the local Chern class map. By Theorem 6.2 the middle vertical map is an isomorphism and the left vertical map is surjective. This implies that the right vertical map is an isomorphism. The niveau spectral sequence

$$E_1^{p,q}(X,j_!\mu_n^{\otimes d+1}) = \bigoplus_{x\in X^p} H_x^{p+q}(X,j_!\mu_n^{\otimes d+1}) \Rightarrow H^{p+q}(X,j_!\mu_n^{\otimes d+1}),$$

in which $\mathrm{coker}\partial \cong E_2^{d,d+1}$, and the same argument for the X_{K_v} therefore implies that there is a commutative diagram with exact rows and collums

$$\begin{tikzcd}
KH_2^{(1)}(X,j_!\mu_n^{\otimes d+1}) \arrow[r, "\cong"] \arrow[d] & \prod_{v\in P_K} KH_2^{(1)}(X_{K_v},j_!\mu_n^{\otimes d+1}) \arrow[d] & \\
H^d_{\mathrm{Nis}}(X,\mathcal{K}^M_{d+1,X|D})/n \arrow[r, "\alpha"] \arrow[d] & \prod'_{v\in P_K} H^d_{\mathrm{Nis}}(X_{K_v},\mathcal{K}^M_{d+1,X_{K_v}|D_{K_v}})/n \arrow[d] & \\
H^{2d+1}_{\text{ét}}(X,j_!\mu_n^{\otimes d+1})) \arrow[r] \arrow[d] & \prod'_{v\in P_K} H^{2d+1}_{\text{ét}}(X_{K_v},j_!\mu_n^{\otimes d+1})) \arrow[r] \arrow[d] & \pi_1^{\mathrm{ab}}(U)/n \\
KH_1^{(1)}(X,j_!\mu_n^{\otimes d+1}) \arrow[r, "\cong"] & \prod_{v\in P_K} KH_1^{(1)}(X_{K_v},j_!\mu_n^{\otimes d+1}). &
\end{tikzcd}$$

The isomorphisms in the first and last row follow from Proposition 6.3. The theorem now follows from a diagram chase. □

Passing to the direct limit over all D, we obtain a description of $\mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}}$, where F is the function field of X. This recovers the following theorem of Kato and Saito:

Theorem 6.5. *(*[22, Thm. 9.1]*) Let X be a projective integral scheme over $\mathbb{Z}$ of dimension $d+1$ with function field F. Assume for simplicity that F contains a totally imaginary field if* $\mathrm{ch}(F) = 0$. *Then there is an isomorphism*

$$\varprojlim_{n,D} H^{d+1}_{\mathrm{Nis}}(X,\mathcal{K}^M_{d+1,X|D})/n \to \mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}}.$$

In fact, it can be shown that taking the inverse limit over the direct sum of the local terms (modulo relations) appearing in the expression of $H^{d+1}_{\mathrm{Nis}}(X,\mathcal{K}^M_{d+1,X|D})/n$ via the coniveau spectral sequence, is isomorphic to the cokernel defined using the restricted product in Theorem 6.4.

7. The p-adic cycle class map

In this section let A be a henselian discrete valuation ring of characteristic zero with residue field of characteristic p and function field K. Let X be smooth and projective of relative dimension d over $\mathrm{Spec}(A)$. Let X_n denote the thickenings of the special X_1. Let X_K denote the generic fiber of X. In [24, Sec. 10], Kerz, Esnault and Wittenberg state the following conjecture:

Conjecture 7.1. *Assume the Gersten conjecture for the Milnor K-sheaf $\mathcal{K}^M_{n,X}$. Then the restriction map*

$$res : \mathrm{CH}^d(X)/p^r \to \text{"}\lim_n\text{"} H^d(X_1,\mathcal{K}^M_{d,X_n}/p^r)$$

is an isomorphism.

The following theorem and its corollary (i.e. Corollary 1.9 in the introduction) give some evidence for this conjecture.

Theorem 7.2. *Let $d = \dim X - 1 \leq 2$. Then there is an isomorphism*

$$\mathrm{CH}^{d+1}(X, 1, \mathbb{Z}/p^r\mathbb{Z}) \to H^{2d+1}_{\text{ét}}(X, \mathcal{T}_r(d+1)).$$

Here $\mathcal{T}_r(d+1)$ are the p-adic étale Tate twists defined in [35].

Proof. We use the homology theory defined in Remark 4.21 and consider the associated spectral sequence

$$E_1^{u,v} = \bigoplus_{x \in X^u} H^{v-u}(x, \mathbb{Z}/p^r\mathbb{Z}(d+1-u)) \Rightarrow H^{u+v}_{\text{ét}}(X, \mathcal{T}_r(d+1)).$$

The case $d = 1$ is clear. Let us therefore assume that $d = 2$. Then

$$E_2^{2,3} = \mathrm{CH}^{d+1}(X, 1, \mathbb{Z}/p^r\mathbb{Z}).$$

We therefore need to show that $E_2^{\bullet,4} = 0$. By [19, Thm. 4.10.2], the complex $E_1^{\bullet,4}$ coincides up to sign with the relevant complex defined by Kato. Consider the localization sequence (4.1)

$$KH^{(0)}_{a+1}(X_1, \mathbb{Z}/p^r\mathbb{Z}) \to KH^{(0)}_{a+1}(X, \mathbb{Z}/p^r\mathbb{Z}) \to KH^{(1)}_a(X_K, \mathbb{Z}/p^r\mathbb{Z})$$

for the homology theory. The group $KH^{(0)}_{a+1}(X_1, \mathbb{Z}/p^r\mathbb{Z})$ vanishes for $a \leq 3$ by the results on the Kato conjectures with p-coefficients in low degrees cited in Remark 4.21. For the vanishing of the group $KH^{(1)}_d(X_K, \mathbb{Z}/p^r\mathbb{Z})$ consider the short exact sequence

$$0 \to KH^{(1)}_{a+1}(X_{K_v}, \mathbb{Q}_p/\mathbb{Z}_p)/p^n \to KH^{(1)}_a(X_{K_v}, \mathbb{Z}/p^n\mathbb{Z}) \to KH^{(1)}_a(X_{K_v}, \mathbb{Q}_p/\mathbb{Z}_p)[p^n] \to 0$$

([25, Lemma 7.6]). Now $KH^{(1)}_3(X_{K_v}, \mathbb{Q}_p/\mathbb{Z}_p))$ vanishes for dimension reasons and for $a \leq 2$ we have that $KH^{(1)}_a(X_{K_v}, \mathbb{Q}_p/\mathbb{Z}_p) \cong KH^{(1)}_a(X_v, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ by [17, Thm. 1.6]. □

Note that due to the twist by $d+1$ we have full purity for the logarithmic deRham-Witt sheaves. This seems to be the main difference to the zero-cycle case.

8. A finiteness theorem for arithmmetic schemes

A version of Bass' finiteness conjecture predicts that for a regular scheme X of finite type over $\mathbb{Z}$, the groups

$$\mathrm{CH}^r(X, q)$$

are finitely generated. This conjecture is known to hold for $r = 1$ or $\dim(X) = 1$ by results of Quillen. In arbitrary dimension d there are few results. If X is proper and flat over $\mathrm{Spec}\mathbb{Z}$, then $\mathrm{CH}^d(X)$ is finite by unramified class field theory (see [33, Thm. 0.6] or for a more general statement [22, Sec. 6]). If X is smooth projective over a finite field, then the groups $\mathrm{CH}^{d+j}(X, j)$ are finitely generated for $j \geq 0$ again by unramified class field theory (see [8, Sec. 2] and [22, Sec. 6] for $j = 0$ and [29, Sec. 6] for $j \geq 1$).

As an application of the Kato conjectures, Kerz and Saito show that for any quasi-projective scheme X of dimension d over a finite field k and n invertible on X, the groups

$$\mathrm{CH}^d(X, q, \mathbb{Z}/n\mathbb{Z})$$

are finite for all $q \geq 0$ (see [25, Cor. 9.4]). In the following theorem we establish some small-degree cases for arithmetic schemes (see also [11, Sec. 7.2]):

Theorem 8.1. *Let K be a global field with ring of integers $\mathcal{O}_K$, let $U \subset \mathrm{Spec}\mathcal{O}_K$ be open, nonempty and let $n \in \mathbb{N}_{>1}$ be invertible on U. Let X be a regular connected scheme, proper and flat over U with smooth generic fiber X_K and let $d = \dim(X) - 1 = \dim(X)$. If K is a number field assume furthermore that either n is odd or that K has no real places.*

(1) *Suppose that for all places v of K dividing n, X_{K_v} satisfies $(\star)$. Then the groups*

$$\mathrm{CH}^{d+1}(X, a, \mathbb{Z}/n\mathbb{Z})$$

are finite for all $1 \geq a \geq 0$.

(2) *Suppose $d = 2$ and for all places v of K dividing n, X_{K_v} satisfies $(\star\star)$. Then the groups*

$$\mathrm{CH}^{d+1}(X, a, \mathbb{Z}/n\mathbb{Z})$$

are finite for all $a \geq 0$.

Proof. By Lemma 4.16(2) there is an exact sequence

$$\begin{aligned} ... \to KH^{(0)}_{q+2}(X, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}^{d+1}(X, q, \mathbb{Z}/n\mathbb{Z}) \to H^{2d+2-q}_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(d+1)) \\ \to KH^{(0)}_{q+1}(X, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}^{d+1}(X, q-1, \mathbb{Z}/n\mathbb{Z}) \to ... \end{aligned} \tag{8.1}$$

By [30, Ch. II, 7.1], the étale cohomology groups $H^{2d+2-q}_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(d))$ are known to be finite for n invertible on U. It therefore suffices to show the finiteness of $KH^{(0)}_{q+2}(X, \mathbb{Z}/n\mathbb{Z})$. By definition we have an exact sequence

$$... \to KH^{(0)}_a(X/U, \mathbb{Z}/n\mathbb{Z}) \to KH^{(0)}_a(X, \mathbb{Z}/n\mathbb{Z}) \to \bigoplus_{v \in \sum_U} KH^{(1)}_{a-1}(X_{K_v}, \mathbb{Z}/n\mathbb{Z})) \to ..$$

By Theorem 4.11, $KH^{(0)}_a(X/U, \mathbb{Z}/n\mathbb{Z})$ vanishes for $a > 0$ and is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ for $a = 0$. The statement now follows from Lemma 4.20. □

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