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Results on Abstract Integro Hybrid Evolution System with Impulses on Time Scales

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Abstract: Our purpose in this manuscript is to study the existence of solution, stability analysis and exact controllability results for an abstract integro hybrid evolution system with impulses on time scales. Non-linear functional analysis, evolution operator theory and Banach contraction theorem have been used to establish these results. In the end, we present some theoretical and numerical examples for different time scales to illustrate the application of these analytical results.

Keywords: Existence; Stability; Controllability; Impulsive evolution system; Time scales.

AMS Subject Classification: 34A12; 03C45; 93B05; 34A37; 34N05.

1 Introduction

In general, one investigates the discrete and continuous dynamical systems separately and most of the results have to be proved for each case (using discrete analysis or continuous analysis). The theory of time scales was introduced by Hilger [1] in 1988 that unifies equations of continuous, discrete, quantum calculus and other calculus. A time scales is an arbitrary non-empty closed subset of real numbers, for example \mathbb{R} , \mathbb{N} , $h\mathbb{Z}$ (where $h > 0$) are the time scales. Therefore, the results obtained on time scales will be true for the continuous (by considering the time scales as the real numbers), discrete (by choosing the time scales to be the set of integers) as well as for any non-uniform time domain (combination between discrete points and continuous intervals or a discrete non-uniform domain) which is very useful in the study of complex dynamical systems. From the above discussion, we may outline that unification and extension are the two primary highlights of the time scales calculus. Hence, the study of dynamical systems on time scales has gained great attention and many researchers have found the numerous applications of time scales in economics [2], control theory [3], population dynamics and heat transfer system [4, 5]. More recently, several authors discussed the existence, uniqueness of periodic, almost periodic solutions and stability of abstract dynamical equations on time scales [6–9]. For the further details on time scales, one can see the monographs [10, 11] and the papers [12–16].

Controllability is the most important concept in the mathematical control theory, which in general means, that it is possible to steer dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. It has a wide range of applications in many fields such as engineering, physics and biology. For more details please see [17–26] and references therein. Moreover, the controllability result on time scales is a very new area of research and few authors studied the controllability result for the linear and nonlinear dynamic system on time scales [27–34]. In particular, Davis et al. [27] considered a finite dimensional linear dynamic system

on time scales and established the controllability and observability results. Bohner et al. [28], established the controllability, reachability, and observability results for the time-invariant systems. Bartosiewicz [29], studied the linear positive control systems on time scales. Duque et al. [30] examined the approximate controllability results for the semilinear dynamic system on time scales.

On the other hand, there are many certifiable issues such as harvesting, heart pulsates and natural disasters that are characterized by rapid changes in their state. These rapid changes are known as the impulsive effect in the system. In the last few years, the impulsive differential equations have become the natural framework for modelling of many evolving processes and phenomena studied in economics, population dynamics, industrial robotics, optimal control theory, physics, engineering and medicine [35, 36]. The relevance of this model emerges from the way that impulses can be viewed as a suitable model for describing a process, which at specific moments changes its state quickly and that cannot be described by ordinary differential equations.

However, some certain dynamics of evolution processes in pharmacotherapy cannot be described by instantaneous impulsive dynamic systems, for example, in the case of a hyperglycemic patient, an insulin can be endorsed. The presentation of the medication in the circulation system causes a sudden change in the system, followed by a continuous process until the medication is totally absorbed. We cannot model this situation by considering an instantaneous impulse because the sudden changes stay active during a finite time interval. In fact, the above circumstance is fallen for another situation of impulsive action, which begins at any discrete fixed point and remains active on a finite time interval. Conclusively, Hernández et al. [37] presented the theory of non-instantaneous impulsive systems and established the existence of solutions. After that, Wang et al. [38, 39] generalized this model to two general classes of impulsive differential equations, which are increasingly important to the study of dynamics of evolutionary processes in pharmacotherapy. In practicality, there are no impulses that happen instantaneously rather it is non-instantaneous howsoever time of the event is small. Along these lines, it is beneficial to examine a class of differential equations with non-instantaneous impulses. For the comprehensive study on non-instantaneous impulsive systems, one may go through the books [40, 41] and the papers [42–44].

On the other side, to address the problems involving like memory and hereditary influence which arise in continuous model nuclear reactor, theory of heat conduction for materials, ecological models, blood cell production models, rheology and biological population models, we need to include generalized Volterra integral terms [45, 46]. Recently, the theory and applications of differential equations with integro terms have been received much attention by many authors. Shin et al. [47], established the existence, uniqueness of solution and controllability results to a neutral functional integro-differential equation with delay. Yong and Zhenhai [48], discussed the controllability of nonlinear impulsive integro-differential fractional time-invariant systems by using the Schauder fixed point theorem. By using the theory of resolvent operator and condensing map, Anguraj and Kanjanadevi [49], established the existence of a mild solution to non-instantaneous impulsive fractional integro-differential equations with non-local conditions. However, these results cannot be easily extended to the case of hybrid evolution systems with non-instantaneous impulses on time scales and to the best of our knowledge, there is no work related to the existence of solution, stability analysis and the exact controllability of an impulsive hybrid evolution system on time scales in the abstract space. Therefore, motivated by the above facts, in this manuscript, we establish the existence, uniqueness and stability analysis of the following abstract integro hybrid evolution system on time scales

$$\begin{aligned}
 x^\Delta(t) &= A(t)x(t) + \mathcal{F}(t, x(t)) + \int_0^t \mathcal{H}(t, r, x(r))\Delta r, \quad t \in \cup_{p=0}^n (r_p, t_{p+1}]_{\mathbb{T}}, \\
 x(t) &= \mathcal{G}_p(t, x(t_p^-)), \quad t \in (t_p, r_p]_{\mathbb{T}}, \quad p = 1, 2, \dots, n, \\
 x(0) &= x_0.
 \end{aligned} \tag{1.1}$$

Moreover, we examine the exact controllability result for the following abstract integro hybrid evolution system

$$\begin{aligned}
x^\Delta(t) &= A(t)x(t) + \mathcal{F}(t, x(t)) + \int_0^t \mathcal{H}(t, r, x(r))\Delta r + Bu(t), \quad t \in \cup_{p=0}^n (r_p, t_{p+1}]_{\mathbb{T}}, \\
x(t) &= \mathcal{G}_p(t, x(t_p^-)), \quad t \in (t_p, r_p]_{\mathbb{T}}, \quad p = 1, 2, \dots, n, \\
x(0) &= x_0,
\end{aligned} \tag{1.2}$$

where the state $x(\cdot)$ takes the value in a Banach space X , \mathbb{T} is a time scales and $t_p, r_p \in \mathbb{T}$ are some points with $0 = r_0 = t_0 < t_1 < r_1 < t_2 < \dots < r_n < t_{n+1} = T$. $x(t_p^+) = \lim_{h \rightarrow 0^+} x(t_p + h)$, $x(t_p^-) = \lim_{h \rightarrow 0^+} x(t_p - h)$ denote the right and left limit of $x(t)$ at $t = t_p$ in the time scales sense respectively, $A(t)$ is a family of linear operators which generates an evolution operator $\{E(t, s) : (t, s) \in I = [0, T]_{\mathbb{T}} \times I : 0 \leq s \leq t \leq T\}$, B is a bounded linear operator from a Banach space U to X and $u \in L^2(I, U)$ is the control input. The functions $\mathcal{G}_p : I_p \times X \rightarrow X$, $I_p = [t_p, r_p]_{\mathbb{T}}$, $p = 1, 2, \dots, n$, represent the non-instantaneous impulses during the intervals $(t_p, r_p]_{\mathbb{T}}$, $p = 1, 2, \dots, n$. The non-linear functions $\mathcal{F} : J_1 \times X \rightarrow X$, $\mathcal{H} : \mathcal{Q} \times X \rightarrow X$ and $\mathcal{G}_p : I_p \times X \rightarrow X$, $p = 1, 2, \dots, n$, are satisfying some suitable conditions which will be specified later, where $J_1 = \cup_{p=0}^n [r_p, t_{p+1}]_{\mathbb{T}}$ and $\mathcal{Q} = \{(t, r) \in I \times I : 0 \leq r \leq t \leq T\}$. The results of this manuscript are completely new even for the continuous case, i.e., when $\mathbb{T} = \mathbb{R}$ as well as for the discrete case, i.e., when $\mathbb{T} = \mathbb{Z}$.

There are many natural processes such as chemical kinetics, electronics, biological models and fluid dynamics in which the partial integro-differential equations with impulsive conditions arise. Most of these phenomena can not be described through classical differential equations and hence, in the last few years, many engineers, physicists and mathematicians worked on the partial integro-differential equations with impulses. A powerful way to study such types of systems is to transfer them into integro-differential evolution systems (like (1.1)) in abstract spaces. As an example, if $Z(t, \eta)$ denotes the temperature of a rod of unit length at space coordinate η and time t , then the heat-transfer in this rod can be described by the following model

$$\begin{aligned}
\frac{\partial}{\Delta_1 t} Z(t, \eta) &= a(t, \eta) \frac{\partial^2}{\Delta_2 \eta^2} Z(t, \eta) + b(\eta)W(t, \eta) + G_1(t, Z(t, \eta)) + \int_0^t G_2(t, r, Z(r, \eta))\Delta r, \\
& \quad t, r \in \cup_{p=0}^n (r_p, t_{p+1}]_{\mathbb{T}}, \quad \eta \in [0, \pi]_{\mathbb{T}}, \\
Z(t, 0) &= Z(t, \pi) = 0, \quad t \in [0, T]_{\mathbb{T}}, \\
Z(t, \eta) &= \mathcal{G}_p(t, Z(t_p^-, \eta)), \quad t \in (t_p, r_p]_{\mathbb{T}}, \quad p = 1, 2, \dots, n, \\
Z(0, \eta) &= x_0, \quad \eta \in [0, \pi]_{\mathbb{T}},
\end{aligned} \tag{1.3}$$

where \mathbb{T} is a time scales and $t_p, r_p \in \mathbb{T}$ are some points with $0 = r_0 = t_0 < t_1 < r_1 < t_2 < \dots < r_n < t_{n+1} = T$. Δ_1 denotes the first order partial derivative while Δ_2 denotes the second order partial derivative. $Z, S, a : T_0 \times [0, \pi]_{\mathbb{T}} \rightarrow \mathbb{R}$, $G_1 : T_0 \times \mathbb{R} \rightarrow \mathbb{R}$, $G_2 : T_0 \times T_0 \times \mathbb{R} \rightarrow \mathbb{R}$, $\mathcal{G}_p : J_p \times \mathbb{R} \rightarrow \mathbb{R}$ are the real valued functions satisfying certain conditions, $T_0 = \cup_{p=0}^n [r_p, t_{p+1}]_{\mathbb{T}}$, $J_p = [t_p, t_p]_{\mathbb{T}}$, $p = 1, 2, \dots, n$. It will be seen in Section 5 that equation (1.3) can be rewritten in the form of abstract equation (1.2).

The primary contribution and advantage of this paper can be foreground as follows. The main results of this manuscript are on arbitrary time domains which are formulated in terms of time scales. It includes the continuous, discrete as well as any combination of these two; henceforth the results of this manuscript are new. We considered a more general hybrid integro-evolution system with non-instantaneous impulses on time scales (1.1) and analyze the existence of a unique mild solution and stability analysis. Further, some sufficient conditions are established in the evolution setting to guarantee that the system (1.2) is exact controllable. Some theoretical as well as numerical examples with simulation on different time scales including continuous time scales, the union of disjoint continuous time scales, and a combination of discrete and continuous time scales are given to verify the proposed theoretical results.

Note that the problems (1.1) and (1.2), considered in this manuscript are new and start the investigation of a hybrid evolution system with impulses in a Banach space X . We trust that the acquired outcomes will give a

significant contribution to the existing literature on the topic. The rest of the manuscript is organized as follows. In Section 2, we give some preliminaries, notations, definitions, lemmas and important results which will be used to prove our main results. In Sections 3, we establish the existence and stability results. Moreover, Section 4, is devoted to the study of exact controllability of the evolution system (1.2). In the last, some theoretical and numerical examples are given to illustrate the effectiveness of the developed analytical results.

2 Preliminaries and Definitons

In this segment, we briefly describe some notations, fundamental definitions and important lemmas which are useful to prove the main results. Let $(X, \|\cdot\|)$ be a Banach space and $\mathbb{B}(X)$ denotes the set of all bounded linear operator from X into X . $C(I, X)$ be the set of all continuous functions.

A time scales interval is defined by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$. In a similar way, we can define $(a, b)_{\mathbb{T}}, [a, b)_{\mathbb{T}}, (a, b]_{\mathbb{T}}$ and so on. Also, we define $\mathbb{T}^k = \mathbb{T} \setminus \{\max \mathbb{T}\}$ if $\max \mathbb{T}$ exists, otherwise $\mathbb{T}^k = \mathbb{T}$. The forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ is defined by $\sigma(t) = \inf\{r \in \mathbb{T} : r > t\}$ and the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is defined by $\rho(t) = \sup\{r \in \mathbb{T} : r < t\}$ with the substitution $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$ respectively. The graininess function $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t, \forall t \in \mathbb{T}$.

Definition 2.1. [10] A point $t \in \mathbb{T}$ is said to be

- right-scattered if $\sigma(t) > t$ and left-scattered if $\rho(t) < t$;
- right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$;
- left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$.

Remark 2.2. A point t is called the dense point if it is right and left dense at the same time.

In the next definition, we define the Δ -derivative.

Definition 2.3. [10] Let $\psi : \mathbb{T} \rightarrow X$ be a function and $t \in \mathbb{T}^k$. Then the delta derivative (or Δ -derivative) of ψ at the point t is defined to be the number $\psi^\Delta(t)$ (provided it exists) with the property that for each $\epsilon > 0$ there is a neighborhood \mathcal{U} of t such that

$$|[\psi(\sigma(t)) - \psi(r)] - \psi^\Delta(t)[\sigma(t) - r]| \leq \epsilon|\sigma(t) - r|, \forall r \in \mathcal{U}.$$

If $\mathbb{T} = \mathbb{R}$, then $\psi^\Delta(t) = \psi'(t)$, which is the usual derivative of $\psi : \mathbb{R} \rightarrow X$. If $\mathbb{T} = \mathbb{Z}$, then $\psi^\Delta(t) = \Delta\psi$, which is the usual forward difference of $\psi : \mathbb{Z} \rightarrow X$ defined by $\Delta\psi(t) = \psi(t+1)\psi(t)$.

Definition 2.4. [10] Let Ψ be a function, it is called the antiderivative of $\psi : \mathbb{T} \rightarrow X$ provided $\Psi^\Delta(t) = \psi(t)$ for each $t \in \mathbb{T}^k$, then the delta integral is given by

$$\int_{t_0}^t \psi(\vartheta)\Delta\vartheta = \Psi(t) - \Psi(t_0).$$

The next properties of functions defined on time scale are often used.

A function $\psi : \mathbb{T} \rightarrow X$ is called regulated if the right-hand limit exists (finite) at all right-dense points of \mathbb{T} and the left-hand limit exists (finite) at all left-dense points of \mathbb{T} . Also, ψ is called rd-continuous, if it is regulated and it is continuous at all right-dense points. Moreover, ψ is piecewise rd-continuous if it is regulated and rd-continuous at all, except possibly at finitely many right-dense points $t \in \mathbb{T}$. The collection of all piecewise rd-continuous functions is denoted by $C_{prd}(\mathbb{T}, X)$.

Definition 2.5. [10] A function $q : \mathbb{T} \rightarrow \mathbb{R}$ is said to be regressive if $1 + \mu(t)q(t) \neq 0, \forall t \in \mathbb{T}$. The set of all regressive functions is denoted by \mathcal{R} . Moreover, q is said to be positive regressive if $1 + \mu(t)q(t) > 0, \forall t \in \mathbb{T}$ and the set of all positive regressive functions is denoted by \mathcal{R}^+ .

Definition 2.6. [10] For $p, q \in \mathcal{R}$, we define the following

$$(i) \ominus p = \frac{-p}{1 + \mu(t)p}. \quad (ii) p \oplus q = p + q + \mu(t)pq. \quad (iii) p \ominus q = p \oplus (\ominus q).$$

The generalized exponential function of scalar function $q \in \mathcal{R}$ is defined hereafter.

Definition 2.7. [10] If $q \in \mathcal{R}$, then we define the exponential function by

$$e_q(t, r) = \exp \left(\int_r^t \zeta_{\mu(\vartheta)}(q(\vartheta)) \Delta \vartheta \right), \text{ for } t, r \in \mathbb{T},$$

where

$$\zeta_{\mu(r)}(q(r)) = \begin{cases} \frac{1}{\mu(r)} \text{Log}(1 + q(r)\mu(r)), & \text{if } \mu(r) \neq 0, \\ q(r), & \text{if } \mu(r) = 0. \end{cases}$$

Lemma 2.8. [10] If $q \in \mathcal{R}$, then

$$(i) e_0(t, r) = 1 \text{ and } e_q(t, t) = 1. \quad (ii) e_q(\sigma(t), r) = (1 + \mu(t)q(t))e_q(t, r). \quad (iii) e_q(t, r)e_q(r, \vartheta) = e_q(t, \vartheta). \\ (iv) e_q(t, r) = \frac{1}{e_q(r, t)} = e_{\ominus q}(r, t). \quad (v) (e_{\ominus q}(t, r))^\Delta = \ominus q(t)e_{\ominus q}(t, r).$$

Lemma 2.9. [10] If $t_0, t_1, a \in \mathbb{T}$ and $q \in \mathcal{R}$, then

$$\int_{t_0}^{t_1} q(\vartheta)e_q(a, \sigma(\vartheta))\Delta\vartheta = e_q(a, t_0) - e_q(a, t_1).$$

Lemma 2.10. [8] If $t, r \in \mathbb{T}$ such that $r < t$, then $e_{\ominus\alpha}(t, r) \leq 1$ for $\alpha > 0$.

Next, we define the evolution operator family which is often used.

Definition 2.11. [9] A two-parameter family $E(t, r) : I \times I \rightarrow \mathbb{B}(X)$ is said to be a linear evolution operator if it satisfies the following conditions:

1. $E(t, t) = \tilde{I}$, where \tilde{I} is the identity operator in X .
2. $E(t, r)E(r, s) = E(t, s)$.
3. $(t, r) \rightarrow E(t, r)x$ is continuous mapping for any fixed $x \in X$.

Definition 2.12. [9] An evolution operator $E(t, s)$ is said to be exponentially stable if there exist constants $K_0 \geq 1$ and $\nu > 0$ such that

$$\|E(t, s)\| \leq K_0 e_{\ominus\nu}(t, s), \quad t \geq s.$$

In order to define the solution of the system (1.1), we define the space of piecewise continuous functions $PC(I, X) = \{x : I \rightarrow X : x \in C((t_p, t_{p+1}]_{\mathbb{T}}, X), p = 0, 1, \dots, n \text{ and there exist } x(t_p^-) \text{ and } x(t_p^+), p = 1, 2, \dots, n, \text{ with } x(t_p^-) = x(t_p)\}$. It can be seen easily that $PC(I, X)$ is a Banach space endowed with the norm

$$\|x\|_\Omega = \sup_{t \in [a, b]} \frac{\|x(t)\|}{e_\Omega(t, a)}, \text{ for some } \Omega \in \mathcal{R}^+.$$

In the next definition, we give the solution of the system (1.1).

Definition 2.13. A function $x \in PC(I, X)$ is said to be a mild solution of the system (1.1), if x satisfies $x(0) = x_0$, $x(t) = \mathcal{G}_p(t, x(t_p^-))$, $p = 1, 2, \dots, n$ and the following integral equations

$$\begin{aligned} x(t) &= E(t, 0)x_0 + \int_0^t E(t, \sigma(\vartheta))\mathcal{F}(\vartheta, x(\vartheta))\Delta\vartheta + \int_0^t E(t, \sigma(\vartheta)) \int_0^{\vartheta} \mathcal{H}(\vartheta, r, x(r))\Delta r \Delta\vartheta, \quad \forall t \in [0, t_1]_{\mathbb{T}}, \\ x(t) &= E(t, r_p)\mathcal{G}_p(r_p, x(t_p^-)) + \int_{r_p}^t E(t, \sigma(\vartheta))\mathcal{F}(\vartheta, x(\vartheta))\Delta\vartheta + \int_{r_p}^t E(t, \sigma(\vartheta)) \int_0^{\vartheta} \mathcal{H}(\vartheta, r, x(r))\Delta r \Delta\vartheta, \\ &\quad \forall t \in (r_p, t_{p+1}]_{\mathbb{T}}, \quad p = 1, 2, \dots, n. \end{aligned}$$

In order to prove the main results, we need the following assumptions:

(H1): The family $\{A(t) : t \in I\}$ of bounded linear operators in X generates an exponentially stable evolution operator $\{E(t, s) : t \geq s\}$. i.e., there exist constants $K_0 \geq 1$ and $\nu > 0$ such that $\|E(t, s)\| \leq K_0 e_{\ominus\nu}(t, s)$.

(H2): Function $\mathcal{F} : J_1 \times X \rightarrow X$ is continuous and there exists a positive constant $L_{\mathcal{F}}$ such that

$$\|\mathcal{F}(t, x) - \mathcal{F}(t, y)\| \leq L_{\mathcal{F}}\|x - y\|, \quad \forall t \in J_1, \quad x, y \in X.$$

Also, there exist positive constants $C_{\mathcal{F}}$ and $M_{\mathcal{F}}$ such that

$$\|\mathcal{F}(t, x)\| \leq C_{\mathcal{F}} + M_{\mathcal{F}}\|x\|, \quad \forall t \in J_1, \quad x \in X.$$

(H3): Function $\mathcal{H} : \mathcal{Q} \times X \rightarrow X$ is continuous and there exists a positive constant $L_{\mathcal{H}}$ such that

$$\|\mathcal{H}(t, r, x) - \mathcal{H}(t, r, y)\| \leq L_{\mathcal{H}}\|x - y\|, \quad \forall t, r \in \mathcal{Q}, \quad x, y \in X.$$

Also, there exist positive constants $C_{\mathcal{H}}$ and $M_{\mathcal{H}}$ such that

$$\|\mathcal{H}(t, r, x)\| \leq C_{\mathcal{H}} + M_{\mathcal{H}}\|x\|, \quad \forall t, r \in \mathcal{Q}, \quad x \in X.$$

(H4): The functions $\mathcal{G}_p : I_p \times X \rightarrow X$, $p = 1, 2, \dots, n$, are continuous and there exists a positive constant $L_{\mathcal{G}}$ such that

$$\|\mathcal{G}_p(t, x) - \mathcal{G}_p(t, y)\| \leq L_{\mathcal{G}}\|x - y\|, \quad \forall x, y \in X, t \in I_p, \quad p = 1, 2, \dots, n.$$

Also, there exists a positive constant $M_{\mathcal{G}}$ such that $\|\mathcal{G}_p(t, x)\| \leq M_{\mathcal{G}}, \forall t \in I_p$ and $x \in X$.

For the notional convenience, we set

$$\mathcal{N}_1 = \frac{K_0(1 + \bar{\mu}\nu)}{\nu} (C_{\mathcal{F}} + C_{\mathcal{H}}T), \quad \mathcal{N}_2 = \frac{K_0(1 + \bar{\mu}\nu)}{\nu \oplus \Omega} \left(M_{\mathcal{F}} + \frac{M_{\mathcal{H}}}{\Omega} \right), \quad \mathcal{N}_3 = \frac{K_0(1 + \bar{\mu}\nu)}{\nu \oplus \Omega} \left(L_{\mathcal{F}} + \frac{L_{\mathcal{H}}}{\Omega} \right),$$

$$\bar{\mu} = \sup_{t \in I} \mu(t).$$

3 Existence and Stability Results

In this section, we establish our main result of existence of a unique solution for the system (1.1). Banach contraction theorem has been used to establish this result.

Theorem 3.1. *If all the assumptions (H1) to (H4) are satisfied, then the system (1.1) has a unique solution.*

Proof: Consider a subset $\mathcal{D} \subseteq PC(I, X)$ such that

$$\mathcal{D} = \{x \in PC(I, X) : \|x\|_{\Omega} \leq \delta\},$$

where

$$\delta = \max \left(\frac{K_0 \|x_0\| + \mathcal{N}_1}{1 - \mathcal{N}_2}, \frac{K_0 M_G + \mathcal{N}_1}{1 - \mathcal{N}_2}, M_G \right).$$

Define an operator $\Pi : \mathcal{D} \rightarrow \mathcal{D}$ as

$$\begin{aligned} (\Pi x)(t) &= E(t, 0)x_0 + \int_0^t E(t, \sigma(\vartheta))\mathcal{F}(\vartheta, x(\vartheta))\Delta\vartheta + \int_0^t E(t, \sigma(\vartheta)) \int_0^{\vartheta} \mathcal{H}(\vartheta, r, x(r))\Delta r \Delta\vartheta, \quad \forall t \in [0, t_1]_{\mathbb{T}}, \\ (\Pi x)(t) &= \mathcal{G}_p \left(t, E(t_p, r_{p-1})\mathcal{G}_{p-1}(r_{p-1}, x(t_{p-1}^-)) + \int_{r_{p-1}}^{t_p} E(t_p, \sigma(\vartheta))\mathcal{F}(\vartheta, x(\vartheta))\Delta\vartheta \right. \\ &\quad \left. + \int_{r_{p-1}}^{t_p} E(t_p, \sigma(\vartheta)) \int_0^{\vartheta} \mathcal{H}(\vartheta, r, x(r))\Delta r \Delta\vartheta \right), \quad \forall t \in (t_p, r_p]_{\mathbb{T}}, \quad p = 1, 2, \dots, n, \\ (\Pi x)(t) &= E(t, r_p)\mathcal{G}_p(r_p, x(t_p^-)) + \int_{r_p}^t E(t, \sigma(\vartheta))\mathcal{F}(\vartheta, x(\vartheta))\Delta\vartheta + \int_{r_p}^t E(t, \sigma(\vartheta)) \int_0^{\vartheta} \mathcal{H}(\vartheta, r, x(r))\Delta r \Delta\vartheta, \\ &\quad \forall t \in (r_p, t_{p+1}]_{\mathbb{T}}, \quad p = 1, 2, \dots, n. \end{aligned}$$

For the sake of simplicity, the proof of the theorem is divided into the following two steps:

Step 1: We prove the operator Π is a well-defined from \mathcal{D} into \mathcal{D} .

For $t \in [0, t_1]_{\mathbb{T}}$ and $x \in \mathcal{D}$, we have

$$\begin{aligned} \|(\Pi x)(t)\| &\leq \|E(t, 0)\| \|x_0\| + \int_0^t \|E(t, \sigma(\vartheta))\| \|\mathcal{F}(\vartheta, x(\vartheta))\| \Delta\vartheta + \int_0^t \|E(t, \sigma(\vartheta))\| \int_0^{\vartheta} \|\mathcal{H}(\vartheta, r, x(r))\| \Delta r \Delta\vartheta \\ &\leq K_0 e_{\ominus\nu}(t, 0) \|x_0\| + \int_0^t K_0 e_{\ominus\nu}(t, \sigma(\vartheta)) (C_{\mathcal{F}} + M_{\mathcal{F}} \|x(\vartheta)\|) \Delta\vartheta \\ &\quad + \int_0^t K_0 e_{\ominus\nu}(t, \sigma(\vartheta)) \int_0^{\vartheta} (C_{\mathcal{H}} + M_{\mathcal{H}} \|x(r)\|) \Delta r \Delta\vartheta \\ &\leq K_0 e_{\ominus\nu}(t, 0) \|x_0\| + K_0 C_{\mathcal{F}} \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) \Delta\vartheta + K_0 M_{\mathcal{F}} \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) \|x(\vartheta)\| \Delta\vartheta \\ &\quad + K_0 C_{\mathcal{H}} t_1 \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) \Delta\vartheta + K_0 M_{\mathcal{H}} \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) \int_0^{\vartheta} \|x(r)\| \Delta r \Delta\vartheta \\ &= K_0 e_{\ominus\nu}(t, 0) \|x_0\| + (K_0 C_{\mathcal{F}} + K_0 C_{\mathcal{H}} t_1) \mathcal{I}_1 + K_0 M_{\mathcal{F}} \mathcal{I}_2 + K_0 M_{\mathcal{H}} \mathcal{I}_3, \end{aligned} \tag{3.1}$$

where

$$\mathcal{I}_1 = \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) \Delta\vartheta = \frac{(1 + \bar{\mu}\nu)(1 - e_{\ominus\nu}(t, 0))}{\nu}, \tag{3.2}$$

$$\begin{aligned} \mathcal{I}_2 &= \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) \|x(\vartheta)\| \Delta\vartheta \leq \|x\|_{\Omega} \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) e_{\Omega}(\vartheta, 0) \Delta\vartheta \\ &\leq \frac{\|x\|_{\Omega} (1 + \bar{\mu}\nu) e_{\ominus\nu}(t, 0) (e_{\nu \oplus \Omega}(t, 0) - 1)}{\nu \oplus \Omega} \\ &\leq \frac{\|x\|_{\Omega} (1 + \bar{\mu}\nu) e_{\Omega}(t, 0)}{\nu \oplus \Omega} \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \mathcal{I}_3 &= \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) \int_0^\vartheta \|x(r)\| \Delta r \Delta\vartheta \leq \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) \|x\|_\Omega \int_0^\vartheta e_\Omega(r, 0) \Delta r \Delta\vartheta \\ &\leq \frac{\|x\|_\Omega (1 + \bar{\mu}\nu) e_\Omega(t, 0)}{\Omega(\nu \oplus \Omega)}. \end{aligned} \quad (3.4)$$

Subsequently, from the inequalities (3.1), (3.2), (3.3) and (3.4), we get

$$\begin{aligned} \|(\Pi x)t\| &\leq K_0 e_{\ominus\nu}(t, 0) \|x_0\| + (K_0 C_{\mathcal{F}} + K_0 C_{\mathcal{H}} t_1) \frac{(1 + \bar{\mu}\nu)(1 - e_{\ominus\nu}(t, 0))}{\nu} + \|x\|_\Omega \frac{K_0 M_{\mathcal{F}}(1 + \bar{\mu}\nu) e_\Omega(t, 0)}{\nu \oplus \Omega} \\ &\quad + \|x\|_\Omega \frac{K_0 M_{\mathcal{H}}(1 + \bar{\mu}\nu) e_\Omega(t, 0)}{\Omega(\nu \oplus \Omega)} \\ &\leq K_0 e_{\ominus\nu}(t, 0) \|x_0\| + \frac{K_0(1 + \bar{\mu}\nu)(1 - e_{\ominus\nu}(t, 0))}{\nu} (C_{\mathcal{F}} + C_{\mathcal{H}} t_1) + \frac{K_0 \delta(1 + \bar{\mu}\nu) e_\Omega(t, 0)}{\nu \oplus \Omega} \left(M_{\mathcal{F}} + \frac{M_{\mathcal{H}}}{\Omega} \right). \end{aligned}$$

Now, divide the above inequality by $e_\Omega(t, 0)$ and take the sup over $t \in [0, t_1]_{\mathbb{T}}$, we get

$$\begin{aligned} \|\Pi x\|_\Omega &\leq K_0 \|x_0\| + \frac{K_0(1 + \bar{\mu}\nu)}{\nu} (C_{\mathcal{F}} + C_{\mathcal{H}} t_1) + \frac{K_0 \delta(1 + \bar{\mu}\nu)}{\nu \oplus \Omega} \left(M_{\mathcal{F}} + \frac{M_{\mathcal{H}}}{\Omega} \right) \\ &\leq K_0 \|x_0\| + \mathcal{N}_1 + \delta \mathcal{N}_2 \leq \delta. \end{aligned} \quad (3.5)$$

For any $t \in (r_p, t_{p+1}]_{\mathbb{T}}$, $p = 1, 2, \dots, n$ and $x \in \mathcal{D}$, we have

$$\begin{aligned} &\|(\Pi x)(t)\| \\ &\leq \|E(t, r_p)\| \|\mathcal{G}_p(r_p, x(t_p^-))\| + \int_{r_p}^t \|E(t, \sigma(\vartheta))\| \|\mathcal{F}(\vartheta, x(\vartheta))\| \Delta\vartheta + \int_{r_p}^t \|E(t, \sigma(\vartheta))\| \int_0^\vartheta \|\mathcal{H}(\vartheta, r, x(r))\| \Delta r \Delta\vartheta \\ &\leq K_0 e_{\ominus\nu}(t, r_p) M_{\mathcal{G}} + \int_{r_p}^t K_0 e_{\ominus\nu}(t, \sigma(\vartheta)) (C_{\mathcal{F}} + M_{\mathcal{F}} \|x(\vartheta)\|) \Delta\vartheta + \int_{r_p}^t K_0 e_{\ominus\nu}(t, \sigma(\vartheta)) \int_0^\vartheta (C_{\mathcal{H}} + M_{\mathcal{H}} \|x(r)\|) \Delta r \Delta\vartheta \\ &\leq K_0 e_{\ominus\nu}(t, r_p) M_{\mathcal{G}} + K_0 C_{\mathcal{F}} \int_{r_p}^t e_{\ominus\nu}(t, \sigma(\vartheta)) \Delta\vartheta + K_0 M_{\mathcal{F}} \int_{r_p}^t e_{\ominus\nu}(t, \sigma(\vartheta)) \|x(\vartheta)\| \Delta\vartheta \\ &\quad + K_0 \int_{r_p}^t e_{\ominus\nu}(t, \sigma(\vartheta)) \int_0^\vartheta C_{\mathcal{H}} \Delta r \Delta\vartheta + K_0 M_{\mathcal{H}} \int_{r_p}^t e_{\ominus\nu}(t, \sigma(\vartheta)) \int_0^\vartheta \|x(r)\| \Delta r \Delta\vartheta \\ &\leq K_0 e_{\ominus\nu}(t, r_p) M_{\mathcal{G}} + \frac{K_0 C_{\mathcal{F}} (e_{\ominus\nu}(t, r_p) - 1)}{\ominus\nu} + K_0 M_{\mathcal{F}} \delta \int_{r_p}^t e_{\ominus\nu}(t, \sigma(\vartheta)) e_\Omega(\vartheta, r_p) \Delta\vartheta \\ &\quad + \frac{K_0 C_{\mathcal{H}} T (e_{\ominus\nu}(t, r_p) - 1)}{\ominus\nu} + \frac{K_0 M_{\mathcal{H}} \delta}{\Omega} \int_{r_p}^t e_{\ominus\nu}(t, \sigma(\vartheta)) e_\Omega(\vartheta, r_p) \Delta\vartheta \\ &\leq K_0 e_{\ominus\nu}(t, r_p) M_{\mathcal{G}} + \frac{K_0(1 + \bar{\mu}\nu)(1 - e_{\ominus\nu}(t, r_p))}{\nu} (C_{\mathcal{F}} + C_{\mathcal{H}} T) + \frac{K_0 \delta(1 + \bar{\mu}\nu) e_\Omega(t, r_p)}{\nu \oplus \Omega} \left(M_{\mathcal{F}} + \frac{M_{\mathcal{H}}}{\Omega} \right). \end{aligned}$$

Hence,

$$\begin{aligned} \|\Pi x\|_\Omega &\leq K_0 M_{\mathcal{G}} + \frac{K_0(1 + \bar{\mu}\nu)}{\nu} (C_{\mathcal{F}} + C_{\mathcal{H}} T) + \frac{K_0 \delta(1 + \bar{\mu}\nu)}{\nu \oplus \Omega} \left(M_{\mathcal{F}} + \frac{M_{\mathcal{H}}}{\Omega} \right) \\ &= K_0 M_{\mathcal{G}} + \mathcal{N}_1 + \delta \mathcal{N}_2 \leq \delta. \end{aligned} \quad (3.6)$$

Similarly, for any $t \in (t_p, r_p]_{\mathbb{T}}$, $p = 1, 2, \dots, n$ and $x \in \mathcal{D}$, we have

$$\|\Pi x\|_{\Omega} \leq M_{\mathcal{G}} \leq \delta. \quad (3.7)$$

After summarizing the above inequalities (3.5), (3.6) and (3.7), for all $t \in I$, we get

$$\|\Pi x\|_{\Omega} \leq \delta.$$

Therefore, Π is well-defined from \mathcal{D} into \mathcal{D} .

Step 2: In this step, we prove the operator Π is a contraction from \mathcal{D} into \mathcal{D} .

For any $t \in [0, t_1]_{\mathbb{T}}$ and $x, y \in \mathcal{D}$, we have

$$\begin{aligned} \|(\Pi x)(t) - (\Pi y)(t)\| &\leq K_0 \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) \|\mathcal{F}(\vartheta, x(\vartheta)) - \mathcal{F}(\vartheta, y(\vartheta))\| \Delta\vartheta \\ &\quad + K_0 \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) \int_0^{\vartheta} \|\mathcal{H}(\vartheta, r, x(r)) - \mathcal{H}(\vartheta, r, y(r))\| \Delta r \Delta\vartheta \\ &\leq K_0 L_{\mathcal{F}} \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) \|x(\vartheta) - y(\vartheta)\| \Delta\vartheta + K_0 L_{\mathcal{H}} \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) \int_0^{\vartheta} \|x(r) - y(r)\| \Delta r \Delta\vartheta \\ &\leq K_0 L_{\mathcal{F}} \|x - y\|_{\Omega} \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) e_{\Omega}(\vartheta, 0) \Delta\vartheta + \frac{K_0 L_{\mathcal{H}} \|x - y\|_{\Omega}}{\Omega} \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) e_{\Omega}(\vartheta, 0) \Delta\vartheta \\ &\leq K_0 \|x - y\|_{\Omega} \left(L_{\mathcal{F}} + \frac{L_{\mathcal{H}}}{\Omega} \right) \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) e_{\Omega}(\vartheta, 0) \Delta\vartheta \\ &\leq K_0 \|x - y\|_{\Omega} \left(L_{\mathcal{F}} + \frac{L_{\mathcal{H}}}{\Omega} \right) \frac{(1 + \bar{\mu}\nu)}{\nu \oplus \Omega} e_{\Omega}(t, 0). \end{aligned}$$

Therefore,

$$\|\Pi x - \Pi y\|_{\Omega} \leq \frac{K_0(1 + \bar{\mu}\nu)}{\nu \oplus \Omega} \left[L_{\mathcal{F}} + \frac{L_{\mathcal{H}}}{\Omega} \right] \|x - y\|_{\Omega} \leq \mathcal{N}_3 \|x - y\|_{\Omega}. \quad (3.8)$$

For any $t \in (r_p, t_{p+1}]_{\mathbb{T}}$, $p = 1, 2, \dots, n$ and $x, y \in \mathcal{D}$, we have

$$\begin{aligned} \|(\Pi x)(t) - (\Pi y)(t)\| &\leq K_0 e_{\ominus\nu}(t, r_p) \|\mathcal{G}_p(r_p, x(t_p^-)) - \mathcal{G}_p(r_p, y(t_p^-))\| + K_0 \int_{r_p}^t e_{\ominus\nu}(t, \sigma(\vartheta)) \|\mathcal{F}(\vartheta, x(\vartheta)) - \mathcal{F}(\vartheta, y(\vartheta))\| \Delta\vartheta \\ &\quad + K_0 \int_{r_p}^t e_{\ominus\nu}(t, \sigma(\vartheta)) \int_0^{\vartheta} \|\mathcal{H}(\vartheta, r, x(r)) - \mathcal{H}(\vartheta, r, y(r))\| \Delta r \Delta\vartheta \\ &\leq K_0 L_{\mathcal{G}} e_{\ominus\nu}(t, r_p) e_{\Omega}(t_p^-, r_p) \|x - y\|_{\Omega} + K_0 L_{\mathcal{F}} \int_{r_p}^t e_{\ominus\nu}(t, \sigma(\vartheta)) \|x(\vartheta) - y(\vartheta)\| \Delta\vartheta \\ &\quad + K_0 L_{\mathcal{H}} \int_{r_p}^t e_{\ominus\nu}(t, \sigma(\vartheta)) \int_0^{\vartheta} \|x(r) - y(r)\| \Delta r \Delta\vartheta \\ &\leq K_0 L_{\mathcal{G}} e_{\ominus\nu}(t, r_p) e_{\Omega}(t_p^-, r_p) \|x - y\|_{\Omega} + K_0 L_{\mathcal{F}} \|x - y\|_{\Omega} \int_{r_p}^t e_{\ominus\nu}(t, \sigma(\vartheta)) e_{\Omega}(\vartheta, r_p) \Delta\vartheta \\ &\quad + \frac{K_0 L_{\mathcal{H}} \|x - y\|_{\Omega}}{\Omega} \int_{r_p}^t e_{\ominus\nu}(t, \sigma(\vartheta)) e_{\Omega}(\vartheta, r_p) \Delta\vartheta \\ &\leq K_0 L_{\mathcal{G}} e_{\ominus\nu}(t, r_p) e_{\Omega}(t_p^-, r_p) \|x - y\|_{\Omega} + K_0 \|x - y\|_{\Omega} \left(L_{\mathcal{F}} + \frac{L_{\mathcal{H}}}{\Omega} \right) \int_{r_p}^t e_{\ominus\nu}(t, \sigma(\vartheta)) e_{\Omega}(\vartheta, r_p) \Delta\vartheta \\ &\leq \frac{K_0 L_{\mathcal{G}} e_{\ominus\nu}(t, r_p)}{e_{\Omega}(r_p, t_p^-)} \|x - y\|_{\Omega} + K_0 \|x - y\|_{\Omega} \left(L_{\mathcal{F}} + \frac{L_{\mathcal{H}}}{\Omega} \right) \frac{(1 + \bar{\mu}\nu)}{\nu \oplus \Omega} e_{\Omega}(t, r_p). \end{aligned}$$

Thus, we have

$$\begin{aligned}\|\Pi x - \Pi y\|_{\Omega} &\leq \left[\frac{K_0 L_{\mathcal{G}}}{e_{\Omega}(r_p, t_p^-)} + \frac{K_0(1 + \bar{\mu}\nu)}{\nu \oplus \Omega} \left(L_{\mathcal{F}} + \frac{L_{\mathcal{H}}}{\Omega} \right) \right] \|x - y\|_{\Omega} \\ &= \left[\frac{K_0 L_{\mathcal{G}}}{e_{\Omega}(r_p, t_p^-)} + \mathcal{N}_3 \right] \|x - y\|_{\Omega}.\end{aligned}\tag{3.9}$$

Similarly, for $t \in (t_p, r_p]_{\mathbb{T}}$, $p = 1, 2, \dots, n$ and $x, y \in \mathcal{D}$, we have

$$\begin{aligned}\|(\Pi x)t - (\Pi y)t\| &\leq L_{\mathcal{G}} K_0 \left[L_{\mathcal{G}} e_{\ominus\nu}(t_p, r_{p-1}) \|x(t_{p-1}^-) - y(t_{p-1}^-)\| + L_{\mathcal{F}} \int_{r_{p-1}}^{t_p} e_{\ominus\nu}(t_p, \sigma(\vartheta)) \|x(\vartheta) - y(\vartheta)\| \Delta\vartheta \right. \\ &\quad \left. + L_{\mathcal{H}} \int_{r_{p-1}}^{t_p} e_{\ominus\nu}(t_p, \sigma(\vartheta)) \int_0^{\vartheta} \|x(r) - y(r)\| \Delta r \Delta\vartheta \right] \\ &\leq L_{\mathcal{G}} K_0 \left[L_{\mathcal{G}} e_{\ominus\nu}(t_p, r_{p-1}) e_{\Omega}(t_{p-1}^-, r_{p-1}) \|x - y\|_{\Omega} \right. \\ &\quad \left. + \|x - y\|_{\Omega} \left(L_{\mathcal{F}} + \frac{L_{\mathcal{H}}}{\Omega} \right) \int_{r_{p-1}}^{t_p} e_{\ominus\nu}(t_p, \sigma(\vartheta)) e_{\Omega}(\vartheta, r_{p-1}) \Delta\vartheta \right].\end{aligned}$$

Therefore,

$$\begin{aligned}\|\Pi x - \Pi y\|_{\Omega} &\leq \left[\frac{K_0 L_{\mathcal{G}}^2}{e_{\Omega}(r_{p-1}, t_{p-1}^-)} + \frac{K_0 L_{\mathcal{G}}(1 + \bar{\mu}\nu)}{(\nu \oplus \Omega) e_{\Omega}(r_{p-1}, t_p)} \left(L_{\mathcal{F}} + \frac{L_{\mathcal{H}}}{\Omega} \right) \right] \|x - y\|_{\Omega} \\ &= \left[\frac{K_0 L_{\mathcal{G}}^2}{e_{\Omega}(r_{p-1}, t_{p-1}^-)} + \frac{L_{\mathcal{G}} \mathcal{N}_3}{e_{\Omega}(r_{p-1}, t_p)} \right] \|x - y\|_{\Omega}.\end{aligned}\tag{3.10}$$

After summarizing the inequalities (3.8), (3.9) and (3.10), for all $t \in I$, we get

$$\|\Pi x - \Pi y\|_{\Omega} \leq L_{\Pi} \|x - y\|_{\Omega},$$

where

$$L_{\Pi} = \max_{1 \leq p \leq n} \left[\frac{K_0 L_{\mathcal{G}}}{e_{\Omega}(r_p, t_p^-)} + \mathcal{N}_3, \frac{K_0 L_{\mathcal{G}}^2}{e_{\Omega}(r_{p-1}, t_{p-1}^-)} + \frac{L_{\mathcal{G}} \mathcal{N}_3}{e_{\Omega}(r_{p-1}, t_p)} \right].$$

Hence, Π is a strict contraction operator for sufficiently large Ω .

Thus, from the step 1 and step 2, we can conclude that the operator Π satisfies all the conditions of Banach contraction theorem. Therefore, Banach contraction theorem immediately gives a unique fixed point of Π , which is the solution of the system (1.1). \square

Next, we give the stability result for the system (1.1). For convenience, the definition is introduced first.

Definition 3.2. A solution x of the system (1.1) is said to be *locally stable*, if for arbitrary $\epsilon > 0$, there exists $\gamma = \gamma(\epsilon) > 0$ such that

$$\|x(t) - \bar{x}(t)\| < \epsilon, \quad \forall t \in I,$$

whenever $\|x(0) - \bar{x}_0\| < \gamma$, where \bar{x} is the solution of the system (1.1) with initial condition $\bar{x}(0) = \bar{x}_0$ and the impulsive condition $\bar{x}(t) = \mathcal{G}_p(t, \bar{x}(t_p^-))$ for $t \in (t_p, r_p]_{\mathbb{T}}$, $p = 1, 2, \dots, n$.

Next theorem give the main stability result for the system (1.1).

Theorem 3.3. Under the assumptions of Theorem 3.1, system (1.1) has a unique solution which is *locally stable*.

Proof: From Theorem 3.1, we find that the system (1.1) has a unique solution $x(t)$. Let $\bar{x}(t)$ be any other solution of (1.1) with the initial condition $\bar{x}(0) = \bar{x}_0$ and impulses $\bar{x}(t) = \mathcal{G}_p(t_p, \bar{x}(t_p^-))$, $t \in (t_p, r_p]_{\mathbb{T}}$, $p = 1, 2, \dots, n$. Then:
For any $t \in [0, t_1]_{\mathbb{T}}$, we have

$$\begin{aligned} \|x(t) - \bar{x}(t)\| &\leq K_0 e_{\ominus\nu}(t, 0) \|x_0 - \bar{x}_0\| + K_0 \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) \|\mathcal{F}(\vartheta, x(\vartheta)) - \mathcal{F}(\vartheta, \bar{x}(\vartheta))\| \Delta\vartheta \\ &\quad + K_0 \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) \int_0^{\vartheta} \|\mathcal{H}(\vartheta, r, x(r)) - \mathcal{H}(\vartheta, r, \bar{x}(r))\| \Delta r \Delta\vartheta \\ &\leq K_0 e_{\ominus\nu}(t, 0) \|x_0 - \bar{x}_0\| + K_0 \|x - \bar{x}\|_{\Omega} \left(L_{\mathcal{F}} + \frac{L_{\mathcal{H}}}{\Omega} \right) \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) e_{\Omega}(\vartheta, 0) \Delta\vartheta \\ &\leq K_0 e_{\ominus\nu}(t, 0) \|x_0 - \bar{x}_0\| + K_0 \|x - \bar{x}\|_{\Omega} \left(L_{\mathcal{F}} + \frac{L_{\mathcal{H}}}{\Omega} \right) \frac{(1 + \bar{\mu}\nu)}{\nu \oplus \Omega} e_{\Omega}(t, 0). \end{aligned}$$

Therefore,

$$\begin{aligned} \|x - \bar{x}\|_{\Omega} &\leq K_0\gamma + \frac{K_0(1 + \bar{\mu}\nu)}{\nu \oplus \Omega} \left[L_{\mathcal{F}} + \frac{L_{\mathcal{H}}}{\Omega} \right] \|x - \bar{x}\|_{\Omega} \\ &\leq K_0\gamma + L_{\Pi} \|x - \bar{x}\|_{\Omega} \leq \frac{K_0\gamma}{1 - L_{\Pi}}. \end{aligned} \tag{3.11}$$

For any $t \in (r_p, t_{p+1}]_{\mathbb{T}}$, $p = 1, 2, \dots, n$, we have

$$\begin{aligned} \|x(t) - \bar{x}(t)\| &\leq K_0 e_{\ominus\nu}(t, r_p) \|\mathcal{G}_p(r_p, x(t_p^-)) - \mathcal{G}_p(r_p, \bar{x}(t_p^-))\| \\ &\quad + K_0 \int_{r_p}^t e_{\ominus\nu}(t, \sigma(\vartheta)) \|\mathcal{F}(\vartheta, x(\vartheta)) - \mathcal{F}(\vartheta, \bar{x}(\vartheta))\| \Delta\vartheta \\ &\quad + K_0 \int_{r_p}^t e_{\ominus\nu}(t, \sigma(\vartheta)) \int_0^{\vartheta} \|\mathcal{H}(\vartheta, r, x(r)) - \mathcal{H}(\vartheta, r, \bar{x}(r))\| \Delta r \Delta\vartheta \\ &\leq K_0\gamma + K_0 \|x - \bar{x}\|_{\Omega} \left(L_{\mathcal{F}} + \frac{L_{\mathcal{H}}}{\Omega} \right) \frac{(1 + \bar{\mu}\nu)}{\nu \oplus \Omega} e_{\Omega}(t, r_p) + \frac{K_0 L_{\mathcal{G}} e_{\ominus\nu}(t, r_p)}{e_{\Omega}(r_p, t_p^-)} \|x - \bar{x}\|_{\Omega}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|x - \bar{x}\|_{\Omega} &\leq K_0\gamma + \left[\frac{K_0 L_{\mathcal{G}}}{e_{\Omega}(r_p, t_p^-)} + \frac{K_0(1 + \bar{\mu}\nu)}{\nu \oplus \Omega} \left(L_{\mathcal{F}} + \frac{L_{\mathcal{H}}}{\Omega} \right) \right] \|x - \bar{x}\|_{\Omega} \\ &\leq K_0\gamma + L_{\Pi} \|x - \bar{x}\|_{\Omega} \leq \frac{K_0\gamma}{1 - L_{\Pi}}. \end{aligned} \tag{3.12}$$

Similarly, for $t \in (t_p, r_p]_{\mathbb{T}}$, $p = 1, 2, \dots, n$, we can easily get

$$\|x - \bar{x}\|_{\Omega} \leq K_0\gamma + L_{\Pi} \|x - \bar{x}\|_{\Omega} \leq \frac{K_0\gamma}{1 - L_{\Pi}}. \tag{3.13}$$

From the above inequalities (3.11), (3.12) and (3.13), for all $t \in I$, we get

$$\|x - \bar{x}\|_{\Omega} \leq \frac{K_0\gamma}{1 - L_{\Pi}}.$$

Hence,

$$\|x - \bar{x}\|_{\Omega} \leq \mathcal{C}\gamma, \quad t \in I,$$

where $\mathcal{C} = \frac{K_0}{1 - L_\Pi}$. Now, choose a $\gamma > 0$ such that $\gamma < \frac{\epsilon}{\mathcal{C}}$, then

$$\|x - \bar{x}\|_\Omega < \epsilon.$$

Thus, from the Definition 3.2, we can conclude that the system (1.1) is locally stable. \square

4 Exact Controllability Result

In this section, we proceed to establish the sufficient criteria for controllability of the impulsive system (1.2). For convenience, the definition and some notations are introduced first.

Lemma 4.1. *The solution of the following linear system*

$$\begin{aligned} x^\Delta(t) &= A(t)x(t) + Bu(t), \quad t \in I, \\ x(0) &= x_0, \end{aligned} \tag{4.1}$$

is given by

$$x(t) = E(t, 0)x_0 + \int_0^t E(t, \sigma(r))Bu(r)\Delta r.$$

Definition 4.2. *Linear system (4.1) is called exact controllable on $[0, T]_\mathbb{T}$, if for every $x_0, x_T \in X$, there exists a rd-continuous function $u \in L^2([0, T]_\mathbb{T}, X)$ such that the corresponding solution of (4.1) satisfies $x(0) = x_0$ and $x(T) = x_T$.*

In the next definition, we generalized the above definition for the system (1.2).

Definition 4.3. *The non-instantaneous impulsive system (1.2) is called exact controllable on $[0, T]_\mathbb{T}$, if for $p = 1, 2, \dots, n$, it is exact controllable on $[0, t_1]_\mathbb{T}$ and $(r_p, t_{p+1}]_\mathbb{T}$. i.e., if for every x_0 and $x_{t_{p+1}} \in X, p = 0, 1, \dots, n$, there exists a piece-wise rd-continuous function $u \in L^2([0, T]_\mathbb{T}, X)$ such that the mild solution of (1.2) satisfies $x(0) = x_0$ and $x(t_{p+1}) = x_{t_{p+1}}$ for $p = 0, 1, \dots, n$.*

Next, we define the solution of the system (1.2).

Definition 4.4. *A function $x \in PC(I, X)$ is said to be a mild solution of the system (1.2), if x satisfies $x(0) = x_0$, $x(t) = \mathcal{G}_p(t, x(t_p^-))$, $\forall t \in (t_p, r_p]_\mathbb{T}$, $p = 1, 2, \dots, n$ and x is the solution of the following integral equations*

$$x(t) = E(t, 0)x_0 + \int_0^t E(t, \sigma(\vartheta))\mathcal{F}(\vartheta, x(\vartheta))\Delta\vartheta + \int_0^t E(t, \sigma(\vartheta)) \int_0^\vartheta \mathcal{H}(\vartheta, r, x(r))\Delta r \Delta\vartheta + \int_0^t E(t, \sigma(\vartheta))Bu(\vartheta)\Delta\vartheta, \quad \forall t \in [0, t_1]_\mathbb{T}, \tag{4.2}$$

$$\begin{aligned} x(t) &= E(t, r_p)\mathcal{G}_p(r_p, x(t_p^-)) + \int_{r_p}^t E(t, \sigma(\vartheta))\mathcal{F}(\vartheta, x(\vartheta))\Delta\vartheta + \int_{r_p}^t E(t, \sigma(\vartheta)) \int_0^\vartheta \mathcal{H}(\vartheta, r, x(r))\Delta r \Delta\vartheta \\ &+ \int_{r_p}^t E(t, \sigma(\vartheta))Bu(\vartheta)\Delta\vartheta, \quad \forall t \in (r_p, t_{p+1}]_\mathbb{T}, \quad p = 1, 2, \dots, n. \end{aligned} \tag{4.3}$$

To prove the controllability result for the impulsive system (1.2), we need the following assumptions:

(H5): The linear operators $\mathcal{W}_{r_p}^{t_{p+1}} : L^2(I, U) \rightarrow X$ defined by

$$\mathcal{W}_{r_p}^{t_{p+1}} u = \int_{r_p}^{t_{p+1}} E(t_{p+1}, \sigma(r))Bu(r)\Delta r, \quad p = 0, 1, \dots, n,$$

have the bounded invertible operators $(\mathcal{W}_{r_p}^{t_p+1})^{-1}$ which take values in $L^2(I, U) \setminus \ker \mathcal{W}_{r_p}^{t_p+1}$ and there exist positive constants $M_W^p, p = 0, 1, \dots, n$, such that $\|(\mathcal{W}_{r_p}^{t_p+1})^{-1}\| \leq M_W^p$.

Also, B is continuous operator from U to X and there exists a positive constant M_B such that $\|B\| \leq M_B$.

We set $\mathcal{N}_4^p = \frac{K_0 M_B (1 + \bar{\mu}\nu) M_W^p}{\nu}$, $p = 0, 1, \dots, n$.

(H6): $\mathcal{N}_2(1 + \mathcal{N}_4^p e_\Omega(t_{p+1}, r_p)) < 1$, $p = 0, 1, \dots, n$.

Next, we give some lemma which will help to prove our main result of this section.

Lemma 4.5. *If all the assumptions (H1) to (H5) are satisfied, then the control input*

$$u(t) = (\mathcal{W}_0^{t_1})^{-1} \left[x_{t_1} - E(t_1, 0)x_0 - \int_0^{t_1} E(t_1, \sigma(\vartheta)) \mathcal{F}(\vartheta, x(\vartheta)) \Delta\vartheta - \int_0^{t_1} E(t_1, \sigma(\vartheta)) \int_0^\vartheta \mathcal{H}(\vartheta, r, x(r)) \Delta r \Delta\vartheta \right] (t),$$

(4.4)

$\forall t \in [0, t_1]_{\mathbb{T}}$,

steers the state $x(t)$ of system (1.2) from x_0 to x_{t_1} at the time t_1 . Moreover, the control estimate $\|u(t)\|$ is M_u^0 , where

$$M_u^0 = M_W^0 [\|x_{t_1}\| + K_0 e_{\ominus\nu}(t_1, 0) \|x_0\| + \mathcal{N}_1 + \|x\|_\Omega \mathcal{N}_2 e_\Omega(t_1, 0)].$$

Proof: Consider the mild solution $x(t)$ of the system (1.2) on $[0, t_1]_{\mathbb{T}}$ defined by (4.2). For $t = t_1$, we get

$$\begin{aligned} x(t_1) &= E(t_1, 0)x_0 + \int_0^{t_1} E(t, \sigma(\vartheta)) \mathcal{F}(\vartheta, x(\vartheta)) \Delta\vartheta + \int_0^{t_1} E(t_1, \sigma(\vartheta)) \int_0^\vartheta \mathcal{H}(\vartheta, r, x(r)) \Delta r \Delta\vartheta + \mathcal{W}_0^{t_1} u(t) \\ &= E(t_1, 0)x_0 + \left[x_{t_1} - E(t_1, 0)x_0 - \int_0^{t_1} E(t_1, \sigma(\vartheta)) \mathcal{F}(\vartheta, x(\vartheta)) \Delta\vartheta - \int_0^{t_1} E(t_1, \sigma(\vartheta)) \int_0^\vartheta \mathcal{H}(\vartheta, r, x(r)) \Delta r \Delta\vartheta \right] \\ &\quad + \int_0^{t_1} E(t, \sigma(\vartheta)) \mathcal{F}(\vartheta, x(\vartheta)) \Delta\vartheta + \int_0^{t_1} E(t_1, \sigma(\vartheta)) \int_0^\vartheta \mathcal{H}(\vartheta, r, x(r)) \Delta r \Delta\vartheta \\ &= x_{t_1}. \end{aligned}$$

Hence, control input (4.4) is suitable for the system (1.2). Moreover, the control estimate is calculated as

$$\begin{aligned} \|u(t)\| &\leq M_W^0 \left[\|x_{t_1}\| + K_0 e_{\ominus\nu}(t_1, 0) \|x_0\| + K_0 \int_0^{t_1} e_{\ominus\nu}(t_1, \sigma(\vartheta)) \|\mathcal{F}(\vartheta, x(\vartheta))\| \Delta\vartheta \right. \\ &\quad \left. + K_0 \int_0^{t_1} e_{\ominus\nu}(t_1, \sigma(\vartheta)) \int_0^\vartheta \|\mathcal{H}(\vartheta, r, x(r))\| \Delta r \Delta\vartheta \right] \\ &= M_W^0 [\|x_{t_1}\| + K_0 e_{\ominus\nu}(t_1, 0) \|x_0\| + K_0 \mathcal{I}_4 + K_0 \mathcal{I}_5], \end{aligned}$$

where

$$\mathcal{I}_4 = \int_0^{t_1} e_{\ominus\nu}(t_1, \sigma(\vartheta)) \|\mathcal{F}(\vartheta, x(\vartheta))\| \Delta\vartheta \leq \frac{C_{\mathcal{F}}(1 + \bar{\mu}\nu)}{\nu} + \frac{M_{\mathcal{F}} \|x\|_\Omega (1 + \bar{\mu}\nu) e_\Omega(t_1, 0)}{\nu \oplus \Omega}$$

and

$$\mathcal{I}_5 = \int_0^{t_1} e_{\ominus\nu}(t_1, \sigma(\vartheta)) \int_0^\vartheta \|\mathcal{H}(\vartheta, r, x(r))\| \Delta r \Delta\vartheta \leq \frac{t_1 C_{\mathcal{H}}(1 + \bar{\mu}\nu)}{\nu} + \frac{M_{\mathcal{H}}(1 + \bar{\mu}\nu) \|x\|_\Omega e_\Omega(t_1, 0)}{\Omega(\nu \oplus \Omega)}.$$

Therefore,

$$\begin{aligned}
\|u(t)\| &\leq M_W^0 \left[\|x_{t_1}\| + K_0 e_{\ominus\nu}(t_1, 0) \|x_0\| + \frac{K_0 C_{\mathcal{F}}(1 + \bar{\mu}\nu)}{\nu} + \frac{K_0 M_{\mathcal{F}} \|x\|_{\Omega}(1 + \bar{\mu}\nu) e_{\Omega}(t_1, 0)}{\nu \oplus \Omega} \right. \\
&\quad \left. + \frac{K_0 C_{\mathcal{H}} T(1 + \bar{\mu}\nu)}{\nu} + \frac{K_0 M_{\mathcal{H}}(1 + \bar{\mu}\nu) \|x\|_{\Omega} e_{\Omega}(t_1, 0)}{\Omega(\nu \oplus \Omega)} \right] \\
&= M_W^0 [\|x_{t_1}\| + K_0 e_{\ominus\nu}(t_1, 0) \|x_0\| + \mathcal{N}_1 + \|x\|_{\Omega} \mathcal{N}_2 e_{\Omega}(t_1, 0)] \\
&= M_u^0.
\end{aligned}$$

□

Lemma 4.6. *If all the assumptions (H1) to (H5) are satisfied, then the control input*

$$\begin{aligned}
u(t) &= (\mathcal{W}_{r_p}^{t_{p+1}})^{-1} \left[x_{t_{p+1}} - E(t_{p+1}, r_p) \mathcal{G}_p(t_{p+1}, x(t_p^-)) - \int_{r_p}^{t_{p+1}} E(t_{p+1}, \sigma(\vartheta)) \mathcal{F}(\vartheta, x(\vartheta)) \Delta\vartheta \right. \\
&\quad \left. - \int_{r_p}^{t_{p+1}} E(t_{p+1}, \sigma(\vartheta)) \int_0^{\vartheta} \mathcal{H}(\vartheta, r, x(r)) \Delta r \Delta\vartheta \right] (t), \quad \forall t \in (r_p, t_{p+1}]_{\mathbb{T}}, \quad p = 1, 2, \dots, n, \quad (4.5)
\end{aligned}$$

steers the state $x(t)$ of the system (1.2) from x_{r_p} to $x_{t_{p+1}}$ at the time t_{p+1} . Moreover, the control estimate $\|u(t)\|$ is M_u^p , $p = 1, 2, \dots, n$, where

$$M_u^p = M_W^p [\|x_{t_{p+1}}\| + K_0 e_{\ominus\nu}(t_{p+1}, r_p) M_{\mathcal{G}} + \mathcal{N}_1 + \|x\|_{\Omega} \mathcal{N}_2 e_{\Omega}(t_{p+1}, r_p)].$$

Proof: Consider the mild solution $x(t)$ of the system (1.2) on $(r_p, t_{p+1}]_{\mathbb{T}}$ defined by (4.3). For $t = t_{p+1}$, we get

$$\begin{aligned}
x(t_{p+1}) &= E(t_{p+1}, r_p) \mathcal{G}_p(r_p, x(t_p^-)) + \int_{r_p}^{t_{p+1}} E(t_{p+1}, \sigma(\vartheta)) \left[\mathcal{F}(\vartheta, x(\vartheta)) + \int_0^{\vartheta} \mathcal{H}(\vartheta, r, x(r)) \Delta r \right] \Delta\vartheta + \mathcal{W}_{r_p}^{t_{p+1}} u(t) \\
&= E(t_{p+1}, r_p) \mathcal{G}_p(r_p, x(t_p^-)) + \int_{r_p}^{t_{p+1}} E(t_{p+1}, \sigma(\vartheta)) \mathcal{F}(\vartheta, x(\vartheta)) \Delta\vartheta + \int_{r_p}^{t_{p+1}} E(t_{p+1}, \sigma(\vartheta)) \int_0^{\vartheta} \mathcal{H}(\vartheta, r, x(r)) \Delta r \Delta\vartheta \\
&\quad + (\mathcal{W}_{r_p}^{t_{p+1}})^{-1} \left[x_{t_{p+1}} - E(t_{p+1}, r_p) \mathcal{G}_p(t_{p+1}, x(t_p^-)) - \int_{r_p}^{t_{p+1}} E(t_{p+1}, \sigma(\vartheta)) \mathcal{F}(\vartheta, x(\vartheta)) \Delta\vartheta \right. \\
&\quad \left. - \int_{r_p}^{t_{p+1}} E(t_{p+1}, \sigma(\vartheta)) \int_0^{\vartheta} \mathcal{H}(\vartheta, r, x(r)) \Delta r \Delta\vartheta \right] (t) \\
&= x_{t_{p+1}}.
\end{aligned}$$

Hence, control input (4.5) is suitable for the system (1.2). Also, the control estimate is calculated as

$$\begin{aligned}
\|u(t)\| &\leq M_W^p \left[\|x_{t_{p+1}}\| + K_0 e_{\ominus\nu}(t_{p+1}, r_p) \|\mathcal{G}_p(t_{p+1}, x(t_p^-))\| + K_0 \int_{r_p}^{t_{p+1}} e_{\ominus\nu}(t_{p+1}, \sigma(\vartheta)) \|\mathcal{F}(\vartheta, x(\vartheta))\| \Delta\vartheta \right. \\
&\quad \left. + K_0 \int_{r_p}^{t_{p+1}} e_{\ominus\nu}(t_{p+1}, \sigma(\vartheta)) \int_0^{\vartheta} \|\mathcal{H}(\vartheta, r, x(r))\| \Delta r \Delta\vartheta \right] \\
&= M_W^p [\|x_{t_{p+1}}\| + K_0 e_{\ominus\nu}(t_{p+1}, r_p) M_{\mathcal{G}} + K_0 \mathcal{I}_6 + K_0 \mathcal{I}_7],
\end{aligned}$$

where

$$\mathcal{I}_6 = \int_{r_p}^{t_{p+1}} e_{\ominus\nu}(t_{p+1}, \sigma(\vartheta)) \|\mathcal{F}(\vartheta, x(\vartheta))\| \Delta\vartheta \leq \frac{C_{\mathcal{F}}(1 + \bar{\mu}\nu)}{\nu} + \frac{M_{\mathcal{F}} \|x\|_{\Omega}(1 + \bar{\mu}\nu) e_{\Omega}(t_{p+1}, r_p)}{\nu \oplus \Omega}$$

and

$$\mathcal{I}_7 = \int_{r_p}^{t_{p+1}} e_{\ominus\nu}(t_{p+1}, \sigma(\vartheta)) \int_0^{\vartheta} \|\mathcal{H}(\vartheta, r, x(r))\| \Delta r \leq \frac{TC_{\mathcal{H}}(1 + \bar{\mu}\nu)}{\nu} + \frac{M_{\mathcal{H}}(1 + \bar{\mu}\nu)\|x\|_{\Omega} e_{\Omega}(t_{p+1}, r_p)}{\Omega(\nu \oplus \Omega)}.$$

Therefore,

$$\begin{aligned} \|u(t)\| &\leq M_W^p \left[\|x_{t_{p+1}}\| + K_0 e_{\ominus\nu}(t_{p+1}, r_p) M_{\mathcal{G}} + \frac{K_0 C_{\mathcal{F}}(1 + \bar{\mu}\nu)}{\nu} + \frac{M_{\mathcal{F}}\|x\|_{\Omega}(1 + \bar{\mu}\nu) e_{\Omega}(t_{p+1}, r_p)}{\Omega(\nu \oplus \Omega)} \right. \\ &\quad \left. + \frac{K_0 TC_{\mathcal{H}}(1 + \bar{\mu}\nu)}{\nu} + \frac{M_{\mathcal{H}}(1 + \bar{\mu}\nu)\|x\|_{\Omega} e_{\Omega}(t_{p+1}, r_p)}{\Omega(\nu \oplus \Omega)} \right] \\ &= M_W^p [\|x_{t_{p+1}}\| + K_0 e_{\ominus\nu}(t_{p+1}, r_p) M_{\mathcal{G}} + \mathcal{N}_1 + \|x\|_{\Omega} \mathcal{N}_2 e_{\Omega}(t_{p+1}, r_p)] \\ &= M_u^p. \end{aligned}$$

□

For the notational convenience, we denote

$$\begin{aligned} L_{\alpha}^0 &= \mathcal{N}_3(1 + \mathcal{N}_4^0 e_{\Omega}(t_1, 0)), \quad L_{\alpha}^p = \left[\frac{K_0 L_{\mathcal{G}}(1 + \mathcal{N}_4^p)}{e_{\Omega}(r_p, t_p^-)} + \mathcal{N}_3(1 + \mathcal{N}_4^p e_{\Omega}(t_{p+1}, r_p)) \right], \quad p = 1, 2, \dots, n, \\ K_{\alpha}^p &= \frac{K_0 L_{\mathcal{G}}^2(1 + \mathcal{N}_4^p)}{e_{\Omega}(r_{p-1}, t_{p-1}^-)} + L_{\mathcal{G}} \mathcal{N}_3(1 + \mathcal{N}_4^p), \quad p = 1, 2, \dots, n, \\ L_{\alpha} &= \max\{L_{\alpha}^0, \max_{p=1,2,\dots,n} (L_{\alpha}^p, K_{\alpha}^p)\}. \end{aligned}$$

Now, we are ready to present and prove the main result of this manuscript, which is the exact controllability of the impulsive system (1.2).

Theorem 4.7. *If all the assumptions (H1) to (H6) are satisfied, then the control system (1.2) is exact controllable on I , provided*

$$L_{\alpha} < 1. \quad (4.6)$$

Proof: Consider a subset $\mathcal{D}_1 \subseteq PC(I, X)$ such that

$$\mathcal{D}_1 = \{x \in PC(I, X) : \|x\|_{\Omega} \leq \delta_1\},$$

where

$$\delta_1 = \max_{1 \leq p \leq n} \left(\frac{\mathcal{N}_4^0 \|x_{t_1}\| + (1 + \mathcal{N}_4^0)(K_0 \|x_0\| + \mathcal{N}_1)}{1 - \mathcal{N}_2(1 + \mathcal{N}_4^0 e_{\Omega}(t_1, 0))}, \frac{\mathcal{N}_4^0 \|x_{t_{p+1}}\| + (1 + \mathcal{N}_4^p)(K_0 M_{\mathcal{G}} + \mathcal{N}_1)}{1 - \mathcal{N}_2(1 + \mathcal{N}_4^p e_{\Omega}(t_{p+1}, r_p))}, M_{\mathcal{G}} \right).$$

Now, define an operator $\Pi_1 : \mathcal{D}_1 \rightarrow \mathcal{D}_1$ given by

$$(\Pi_1 x)(t) = E(t, 0)x_0 + \int_0^t E(t, \sigma(\vartheta)) \mathcal{F}(\vartheta, x(\vartheta)) \Delta \vartheta + \int_0^t E(t, \sigma(\vartheta)) B u(\vartheta) \Delta \vartheta + \int_0^t E(t, \sigma(\vartheta)) \int_0^{\vartheta} \mathcal{H}(\vartheta, r, x(r)) \Delta r \Delta \vartheta, \quad \forall t \in [0, t_1]_{\mathbb{T}},$$

$$\begin{aligned} (\Pi_1 x)(t) &= \mathcal{G}_p \left(t, E(t_p, r_{p-1}) \mathcal{G}_{p-1}(r_{p-1}, x(t_{p-1}^-)) + \int_{r_{p-1}}^{t_p} E(t_p, \sigma(\vartheta)) \mathcal{F}(\vartheta, x(\vartheta)) \Delta \vartheta + \int_{r_{p-1}}^{t_p} E(t_p, \sigma(\vartheta)) B u(\vartheta) \Delta \vartheta \right. \\ &\quad \left. + \int_{r_{p-1}}^{t_p} E(t_p, \sigma(\vartheta)) \int_0^{\vartheta} \mathcal{H}(\vartheta, r, x(r)) \Delta r \Delta \vartheta \right), \quad \forall t \in (t_p, r_p]_{\mathbb{T}}, \quad p = 1, 2, \dots, n, \end{aligned}$$

$$\begin{aligned} (\Pi_1 x)(t) &= E(t, r_p) \mathcal{G}_p(r_p, x(t_p^-)) + \int_{r_p}^t E(t, \sigma(\vartheta)) \mathcal{F}(\vartheta, x(\vartheta)) \Delta \vartheta + \int_{r_p}^t E(t, \sigma(\vartheta)) B u(\vartheta) \Delta \vartheta \\ &\quad + \int_{r_p}^t E(t, \sigma(\vartheta)) \int_0^{\vartheta} \mathcal{H}(\vartheta, r, x(r)) \Delta r \Delta \vartheta, \quad \forall t \in (r_p, t_{p+1}]_{\mathbb{T}}, \quad p = 1, 2, \dots, n, \end{aligned}$$

where $u(t)$ is given by the equations (4.4) and (4.5) in the intervals $[0, t_1]_{\mathbb{T}}$ and $(r_p, t_{p+1}]_{\mathbb{T}}$, $p = 1, 2, \dots, n$, respectively. From the Lemma 4.4 and 4.5, we can see that $x(t)$ satisfies $x(t_1) = x_{t_1}$ and $x(t_{p+1}) = x_{t_{p+1}}$, $p = 1, 2, \dots, n$. Therefore, to prove the exact controllability of the system (1.2), it remains to show that the operator Π_1 has a fixed point. For better readability, we break the proof into a sequence of step.

Step 1: We show that the operator Π_1 is well-defined from \mathcal{D}_1 into \mathcal{D}_1 .

For any $t \in [0, t_1]_{\mathbb{T}}$ and $x \in \mathcal{D}_1$, we have

$$\begin{aligned} \|(\Pi_1 x)(t)\| &\leq K_0 e_{\ominus\nu}(t, 0) \|x_0\| + K_0 \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) \|\mathcal{F}(\vartheta, x(\vartheta))\| \Delta\vartheta \\ &\quad + K_0 \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) M_B \|u(\vartheta)\| \Delta\vartheta + K_0 \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) \int_0^{\vartheta} \|\mathcal{H}(\vartheta, r, x(r))\| \Delta r \Delta\vartheta \\ &\leq K_0 \|x_0\| + \mathcal{N}_1 + \delta_1 \mathcal{N}_2 + \mathcal{N}_4^0 (\|x_{t_1}\| + K_0 \|x_0\| + \mathcal{N}_1 + \delta_1 \mathcal{N}_2 e_{\Omega}(t_1, 0)). \end{aligned}$$

Hence,

$$\|\Pi_1 x\|_{\Omega} \leq K_0 \|x_0\| + \mathcal{N}_1 + \delta_1 \mathcal{N}_2 + \mathcal{N}_4^0 (\|x_{t_1}\| + K_0 \|x_0\| + \mathcal{N}_1 + \delta_1 \mathcal{N}_2 e_{\Omega}(t_1, 0)) \leq \delta_1. \quad (4.7)$$

Also, for any $t \in (r_p, t_{p+1}]_{\mathbb{T}}$, $p = 1, 2, \dots, n$ and $x \in \mathcal{D}_1$, we have

$$\begin{aligned} \|(\Pi_1 x)(t)\| &\leq K_0 e_{\ominus\nu}(t, r_p) \|\mathcal{G}_p(r_p, x(t_p^-))\| + K_0 \int_{r_p}^t e_{\ominus\nu}(t, \sigma(\vartheta)) \|\mathcal{F}(\vartheta, x(\vartheta))\| \Delta\vartheta \\ &\quad + K_0 \int_{r_p}^t e_{\ominus\nu}(t, \sigma(\vartheta)) M_B \|u(\vartheta)\| \Delta\vartheta + K_0 \int_{r_p}^t e_{\ominus\nu}(t, \sigma(\vartheta)) \int_0^{\vartheta} \|\mathcal{H}(\vartheta, r, x(r))\| \Delta r \Delta\vartheta \\ &\leq K_0 M_G + \mathcal{N}_1 + \delta_1 \mathcal{N}_2 + \mathcal{N}_4^p (\|x_{t_{p+1}}\| + K_0 M_G + \mathcal{N}_1 + \delta_1 \mathcal{N}_2 e_{\Omega}(t_{p+1}, r_p)). \end{aligned}$$

Hence,

$$\|\Pi_1 x\|_{\Omega} \leq K_0 M_G + \mathcal{N}_1 + \delta_1 \mathcal{N}_2 + \mathcal{N}_4^p (\|x_{t_{p+1}}\| + K_0 M_G + \mathcal{N}_1 + \delta_1 \mathcal{N}_2 e_{\Omega}(t_{p+1}, r_p)) \leq \delta_1. \quad (4.8)$$

Similarly, for any $t \in (t_p, r_p]_{\mathbb{T}}$, $p = 1, 2, \dots, n$ and $x \in \mathcal{D}_1$, we have

$$\|\Pi_1 x\|_{\Omega} \leq M_G \leq \delta_1. \quad (4.9)$$

After summarizing the above inequalities (4.7), (4.8) and (4.9), for all $t \in I$, we get

$$\|\Pi_1 x\|_{\Omega} \leq \delta_1.$$

Therefore, Π_1 is well-defined from \mathcal{D}_1 into \mathcal{D}_1 .

Step 2: We prove the operator Π_1 is a contraction from \mathcal{D}_1 into \mathcal{D}_1 .

For any $t \in [0, t_1]_{\mathbb{T}}$ and $x, y \in \mathcal{D}_1$, we have

$$\begin{aligned} \|(\Pi_1 x)(t) - (\Pi_1 y)(t)\| &\leq K_0 L_{\mathcal{F}} \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) \|x(\vartheta) - y(\vartheta)\| \Delta\vartheta \\ &\quad + K_0 L_{\mathcal{H}} \int_0^t e_{\ominus\nu}(t, \sigma(\vartheta)) \int_0^{\vartheta} \|x(r) - y(r)\| \Delta r \Delta\vartheta + K_0 M_B M_W^0 \int_0^t e_{\ominus\nu}(t, \sigma(r)) \\ &\quad \times \left[K_0 L_{\mathcal{F}} \int_0^{t_1} e_{\ominus\nu}(t, \sigma(\vartheta)) \|x(\vartheta) - y(\vartheta)\| \Delta\vartheta \right. \\ &\quad \left. + K_0 L_{\mathcal{H}} \int_0^{t_1} e_{\ominus\nu}(t, \sigma(\vartheta)) \int_0^{\vartheta} \|x(r) - y(r)\| \Delta r \Delta\vartheta \right] \Delta r \\ &\leq \mathcal{N}_3 \|x - y\|_{\Omega} (1 + \mathcal{N}_4^0 e_{\Omega}(t_1, 0)). \end{aligned}$$

Thus,

$$\|\Pi_1 x - \Pi_1 y\|_\Omega \leq \mathcal{N}_3(1 + \mathcal{N}_4^0 e_\Omega(t_1, 0)) \|x - y\|_\Omega \leq L_\alpha \|x - y\|_\Omega. \quad (4.10)$$

For any $t \in (r_p, t_{p+1}]_{\mathbb{T}}$, $p = 1, 2, \dots, n$ and $x, y \in \mathcal{D}_1$, we have

$$\begin{aligned} \|(\Pi_1 x)(t) - (\Pi_1 y)(t)\| &\leq K_0 L_{\mathcal{G}} e_\Omega(t_p^-, r_p) \|x - y\|_\Omega + K_0 L_{\mathcal{F}} \int_{r_p}^t e_{\ominus\nu}(t, \sigma(\vartheta)) \|x(\vartheta) - y(\vartheta)\| \Delta\vartheta \\ &\quad + K_0 L_{\mathcal{H}} \int_{r_p}^t e_{\ominus\nu}(t, \sigma(\vartheta)) \int_0^{\vartheta} \|x(r) - y(r)\| \Delta r \Delta\vartheta + K_0 M_B M_W^p \int_{r_p}^t e_{\ominus\nu}(t, \sigma(r)) \\ &\quad \times \left[K_0 L_{\mathcal{G}} e_\Omega(t_p^-, r_p) \|x - y\|_\Omega + K_0 L_{\mathcal{F}} \int_{r_p}^{t_{p+1}} e_{\ominus\nu}(t, \sigma(\vartheta)) \|x(\vartheta) - y(\vartheta)\| \Delta\vartheta \right. \\ &\quad \left. + K_0 L_{\mathcal{H}} \int_{r_p}^{t_{p+1}} e_{\ominus\nu}(t, \sigma(\vartheta)) \int_0^{\vartheta} \|x(r) - y(r)\| \Delta r \Delta\vartheta \right] \Delta r \\ &\leq \frac{K_0 L_{\mathcal{G}}(1 + \mathcal{N}_4^p)}{e_\Omega(r_p, t_p^-)} \|x - y\|_\Omega + \mathcal{N}_3(1 + \mathcal{N}_4^p(e_\Omega(t_{p+1}, r_p))) \|x - y\|_\Omega. \end{aligned}$$

Therefore,

$$\|\Pi_1 x - \Pi_1 y\|_\Omega \leq \left[\frac{K_0 L_{\mathcal{G}}(1 + \mathcal{N}_4^p)}{e_\Omega(r_p, t_p^-)} + \mathcal{N}_3(1 + \mathcal{N}_4^p e_\Omega(t_{p+1}, r_p)) \right] \|x - y\|_\Omega \leq L_\alpha \|x - y\|_\Omega. \quad (4.11)$$

Similarly, for any $t \in (t_p, r_p]_{\mathbb{T}}$, $p = 1, 2, \dots, n$ and $x, y \in \mathcal{D}_1$, we can easily find

$$\|\Pi_1 x - \Pi_1 y\|_\Omega \leq L_{\mathcal{G}} \left[\frac{K_0 L_{\mathcal{G}}(1 + \mathcal{N}_4^p)}{e_\Omega(r_{p-1}, t_{p-1}^-)} + \mathcal{N}_3(1 + \mathcal{N}_4^p) \right] \|x - y\|_\Omega \leq L_\alpha \|x - y\|_\Omega. \quad (4.12)$$

After summarizing the inequalities (4.10), (4.11) and (4.12), for all $t \in I$, we get

$$\|\Pi_1 x - \Pi_1 y\|_\Omega \leq L_\alpha \|x - y\|_\Omega.$$

Hence, from the equation (4.6), Π_1 is a strict contraction operator.

Therefore, from the step 1 and step 2 along with Banach contraction theorem, Π_1 has a unique fixed point which is the solution of the system (1.2). Thus, the system (1.2) is exact controllable on I . \square

5 Illustrative Examples

In this section, we provide some examples to illustrate the obtained results.

Example 5.1. We consider the partial differential equation on time scales \mathbb{T} in the following form

$$\begin{aligned} \frac{\partial}{\Delta_1 t} Z(t, \eta) &= a(t, \eta) \frac{\partial^2}{\Delta_2 \eta^2} Z(t, \eta) + b(\eta) W(t, \eta) + G_1(t, Z(t, \eta)) + \int_0^t G_2(t, r, Z(r, \eta)) \Delta r, \\ &\quad t, r \in \cup_{p=0}^n (r_p, t_{p+1}]_{\mathbb{T}}, \quad \eta \in [0, \pi]_{\mathbb{T}}, \\ Z(t, 0) &= Z(t, \pi) = 0, \quad t \in [0, T]_{\mathbb{T}}, \\ Z(t, \eta) &= G_p(t, Z(t_p^-, \eta)), \quad t \in (t_p, r_p]_{\mathbb{T}}, \quad p = 1, 2, \dots, n, \\ Z(0, \eta) &= x_0, \quad \eta \in [0, \pi]_{\mathbb{T}}, \end{aligned} \quad (5.13)$$

where \mathbb{T} is a time scales and $t_p, r_p \in \mathbb{T}$ are some points with $0 = r_0 = t_0 < t_1 < r_1 < t_2 < \dots < r_n < t_{n+1} = T$. Δ_1 denotes the first order partial derivative while Δ_2 denotes the second order partial derivative. $Z, S, a : T_0 \times [0, \pi]_{\mathbb{T}} \rightarrow \mathbb{R}$, $G_1 : T_0 \times \mathbb{R} \rightarrow \mathbb{R}$, $G_2 : T_0 \times T_0 \times \mathbb{R} \rightarrow \mathbb{R}$, $G_p : J_p \times \mathbb{R} \rightarrow \mathbb{R}$ are the real valued functions satisfying certain conditions to be specified later where $T_0 = \cup_{p=0}^n [r_p, t_{p+1}]_{\mathbb{T}}$, $J_p = [t_p, t_p]_{\mathbb{T}}$, $p = 1, 2, \dots, n$.

Let $X = L^2[0, \pi]_{\mathbb{T}}$. Define an operator $A(t)$ by $A(t)x = a(t, \eta) \frac{\partial^2}{\Delta_2 \eta^2} x$, $\forall x \in D(A) = \{x \in H_0^1[0, \pi]_{\mathbb{T}} \cap H^2[0, \pi]_{\mathbb{T}}\}$. Further, it is known that $A(t)$ generates an evolution operator $\{E(t, s) : t \geq s\}$ which satisfies $E(t, s) \leq K_0 e_{\ominus \nu}(t, s)$, $\forall (t, s)$ ($t \geq s$) with $K_0 = 1$ and $\nu = \frac{1}{2}$ (please see [8]). Define $B \in \mathbb{B}(U, X)$ by $Bu(t)(\eta) = b(\eta)W(t, \eta)$, $\eta \in [0, \pi]_{\mathbb{T}}$, $b(\eta) \in L^2[0, \pi]_{\mathbb{T}}$. With the above formulations, the equation (5.13) can be rewritten as the following abstract equation $X = L^2[0, \pi]_{\mathbb{T}}$,

$$\begin{aligned} x^\Delta(t) &= A(t)x(t) + \mathcal{F}(t, x(t)) + \int_0^t \mathcal{H}(t, r, x(r))\Delta r + Bu(t), \quad t \in \cup_{p=0}^n (r_p, t_{p+1}]_{\mathbb{T}}, \\ x(t) &= \mathcal{G}_p(t, x(t_p^-)), \quad t \in (t_p, r_p]_{\mathbb{T}}, \quad p = 1, 2, \dots, n, \\ x(0) &= x_0, \end{aligned} \tag{5.14}$$

where $x(t) = Z(t, \cdot)$ that is $x(t)(\eta) = Z(t, \eta)$, $\mathcal{F}(t, x(t))\eta = G_1(t, Z(t, \eta))$, $\mathcal{H}(t, r, x)\eta = G_2(t, r, Z(t, \eta))$, $\mathcal{G}_p(t, x(t_p^-))(\eta) = G_p(t, Z(t_p^-, \eta))$, $\eta \in [0, \pi]_{\mathbb{T}}$. Thus, the results of this manuscript for the system (5.14) and (5.13) are same. Now, let $\mathcal{F}(t, x(t)) = \frac{t + \sin(x(t))}{25e^{t+5}}$, $\mathcal{H}(t, r, x(t)) = \frac{\sin(t)(r+2)^2(5+|x(t)|)}{e^{t+7}(1+|x(t)|)}$ and $\mathcal{G}_p(t, x(t_p^-)) = \frac{3+t\sin(x(t_p^-))}{20e^{pt^2+1}}$, $p = 1, 2, \dots, n$. Then, we can easily verify that the non-linear functions $\mathcal{F}, \mathcal{H}, \mathcal{G}_p$, $p = 1, 2, \dots, n$, satisfy all the assumptions of Theorem 4.7. Therefore, based on the Theorem 3.1, Theorem 3.3 and Theorem 4.7, it can be concluded that the equation (5.13) has a unique solution which is stable and exact controllable. \square

Next, we give a numerical example to illustrate the application of the obtained analytical results.

Example 5.2. We consider the following linear regressive system with non instantaneous impulses in the space $X = \mathbb{R}$

$$\begin{aligned} x^\Delta(t) &= \frac{-1}{1+2\mu(t)}x(t) + \frac{3+|x(t)|}{40e^{(t+1)}(1+|x(t)|)} + \int_0^t \frac{tr \sin(x(r))}{60e^{(t+2)}}\Delta r, \quad t \in \cup_{p=0}^n (r_p, t_{p+1}]_{\mathbb{T}}, \\ x(t) &= \frac{1 + \sin(px(t_p^-))}{40e^{t+2}} + \frac{p}{t}, \quad t \in (t_p, r_p]_{\mathbb{T}}, \quad p = 1, 2, \dots, n, \\ x(0) &= 1. \end{aligned} \tag{5.15}$$

Here we set, $A(t) = \frac{-1}{1+2\mu(t)}$, $\mathcal{F}(t, x(t)) = \frac{3+|x(t)|}{40e^{(t+1)}(1+|x(t)|)}$, $\mathcal{H}(t, r, x(t)) = \frac{tr \sin(x(t))}{60e^{(t+2)}}$, $\mathcal{G}_p(t, x(t_p^-)) = \frac{1 + \sin(px(t_p^-))}{40e^{t+2}} + \frac{p}{t}$, $p = 1, 2, \dots, n$. Also, we can see that $E(t, s) = e_{\ominus 1}(t, s)$ which satisfies the relation $\|E(t, s)\| \leq e_{\ominus 1}(t, s)$. i.e., it is exponentially stable with $K_0 = 1$ and $\nu = 1$. Now, we consider the following three cases for different time scales:

• **Case 1:** We take $\mathbb{T} = \mathbb{R}$, $I = [0, 2]_{\mathbb{R}} = [0, 2]$, $n = 1$, $t_0 = 0$, $t_1 = 0.6$, $r_1 = 0.8$, $t_2 = T = 2$, $x(t_1) = 3$ and $x(t_2) = 5$. Clearly, we can see that the trajectory of the system (5.15) shown in Figure 1 does not pass through the desired points $x(t_1) = 3$ and $x(t_2) = 5$. But if we add a control function $u(t)$ given by

$$u(t) = \begin{cases} (\mathcal{W}_0^{t_1})^{-1} \left[3 - E(t_1, 0) - \int_0^{t_1} E(t_1, \sigma(\vartheta))\mathcal{F}(\vartheta, x(\vartheta))\Delta\vartheta - \int_0^{t_1} E(t_1, \sigma(\vartheta)) \int_0^\vartheta \mathcal{H}(\vartheta, r, x(r))\Delta r \Delta\vartheta \right] (t), & t \in [0, t_1]_{\mathbb{R}} \\ (\mathcal{W}_{r_1}^T)^{-1} \left[5 - E(T, r_1)\mathcal{G}_1(T, x(t_1^-)) - \int_{r_1}^T E(T, \sigma(\vartheta))\mathcal{F}(\vartheta, x(\vartheta))\Delta\vartheta - \int_{r_1}^T E(T, \sigma(\vartheta)) \int_0^\vartheta \mathcal{H}(\vartheta, r, x(r))\Delta r \Delta\vartheta \right] (t), & t \in (r_1, T]_{\mathbb{R}}, \end{cases}$$

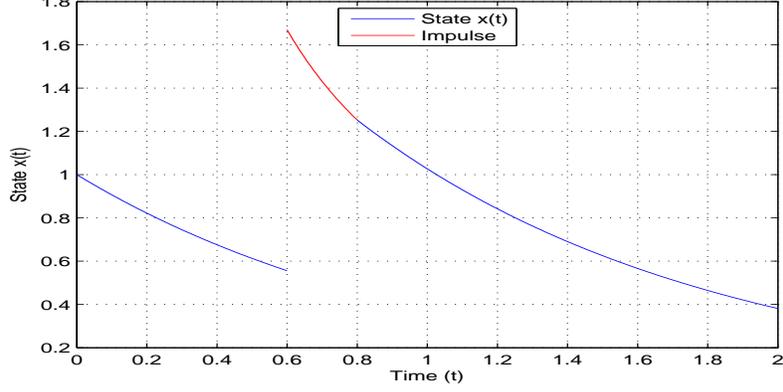


Figure 1: Trajectory of the system (5.15) when $\mathbb{T} = \mathbb{R}$.

where

$$\mathcal{W}_0^{t_1} = \int_0^{t_1} e_{\ominus 1}(t_1, \sigma(r)) \Delta r \text{ and } \mathcal{W}_{r_1}^T = \int_{r_1}^T e_{\ominus 1}(T, \sigma(r)) \Delta r,$$

with $B = 1$ in the control system (5.15), it becomes

$$\begin{aligned} x^\Delta(t) &= A(t)x(t) + \mathcal{F}(t, x(t)) + \int_0^t \mathcal{H}(t, r, x(r)) \Delta r + u(t), \quad t \in [0, 2]_{\mathbb{R}} \setminus (0.6, 0.8]_{\mathbb{R}}, \\ x(t) &= \mathcal{G}_p(t, x(t_p^-)), \quad t \in (0.6, 0.8]_{\mathbb{R}}, \quad p = 1, \\ x(0) &= 1. \end{aligned} \tag{5.16}$$

Now, using the MATLAB, we find

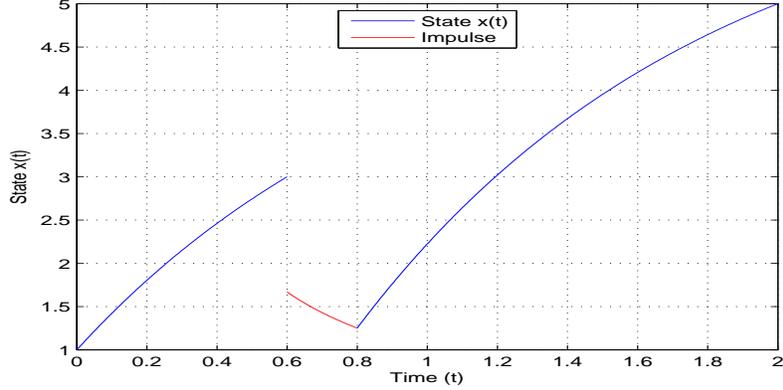


Figure 2: Trajectory of the controlled system (5.16) when $\mathbb{T} = \mathbb{R}$, $x(t_1) = 3$ and $x(T) = 5$.

$$\begin{aligned} \mathcal{W}_0^{t_1} &= \int_0^{0.6} E(t_1, \sigma(r)) B \Delta r = \int_0^{0.6} e_{\ominus 1}(t_1, \sigma(r)) \Delta r = \int_0^{0.6} e^{-(0.6-r)} dr = 0.4512, \\ \mathcal{W}_{r_1}^T &= \int_{0.8}^2 E(t_2, \sigma(r)) B \Delta r = \int_{0.8}^2 e_{\ominus 1}(t_2, \sigma(r)) \Delta r = \int_{0.8}^2 e^{-(2-r)} dr = 0.6988, \end{aligned}$$

$$\begin{aligned} \mathcal{N}_4^0 &= \frac{K_0 M_B (1 + \bar{\mu}\nu) M_W^0}{\nu} = 2.2164, \quad \mathcal{N}_4^1 = \frac{K_0 M_B (1 + \bar{\mu}\nu) M_W^1}{\nu} = 1.4310, \\ \mathcal{N}_2(1 + \mathcal{N}_4^0 e_\Omega(t_1, 0)) &= 0.0743, \quad \mathcal{N}_2(1 + \mathcal{N}_4^1 e_\Omega(T, r_1)) = 0.0848, \\ L_\alpha^0 &= \frac{K_0 L_G (1 + \mathcal{N}_4^0)}{e_\Omega(0, t_0)} + \mathcal{N}_3(1 + \mathcal{N}_4^0 e_\Omega(t_1, 0)) = 0.0961, \quad L_\alpha^1 = \frac{K_0 L_G (1 + \mathcal{N}_4^1)}{e_\Omega(r_1, t_2)} + \mathcal{N}_3(1 + \mathcal{N}_4^1 e_\Omega(t_2, r_1)) = 0.1026, \\ K_\alpha^1 &= \frac{K_0 L_G^2 (1 + \mathcal{N}_4^1)}{e_\Omega(0, t_0)} + L_G \mathcal{N}_3(1 + \mathcal{N}_4^1) = 0.0167. \end{aligned}$$

Therefore, all the assumptions of Theorem 4.7 are satisfied. Hence, the system (5.16) is controllable and the controlled trajectory is shown in Figure 2.

• **Case 2:** We take $\mathbb{T} = \mathbb{P}_{1,1} = \cup_{j=0}^\infty [2j, 2j+1]$, $I = [0, 3]_{\mathbb{T}} = [0, 1] \cup [2, 3]$, $n = 1$, $t_0 = r_0 = 0$, $t_1 = 0.4$, $r_1 = 0.6$, $t_2 = T = 3$, $x(t_1) = 2$ and $x(t_2) = 4$. Clearly, we can see that the trajectory of the system (5.15) shown in Figure 3 does not pass through the desired points $x(t_1) = 2$ and $x(t_2) = 4$. But if we add a control function $u(t)$ given by

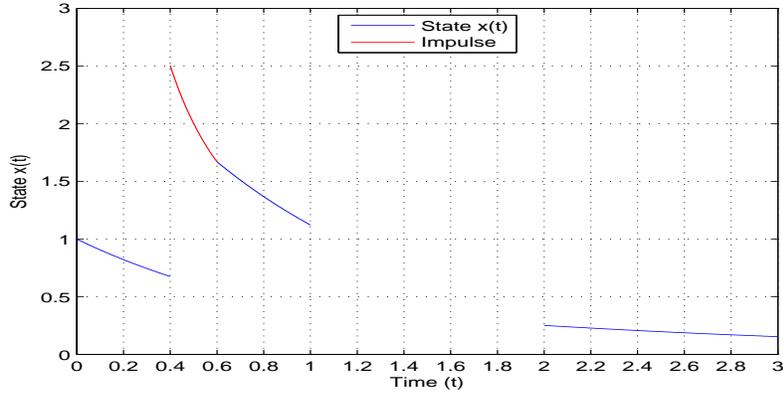


Figure 3: Trajectory of the system (5.15) when $\mathbb{T} = \mathbb{P}_{1,1}$.

$$u(t) = \begin{cases} (\mathcal{W}_0^{t_1})^{-1} \left[2 - E(t_1, 0) - \int_0^{t_1} E(t_1, \sigma(\vartheta)) \mathcal{F}(\vartheta, x(\vartheta)) \Delta\vartheta - \int_0^{t_1} E(t_1, \sigma(\vartheta)) \int_0^\vartheta \mathcal{H}(\vartheta, r, x(r)) \Delta r \Delta\vartheta \right] (t), & t \in [0, t_1]_{\mathbb{P}_{1,1}} \\ (\mathcal{W}_{r_1}^T)^{-1} \left[4 - E(T, r_1) \mathcal{G}_1(T, x(t_1^-)) - \int_{r_1}^T E(T, \sigma(\vartheta)) \mathcal{F}(\vartheta, x(\vartheta)) \Delta\vartheta - \int_{r_1}^T E(T, \sigma(\vartheta)) \int_0^\vartheta \mathcal{H}(\vartheta, r, x(r)) \Delta r \Delta\vartheta \right] (t), & t \in (r_1, T]_{\mathbb{P}_{1,1}}, \end{cases}$$

where

$$\mathcal{W}_0^{t_1} = \int_0^{t_1} e_{\ominus 1}(t_1, \sigma(r)) \Delta r \quad \text{and} \quad \mathcal{W}_{r_1}^T = \int_{r_1}^T e_{\ominus 1}(T, \sigma(r)) \Delta r,$$

in the control system (5.15), it becomes

$$\begin{aligned} x^\Delta(t) &= A(t)x(t) + \mathcal{F}(t, x(t)) + \int_0^t \mathcal{H}(t, r, x(r)) \Delta r + u(t), \quad t \in [0, 3]_{\mathbb{P}_{1,1}} \setminus (0.4, 0.6]_{\mathbb{P}_{1,1}}, \\ x(t) &= \mathcal{G}_p(t, x(t_p^-)), \quad t \in (0.4, 0.6]_{\mathbb{P}_{1,1}}, \quad p = 1, \\ x(0) &= 1. \end{aligned} \tag{5.17}$$

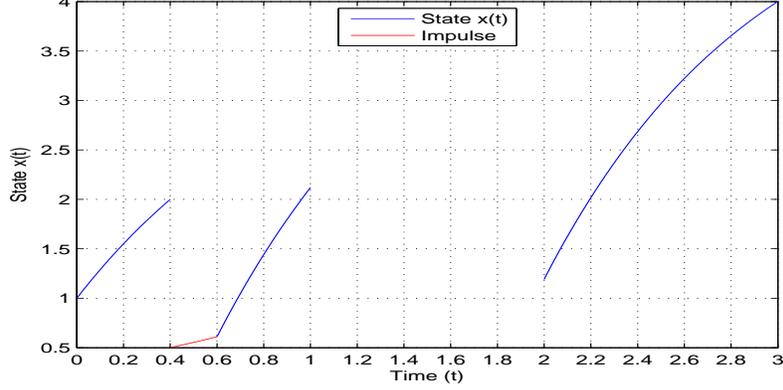


Figure 4: Trajectory of the controlled system (5.17) when $\mathbb{T} = \mathbb{P}_{1,1}$, $x(t_1) = 2$ and $x(T) = 4$.

Now, using the MATLAB, we find

$$\begin{aligned} \mathcal{W}_0^{t_1} &= \int_0^{0.4} E(t_1, \sigma(r))B\Delta r = 0.3297, \quad \mathcal{W}_{r_1}^T = \int_{0.6}^3 E(t_2, \sigma(r))B\Delta r = 0.7421, \\ \mathcal{N}_4^0 &= \frac{K_0 M_B(1 + \bar{\mu}\nu)M_W^0}{\nu} = 6.0665, \quad \mathcal{N}_4^1 = \frac{K_0 M_B(1 + \bar{\mu}\nu)M_W^0}{\nu} = 2.6952, \\ \mathcal{N}_2(1 + \mathcal{N}_4^0 e_{\Omega}(t_1, 0)) &= 0.2965, \quad \mathcal{N}_2(1 + \mathcal{N}_4^1 e_{\Omega}(T, r_1)) = 0.9058, \\ L_{\alpha}^0 &= \frac{K_0 L_{\mathcal{G}}(1 + \mathcal{N}_4^0)}{e_{\Omega}(0, t_0)} + \mathcal{N}_3(1 + \mathcal{N}_4^0 e_{\Omega}(t_1, 0)) = 0.3443, \quad L_{\alpha}^1 = \frac{K_0 L_{\mathcal{G}}(1 + \mathcal{N}_4^1)}{e_{\Omega}(r_1, t_2)} + \mathcal{N}_3(1 + \mathcal{N}_4^1 e_{\Omega}(t_2, r_1)) = 0.9450, \\ K_{\alpha}^1 &= \frac{K_0 L_{\mathcal{G}}^2(1 + \mathcal{N}_4^1)}{e_{\Omega}(0, t_0)} + L_{\mathcal{G}}\mathcal{N}_3(1 + \mathcal{N}_4^1) = 0.0257. \end{aligned}$$

Therefore, all the assumptions of Theorem 4.7 are satisfied. Hence, the system (5.17) is controllable and the controlled trajectory is shown in Figure 4.

• **Case 3:** We take $\mathbb{T} = [0, 3]_{\frac{1}{2}\mathbb{Z}} \cup [4, 6]_{\mathbb{R}} = D$ (say), $I = [0, 6]_{\mathbb{T}} = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\} \cup [4, 6]$, $n = 1$, $t_0 = r_0 = 0$, $t_1 = 4.8$, $r_1 = 5.2$, $t_2 = T = 6$, $x(t_1) = 2$ and $x(t_2) = 3$. Clearly, we can see that the trajectory of the system (5.15) shown in Figure 5 does not pass through the desired points $x(t_1) = 2$ and $x(t_2) = 3$. But if we add a control function

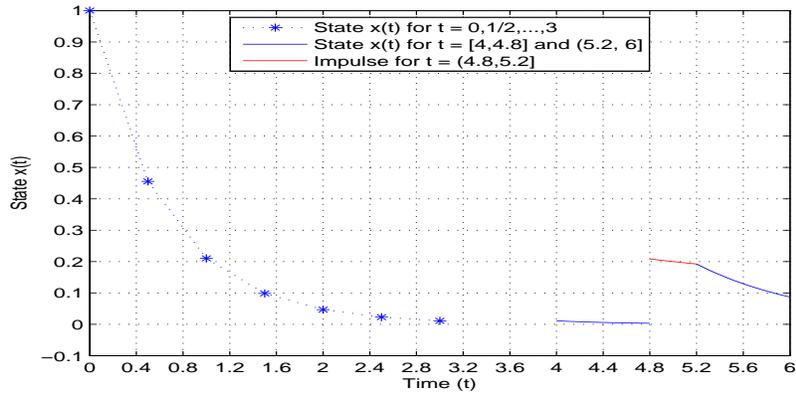


Figure 5: Trajectory of the system (5.15) when $\mathbb{T} = D$.

$u(t)$ given by

$$u(t) = \begin{cases} (\mathcal{W}_0^{t_1})^{-1} \left[2 - E(t_1, 0) - \int_0^{t_1} E(t_1, \sigma(\vartheta)) \mathcal{F}(\vartheta, x(\vartheta)) \Delta\vartheta - \int_0^{t_1} E(t_1, \sigma(\vartheta)) \int_0^\vartheta \mathcal{H}(\vartheta, r, x(r)) \Delta r \Delta\vartheta \right] (t), & t \in [0, t_1]_D \\ (\mathcal{W}_{r_1}^T)^{-1} \left[3 - E(T, r_1) \mathcal{G}_1(T, x(t_1^-)) - \int_{r_1}^T E(T, \sigma(\vartheta)) \mathcal{F}(\vartheta, x(\vartheta)) \Delta\vartheta - \int_{r_1}^T E(T, \sigma(\vartheta)) \int_0^\vartheta \mathcal{H}(\vartheta, r, x(r)) \Delta r \Delta\vartheta \right] (t), & t \in (r_1, T]_D, \end{cases}$$

where

$$\mathcal{W}_0^{t_1} = \int_0^{t_1} e_{\ominus 1}(t_1, \sigma(r)) \Delta r \text{ and } \mathcal{W}_{r_1}^T = \int_{r_1}^T e_{\ominus 1}(T, \sigma(r)) \Delta r,$$

in the control system (5.15), it becomes

$$\begin{aligned} x^\Delta(t) &= A(t)x(t) + \mathcal{F}(t, x(t)) + \int_0^t \mathcal{H}(t, r, x(r)) \Delta r + u(t), & t \in [0, 6]_D \setminus (4.8, 5.2]_D, \\ x(t) &= \mathcal{G}_p(t, x(t_p^-)), & t \in (4.8, 5.2]_D, \quad p = 1, \\ x(0) &= 1. \end{aligned} \tag{5.18}$$

Now, using the MATLAB, we find

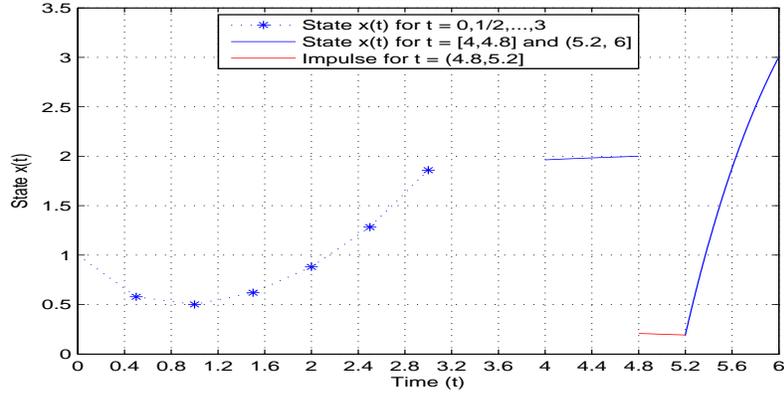


Figure 6: Trajectory of the controlled system (5.18) when $\mathbb{T} = D, x(t_1) = 2$ and $x(T) = 3$.

$$\begin{aligned} \mathcal{W}_0^{t_1} &= \int_0^{4.8} E(t_1, \sigma(r)) B \Delta r = 15.67710, \quad \mathcal{W}_{r_1}^T = \int_{5.2}^6 E(t_2, \sigma(r)) B \Delta r = 0.5507, \\ \mathcal{N}_4^0 &= \frac{K_0 M_B (1 + \bar{\mu}\nu) M_W^0}{\nu} = 0.0957, \quad \mathcal{N}_4^1 = \frac{K_0 M_B (1 + \bar{\mu}\nu) M_W^0}{\nu} = 2.7238, \\ \mathcal{N}_2(1 + \mathcal{N}_4^0 e_{\Omega}(t_1, 0)) &= 0.2793, \quad \mathcal{N}_2(1 + \mathcal{N}_4^1 e_{\Omega}(T, r_1)) = 0.1562, \\ L_\alpha^0 &= \frac{K_0 L_G (1 + \mathcal{N}_4^0)}{e_{\Omega}(0, t_0)} + \mathcal{N}_3(1 + \mathcal{N}_4^0 e_{\Omega}(t_1, 0)) = 0.2867, \quad L_\alpha^1 = \frac{K_0 L_G (1 + \mathcal{N}_4^1)}{e_{\Omega}(r_1, t_2)} + \mathcal{N}_3(1 + \mathcal{N}_4^1 e_{\Omega}(t_2, r_1)) = 0.1612, \\ K_\alpha^1 &= \frac{K_0 L_G^2 (1 + \mathcal{N}_4^1)}{e_{\Omega}(0, t_0)} + L_G \mathcal{N}_3(1 + \mathcal{N}_4^1) = 0.0258. \end{aligned}$$

Therefore, all the assumptions of Theorem 4.7 are satisfied. Hence, the system (5.18) is controllable and the controlled trajectory is shown in Figure 6. \square

6 Conclusion

In this manuscript, we have successfully established the existence of a unique solution and stability result for an impulsive hybrid evolution system (1.1) on time scales in a Banach space X . In addition, we have studied the exact controllability result for the hybrid evolution system (1.2). These results have been established by using the evolution operator theory, nonlinear analysis and the Banach contraction theorem. Moreover, to illustrate the application of obtained analytical outcomes, we provided one theoretical and with the help of MATLAB, we have given a numerical example for the different time scales including $\mathbb{T} = \mathbb{R}$ (case-1 of example 6.2), $\mathbb{T} = \mathbb{P}_{1,1}$ (case-2 of example 6.2) and $\mathbb{T} = [0, 3]_{\frac{1}{2}\mathbb{Z}} \cup [4, 6]_{\mathbb{R}}$ (case-3 of example 6.2).

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