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**Adaptive Observer Design for Wave PDEs With Nonlinear Dynamics and Parameter Uncertainty**A. Benabdelhadi <sup>1</sup>, F. Giri <sup>2</sup>, T. Ahmed-Ali <sup>2</sup>, M. Krstic <sup>3</sup>, H. El Fadil <sup>1</sup>, F.Z. Chaoui <sup>4</sup><sup>1</sup> ISA Lab, ENSA, Ibn Tofail University, 14000 Kénitra, Morocco<sup>2</sup> Normandie University, UNICAEN, ENSICAEN, LAC, 14000 Caen, France<sup>3</sup> University of California at San Diego, La Jolla, CA 92093-0411, USA.<sup>4</sup> University Mohammed V, ENSET, 10000 Rabat, Morocco.

**Abstract.** We consider the problem of state observer design for wave PDEs containing Lipschitz nonlinearities in the domain and parameter uncertainties in the domain and at the boundaries. Using the decoupling-transformation design approach, we develop an adaptive boundary observer consisting of a state observer, a least-squares type parameter adaptive law, and a hyperbolic auxiliary filter. Using Lyapunov stability analysis, we show that the observer is exponentially convergent under a persistent excitation condition. The novelty is twofold: (i) the class of systems is much wider than those studied in previous works, it particularly accounts for structured disturbances acting on the domain and all boundaries; (ii) the proposed adaptive observer is quite different from existing ones for wave-type PDEs.

**1. INTRODUCTION**

Designing state observers for PDEs has received an increasing interest over the last decade, e.g. (Smyshlyaev and Krstic, 2009; Vazquez et al., 2008, 2011; Guo and Guo, 2013; Fridman, 2013; Meurer, 2013; Fridman and Terushkin, 2016; Ahmed-Ali et al., 2015a; 2016a-b, 2017a-b, 2018a-b, Karafyllis et al., 2017). Most existing works proposed nonadaptive observers assuming the system model to be fully known. Results on adaptive observer design for PDEs are much less common (compared to nonadaptive observers). Furthermore, a certain portion of the available results were implicitly derived in the framework of output-feedback adaptive controllers (Smyshlyaev and Krstic, 2005, 2007a-b, 2010; Krstic, 2010; Aamo, 2013; Guo and Guo, 2013; Guo, 2014; Bernard and Krstic, 2014; Anfinsen and Aamo, 2015, 2018; Guo et al., 2018). In that context, state and/or parameter online estimations are needed because the control is performed at the boundary. Also, the convergence of the parameter estimates to their true values is not required for the achievement of the control objectives (e.g. typically output-reference tracking). Wave PDEs were considered in (Guo and Guo, 2013) and (Guo et al., 2018) but only the former involved an adaptive observer.

The problem of designing adaptive observers for PDEs, outside the framework of output feedback control, has recently received interest. The problem has been addressed in (Ahmed-Ali et al., 2016c-d) where exponentially convergent adaptive observers have been designed for linear and semilinear parabolic PDEs involving unknown parameters both in the domain and the boundary. In (Ahmed-Ali et al., 2017b), an adaptive observer design has been developed for cascade systems with parabolic PDE sensing. A similar result has been achieved in (Ahmed-Ali et al., 2016e) in the case of 1<sup>st</sup> order hyperbolic PDE sensing. In (Aamo, 2016), (Anfinsen et al., 2016), adaptive observers have been designed for coupled first order linear hyperbolic PDEs involving unknown parameters in the boundaries. The case of 2 × 2 first-order hyperbolic PDEs has been addressed in (Anfinsen and Aamo, 2017).

In the light of the above description, it is seen that quite a few previous works have been devoted to adaptive observer design for hyperbolic PDEs of wave type. In the present study, we develop a new adaptive observer design that applies to wave PDEs with Lipschitz nonlinear dynamics in the domain and unknown parameters in both the domain and the boundaries. The introduction of parameter uncertainty makes it possible to account for uncertain characteristics of the system or of external disturbances acting on its domain or its boundaries. We extend the

(finite-dimensional) decoupling-transformation design technique (Zhang, 2002) to (infinite-dimensional) systems described by hyperbolic PDEs. The first generalization to infinite-dimensional systems has been made in (Ahmed-Ali et al., 2015b, 2016a,c) for systems involving heat type linear PDEs. The novelty of the present study is not only the extension of the approach to wave type PDEs, it is also its generalization to the case where the PDE includes Lipschitz nonlinearities and parameter uncertainty entering all (domain and boundary) equations. The resulting observers include three main components: (i) a state observer which is a copy of the system, augmented with domain and boundary feedback actions; (ii) a parameter adaptive law of the least-squares type; (iii) an auxiliary filter defined by a hyperbolic PDE, that generates a vector signal defining the search direction for the parameter adaptive law. Finally, we show that the adaptive observer is exponentially stable, provided that the vector signal generated by the filter is persistently exciting. Compared to the conference version (Benabdelhadi et al., 2019), the present paper includes all technical proofs which were missing in the former.

The paper is organized as follows: the considered observation problem is stated in Section 2; the adaptive observer design is presented in Section 3 and its analysis is dealt with in Section 4. To alleviate the exposition, all technical proofs are given in the Appendix.

**Notation.** Euclidean norms in  $\mathbf{R}^n$  and  $\mathbf{R}^{n \times m}$  (with  $n > 0$ ) are denoted  $\|\cdot\|$ .  $L^2[0, D]$  is the Hilbert space of square integrable functions  $\eta : [0, D] \rightarrow \mathbf{R}^n$ , for some integer  $n > 0$  and real  $D > 0$ , and  $\|\cdot\|$  is the associated  $L^2$ -norm.  $H^1(0,1)$  (resp.  $H^2(0,1)$ ) is the Sobolev space of absolutely continuous functions  $\eta : [0, D] \rightarrow \mathbf{R}^n$ , some integer  $n > 0$ , with  $d^2\eta/d\zeta^2 \in L^2[0, D]$  (resp.  $d\eta/d\zeta \in L^2[0, D]$ ). The following Wirtinger's inequality holds, for any scalar function  $\varphi \in H^1(0,1)$  such that  $\varphi(0) = 0$  or  $\varphi(1) = 0$ :

$$\sup_{0 \leq x \leq D} \varphi^2(x) \leq D \int_0^D \varphi_x^2(x) dx \quad (1)$$

**2. OBSERVATION PROBLEM STATEMENT**

We are considering nonlinear systems that can be modelled

by the following class of wave PDEs:

$$u_{tt}(x,t) = u_{xx}(x,t) + f(u(x,t), u_t(x,t), u_x(x,t), x, t) + \phi_1(x,t)q_1, \quad \text{for } 0 < x < D, t > 0 \quad (2a)$$

$$u_x(0,t) = -q_2 u(0,t) + \phi_3(t)q_3, \quad t \geq 0 \quad (2b)$$

$$u(D,t) = U(t) + \phi_4(t)q_4, \quad t \geq 0 \quad (2c)$$

with initial condition  $u(\cdot, 0) \in H_u$ ,  $u_t(\cdot, 0) \in L^2(0, D)$ , and

$$H_u = \left\{ \zeta \in H^1(0, D) / \zeta(D) = U(0) + \phi_4(0)q_4(0); \right. \\ \left. \zeta_x(0) = -q_2(0)\zeta(0) + \phi_3(0)q_3(0) \right\}$$

The various notation in (2a-c) have the following meaning:

- .  $D > 0$  is the known domain length;
- .  $u(x,t)$  ( $0 < x < D, t \geq 0$ ) is the unknown distributed state;
- .  $U \in C^0([0, \infty), \mathbf{R})$  is a known input signal;
- .  $f \in C^0(\mathbf{R}^3 \times [0, D] \times [0, \infty); \mathbf{R})$  is known globally Lipschitz with respect to  $u$ ,  $u_x$  and  $u_t$ , uniformly in  $(x, t)$ .
- .  $\phi_i \in C^1([0, D] \times [0, \infty); \mathbf{R}^{1 \times m_i})$  ( $i = 1, 3, 4$ ) are known bounded functions; we will also need  $\dot{\phi}_4$  to be bounded;
- .  $q_i \in \mathbf{R}^{m_i}$  ( $i = 1, 2, 3, 4$ ) are unknown vector parameters.
- .  $m_2 = 1$  and  $1 \leq m_i < \infty$  ( $i = 1, 3, 4$ ) are known integers.

For convenience, we introduce the following augmented (unknown) parameter vector:

$$\theta = [q_1^T \quad q_2^T \quad q_3^T \quad q_4^T]^T \in \mathbf{R}^n, \quad \text{with } n = m_1 + m_2 + m_3 + m_4 \quad (2e)$$

The goal of the study is to design an adaptive observer that provides accurate online estimates,  $\hat{u}(x,t)$  and  $\hat{\theta}(t)$ , of the system distributed state  $u(x,t)$  ( $0 < x < D; t \geq 0$ ) and the parameter vector  $\theta$ , based only on the input and output measurements ( $U(t), y(t); t \geq 0$ ) with:

$$y(t) = [u(0,t), u_t(0,t)], \quad t \geq 0 \quad (\text{system output}) \quad (3)$$

**Remark 1.** 1) As pointed out in (Guo and Guo, 2013), in the case where  $f(u, u_t, u_x, x, t) = 0, q_1 = 0$ , equations (2a-c) model the classical Rijke tube control experiment in combustion dynamics with control of a loudspeaker, and a pressure sensor (microphone) near the speaker, with  $\phi_i(t)q_i$  ( $i = 3, 4$ ) are external disturbances. On the other hand, in the case where  $f(u, u_t, u_x, x, t) = \alpha_0 u - \alpha_1 u_t$  and  $q_i = 0$  ( $i = 1 \dots 4$ ), equation (2a) reduces to the so-called damped sine-Gordon equation (Kobayash, 2003).

2) Compared to (Guo and Guo, 2013) where an adaptive observer has been proposed for linear hyperbolic PDEs of wave type, the present study is more general in the sense that the domain equation includes nonlinear dynamics and parameter uncertainty is in the domain and both boundaries. In (Guo et al., 2018) an adaptive output-feedback controller was proposed that only involved a parameter adaptive law (no adaptive state estimation was required there)

### 3. ADAPTIVE OBSERVER DESIGN

#### 3.1 Observer structure and error system

Consider the following observer structure, which is a copy of the system model (2a-c) with additional inputs:

$$\hat{u}_{tt}(x,t) = \hat{u}_{xx}(x,t) + f(\hat{u}(x,t), \hat{u}_t(x,t), \hat{u}_x(x,t), x, t) + \phi_1(x,t)\hat{q}_1(t) + v_1(x,t) \quad (4a)$$

$$\hat{u}_x(0,t) = -\hat{q}_2(t)u(0,t) + \phi_3(t)\hat{q}_3(t) + a(\hat{u}_t(0,t) - u_t(0,t)) + v_2(t) \quad (4b)$$

$$\hat{u}(D,t) = U(t) + \phi_4(t)\hat{q}_4(t) \quad (4c)$$

where  $\hat{u}(x,t)$  denotes the state estimate;  $\hat{q}_0 \in \mathbf{R}$  and  $\hat{q}_1 \in \mathbf{R}^n$  are parameter estimates yet to be defined;  $a > 0$  is an observer gain;  $v_1(x,t)$  and  $v_2(t)$  are additional feedback functions, introduced to cope with the parameter uncertainty (i.e. they are set to zero in the case of known  $\theta$ ). The gain  $a$ , the couple of feedback functions, and the parameter adaptive law will be selected so that the following state and parameter estimation errors vanish exponentially:

$$\tilde{u}(x,t) = \hat{u}(x,t) - u(x,t) \quad (5a)$$

$$\tilde{q}_i(t) = \hat{q}_i(t) - q_i, \quad (i = 1, 2, 3, 4) \quad (5b)$$

For convenience, introduce the following notations:

$$\hat{\theta}(t) = [\hat{q}_1^T(t) \quad \hat{q}_2^T(t) \quad \hat{q}_3^T(t) \quad \hat{q}_4^T(t)]^T, \quad (5c)$$

$$\tilde{\theta}(t) = \hat{\theta}(t) - \theta \quad (5d)$$

Then, it readily follows from (2e) and (5b-c) that:

$$\tilde{\theta}(t) = [\tilde{q}_1^T(t) \quad \tilde{q}_2^T(t) \quad \tilde{q}_3^T(t) \quad \tilde{q}_4^T(t)]^T \quad (5e)$$

The error system is obtained by subtracting, side by side, equations (2a-c) and the corresponding ones in (4a-c). Doing so, we obtain the following error system:

$$\tilde{u}_{tt}(x,t) = \tilde{u}_{xx}(x,t) + \tilde{f}(x,t) + \phi_1(x,t)\tilde{q}_1(t) + v_1(x,t) \quad (6a)$$

$$\tilde{u}_x(0,t) = -\tilde{q}_2(t)u(0,t) + a\tilde{u}_t(0,t) + \phi_3(t)\tilde{q}_3(t) + v_2(t) \quad (6b)$$

$$\tilde{u}(D,t) = \phi_4(t)\tilde{q}_4(t) \quad (6c)$$

$$\tilde{f}(x,t) = f(\hat{u}(x,t), \hat{u}_t(x,t), \hat{u}_x(x,t), x, t) - f(u(x,t), u_t(x,t), u_x(x,t), x, t) \quad (6d)$$

#### 3.2. Decoupling transformation

Consider the following  $\tilde{u} \rightarrow z$  transformation:

$$z(x,t) = \tilde{u}(x,t) - \lambda(x,t)\tilde{\theta}(t) \quad (7a)$$

with

$$\lambda(x,t) = [\lambda_1(x,t) \quad \lambda_2(x,t) \quad \lambda_3(x,t) \quad \lambda_4(x,t)] \in \mathbf{R}^{1 \times n} \quad (7b)$$

where  $\lambda_i(x,t) \in \mathbf{R}^{m_i}$  ( $i = 1, 2, 3, 4$ ) are auxiliary signals yet to be defined. To this end, the dynamics of the  $z(x,t)$  are identified first. These are obtained from (7a) using (6a):

$$z_{tt} = \tilde{u}_{xx} + \tilde{f} + \phi_1\tilde{q}_1(t) + v_1 - \lambda_{tt}\tilde{\theta} - 2\lambda_t\dot{\tilde{\theta}} - \lambda\ddot{\tilde{\theta}}, \quad t \geq 0$$

where the arguments  $(x,t)$  are, and will continue to be, omitted to alleviate the expressions. Substituting  $z(x,t) + \lambda(x,t)\tilde{\theta}(t)$  to  $\tilde{u}(x,t)$ , the above equality develops as follows:

$$z_{tt} = z_{xx} + \tilde{f} + (\lambda_{xx} + \psi_t - \lambda_{tt})\tilde{\theta} - 2\lambda_t\dot{\tilde{\theta}}(t) - \lambda\ddot{\tilde{\theta}}(t) + v_1 \quad (8a)$$

with

$$\psi_t(x,t) = [\phi_1(x,t) \quad 0_{n-m_1}^T] \in \mathbf{R}^{1 \times n} \quad (8b)$$

with  $0_{n-m_1}$  the zero vector in  $\mathbf{R}^{n-m_1}$ . In view of (8a), we let  $v_1(x, t)$  and  $\lambda(x, t)$  be defined as follows:

$$v_1(t) = 2\lambda_r(x, t)\dot{\tilde{\theta}}(t) + \lambda(x, t)\ddot{\tilde{\theta}}(t) \quad (9)$$

$$\lambda_u(x, t) = \lambda_{xx}(x, t) + \psi_1(x, t) \quad (10)$$

Doing so, equation (8a) boils down to:

$$z_u(x, t) = z_{xx}(x, t) + \tilde{f}(x, t) \quad (11)$$

Using the mean value theorem, (6b), (6d) and (7a) give:

$$\begin{aligned} \tilde{f}(x, t) &= f_1(x, t)\tilde{u}(x, t) + f_2(x, t)\tilde{u}_t(x, t) + f_3(x, t)\tilde{u}_x(x, t) \\ &= f_1(x, t)\left(z(x, t) + \lambda(x, t)\tilde{\theta}(t)\right) \\ &\quad + f_2(x, t)\left(z_t(x, t) + \lambda_t(x, t)\tilde{\theta}(t) + \lambda(x, t)\dot{\tilde{\theta}}(t)\right) \\ &\quad + f_3(x, t)\left(z_x(x, t) + \lambda_x(x, t)\tilde{\theta}(t)\right) \end{aligned} \quad (12a)$$

with

$$f_1(x, t) = \int_0^1 \frac{\partial f}{\partial u}(u(x, t) + (s-1)\tilde{u}(x, t), \tilde{u}_t(x, t), \tilde{u}_x(x, t), x, t) ds$$

$$f_2(x, t) = \int_0^1 \frac{\partial f}{\partial u_t}(u(x, t), u_t(x, t) + (s-1)\tilde{u}_t(x, t), \tilde{u}_x(x, t), x, t) ds$$

$$f_3(x, t) = \int_0^1 \frac{\partial f}{\partial u_x}(u(x, t), u_t(x, t), u_x(x, t) + (s-1)\tilde{u}_x(x, t), x, t) ds$$

Since  $f$  is globally Lipschitz in  $(u, u_t, u_x)$ , one has:

$$|f_i(x, t)| \leq \beta_i \quad (i=1,2,3), \quad \forall x \in [0, D], \forall t \geq 0, \quad (12b)$$

where the  $\beta_i$ 's denote the Lipschitz coefficients of  $f$  with respect to the arguments  $u, u_t$ , and  $u_x$ , respectively. To set the boundary conditions for PDEs (8) and (10), let  $x = D$  in equation (7a):

$$z(D, t) = (\psi_4(t) - \lambda(D, t))\tilde{\theta}(t)$$

using (6c), where  $\psi_4(t) = [0_{n-m_4}^T \quad \phi_4(t)]$ . Then, we let:

$$\lambda(D, t) = \psi_4(t), \quad \forall t \geq 0 \quad (13a)$$

so that we obtain:

$$z(D, t) = 0, \quad \forall t \quad (13b)$$

For  $x = 0$ , equation (7a) and (6b) yield:

$$\begin{aligned} z_x(0, t) &= \tilde{u}_x(0, t) - \lambda_x(0, t)\tilde{\theta}(t) \\ &= az_t(0, t) + (\psi_2(t) + a\lambda_t(0, t) - \lambda_x(0, t))\tilde{\theta}(t) \\ &\quad + v_2(t) + a\lambda(0, t)\dot{\tilde{\theta}}(t) \end{aligned} \quad (14a)$$

with  $\psi_2(t) = [0_{m_1}^T \quad -u(0, t) \quad \phi_3(t) \quad 0_{m_4}^T] \in \mathbf{R}^{1 \times n}$ . Equality (14a) suggests the following definition and boundary condition:

$$v_2(t) = -a\lambda(0, t)\dot{\tilde{\theta}}(t) \quad (14b)$$

$$\lambda_x(0, t) = \psi_2(t) + a\lambda_t(0, t) \quad (14c)$$

Doing so, equation (14a) reduces to:

$$z_x(0, t) = az_t(0, t) \quad (14d)$$

### 3.3. Adaptive parameter law

Consider equation (7a) and write it at  $x=0$  so that we get  $\tilde{u}(0, t) = \lambda(0, t)\tilde{\theta}(t) + z(0, t)$ . Since  $\tilde{u}(0, t)$  and  $\lambda(0, t)$  are known

quantities, the above equations turns out to be affine in the unknown vector  $\tilde{\theta}(t)$ . Therefore, the following least-squares algorithm is a suitable estimator:

$$\dot{\hat{\theta}}(t) = \dot{\tilde{\theta}}(t) = -\frac{R(t)\lambda(0, t)^T \tilde{u}(0, t)}{1 + \lambda(0, t)\lambda(0, t)^T} \quad (15a)$$

$$\dot{R}(t) = \alpha R(t) - \frac{R(t)\lambda(0, t)^T \lambda(0, t)R(t)}{1 + \lambda(0, t)\lambda(0, t)^T} \quad (15b)$$

with initial value  $\hat{\theta}(0)$  and  $\alpha > 0$  a forgetting factor both set by the user, where  $R(t) \in \mathbf{R}^{n \times n}$  is a matrix gain that is expected to be positive definite. To this end, we let  $R(0) = R(0)^T > 0$  and assume the following persistent excitation condition to be true:

**PE assumption.** The vector signal  $\lambda(0, t)$  is persistently exciting (PE), in the sense that,

$$\exists \delta, \varepsilon_0 > 0, \forall t > 0: \int_t^{t+\delta} \frac{\lambda(0, s)^T \lambda(0, s)}{1 + \lambda(0, s)\lambda(0, s)^T} ds > \varepsilon_0 I \quad (16)$$

where  $I$  denotes the identity matrix of the matrix space  $\mathbf{R}^{n \times n}$ . Technically, (16) ensures the existence and positive definiteness of the inverse  $R(t)^{-1}$  (Ioannou and Sun, 2006). Specifically, there exist scalars  $0 < r_0 < r_1 < \infty$  such that:

$$r_0 I < R^{-1}(t) < r_1 I, \quad \forall t \quad (17)$$

It is checked using (15b) that  $R(t)^{-1}$  satisfies the ODE:

$$\dot{R}^{-1}(t) = -\alpha R^{-1}(t) + \frac{\lambda(0, t)^T \lambda(0, t)}{1 + \lambda(0, t)\lambda(0, t)^T} \quad (18)$$

where  $\dot{R}^{-1}(t)$  denotes the time derivative of  $R^{-1}(t)$ .

### 3.4. Complete adaptive observer

The full adaptive observer is constituted of the state estimator defined by (4a-c), (9) and (14b), the parameter adaptive law defined by (15a-b), and the filter defined by (10) and the boundary conditions (13a) and (14c). For convenience, the whole observer thus defined is recapitulated in Table 1.

**Table 1.** Adaptive state observer

*State observer:*

$$\begin{aligned} \hat{u}_u(x, t) &= \hat{u}_{xx}(x, t) + f(\hat{u}(x, t), \hat{u}_t(x, t), \hat{u}_x(x, t), x, t) \\ &\quad + \phi_1(x, t)\hat{q}_1(t) - 2\lambda_r(x, t)\dot{\tilde{\theta}}(t) - \lambda(x, t)\ddot{\tilde{\theta}}(t) \end{aligned} \quad (19a)$$

$$\begin{aligned} \hat{u}_x(0, t) &= -\hat{q}_2(t)u(0, t) + \phi_3(t)\hat{q}_3(t) \\ &\quad + a\tilde{u}_t(0, t) - a\lambda(0, t)\dot{\tilde{\theta}}(t) \end{aligned} \quad (19b)$$

$$\hat{u}(D, t) = U(t) + \phi_4(t)\hat{q}_4(t) \quad (19c)$$

where  $\tilde{u}(0, t) = \hat{u}(0, t) - u(0, t)$ , with any  $a > 0$  and initial condition satisfying  $\hat{u}(\cdot, 0) \in H_{\hat{u}}$ ,  $\hat{u}_t(\cdot, 0) \in L^2(0, D)$  with

$$\begin{aligned} H_{\hat{u}} &= \left\{ \zeta \in H^1(0, D) / \zeta(D) = U(0) + \phi_4(0)\hat{q}_4(0); \right. \\ &\quad \left. \zeta_x(0) = -\hat{q}_2(0)\zeta(0) + \phi_3(0)\hat{q}_3(0) \right\} \end{aligned}$$

Parameter estimator:

$$\dot{\hat{\theta}}(t) = -\frac{R(t)\lambda^T(0,t)}{1+\lambda(0,t)\lambda^T(0,t)}\tilde{u}(0,t); \quad \hat{\theta}(t) \in \mathbf{R}^n \quad (19d)$$

$$\dot{R}(t) = \alpha R(t) - \frac{R(t)\lambda^T(0,t)\lambda(0,t)R(t)}{1+\lambda(0,t)\lambda^T(0,t)}; \quad R(t) \in \mathbf{R}^{n \times n} \quad (19e)$$

with  $\alpha > 0$ ,  $\hat{\theta}(0)$ ,  $R(0) = R(0)^T > 0$  arbitrary.

Filter:

$$\lambda_u(x,t) = \lambda_{xx}(x,t) + \psi_1(x,t); \quad \lambda \in \mathbf{R}^{1 \times n} \quad (19f)$$

$$\lambda_x(0,t) = \psi_2(t) + a\lambda_t(0,t), \quad (19g)$$

$$\lambda(D,t) = \psi_4(t), \quad \forall t \geq 0 \quad (19h)$$

with

$$\psi_1(t) = [\phi_1(x,t) \ 0_{n-m_1}^T] \in \mathbf{R}^{1 \times n} \quad (19i)$$

$$\psi_2(t) = [0_{m_1}^T \ -u(0,t) \ \phi_3(t) \ 0_{m_4}^T] \in \mathbf{R}^{1 \times n} \quad (19j)$$

$$\psi_4(t) = [0_{n-m_4}^T \ \phi_4(t)] \in \mathbf{R}^{1 \times n} \quad (19k)$$

and  $\lambda(\cdot, 0) \in H_\lambda$ ,  $\lambda_t(\cdot, 0) \in L^2(0, D)$  with

$$H_\lambda = \left\{ \zeta \in H^1(0, D) / \zeta(D) = \psi_4(0); \zeta_x(0) = \psi_4(0)\hat{q}_3(0) \right\}$$

**Remark 2.** Like the adaptive observers in (Guo and Guo, 2013) and (Guo et al., 2018), the present adaptive observer requires both  $u(0,t)$  and  $u_t(0,t)$ . Except for this similarity, the present adaptive observer is quite novel. In effect, the main objective in (Guo et al., 2018) was to design a boundary output-feedback control of a wave PDE where parameter uncertainty came in solely through a structured boundary disturbance. Accordingly, the proposed controller only involved a parameter estimator (as no state observer was necessary). A nearly similar control problem was studied in (Guo and Guo, 2013), but the output-feedback controller proposed did involve an adaptive state observer. Again, parameter uncertainty entered only one boundary disturbance, unlike in the present study where parameter uncertainty comes in the domain and both boundary equations. Furthermore, Table 1 shows that the present observer is quite different from existing ones because it involves two innovation actions entering the domain and one boundary equation, while the adaptive observer in e.g. (Guo and Guo, 2013) only involved an innovation action in one of the boundary equations

#### 4. ADAPTIVE OBSERVER ANALYSIS

The analysis will mainly consist in proving that the observer error system  $(z, \tilde{\theta})$  is exponentially stable. As a preliminary result, we establish bounded properties of  $\lambda(x,t)$ .

**Lemma 1.** The signal  $\lambda(x,t)$  generated by the filter (19f-h) is bounded. Furthermore,  $\limsup_{t \rightarrow \infty} \left( \max_{0 < x < 1} \|\lambda(x,t)\| \right) \leq \lambda_M(D)/2$  with

$$\lambda_M(D) \text{ a polynomial satisfying } \lambda_M(0) = 0$$

See Appendix A for the proof of this lemma.

For convenience, let us rewrite equations (11), (13b), (14d), (15a) and (18) which describe the  $(z, \tilde{\theta})$ -system:

$$z_{tt}(x,t) = z_{xx}(x,t) + \tilde{f}(x,t) \quad (20a)$$

$$z(D,t) = 0, \quad \forall t \quad (20b)$$

$$z_x(0,t) = az_t(0,t) \quad (20c)$$

$$\dot{\tilde{\theta}}(t) = -\frac{R(t)\lambda(0,t)^T \lambda(0,t)\tilde{\theta}(t)}{1+\lambda(0,t)\lambda(0,t)^T} - \frac{R(t)\lambda(0,t)^T z(0,t)}{1+\lambda(0,t)\lambda(0,t)^T} \quad (20d)$$

$$\dot{R}^{-1}(t) = -\alpha R^{-1}(t) + \frac{\lambda(0,t)^T \lambda(0,t)}{1+\lambda(0,t)\lambda(0,t)^T} \quad (20e)$$

where (20d) is obtained from (15a), using (7a). The analysis of the above error system is performed by making use of the following Lyapunov functional and function:

$$V_z(t) = \frac{1}{2} \int_0^D z_x^2(x,t) dx + \frac{1}{2} \int_0^D z_t^2(x,t) dx + \delta \int_0^D \left(-2 + \frac{x}{D}\right) z_x(x,t) z_t(x,t) dx \quad (21)$$

$$V_{\tilde{\theta}}(t) = \frac{1}{2} \tilde{\theta}^T(t) R^{-1}(t) \tilde{\theta}(t) \quad (22)$$

with any scalar  $0 < \delta < 1/2$ . It is readily checked that the functional (21) and the function (22) has the following properties:

$$\begin{aligned} & \left( \frac{1}{2} + \delta \right) \left( \int_0^D z_x^2(x,t) dx + \int_0^D z_t^2(x,t) dx \right) \\ & \geq V_z(t) \geq \left( \frac{1}{2} - \delta \right) \left( \int_0^D z_x^2(x,t) dx + \int_0^D z_t^2(x,t) dx \right) \end{aligned} \quad (23)$$

$$\frac{r_0}{2} \|\tilde{\theta}(t)\|^2 \leq V_{\tilde{\theta}}(t) \leq \frac{r_1}{2} \|\tilde{\theta}(t)\|^2 \quad (\text{using 18b}) \quad (24)$$

To state the main result, we also introduce the following notations:

$$\gamma_1 \stackrel{\text{def}}{=} \frac{\delta}{2D} - \frac{\beta_1}{2} - \frac{\beta_1 \lambda_M(D)}{2} - \frac{3\beta_2}{2} - \beta_3 - \delta\beta_2 \quad (25)$$

$$\begin{aligned} \gamma_2 \stackrel{\text{def}}{=} & \frac{\delta}{2D} - \frac{2D\beta_1}{\pi^2} - \beta_2 D r_0^{-2} \lambda_M^2(D) - \delta\beta_1 \\ & - \frac{\beta_3}{2} - \delta\beta_1 \frac{4D}{\pi^2} - \delta\beta_1 \lambda_M(D) - 2\delta\beta_2 \\ & - \delta\beta_2 (1 + 2r_0^{-2} D \lambda_M^2(D)) - 3\delta\beta_3 \end{aligned} \quad (26)$$

$$\begin{aligned} \gamma_3 \stackrel{\text{def}}{=} & \frac{\beta_1 D \lambda_M(D)}{2} + \beta_2 D \lambda_M^3(D) r_0^{-2} \\ & + \delta\beta_1 D \lambda_M(D) + \delta\beta_2 \lambda_M(D) + \frac{\beta_3 \lambda_M(D)}{2} \\ & + 2\delta\beta_2 r_0^{-2} D \lambda_M^3(D) + \frac{\beta_2 \lambda_M(D)}{2} + \delta\beta_3 \lambda_M(D) \end{aligned} \quad (27)$$

where  $\delta$  is as in (21). Then, the main result states as follows:

**Theorem 1.** Consider the system described by (2a-d) subject to PE assumption (16) and the adaptive observer of Table 1. Let us further assume that the system parameters  $(D, \beta_1, \beta_2, \beta_3)$  and the parameter  $a$  are such that the following inequalities hold:

$$\frac{a}{1+a^2} \geq \delta, \quad \gamma_1 > 0, \quad \gamma_2 > 0, \quad 0 < \gamma_4 < 1 \quad (28)$$

with

$$\gamma_4 = \sqrt{\frac{4\gamma_3}{\gamma_0(1-2\delta)}} \sqrt{\frac{2D}{r_0\alpha}}, \quad \gamma_0 = \min(\gamma_1, \gamma_2) \quad (29)$$

Then, the estimation errors  $|\tilde{u}(x,t)|$  ( $0 \leq x \leq D$ ) and  $\tilde{\theta}(t)$  are exponentially converging to zero, as  $t \rightarrow \infty$ , for any initial conditions  $\tilde{\theta}(0)$ ,  $\tilde{u}(\cdot, 0) \in H_{\tilde{u}}$ ,  $\tilde{u}_t(\cdot, 0) \in L^2(0, D)$ , with

$$H_{\tilde{u}} = \left\{ \zeta \in H^1(0, D) / \zeta(D) = \phi_4(0)\tilde{q}_4(0); \right. \\ \left. \zeta_x(0) = -\tilde{q}_2(0)\zeta(0) + \phi_3(0)\tilde{q}_3(0) \right\}$$

The proof of Theorem 1 includes three parts. First, using (21) and (23), we show that the mapping  $\|\tilde{\theta}(t)\| \rightarrow \|[z_x(\cdot, t) z_t(\cdot, t)]\|$  is input-to-state-stable (ISS), where  $z$  and  $\tilde{\theta}$  are governed by equations (20a-e) and using the following notations:

$$\|[z_x(\cdot, t) z_t(\cdot, t)]\|^2 = \|z_x(\cdot, t)\|^2 + \|z_t(\cdot, t)\|^2 \quad (30)$$

$$\|z_x(\cdot, t)\|^2 = \int_0^D z_x^2(x, t) dx, \quad \|z_t(\cdot, t)\|^2 = \int_0^D z_t^2(x, t) dx \quad (31)$$

Then, using (22) and (24), we show that the mapping (inverse)  $\|[z_x(\cdot, t) z_t(\cdot, t)]\| \rightarrow \|\tilde{\theta}(t)\|$  is also ISS. Finally, invoking a small-gain-theorem like argument, we establish the result of Theorem 1. Being a bit long, the detailed proof is placed in Appendix B.

**Remark 3.** 1) In view of (25)-(27), the last three inequalities in (28) entail conditions on the system characteristics  $(D, \beta_1, \beta_2, \beta_3)$ . Accordingly, we cannot have  $D$  and  $\max(\beta_1, \beta_2, \beta_3)$  simultaneously large. If the former is large, the latter must be small and vice-versa. In the particular case of linear systems, i.e. when  $f$  is null in (2a-c), then  $\max(\beta_1, \beta_2, \beta_3)$  is zero and, consequently,  $D$  is allowed to be arbitrarily large. Of course, Theorem 1 provides sufficient conditions for the observer exponential convergence. No formal analysis is currently available of whether the conditions are necessary.

2) The above comment highlights the fact that the presence of the nonlinearity  $f$  entails a limitation on the admissible domain length  $D$ . This limitation stands even in the absence of parameter uncertainty, see e.g. the LMIs in (Fridman, 2013; Fridman and Terushkin, 2016; Terushkin and Fridman, 2019).

3) The first inequality in (28) entails a condition on the observer gain  $a$ . Accordingly, the gain must be positive. The value  $a = 1$  is particularly interesting as then the condition is satisfied for all  $0 < \delta < 1/2$

## 5. SIMULATION

Through this section, the finite difference method is used to simulate both the system and observer, with time step and space step of 0.01 and 0.011, respectively. We used spatial grid size  $N = 91$ . To illustrate the theoretical result of Theorem 1, we consider the following system, which is of the form (2a-e):

$$u_{tt}(x, t) = u_{xx}(x, t) + f(u(x, t) + \phi_1(x, t)q_1), \quad 0 < x < 1, \quad t > 0$$

with the mixed boundary conditions:

$$u_x(0, t) = -q_2u(0, t) + \phi_3(t)q_3, \quad t \geq 0$$

$$u(1, t) = U(t) + \phi_4(t)q_4, \quad t \geq 0$$

where

$$f(u(x, t)) = \beta_1 \sin(u(x, t)),$$

$$\phi_1(x, t) = A_1 \left( \sin(\omega_{11} \frac{x}{D} + \omega_{12})t + \sin(\omega_{13})t \right),$$

$$\phi_i(t) = A_i (\sin(\omega_{i1}t) + \sin(\omega_{i2}t)) \quad (i = 3, 4),$$

$$U(t) = A_0 (\sin(\varpi_1 t) + \sin(\varpi_2 t)).$$

For simulation purpose, the parameters of the system and associated signals are given the following values:

$$q_1 = 0.8, \quad q_2 = 1, \quad q_3 = 0.5, \quad q_4 = 2, \quad \varpi_1 = 3\pi, \quad \varpi_2 = 1.2\pi$$

$$\beta_1 = 2, \quad A_1 = 2, \quad A_0 = A_3 = A_4 = 1,$$

$$\omega_{11} = 0.2\pi, \quad \omega_{12} = 0.24\pi, \quad \omega_{13} = 0.2\pi,$$

$$\omega_{31} = 0.2\pi, \quad \omega_{32} = 0.5\pi, \quad \omega_{41} = 2\pi, \quad \omega_{42} = 1.5\pi.$$

The system initial conditions are  $u(x, 0) = u_0(x) = 2$ ,  $u_t(x, 0) = u_x(x, 0) = 0$ , for all  $x \in [0, 1]$  and  $u(1, 0) = U(0) + \phi_4(0)q_4 = 0$ .

We apply the adaptive observer of Table 1 to get state estimates  $\hat{u}(x, t)$  ( $0 < x < 1$ ) and parameter estimates  $\hat{q}_i$  ( $i = 1 \dots 4$ ). The observer gain is given the value  $a = 1$  and  $\alpha = 0.8$  which proves to be convenient. The observer initial conditions are set as follows:

$$\hat{u}(x, 0) = \hat{u}_x(x, 0) = 0, \quad \hat{u}_t(x, 0) = 0, \quad \text{for all } x \in [0, 1]$$

$$\hat{\theta}(0) = [5 \ 5 \ 5 \ 5]^T \quad \text{and} \quad R(0) = \text{diag}(100, 25, 25, 25).$$

Fig. 1 shows the time evolution of the state estimate  $u(x, t)$  at two particular positions in the domain, as well as the corresponding estimates. Clearly, the estimates get very close to their true variables after a transient period of 4 (s). Fig. 2 shows that the result holds for the state  $u(x, t)$ , for all  $0 < x < 1$ . Fig. 3 shows that the parameter estimation quality is also quite satisfactory as the estimates are accurate after just a transient period of 4 (s). The above observations confirm the theoretical asymptotic performance stated in Theorem 1.

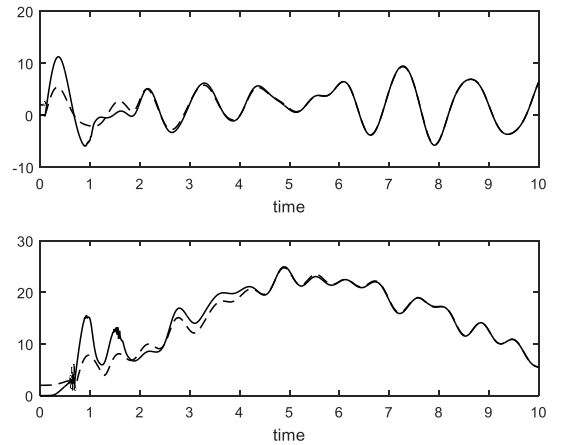


Fig. 1. State  $u(x, t)$  (dashed) and its estimate  $\hat{u}(x, t)$  (solid) for  $x = 0.1$  (top) and  $x = 0.7$  (bottom)

## 6. CONCLUSION

We have investigated the problem of adaptive observer design for the class of hyperbolic systems of wave type described by (2a-c). The complexity lies in the nonlinearity entering the domain and the parameter uncertainty entering the domain equation and both boundary equations. The proposed

observer involves both domain and boundary output injection, and so differs from the other existing adaptive observers which only involve boundary correction. The present research can be pursued in many directions. One topic for possible future investigations is the extension of the proposed adaptive observer design approach to n-D domain PDEs. The case of n-D domains of regular shapes (e.g. rectangular) should not be an issue. However, it is not clear whether the approach can be extended to more general shape domains (of higher dimension).

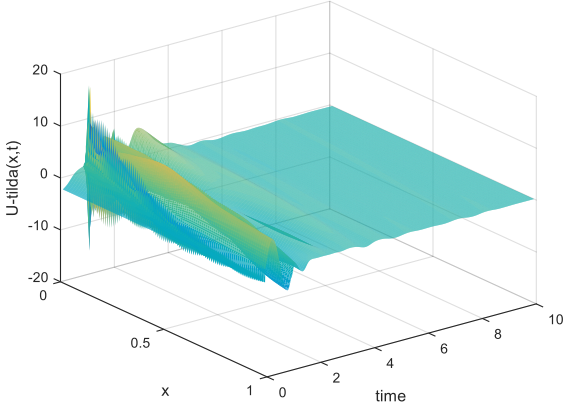


Fig. 2. State estimation error  $\tilde{u}(x,t) = \hat{u}(x,t) - u(x,t)$ ,  $0 < x < 1$

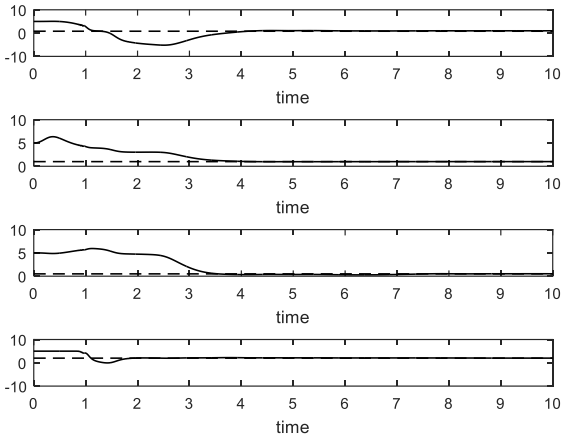


Fig. 3. Fixed value of the parameters  $q_1 \dots q_4$  (dashed), and their estimates (solid):  $q_1$  (top) and  $q_4$  (bottom).

#### APPENDIX A. PROOF OF LEMMA 1

To analyze the system (19f-h), we consider the following Lyapunov functional candidate (similar to (20)):

$$V_\lambda(t) = \frac{1}{2} \int_0^D \|\lambda_x(x,t)\|^2 dx + \frac{1}{2} \int_0^D \|\lambda_t(x,t)\|^2 dx + \eta \int_0^D \left(-2 + \frac{x}{D}\right) \lambda_x^T(x,t) \lambda_t(x,t) dx \quad (A1)$$

with any  $0 < \eta < 1/2$ . It is readily checked that:

$$\left(\frac{1}{2} + \eta\right) \int_0^D \|\lambda_x(x,t)\|^2 dx + \left(\frac{1}{2} + \eta\right) \int_0^D \|\lambda_t(x,t)\|^2 dx \geq V_\lambda(t) \geq \left(\frac{1}{2} - \eta\right) \int_0^D \|\lambda_x(x,t)\|^2 dx + \left(\frac{1}{2} - \eta\right) \int_0^D \|\lambda_t(x,t)\|^2 dx \quad (A2)$$

The time-derivation of  $V_\lambda$  gives, using (19f):

$$\begin{aligned} \dot{V}_\lambda(t) &= \int_0^D \lambda_x^T(x,t) \lambda_{xt}(x,t) dx + \int_0^D \lambda_t^T(x,t) \lambda_{tt}(x,t) dx \\ &\quad + \eta \int_0^D \left(-2 + \frac{x}{D}\right) \lambda_{xt}^T(x,t) \lambda_t(x,t) dx \\ &\quad + \eta \int_0^D \left(-2 + \frac{x}{D}\right) \lambda_x^T(x,t) \lambda_{tt}(x,t) dx \end{aligned} \quad (A3)$$

Applying the integration by part to the first term on the right side of (A3) becomes:

$$\begin{aligned} \int_0^D \lambda_x^T(x,t) \lambda_{xt}(x,t) dx &= \int_0^D \lambda_x^T(x,t) \lambda_{tx}(x,t) dx \\ &= \lambda_x^T(D,t) \psi_4(t) - (\psi_2(t) + a \lambda_t(0,t))^T \lambda_t(0,t) \\ &\quad - \int_0^D \lambda_{xx}^T(x,t) \lambda_t(x,t) dx \\ &\leq \frac{\bar{\zeta}_0}{2} \|\lambda_x(D,t)\|^2 - (a - \frac{\bar{\zeta}_0}{2}) \|\lambda_t(0,t)\|^2 + \frac{\|\dot{\psi}_4(t)\|^2}{2\bar{\zeta}_0} \\ &\quad + \frac{\|\psi_2(t)\|^2}{2\bar{\zeta}_0} - \int_0^D \lambda_{xx}^T(x,t) \lambda_t(x,t) dx \end{aligned} \quad (A4)$$

for any real scalar  $\bar{\zeta}_0, \bar{\zeta}_0 > 0$ , where the first equality is obtained using similar argument as in (Fridman and Terushkin, 2016), the second equality is obtained using an integration by part and (19g-h), the last inequality is obtained applying Young's inequality. Similarly, the remaining terms on the right side of (A3) are bounded as follows:

$$\begin{aligned} \int_0^D \lambda_t(x,t) \lambda_{tt}(x,t) dx &\leq \int_0^D \lambda_t(x,t) \lambda_{xxx}(x,t) dx \\ &\quad + \frac{\zeta_1}{2} \int_0^D \lambda_t^2(x,t) dx + \frac{1}{2\zeta_1} \int_0^D \psi_1^2(x,t) dx \end{aligned} \quad (A5)$$

for any scalar  $\zeta_1 > 0$ , where the last inequality is obtained using Young's inequality.

In turn, the third term on the right side of (A3) develops as follows, applying integration by part:

$$\begin{aligned} \eta \int_0^D \left(-2 + \frac{x}{D}\right) \lambda_{xt}(x,t) \lambda_t(x,t) dx \\ = -\eta \dot{\psi}_4^2(t) + \eta \lambda_t^2(0,t) - \frac{\eta}{D} \int_0^D \lambda_t^2(x,t) dx \end{aligned} \quad (A6)$$

The last term on the right side of (A3) gives, using (19f):

$$\begin{aligned} \eta \int_0^D \left(-2 + \frac{x}{D}\right) \lambda_x(x,t) \lambda_{tt}(x,t) dx \\ = \eta \int_0^D \left(-2 + \frac{x}{D}\right) \lambda_x(x,t) \lambda_{xxx}(x,t) dx \\ - \alpha \eta \int_0^D \left(-2 + \frac{x}{D}\right) \lambda_x(x,t) \lambda_t(x,t) dx \\ + \eta \int_0^D \left(-2 + \frac{x}{D}\right) \lambda_x(x,t) \psi_1(x,t) dx \\ \leq -\frac{\eta}{2} \lambda_x^2(D,t) + 2\eta a^2 \lambda_t^2(0,t) + 2\eta \psi_2^2(t) \\ - \frac{\eta}{2D} \int_0^D \lambda_x^2(x,t) dx \\ + \alpha \eta \zeta_2 \int_0^D \lambda_x^2(x,t) dx + \frac{\alpha \eta}{\zeta_2} \int_0^D \lambda_t^2(x,t) dx \\ + \eta \zeta_3 \int_0^D \lambda_x^2(x,t) dx + \frac{\eta}{\zeta_3} \int_0^D \psi_1^2(x,t) dx \end{aligned} \quad (A7)$$

For any real scalar  $\varsigma_2, \varsigma_3 > 0$ . Using (A4) to (A7) and rearranging terms, inequality (A3) gives:

$$\begin{aligned} \dot{V}_\lambda(t) \leq & -\left(\frac{\eta}{2} - \frac{\varsigma_0}{2}\right) \lambda_x^2(D, t) - \left(a - \frac{\bar{\varsigma}_0}{2} - \eta - 2\eta a^2\right) \lambda_t^2(0, t) \\ & - \left(\frac{\eta}{D} - \frac{\varsigma_1}{2}\right) \int_0^D \lambda_t^2(x, t) dx - \left(\frac{\eta}{2D} - \eta \varsigma_3\right) \int_0^D \lambda_x^2(x, t) dx \\ & + \left(2\eta + \frac{1}{2\bar{\varsigma}_0}\right) \psi_2^2(t) + \left(\frac{1}{2\varsigma_0} - \eta\right) \psi_4^2(t) \\ & + \left(\frac{1}{2\varsigma_1} + \frac{\eta}{\varsigma_3}\right) \int_0^D \psi_1^2(x, t) dx \end{aligned} \quad (\text{A8})$$

Let the free scalar  $\eta$  be small so that:

$$a - \eta - 2\eta a^2 > 0 \quad (\text{A9})$$

Then, we let the remaining free positive scalars be as follows (this is not a unique choice):

$$\varsigma_0 = \frac{\eta}{2}, \quad \bar{\varsigma}_0 = a - \eta - 2\eta a^2, \quad \varsigma_1 = \frac{\eta}{D}, \quad \varsigma_2 = 2\eta, \quad \varsigma_3 = \frac{1}{4D}.$$

Doing so, it is readily checked that:

$$\frac{\eta}{2} - \frac{\varsigma_0}{2} > 0, \quad a - \frac{\bar{\varsigma}_0}{2} - \eta - 2\eta a^2 > 0 \quad (\text{due to (A9)}) \quad (\text{A10})$$

$$\frac{\eta}{D} - \frac{\varsigma_1}{2} = \frac{\eta}{2D} > 0, \quad \frac{\eta}{2D} - \eta \varsigma_3 = \frac{\eta}{4D} > 0 \quad (\text{A11})$$

where the last inequality is a consequence of the second part of (A12). In view of (13)-(A5), it follows from (A11) that:

$$\begin{aligned} \dot{V}_\lambda(t) \leq & -\frac{\eta}{2D} \int_0^D \lambda_t^2(x, t) dx - \frac{\eta}{4D} \int_0^D \lambda_x^2(x, t) dx + \Psi_M(D) \\ \leq & -\frac{\eta}{D(1-2\eta)} V_\lambda(t) + \Psi_M(D) \end{aligned} \quad (\text{A12})$$

using (A2), with  $\Psi_M$  a positive real constant defined as follows:

$$\begin{aligned} \Psi_M(D) = & \left( \left( 2\eta + \frac{1}{2(a - \eta - 2\eta a^2)} \right) \psi_{2M}^2 + \left( \frac{1}{\eta} - \eta \right) \psi_{4d}^2 \right. \\ & \left. + \left( \frac{D}{2\eta} + 4D\eta \right) \psi_{1M}^2 \right) \end{aligned} \quad (\text{A13})$$

$$\psi_{2M}^2 = \sup_t \psi_2^2(t), \quad \psi_{4d}^2 = \sup_t \psi_4^2(t), \quad \psi_{1M}^2 = \sup_{\substack{0 \leq x \leq D \\ t \geq 0}} \psi_1^2(x, t) \quad (\text{A14})$$

Applying the comparison theorem to (A12), we get that

$$\limsup_{t \rightarrow \infty} V_\lambda(t) \leq \frac{D(1-2\eta)}{\eta} \Psi_M(D). \quad \text{The rest of the proof is straightforward using (A2) and (1) ■}$$

## APPENDIX B. PROOF OF THEOREM 1

**Part 1. Proof that  $\|\tilde{\theta}(t)\| \rightarrow \| [z_x(\cdot, t) z_t(\cdot, t) ] \|$  is ISS.**

Time-derivation of  $V_z$  gives, using (20a-c):

$$\begin{aligned} \dot{V}_z(t) = & \int_0^D z_x(x, t) z_{xt}(x, t) dx + \int_0^D z_t(x, t) z_{tt}(x, t) dx \\ & + \delta \int_0^D \left(-2 + \frac{x}{D}\right) z_{xt}(x, t) z_t(x, t) dx \\ & + \delta \int_0^D \left(-2 + \frac{x}{D}\right) z_x(x, t) z_{tt}(x, t) dx \end{aligned} \quad (\text{B1})$$

In the next lines, the different terms on the right side of (B1) will be focused on. The aim is to show that the right side of (B3) is bounded by  $-V_z(t)$ . According to Lemma 1 we have

$\max_{0 < x < 1} \|\lambda(x, t)\| \leq \lambda_M(D)$ , for  $t$  sufficiently large. This result will be used several times in the sequel but, to alleviate the text, it will not be recalled each time that the result holds for  $t$  sufficiently large. The first term on the right side of (B1) is processed as follows:

$$\begin{aligned} \int_0^D z_x(x, t) z_{xt}(x, t) dx &= \int_0^D z_x(x, t) z_{tx}(x, t) dx \\ &= -a z_t^2(0, t) - \int_0^D z_{xx}(x, t) z_t(x, t) dx \end{aligned} \quad (\text{B2})$$

where the first equality is obtained using similar argument as in (Fridman and Terushkin, 2016) and the second by applying an integration by part and the boundary conditions (20b-c). The second term on the right side of (B3) develops as follows, using (20a), (12a-b) and Lemma 1 and applying Young's inequality several times:

$$\begin{aligned} \int_0^D z_t(x, t) z_{tt}(x, t) dx &= \int_0^D z_t(x, t) z_{xx}(x, t) dx \\ &+ \int_0^D f_1(\cdot) z(x, t) z_t(x, t) dx + \int_0^D f_1(\cdot) z_t(x, t) \lambda(x, t) \tilde{\theta}(t) dx \\ &+ \int_0^D z_t(x, t) f_2(x, t) \left( z_t(x, t) + \lambda_t(x, t) \tilde{\theta}(t) + \lambda(x, t) \dot{\tilde{\theta}}(t) \right) dx \\ &+ \int_0^D z_t(x, t) f_3(x, t) \left( z_x(x, t) + \lambda_x(x, t) \tilde{\theta}(t) \right) dx \\ \leq & \int_0^D z_t(x, t) z_{xx}(x, t) dx \\ &+ \frac{\beta_1}{2} \int_0^D z_t^2(x, t) dx + \frac{\beta_1}{2} \int_0^D z^2(x, t) dx \\ &+ \frac{\beta_1 \lambda_M}{2} \int_0^D z_t^2(x, t) dx + \frac{\beta_1 D \lambda_M(D)}{2} \|\tilde{\theta}(t)\|^2 \\ &+ \beta_2 \int_0^D z_t^2(x, t) dx + \frac{\beta_2}{2} \int_0^D z_t^2(x, t) dx + \frac{\beta_2}{2} \lambda_M(D) \|\tilde{\theta}(t)\|^2 \\ &+ \frac{\beta_2}{2} \int_0^D z_t^2(x, t) dx + \frac{\beta_2}{2} D \lambda_M(D) \left( 2\lambda_M^2(D) r_0^{-2} \|\tilde{\theta}(t)\|^2 \right. \\ &\quad \left. + 2\lambda_M(D) r_0^{-2} \int_0^D z_x^2(x, t) dx \right) \\ &+ \frac{\beta_3}{2} \int_0^D z_t^2(x, t) dx + \frac{\beta_3}{2} \int_0^D z_x^2(x, t) dx \\ &+ \frac{\beta_3}{2} \int_0^D z_t^2(x, t) dx + \frac{\beta_3}{2} \lambda_M(D) \|\tilde{\theta}(t)\|^2 \end{aligned} \quad (\text{B3})$$

with  $\lambda_M(D)$  as in Lemma 1, where we have used the inequality:

$$\|\dot{\tilde{\theta}}(t)\|^2 \leq 2r_0^{-2} \lambda_M^2(D) \|\tilde{\theta}(t)\|^2 + 2r_0^{-2} \lambda_M(D) \int_0^D z_x^2(x, t) dx \quad (46)$$

where we have used (20d), (20b) and (1). Applying integration by part, the third term on the right side of (25) writes as follows:

$$\begin{aligned} \delta \int_0^D \left(-2 + \frac{x}{D}\right) z_{xt}(x, t) z_t(x, t) dx \\ = \delta z_t^2(0, t) - \frac{\delta}{2D} \int_0^D z_t^2(x, t) dx \end{aligned} \quad (\text{B5})$$

Using (20a), the last term on the right side of (B1) is bounded from above as follows, where the arguments of  $z$ ,  $\lambda$  and the  $f_i$ 's are omitted to alleviate the expressions:

$$\begin{aligned} & \delta \int_0^D \left(-2 + \frac{x}{D}\right) z_x z_{xx} dx \\ &= \delta \int_0^D \left(-2 + \frac{x}{D}\right) z_x z_{xx} dx + \delta \int_0^D \left(-2 + \frac{x}{D}\right) z_x f_1(\cdot) z dx \\ & \quad + \delta \int_0^D \left(-2 + \frac{x}{D}\right) z_x f_1(\cdot) \lambda \tilde{\theta}(t) dx + \delta \int_0^D \left(-2 + \frac{x}{D}\right) z_x f_2(\cdot) z_t dx \\ & \quad + \delta \int_0^D \left(-2 + \frac{x}{D}\right) z_x f_2(\cdot) \lambda_t \tilde{\theta}(t) dx + \delta \int_0^D \left(-2 + \frac{x}{D}\right) z_x f_2(\cdot) \lambda \tilde{\theta}'(t) dx \\ & \quad + \delta \int_0^D \left(-2 + \frac{x}{D}\right) z_x^2 f_3(\cdot) dx + \delta \int_0^D \left(-2 + \frac{x}{D}\right) z_x f_3(\cdot) \lambda_x \tilde{\theta}(t) dx \quad (B6) \end{aligned}$$

The various terms on the right side of (B8) are further processed in the next lines. Applying an integration by part to the first term one gets, using the boundary conditions (20b-c):

$$\begin{aligned} & \delta \int_0^D \left(-2 + \frac{x}{D}\right) z_x(x,t) z_{xx}(x,t) dx \\ &= -\frac{\delta}{2} z_x^2(D,t) + \delta a^2 z_t^2(0,t) - \frac{\delta}{2D} \int_0^D z_x^2(x,t) dx \quad (B7) \end{aligned}$$

Applying Young's inequality to the two terms containing  $f_1(\cdot)$  on the right side of (B8) and using (12b) and Lemma 1, we respectively get:

$$\begin{aligned} & \delta \int_0^D \left(-2 + \frac{x}{D}\right) z_x(x,t) f_1(\cdot) z(x,t) dx \\ & \leq \delta \beta_1 \int_0^D z_x^2(x,t) dx + \delta \beta_1 \int_0^D z^2(x,t) dx \quad (B8) \end{aligned}$$

$$\begin{aligned} & \delta \int_0^D \left(-2 + \frac{x}{D}\right) z_x(x,t) f_1(\cdot) \lambda(x,t) \tilde{\theta}(t) dx \\ & \leq \delta \beta_1 \lambda_M(D) \int_0^D z_x^2(x,t) dx + \delta \beta_1 D \lambda_M(D) \|\tilde{\theta}(t)\|^2 \quad (B9) \end{aligned}$$

The next three terms containing  $f_2(\cdot)$  on the right side of (B6) are similarly handled. The first of the three gives, applying Young's inequality and using (12b):

$$\begin{aligned} & \delta \int_0^D \left(-2 + \frac{x}{D}\right) z_x(x,t) f_2(\cdot) z_t(x,t) dx \\ & \leq \delta \beta_2 \int_0^D z_x^2(x,t) dx + \delta \beta_2 \int_0^D z_t^2(x,t) dx \quad (B9) \end{aligned}$$

The second term in  $f_2(\cdot)$  on the right side of (B6) yields, using Young's inequality:

$$\begin{aligned} & \delta \int_0^D \left(-2 + \frac{x}{D}\right) z_x(x,t) f_2(\cdot) \lambda_t(x,t) \tilde{\theta}(t) dx \\ & \leq \delta \beta_2 \int_0^D z_x^2(x,t) dx + \delta \beta_2 \lambda_M(D) \|\tilde{\theta}(t)\|^2 \quad (B10) \end{aligned}$$

The last term containing  $f_2(\cdot)$  on the right side of (B6) implies:

$$\begin{aligned} & \delta \int_0^D \left(-2 + \frac{x}{D}\right) z_x(x,t) f_2(\cdot) \lambda(x,t) \tilde{\theta}'(t) dx \\ & \leq \delta \beta_2 \int_0^D z_x^2(x,t) dx + \delta \beta_2 \lambda_M D \|\tilde{\theta}'(t)\|^2 \\ & \leq \delta \beta_2 \left(1 + 2\lambda_M^2 r_0^{-2} D\right) \int_0^D z_x^2(x,t) dx \end{aligned}$$

$$+ 2\delta \beta_2 \lambda_M^3 r_0^{-2} D \|\tilde{\theta}(t)\|^2 \quad (B11)$$

where we have used (B4) in the last inequality.

The last couple terms containing  $f_3(\cdot)$ , on the right side of (B6), are next processed in order. First, the penultimate term is simply bounded from above, using (12b):

$$\delta \int_0^D \left(-2 + \frac{x}{D}\right) z_x^2(x,t) f_3(x,t) dx \leq 2\delta \beta_3 \int_0^D z_x^2(x,t) dx \quad (B12)$$

The last term, on the right side of (B8), is bounded as follows, using (12b) and applying Young's inequality:

$$\begin{aligned} & \delta \int_0^D \left(-2 + \frac{x}{D}\right) z_x(x,t) f_3(x,t) \lambda_x(x,t) \tilde{\theta}(t) dx \\ & \leq \delta \beta_3 \int_0^D z_x^2(x,t) dx + \delta \beta_3 \lambda_M(D) \|\tilde{\theta}(t)\|^2 \quad (B13) \end{aligned}$$

Using (B4) to (B13), we obtain from (B1), rearranging terms:

$$\begin{aligned} \dot{V}_z(t) & \leq -(a - \delta - \delta a^2) z_t^2(0,t) - \frac{\delta}{2} z_x^2(D,t) - \gamma_1 \int_0^D z_t^2(x,t) dx \\ & \quad - \gamma_2 \int_0^D z_x^2(x,t) dx + \gamma_3 \|\tilde{\theta}(t)\|^2 \\ & \leq -\gamma_0 V_z(t) + \gamma_3 \|\tilde{\theta}(t)\|^2 \quad (B14) \end{aligned}$$

with  $\gamma_0 = \min(\gamma_1, \gamma_2)$ , where the last inequality is obtained using (28). It follows from (B14) that:

$$\begin{aligned} V_z(t) & \leq e^{-\gamma_0 t} V_z(0) + \gamma_3 \int_0^t e^{-\gamma_0(t-s)} \|\tilde{\theta}(s)\|^2 ds \\ & \text{which together with (23), (30) and (31) gives} \\ \|[z_x(\cdot, t) \ z_t(\cdot, t)]\|^2 & \leq e^{-\gamma_0 t} \frac{1+2\delta}{1-2\delta} \|[z_x(\cdot, 0) \ z_t(\cdot, 0)]\|^2 \\ & \quad + \frac{2\gamma_3}{1-2\delta} \int_0^t e^{-\gamma_0(t-s)} \|\tilde{\theta}(s)\|^2 ds \quad (B15) \end{aligned}$$

Following a well known argument (see e.g. Appendix C in Folin et al., 2016), it follows from (B15) that, for all  $t > 0$ :

$$\begin{aligned} & \sup_{0 \leq s \leq t} \left( e^{\sigma s/2} \|[z_x(\cdot, s) \ z_t(\cdot, s)]\| \right) \\ & \leq M_0 + \sqrt{\frac{4\gamma_3}{\gamma_0(1-2\delta)}} \sup_{0 \leq s \leq t} \left( e^{\sigma s} \|\tilde{\theta}(s)\| \right) \quad (B16) \end{aligned}$$

for any  $0 < \sigma < \gamma_0/2$ , with

$$M_0 = \sqrt{\frac{1+2\delta}{1-2\delta}} \|[z_x(\cdot, 0) \ z_t(\cdot, 0)]\|. \text{ This proves Part 1.}$$

**Part 2. Proof that the mapping  $\|[z_x(\cdot, t) \ z_t(\cdot, t)]\| \rightarrow \|\tilde{\theta}(t)\|$  is ISS.**

Time-derivation of  $V_{\tilde{\theta}}$  gives, using (23) and (20d-e):

$$\begin{aligned} \dot{V}_{\tilde{\theta}}(t) & = -\alpha V_{\tilde{\theta}}(t) - \frac{1}{2} \frac{(\lambda(0,t) \tilde{\theta}(t))^2}{1 + \lambda(0,t) \lambda(0,t)^T} \\ & \quad - \frac{\tilde{\theta}^T(t) \lambda(0,t)^T}{1 + \lambda(0,t) \lambda(0,t)^T} z(0,t) \quad (\text{using (7a)}) \\ & \leq -\alpha V_{\tilde{\theta}}(t) + \frac{D}{2} \int_0^D z_x^2(x,t) dx \quad (\text{using (20d)}) \quad (B17) \end{aligned}$$

where the last inequality is obtained using successively Young's and Schwarz inequalities. Inequality (B17), together

with (24), implies the ISS property for the mapping  $\| [z_x(\cdot, t) z_t(\cdot, t)] \| \rightarrow \| \tilde{\theta}(t) \|$ , just as (B14) did for the mapping  $\| \tilde{\theta}(t) \| \rightarrow \| [z_x(\cdot, t) z_t(\cdot, t)] \|$ . Accordingly, we get the following inequality, for any  $0 < \sigma < \alpha/2$ :

$$\sup_{0 \leq s \leq t} e^{\sigma s/2} \| \tilde{\theta}(s) \| \leq M_1 + \sqrt{\frac{2D}{r_0 \alpha}} \sup_{0 \leq s \leq t} e^{\alpha s/2} \| [z_x(\cdot, s) z_t(\cdot, s)] \| \quad (\text{B18})$$

with  $M_1 = \sqrt{\frac{r_1}{r_0}} \| \tilde{\theta}(0) \|$ . This proves Part 2.

### Part 3. Proof of the theorem result.

Using (28) and (29), it follows from (B16) and (B18) that for any  $0 < \sigma < \min(\gamma_0/2, \alpha/2)$ :

$$\sup_{0 \leq s \leq t} \left( e^{\sigma s/2} \| [z_x(\cdot, s) z_t(\cdot, s)] \| \right) \leq \frac{1}{1-\gamma_4} \left( M_0 + \sqrt{\frac{4\gamma_3}{\gamma_0(1-2\delta)}} M_1 \right) \quad (\text{B19})$$

$$\sup_{0 \leq s \leq t} e^{\sigma s/2} \| \tilde{\theta}(s) \| \leq \frac{1}{1-\gamma_4} \left( M_1 + \sqrt{\frac{2D}{r_0 \alpha}} M_0 \right) \quad (\text{B20})$$

It readily follows from (B19) and (B20) that  $\| \tilde{\theta}(t) \|$ ,  $\int_0^D z_t^2(x, t) dx$  and  $\int_0^D z_x^2(x, t) dx$  converge exponentially to zero as  $t \rightarrow \infty$ . By (1), it turns out that  $\sup_{0 \leq x \leq D} |z(x, t)|$  also converges to zero as  $t \rightarrow \infty$ . On the other hand, one immediately gets from (7a) using Lemma 1:

$$\sup_{0 \leq x \leq D} |\tilde{u}(x, t)| \leq \sup_{0 \leq x \leq D} |z(x, t)| + \sqrt{\lambda_M} \| \tilde{\theta}(t) \|, \quad \forall t \geq 0 \quad (\text{B21})$$

using Lemma 1. Inequality (B21) implies that  $\sup_{0 \leq x \leq D} |\tilde{u}(x, t)|$  also goes to zero as  $t \rightarrow \infty$ . Theorem 1 is established ■

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